Consideremos el estado inicial de la DTQW:

$$\rho_0 = \frac{1}{2}|\psi_1\rangle\langle\psi_1|\otimes|0\rangle\langle 0| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|\otimes|0\rangle\langle 0| \quad , \quad \langle\psi_1|\psi_2\rangle = 0$$

$$\rho(t) = \frac{1}{2}\left\{ U^t|\psi_1\rangle\langle\psi_1|\otimes|0\rangle\langle 0|(U^\dagger)^t + U^t|\psi_2\rangle\langle\psi_2|\otimes|0\rangle\langle 0|(U^\dagger)^t \right\}$$

$$p(\hat{x}=x_i) = \frac{1}{2}\Big\{\underbrace{\langle x_i|Tr_c(U^t|\psi_1\rangle\langle\psi_1|\otimes|0\rangle\langle 0|(U^\dagger)^t|x_i\rangle}_{\alpha} + \underbrace{\langle x_i|Tr_c(U^t|\psi_2\rangle\langle\psi_2|\otimes|0\rangle\langle 0|(U^\dagger)^t|x_i\rangle}_{\beta}\Big\},$$

$$\theta \mapsto \pi-\theta \qquad \phi \mapsto \phi+\pi$$

$$|\psi_1\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

$$|\psi_2\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\phi+\pi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$$
$$= \sin(\theta/2)|0\rangle - e^{i\phi}\cos(\theta/2)|1\rangle$$

$$\langle\psi_2|\psi_1\rangle = \cos(\theta/2)\sin(\theta/2) - \cos(\theta/2)\sin(\theta/2) = 0$$

$$U|\psi_2\rangle|0\rangle = \sin(\theta/2)U^t|0\rangle|0\rangle + e^{i\phi}\cos(\theta/2)U^t|1\rangle$$

$$U|\psi_2\rangle|0\rangle = \cos(\theta/2)U^t|0\rangle|0\rangle - e^{i\phi}\sin(\theta/2)U^t|1\rangle$$

$$\begin{aligned}\alpha = {} & \cos^2(\theta/2)\langle x_i|Tr_c(U^t|0,0\rangle\langle 0,0|(U^\dagger)^t)|x_i\rangle \\ & + \sin^2(\theta/2)\langle x_i|Tr_c(U^t|1,0\rangle\langle 1,0|(U^\dagger)^t)|x_i\rangle \\ & + \cos(\theta/2)\sin(\theta/2)e^{i\phi}\langle x_i|Tr_c(U^t|0,0\rangle\langle 1,0|(U^\dagger)^t)|x_i\rangle \\ & + \cos(\theta/2)\sin(\theta/2)e^{i\phi}\langle x_i|Tr_c(U^t|0,0\rangle\langle 1,0|(U^\dagger)^t)|x_i\rangle\end{aligned}$$

$$\begin{aligned}\beta = {} & \sin^2(\theta/2)\langle x_i|Tr_c(U^t|0,0\rangle\langle 0,0|(U^\dagger)^t)|x_i\rangle \\ & \cos^2(\theta/2)\langle x_i|Tr_c(U^t|1,0\rangle\langle 1,0|(U^\dagger)^t)|x_i\rangle \\ & - \cos(\theta/2)\sin(\theta/2)e^{i\phi}\langle x_i|Tr_c(U^t|0,0\rangle\langle 1,0|(U^\dagger)^t)|x_i\rangle \\ & - \cos(\theta/2)\sin(\theta/2)e^{i\phi}\langle x_i|Tr_c(U^t|0,0\rangle\langle 1,0|(U^\dagger)^t)|x_i\rangle\end{aligned}$$

$$p(\hat{x}=x_i)=\frac{1}{2}\{\alpha+\beta\}$$
$$=\frac{1}{2}\{(\cos^2\theta/_2+\sin^2\theta/_2)\langle x_i|Tr_c(U^t|0,0\rangle\langle 0,0|(U^t)^t)|x_i\rangle$$
$$+(\sin^2\theta/_2+\cos^2\theta/_2)\langle x_i|Tr_c(U^t|1,0\rangle\langle 1,0|(U^t)^t)|x_i\rangle\}$$
$$=\frac{1}{2}\{\langle x_i|Tr_c(U^t|0,0\rangle\langle 0,0|(U^t)^t)|x_i\rangle$$
$$+\langle x_i|Tr_c(U^t|1,0\rangle\langle 1,0|(U^t)^t)|x_i\rangle\}$$

Esta es una distribución balanceada. Por lo tanto, siempre que el estado inicial de la moneda sea 1/2, la distribución será balanceada. □

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle)$$

$$p(\hat{x}=x_i)=\frac{1}{2}(U|0,0\rangle\langle 0,0|U^t-iU|0,0\rangle\langle 1,0|U^t$$
$$+iU|1,0\rangle\langle 0,0|U^t+U|1,1\rangle\langle 1,1|U^t)$$
$$=\frac{1}{2}(U|0,0\rangle\langle 0,0|U^t+U|1,1\rangle\langle 1,1|U^t)$$

$$Tr_c(U|1,0\rangle\langle 0,0|U^t)=\langle c,x_i|U|1,0\rangle\langle 0,0|U^t|c,x_i\rangle$$
$$=\overline{\langle c,x_i|U|0,0\rangle}\,\overline{\langle 1,0|U^t|c,x_i\rangle}$$

$$-ia+i\bar{a}=i(\bar{a}-a)=i(x-iy-x-iy)=2y=2Im(a)$$

$$\overline{\langle c,q|U|0,0\rangle\langle 1,0|U^t|c',q'\rangle}=\langle 0,0|U^t|c,q\rangle\langle c',q'|U|1,0\rangle$$

$$\overline{\langle c,q|U|c',q'\rangle}=\overline{\langle c,q|SC|c',q'\rangle}$$
$$=\langle c',q'|C^tS^t|c,q\rangle$$

$$C^t=C$$

$$CS^t$$

$$S=\sum|0,i+1\rangle\langle 0,i|+|1,i-1\rangle\langle 1,i|$$
$$S^t=\sum|0,i\rangle\langle 0,i+1|+|1,i\rangle\langle 1,i-1|$$

$$C|i\rangle=|0\rangle+(-1)^i|1\rangle$$

$$=\frac{1}{\sqrt{2}}(\langle 0|+(-1)^{c'}\langle 1|)\langle q'|S^t|c,q\rangle$$
$$=\frac{1}{\sqrt{2}}(\langle 0,q'+1|+(-1)^{c'}\langle 1,q'-1|)|c,q\rangle$$
$$=\frac{1}{\sqrt{2}}(\delta_{c,0}\delta_{q'+1,q}+(-1)^{c'}\delta_{1,c}\delta_{q'-1,q})$$
$$=\langle c,q|SC|c',q\rangle$$