$U = S(C \otimes \mathbb{1})$ $\quad |\psi_0\rangle = |c_0\rangle \otimes |p_0\rangle$ $\quad (AB)^\dagger = B^\dagger A^\dagger$

$\mathrm{Tr}_c\, U|\psi_0\rangle\langle\psi_0|U^\dagger = \mathrm{Tr}_c\left(S C\otimes\mathbb{1}\, |c_0\rangle\langle c_0| \otimes |p_0\rangle\langle p_0|\, C^\dagger \otimes \mathbb{1}\, S^\dagger\right)$

$S = |0\rangle\langle 0| \otimes \sum_i |i+1\rangle\langle i| + |1\rangle\langle 1| \sum_i |i-1\rangle\langle i| \qquad S^\dagger = |0\rangle\langle 0| \otimes \sum_i |i\rangle\langle i+1| + |1\rangle\langle 1| \otimes \sum_i |i\rangle\langle i-1|$

$C = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)$

$C \otimes \mathbb{1} = C \otimes \sum_i |i\rangle\langle i|$

$S(C\otimes\mathbb{1}) = \frac{1}{\sqrt{2}}\left\{\left(|0\rangle\langle 0| + |0\rangle\langle 1|\right) \otimes \sum_i |i+1\rangle\langle i| + \left(|1\rangle\langle 0| - |1\rangle\langle 1|\right) \otimes \sum_i |i-1\rangle\langle i|\right\}$

$[S(C\otimes\mathbb{1})]^\dagger = (C^\dagger \otimes \mathbb{1})\, S^\dagger = \frac{1}{\sqrt{2}}\left\{\left(|0\rangle\langle 0| + |1\rangle\langle 0|\right) \otimes \sum_i |i\rangle\langle i+1| + \left(|0\rangle\langle 1| - |1\rangle\langle 1|\right) \otimes \sum_i |i\rangle\langle i-1|\right\}$

$\mathrm{Tr}_c\, U|\psi_0\rangle\langle\psi_0|U^\dagger = \langle 0|_c\, U|\psi_0\rangle\langle\psi_0|U^\dagger|0\rangle_c + \langle 1|_c\, U|\psi_0\rangle\langle\psi_0|U^\dagger|1\rangle_c$

$\langle 0|S(C\otimes\mathbb{1})|c\rangle = \frac{1}{\sqrt{2}}\left(\langle 0|c_0\rangle + \langle 1|c_0\rangle\right) \sum_i |i+1\rangle\langle i| = K_0$

$\langle 1|_c S(C\otimes\mathbb{1})|c_0\rangle = \frac{1}{\sqrt{2}}\left(\langle 0|c_0\rangle - \langle 1|c_0\rangle\right) \sum_i |i-1\rangle\langle i| = K_1$

$\langle 0|_c\, S(C\otimes\mathbb{1})|c_0\rangle |p_0\rangle\langle p_0| \langle c_0|\, (C^\dagger\otimes\mathbb{1}) S^\dagger |0\rangle_c = \frac{1}{2}\left(\langle 0|c_0\rangle + \langle 1|c_0\rangle\right) \sum_{ij} |i+1\rangle\langle i|$

$K_0^\dagger K_0 = \frac{1}{2}\left(\langle c_0|0\rangle + \langle c_0|1\rangle\right)\left(\langle 0|c_0\rangle + \langle 1|c_0\rangle\right) \sum_{ij} |i\rangle\langle i+1|j+1\rangle\langle j|$

$= \frac{1}{2}\left(|\langle 0|c_0\rangle|^2 + |\langle 1|c_0\rangle|^2 + \langle c_0|1\rangle\langle 0|c_0\rangle + \langle c_0|0\rangle\langle 1|c_0\rangle\right) \sum |i\rangle\langle i|$

$K_1^\dagger K_1 = \frac{1}{2}\left(\langle c_0|0\rangle - \langle c_0|1\rangle\right)\left(\langle 0|c_0\rangle - \langle 1|c_0\rangle\right) \sum_{ij} |i\rangle\langle i-1|j-1\rangle\langle j|$

$= \frac{1}{2}\left(|\langle 0|c_0\rangle|^2 + K_1|c_0\rangle|^2 - \langle c_0|1\rangle\langle 0|c_0\rangle - \langle c_0|0\rangle\langle 1|c_0\rangle\right) \sum_i |i\rangle\langle i|$

$K_0^\dagger K_0 + K_1^\dagger K_1 = \mathbb{1} \qquad K_i = \langle i|_c S(C\otimes\mathbb{1})|c_0\rangle$

$U|\psi_0\rangle = S(C\otimes\mathbb{1})\, \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)\otimes|0\rangle = \frac{1}{2} S\left(|0\rangle + |1\rangle + i|0\rangle - i|1\rangle\right)\otimes|0\rangle$

$= \frac{1}{2}\left(|01\rangle + |1,-1\rangle + i|01\rangle - i|1,-1\rangle\right)$

$= \frac{1}{2}\left\{(1+i)|01\rangle + (1-i)|1,-1\rangle\right\} = \frac{1}{\sqrt{2}}\left(e^{i\pi/4}|0,1\rangle + e^{-i\pi/4}|1,-1\rangle\right)$

$= e^{i\pi/4}\frac{1}{\sqrt{2}}\left(|0,1\rangle + e^{-i\pi/2}|1,-1\rangle\right) = e^{i\pi/4}\frac{1}{\sqrt{2}}\left(|0,1\rangle - i|1,-1\rangle\right)$

$|\psi_1\rangle = e^{i\pi/4}/\sqrt{2}\,(|0,1\rangle - i|1,-1\rangle)$ $\quad |\psi_1\rangle\langle\psi_1| = \frac{1}{2}(|0,1\rangle\langle 0,1| - i|1,-1\rangle\langle 0,1|$

$+ i|0,1\rangle\langle 1,-1| + |1,-1\rangle\langle 1,-1|)$

$\mathrm{Tr}_c|\psi_2\rangle\langle\psi_2| = \mathrm{Tr}_c\, U|\psi_1\rangle\langle\psi_1|\,U^\dagger$

$|\psi_2\rangle = S(C\otimes\mathbb{1})\,e^{i\pi/4}/\sqrt{2}\,(|0,1\rangle - i|1,-1\rangle)$

$= e^{i\pi/4}/2\; S(|0,1\rangle + |1,1\rangle - i|0,-1\rangle + i|1,-1\rangle)$

$= e^{i\pi/4}/2\,(|0,2\rangle + |1,0\rangle - i|0,0\rangle + i|1,-2\rangle)$

$\mathrm{Tr}_c(U|\psi_1\rangle\langle\psi_1|U^\dagger) = \Lambda(\rho_p^{(1)})$, $\quad \rho_p^{(1)} = \mathrm{Tr}_c|\psi_1\rangle\langle\psi_1|$ $\quad$ ∃Λ?

$$\begin{array}{ccc} \mathrm{Tr}_c|\psi_1\rangle\langle\psi_1| & \longrightarrow & \mathrm{Tr}_c|\psi_2\rangle\langle\psi_2| \\ \uparrow & \overset{\Lambda}{\nearrow} & \uparrow \\ |\psi_1\rangle\langle\psi_1| & \underset{U}{\longmapsto} & |\psi_2\rangle\langle\psi_2| \end{array}$$

$|\psi_1\rangle = a|0,-1\rangle + b|0,0\rangle + c|0,1\rangle + d|1,-1\rangle + e|1,0\rangle + f|1,1\rangle$

$U|\psi_1\rangle = 1/\sqrt{2}\,\{a(|0,0\rangle + |1,-2\rangle) + b(|0,1\rangle + |1,-1\rangle) + c(|0,2\rangle + |1,0\rangle)$

$+ d(|0,0\rangle - |1,-2\rangle) + e(|0,1\rangle - |1,-1\rangle) + f(|0,2\rangle - |1,0\rangle) = |\psi_2\rangle$

$\mathrm{Tr}_c|\psi_2\rangle\langle\psi_2| =$

$|\psi_t\rangle = U^t|\psi_0\rangle$

$$\rho_0 \overset{\mathcal{E}}{\longmapsto} \rho_1 \overset{\mathcal{E}}{\longmapsto} \rho_2$$

$|\psi_0\rangle = 1/\sqrt{2}\,(|0\rangle + i|1\rangle)\otimes|0\rangle$ $\quad K_0 = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + i/\sqrt{2}\right)\sum_i |i+1\rangle\langle i|$

$= \frac{1}{2}(1+i)\sum_i |i+1\rangle\langle i|$

$K_1 = \frac{1}{2}(1-i)\sum_i |i-1\rangle\langle i|$

$|\psi_1\rangle = U|\psi_0\rangle = \frac{1}{2}(|0,1\rangle + |1,-1\rangle + i|0,1\rangle - i|1,-1\rangle)$

$\mathrm{Tr}_c|\psi_1\rangle\langle\psi_1| = \frac{1}{2}(|-1\rangle\langle -1| + |1\rangle\langle 1|)$

$K_0|0\rangle = \frac{1}{2}(1+i)|1\rangle$ $\quad K_1 = \frac{1}{2}(1-i)|-1\rangle$

$\langle 0|K_0^\dagger = \frac{1}{2}(1-i)\langle 1|$ $\quad K_1 = \frac{1}{2}(1+i)\langle -1|$

$$\mathcal{E}(|0\rangle\langle 0|) = \frac{1}{2}(|-1\rangle\langle -1| + |1\rangle\langle 1|)$$

$$U|\psi_1\rangle = \frac{1}{2}U\{(1+i)|0,1\rangle + (1-i)|1,-1\rangle\}$$
$$= \frac{1}{2\sqrt{2}}\{(1+i)(|0,2\rangle + |1,0\rangle) + (1-i)(|0,0\rangle - |1,-2\rangle)\}$$
$$= \frac{1}{2\sqrt{2}}$$

$K_0|-1\rangle = \frac{1}{2}(1+i)|0\rangle$ $\quad K_1|-1\rangle = \frac{1}{2}(1-i)|-2\rangle$

$K_0|1\rangle = \frac{1}{2}(1+i)|2\rangle$ $\quad K_1|1\rangle = \frac{1}{2}(1-i)|0\rangle$

$$\frac{1}{2}\{K_0|1\rangle\langle 1|K_0 + K_0|-1\rangle\langle -1|K_0 + K_1|1\rangle\langle 1|K_1^\dagger + K_1|-1\rangle\langle -1|K_1^\dagger\}$$
$$= \frac{1}{2}(|2\rangle\langle 2| + |0\rangle\langle 0| + |0\rangle\langle 0| + |-2\rangle\langle -2|)$$

$$\rho_0 \xrightarrow{\mathcal{E}} \rho_1 \xrightarrow{\cancel{\mathcal{E}}} \rho_2$$