2874#

CHAÎNE D'ISING en CHAMPS TRANSVERSE

$$H = 4 \sum_m \left[S_m^x S_{m+1}^x + \frac{\lambda}{2} S_m^z \right] = \sum_m \left(\sigma_m^x \sigma_{m+1}^x + \lambda \sigma_m^z \right)$$

$$= 4 \sum_m \left[\frac{1}{4} \left(S_m^+ + S_m^- \right) \left(S_{m+1}^+ + S_{m+1}^- \right) + \frac{\lambda}{2} S_m^z \right] \qquad S_+ = S^x + i S^y$$

$$= \sum_m \left[\left(S_m^+ S_{m+1}^+ + h.c \right) + \left(S_m^+ S_{m+1}^- + h.c \right) + 2\lambda S_m^z \right]$$

Jordan-Wigner : $S_m^+ = c_m^+ e^{i\pi N_m}$, $\quad N_m = \sum_{i<m} c_i^+ c_i$

$$S_m^- = e^{-i\pi N_m} c_m$$

$$S_m^+ S_{m+1}^+ = c_m^+ e^{i\pi N_m} c_{m+1}^+ e^{i\pi N_{m+1}} = c_m^+ c_{m+1}^+ \quad \text{0 si } N_n \neq N_{n+1}$$

$$S_m^+ S_{m+1}^- = c_m^+ e^{i\pi N_m} e^{-i\pi N_{m+1}} c_{m+1} = c_m^+ c_{m+1}$$

Rem : on n'aurait une telle simplification avec couplage aux 2ème voisins.

Chaîne de longueur L $\quad m = 0, \dots, L-1$.

$$S_{L-1}^+ S_0^+ = c_{L-1}^+ e^{i\pi N_{L-1}} c_0^+ = c_{L-1}^+ c_0^+ \left(-e^{i\pi N_L} \right)$$

$$S_{L-1}^+ S_0^- = c_{L-1}^+ e^{i\pi N_{L-1}} c_0 = c_{L-1}^+ c_0 \underbrace{\left(-e^{i\pi N_L} \right)}$$

← opérateur en fait

N_L : nb total de fermion

⚠ effet de taille finie... négligé dans la suite

$$H = \sum_m \left[\left(c_m^+ c_{m+1}^+ + h.c \right) + \left(c_m^+ c_{m+1} + h.c \right) + 2\lambda \left(c_m^+ c_m - \frac{1}{2} \right) \right]$$

$$c_m^+ = \frac{1}{\sqrt{L}} \sum_k e^{-ikm} c_k^+$$

$$H = \sum_k \left[\left(e^{ik} c_k^+ c_{-k}^+ + h.c \right) + \left(e^{ik} c_k^+ c_k + h.c \right) + 2\lambda \left(c_k^+ c_k - \frac{1}{2} \right) \right]$$

moyenne des contributions à k et $-k$

$$H = \sum_k \Bigg[\overbrace{\frac{1}{2}\left(e^{ik} c^+_k c^+_{-k} + e^{-ik} \underbrace{c^+_{-k} c^+_k}_{-c^+_k c^+_{-k}} + h.c\right)}^{} + \underbrace{\left(e^{ik} c^+_k c_k + e^{-ik} c^+_k c_k\right)}_{2\cos k\, c^+_k c_k} + 2\lambda\left(c^+_k c_k - \frac{1}{2}\right)\Bigg]$$

$$\frac{1}{2}\left(e^{ik} - e^{-ik}\right) c^+_k c^+_{-k} + h.c$$

$$\boxed{H = \sum_k \Big[\left(i \sin(k)\, c^+_k c^+_{-k} + h.c\right) + 2 c^+_k c_k \left(\cos(k) + \lambda\right) - \lambda \Big]}$$

Rotation de Bogolioubov

$$\begin{cases} c_k = \left(\cos\theta_k\, d_k + \sin\theta_k\, d^+_{-k}\right) e^{i\pi/4} \\ c^+_{-k} = \left(-\sin\theta_k\, d_k + \cos\theta_k\, d^+_{-k}\right) e^{-i\pi/4} \end{cases}$$

$$M = \begin{bmatrix} c e^{i\pi/4} & s e^{i\pi/4} \\ -s e^{-i\pi/4} & c e^{-i\pi/4} \end{bmatrix}$$

$$\begin{cases} d_k = \cos\theta_k\, c_k e^{-i\pi/4} - \sin\theta_k\, c^+_{-k} e^{i\pi/4} \\ d^+_{-k} = \sin\theta_k\, c_k e^{-i\pi/4} + \cos\theta_k\, c^+_{-k} e^{i\pi/4} \end{cases} \qquad \theta_{-k} = -\theta_k$$

$$d^+_k = \cos\theta_k\, c^+_k e^{i\pi/4} - \sin\theta_k\, c_{-k} e^{-i\pi/4}$$

$$M^{-1} = \begin{bmatrix} c e^{-i\pi/4} & -s e^{+i\pi/4} \\ s e^{-i\pi/4} & c e^{i\pi/4} \end{bmatrix}$$

$$c^+_k c^+_{-k} = \left(c d^+_k + s d_{-k}\right) e^{-i\pi/4} \left(-s d_k + c d^+_{-k}\right) e^{-i\pi/4} = (-i)\left[c^2 d^+_k d^+_{-k} - s^2 d_{-k} d_k + cs\left(d_{-k} d^+_{-k} - d^+_k d_k\right)\right]$$

$$c^+_k c_k = \left(c d^+_k + s d_{-k}\right)\left(c d_k + s d^+_{-k}\right) = \left[cs\left(d^+_k d^+_{-k} + d_{-k} d_k\right) + c^2 d^+_k d_k + s^2 d_{-k} d^+_{-k}\right]$$

$$H = \sum_k \Bigg[d_k^\dagger d_{-k}^\dagger \Big\{ \overbrace{[(-i)c^2]\, i\sin(k) + [(-i)(-s^2)]\, i\sin(k)}^{\sin(k)(c^2 - s^2)} + [cs]\, 2(\cos k + \lambda) \Big\} + \underset{\uparrow\atop d_{-k}d_k}{h.c}$$

$$+ d_k^\dagger d_k \Big\{ 2[(-i)(-cs)]\, i\sin(k) + 2(\cos k + \lambda) c^2 \Big\}$$

$$+ d_{-k} d_{-k}^\dagger \Big\{ 2[(-i)\, cs]\, i\sin(k) + 2(\cos k + \lambda) s^2 \Big\}$$

$$- \lambda \Bigg]$$

$$H = \sum_k \Bigg[d_k^\dagger d_{-k}^\dagger \Big\{ \sin(k)\cos(2\theta_k) + (\cos(k) + \lambda)\sin(2\theta_k) \Big\} + h.c$$

$$+ d_k^\dagger d_k \Big\{ -\sin(k)\sin(2\theta_k) + 2(\cos k + \lambda)\cos^2\theta_k \Big\}$$

$$+ (1 - d_{-k}^\dagger d_{-k}) \Big\{ \sin(k)\sin(2\theta_k) + 2(\cos k + \lambda)\sin^2\theta_k \Big\} - \lambda \Bigg]$$

$$\underset{k \to -k}{\rightsquigarrow} (1 - d_k^\dagger d_k) \Big\{ \sin(k)\sin(2\theta_k) + 2(\cos k + \lambda)\sin^2\theta_k$$

$$H = \sum_k \Bigg[d_k^\dagger d_k \Big\{ -2\sin(k)\sin(2\theta_k) + 2(\cos k + \lambda)\overbrace{(\cos^2\theta_k - \sin^2\theta_k)}^{\cos 2\theta_k} \Big\}$$

$$+ \sin(k)\sin(2\theta_k) + 2(\cos k + \lambda)\sin^2\theta_k - \lambda \Bigg]$$

Avec : $\sin(k)\cos(2\theta_k) + (\cos(k) + \lambda)\sin(2\theta_k) = 0$

$$\sin(k) + (\cos(k) + \lambda)\tan 2\theta_k = 0$$

$$\Rightarrow \tan 2\theta_k = -\frac{\sin(k)}{\cos(k) + \lambda} = \kappa(k)$$

Relation de dispersion :

$$\frac{1}{2}\varepsilon(k) = -\sin(k)\sin(2\theta_k) + (\cos k + \lambda)\cos 2\theta_k .$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$1 + \tan^2 x = \frac{1}{\cos^2 x} \qquad \cos x = \pm \frac{1}{\sqrt{1 + \tan^2 x}} \qquad \sin x = \pm \frac{\tan x}{\sqrt{1 + \tan^2 x}} = \tan x \cdot \cos x$$

$$1 + \frac{1}{\tan^2 x} = \frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$\sin(2\theta_k) = \pm \frac{\alpha(k)}{\sqrt{1+\alpha^2(k)}} \qquad \cos(2\theta_k) = \pm \frac{1}{\sqrt{1+\alpha^2(k)}}$$

$$\frac{1}{2}\varepsilon(k) = \pm \frac{1}{\sqrt{1+\frac{\sin^2(k)}{(\cos(k)+\lambda)^2}}}\left(+ \frac{\sin^2(k)}{\cos(k)+\lambda} + \cos(k)+\lambda \right)$$

$$= \text{sign}\left(\cos(2\theta_k)\right)$$

$$\frac{1}{2}\varepsilon(k) = \text{sign}\left(\cos(2\theta_k)\right)\,\text{sign}\left(\cos(k)+\lambda\right)\frac{(\cos(k)+\lambda)^2+\sin^2(k)}{\sqrt{(\cos(k)+\lambda)^2+\sin^2(k)}}$$

$$\frac{1}{2}\varepsilon(k) = \text{sign}\left(\cos(2\theta_k)\right)\,\text{sign}\left(\cos(k)+\lambda\right)\sqrt{2\cos(k)\lambda+1+\lambda^2}$$

$$\tan 2\theta_k = -\frac{\sin(k)}{\cos(k)+\lambda} = \alpha_k \qquad 2\theta_k = \text{Arctan}\left(\alpha_k\right)$$

$$\Downarrow$$

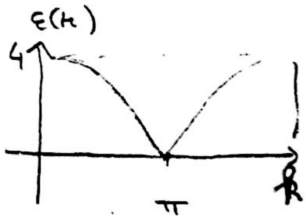
$\exists$ solution $2\theta_k \in \left]-\pi/2, \pi/2\right[$

$$\Downarrow$$

$\cos 2\theta_k > 0.$

$$\boxed{\frac{1}{2}\varepsilon(k) = \text{sign}\left(\cos(k)+\lambda\right)\sqrt{2\cos(k)\lambda+1+\lambda^2}}$$

Au point critique $\underline{\lambda = 1}$. $\quad \frac{1}{2}\varepsilon(k) = \sqrt{2}\sqrt{\cos(k)+1}$



$\lambda = 0$: modèle Ising

$\lambda = \pm 1$: transition phase

$\lambda \in \left]-1;1\right[$ = "~~pseudo~~ phase du modèle Ising"

$|\lambda| > 1$: "phase du champ transverse" cad 0 fermion par état

$$\boxed{\text{Energie du fondamental}} \quad |0\rangle$$

Cas $\lambda \geq 1$ $\qquad \varepsilon(k)$ toujours $\geq 0 \Rightarrow \langle 0| d^+ d |0\rangle = 0$

provient de la cte seulement

$$E_0 = \sum_k \left[\sin(k)\sin(2\theta_k) + 2(\cos k + \lambda)\sin^2\theta_k - \lambda \right]$$

$$\sin(2\theta_k) = \frac{\frac{-\sin(k)}{\cos(k)+\lambda}}{\sqrt{1 + \frac{\sin^2(k)}{(\cos(k)+\lambda)^2}}}$$

$$= \operatorname{sign}(\cos(k)+\lambda) \frac{-\sin(k)}{\sqrt{(\cos(k)+\lambda)^2 + \sin^2(k)}}$$

$$= \operatorname{sign}(\cos(k)+\lambda) \frac{-\sin(k)}{\sqrt{2\lambda\cos(k) + \lambda^2 + 1}}$$

$$\sin(2\theta_k) = \frac{-2\sin(k)}{\varepsilon(k)}$$

$$\sin^2\theta_k = \frac{1}{2}\left(1 - \cos(2\theta_k)\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + \frac{\sin^2 k}{(\cos k + \lambda)^2}}}\right) = \frac{1}{2}\left(1 - \operatorname{sign}(\cos k + \lambda)\frac{\cos(k)+\lambda}{\sqrt{2\lambda\cos(k) + \lambda^2 + 1}}\right)$$

$$= \frac{1}{2}\left(1 - 2\frac{\cos(k)+\lambda}{\varepsilon(k)}\right)$$

$$E_0 = \sum_k \left[-2\frac{\sin^2(k)}{\varepsilon(k)} + \cos(k) + \cancel{\lambda} - 2\frac{(\cos(k)+\lambda)^2}{\varepsilon(k)} - \cancel{\lambda} \right] = \sum_k \left[\cos(k) - \frac{|2\lambda\cos(k)+\lambda^2+1|}{\sqrt{2\lambda\cos(k)+\lambda^2+1}} \right]$$

$$E_0 = \sum_k \left[\cancel{\cos(k)} - \sqrt{2\lambda\cos(k) + \lambda^2 + 1} \right]$$

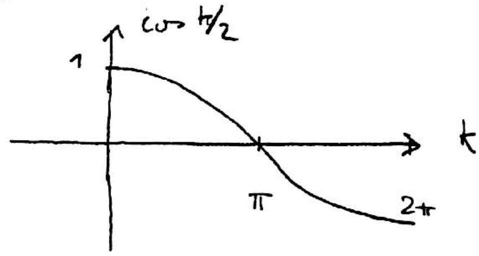
si $N \gg 1$

$$\boxed{\frac{E_0}{L} = -\int_0^{2\pi} \frac{dk}{2\pi} \sqrt{2\lambda\cos(k) + \lambda^2 + 1} = -\frac{1}{2}\int_0^{2\pi} \frac{dk}{2\pi} |\varepsilon(k)|}$$

Energie du fondamental au point critique $\lambda = 1$

$$\frac{E_0}{L} = -\sqrt{2}\int_0^{2\pi} \frac{dk}{2\pi} \sqrt{\cos(k) + 1}$$

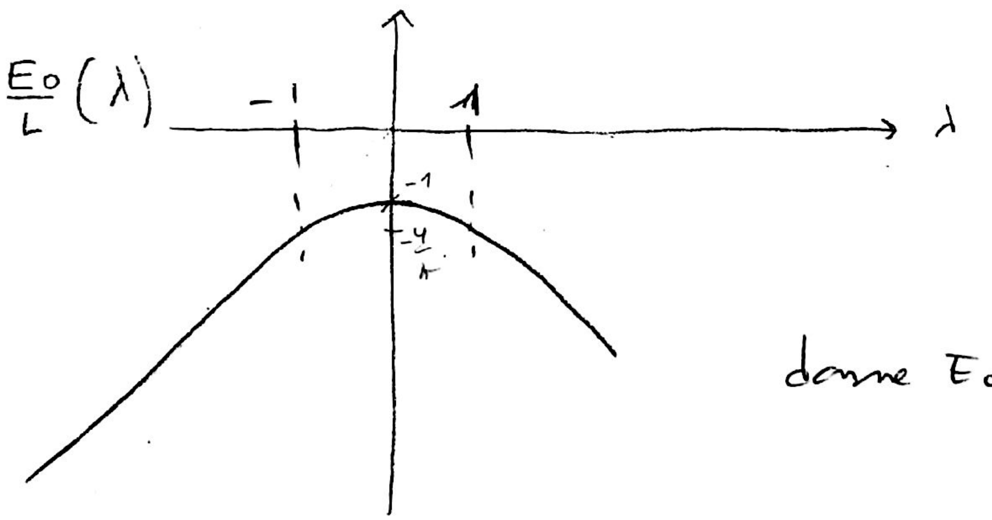
$$2\cos^2\left(\frac{x}{2}\right) = 1 + \cos x$$



$$\frac{E_0}{L} = -\sqrt{2}\int^{2\pi} \frac{dk}{2\pi} \sqrt{2\cos^2\frac{k}{2}}$$

$$= -4\int_0^{\pi} \frac{dk}{2\pi} \cos\left(\frac{k}{2}\right)$$

$$= -\frac{2}{\pi}\left[2\sin\left(\frac{k}{2}\right)\right]_0^{\pi} = \boxed{-\frac{4}{\pi}} =$$



donne E_0 si $|\lambda| > 1$