Model of spins on Lattice

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Classement: Spins sur réseau

Contents

References:

- Book of Sachdev: model of Ising spin on 1Dim lattice in transverse magnetic field. This is an exact soluble model.

1 Definitions

Consider a **périodique 1D lattice with N discrete site:**

$$i = 1 \to N$$

On each site there is a spin 1/2.

The total Hilbert space is then

$$\mathcal{H}_{tot} = \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{N}$$

One has $\dim \mathcal{H}_{tot} = 2^N$.

The standard basis is composed by states of the form:

$$|s_1, s_2, \ldots, s_N\rangle, \qquad s_i = 0 \text{ or } 1$$

In particular the state $|0\rangle = |0, \ldots, 0\rangle$ corresponds to all the spin down.

1.0.1 Relations of SU(2) algebra

$$|z\rangle = \exp\left(z\hat{J}_+\right)|0\rangle : \text{ coherent state}$$

$$\langle z|z\rangle = \left(1 + |z|^2\right)$$

$$\begin{aligned}
\langle z|\hat{J}_0|z\rangle &= \frac{1}{2}\left(|z|^2 - 1\right) \\
{}_n\langle z|\hat{J}_0|z\rangle_n &= \frac{1}{2}\frac{\left(|z|^2 - 1\right)}{\left(1 + |z|^2\right)} = 2\beta\overline{\beta} - 1 = \frac{1}{2}\cos\theta \\
{}_n\langle z|\hat{J}_x|z\rangle_n &= \frac{z + \overline{z}}{1 + |z|^2} = \left(\beta + \overline{\beta}\right)\sqrt{1 - \beta\overline{\beta}} = \frac{1}{2}\sin\theta\cos\varphi
\end{aligned}$$

$$\beta = \frac{z}{\sqrt{1 + |z|^2}}; \qquad |\beta| < 1$$

Remark (for after):

$$\partial_{\overline{\beta}}\langle J_0\rangle_n = 2\beta$$

$$\partial_{\overline{\beta}}\langle J_x\rangle_n = \frac{1}{2\sqrt{1 - \beta\overline{\beta}}}\left(2 - 3\beta\overline{\beta} - \beta^2\right)$$

2 Coherent States

2.1 Field coherent states

Definition 2.1. For a given $\psi \equiv (\psi_1, \ldots, \psi_n) \in \mathbb{C}^N$, one associates a **field coherent state**:

$$\begin{aligned} |\psi\rangle &= \exp\left(\sum_{i=1}^{N} \psi_i \hat{J}_{+,i}\right) |0\rangle \\ &= \bigotimes_{i=1}^{N} \exp\left(\psi_i \hat{J}_{+,i}\right) |0_i\rangle = \bigotimes_{i=1}^{N} |\psi_i\rangle \end{aligned}$$

Think of $|\psi\rangle$ as a classical field $\psi(i)$, noted $\psi_i = (i|\psi)$, on the lattice with values in $\mathbb{C}$.

We note

$$\mathcal{H}_N = \mathbb{C}^N$$

the space of classical fields and $|\psi) \in \mathcal{H}_N = \mathbb{C}^N$ a classical field.

This space is the quantized space of the torus $\mathcal{H}_N\,(\theta = 0)$.

The standard (position) basis is written:

$$|i) = (0, \ldots, 1, \ldots 0) \in \mathcal{H}_N = \mathbb{C}^N$$

with $i = 1 \to N$.

For example, for $\phi \in \mathbb{C}$,

$$|\phi|i)\rangle = \exp\left(\phi \hat{J}_{+,i}\right) |0\rangle$$

is a spin coherent state at site i.

2.1.1 Field operators

We consider and note $(\hat{J}_+| = \left(\hat{J}_{+,1}, \ldots \hat{J}_{+,i}, \ldots \hat{J}_{+,N}\right)$ as a field operator. Similarly for every operator.

For instance

$$\hat{J}_{+,i} = (\hat{J}_+|i)$$

we write also:

$$\hat{J}_{+,\psi} = (\hat{J}_+|\psi)$$

One has

$$\begin{aligned} |\psi\rangle &= \exp\left(\sum_{i=1}^{N} \psi_i \hat{J}_{+,i}\right) |0\rangle \\ &= \exp\left(\sum_{i=1}^{N} (i|\psi)(\hat{J}_+|i)\right) |0\rangle \\ &= \exp\left((\hat{J}_+|\psi)\right) |0\rangle \end{aligned}$$

2.1.2 Scalar product

One has

$$\begin{aligned}\langle\psi|\psi\rangle &= \prod_{i=1}^{N}\langle\psi_i|\psi_i\rangle = \prod_{i=1}^{N}\left(1+|\psi_i|^2\right)\\ &= \exp\left(\sum_{i=1}^{N}\ln\left(1+|\psi_i|^2\right)\right)\end{aligned}$$

So only if $|\psi_i|^2 \simeq 0$, then one has $\langle\psi|\psi\rangle \simeq \exp\left(\sum_{i=1}^{N}|\psi_i|^2\right) = \exp\left((\psi|\psi)\right)$.

2.1.3 Canonical variables; Classical equation of motion

As for SU(2) coherent states, one define

$$\beta_i = \frac{\psi_i}{\sqrt{1+|\psi_i|^2}}$$

so the classical equation of motion are

$$\frac{d\beta_i}{dt} = -i\frac{\partial H}{\partial\overline{\beta}_i}$$

with

$$H =_n \langle\psi|\hat{H}|\psi\rangle_n$$

2.1.4 Closure relation

$$\hat{I}_{\mathcal{H}_{tot}} = \int d\left[\psi\right]\ |\psi\rangle_{nn}\langle\psi|$$

2.1.5 Operator Magnetization

$$\begin{aligned}\hat{N} &= \sum_{i=1}^{N}\hat{N}_i\\ \hat{N}_i &= \left(\hat{J}_{0,i}+\frac{1}{2}\right)\end{aligned}$$

has mean value:

$$\begin{aligned}\langle\psi|\hat{N}|\psi\rangle &= \sum_{i=1}^{N}\langle\psi_i|\hat{N}_i|\psi_i\rangle\\ &= \frac{1}{2}\sum_{i=1}^{N}|\psi_i|^2 = \frac{1}{2}(\psi|\psi)\end{aligned}$$

or

$$ {}_n\langle\psi|\hat{N}|\psi\rangle_n = \sum_{i=1}^{N} |\beta_i|^2 $$

Finally if $\hat{H} = \hat{N}$ is the Hamiltonian then

$$ \frac{d\beta_i}{dt} = -i\frac{\partial H}{\partial \overline{\beta}_i} = -i\beta_i $$

so as expected

$$ \beta_i(t) = \beta_i(0)\, e^{-it} $$

2.2 Localized coherent states

There are well defined coherent states of the Torus denoted by

$$ |z) \in \mathcal{H}_N, \quad z \in \mathbb{C} $$

With a given $\phi \in \mathbb{C}$, this gives special "field coherents states":

$$ |\phi|z)\rangle \in \mathcal{H}_{tot} $$

called **localized coherent states.**

Explicitely:

$$ \begin{aligned} |\phi|z)\rangle &= \exp\left(\phi\sum_{i=1}^{N}(i|z)\,\hat{J}_{+,i}\right)|0\rangle \\ &= \exp\left(\phi(\hat{J}_+|z)\right)|0\rangle \end{aligned} $$

from this last expression, because it depends algebraically on ϕ, z, it is clear that the familly of states $(|\phi|z)\rangle)_{(\phi,z)\in\mathbb{C}^2}$ form a 2 dim algebraic submanifold $\mathcal{F}$ of $P(\mathcal{H}_{tot})$.

$$ |\psi_i\rangle = \exp\left(\phi(i|z)\,\hat{J}_{+,i}\right)|0_i\rangle $$

There norm is

$$ \langle\phi|z)|\phi|z)\rangle = \prod_{i=1}^{N}\left(1 + |\phi|^2 |(i|z)|^2\right) $$

$$ {}_n\langle\phi, z|\hat{N}|\phi, z\rangle_n = \left(\prod_{i=1}^{N}\left(1 + |\phi|^2 |(i|z)|^2\right)^{-1}\right)\left(\frac{1}{2}|\phi|^2 (z|z)\right) $$

Two questions arise

Questions

1. Is the Manifold $\mathcal{F}$ complete? (i.e. span all $\mathcal{H}_{tot}$)?
2. Find a closure relation?

Elements to answer:

- One can look for a state ortohognal to every localized coherent state.
- Compute the scalar product $\langle |\phi|z)|\phi|z)\rangle$ which gives the

3 Model

3.1 Hamiltonian

Proposed by Gregoire Misguish:

$$\begin{aligned}
\hat{H}_t &= (1-t)\,\hat{H}_0 + t\hat{H}_1 \\
\hat{H}_0 &= 2\sum_{i=1}^{N} \hat{J}_{0,i} : \text{ Magnetic } z\text{−Field} \\
\hat{H}_1 &= -4\sum_{i=1}^{N} \hat{J}_{x,i}\hat{J}_{x,i+1} \text{: Ising} - x
\end{aligned}$$

An other one:

$$\hat{H}_{XYZ} = 4C_x \sum_{i=1}^{N} \hat{J}_{x,i}\hat{J}_{x,i+1} + 4C_y \sum_{i=1}^{N} \hat{J}_{y,i}\hat{J}_{y,i+1} + 4C_z \sum_{i=1}^{N} \hat{J}_{z,i}\hat{J}_{z,i+1}$$

H_{XYZ}if the **Heisenberg model** if $C_x = C_y = C_z$, or the **XXZ model** if $C_x = C_y \neq C_z$.
A more general one:

$$H_{C_x,C_y,C_z,C_0} = \hat{H}_{XYZ} + C_0\hat{H}_0$$

for example $\hat{H}_t$ is obtained with $C_x = -t$, $C_y = C_z = 0$, $C_0 = 1 - t$.

Remark: one can write:

$$\hat{J}_x = \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right)$$

$$\hat{J}_y = \frac{i}{2}\left(\hat{J}_- - \hat{J}_+\right)$$

.

3.1.1 Hamiltonian $\hat{H}_0$

$$\hat{H}_0 = 2\sum_{i=1}^{N} \hat{J}_{0,i} : \text{ Magnetic } z\text{−Field}$$

its spectrum is

$$\begin{aligned} E_i &= 2i - N = -N, -N+2, \dots, N, \\ i &= 0, 1, 2, \dots, N \end{aligned}$$

Level E_i corresponds to a number i of spins up. There is C_N^i possibilities. So the multiplicity is

$$d_i = C_N^i = 1, N, ..., N, 1$$

3.1.2 Hamiltonian $\hat{H}_1$

$$\hat{H}_1 = -4\sum_{i=1}^{N} \hat{J}_{x,i}\hat{J}_{x,i+1}\text{: Ising} - x$$

Level E_j corresponds to $2j$ frontiers (different closed spins), with

$$j = 0, 1, 2, \dots, [N/2]$$

Its energy is 1 for one frontier, and there is an even number of frontiers, so

$$E_j = -N + 4j = -N, -N+4, \dots, \left(-N + 4\left[\frac{N}{2}\right]\right)$$

with multiplicity:

$$d_j = 2C_N^j$$

3.1.3 Matrix elements

In the standard basis $|s\rangle = |s_1, \dots, s_N\rangle$, with dimension 2^N.

On has

$$\hat{J}_x = \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right) \equiv_{basis(+,-)} \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{J}_y = \frac{i}{2}\left(\hat{J}_- - \hat{J}_+\right) \equiv \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{J}_z \equiv \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So if:

$$\begin{aligned} s &= 0 \quad for \quad |-\rangle \\ &= 1 \quad for \quad |+\rangle \end{aligned}$$

and

$$\overline{s} = 1 - s$$

then

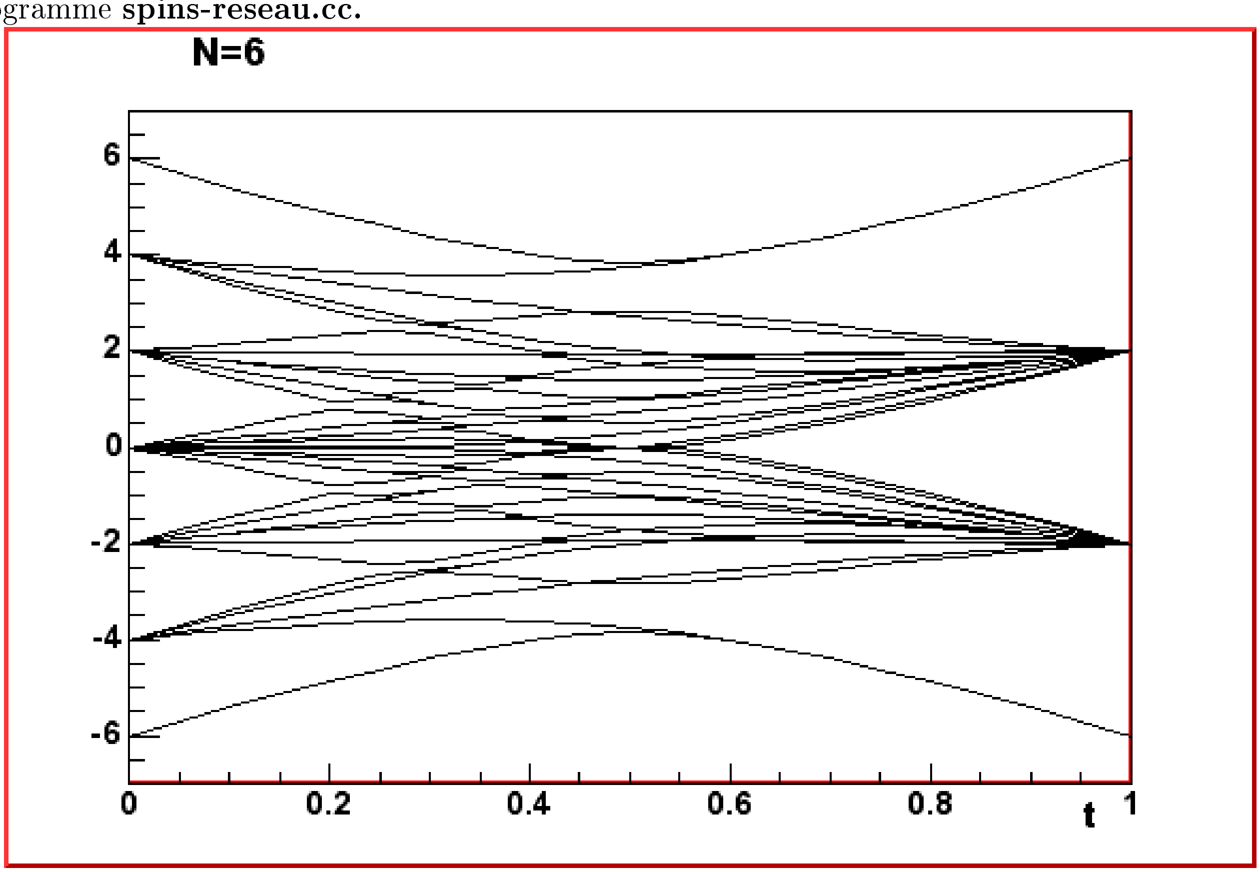
$$2\hat{J}_x|\overline{s}\rangle = |s\rangle$$

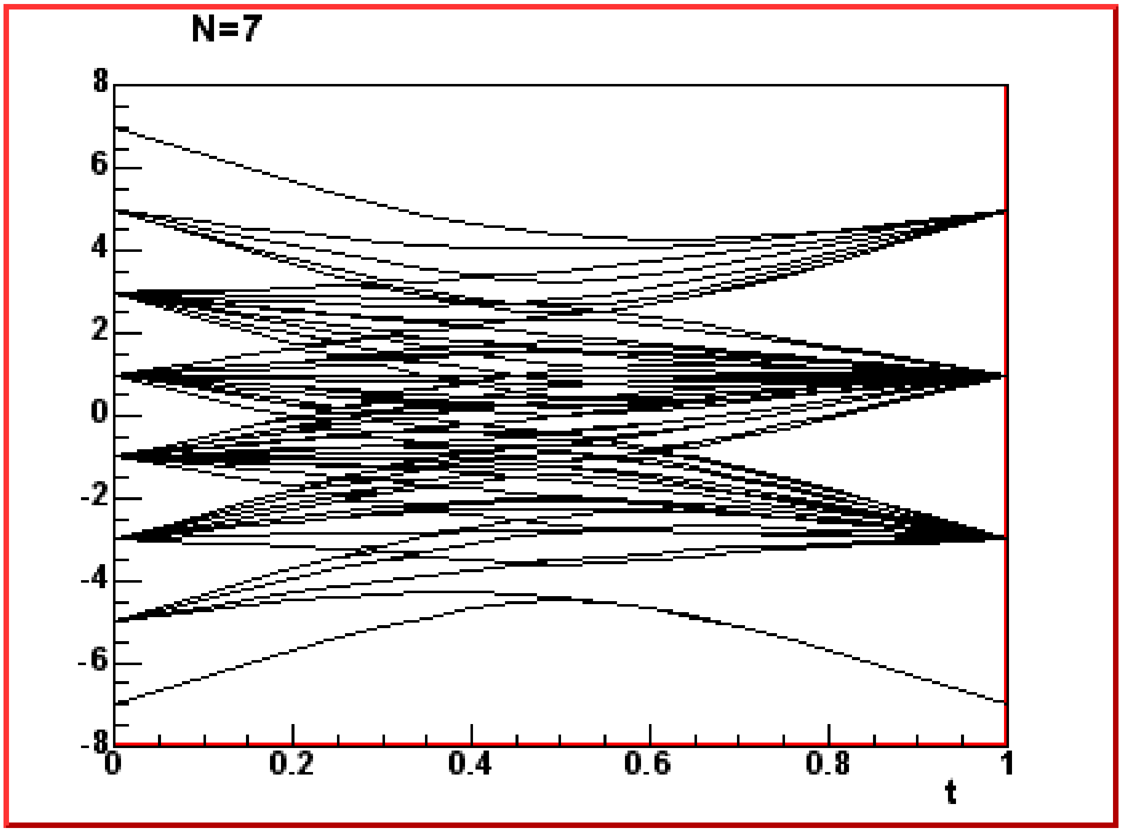
$$2\hat{J}_y|\overline{s}\rangle = i\,(1-2s)\,|s\rangle$$

$$2\hat{J}_z|s\rangle = (2s-1)\,|s\rangle$$

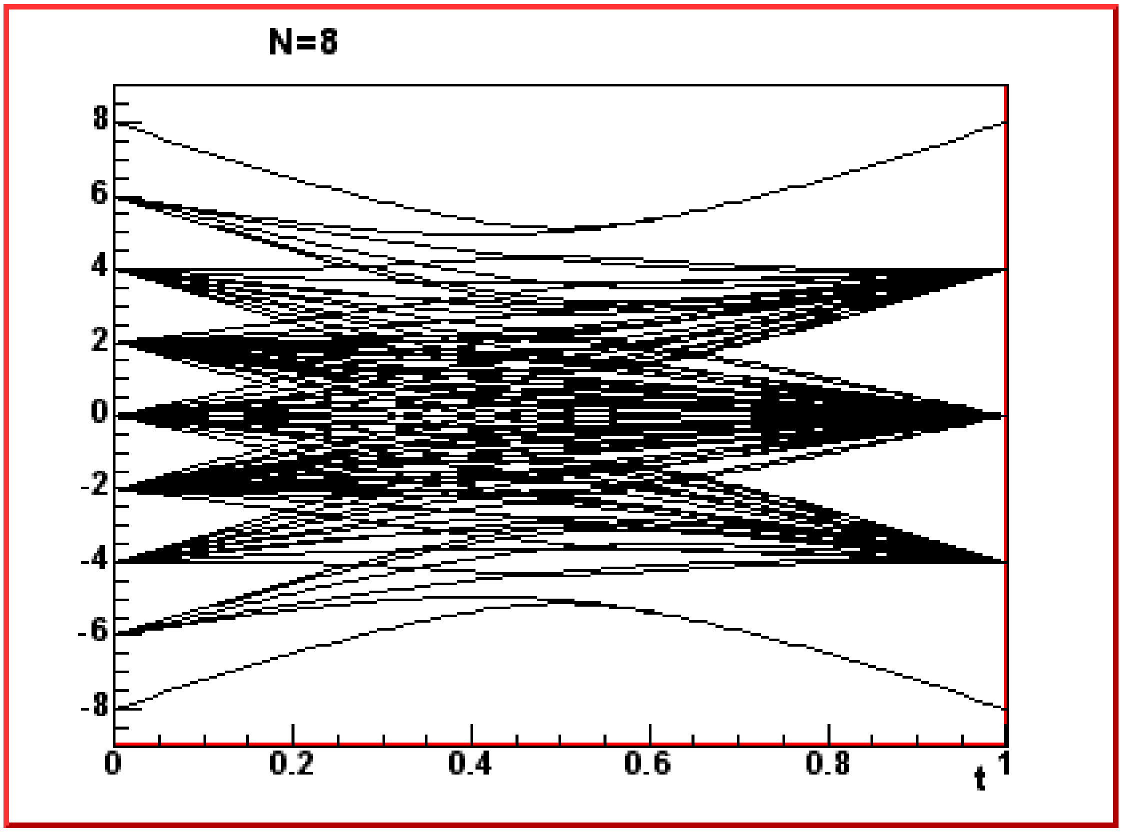
3.1.4 Numerical spectrum

programme **spins-reseau.cc.**





N=7
8
6
4
2
0
-2
-4
-6
-8
0
0.2
0.4
0.6
0.8
1
t



3.2 Classical equation of motion for field coherent states

As explained above,

$$\frac{d\beta_i}{dt} = -i\frac{\partial H}{\partial \overline{\beta}_i}$$

with

$$\begin{aligned} H &= {}_n\langle\psi|\hat{H}|\psi\rangle_n \\ &= (1-t)H_0 + tH_1 \end{aligned}$$

with

$$H_0 = 2\sum_{i=1}^{N} \langle J_{0,i}\rangle$$

$$H_1 = -4\sum_{i=1n}^{N} \langle\psi|\hat{J}_{x,i}\hat{J}_{x,i+1}|\psi\rangle_n = -4\sum_{i=1}^{N} \left\langle \hat{J}_{x,i}\right\rangle \left\langle \hat{J}_{x,i+1}\right\rangle$$

So

$$\frac{d\beta_i}{dt} = (-i)\,(1-t)\,\partial_{\overline{\beta}_i}\langle J_{0,i}\rangle + 4it\left(\left\langle \hat{J}_{x,i-1}\right\rangle + \left\langle \hat{J}_{x,i+1}\right\rangle\right)\partial_{\overline{\beta}_i}\langle J_{x,i}\rangle$$

3.3 Minimum of energy

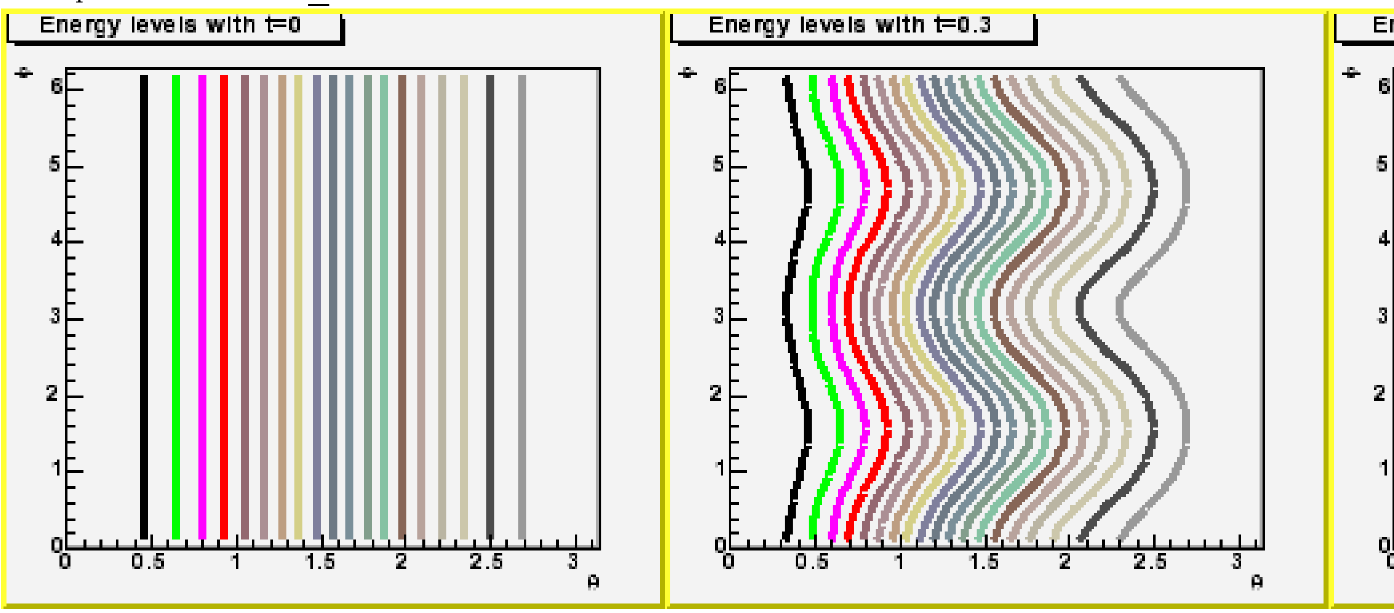
First suppose that the **coherent state field is uniform**:

$$\beta_i = \beta \quad \forall i$$

then in spherical coordinates,

$$\langle H \rangle = (1-t)\, N \cos\theta - tN \sin^2\theta \cos^2\varphi$$

which is plot with **dessin_H.C**



For $t=0$, the minimum is at $\theta = \pi$ (spin down $|-_z>$).

For $t=1$, there are two equal minima at $\theta = \pi/2$ and $\varphi = 0$ and $\varphi = \pi$ (spin $|+_x>$ or $|-_x>$).

Analitically, we have to solve

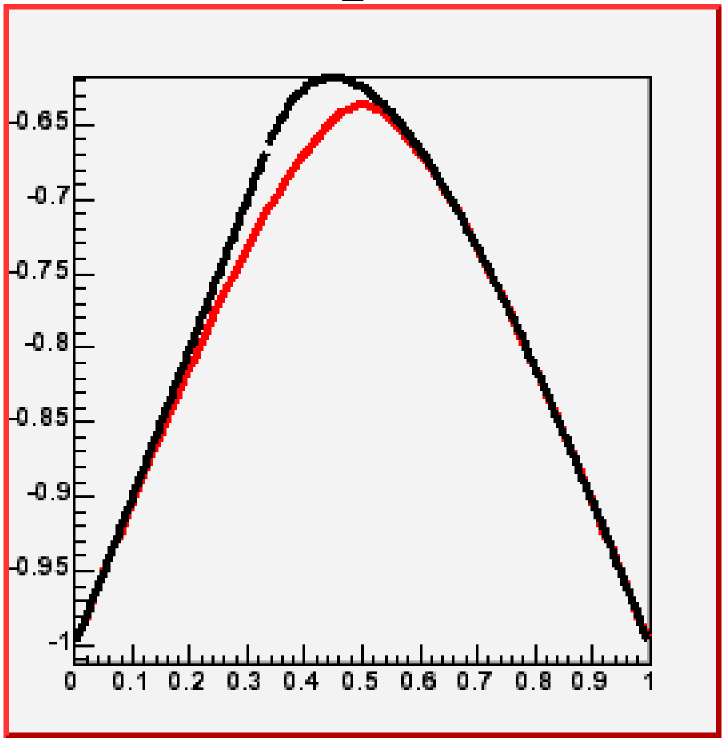
$$\begin{aligned} 0 &= \partial_\theta \langle H \rangle = -(1-t)\, N \sin\theta - 2tN \sin\theta \cos\theta \cos^2\varphi \\ \Leftrightarrow -\frac{1-t}{2t} \sin\theta &= \sin\theta \quad \cos\theta \cos^2\varphi \end{aligned}$$

$$0 = \partial_\varphi \langle H \rangle = tN \sin^2\theta \sin 2\varphi$$

The minimum is then:
for $t < 1/3$, at $\theta = \pi$
for $t > 1/3$, at $\varphi = 0, \pi$ and $\cos\theta = -\frac{1-t}{2t}$.
The value of energy is then

$$\begin{aligned} E_{min} &= -(1-t)\, N \qquad \text{for } t < 1/3 \\ &= -\left(\frac{1-2t+5t^2}{4t}\right) N \qquad \text{for } t > 1/3 \end{aligned}$$

Plot in black with **spins_reseau.cc.**



Compared with exact result in red (from Jordan-Wigner trick):

$$E_0(t) = -tN\left(\int_0^{2\pi} \frac{dk}{2\pi}\sqrt{2\lambda\cos(k)+\lambda^2+1}\right), \quad \lambda = \frac{1-t}{t} = 0 \to \infty$$

Remark: there is a symmetry

$$\begin{aligned} E_0(1-t) &= -(1-t)N\left(\int_0^{2\pi} \frac{dk}{2\pi}\sqrt{2\lambda^{-1}\cos(k)+\lambda^{-2}+1}\right) \\ &= -t\lambda N\left(\int_0^{2\pi} \frac{dk}{2\pi}\sqrt{2\lambda^{-1}\cos(k)+\lambda^{-2}+1}\right) \\ &= E_0(t) \end{aligned}$$

3.4 Maximum of energy

is for $\theta = 0$, and gives

$$E_{max}(t) = (1-t)N$$

which shows that the upper spectrum is bad described for $t \simeq 1$ (Ising dynamics). The reason is that the eigenstate of maximum energy is an anti-ferro state, bad described by our trial state.

3.5 Fourier basis @@

Because $\hat{H}$ is invariant by translation, we try to express it after a Fourier transform $(z_n) \to (z_m)$, in order to decouple sectors of different momentum m:

$$\begin{aligned}
(m|\psi) &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \exp(-i\frac{2\pi}{N} nm)\ \psi_n && (1)\\
&= \sum_{n=0}^{N-1} (m|n)(n|\psi) && (2)
\end{aligned}$$

Recall of notations for operators:

$$(\hat{A}|i) \hat{=} \hat{A}_i$$

and for $|\psi) \in \mathbb{C}^N$,

$$\hat{A}_\psi = (\hat{A}|\psi)$$

And define:

$$(i|\hat{A}) \hat{=} \left((\hat{A}|i)\right)^+ = \hat{A}_i^+ = (\hat{A}^+|i)$$

So that

$$\begin{aligned}
(\hat{A}|\psi)^+ &= \left(\sum_i \psi_i \hat{A}_i\right)^+ \\
&= \left(\sum_i \overline{\psi_i} \left(\hat{A}_i\right)^+\right) \\
&= \sum_i (\psi|i)(i|\hat{A}) \\
&= (\psi|\hat{A})
\end{aligned}$$

We define:

$$\hat{J}_{+,m} = \sum_n (\hat{J}_+|n)\,(n|m)$$

Proposition 3.1. *The operators basis $\hat{J}_{0,m}$,$\hat{J}_{\pm,m}$, for $m = 1 \to N$, have algebra relations:*

$$\left[\hat{J}_{+,m}, \hat{J}_{-,m'}\right] = 2\hat{J}_{0,m+m'}$$

and give a new decomposition of the whole space:

$$\mathcal{H}_{tot} = \bigotimes_{i=1}^{N} \mathcal{H}_{m,i} = \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{N}$$

Proof. One has

$$\begin{aligned}
\left[\hat{J}_{+,m},\hat{J}_{-,m'}\right] &= \hat{J}_{+,m}\hat{J}_{-,m'}-\hat{J}_{-,m'}\hat{J}_{+,m}\\
&= \sum_{n,n'}(J_+|n)(n|m)(J_-|n')(n'|m')-(J_-|n')(n'|m')(J_+|n)(n|m)\\
&= \sum_{n,n'}\left((J_-|n')(J_+|n)+2\delta_{n,n'}J_{0,n}\right)(n|m)(n'|m')-(J_-|n')(n'|m')(J_+|n)(n|m)\\
&= \sum_{n}2J_{0,n}(n|m)(n|m')\\
&= \sum_{n}2(J_0|n)(n|m+m')\\
&= 2J_{0,m+m'}
\end{aligned}$$

Remak: in case of algebra $[a_n^+,a_{n'}]=\delta_{n,n'}I$, one obtains instead

$$\left[a_m^+,a_{m'}\right]=...=\sum_n 2(I|n)(n|m+m')=2\sum_n(n|m+m')=2\sqrt{N}\delta_{m,(-m')}$$

□

References