

Concentration of eigenfunctions for semiclassical Toeplitz operators

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Plan

1 Toeplitz operators

- Toeplitz operators on \mathbb{C}^n
- Toeplitz operators on compact manifolds

2 Spin systems in the semiclassical limit

3 Melin estimate and localization

- General result
- Localization of the ground state
- Future work

Bargmann spaces

- Original idea: express Quantum Mechanics directly in phase space.

[1] Bargmann, V. Comm. Pure Appl. Math. 14, no. 3 (1961): 187–214.

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- The standard $L^2(\mathbb{R}^n)$ is replaced with the *Bargmann space*, with parameter $N > 0$:

$$B_N = L^2(\mathbb{C}^n) \cap \left\{ e^{-\frac{N}{2}|\cdot|^2} f, f \text{ is holomorphic on } \mathbb{C}^n \right\}.$$

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- This is a closed subspace of $L^2(\mathbb{C}^n)$, with reproducing kernel

$$\Pi_N(x, y) = \left(\frac{N}{\pi} \right)^n \exp \left(-\frac{N}{2} |x - y|^2 + iN \Im(x \cdot \bar{y}) \right).$$

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Szegő kernel

Hilbert basis indexed by \mathbb{N}^n :

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From there one recovers Π_N with

$$\Pi_N(x, y) = \sum_{v \in \mathbb{N}^n} e_v(x) \overline{e_v(y)}.$$

The Szegő kernel decays exponentially fast away from the diagonal.

Toeplitz quantization

Let $f \in C^\infty(\mathbb{C}^n, \mathbb{C})$ bounded. The Toeplitz operator associated with f is the bounded operator

$$\begin{aligned} T_N(f) : B_N(\mathbb{C}^n) &\mapsto B_N(\mathbb{C}^n) \\ u &\mapsto fu . \end{aligned}$$

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- If f is real-valued then $T_N(f)$ is ess. self-adjoint.
- If moreover $f \geq 0$ then $T_N(f) \geq 0$.

Composition of Toeplitz operators

■ Recipe:

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- More generally, composition yields a formal series:

$$T_N(f)T_N(g) = T_N(fg + N^{-1}C_1(f, g) + N^{-2}C_2(f, g) + \cdots).$$

C_j is a bidifferential operator of total order $2j$.

Toeplitz operators versus Ψ DOs

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- Formal equivalence between Toeplitz and Ψ DO calculus.
- Toeplitz quantization is formulated directly in phase space, and it is positive.

Hardy spaces and Szegő kernel

- Geometrical setting: compact Kähler manifold M .
 - Symplectic form
 - Complex structure

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The spaces $H_N(M, L)$ are finite-dimensional in that case. The line bundles $L^{\otimes N}$ correspond to the weights $e^{-\frac{N}{2}|\cdot|^2}$ in the flat case.

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Algebra of Toeplitz operators

- The Szegő kernel S_N has a full expansion near the diagonal, and decays far from it.

[3] Boutet de Monvel, L, Sjöstrand, J. Journées EDP 34–35 (1975): 123–64.

[4] Charles, L. Comm. Math. Phys. 239, no. 1–2 (2003): 1–28.

[5] Berman, R., Berndtsson, B., Sjöstrand, J., Arkiv För Matematik 46, no. 2 (2008).

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- Toeplitz operators form a C^* -algebra as previously.

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An example: the 2D sphere

Here $M = \mathbb{S}^2$. In the stereographic projection, L corresponds to the weight $z \mapsto \frac{1}{1+|z|^2}$, so that

$$\begin{aligned} H_N(M, L) &\simeq \left\{ f \text{ holomorphic in } \mathbb{C}, \int_{\mathbb{C}} \frac{|f|^2}{(1+|z|^2)^{N+2}} < \infty \right\} \\ &= \mathbb{C}_N[X]. \end{aligned}$$

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In the canonical basis $\binom{N}{k}^{-\frac{1}{2}} X^k$, the Toeplitz quantization of the three base coordinates on \mathbb{S}^2 are the Spin matrices with spin $S = \frac{N-1}{2}$.

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General spin systems

- Systems with n spins correspond to the Kähler manifold $(\mathbb{S}^2)^n$.
- We are interested in *antiferromagnetic* systems. Let $G = (V, E)$ a finite graph, the antiferromagnetic symbol on $(\mathbb{S}^2)^{|V|}$ is set to

$$h_{AF} = \sum_{(i,j) \in E} x_i x_j + y_i y_j + z_i z_j.$$

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$$h_{AF} = \sum_{(i,j) \in E} x_i x_j + y_i y_j + z_i z_j.$$

- If the graph is bipartite, then the minimum is reached when two neighbours always have opposite values.

Frustrated systems

- In frustrated systems, the previous solution is not possible.

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Frustrated systems

- In frustrated systems, the previous solution is not possible.
- If the graph is “made with triangles”, then on each triangle the sum of the vectors should be zero.
- This yields a *degenerate* minimal set, which is not a manifold.
- What behaviour should one expect for the eigenvectors with minimal eigenvalue of $T_N(h_{AF})$?

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Characteristic value

Can one improve the lower bound $f \geq 0 \Rightarrow T_N(f) \geq 0$?

- If q is a quadratic form in \mathbb{C}^n , the minimal eigenvalue of $T_N(q)$ is $N^{-1}\mu(q)$ with

$$\mu(q) = N^{-1}(\text{Tr}^+(q) + \frac{1}{2} \text{tr}(q)).$$

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- Here, up to a symplectomorphism,

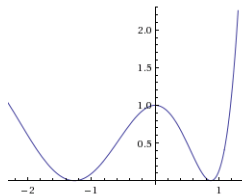
$$q = \sum_{i=1}^r \lambda_i (q_i^2 + p_i^2) + \sum_{i=r+1}^{r+r'} p_i^2,$$

so

$$\text{Tr}^+(q) = \sum_{i=1}^r \lambda_i.$$

Case of several wells

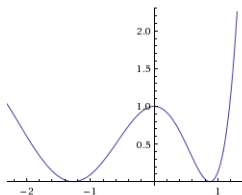
What can be said if h is minimal on non-degenerate critical points?



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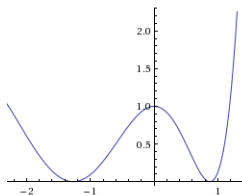
Theorem

The eigenvectors of minimal eigenvalue concentrate only on “minimal” points. Eigenvectors and eigenvalues have an asymptotical expansion in powers of $N^{-\frac{1}{2}}$.

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Theorem

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What is minimized? The μ of the Hessian at this point.

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Melin estimate

- Local result: for sections u sufficiently concentrated around a minimal point of f where the Hessian matrix is q , one has $\langle u, T_N(f)u \rangle \geq N^{-1}\mu(q) + CN^{-1-\epsilon}$.

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- Global result: if μ_{\inf} is the infimum of μ over all minimal points, then

$$T_N(f) \geq N^{-1}\mu_{\inf} + N^{-1-\epsilon}.$$

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Ideas for the proof

- Local result: use the fact that the Szegő kernel is equivalent to the \mathbb{C}^n case near the diagonal, as $N \rightarrow +\infty$.

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- Global result: pick a covering of the manifold with small open sets corresponding to the section, and ask that the section is relatively smaller at the intersection of the open sets than elsewhere.

A general localization result

Proposition

Let $f \in C^\infty(M, \mathbb{R})$ with $\min(f) = 0$. Then any sequence (u_N) of normalized eigenstates of $T_N(f)$ with eigenvalues $O(N^{-\epsilon})$ is such that

$$\int_{\{f(x) \geq N^{-1+\delta}\}} |u_N(x)|^2 d\text{Vol} = O(N^{-\infty}).$$

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Indeed, the first eigenvalue is $O(N^{-1})$, so that

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By induction, $\langle u_N, f^k u_N \rangle = O(N^{-k})$, hence the claim.

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Subprincipal effects on localization

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Let $f \in C^\infty(M, \mathbb{R})$ with $\min(f) = 0$. Let V at positive distance from $\{\mu = \mu_{\min}\}$. Then, with (u_N) as previously one has

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The proof uses the Melin estimates.

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Expansions in the regular case

The regular case is a generalization of Helffer-Sjöstrand's "miniwells". If μ is only minimal at one point x_0 near which $\{f = 0\}$ is an isotropic submanifold, and the minimum is non-degenerate, then

- The first eigenvalue is simple and admits an expansion in powers of $N^{-\frac{1}{4}}$.
- The first eigenvector concentrates at speed $N^{-\frac{1}{4}+}$ on x_0 along $\{f = 0\}$.
- Spectral gap $CN^{-\frac{3}{2}}$.

[11] Helffer, B., Sjöstrand, J. Current Topics in PDEs, 1986, 133–186

[10] Deleporte, A., Arxiv preprint.

Crossing points

We treat the case where $\{f = 0\}$ is a union of two submanifolds with a “crossing point”, where μ is minimal.

Toy model: $h(q_1, q_2, p_1, p_2) = p_1^2 + p_2^2 + q_1^2 q_2^2$.

- The first eigenvalue is simple and admits an expansion in powers of $N^{-\frac{1}{6}}$.
- The first eigenvector concentrates at speed $N^{-\frac{1}{3}+}$ on x_0 along $\{f = 0\}$.
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Tools for the two cases

We constructed symplectic normal forms, which are new even in the context of pseudodifferential calculus.

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Proposition

If $\sigma : \mathcal{U} \rightarrow \mathcal{V}$ is a local symplectomorphism between two Kähler manifolds M and N , then there is a sequence of maps $\mathfrak{S}_N : M \rightarrow N$, such that, when acting on sequences of sections microlocalising on \mathcal{U} ,

- *\mathfrak{S}_N is an isometry modulo $O(N^{-\infty})$ error.*

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At the minimal points, the subprincipal symbol is prescribed by the Melin estimates on both sides.

Perspective

- 1 Work in progress:
 - Exponential estimates in the analytic case,
- 2 Conjectures:

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- Where is μ minimal for the Kagome lattice?
- The Scottish flag: $T_N(\cos(q) + i \cos(p))$ on the two-torus.