Szegő kernels and Toeplitz operators

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MSRI

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Plan

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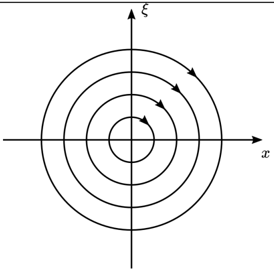

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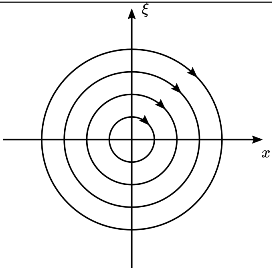		

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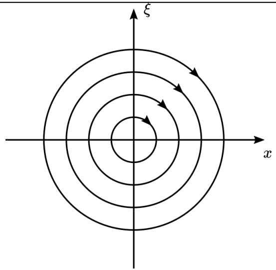		
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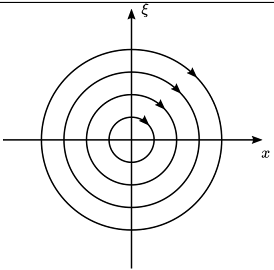		???
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- The standard $L^2(\mathbb{R}^n)$ is replaced with the *Bargmann space,* with parameter $N > 0$ (think of $N = \hbar^{-1}$):

$$B_N = L^2(\mathbb{C}^n) \cap \left\{ e^{-\frac{N}{2}|\cdot|^2} f,\ f \text{ is holomorphic on } \mathbb{C}^n \right\}.$$

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- This is a closed subspace of $L^2(\mathbb{C}^n)$, with reproducing kernel

$$\Pi_N(x, y) = \left(\frac{N}{\pi}\right)^n \exp\left(-\frac{N}{2}|x - y|^2 + iN\Im(x \cdot \overline{y})\right).$$

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From there one recovers Π_N with

$$\Pi_N(x,y) = \sum_{\nu\in\mathbb{N}^n} e_\nu(x)\overline{e_\nu}(y).$$

The Szegő kernel decays exponentially fast away from the diagonal.

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- $\langle u, \mathfrak{d}_N z v\rangle = \langle zu, zv\rangle = \langle u, |z|^2 v\rangle$.
- Spectrum: $N^{-1}\mathbb{N}$; eigenfunctions: monomials.

Toeplitz quantization

Let $f \in C^\infty(\mathbb{C}^n, \mathbb{C})$ bounded. The Toeplitz operator associated with f is the bounded operator

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- If f is real-valued then $T_N(f)$ is ess. self-adjoint.
- If moreover $f \geqslant 0$ then $T_N(f) \geqslant 0$.

Composition of Toeplitz operators

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- More generally, composition yields a formal series:

$$T_N(f)T_N(g) = T_N\left(fg + N^{-1}C_1(f,g) + N^{-2}C_2(f,g) + \cdots\right).$$

C_j is a bidifferential operator of total order $2j$.

Toeplitz operators versus ΨDOs

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- Formal equivalence between Toeplitz and ΨDO calculus.
- Toeplitz quantization is formulated **directly in phase space**, and **positive**.

Plan

Generalized Bargmann spaces

- Changing the positive quadratic weight in the Bargmann space:

$$B_N^{\psi} = \left\{ f \in L^2(\mathbb{C}^n), e^{-N\psi} f \text{ is holomorphic} \right\}$$

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- Note: one cannot simplify both symplectic and complex structure at the same time!

Hardy spaces and Szegő kernel

- Geometrical setting: compact Kähler manifold M.
  - Symplectic form
  - Complex structure

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- Szegő projector $S_N : L^2(M, L^{\otimes N}) \to H_N(M, L)$.

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Algebra of Toeplitz operators

- The Szegő kernel S_N has a full expansion near the diagonal, and decays far from it.

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[5] Berman, R., Berndtsson, B., Sjöstrand, J., Arkiv För Matematik 46, no. 2 (2008).
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- The dominant term is always Π_N.
- Toeplitz operators form a C^*-algebra as previously.

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An example: the 2D sphere

Here $M = \mathbb{S}^2$. In the stereographic projection, L corresponds to the weight $z \mapsto \frac{1}{1+|z|^2}$, so that

$$
\begin{aligned}
H_N(M, L) &\simeq \left\{ \text{f holomorphic in } \mathbb{C}, \int_{\mathbb{C}} \frac{|f|^2}{(1+|z|^2)^{N+2}} < \infty \right\} \\
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In the canonical basis $\binom{N}{k}^{-\frac{1}{2}} X^k$, the Toeplitz quantization of the three base coordinates on $\mathbb{S}^2$ are the Spin matrices with spin $S = \frac{N-1}{2}$.

Plan

Characteristic value

Can one improve the lower bound $f \geqslant 0 \Rightarrow T_N(f) \geqslant 0$?

- If q is a quadratic form in $\mathbb{C}^n$, the minimal eigenvalue of $T_N(q)$ is $N^{-1}\mu(q)$ with

$$\mu(q) = N^{-1}(\mathrm{Tr}^{+}(q) + \frac{1}{2}\mathrm{tr}(q)).$$

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- Here, up to a symplectomorphism,

$$q = \sum_{i=1}^{r} \lambda_i(q_i^2 + p_i^2) + \sum_{i=r+1}^{r+r'} p_i^2,$$

so

$$\mathrm{Tr}^+(q) = \sum_{i=1}^{r} \lambda_i.$$

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1. **Upper estimate**: try states localised near a point x where f is minimal: contribution $\min(f) + N^{-1}\mu((f)(x)) + O(N^{-2})$.

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2. Corresponding lower bound for **states localised near a point.**
3. Proof in three steps:
   - Small energy eigenfunctions localize where f is minimal.
   - Cut into pieces of size $N^{-\frac{1}{2}+\epsilon}$ **corresponding to the eigenfunction** (see next slide)
   - Apply the lower bound on each piece.

A cutting lemma

A function cannot be too large everywhere!

Example: given $t < a$ and $u : \mathbb{S}^1 \to \mathbb{R}$, then $\mathbb{S}^1$ can be cut into pieces U_j of size between a and $2a$, with overlap t, such that

$$\sum_{i,j} \int_{U_i \cap U_j} |u| \leqslant C \frac{t}{a} \sum_i \int_{U_i} |u|.$$

Melin estimate

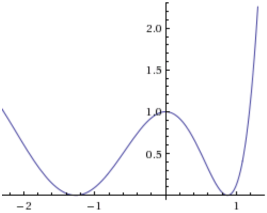
Theorem

If $f \in C^\infty(M, \mathbb{R})$ *and if* μ_{inf} *is the infimum of* μ *over all minimal points, then*

$$T_N(f) \geqslant \min(f) + N^{-1}\mu_{inf} + N^{-1-\epsilon}.$$

[8] Melin, A. Arkiv För Matematik 9, no. 1 (1971): 117–140
[9] Deleporte, A. Comm. Math. Phys (accepted)

Consequence: localization of the ground state



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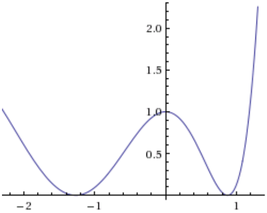

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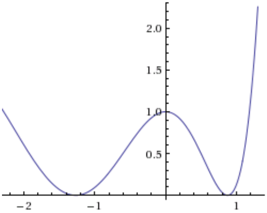

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What is minimized ? The μ of the Hessian at this point.

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