Introduction

- Quantum spins: triplet of self-adjoint matrices $S_x, S_y, S_z \in M_{2S+1}(\mathbb{C})$, with

$$[S_a, S_b] = \frac{i}{S}\epsilon_{abc} S_c.$$

- For $S = \frac{1}{2}$, one finds the Pauli matrices

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; S_y = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \; S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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- For any finite graph G, we wish to study the following operator acting on $(\mathbb{C}^{2S+1})^{\otimes |G|}$:

$$H = \sum_{e \sim f} S_{x,e}S_{x,f} + S_{y,e}S_{y,f} + S_{z,e}S_{z,f}$$

as $S \to +\infty$.

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Classical mechanics	Quantum mechanics
Symplectic manifold M	Hilbert Space H
Function $a \in C^\infty(M, \mathbb{R})$	Self-adjoint operator $A \in L(H)$
Hamiltonien flow of a	Flow of $e^{itA/\hbar}$
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- Quantization : for a given classical model, how to construct an associated quantum model ?
- Semiclassics : the quantum model is $\hbar$-dependent. What can be said in the $\hbar \to 0$ limit ?

Concentration of eigenfunctions for semiclassical Toeplitz operators

Alix Deleporte
Advisor : Nalini Anantharaman

Institut de Recherche Mathématique Avancée
Université de Strasbourg

August 26, 2018

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- Toeplitz operators on $\mathbb{C}^n$
- Toeplitz operators on compact manifolds

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3 Melin estimate and localization
- General result
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- Future work

Bargmann spaces

- Original idea: express Quantum Mechanics directly in phase space.

[1] Bargmann, V. Comm. Pure Appl. Math. 14, no. 3 (1961): 187–214.

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- The standard $L^2(\mathbb{R}^n)$ is replaced with the *Bargmann space*, with parameter $N > 0$:

$$B_N = L^2(\mathbb{C}^n) \cap \left\{ e^{-\frac{N}{2}|\cdot|^2} f,\ f \text{ is holomorphic on } \mathbb{C}^n \right\}.$$

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- This is a closed subspace of $L^2(\mathbb{C}^n)$, with reproducing kernel

$$\Pi_N(x,y) = \left(\frac{N}{\pi}\right)^n \exp\left(-\frac{N}{2}|x-y|^2 + iN\Im(x\cdot\overline{y})\right).$$

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Szegő kernel

Hilbert basis indexed by $\mathbb{N}^n$:

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From there one recovers Π_N with

$$\Pi_N(x,y) = \sum_{\nu \in \mathbb{N}^n} e_\nu(x)\overline{e_\nu}(y).$$

The Szegő kernel decays exponentially fast away from the diagonal.

Toeplitz quantization

Let $f \in C^\infty(\mathbb{C}^n, \mathbb{C})$ bounded. The Toeplitz operator associated with f is the bounded operator

$$\begin{aligned} T_N(f) : B_N(\mathbb{C}^n) &\mapsto B_N(\mathbb{C}^n) \\ u &\mapsto fu \,. \end{aligned}$$

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If f has polynomial growth then $T_N(f)$ is an unbounded operator with dense domain.

- If f is real-valued then $T_N(f)$ is ess. self-adjoint.
- If moreover $f \geqslant 0$ then $T_N(f) \geqslant 0$.

Composition of Toeplitz operators

- Recipe:

$$T_N(z \mapsto \bar{z}^{\alpha} z^{\beta}) = N^{-|\alpha|}\partial^{\alpha} z^{\beta}.$$

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$$T_N(fgh) = T_N(f)T_N(g)T_N(h).$$

- More generally, composition yields a formal series:

$$T_N(f)T_N(g) = T_N\left(fg + N^{-1}C_1(f,g) + N^{-2}C_2(f,g) + \cdots\right).$$

C_j is a bidifferential operator of total order $2j$.

Toeplitz operators versus ΨDOs

- The Bargmann transform $\mathcal{B}_N$ conjugates B_N and $L^2(\mathbb{R}^n)$.

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- Formal equivalence between Toeplitz and ΨDO calculus.
- Toeplitz quantization is formulated directly in phase space, and it is positive.

Hardy spaces and Szegő kernel

- Geometrical setting: compact Kähler manifold M.
  - Symplectic form
  - Complex structure

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- Szegő projector $S_N : L^2(M, L^{\otimes N}) \to H_N(M, L)$.

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- Szegő projector $S_N : L^2(M, L^{\otimes N}) \to H_N(M, L)$.

The spaces $H_N(M, L)$ are finite-dimensional in that case. The line bundles $L^{\otimes N}$ correspond to the weights $e^{-\frac{N}{2}|\cdot|^2}$ in the flat case.

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Algebra of Toeplitz operators

- The Szegő kernel S_N has a full expansion near the diagonal, and decays far from it.

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[5] Berman, R., Berndtsson, B., Sjöstrand, J., Arkiv För Matematik 46, no. 2 (2008).
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- The dominant term is always Π_N.
- Toeplitz operators form a C^*-algebra as previously.

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An example: the 2D sphere

Here $M = \mathbb{S}^2$. In the stereographic projection, L corresponds to the weight $z \mapsto \frac{1}{1+|z|^2}$, so that

$$H_N(M, L) \simeq \left\{ f \text{ holomorphic in } \mathbb{C}, \int_{\mathbb{C}} \frac{|f|^2}{(1+|z|^2)^{N+2}} < \infty \right\}$$
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In the canonical basis $\binom{N}{k}^{-\frac{1}{2}} X^k$, the Toeplitz quantization of the three base coordinates on $\mathbb{S}^2$ are the Spin matrices with spin $S = \frac{N-1}{2}$.

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General spin systems

- Systems with n spins correspond to the Kähler manifold $(\mathbb{S}^2)^n$.
- We are interested in *antiferromagnetic* systems. Let $G = (V, E)$ a finite graph, the antiferromagnetic symbol on $(\mathbb{S}^2)^{|V|}$ is set to

$$h_{AF} = \sum_{(i,j)\in E} x_i x_j + y_i y_j + z_i z_j.$$

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- If the graph is bipartite, then the minimum is reached when two neighbours always have opposite values.

Frustrated systems

- In frustrated systems, the previous solution is not possible.

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- If the graph is "made with triangles", then on each triangle the sum of the vectors should be zero.

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Frustrated systems

- In frustrated systems, the previous solution is not possible.
- If the graph is “made with triangles”, then on each triangle the sum of the vectors should be zero.
- This yields a *degenerate* minimal set, which is not a manifold.
- What behaviour should one expect for the eigenvectors with minimal eigenvalue of $T_N(h_{AF})$?

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Characteristic value

Can one improve the lower bound $f \geqslant 0 \Rightarrow T_N(f) \geqslant 0$?

- If q is a quadratic form in $\mathbb{C}^n$, the minimal eigenvalue of $T_N(q)$ is $N^{-1}\mu(q)$ with

$$\mu(q) = N^{-1}(\mathrm{Tr}^{+}(q) + \frac{1}{2}\,\mathrm{tr}(q)).$$

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$$\mu(q) = N^{-1}(\mathrm{Tr}^+(q) + \frac{1}{2}\mathrm{tr}(q)).$$

- Here, up to a symplectomorphism,

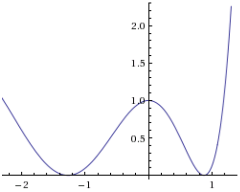
$$q = \sum_{i=1}^{r} \lambda_i(q_i^2 + p_i^2) + \sum_{i=r+1}^{r+r'} p_i^2,$$

so

$$\mathrm{Tr}^+(q) = \sum_{i=1}^{r} \lambda_i.$$

Case of several wells

What can be said if h is minimal on non-degenerate critical points?



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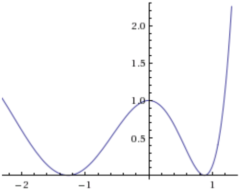

Theorem

The eigenvectors of minimal eigenvalue concentrate only on "minimal" points. Eigenvectors and eigenvalues have an asymptotical expansion in powers of $N^{-\frac{1}{2}}$.

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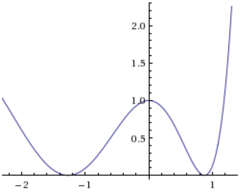

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What is minimized ? The μ of the Hessian at this point.

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Melin estimate

- Local result: for sections u sufficiently concentrated around a minimal point of f where the Hessian matrix is q, one has $\langle u, T_N(f)u\rangle \geqslant N^{-1}\mu(q) + CN^{-1-\epsilon}$.

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- Global result: if μ_{inf} is the infimum of μ over all minimal points, then

$$T_N(f) \geqslant N^{-1}\mu_{inf} + N^{-1-\epsilon}.$$

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Ideas for the proof

- Local result: use the fact that the Szegő kernel is equivalent to the $\mathbb{C}^n$ case near the diagonal, as $N \to +\infty$.

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- Local result: use the fact that the Szegő kernel is equivalent to the $\mathbb{C}^n$ case near the diagonal, as $N \to +\infty$.
- Global result: pick a covering of the manifold with small open sets corresponding to the section, and ask that the section is relatively smaller at the intersection of the open sets than elsewhere.

A general localization result

Proposition

Let $f \in C^{\infty}(M, \mathbb{R})$ *with* $\min(f) = 0$. *Then any sequence* (u_N) *of normalized eigenstates of* $T_N(f)$ *with eigenvalues* $O(N^{-\epsilon})$ *is such that*

$$\int_{\{f(x) \geqslant N^{-1+\delta}\}} |u_N(x)|^2 dVol = O(N^{-\infty}).$$

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Indeed, the first eigenvalue is $O(N^{-1})$, so that
$\langle u_N, f u_N \rangle = O(N^{-1})$.
By induction, $\langle u_N, f^k u_N \rangle = O(N^{-k})$, hence the claim.

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Subprincipal effects on localization

Theorem

Let $f \in C^{\infty}(M, \mathbb{R})$ *with* $\min(f) = 0$. *Let* V *at positive distance from* $\{\mu = \mu_{min}\}$. *Then, with* (u_N) *as previously one has*

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The proof uses the Melin estimates.

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Expansions in the regular case

The regular case is a generalization of Helffer-Sjöstrand's "miniwells". If μ is only minimal at one point x_0 near which $\{f = 0\}$ is an isotropic submanifold, and the minimum is non-degenerate, then

- The first eigenvalue is simple and admits an expansion in powers of $N^{-\frac{1}{4}}$.
- The first eigenvector concentrates at speed $N^{-\frac{1}{4}+}$ on x_0 along $\{f = 0\}$.
- Spectral gap $CN^{-\frac{3}{2}}$.

[11] Helffer, B., Sjöstrand, J. Current Topics in PDEs, 1986, 133–186
[10] Deleporte, A., Arxiv preprint.

Crossing points

We treat the case where $\{f = 0\}$ is a union of two sumbanifolds with a "crossing point", where μ is minimal.
Toy model: $h(q_1, q_2, p_1, p_2) = p_1^2 + p_2^2 + q_1^2 q_2^2$.

- The first eigenvalue is simple and admits an expansion in powers of $N^{-\frac{1}{6}}$.
- The first eigenvector concentrates at speed $N^{-\frac{1}{3}+}$ on x_0 along $\{f = 0\}$.
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Tools for the two cases

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If $\sigma : U \to V$ is a local symplectomorphism between two Kähler manifolds M and N, then there is a sequence of maps $\mathfrak{S}_N : M \to N$, such that, when acting on sequences of sections microlocalising on U,

- *$\mathfrak{S}_N$ is an isometry modulo $O(N^{-\infty})$ error.*

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- $\mathfrak{S}_N T_N(h) \mathfrak{S}_N^{-1} = T_N(\sigma \circ h + N^{-1} g_1 + N^{-2} g_2 + \cdots)$.

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- $\mathfrak{S}_N$ *is an isometry modulo $O(N^{-\infty})$ error.*
- $\mathfrak{S}_N T_N(h) \mathfrak{S}_N^{-1} = T_N(\sigma \circ h + N^{-1} g_1 + N^{-2} g_2 + \cdots)$.

At the minimal points, the subprincipal symbol is prescribed by the Melin estimates on both sides.

Perspective

1. Work in progress:
   - Exponential estimates in the analytic case,
2. Conjectures:

Perspective

1. Work in progress:
   - Exponential estimates in the analytic case, Tunnelling
2. Conjectures:

Perspective

1. Work in progress:
   - Exponential estimates in the analytic case, Tunnelling
   - Low-energy time evolution
2. Conjectures:

Perspective

1. Work in progress:
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   - Low-energy time evolution
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   - Where is μ minimal for the Kagome lattice?

Perspective

1. Work in progress:
   - Exponential estimates in the analytic case, Tunnelling
   - Low-energy time evolution
2. Conjectures:
   - Where is μ minimal for the Kagome lattice?
   - The Scottish flag: $T_N(\cos(q) + i\cos(p))$ on the two-torus.