Localization of low-energy states for semiclassical Toeplitz Operators

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Introduction

Classical mechanics	Quantum mechanics
Symplectic manifold M	Hilbert Space H
Function $a \in C^\infty(M, \mathbb{R})$	Self-adjoint operator $A \in L(H)$
Hamiltonien flow of a	Flow of $e^{itA/\hbar}$
Poisson Bracket	Lie Bracket

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- Quantization : for a given classical model, how to construct an associated quantum model ?
- Semiclassics : the quantum model is $\hbar$-dependent. What can be said in the $\hbar \to 0$ limit ?

Introduction

- Quantum spins: triplet of self-adjoint matrices $S_x, S_y, S_z \in M_{2S+1}(\mathbb{C})$, with

$$[S_a, S_b] = \frac{i}{S}\epsilon_{abc}S_c.$$

- For $S = \frac{1}{2}$, one finds the Pauli matrices

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; S_y = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \; S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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- For any finite graph G, we wish to study the following operator acting on $(\mathbb{C}^{2S+1})^{\otimes |G|}$:

$$H = \sum_{e \sim f} S_{x,e}S_{x,f} + S_{y,e}S_{y,f} + S_{z,e}S_{z,f}$$

as $S \to +\infty$.

Plan

Bargmann spaces

The Bargmann spaces are L^2 spaces of holomorphic functions on $\mathbb{C}^n$, with a weight.

$$B_N(\mathbb{C}^n) = \left\{ z \mapsto \exp\left(-\frac{N}{2}|z|^2\right) f(z),\ f \text{ holomorphic} \right\} \cap L^2(\mathbb{C}^n)$$

Those are closed subspaces of $L^2(\mathbb{C}^n)$.

The Szegő projector

Let Π_N be the orthogonal projector from $L^2(\mathbb{C}^n)$ onto $B_N(\mathbb{C}^n)$. Π_N. It admits a Schwartz kernel:

$$\Pi_N(z,w) = \frac{N^n}{\pi^n} \exp\left[N(-\frac{1}{2}|z-w|^2 + i\Im(z \cdot \overline{w}))\right].$$

As $N \to +\infty$, the kernel is exponentially decreasing far from the diagonal. The typical interaction scale is $N^{-1/2}$.

Toeplitz operators

Definition

Let $h \in C^\infty(\mathbb{C}^n)$ a smooth bounded function, and $N \in \mathbb{N}$. We denote by $T_N(h)$ the Toeplitz operator associated to h:

$$\begin{aligned} T_N(h) : B_N(\mathbb{C}^n) &\mapsto B_N(\mathbb{C}^n) \\ u &\mapsto \Pi_N(hu). \end{aligned}$$

If h is not bounded, we can construct $T_N(h)$ as an unbounded operator on $B_N(\mathbb{C}^n)$.
The mapping $h \mapsto T_N(h)$ is linear and adjoint-preserving. If h is real-valued, then $T_N(h)$ is formally self-adjoint.

Anti-Wick quantization

- There holds $T_N(1) = 1$, and if h is holomorphic, then$T_N(h) = h$; for instance $T_N(z_i) = z_i$.

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- Moreover $T_N(\bar{z}_i) = N^{-1}\partial_i$.
- If $h : z \mapsto \bar{z}^\alpha z^\beta$, then $T_N(h) = N^{-|\alpha|}\partial^\alpha z^\beta$.
- If q is a definite quadratic form on $\mathbb{R}^{2n}$, then $T_N(q)$ has a compact resolvent. The first eigenvalue $\mu_N(q) = N^{-1}\mu_1(q)$ is positive.

$$\mu_1(q) = \min \mathrm{Sp}(\mathrm{Op}_1^W(q)) + \frac{1}{2}\mathrm{tr}(q)$$

Composition and bracket

Proposition

Let a *and* b *two smooth bounded functions on* $\mathbb{C}^n$*. Then there is a sequence* $(c_i)_{i\in\mathbb{N}}$ *of smooth bounded functions on* $\mathbb{C}^n$*, with* $c_0 = ab$ *so that, as* $N \to +\infty$*, there holds:*

$$T_N(a)T_N(b) = T_N(c_0) + N^{-1}T_N(c_1) + N^{-2}T_N(c_2) + \dots$$

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In particular,

$$[T_N(a), T_N(b)] = \frac{i}{N}T_N(\{a,b\}) + O(N^{-2}).$$

Plan

- We wish to generalize Bargmann spaces to other complex manifolds.
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- Instead of considering weighted spaces, we will consider spaces of holomorphic sections.

Notations

- M is a compact Kähler manifold, with symplectic form ω.
- L is a complex line bundle on M, endowed with a hermitian structure h, so that the curvature of the Chern connexion is ω.
- $N \geqslant 1$ is an integer.

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Then if s is a (continuous) section of $L^{\otimes N}$, one can compute

$$\|s\|_{L^2} := \int_M h_N(s(m)) \frac{\omega^{\wedge n}}{n!}.$$

By completion, one defines the Hilbert space of square-integrable sections of $L^{\otimes N}$.

Hardy spaces

Definition

The *N-equivariant Hardy space* is the space $H_N(M, L)$ of L^2 and holomorphic sections of $L^{\otimes N}$.

This space is finite-dimensional, the dimension is polynomial in N (Riemann-Roch).

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The *Szegő projector* S_N is the orthogonal projector from $L^2(M,L^{\otimes N})$ onto $H_N(M,L)$.

It always admits a Schwartz kernel (as a section of $L^{\otimes N} \boxtimes L^{\otimes -N}$) because $H_N(M,L)$ is finite-dimensional.

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$T_N(h)$ acts on a finite-dimensional space, and it is symmetric when h is real-valued.
Observe that, for any $u, v \in H_N(M,L)$, there holds

$$\langle u, T_N(h)v \rangle = \langle u, hv \rangle.$$

Asymptotics for the Szegő projector

Proposition (Boutet-Sjostrand 74)

For every $\epsilon > 0$ *and* $k \in \mathbb{N}$ *there exists* C *such that, for every* $N \in \mathbb{N}$*:*

$$d(x,y) > \epsilon \Rightarrow |S_N(x,y)| \leqslant CN^{-k}$$

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Proposition (Charles 00, Zelditch 02, Ma 06)

In a convenient system of local coordinates, near any point of the diagonal, there holds:

$$S_N(z,w) \simeq \Pi_N(z,w)\left[1 + \sum_{k=1}^{K} N^{-k/2} b_k(\sqrt{N}z, \sqrt{N}w)\right]$$

Composition and bracket

Proposition (Charles 00, Schlichenmaier 02)

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$$T_N(a)T_N(b) = T_N(c_0) + N^{-1}T_N(c_1) + N^{-2}T_N(c_2) + \dots$$

In particular,

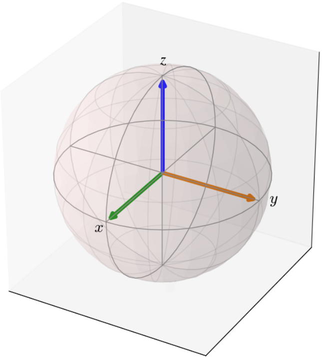
$$[T_N(a), T_N(b)] = \frac{i}{N}T_N(\{a,b\}) + O(N^{-2}).$$

Hardy spaces on the sphere

- $H_N(\mathbb{CP}^1, L)$ corresponds to the the set of meromorphic functions on the sphere, with one pole of order at most N.
- Hence $H_N(\mathbb{CP}^1, L) \simeq \mathbb{C}_N[X]$, with dimension $N + 1$.
- One Hilbert base is:

$$e_{k,N}(X) = \frac{\binom{k}{N}^{1/2}}{N} X^k.$$

Coordinate functions



- There are three coordinate functions on the sphere: x, y and z.
- The Toeplitz quantizations of these three functions are the spin operators, with $S = \frac{N}{2}$.

Plan

A priori localization

- In the classical model, in order to minimize the energy, one picks any point where the energy is minimal.
- What happens for an eigenvector associated with the smallest eigenvalue of $T_N(h)$, as $N \to +\infty$?

Proposition (Charles 00)

An eigenvector with minimal eigenvalue is uniformly $O(N^{-\infty})$ *outside any fixed neighbourhood of* $\{h = \min(h)\}$.

Can we get a more precise result?

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Proposition (D. 16)

If the minimal set is non-degenerate, then for every $\delta \in [0, 1/2)$, an eigenvector with minimal eigenvalue is uniformly $O(N^{-\infty})$ outside a neighbourhood of $\{h = \min(h)\}$ with size $N^{-\delta}$.

Proof for localization speed

Let (u_N) be a sequence of unit eigenfunctions with minimal energy (λ_N). Assume $\min(h) = 0$.
We prove by induction on k that

$$\langle u_n, h^k u_n \rangle = O(N^{-k}).$$

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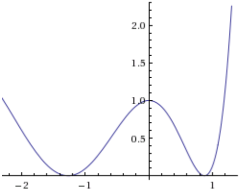
- Hard part: $k = 1$ (test $T_N(h)$ against a coherent state centred on a minimal point).
- Easy part: induction.

$$\langle u_n, h \star h u_n \rangle = \lambda_N^2 + O(N^{-\infty}),$$

where $h \star h = h^2 + N^{-1} c_1(h, h) + O(N^{-2})$.
Now $c_1(h, h) \leqslant Ch$.

Case of several wells



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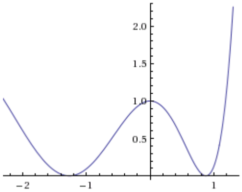

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The eigenvectors of minimal eigenvalue concentrate only on "minimal" points. Eigenvectors and eigenvalue have an asymptotical expansion.

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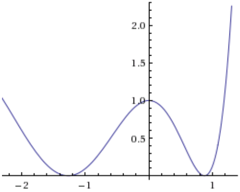

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What is minimized ? The μ_1 of the hessian at this point.

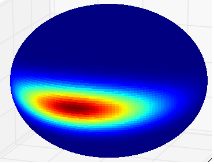
Case of wells: idea of proof

- By making more precise the previous argument, we have a lower bound for the first eigenvalue.
- The upper bound and a spectral gap are obtained by N^{-K}-quasimode for fixed K.

We remark that the quasimodes are exponentially localized, but this does not imply that the true eigenfunction is also localized.

Case of submanifold wells

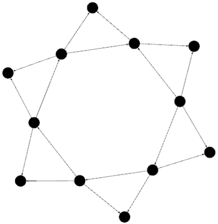
- What can be said if h is minimal on a submanifold, with non-degenerate transverse hessian?
- $\Rightarrow$ Same conclusion. (D.)



As N grows, the state concentrates on the miniwell and is more and more squeezed.

Miniwells in physics

It really happens in physics ! For instance, with antiferromagnetic spins on a triangle graph.



It is conjectured that the minimal configurations are planar, in some cases.

Conjectures

Exponential Localization For now we only have $O(N^{-\infty})$ estimates for localisation. Can we hope for $O(\exp(-cN))$ estimates ?

Thermodynamical limit Instead of considering a fixed manifold M, we look at a particular symbol on M^n, and we let $n \to +\infty$. What is the behaviour vis-à-vis the semiclassical limit?

These two questions should be linked with each other.