Low-energy spectrum of Toeplitz Operators

Alix Deleporte
Advisor : Nalini Anantharaman

Institut de Recherche Mathématique Avancée
Université de Strasbourg

October 5, 2017

Plan

Bargmann spaces

- Original idea: express Quantum Mechanics directly in phase space.

[1] Bargmann, V. Comm. Pure Appl. Math. 14, no. 3 (1961): 187–214.

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- The standard $L^2(\mathbb{R}^n)$ is replaced with the *Bargmann space*, with parameter $N > 0$:

$$B_N = L^2(\mathbb{C}^n) \cap \left\{ e^{-\frac{N}{2}|\cdot|^2} f, \text{ f is holomorphic on } \mathbb{C}^n \right\}.$$

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- This is a closed subspace of $L^2(\mathbb{C}^n)$, with reproducing kernel

$$\Pi_N(x,y) = \left(\frac{N}{\pi}\right)^n \exp\left(-\frac{N}{2}|x-y|^2 + iN\Im(x\cdot\overline{y}) \right).$$

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Szegő kernel

Hilbert basis indexed by $\mathbb{N}^n$.

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From there one recovers Π_N with

$$\Pi_N(x,y) = \sum_{\nu\in\mathbb{N}^n} e_\nu(x)\overline{e_\nu}(y).$$

The Szegő kernel decays exponentially fast away from the diagonal.

Toeplitz quantization

Let $f \in C^\infty(\mathbb{C}^n, \mathbb{C})$ bounded. The Toeplitz operator associated with f is the bounded operator

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- If f is real-valued then $T_N(f)$ is ess. self-adjoint.
- If moreover $f \geqslant 0$ then $T_N(f) \geqslant 0$.

Composition of Toeplitz operators

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- More generally, composition yields a formal series:

$$T_N(f)T_N(g) = T_N\left(fg + N^{-1}C_1(f,g) + N^{-2}C_2(f,g) + \cdots\right)$$

C_j is a bidifferential operator of total order $2j$.

Hardy spaces and Szegő kernel

- Geometrical setting: compact Kähler manifold M.
  - Symplectic form
  - Complex structure

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- Szegő projector $S_N : L^2(M, L^{\otimes N}) \to H_N(M, L)$.

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The spaces $H_N(M, L)$ are finite-dimensional in that case. The line bundles $L^{\otimes N}$ correspond to the weights $e^{-\frac{N}{2}|\cdot|^2}$ in the flat case.

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Algebra of Toeplitz operators

- The Szegő kernel S_N has a full expansion near the diagonal, and decays far from it.

[3] Boutet de Monvel, L, Sjöstrand, J. Journées EDP 34–35 (1975): 123–64.
[4] Charles, L. Comm. Math. Phys. 239, no. 1–2 (2003): 1–28.
[5] Berman, R., Berndtsson, B., Sjöstrand, J., Arkiv För Matematik 46, no. 2 (2008).
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- Toeplitz operators form a C^*-algebra as previously.

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An example: the 2D sphere

Here $M = \mathbb{S}^2$. In the stereographic projection, L corresponds to the weight $z \mapsto \frac{1}{1+|z|^2}$, so that

$$H_N(M, L) \simeq \left\{ f \text{ holomorphic in } \mathbb{C}, \int_{\mathbb{C}} \frac{|f|^2}{(1+|z|^2)^{N+2}} < \infty \right\}$$
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In the canonical basis $\binom{N}{k}^{-\frac{1}{2}} X^k$, the Toeplitz quantization of the three base coordinates on $\mathbb{S}^2$ are the Spin matrices with spin $S = \frac{N-1}{2}$.

Plan

General spin systems

- Systems with n spins correspond to the Kähler manifold $(\mathbb{S}^2)^n$.
- We are interested in *antiferromagnetic* systems. Let $G = (V, E)$ a finite graph, the antiferromagnetic symbol on $(\mathbb{S}^2)^{|V|}$ is set to

$$h_{AF} = \sum_{(i,j) \in E} x_i x_j + y_i y_j + z_i z_j.$$

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- If the graph is bipartite, then the minimum is reached when two neighbours always have opposite values.

Frustrated systems

- In frustrated systems, the previous solution is not possible.

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Frustrated systems

- In frustrated systems, the previous solution is not possible.
- If the graph is "made with triangles", then on each triangle the sum of the vectors should be zero.
- This yields a *degenerate* minimal set, which is not a manifold.
- What behaviour should one expect for the eigenvectors with minimal eigenvalue of $T_N(h_{AF})$?

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Characteristic value

Can one improve the lower bound $f \geqslant 0 \Rightarrow T_N(f) \geqslant 0$?

- If q is a quadratic form in $\mathbb{C}^n$, the minimal eigenvalue of $T_N(q)$ is $N^{-1}\mu(q)$ with

$$\mu(q) = N^{-1}(\mathrm{Tr}^+(q) + \frac{1}{2}\mathrm{tr}(q)).$$

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- Here, up to a symplectomorphism,

$$q = \sum_{i=1}^{r} \lambda_i(q_i^2 + p_i^2) + \sum_{i=r+1}^{r+r'} p_i^2,$$

so

$$\mathrm{Tr}^+(q) = \sum_{i=1}^{r} \lambda_i.$$

Melin estimate

- Local result: for sections u sufficiently concentrated around a minimal point of f where the Hessian matrix is q, one has $\langle u, T_N(f)u\rangle \geqslant N^{-1}\mu(q) + CN^{-1-\epsilon}$.

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- Local result: for sections u sufficiently concentrated around a minimal point of f where the Hessian matrix is q, one has $\langle u, T_N(f)u\rangle \geqslant N^{-1}\mu(q) + CN^{-1-\epsilon}$.
- Global result: if μ_{inf} is the infimum of μ over all minimal points, then

$$T_N(f) \geqslant N^{-1}\mu_{inf} + N^{-1-\epsilon}.$$

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Ideas for the proof

- Local result: use the fact that the Szegő kernel is equivalent to the $\mathbb{C}^n$ case near the diagonal, as $N \to +\infty$.

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- Global result: pick a covering of the manifold with small open sets corresponding to the section, and ask that the section is relatively smaller at the intersection of the open sets than elsewhere.

A general localization result

Theorem

Let $f \in C^\infty(M,\mathbb{R})$ *with* $\min(f) = 0$. *Then any sequence* (u_N) *of normalized ground states of* $T_N(f)$ *is such that*

$$\int_{f(x)\geqslant N^{-1+\delta}} |u_N(x)|^2 dVol = O(N^{-\infty}).$$

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$\langle u_N, f u_N \rangle = O(N^{-1})$.
By induction, $\langle u_N, f^k u_N \rangle = O(N^{-k})$, hence the claim.

Subprincipal effects on localization

Theorem

Let $f \in C^{\infty}(M, \mathbb{R})$ *with* $\min(f) = 0$. *Let* V *at positive distance from* $\{\mu = \mu_{min}\}$. *Then, with* (u_N) *as previously one has*

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The proof uses the Melin estimates.

Expansions in the regular case

The regular case is a generalization of Helffer-Sjöstrand's "miniwells". If μ is only minimal at one point x_0 near which $\{f = 0\}$ is an isotropic submanifold, and the minimum is non-degenerate, then

- The first eigenvalue is simple and admits an expansion in powers of $N^{-\frac{1}{4}}$.
- The first eigenvector concentrates at speed $N^{-\frac{1}{2}+}$ on $\{f = 0\}$ and at speed $N^{-\frac{1}{4}+}$ on x_0 along $\{f = 0\}$.

[9] Helffer, B., Sjöstrand, J. Current Topics in PDEs, 1986, 133–186.

Tools for the regular case

For the regular case we use quantizations of symplectic maps: If $\sigma : U \to V$ is a local symplectomorphism between two Kähler manifolds M and N, then there is a sequence of maps $\mathfrak{S}_N : M \to N$, such that, when it acts on sequences of sections microlocalizing on U,

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At the minimal points, the subprincipal symbol g_1 is prescribed by the Melin estimates on both sides.

A singular toy model

- Consider the following symbol on $\mathbb{C}^n$: $y_1^2 + y_2^2 + x_1^2 x_2^2$.
- The minimal set is a union of two lines $(x_1, 0, 0, 0)$ and $(0, x_2, 0, 0)$. Along the first, $\mu = |x_1|$.

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- What is the rate of decay along the minimal set?