

Concentration of eigenfunctions for semiclassical Toeplitz operators

Alix Deleporte
Advisor : Nalini Anantharaman

Institut de Recherche Mathématique Avancée
Université de Strasbourg

April 5, 2018

Plan

1 Toeplitz operators

- Toeplitz operators on \mathbb{C}^n
- Toeplitz operators on compact manifolds

2 Spin systems in the semiclassical limit

3 Melin estimate and localization

- General result
- Localization of the ground state
- Future work

Bargmann spaces

- Original idea: express Quantum Mechanics directly in phase space.

[1] Bargmann, V. Comm. Pure Appl. Math. 14, no. 3 (1961): 187–214.

Bargmann spaces

- Original idea: express Quantum Mechanics directly in phase space.
- The standard $L^2(\mathbb{R}^n)$ is replaced with the *Bargmann space*, with parameter $N > 0$:

$$B_N = L^2(\mathbb{C}^n) \cap \left\{ e^{-\frac{N}{2}|\cdot|^2} f, f \text{ is holomorphic on } \mathbb{C}^n \right\}.$$

[1] Bargmann, V. Comm. Pure Appl. Math. 14, no. 3 (1961): 187–214.

Bargmann spaces

- Original idea: express Quantum Mechanics directly in phase space.
- The standard $L^2(\mathbb{R}^n)$ is replaced with the *Bargmann space*, with parameter $N > 0$:

$$B_N = L^2(\mathbb{C}^n) \cap \left\{ e^{-\frac{N}{2}|\cdot|^2} f, f \text{ is holomorphic on } \mathbb{C}^n \right\}.$$

- This is a closed subspace of $L^2(\mathbb{C}^n)$, with reproducing kernel

$$\Pi_N(x, y) = \left(\frac{N}{\pi} \right)^n \exp \left(-\frac{N}{2} |x - y|^2 + iN \Im(x \cdot \bar{y}) \right).$$

[1] Bargmann, V. Comm. Pure Appl. Math. 14, no. 3 (1961): 187–214.

Szegő kernel

Hilbert basis indexed by \mathbb{N}^n .

$$e_{\nu} = \frac{N^{|\nu|}}{\nu_1! \nu_2! \dots \nu_n!} z^{\nu} e^{-\frac{N|z|^2}{2}}.$$

Szegő kernel

Hilbert basis indexed by \mathbb{N}^n .

$$e_{\nu} = \frac{N^{|\nu|}}{\nu_1! \nu_2! \dots \nu_n!} z^{\nu} e^{-\frac{N|z|^2}{2}}.$$

From there one recovers Π_N with

$$\Pi_N(x, y) = \sum_{\nu \in \mathbb{N}^n} e_{\nu}(x) \overline{e_{\nu}(y)}.$$

The Szegő kernel decays exponentially fast away from the diagonal.

Toeplitz quantization

Let $f \in C^\infty(\mathbb{C}^n, \mathbb{C})$ bounded. The Toeplitz operator associated with f is the bounded operator

$$\begin{aligned} T_N(f) : B_N(\mathbb{C}^n) &\mapsto B_N(\mathbb{C}^n) \\ u &\mapsto fu . \end{aligned}$$

Toeplitz quantization

Let $f \in C^\infty(\mathbb{C}^n, \mathbb{C})$ bounded. The Toeplitz operator associated with f is the bounded operator

$$\begin{aligned} T_N(f) : B_N(\mathbb{C}^n) &\mapsto B_N(\mathbb{C}^n) \\ u &\mapsto \Pi_N(fu). \end{aligned}$$

Toeplitz quantization

Let $f \in C^\infty(\mathbb{C}^n, \mathbb{C})$ bounded. The Toeplitz operator associated with f is the bounded operator

$$\begin{aligned} T_N(f) : B_N(\mathbb{C}^n) &\mapsto B_N(\mathbb{C}^n) \\ u &\mapsto \Pi_N(fu). \end{aligned}$$

If f has polynomial growth then $T_N(f)$ is an unbounded operator with dense domain.

Toeplitz quantization

Let $f \in C^\infty(\mathbb{C}^n, \mathbb{C})$ bounded. The Toeplitz operator associated with f is the bounded operator

$$\begin{aligned} T_N(f) : B_N(\mathbb{C}^n) &\mapsto B_N(\mathbb{C}^n) \\ u &\mapsto \Pi_N(fu). \end{aligned}$$

If f has polynomial growth then $T_N(f)$ is an unbounded operator with dense domain.

- If f is real-valued then $T_N(f)$ is ess. self-adjoint.
- If moreover $f \geq 0$ then $T_N(f) \geq 0$.

Composition of Toeplitz operators

■ Recipe:

$$T_N(z \mapsto \bar{z}^\alpha z^\beta) = N^{-|\alpha|} \partial^\alpha \bar{\partial}^\beta.$$

Composition of Toeplitz operators

- Recipe:

$$T_N(z \mapsto \bar{z}^\alpha z^\beta) = N^{-|\alpha|} \partial^\alpha \bar{\partial}^\beta.$$

- The Toeplitz quantization is *anti-Wick*: if f is anti-holomorphic and h is holomorphic then

$$T_N(fgh) = T_N(f)T_N(g)T_N(h).$$

Composition of Toeplitz operators

- Recipe:

$$T_N(z \mapsto \bar{z}^\alpha z^\beta) = N^{-|\alpha|} \partial^\alpha z^\beta.$$

- The Toeplitz quantization is *anti-Wick*: if f is anti-holomorphic and h is holomorphic then

$$T_N(fgh) = T_N(f)T_N(g)T_N(h).$$

- More generally, composition yields a formal series:

$$T_N(f)T_N(g) = T_N(fg + N^{-1}C_1(f, g) + N^{-2}C_2(f, g) + \cdots).$$

C_j is a bidifferential operator of total order $2j$.

Toeplitz operators versus Ψ DOs

- The Bargmann transform \mathcal{B}_N conjugates \mathcal{B}_N and $L^2(\mathbb{R}^n)$.

Toeplitz operators versus Ψ DOs

- The Bargmann transform \mathcal{B}_N conjugates \mathcal{B}_N and $L^2(\mathbb{R}^n)$.
- This Bargmann transform is related to the FBI transform.

Toeplitz operators versus Ψ DOs

- The Bargmann transform \mathcal{B}_N conjugates \mathcal{B}_N and $L^2(\mathbb{R}^n)$.
- This Bargmann transform is related to the FBI transform.
- With $g_N = (N/\pi)^n e^{-N|z|^2}$ one has

$$\mathcal{B}_N^{-1} T_N(f) \mathcal{B}_N = \text{Op}_W^{N^{-1}}(f * g_N).$$

Toeplitz operators versus Ψ DOs

- The Bargmann transform \mathcal{B}_N conjugates \mathcal{B}_N and $L^2(\mathbb{R}^n)$.
- This Bargmann transform is related to the FBI transform.
- With $g_N = (N/\pi)^n e^{-N|z|^2}$ one has

$$\mathcal{B}_N^{-1} T_N(f) \mathcal{B}_N = \text{Op}_W^{N^{-1}}(f * g_N).$$

- Formal equivalence between Toeplitz and Ψ DO calculus.

Toeplitz operators versus Ψ DOs

- The Bargmann transform \mathcal{B}_N conjugates \mathcal{B}_N and $L^2(\mathbb{R}^n)$.
- This Bargmann transform is related to the FBI transform.
- With $g_N = (N/\pi)^n e^{-N|z|^2}$ one has

$$\mathcal{B}_N^{-1} T_N(f) \mathcal{B}_N = \text{Op}_W^{N^{-1}}(f * g_N).$$

- Formal equivalence between Toeplitz and Ψ DO calculus.
- Toeplitz quantization is formulated directly in phase space!

Hardy spaces and Szegő kernel

- Geometrical setting: compact Kähler manifold M .
 - Symplectic form
 - Complex structure

[2] Woodhouse, N. Geometric Quantization. Oxford University Press, 1997.

Hardy spaces and Szegő kernel

- Geometrical setting: compact Kähler manifold M .
 - Symplectic form
 - Complex structure } Compatibility condition

[2] Woodhouse, N. Geometric Quantization. Oxford University Press, 1997.

Hardy spaces and Szegő kernel

- Geometrical setting: compact Kähler manifold M .
 - Symplectic form
 - Complex structure } Compatibility condition
- Complex line bundle $L \rightarrow M$ with curvature $-i\omega$.

[2] Woodhouse, N. Geometric Quantization. Oxford University Press, 1997.

Hardy spaces and Szegő kernel

- Geometrical setting: compact Kähler manifold M .
 - Symplectic form
 - Complex structure
 } Compatibility condition
- Complex line bundle $L \rightarrow M$ with curvature $-i\omega$.
- Hardy space $H_N(M, L)$ of holomorphic sections of $L^{\otimes N}$.

[2] Woodhouse, N. Geometric Quantization. Oxford University Press, 1997.

Hardy spaces and Szegő kernel

- Geometrical setting: compact Kähler manifold M .
 - Symplectic form
 - Complex structure
 } Compatibility condition
- Complex line bundle $L \rightarrow M$ with curvature $-i\omega$.
- Hardy space $H_N(M, L)$ of holomorphic sections of $L^{\otimes N}$.
- Szegő projector $S_N : L^2(M, L^{\otimes N}) \rightarrow H_N(M, L)$.

[2] Woodhouse, N. Geometric Quantization. Oxford University Press, 1997.

Hardy spaces and Szegő kernel

- Geometrical setting: compact Kähler manifold M .
 - Symplectic form
 - Complex structure
 } Compatibility condition
- Complex line bundle $L \rightarrow M$ with curvature $-i\omega$.
- Hardy space $H_N(M, L)$ of holomorphic sections of $L^{\otimes N}$.
- Szegő projector $S_N : L^2(M, L^{\otimes N}) \rightarrow H_N(M, L)$.

The spaces $H_N(M, L)$ are finite-dimensional in that case. The line bundles $L^{\otimes N}$ correspond to the weights $e^{-\frac{N}{2}|\cdot|^2}$ in the flat case.

[2] Woodhouse, N. Geometric Quantization. Oxford University Press, 1997.

Algebra of Toeplitz operators

- The Szegő kernel S_N has a full expansion near the diagonal, and decays far from it.

[3] Boutet de Monvel, L, Sjöstrand, J. Journées EDP 34–35 (1975): 123–64.

[4] Charles, L. Comm. Math. Phys. 239, no. 1–2 (2003): 1–28.

[5] Berman, R., Berndtsson, B., Sjöstrand, J., Arkiv För Matematik 46, no. 2 (2008).

[6] Kordyukov, Y. ArXiv Preprint ArXiv:1703.04107, 2017.

Algebra of Toeplitz operators

- The Szegő kernel S_N has a full expansion near the diagonal, and decays far from it.
- Indeed S_N can be seen as the N -th Fourier mode of a Fourier Integral Operator with complex phase; the critical set is the diagonal.

[3] Boutet de Monvel, L, Sjöstrand, J. Journées EDP 34–35 (1975): 123–64.

[4] Charles, L. Comm. Math. Phys. 239, no. 1–2 (2003): 1–28.

[5] Berman, R., Berndtsson, B., Sjöstrand, J., Arkiv För Matematik 46, no. 2 (2008).

[6] Kordyukov, Y. ArXiv Preprint ArXiv:1703.04107, 2017.

Algebra of Toeplitz operators

- The Szegő kernel S_N has a full expansion near the diagonal, and decays far from it.
- Indeed S_N can be seen as the N -th Fourier mode of a Fourier Integral Operator with complex phase; the critical set is the diagonal.
- The dominant term is always Π_N .

[3] Boutet de Monvel, L, Sjöstrand, J. Journées EDP 34–35 (1975): 123–64.

[4] Charles, L. Comm. Math. Phys. 239, no. 1–2 (2003): 1–28.

[5] Berman, R., Berndtsson, B., Sjöstrand, J., Arkiv För Matematik 46, no. 2 (2008).

[6] Kordyukov, Y. ArXiv Preprint ArXiv:1703.04107, 2017.

Algebra of Toeplitz operators

- The Szegő kernel S_N has a full expansion near the diagonal, and decays far from it.
- Indeed S_N can be seen as the N -th Fourier mode of a Fourier Integral Operator with complex phase; the critical set is the diagonal.
- The dominant term is always Π_N .
- Toeplitz operators form a C^* -algebra as previously.

[3] Boutet de Monvel, L, Sjöstrand, J. Journées EDP 34–35 (1975): 123–64.

[4] Charles, L. Comm. Math. Phys. 239, no. 1–2 (2003): 1–28.

[5] Berman, R., Berndtsson, B., Sjöstrand, J., Arkiv För Matematik 46, no. 2 (2008).

[6] Kordyukov, Y. ArXiv Preprint ArXiv:1703.04107, 2017.

An example: the 2D sphere

Here $M = \mathbb{S}^2$. In the stereographic projection, L corresponds to the weight $z \mapsto \frac{1}{1+|z|^2}$, so that

$$\begin{aligned} H_N(M, L) &\simeq \left\{ f \text{ holomorphic in } \mathbb{C}, \int_{\mathbb{C}} \frac{|f|^2}{(1+|z|^2)^{N+2}} < \infty \right\} \\ &= \mathbb{C}_N[X]. \end{aligned}$$

An example: the 2D sphere

Here $M = \mathbb{S}^2$. In the stereographic projection, L corresponds to the weight $z \mapsto \frac{1}{1+|z|^2}$, so that

$$\begin{aligned} H_N(M, L) &\simeq \left\{ f \text{ holomorphic in } \mathbb{C}, \int_{\mathbb{C}} \frac{|f|^2}{(1+|z|^2)^{N+2}} < \infty \right\} \\ &= \mathbb{C}_N[X]. \end{aligned}$$

In the canonical basis $\binom{N}{k}^{-\frac{1}{2}} X^k$, the Toeplitz quantization of the three base coordinates on \mathbb{S}^2 are the Spin matrices with spin $S = \frac{N-1}{2}$.

Plan

1 Toeplitz operators

- Toeplitz operators on \mathbb{C}^n
- Toeplitz operators on compact manifolds

2 Spin systems in the semiclassical limit

3 Melin estimate and localization

- General result
- Localization of the ground state
- Future work

General spin systems

- Systems with n spins correspond to the Kähler manifold $(\mathbb{S}^2)^n$.
- We are interested in *antiferromagnetic* systems. Let $G = (V, E)$ a finite graph, the antiferromagnetic symbol on $(\mathbb{S}^2)^{|V|}$ is set to

$$h_{AF} = \sum_{(i,j) \in E} x_i x_j + y_i y_j + z_i z_j.$$

General spin systems

- Systems with n spins correspond to the Kähler manifold $(\mathbb{S}^2)^n$.
- We are interested in *antiferromagnetic* systems. Let $G = (V, E)$ a finite graph, the antiferromagnetic symbol on $(\mathbb{S}^2)^{|V|}$ is set to

$$h_{AF} = \sum_{(i,j) \in E} x_i x_j + y_i y_j + z_i z_j.$$

- If the graph is bipartite, then the minimum is reached when two neighbours always have opposite values.

Frustrated systems

- In frustrated systems, the previous solution is not possible.

[7] Douçot, B., and Simon, P., J. Physics A: Math and General 31, no. 28 (1998)

Frustrated systems

- In frustrated systems, the previous solution is not possible.
- If the graph is “made with triangles”, then on each triangle the sum of the vectors should be zero.

[7] Douçot, B., and Simon, P., J. Physics A: Math and General 31, no. 28 (1998)

Frustrated systems

- In frustrated systems, the previous solution is not possible.
- If the graph is “made with triangles”, then on each triangle the sum of the vectors should be zero.
- This yields a *degenerate* minimal set, which is not a manifold.
- What behaviour should one expect for the eigenvectors with minimal eigenvalue of $T_N(h_{AF})$?

[7] Douçot, B., and Simon, P., J. Physics A: Math and General 31, no. 28 (1998)

Plan

1 Toeplitz operators

- Toeplitz operators on \mathbb{C}^n
- Toeplitz operators on compact manifolds

2 Spin systems in the semiclassical limit

3 Melin estimate and localization

- General result
- Localization of the ground state
- Future work

Characteristic value

Can one improve the lower bound $f \geq 0 \Rightarrow T_N(f) \geq 0$?

- If q is a quadratic form in \mathbb{C}^n , the minimal eigenvalue of $T_N(q)$ is $N^{-1}\mu(q)$ with

$$\mu(q) = N^{-1}(\text{Tr}^+(q) + \frac{1}{2} \text{tr}(q)).$$

Characteristic value

Can one improve the lower bound $f \geq 0 \Rightarrow T_N(f) \geq 0$?

- If q is a quadratic form in \mathbb{C}^n , the minimal eigenvalue of $T_N(q)$ is $N^{-1}\mu(q)$ with

$$\mu(q) = N^{-1}(\text{Tr}^+(q) + \frac{1}{2} \text{tr}(q)).$$

- Here, up to a symplectomorphism,

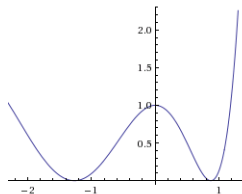
$$q = \sum_{i=1}^r \lambda_i (q_i^2 + p_i^2) + \sum_{i=r+1}^{r+r'} p_i^2,$$

so

$$\text{Tr}^+(q) = \sum_{i=1}^r \lambda_i.$$

Case of several wells

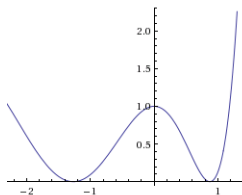
What can be said if h is minimal on non-degenerate critical points?



[9] Deleporte, A., Journal of Spectral Theory (accepted)

Case of several wells

What can be said if h is minimal on non-degenerate critical points?



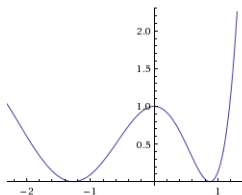
Theorem

The eigenvectors of minimal eigenvalue concentrate only on “minimal” points. Eigenvectors and eigenvalues have an asymptotical expansion in powers of $N^{-\frac{1}{2}}$.

[9] Deleporte, A., Journal of Spectral Theory (accepted)

Case of several wells

What can be said if h is minimal on non-degenerate critical points?



Theorem

The eigenvectors of minimal eigenvalue concentrate only on “minimal” points. Eigenvectors and eigenvalues have an asymptotical expansion in powers of $N^{-\frac{1}{2}}$.

What is minimized ? The μ of the Hessian at this point.

[9] Deleporte, A., Journal of Spectral Theory (accepted)

Melin estimate

- Local result: for sections u sufficiently concentrated around a minimal point of f where the Hessian matrix is q , one has $\langle u, T_N(f)u \rangle \geq N^{-1}\mu(q) + CN^{-1-\epsilon}$.

[8] Melin, A. Arkiv För Matematik 9, no. 1 (1971): 117–140

Melin estimate

- Local result: for sections u sufficiently concentrated around a minimal point of f where the Hessian matrix is q , one has $\langle u, T_N(f)u \rangle \geq N^{-1}\mu(q) + CN^{-1-\epsilon}$.
- Global result: if μ_{\inf} is the infimum of μ over all minimal points, then

$$T_N(f) \geq N^{-1}\mu_{\inf} + N^{-1-\epsilon}.$$

[8] Melin, A. Arkiv För Matematik 9, no. 1 (1971): 117–140

Ideas for the proof

- Local result: use the fact that the Szegő kernel is equivalent to the \mathbb{C}^n case near the diagonal, as $N \rightarrow +\infty$.

Ideas for the proof

- Local result: use the fact that the Szegő kernel is equivalent to the \mathbb{C}^n case near the diagonal, as $N \rightarrow +\infty$.
- Global result: pick a covering of the manifold with small open sets corresponding to the section, and ask that the section is relatively smaller at the intersection of the open sets than elsewhere.

A general localization result

Proposition

Let $f \in C^\infty(M, \mathbb{R})$ with $\min(f) = 0$. Then any sequence (u_N) of normalized eigenstates of $T_N(f)$ with eigenvalues $O(N^{-\epsilon})$ is such that

$$\int_{\{f(x) \geq N^{-1+\delta}\}} |u_N(x)|^2 d\text{Vol} = O(N^{-\infty}).$$

[9] Deleporte, A., Journal of Spectral Theory (accepted)

A general localization result

Proposition

Let $f \in C^\infty(M, \mathbb{R})$ with $\min(f) = 0$. Then any sequence (u_N) of normalized eigenstates of $T_N(f)$ with eigenvalues $O(N^{-\epsilon})$ is such that

$$\int_{\{f(x) \geq N^{-1+\delta}\}} |u_N(x)|^2 d\text{Vol} = O(N^{-\infty}).$$

Indeed, the first eigenvalue is $O(N^{-1})$, so that $\langle u_N, fu_N \rangle = O(N^{-1})$.

[9] Deleporte, A., Journal of Spectral Theory (accepted)

A general localization result

Proposition

Let $f \in C^\infty(M, \mathbb{R})$ with $\min(f) = 0$. Then any sequence (u_N) of normalized eigenstates of $T_N(f)$ with eigenvalues $O(N^{-\epsilon})$ is such that

$$\int_{\{f(x) \geq N^{-1+\delta}\}} |u_N(x)|^2 d\text{Vol} = O(N^{-\infty}).$$

Indeed, the first eigenvalue is $O(N^{-1})$, so that

$$\langle u_N, fu_N \rangle = O(N^{-1}).$$

By induction, $\langle u_N, f^k u_N \rangle = O(N^{-k})$, hence the claim.

[9] Deleporte, A., Journal of Spectral Theory (accepted)

Subprincipal effects on localization

Theorem

Let $f \in C^\infty(M, \mathbb{R})$ with $\min(f) = 0$. Let V at positive distance from $\{\mu = \mu_{\min}\}$. Then, with (u_N) as previously one has

$$\int_V |u_N|^2 d\text{Vol} = O(N^{-\infty}).$$

[9] Deleporte, A., Journal of Spectral Theory (accepted)

[10] Deleporte, A., Arxiv preprint.

Subprincipal effects on localization

Theorem

Let $f \in C^\infty(M, \mathbb{R})$ with $\min(f) = 0$. Let V at positive distance from $\{\mu = \mu_{\min}\}$. Then, with (u_N) as previously one has

$$\int_V |u_N|^2 d\text{Vol} = O(N^{-\infty}).$$

The proof uses the Melin estimates.

[9] Deleporte, A., Journal of Spectral Theory (accepted)

[10] Deleporte, A., Arxiv preprint.

Expansions in the regular case

The regular case is a generalization of Helffer-Sjöstrand's "miniwells". If μ is only minimal at one point x_0 near which $\{f = 0\}$ is an isotropic submanifold, and the minimum is non-degenerate, then

- The first eigenvalue is simple and admits an expansion in powers of $N^{-\frac{1}{4}}$.
- The first eigenvector concentrates at speed $N^{-\frac{1}{4}+}$ on x_0 along $\{f = 0\}$.
- Spectral gap $CN^{-\frac{3}{2}}$.

[11] Helffer, B., Sjöstrand, J. Current Topics in PDEs, 1986, 133–186

[10] Deleporte, A., Arxiv preprint.

Crossing points

We treat the case where $\{f = 0\}$ is a union of two submanifolds with a “crossing point”, where μ is minimal.

Toy model: $h(q_1, q_2, p_1, p_2) = p_1^2 + p_2^2 + q_1^2 q_2^2$.

- The first eigenvalue is simple and admits an expansion in powers of $N^{-\frac{1}{6}}$.
- The first eigenvector concentrates at speed $N^{-\frac{1}{3}+}$ on x_0 along $\{f = 0\}$.
- Spectral gap $CN^{-\frac{4}{3}}$.

[10] Deleporte, A., Arxiv preprint.

Tools for the two cases

We constructed symplectic normal forms, which are new even in the context of pseudodifferential calculus.

Tools for the two cases

We constructed symplectic normal forms, which are new even in the context of pseudodifferential calculus.

Proposition

If $\sigma : \mathcal{U} \rightarrow V$ is a local symplectomorphism between two Kähler manifolds M and N , then there is a sequence of maps $\mathfrak{S}_N : M \rightarrow N$, such that, when acting on sequences of sections microlocalising on \mathcal{U} ,

- *\mathfrak{S}_N is an isometry modulo $O(N^{-\infty})$ error.*

Tools for the two cases

We constructed symplectic normal forms, which are new even in the context of pseudodifferential calculus.

Proposition

If $\sigma : \mathcal{U} \rightarrow V$ is a local symplectomorphism between two Kähler manifolds M and N , then there is a sequence of maps $\mathfrak{S}_N : M \rightarrow N$, such that, when acting on sequences of sections microlocalising on \mathcal{U} ,

- \mathfrak{S}_N is an isometry modulo $O(N^{-\infty})$ error.
- $\mathfrak{S}_N T_N(h) \mathfrak{S}_N^{-1} = T_N(\sigma \circ h + N^{-1}g_1 + N^{-2}g_2 + \dots).$

Tools for the two cases

We constructed symplectic normal forms, which are new even in the context of pseudodifferential calculus.

Proposition

If $\sigma : \mathcal{U} \rightarrow \mathcal{V}$ is a local symplectomorphism between two Kähler manifolds M and N , then there is a sequence of maps $\mathfrak{S}_N : M \rightarrow N$, such that, when acting on sequences of sections microlocalising on \mathcal{U} ,

- \mathfrak{S}_N is an isometry modulo $O(N^{-\infty})$ error.
- $\mathfrak{S}_N T_N(h) \mathfrak{S}_N^{-1} = T_N(\sigma \circ h + N^{-1}g_1 + N^{-2}g_2 + \dots).$

At the minimal points, the subprincipal symbol is prescribed by the Melin estimates on both sides.

Perspective

- 1 Work in progress:
 - Exponential estimates in the analytic case,
- 2 Conjectures:

Perspective

- 1 Work in progress:
 - Exponential estimates in the analytic case, Tunnelling
- 2 Conjectures:

Perspective

1 Work in progress:

- Exponential estimates in the analytic case, Tunnelling
- Low-energy time evolution

2 Conjectures:

Perspective

1 Work in progress:

- Exponential estimates in the analytic case, Tunnelling
- Low-energy time evolution

2 Conjectures:

- Where is μ minimal for the Kagome lattice?

Perspective

1 Work in progress:

- Exponential estimates in the analytic case, Tunnelling
- Low-energy time evolution

2 Conjectures:

- Where is μ minimal for the Kagome lattice?
- The Scottish flag: $T_N(\cos(q) + i \cos(p))$ on the two-torus.