ANALYSIS OF ECOLOGICAL MODELS

ECOLOGY

Before we start the main question that arises is “what is ecology?”

Our answer would be ‘*Ecology is the study of the relationships between living organisms, including humans, and their physical environment; it seeks to understand the vital connections between plants and animals and the world around them. Ecology also provides information about the benefits of ecosystems and how we can use Earth’s resources in ways that leave the environment healthy for future generations.*’

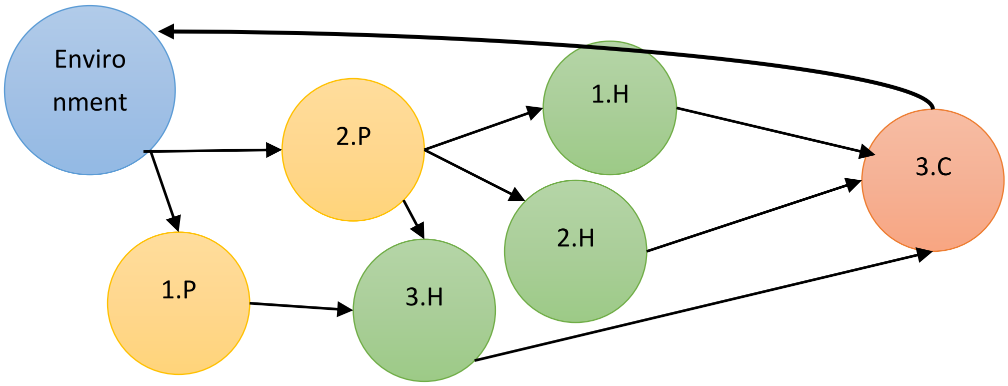
Ecosystem

'An ecosystem is a structural and functional unit of ecology where the living organisms interact with each other and the surrounding environment. In other words, an ecosystem is a chain of interactions between organisms and their environment’

There are different ways for representing an ecosystem. For pointing out the use of mathematics here, we use a directed graph.

Graph and adjacency matrix representation

In this graph each node presents a species and the environment itself is presented by a node. And a directed line going from node i to node j represents the relationships. These arrows can be viewed as different resources. E.g., different substances such as calcium or even energy flow.



P stands for photosynthetic, H for herbaceous and C for consumer

Because the graphical models can’t be used in such a way by math; we will use adjacency matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Now by use of this matrix we can get some useful information:

- The sum of rows will give us number of species, each specie has as food.
- The sum of columns will give us the number of species, each specie serves.
- And a lot more information which the eigenvalues and their eigenvectors tell us.

The dominant eigenvalue of this matrix represents the structural diversity of the ecosystem. Or casually said, it shows how well related the species are. E.g., if we had two different ecosystems with the same number of species but different eigenvalue, the one with the larger eigenvalue is linked better and when the length of path is increased, number of paths will grow faster.

Another useful piece of information is the powers of the matrix. By a graph theorem we get that for A^k, its i,jth entry is the number of different paths from node i to j of length k. In a healthy system, the total number of possible pathways should increase as path length increases. A decrease in path lengths indicates that energy is either leaking or being transported out of the system and collapse is possible.

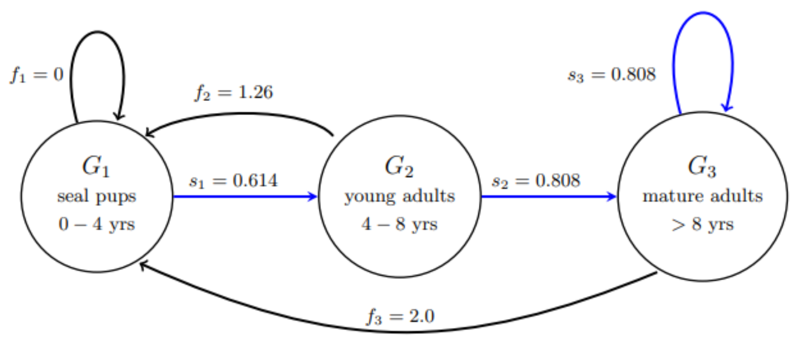
The dominant eigenvector also gives us some interesting information. By the notes told above, we can quickly get that, for u^T = (1,..,1),

$\lim_{k\to\infty} A^k u$ will show us the number of paths going to each specie of any length. And by power method we know that A^ku becomes the dominant eigenvector. Which would tell us, the flow of each node. E.g., in our case (.592 , .338,.338,.194,.194,.387,.443)t shows that the species 6 has twice the feeding opportunities that species 4 and 5 do.

Leslie matrix

Another way of analyzing an ecosystem is to consider a single specie's population.

These populations can be shown as a graph like this:



Which shows the population of a desired female specie like seals otherwise if we want to use both genders, sex ratio or two-sex Leslie matrix should be employed.

G_i represents the group. And f_i represents the fertility rate which you can see is zero for seal pups.

And s_i represents the survival rate. E.g., 60% of our pups will survive till youth.

As mentioned above, for applying math to models we represent them with a matrix which is called a Leslie matrix.

$$L = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ s_1 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix}$$

We can assume $L_{nn} = S_n$ to be nonzero and it that case it is called a generalized Leslie matrix.

As it's clear, the characteristic polynomial of L is:

$$c(x) = \lambda^n - f_1\lambda^{n-1} - s_1 f_2 \lambda^{n-2} - s_2 s_1 f_3 \lambda^{n-3} - \cdots - s_{n-1} \ldots s_3 s_2 s_1 f_n$$

At the end the equation we have is

$$\begin{bmatrix} p_1^{(t+1)} \\ p_2^{(t+1)} \\ p_3^{(t+1)} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} p_1^t \\ p_2^t \\ p_3^t \end{bmatrix}$$

And as a nonmatrix equation: $P^{(t+1)} = LP^{(t)}$.

Perron-Frobenius theorem states that for a primitive and irreducible matrix, there exists a real and positive eigenvalue, which is greater than any other eigenvalues in modulus.

The dominant eigenvalue of L, denoted λ gives the population's asymptotic growth rate (growth rate at the stable age distribution). The corresponding eigenvector provides the stable age distribution, the proportion of individuals of each age within the population, which remains constant at this point of asymptotic growth barring changes to vital rates. When a population reaches a stable age distribution, the population continues to grow in the rate of the dominant eigenvalue and remains in the same stable age distribution structure.

For the mentioned example the eigenvalue is 1.49 which means the population is growing rapidly.

$\lambda < 1$ means the population will decline exponentially.
$\lambda > 1$ means the population will grow exponentially.
$\lambda = 1$ means the population is stable, it does not change.

And as the steps become greater $(k\ becomes\ larger)\frac{p(k+1)}{p(k)} = \lambda$. No matter what the starting state is.

Harvesting

As an example, we can consider a farm of sheep. After constructing a Leslie matrix for a sample, sheep population will increase by about 17.6% per year and will reach a stable distribution. But considering the real life; the farmers cannot live on the income of the wool! They need to harvest some of the sheep but also maintain a growth rate that can become stable. Or in other words a sustainable harvesting policy is a plan for harvesting on a regular schedule in such a way that the harvest is always the same and the state of the population after harvesting is always the same.

Although harvesting and natural death can happen simultaneously in the harvest season, we assume that harvesting plays a dominant role and thus, natural death is negligible in the harvest season. Therefore, harvesting and natural death can be considered as independent events. By probability theory if h_i stands for harvest survival and s_i for natural survival the chance of surviving both would be $h_i s_i$.

Now we construct the harvest matrix as:

$$\mathbf{H} = \begin{bmatrix} H_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & H_2 & \ddots & & & \vdots \\ \vdots & \ddots & H_3 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & H_{A-1} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & H_A \end{bmatrix},$$

And now our equation changes to $P(t+1) = HLP(t)$. Which is the population that lives a whole cycle and survives harvesting. For the population to be sustainably harvested (λ = 1), M = HL must satisfy Mu = u.

There are infinitely many ways to construct a sustainable harvesting policy. We share three different ways of considering matrix H.

I. Harvesting the same amount from all the age groups:

$$h_1 = h_2 = h_3 = h_4 = h_5 = h = \frac{\lambda_0 - 1}{\lambda_0}.$$

II. In some experiments we want to harvest only a certain group. E.g., in our case, lambs' meat is much more popular. So, it would be a wise choice to harvest lambs and let the mature ones breed. Therefor our H matrix will only have one diagonal entry h_1. So:

$$(1-h)(f_1 + f_2 s_1 + f_3 s_1 s_2 + \cdots) = 1$$

$R = (f_1 + f_2 s_1 + f_3 s_1 s_2 + \cdots)$ is called **net reproduction rate.** Which actually is the average number of offspring (often specifically daughters) that would be born to a female if she passed through her lifetime conforming to the age-specific fertility and mortality rates of a given year.

III. The last method we want to introduce is the largest possible harvest. As we could imagine there is a bound to it which is concluded from the following theorem.

If a sustainable harvesting policy is optimal, it harvests only from one or two age classes. If two age classes are harvested, then the older class is completely harvested.

In our farm example, the optimal harvesting will completely omit the oldest class.