Section 5D:

Definition: A diagonal matrix is a square matrix that is 0 everywhere except possibly on the diagonal.

Definition: An operator on V is called diagonalizable if the operator has a diagonal matrix with respect to some basis of V.

Definition: $T \in$ L(V) and $\lambda \in \mathbf{F}$. The eigenspace of T corresponding to λ is the subspace $E(\lambda, T)$ of V defined by $E(\lambda, T)$ = null($T - \lambda I$) = $\{v \in V : Tv = \lambda v\}$. Hence $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector

Result: Sum of eigenvalues is a direct sum -> $T \in$ L(V) and $\lambda 1, \ldots, \lambda m$ are distinct eigenvalues of T. Then $E(\lambda 1, T) + \cdots + E(\lambda m, T)$ is a direct sum. Furthermore, if V is finite-dimensional, then dim $E(\lambda 1, T) + \cdots +$ dim $E(\lambda m, T) \leq$ dim V.

Result: Conditions equivalent to diagonalizability

V is finite-dimensional and $T \in$ L(V). Let $\lambda 1, \ldots, \lambda m$ denote the distinct eigenvalues of T. Then TFAE

(a) T is diagonalizable.

(b) V has a basis consisting of eigenvectors of T.

(c) $V = E(\lambda 1, T) \oplus \cdots \oplus E(\lambda m, T)$.

(d) dim V = dim $E(\lambda 1, T) + \cdots +$ dim $E(\lambda m, T)$.

Result: Enough eigenvalues implies diagonalizability -> V is finite-dimensional and $T \in$ L(V) has dim V distinct eigenvalues. Then T is diagonalizable.

Result: Restriction of diagonalizable operator to invariant subspace -> $T \in$ L(V) is diagonalizable and U is a subspace of V that is invariant under T. Then $T|_U$ is a diagonalizable operator on U.

Section 5E:

Definition: commute

• Two operators S and T on the same vector space commute if $ST = TS$.

• Two square matrices A and B of the same size commute if $AB = BA$.

Result: Eigenspace is invariant under commuting operator - $S, T \in$ L(V) commute and $\lambda \in \mathbf{F}$. Then $E(\lambda, S)$ is invariant under T.

Result: Simultaneous diagonalizablity ⇔ commutativity -> Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis iff the two operators commute.

Result: Eigenvalues of sum and product of commuting operators

V is a finite-dimensional complex vector space and S, T are commuting operators on V.

• every eigenvalue of $S + T$ is an eigenvalue of S plus an eigenvalue of T,

• every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T

Section 6A:

Definition: For $x, y \in \mathbf{R}n$, the dot product of x and y, denoted by $x \cdot y$, is defined by $x \cdot y = x1y1 + \cdots + xnyn$, where $x = (x1, \ldots, xn)$ and $y = (y1, \ldots, yn)$.

Definition: An inner product on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbf{F}$ and has the following properties.

positivity: $\langle v, v\rangle \geq 0$ for all $v \in V$.

definiteness: $\langle v, v\rangle = 0$ iff $v = 0$.

additivity in first slot: $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.

homogeneity in first slot: $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$.

conjugate symmetry: $\langle u, v\rangle = \overline{< v, u >}$ for all $u, v \in$ V

Definition: An inner product space is a vector space V along with an inner product on V

Result: Basic properties of an inner product

(a) For each fixed $v \in V$, the function that takes $u \in V$ to $\langle u, v\rangle$ is a lin map from V to $\mathbf{F}$.

(b) $\langle 0, v\rangle = 0$ for every $v \in V$.

(c) $\langle v, 0\rangle = 0$ for every $v \in V$.

(d) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$.

(e) $\langle u, \lambda v\rangle = \bar{\lambda}\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in$ V

Definition: For $v \in V$, the norm of v is defined by $\|v\| = \sqrt{< v, v >}$.

Result: Basic properties of the norm

Suppose $v \in V$.

(a) $\|v\| = 0$ iff $v = 0$.

(b) $\|\lambda v\| = |\lambda|\ \|v\|$ for all $\lambda \in$ F

Definition: Two vectors $u, v \in V$ are called orthogonal if $\langle u, v\rangle = 0$.

Result: orthogonality and 0

(a) 0 is orthogonal to every vector in V.

(b) 0 is the only vector in V that is orthogonal to itself.

Result: Pythagorean theorem

$u, v \in V$. If u and v are orthogonal, then

$$||u + v||^2 \leq ||u||^2 + ||v||^2$$

Result: an orthogonal decomposition

$u, v \in V$, with $v \neq 0$. Set $c = \frac{<u,v>}{||v||^2}$ and $w = u - \frac{<u,v>}{||v||^2}v$. Then $u = cv + w$ and $\langle w, v\rangle = 0$.

Result: Cauchy–Schwarz inequality

$u, v \in V$. Then $|\langle u, v\rangle| \leq \|u\|\ \|v\|$. This inequality is an equality iff one of u, v is a scalar multiple of the other

Result: triangle inequality

$u, v \in V$. Then $\|u + v\| \leq \|u\| + \|v\|$. This inequality is an equality iff one of u, v is a noneg real multiple of the other.

Result: parallelogram equality

$u, v \in V$. Then $||u + v||^2 + ||u - v||^2 = 2(||u||^2 + ||v||^2)$

Section 6B:

Definition: orthonormal

- A list of vectors is called orthonormal if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.
- In other words, a list $e1, \ldots, em$ of vectors in V is orthonormal if $< e_j, e_k >= \{ {}^{1\ if\ j=k}_{0\ if\ j\neq k}$ for all j,k $\in$ {1,...,m}

Result: norm of an orthonormal lin combination

$e1, \ldots, em$ is an orthonormal list of vectors in V. Then

$||a_1e_1 + ... + a_me_m||^2 = |a_1|^2 + ... + |a_m|^2$ for all a1,...,am $\in$ F.

Result: Every orthonormal list of vectors is linearly independ

Result: Bessel's inequality

$e1, \ldots, em$ is orthonormal list of vectors in V. If $v \in V$ then

$| < v, e_1 > |^2 + ... + | < v, e_m > |^2 \leq ||v||^2$

Definition: An orthonormal basis of V is an orthonormal list of vectors in V that is also a basis of V.

Result: orthonormal lists of the right length are orthonormal bases -> Suppose V is finite-dimensional. Then every orthonormal list of vectors in V of length dim V is an orthonormal basis of V.

Result: writing a vector as a lin combination of an orthonormal basis

$e1, \ldots, en$ is an orthonormal basis of V and $u, v \in V$. Then

(a) $v =< v, e_1 > e_1 + ... + < v, e_n > e_n$

(b) $||v||^2 = | < v, e_1 > |^2 + ... + | < v, e_n > |^2$

(c) $< u, v >=< u, e_1 > \overline{< v, e_1 >} + ... + < u, e_n > \overline{< v, e_n >}$

Result: Gram–Schmidt procedure

$v1, \ldots, vm$ is a linearly independ list of vectors in V. Let $f1 = v1$. For k=2,..., m, define fk inductively by $f_k = v_k - \frac{<v_k,f_1>}{||f_1||^2}f_1 - ... - \frac{<v_k,f_{k-1}>}{||f_{k-1}||^2}f_{k-1}$. For each k=1,...,m. $e_k = \frac{f_k}{||f_k||}$. Then $e1,\ldots,em$ is an orthonormal list of vectors in V such that span($v1, \ldots, vk$) = span($e1, \ldots, ek$) for each k = 1, ..., m

Result: Every finite-dimensional inner product space has an orthonormal basis.

Result: V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

Definition: linear functional, dual space, V'

• A lin functional on V is a lin map from V to $\mathbf{F}$.

• The dual space of V, denoted by V', is the vector space of all lin functionals on V. In other words, $V' =$ L(V, $\mathbf{F}$).

Result: Riesz representation theorem -> V is finite-dimensional and φ is a lin functional on V. Then there is a unique vector $v \in V$ such that $\varphi(u) = \langle u, v\rangle$ for every $u \in$ V.

Section 6C:

Definition: If U is a subset of V, then the orthogonal complement of U, denoted by $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U: $U^\perp = \{v \in V : \langle u, v\rangle = 0$ for every $u \in U\}$.

Result: properties of orthogonal complement

(a) If U is a subset of V, then $U^\perp$ is a subspace of V.

(b) $\{0\}^\perp = V$.

(c) $V^\perp = \{0\}$.

(d) If U is a subset of V, then $U \cap U^\perp \subseteq \{0\}$.

(e) If G and H are subsets of V and $G \subseteq H$, then $H^\perp \subseteq G^\perp$.

Result: direct sum of a subspace and its orthogonal complement -> U is a finite-dimensional subspace of V. Then $V = U \oplus U^\perp$.

Result: dimension of orthogonal complement -> V is finite-dimensional and U is a subspace of V. Then dim $U^\perp$ = dim V − dim U.

Result: Suppose U is a finite-dimensional subspace of V. Then $U = (U^\perp)^\perp$.

Definition: U is a finite-dimensional subspace of V. The orthogonal projection of V onto U is the operator $P_U \in$ L(V) defined as follows: For each $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U\perp$. Then let $P_Uv = u$.

Result: properties of orthogonal projection P_U

U is a finite-dimensional subspace of V. Then

a) $P_U \in L(V)$

b) $P_Uu = u\ for\ every\ u \in U$

c) $P_Uw = 0\ for\ every\ w \in U^\perp$

d) range $P_U = U$

e) null $P_U = U^\perp$

f) $v - P_Uv \in U^\perp\ for\ every\ v \in V$

g) $P_U{}^2 = P_U$

h) $||P_Uv|| \leq ||v||\ for\ every\ v \in V$

i) if $e_1,..., e_m$ is an orthonormal basis of U and $v \in V$, then $P_Uv =< v, e_1 > e_1 + ... + < v, e_m > e_m$

Section 7A:

Definition: Suppose $T \in$ L(V, W). The adjoint of T is the function $T^* : W \to V$ such that $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for every $v \in V$ and every $w \in W$.

Result: adjoint of a lin map is a lin map -> If $T \in$ L(V, W), then $T^* \in$ L(W, V).

Result: Properties of the Adjoint

$T \in$ L(V, W). Then

a) $(S + T)^* = S^* + T^*$ for all S $\in$ L(V, W).

b) $(\lambda T)^* = \bar{\lambda}T^*$ for all $\lambda \in \mathbf{F}$;

c) $(T^*)^* = T$

d) $(ST)^* = T^*S^*$ for all $S \in$ L(W, U) (here U is a finite-dimensional inner product space over $\mathbf{F}$);

e) $I^* = I$, where I is the identity operator on V;

f) if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$

Result: null space and range of T^*

$T \in$ L(V, W). Then

(a) $null\ T^* = (range\ T)^\perp$

(b) $range\ T^* = (null\ T)^\perp$

(c) $null\ T = (range\ T^*)^\perp$

(d) $range\ T = (null\ T^*)^\perp$

Definition: The conjugate transpose of an m-by-n matrix A is the n-by-m matrix A^* obtained by interchanging the rows and columns and then taking the complex conjugate of each entry. In other words, if $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, then $(A^*)_{j,k} = \overline{A_{k,j}}$

Result: matrix of $T*$ equals conjugate transpose of matrix of T

Let $T \in$ L(V, W). $e1, \ldots, en$ is an orthonormal basis of V and $f1, \ldots, fm$ is an orthonormal basis of W. Then $\mathcal{M}(T^*, (f1, \ldots, fm), (e1, \ldots, en))$ is the conjugate transpose of $\mathcal{M}(T, (e1, \ldots, en), (f1, \ldots, fm))$. In other words, $\mathcal{M}(T^*) = (\mu(T))^*$.

Definition: An operator $T \in L(V)$ is called self-adjoint if $T = T^*$.
Result: Every eigenvalue of a self-adjoint operator is real.
Result: T self-adjoint and $\langle Tv, v\rangle = 0$ for all $v \Leftrightarrow T = 0$ -> T is a self-adjoint operator on V. Then $\langle Tv, v\rangle = 0$ for every $v \in V \Leftrightarrow T = 0$.
Definition: normal

- An operator on an inner product space is called normal if it commutes with its adjoint.
- In other words, $T \in L(V)$ is normal if $TT^* = T^*T$.

Result: T is normal iff Tv and T^*v have the same norm -> $T \in L(V)$. Then T is normal $\Leftrightarrow \|Tv\| = \|T^*v\|$ for every $v \in V$.
Result: range, null space, and eigenvectors of a normal operator ($T \in L(V)$ is normal)

(a) null T = null T^* ;
(b) range T = range T^* ;
(c) V = null $T \oplus$ range T;
(d) $T - \lambda I$ is normal for every $\lambda \in \mathbf{F}$;
(e) if $v \in V$ and $\lambda \in \mathbf{F}$, then $Tv = \lambda v$ iff $T^*v = \lambda v$.