Ćirić type fixed point theorems

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Connections with other results given in:

L.B. Ćirić: *Generalized contraction and fixed point theorems*, Publ. Inst. Math., 12 (1971), 19-26.

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- **Definition.** Let X be a nonempty set. Then, by definition $(X, \to, \leq)$ is an ordered L-space if and only if:
  (i) $(X, \to)$ is an L-space;
  (ii) $(X, \leq)$ is a partially ordered set;
  (iii) $(x_n)_{n\in\mathbb{N}} \to x$, $(y_n)_{n\in\mathbb{N}} \to y$ and $x_n \leq y_n$, for each $n \in \mathbb{N} \ \Rightarrow x \leq y$.
- If (X, d) is a metric space, then the triple $(X, d, \leq)$ will be called an ordered metric space.

- Let $(X, \leq)$ be a partially ordered set.
  Denote
  $$X_{\leq} := \{(x, y) \in X \times X | x \leq y \text{ or } y \leq x\}.$$

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- In the same setting, consider $f : X \to X$. Then:
  $(LF)_f := \{x \in X | x \leq f(x)\}$ is the lower fixed point set of f,
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- If $f : X \to X$ and $g : Y \to Y$, then the cartesian product of f and g is denoted by $f \times g$ and it is defined in the following way:
  $$f \times g : X \times Y \to X \times Y, (f \times g)(x, y) := (f(x), g(y)).$$

Definition. (I.A. Rus) Let $(X, \to)$ be an L-space.
An operator $f : X \to X$ is, by definition, a Picard operator if:
(i) $F_f = \{x^*\}$;
(ii) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Theorem. (Ran and Reurings-2004) *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Let d be a metric on X such that the metric space (X, d) is complete. Let $f : X \to X$ be a continuous and monotone (i. e., either decreasing or increasing) operator. Suppose that the following two assertions hold:*

1) there exists $a \in]0, 1[$ such that $d(f(x), f(y)) \leq a \cdot d(x, y)$, for each $x, y \in X$ with $x \leq y$

2) $(LF)_f \cup (UF)_f \neq \emptyset$.

Then f is a Picard operator.

Theorem. (Nieto and Rodríguez-López, 2005) *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower or an upper bound. Let d be a metric on X such that the metric space (X, d) is complete. Let $f : X \to X$ be an increasing operator. Suppose that the following two assertions hold:*

1) there exists $a \in]0, 1[$ such that $d(f(x), f(y)) \leq a \cdot d(x, y)$, for each $x, y \in X$ with $x \leq y$;

2) there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$;

3) if an increasing sequence (x_n) converges to x in X, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Then f is a Picard operator.

Lemma.

(A. Petrușel and Rus, 2006)
Let $(X, \to)$ be an L-space and U a symmetric subset of $X \times X$ such that $\Delta(X) \subset U$. Let $f : X \to X$ be an operator. Suppose that:
(i) for each $x, y \in X$ with $(x, y) \notin U$ there exists $z \in X$ such that $(x, z) \in U$ and $(y, z) \in U$;
(ii) there exist $x_0, x^* \in X$ such that $x_0 \in A_f(x^*)$;
(iii) $(x, y) \in U$ and $x \in A_f(x^*)$ implies $y \in A_f(x^*)$.
Then $A_f(x^*) = X$.
Moreover, if f is orbitally continuous, then f is a Picard operator.

where

$$A_f(x^*) := \{x \in X \ : \ (f^n(x))_{n \in \mathbb{N}} \to x^* \text{ as } n \to \infty\}.$$

A natural consequence of the above result follows by choosing $U := X_{\leq}$.

Lemma.

(A. Petrușel and Rus, 2006)
Let $(X, \to, \leq)$ *be an ordered L-space and* $f : X \to X$ *be an operator.*
Suppose that:
(i) for each $x, y \in X$ *with* $(x, y) \notin X_{\leq}$ *there exists* $z \in X$ *such that* $(x, z) \in X_{\leq}$ *and* $(y, z) \in X_{\leq}$*;*
(ii) there exist $x_0, x^* \in X$ *such that* $x_0 \in A_f(x^*)$*;*
(iii) $(x, y) \in X_{\leq}$ *and* $x \in A_f(x^*)$ *implies* $y \in A_f(x^*)$*;*
(iv) f *is orbitally continuous*
Then f *is a Picard opeartor.*

Theorem.

(O'Regan and A. Petrușel, 2008)
Let $(X, d, \leq)$ be an ordered metric space and $f : X \to X$ be an operator. We suppose that:

(i) For each $x, y \in X$ with $(x, y) \notin X_{\leq}$ there exists $c(x, y) \in X$ such that $(x, c(x, y)) \in X_{\leq}$ and $(y, c(x, y)) \in X_{\leq}$;

(ii) $f : (X, \leq) \to (X, \leq)$ is increasing;

(iii) there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$;

$(iv)_a$ f is orbitally continuous

or

$(iv)_b$ if an increasing sequence (x_n) converges to x in X, then $x_n \leq x$ for all $n \in \mathbb{N}$;

(v) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d(f(x), f(y)) \leq \varphi(d(x, y))$, for each $(x, y) \in X_{\leq}$;

(vi) the metric d is complete.

Then f is a Picard operator.

Let (X, d) be a metric space and Δ the diagonal of $X \times X$.

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Let G be a directed graph such that one can identify G with the pair $(V(G), E(G))$, where the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of the edges of the graph contains Δ.

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If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $(x_n)_{n\in\{0,1,2,\cdots,k\}}$ of vertices such that

$$x_0 = x, x_k = y \text{ and } (x_{i-1}, x_i) \in E(G), \text{ for } i \in \{1, 2, \cdots, k\}.$$

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A graph G is connected if there is a path between any two vertices and it is weakly connected if $\tilde{G}$ is connected, where $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges.

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$$(x, y) \in E(G) \text{ implies } d\left(f(x), f(y)\right) \leq \alpha d(x, y).$$

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Examples:
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2) Let $\leq$ be a partial order in X. Define the graph G_1 by

$$E(G_1) := \{(x, y) \in X \times X \ : \ x \leq y\}.$$

3) Let $\leq$ be a partial order in X. Define the graph G_2 by

$$E(G_2) := \{(x, y) \in X \times X \ : \ x \leq y \text{ or } y \leq x\}.$$

Theorem. (J. Jachymski-2008)

Let (X, d) be a complete metric space and let G be a directed graph G such that the triple (X, d, G) has property (P):

$$(P)\quad \begin{array}{l} \text{for any sequence } (x_n)_{n\in\mathbb{N}} \subset X, \text{ if } x_n \to x \text{ as } n \to +\infty \text{ and} \\ (x_n, x_{n+1}) \in E(G), \text{ for each } n \in \mathbb{N}, \text{ then there exists a subsequence} \\ (x_{k_n})_{n\in\mathbb{N}} \text{ of } (x_n)_{n\in\mathbb{N}} \text{ such that } (x_{k_n}, x) \in E(G), \text{ for each } n \in \mathbb{N}. \end{array}$$

Let $f : X \to X$ be a G-contraction. Then the following statments hold:

1) $F_f \neq \emptyset$ if and only if $X_f \neq \emptyset$,

$$\text{where } X_f := \{x \in X \ : \ (x, f(x)) \in E(G)\};$$

2) if $X_f \neq \emptyset$ and G is weakly connected, then f is a Picard operator.

Theorem.

(Nicolae-O'Regan-A. Petrușel, 2011)
Let (X, d) be a complete metric space and G be a directed graph such that the triple (X, d, G) satisfy property (P). Let $T : X \to P_{cl}(X)$ be a multi-valued operator. Suppose the following assertions hold:

(i) there exists $\alpha \in (0, 1)$ such that

$$H(T(x), T(y)) \leq \alpha d(x, y) \text{ for all } (x, y) \in E(G).$$

(ii) for each $(x, y) \in E(G)$, each $u \in T(x)$ and $v \in T(y)$ satisfying the condition $d(u, v) \leq ad(x, y)$, for some $a \in (0, 1)$, we have $(u, v) \in E(G)$;
Then $F_T \neq \emptyset$ if and only if $X_T \neq \emptyset$, where

$$X_T := \{x \in X;\ \text{there exists } y \in T(x) \text{ such that } (x, y) \in E(G)\},$$

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