Mathematical Models applied in Economy

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Chapter 1

Ordinary Differential Equations and Linear Systems

1.1 Equations with separable variables

This chapter presents the general method of solving separable equations and a few examples.

1.1.1 The general form of an equation with separable variables

Let us consider the equation:

$$x' = f(t)g(x) \tag{SE}$$

where $f : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ and $g : (x_1, x_2) \subset \mathbb{R} \to \mathbb{R}$ are continuous functions and g does not cancel on (x_1, x_2). Before showing how to solve (SE), we recall the following definition:

Definition 1.1.1.1. *We call a solution on the interval $I \subset \mathbb{R}$ for the differential equation $F(t, x, x') = 0$ (where F is a real function defined on an open set from $\mathbb{R}^3$) a function $x : I \to \mathbb{R}$, differentiable on I and which verifies the equation on I, meaning:*

$$F(t, x(t), x'(t)) = 0, \forall t \in I$$

It is understood that x is such as $(t, x(t), x'(t))$ is in the domain of function F for $t \in I$. When reffering to a solution, we will usually point the interval on which it is defined (even the maximal interval if possible).

We return now to (SE). We assume that $x = x(t)$,$t \in (t_1, t_2)$ is a solution for (SE). Then

$$\int_{x_0}^{x(t)} \frac{d\xi}{g(\xi)} = \int_{t_0}^{t} f(s)ds, t \in (t_1, t_2) \tag{1.1}$$

where t_0 is a random point from the interval (t_1, t_2) and $x_0 = x(t_0)$. We define

$$G(y) = \int_{x_0}^{y} \frac{d\xi}{g(\xi)}, y \in (x_1, x_2) \tag{1.2}$$

knowing that G is a differentiable function (with continuous derivative) on (x_1, x_2) and strictly monotone. Therefore we can talk about G^{-1}, defined on the set $G((x_1, x_2))$ which has the same properties as function G. Since relation (1.1) can be written:

$$G(x(t)) = \int_{t_0}^{t} f(s)ds, t \in (t_1, t_2) \tag{1.3}$$

results that solution x has the following expression:

$$x(t) = G^{-1}\left(\int_{t_0}^{t} f(s)ds\right), t \in (t_1, t_2) \tag{1.4}$$

Mutually, a function $x = x(t)$ defined by relation (1.4) (where x_0 is arbitrary in (x_1, x_2) and t goes through a neighbourhood of point t_0 such that $\int_{t_0}^{t} f(s)ds$ is in the domain of function G^{-1}) is a solution for (SE), also checking the Cauchy condition $x(t_0) = x_0$.

1.1.2 Examples

1) $\begin{cases} xx' = e^{-t} \\ x(0) = e \end{cases}$

Solution: Let us consider first the equation $xx' = e^{-t}$. Then we have

$$x\frac{dx}{dx} = e^{-t}.$$

Separating the variables, we obtain

$$xdx = e^{-t}dt.$$

By integration, we get:

$$\int xdx = \int e^{-t}dt \Leftrightarrow$$

$$\frac{x^2}{2} = -e^{-t} + c_1, c_1 \in \mathbb{R}.$$

$$x^2 = -2e^{-t} + c, c \in \mathbb{R}.$$

This form represents the general solution in implicit form. We impose the Cauchy condition $x(0) = e$. Then

$$x^2 = -2e^0 + c.$$

Hence $e^2 = -2 + c$. Thus $c = e^2 + 2$. Hence the solution in implicit form is

$$x^2 = -2e^{-t} + e^2 + 2.$$

Hence

$$x = \pm\sqrt{-2e^{-t} + e^2 + 2}.$$

We choose $x = \sqrt{-2e^{-t} + e^2 + 2}$. Thus, for $t > 0$ we have the solution:

$$x = \sqrt{2(1 - e^{-t}) + e^2}.$$

2) $x' = x^2 - x$, $t \in \mathbb{R}$. We consider the equation:

$$x' = x(x - 1).$$

For $x \neq 0$ and $x \neq 1$ we can separate the variables:

$$\frac{1}{x(x-1)}dx = dt.$$

By integrating, we obtain:

$$\int \frac{1}{x(x-1)}dx = \int dt.$$

$\Leftrightarrow$

$$\int \left(\frac{1}{x-1} - \frac{1}{x}\right)dx = t + c_1, c_1 \in \mathbb{R}.$$

Next, resolving the integrals, we get:

$$ln|x - 1| - ln|x| = t + c_1,$$

$$ln|\frac{x-1}{x}| = t + c_1$$

$\Leftrightarrow$

$$|\frac{x-1}{x}| = c_2 e^t, c_2 \in \mathbb{R}.$$

$$\frac{x-1}{x} = \pm c_2 e^t.$$

We can consider $\pm c_2$ a constant $c \in \mathbb{R}^*$. Then, the implicit form of the solution is:

$$\frac{x-1}{x} = ce^t, c \in \mathbb{R}^*.$$

Next, $x - 1 = xce^t$, resulting that

$$x = \frac{1}{1 - ce^t}, c \in \mathbb{R}^*. \tag{1.1}$$

We also notice that $x(t) = 0$ and $x(t) = 1$, where $t \in \mathbb{R}$ are solution for our equation. For $c = 0$ in (1.1) we get $x(t) = 1$. Hence the solutions are

$$x(t) = 0, t \in \mathbb{R}$$

$$x(t) = \frac{1}{1 - ce^t}, c \in \mathbb{R}, t \in J$$

where J is defined by the restriction $1 - ce^t = 0$.

(i) $c \in R \Rightarrow e^t = \frac{1}{c} \Rightarrow$

  (a) $c > 0 \Rightarrow e^t = \frac{1}{c} \Leftrightarrow t = \ln \frac{1}{c} = -\ln c \Rightarrow x(t) = \frac{1}{1-ce^t}$, $t \in (-\infty, -\ln c)$ or $t \in (-\ln c, +\infty)$.

  (b) $c < 0 \Rightarrow e^t = \frac{1}{c} \Leftrightarrow t \in \emptyset \Rightarrow x(t) = \frac{1}{1-ce^t}, t \in \mathbb{R}$

(ii) $c = 0 \Rightarrow x(t) = 1, t \in \mathbb{R}$.

For example, if we have a Cauchy problem of the following form: $\begin{cases} x' = x^2 - x \\ x(0) = 2 \end{cases}$
then $x(t) = 0$ and $x(t) = 1$, $t \in \mathbb{R}$ are not solutions.
For $x(t) = \frac{1}{1-ce^t}$:

$$x(0) = 2 \Leftrightarrow \frac{1}{1 - c} = 2 \Leftrightarrow 1 - c = \frac{1}{2} \Leftrightarrow c = \frac{1}{2}.$$

In conclusion, the solution for the given Cauchy problem is:

$$x(t) = \frac{1}{1 - \frac{1}{2}e^t}, t \in (-\infty, -\ln 2).$$

For the phase portrait, we have $x' = f(x)$. $f(x) = 0 \Leftrightarrow x = 0$ or $x = 1$. The function $f(x) = x^2 - x$ has the following signs on $\mathbb{R}$:

x	$-\infty$		0		1		∞
$x^2 - x$		$+$	0	$-$	0	$+$	

1.2 Linear Differential Equations of First Order

The following chapter introduces the reader to the general method of solving Linear Differential Equations of First Order and a few examples.

1.2.1 The General Form of a linear differential equation of first order

A Linear differential equation has the following expression:

$$x' = a(t)x + b(t) \tag{LE}$$

where $a, b : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ are continous on (t_1, t_2) (bounded or not). If $x = x(t)$, $t_1 < t < t_2$ is a solutions for (LE), then multiplying with $exp(-\int_{t_0}^{t} a(s)ds)$, where t_0 is arbitrary chosen from (t_1, t_2), the following equation is obtained

$$\frac{d}{dt}\left[e^{-\int_{t_0}^{t} a(s)ds} x(t)\right] = b(t)e^{-\int_{t_0}^{t} a(s)ds}, t \in (t_1, t_2)$$

So

$$x(t) = e^{\int_{t_0}^{t} a(s)ds}(x_0 + \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a(\sigma)d\sigma}), t \in (t_1, t_2) \tag{SOL}$$

where x_0 is an arbitrary real number. Reciprocally, we can easily agree that any function $x = x(t), t \in (t_1, t_2)$ given by the formula (SOL) is a solution for (LE). Actually, (SOL) asserts the solution of (LE) with the Cauchy condition $x(t_0) = x_0$.

Sometimes, it is more convienent to use the following form of (SOL):

$$x(t) = e^{\int a(s)ds} * \int b(t)e^{-\int a(t)dt}dt \tag{SOL*}$$

with the convention that $\int a(t)dt$ is a fixed primitive of $a = a(t)$ (the same in (SOL) and (SOL*)).

1.3 Linear Differential Systems

1.3.1 The general form of a linear differential system

Many evolutive processes from the real world can't be described by only one variable. Therefore, for two or more variables, we must consider a system of two or more differential equations. In the following chapter, we will consider differential systems of the first order:

$$\begin{cases} x' = f(t, x, y) \\ y' = g(t, x, y) \end{cases} \tag{DS}$$

where f,g are given functions and the unknowns are the functions x, y of variable t.

Definition 1.3.1.1. *A solution for* (DS) *is a pair* $(x, y) \in C^1(J)$*, where* $J \subseteq \mathbb{R}$*, which satisfy on* J *the two equations from* (DS)*, for any* $t \in J$*.*

(DS) is linear if functions f and g depend affinely on x and y. Then (DS) has the following form:

$$\begin{cases} x' = a_{11}(t)x + a_{12}(t)y + b_1(t) \\ y' = a_{21}(t)x + a_{22}(t)y + b_2(t) \end{cases} \tag{DS*}$$

Coefficients a_{ij} and free terms b_i are alleged to be continuous functions on an interval J. If, in particular, coefficients a_{ij} are constants, we say that the linear system is with constant coefficients. If $b_i = 0$, we say that the system is homogeneous. If both situations take place, the system is linear and homogeneous, with constant coefficients.

1.3.2 Matrix Analysis Theory

Let us define $\mathcal{M}_{nm}(\mathbb{K})$ the set of matrices with n rows and m columns, with elements from the field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$. It is known that $(\mathcal{M}_{nm}(\mathbb{K}), +, \cdot, \mathbb{R})$ is a linear space of dimension $n \cdot m$. It can be organized as a normalized space using the following norm:

$$\|A\| = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}$$

Then $(\mathcal{M}_{nm}(\mathbb{K}), \|\cdot\|)$ is a Banach space. The defined norm has the following properties on the Banach space presented earlier:

1. $\|A + B\| \leq \|A\| + \|B\|$
2. $\|\lambda A\| = |\lambda| \|A\|$
3. $\|Ax\|_{\mathbb{R}^n} = \|A\| \|x\|_{\mathbb{R}^n}$
4. $\|A \cdot B\| \leq \|A\| \cdot \|B\|$

Next, let $M \in \mathcal{M}_{nn}(\mathbb{K})$. Then:

$$e^M = \sum_{k \geq 0} \frac{1}{k!} M^k \tag{3.1}$$

Proof. Let $S_n = \sum_{k=0}^{n} \frac{1}{k!} M^k$ the partial sum of serie (3.1). Next, we prove that S_n is a Cauchy sequence in the Banach space $(\mathcal{M}_{nm}(\mathbb{K}), \|\cdot\|)$.

$$\|S_{n+p} - S_n\| = \| \frac{1}{(n+1!)} A^{n+1} + ... + \frac{1}{(n+p!)} A^{n+p} \|$$

$$\leq \frac{1}{(n+1)!}\|A^{n+1}\| + ... + \frac{1}{(n+p)!}\|A^{n+p}\| = |a_{n+p} - a_n|$$

where

$$a_n = \sum_{k=0}^{n} \frac{1}{k!}\|M\|^k$$

.

Due to the fact that

$$\sum_{k\geq 0} \frac{1}{k!}x^k$$

is a convergent series and the sum of it is equal to e^x, $\forall x \in \mathbb{R} \Rightarrow$

$$\sum_{k\geq 0} \frac{1}{k!}\|M\|^k$$

is convergent, therefore a_n is convergent, which means that a_n is Cauchy $\Rightarrow$ $\forall \varepsilon > 0$, $\exists n(\varepsilon) \in \mathbb{N}$ so that

$$\|S_{n+p} - S_n\| \leq \varepsilon$$

$\forall n \geq n(\varepsilon)$, $\forall p \in \mathbb{N} \Rightarrow (S_n)$ is a Cauchy sequence $\Rightarrow$ the existence of (3.1) is proved.

□

1.3.3 Existence and uniqueness Theorems

Let us consider the system:

$$\begin{cases} u' = A(t)u + B(t) \\ u(t_0) = u_0 \end{cases} \tag{Sys1}$$

where $t \in J = [t_0 - a, t_0 + a], a > 0$ $t_0 \in J$ and $u_0 \in \mathbb{R}^2$.
Regarding the existence and uniqueness of the solution of the previous system, we present the following result:

Theorem 1.3.3.1. *Considering* (Sys1)*, we assume that* $A \in C(J, \mathcal{M}_2(\mathbb{R}))$, $B \in C(J, \mathbb{R}^2)$. *Then, there exists a unique solution* $u^* \in C^1(J, \mathbb{R}^2)$.

Proof. We will use the following Lemma:

Lemma 1.3.3.2. *The solution* u^* *is equivalent to the following system of integral equations:*

$$u(t) = \int_{t_0}^{t} \Big[A(s)u(s) + B(s)\Big] ds + u_0 \tag{IntSol}$$

Proof. "$\Rightarrow$ "If u satisfies (Sys1), then by integrating from t_0 to $t \in J$, we get:

$$u(t) - u(t_0) = \int_{t_0}^{t} \Big[A(s)u(s) + B(s) \Big] ds$$

$u(t_0) = u_0 \Rightarrow$ (IntSol).
"$\Rightarrow$"Let u be a solution of (IntSol). Then $u'(t) = A(t)u + B(t)$. Since the right side is continuous, we obtain that $u \in C^1(J, \mathbb{R})$. Moreover, $u(t_0) = u_0$. The lemma is proved. □

Resuming the proof of the theorem, let us denote $A : C(J, \mathbb{R}^2) \to C(J, \mathbb{R}^2)$, $u \longmapsto Au$, where: $Au(t) := \int_{t_0}^{t} \Big[A(s)u(s) + B(s) \Big] ds + u_0$

"(IntSol) $\Leftrightarrow u = Au$"

We consider on $C(J, \mathbb{R}^2)$ the Bielecki norm:

$$\|u\|_B = max_{t \in J}(\|u\|_{\mathbb{R}^2} e^{-\tau|t-t_0|}), \tau > 0$$

$(C(J, \mathbb{R}^2), \|.\|)$ is a Banach space.
Then we have that $\|Au - Av\|_B \leq L_A \|u - v\|_B, \forall u, v \in C(J, \mathbb{R}^2, L_A \in (0, 1))$.
For $t \geq t_0$:

$$\begin{aligned}
\|Au(t) - Av(t)\|_{\mathbb{R}^2} &= \| \int_{t_0}^{t} A(s)(u(s) - v(s))ds \| \\
&\leq \int_{t_0}^{t} \|A(s)(u(s) - v(s))\| ds \\
&\leq \int_{t_0}^{t} \|A(s)\|_{\mathcal{M}_2(\mathbb{R}^2)} \|u(s) - v(s)\|_{\mathbb{R}^2} ds \\
&\leq M_A \int_{t_0}^{t} \|u(s) - v(s)\| e^{-\tau(s-t_0)} e^{\tau(s-t_0)} ds \\
&\leq \int_{t_0}^{t} \max_{(s \in J)} (\|u(s) - v(s)\| e^{-\tau(s-t_0)}) e^{\tau(s-t_0)} ds \\
&\leq M_A \|u - v\|_B \frac{1}{\tau} (e^{\tau(t-t_0)} - 1) \\
&\leq \frac{M_A}{\tau} \|u - v\|_B e^{\tau(t-t_0)} \\
&\Rightarrow \|Au - Av\| \leq \frac{M_A}{\tau} \|u - v\|_B.
\end{aligned}$$

Let $L_A = \frac{M_A}{\tau}$ such as $\tau > M_A$. Then $L_A < 1$.

□

1.3.4 Representations of the solution

Let us remark that (DS) can be written as a single vectorial equation:

$$u' = F(t, u) \tag{VDS}$$

where u and F are vectorials, with two real components, more exactly column matrix:

$$u = \begin{bmatrix} x \\ y \end{bmatrix}, \quad F(t, u) = \begin{bmatrix} f(t, x, y) \\ g(t, x, y) \end{bmatrix}$$

The condition for the Cauchy Problem of the system can be written:

$$u(t_0) = u_0, u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Also, (DS*) can be written:

$$u' = A(t)u + B(t) \tag{MDS}$$

where

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \quad B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

Theorem 1.3.4.1 (of representing the solutions of linear systems). *Let $A \in C(J, \mathcal{M}_{\in}))$ and $B \in (J, R^2)$. Then the solutions of the linear system* (MDS) *are defined by the formula*

$$u = e^{\int_{t_0}^t A(\sigma)d\sigma} C + \int_{t_0}^t e^{\int_s^t A(\sigma)d\sigma} B(s)ds$$

where $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ *and* $C_1, C_2 \in \mathbb{R}$.

Theorem 1.3.4.2 (of existence, uniqueness and representation of the solution). *Let $A \in C(J, M_2(\mathbb{R})), B \in C(J, \mathbb{R}), t_0 \in J and u_0 \in \mathbb{R}^2$. Then the Cauchy Problem has a unique solution defined on J, given by the formula:*

$$u = e^{\int_{t_0}^t A(\sigma)d\sigma} u_0 + \int_{t_0}^t e^{\int_s^t A(\sigma)d\sigma} B(s)ds$$

Theorem 1.3.4.3 (the structure of the set of solutions). *(a) The set of solutions of a bidimensional linear and homogeneous system is a linear bidimensional space.*
(b) If u_p is a particular solution of a linear non-homogeneous system (MDS), *then any other solution u of the system* (MDS) *is the sum between the solution of the homogeneous system (u_o) with the particular solution u_p:*

$$u = u_o + u_p$$

Definition 1.3.4.1. *A matrix in which the columns are the linearly independent solutions of the homogeneous system is called fundamental matrix of the system.*

Lemma 1.3.4.4. *(a) Any fundamental matrix is unsingular for any $t \in J$.*
(b) Any fundamental matrix $U(t)$ satisfies the differential matricial equation:

$$U'(t) = A(t)U(t)$$