Mathematical Models applied in Economy

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Chapter 1

Ordinary Differential Equations and Linear Systems

1.1 Equations with separable variables

The following section presents the general method of solving separable equations and a few examples.

1.1.1 The general form of an equation with separable variables

Let us consider the equation:

$$x' = f(t)g(x) \tag{SE}$$

where $f : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ and $g : (x_1, x_2) \subset \mathbb{R} \to \mathbb{R}$ are continuous functions and g does not cancel on (x_1, x_2). Before showing how to solve (SE), we recall the following definition:

Definition 1.1.1.1. *We call a solution on the interval $I \subset \mathbb{R}$ for the differential equation $F(t, x, x') = 0$ (where F is a real function defined on an open set from $\mathbb{R}^3$) a function $x : I \to \mathbb{R}$, differentiable on I and which verifies the equation on I, meaning:*

$$F(t, x(t), x'(t)) = 0, \forall t \in I$$

It is understood that x is such as $(t, x(t), x'(t))$ is in the domain of function F for $t \in I$. When reffering to a solution, we will usually point the interval on which it is defined (even the maximal interval if possible).

We return now to (SE). We assume that $x = x(t), t \in (t_1, t_2)$ is a solution for (SE). Then

$$\int_{x_0}^{x(t)} \frac{d\xi}{g(\xi)} = \int_{t_0}^{t} f(s)ds, t \in (t_1, t_2) \tag{1.1}$$

where t_0 is a random point from the interval (t_1, t_2) and $x_0 = x(t_0)$. We define

$$G(y) = \int_{x_0}^{y} \frac{d\xi}{g(\xi)}, y \in (x_1, x_2) \tag{1.2}$$

knowing that G is a differentiable function (with continuous derivative) on (x_1, x_2) and strictly monotone. Therefore we can talk about G^{-1}, defined on the set $G((x_1, x_2))$ which has the same properties as function G. Since relation (1.1) can be written:

$$G(x(t)) = \int_{t_0}^{t} f(s)ds, t \in (t_1, t_2) \tag{1.3}$$

results that solution x has the following expression:

$$x(t) = G^{-1}\left(\int_{t_0}^{t} f(s)ds \right), t \in (t_1, t_2) \tag{1.4}$$

Mutually, a function $x = x(t)$ defined by relation (1.4) (where x_0 is arbitrary in (x_1, x_2) and t goes through a neighbourhood of point t_0 such that $\int_{t_0}^{t} f(s)ds$ is in the domain of function G^{-1}) is a solution for (SE), also checking the Cauchy condition $x(t_0) = x_0$.

1.1.2 Examples

1) $\begin{cases} xx' = e^{-t} \\ x(0) = e \end{cases}$

Solution: Let us consider first the equation $xx' = e^{-t}$. Then we have

$$x\frac{dx}{dx} = e^{-t}.$$

Separating the variables, we obtain

$$xdx = e^{-t}dt.$$

By integration, we get:

$$\int xdx = \int e^{-t}dt \Leftrightarrow$$

$$\frac{x^2}{2} = -e^{-t} + c_1, c_1 \in \mathbb{R}.$$

$$x^2 = -2e^{-t} + c, c \in \mathbb{R}.$$

This form represents the general solution in implicit form. We impose the Cauchy condition $x(0) = e$. Then

$$x^2 = -2e^0 + c.$$

Hence $e^2 = -2 + c$. Thus $c = e^2 + 2$. Hence the solution in implicit form is

$$x^2 = -2e^{-t} + e^2 + 2.$$

Hence

$$x = \pm\sqrt{-2e^{-t} + e^2 + 2}.$$

We choose $x = \sqrt{-2e^{-t} + e^2 + 2}$. Thus, for $t > 0$ we have the solution:

$$x = \sqrt{2(1 - e^{-t}) + e^2}.$$

2) $x' = x^2 - x$, $t \in \mathbb{R}$. We consider the equation:

$$x' = x(x - 1).$$

For $x \neq 0$ and $x \neq 1$ we can separate the variables:

$$\frac{1}{x(x-1)} dx = dt.$$

By integrating, we obtain:

$$\int \frac{1}{x(x-1)} dx = \int dt.$$

$\Leftrightarrow$

$$\int \left(\frac{1}{x-1} - \frac{1}{x} \right) dx = t + c_1, c_1 \in \mathbb{R}.$$

Next, resolving the integrals, we get:

$$ln|x - 1| - ln|x| = t + c_1,$$

$$ln\left|\frac{x-1}{x}\right| = t + c_1$$

$\Leftrightarrow$

$$\left|\frac{x-1}{x}\right| = c_2 e^t, c_2 \in \mathbb{R}.$$

$$\frac{x-1}{x} = \pm c_2 e^t.$$

We can consider $\pm c_2$ a constant $c \in \mathbb{R}^*$. Then, the implicit form of the solution is:

$$\frac{x-1}{x} = ce^t, c \in \mathbb{R}^*.$$

3

Next, $x - 1 = xce^t$, resulting that

$$x = \frac{1}{1 - ce^t}, c \in \mathbb{R}^*. \tag{1.1}$$

We also notice that $x(t) = 0$ and $x(t) = 1$, where $t \in \mathbb{R}$ are solution for our equation. For $c = 0$ in (1.1) we get $x(t) = 1$. Hence the solutions are

$$x(t) = 0, t \in \mathbb{R}$$

$$x(t) = \frac{1}{1 - ce^t}, c \in \mathbb{R}, t \in J$$

where J is defined by the restriction $1 - ce^t = 0$.

(i) $c \in R \Rightarrow e^t = \frac{1}{c} \Rightarrow$

   (a) $c > 0 \Rightarrow e^t = \frac{1}{c} \Leftrightarrow t = \ln \frac{1}{c} = -\ln c \Rightarrow x(t) = \frac{1}{1-ce^t}$, $t \in (-\infty, -\ln c)$ or $t \in (-\ln c, +\infty)$.

   (b) $c < 0 \Rightarrow e^t = \frac{1}{c} \Leftrightarrow t \in \emptyset \Rightarrow x(t) = \frac{1}{1-ce^t}, t \in \mathbb{R}$

(ii) $c = 0 \Rightarrow x(t) = 1, t \in \mathbb{R}$.

For example, if we have a Cauchy problem of the following form: $\begin{cases} x' = x^2 - x \\ x(0) = 2 \end{cases}$

then $x(t) = 0$ and $x(t) = 1$, $t \in \mathbb{R}$ are not solutions.
For $x(t) = \frac{1}{1-ce^t}$:

$$x(0) = 2 \Leftrightarrow \frac{1}{1 - c} = 2 \Leftrightarrow 1 - c = \frac{1}{2} \Leftrightarrow c = \frac{1}{2}.$$

In conclusion, the solution for the given Cauchy problem is:

$$x(t) = \frac{1}{1 - \frac{1}{2}e^t}, t \in (-\infty, -\ln 2).$$

For the phase portrait, we have $x' = f(x)$. $f(x) = 0 \Leftrightarrow x = 0$ or $x = 1$. The function $f(x) = x^2 - x$ has the following signs on $\mathbb{R}$:

x	$-\infty$		0		1		∞
$x^2 - x$		+	0	−	0	+	

1.1.3 Conclusions

The section introduced the general method of solving equations with separable variables, presented in the classic differential equations books mentioned in the bibliography. The owner's contribution were the selection and solving of the examples.

Figure 1.1: The phase portrait for the equation at 2) exercise.
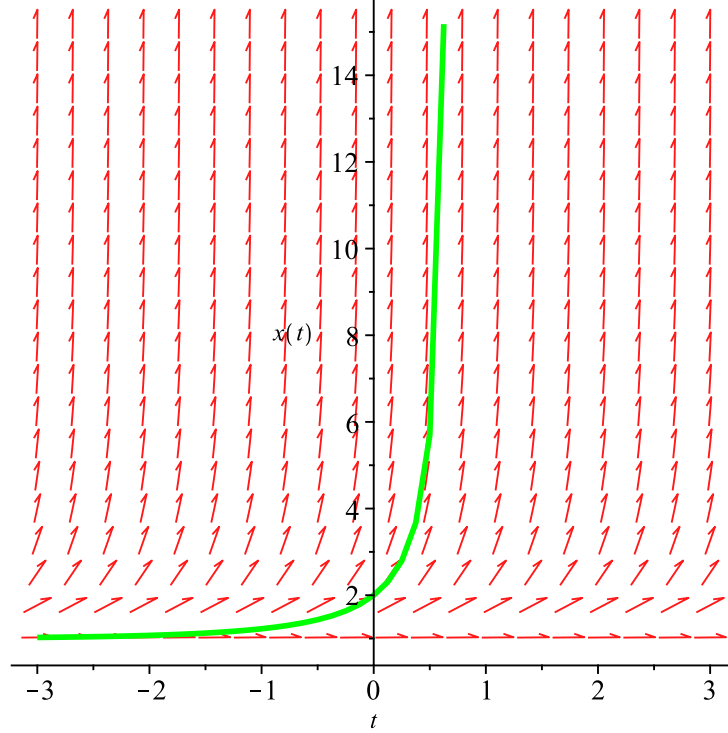
1.2 Linear Differential Equations of First Order

The following section introduces the reader to the general method of solving Linear Differential Equations of First Order, the solving method used in practice and a few examples.

1.2.1 The General Form of a linear differential equation of first order

A Linear differential equation has the following expression:

$$x' = a(t)x + b(t) \tag{LE}$$

where $a, b : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ are continous on (t_1, t_2) (bounded or not). If $x = x(t)$, $t_1 < t < t_2$ is a solutions for (LE), then multiplying with $exp(-\int_{t_0}^{t} a(s)ds)$, where t_0 is arbitrary chosen from (t_1, t_2), the following equation is obtained

$$\frac{d}{dt}\left[e^{-\int_{t_0}^{t} a(s)ds} x(t) \right] = b(t) e^{-\int_{t_0}^{t} a(s)ds}, t \in (t_1, t_2)$$

So

$$x(t) = e^{\int_{t_0}^{t} a(s)ds} x_0 + \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a(\sigma)d\sigma} ds, t \in (t_1, t_2) \qquad \text{(SOL)}$$

where x_0 is an arbitrary real number. Reciprocally, we can easily agree that any function $x = x(t), t \in (t_1, t_2)$ given by the formula (SOL) is a solution for (LE). Actually, (SOL) asserts the solution of (LE) with the Cauchy condition $x(t_0) = x_0$.

Sometimes, it is more convienent to use the following form of (SOL):

$$x(t) = e^{\int a(s)ds} \cdot \int b(t)e^{-\int a(t)dt} dt \qquad \text{(SOL*)}$$

with the convention that $\int a(t)dt$ is a fixed primitive of $a = a(t)$ (the same in (SOL) and (SOL*)).

In practice, a method that does not require the use of (SOL) formula is based on the algebraic link which exists between the set of the (LE) solutions and the set of the associated homogeneous equation solutions:

$$x' = a(t)x.$$

This link is contained in the following theorem:

Theorem 1.2.1.1. *If x_p is a particular solution of* (LE), *than any other solution x of the equation is the sum between a certain solution x_o of the homogeneous equation and x_p, meaning:*

$$x = x_o + x_p.$$

Reciprocally, any solution x_o of the homogeneous equation is the difference between a certain solution x of (LE) *and x_p.*

The theorem sustains that for solving a (LE), we have to go through two stages:

1. to solve the associated homogeneous equation;

2. to determine a particular solution of (LE).

1.2.2 Examples

1) $x' + x\tan t = \frac{1}{\cos t}$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

<u>Solution</u>: For solving this equation, we will use the result from 1.2.1.1. The associated homogeneous equation is:

$$xo' = -x_o \tan t.$$

Separating the variables, we get:

$$\frac{dx_o}{x_o} = -\tan t \, dt.$$

By integration, we obtain:

$$\ln x_o = \ln \cos t + c_1, c_1 \in \mathbb{R} \Leftrightarrow$$

$$x_o = c \cos t, c \in \mathbb{R}.$$

The particular solution for the given linear differential equation of first order is:

$$x_p = c(t) \cos t.$$

By replacing in the given equation, we obtain:

$$c'(t) \cos t - c(t) \sin t + c(t) \tan t \cos t = \frac{1}{\cos t} \Leftrightarrow$$

$$c'(t) \cos t = \frac{1}{\cos t} \Leftrightarrow$$

$$c'(t) = \frac{1}{\cos^2 t} \Leftrightarrow$$

$$c(t) = \tan t.$$

Meaning that the particular solution is:

$$x_p = \tan t \cos t \Leftrightarrow x_p = \sin t.$$

Hence, the solution is:

$$x = x_p + x_o \Leftrightarrow x = \sin t + c \cos t.$$

2) $\begin{cases} x' + x = e^{2t} \\ x(0) = 1 \end{cases}$

<u>Solution:</u> The associated homogeneous equation is:

$$x'_o = -x_o.$$

Separating the variables, we get:

$$\frac{dx_o}{x_o} = -dt.$$

By integration, we obtain:

$$\ln x_o = -t + c_1, c_1 \in \mathbb{R} \Leftrightarrow$$

$$x_o = ce^{-t}, c \in \mathbb{R}.$$

7

Hence, the particular solution for the given linear differential quation of first order has the following form:

$$x_p = c(t)e^{-t}.$$

Further, replacing in the given equation we get:

$$c'(t)e^{-t} - e^{-t}c(t) + e^{-t}c(t) = e^{2t} \Leftrightarrow$$

$$c'(t) = e^{3t} \Leftrightarrow c(t) = \frac{1}{3}e^{3t}.$$

Thus, the general solution has the following form:

$$x = ce^{-t} + \frac{1}{3}e^{2t}, c \in \mathbb{R}.$$

Then, by attaching the Cauchy condition, we get that the constant $c = \frac{2}{3}$, which means that the solution for the given Cauchy problem is:

$$x = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.$$

1.2.3 Conclusions

The section introduced the notion of linear differential equation of first order and two methods of solving it, including the one used in practice, from the classic differential equations books mentioned in the bibliography. The owner's contribution was the selection and solving of the examples.

1.3 Linear Differential Systems

1.3.1 The general form of a linear differential system

Many evolutive processes from the real world can not be described by only one variable. Therefore, for two or more variables, we must consider a system of two or more differential equations. In the following chapter, we will consider differential systems of the first order:

$$\begin{cases} x' = f(t, x, y) \\ y' = g(t, x, y) \end{cases} \tag{DS}$$

where f,g are given functions and the unknowns are the functions x, y of variable t.

Definition 1.3.1.1. *A solution for* (DS) *is a pair* $(x, y) \in C^1(J)$, *where* $J \subseteq \mathbb{R}$, *which satisfy on* J *the two equations from* (DS), *for any* $t \in J$.

(DS) is linear if functions f and g depend affinely on x and y. Then (DS) has the following form:

$$\begin{cases} x' = a_{11}(t)x + a_{12}(t)y + b_1(t) \\ y' = a_{21}(t)x + a_{22}(t)y + b_2(t) \end{cases} \tag{DS*}$$

Coefficients a_{ij} and free terms b_i are alleged to be continuous functions on an interval J. If, in particular, coefficients a_{ij} are constants, we say that the linear system is with constant coefficients. If $b_i = 0$, we say that the system is homogeneous. If both situations take place, the system is linear and homogeneous, with constant coefficients.

1.3.2 Matrix Analysis Theory

Let us define $\mathcal{M}_{nm}(\mathbb{K})$ the set of matrices with n rows and m columns, with elements from the field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$. It is known that $(\mathcal{M}_{nm}(\mathbb{K}), +, \cdot, \mathbb{R})$ is a linear space of dimension $n \cdot m$. It can be organized as a normalized space using the following norm:

$$\|A\| = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}$$

Then $(\mathcal{M}_{nm}(\mathbb{K}), \|\cdot\|)$ is a Banach space. The defined norm has the following properties on the Banach space presented earlier:

1. $\|A + B\| \leq \|A\| + \|B\|$

2. $\|\lambda A\| = |\lambda| \|A\|$

3. $\|Ax\|_{\mathbb{R}^n} = \|A\| \|x\|_{\mathbb{R}^n}$

4. $\|A \cdot B\| \leq \|A\| \cdot \|B\|$

Next, let $M \in \mathcal{M}_{nn}(\mathbb{K})$. Then:

$$e^M = \sum_{k \geq 0} \frac{1}{k!} M^k \tag{3.1}$$

Proof. Let $S_n = \sum_{k=0}^{n} \frac{1}{k!} M^k$ the partial sum of serie (3.1). Next, we prove that S_n is a Cauchy sequence in the Banach space $(\mathcal{M}_{nm}(\mathbb{K}), \|\cdot\|)$.

$$\|S_{n+p} - S_n\| = \|\frac{1}{(n+1!)} A^{n+1} + ... + \frac{1}{(n+p!)} A^{n+p}\|$$

$$\leq \frac{1}{(n+1)!} \|A^{n+1}\| + ... + \frac{1}{(n+p)!} \|A^{n+p}\| = |a_{n+p} - a_n|$$

where
$$a_n = \sum_{k=0}^{n} \frac{1}{k!} \|M\|^k.$$

Due to the fact that
$$\sum_{k\geq 0} \frac{1}{k!} x^k$$

is a convergent series and the sum of it is equal to e^x, $\forall x \in \mathbb{R} \Rightarrow$

$$\sum_{k\geq 0} \frac{1}{k!} \|M\|^k$$

is convergent, therefore a_n is convergent, which means that a_n is Cauchy $\Rightarrow$ $\forall \varepsilon > 0$, $\exists n(\varepsilon) \in \mathbb{N}$ so that

$$\|S_{n+p} - S_n\| \leq \varepsilon$$

$\forall n \geq n(\varepsilon)$, $\forall p \in \mathbb{N} \Rightarrow (S_n)$ is a Cauchy sequence $\Rightarrow$ the existence of (3.1) is proved.

$\square$

1.3.3 Existence and uniqueness theorems

Let us consider the system:

$$\begin{cases} u' = A(t)u + B(t) \\ u(t_0) = u_0 \end{cases} \tag{Sys1}$$

where $t \in J = [t_0 - a, t_0 + a], a > 0$ $t_0 \in J$ and $u_0 \in \mathbb{R}^2$.

Regarding the existence and uniqueness of the solution of the previous system, we present the following result:

Theorem 1.3.3.1. *Considering* (Sys1), *we assume that* $A \in C(J, \mathcal{M}_2(\mathbb{R}))$, $B \in C(J, \mathbb{R}^2)$. *Then, there exists a unique solution* $u^* \in C^1(J, \mathbb{R}^2)$.

Proof. We will use the following Lemma:

Lemma 1.3.3.2. *The solution* u^* *is equivalent to the following system of integral equations:*

$$u(t) = \int_{t_0}^{t} \left[A(s)u(s) + B(s) \right] ds + u_0 \tag{IntSol}$$

Proof. "$\Rightarrow$ "If u satisfies (Sys1), then by integrating from t_0 to $t \in J$, we get:

$$u(t) - u(t_0) = \int_{t_0}^{t} \left[A(s)u(s) + B(s) \right] ds$$

10

$u(t_0) = u_0 \Rightarrow$ (IntSol).

"$\Rightarrow$" Let u be a solution of (IntSol). Then $u'(t) = A(t)u + B(t)$. Since the right side is continuous, we obtain that $u \in C^1(J, \mathbb{R})$. Moreover, $u(t_0) = u_0$. The lemma is proved. $\square$

Resuming the proof of the theorem, let us denote $A : C(J, \mathbb{R}^2) \to C(J, \mathbb{R}^2)$, $u \longmapsto Au$, where: $Au(t) := \int_{t_0}^{t} \left[A(s)u(s) + B(s) \right] ds + u_0$

"(IntSol) $\Leftrightarrow u = Au$"

We consider on $C(J, \mathbb{R}^2)$ the Bielecki norm:

$$\|u\|_B = max_{t \in J}(\|u\|_{\mathbb{R}^2} e^{-\tau|t-t_0|}), \tau > 0$$

$(C(J, \mathbb{R}^2), \|.\|)$ is a Banach space.

Then we have that $\|Au - Av\|_B \leq L_A \|u - v\|_B, \forall u, v \in C(J, \mathbb{R}^2, L_A \in (0, 1))$. For $t \geq t_0$:

$$\|Au(t) - Av(t)\|_{\mathbb{R}^2} = \| \int_{t_0}^{t} A(s)(u(s) - v(s))ds \|$$

$$\leq \int_{t_0}^{t} \|A(s)(u(s) - v(s))\| ds$$

$$\leq \int_{t_0}^{t} \|A(s)\|_{\mathcal{M}_2(\mathbb{R}^2)} \|u(s) - v(s)\|_{\mathbb{R}^2} ds$$

$$\leq M_A \int_{t_0}^{t} \|u(s) - v(s)\| e^{-\tau(s-t_0)} e^{\tau(s-t_0)} ds$$

$$\leq \int_{t_0}^{t} \max_{(s \in J)} (\|u(s) - v(s)\| e^{-\tau(s-t_0)}) e^{\tau(s-t_0)} ds$$

$$\leq M_A \|u - v\|_B \frac{1}{\tau} (e^{\tau(t-t_0)} - 1)$$

$$\leq \frac{M_A}{\tau} \|u - v\|_B e^{\tau(t-t_0)}$$

$$\Rightarrow \|Au - Av\| \leq \frac{M_A}{\tau} \|u - v\|_B.$$

Let $L_A = \frac{M_A}{\tau}$ such as $\tau > M_A$. Then $L_A < 1$.

$\square$

1.3.4 Representations of the solution

Let us remark that (DS) can be written as a single vectorial equation:

$$u' = F(t, u) \tag{VDS}$$

where u and F are vectorials, with two real components, more exactly column matrix:

$$u = \begin{bmatrix} x \\ y \end{bmatrix}, \quad F(t, u) = \begin{bmatrix} f(t, x, y) \\ g(t, x, y) \end{bmatrix}$$

The condition for the Cauchy Problem of the system can be written:

$$u(t_0) = u_0, u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Also, (DS*) can be written:

$$u' = A(t)u + B(t) \tag{MDS}$$

where

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \quad B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

Theorem 1.3.4.1 (of representing the solutions of linear systems). *Let $A \in C(J, \mathcal{M}_\in))$ and $B \in (J, R^2)$. Then the solutions of the linear system* (MDS) *are defined by the formula*

$$u = e^{\int_{t_0}^{t} A(\sigma)d\sigma} C + \int_{t_0}^{t} e^{\int_{s}^{t} A(\sigma)d\sigma} B(s)ds \tag{1.2}$$

where $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ *and* $C_1, C_2 \in \mathbb{R}$.

Theorem 1.3.4.2 (of existence, uniqueness and representation of the solution). *Let $A \in C(J, M_2(\mathbb{R})), B \in C(J, \mathbb{R}), t_0 \in J and u_0 \in \mathbb{R}^2$. Then the Cauchy Problem has a unique solution defined on J, given by the formula:*

$$u = e^{\int_{t_0}^{t} A(\sigma)d\sigma} u_0 + \int_{t_0}^{t} e^{\int_{s}^{t} A(\sigma)d\sigma} B(s)ds \tag{1.3}$$

Theorem 1.3.4.3 (the structure of the set of solutions). *(a) The set of solutions of a bidimensional linear and homogeneous system is a linear bidimensional space.*
(b) If u_p is a particular solution of a linear non-homogeneous system (MDS)*, then any other solution u of the system* (MDS) *is the sum between the solution of the homogeneous system (u_o) with the particular solution u_p:*

$$u = u_o + u_p$$

Definition 1.3.4.1. *A matrix in which the columns are the linearly independent solutions of the homogeneous system is called fundamental matrix of the system.*

Lemma 1.3.4.4. *(a) Any fundamental matrix is unsingular for any $t \in J$.*
(b) Any fundamental matrix $U(t)$ satisfies the differential matricial equation:

$$U'(t) = A(t)U(t)$$

1.3.5 Representation with fundamental matrix

In this paragraph, let us submit the possibility of representing the solutions of the non-homogeneous system in terms of the fundamental matrix $U(t)$, of which purpose is to replace the matrix $e^{\int_{t_0}^{t} A(\sigma)d\sigma}$ from (1.2).
Firstly, let

$$u_p = U(t)C(t)$$

be a particular solution of the non-homogeneous system, where the vectorial function

$$C(t) = \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix}$$

is to be determined with the condition that u_p satisfies the given system. Hence

$$u_p' = U'(t)C(t)U(t)C'(t)$$

and by replacing we find that

$$U'(t)C(t) + U(t)C'(t) = A(t)U(t)C(t) + B(t).$$

Point b) from lemma 1.3.4.4 guarantees that

$$U'(t) = A(t)U(t).$$

Thus $U(t)C'(t) = B(t)$, which means that $C'(t) = U^{-1}(t)B(t)$. Then, according to point a) from 1.3.4.4, matrix $U(t)$ is invertible. Let us choose

$$C(t) = \int_{t_0}^{t} U^{-1}(s)B(s)ds$$

and so we get a particular solution of the non-homogeneous system:

$$u_p = \int_{t_0}^{t} U(t)U^{-1}(s)B(s)ds.$$

Thus, an equivalent of (1.2) is:

$$u = U(t)C + \int_{t_0}^{t} U(t)U^{-1}(s)B(s)ds. \tag{1.4}$$

The analogue of (1.3) results immediately:

$$u = U(t)U^{-1}(t_0)u_0 + \int_{t_0}^t U(t)U^{-1}(s)B(s)ds. \qquad (1.5)$$

Hence, we can present the following theorem:

Theorem 1.3.5.1 (of representation with fundamental matrix)**.** *If $U(t)$ is a fundamental matrix of* (MDS)*, then:*

 (i) *The solutions of the homogeneous system are the functions $u_o = U(t)C$, where C is a random vector from $\mathbb{R}^2$;*

 (ii) *The solutions of the non-homogeneous system are the functions defined by* (1.4)*, where $C \in \mathbb{R}^2$;*

 (iii) *The solution of the Cauchy problem for* (Sys1) *is the function defined by formula* (1.5)*.*

According to 1.3.5.1, solving a linear system depends on finding a fundamental matrix. Generally, this is not possible, but for the systems with constant coefficients is possible.

1.3.6 Linear systems with constant coefficients

In this paragraph, we consider the system (DS*), where the coefficients $a_{ij}(i, j = 1, 2)$ are constant, and $b_1, b_2 \in C(J)$. By using matrices, we can rewrite the system as:

$$u' = Au + B(t),$$

where

$$u = \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}.$$

Due to the fact that all the previous results stay valid, the solutions of the system have the following form:

$$u = u_o + u_p.$$

The explicit form of the solution is:

$$u = e^{(t-t_0) \cdot A} \cdot C + \int_{t_0}^t e^{(t-s) \cdot A} B(s)ds,$$

where the constant vector C is random. Also, the Cauchy problem, with $u(t_0) = u_0$ as the initial condition admits only one solution, which is

$$u = e^{(t-t_0) \cdot A} \cdot u_0 + \int_{t_0}^t e^{(t-s) \cdot A} B(s)ds \qquad (1.6)$$

If by the replacement of $e^{(t-t_0)\cdot A}$ it is considered another fundamental matrix $U(t)$, then the solutions of the system and also for the Cauchy problem are given by formulas (1.4) and (1.5). The question of how we can determinate the fundamental matrix and how we can directly solve the system still remains.

For this, let us search for solutions for the associated homogeneous system based on the following form:

$$u = e^{rt}V,$$

where $r \in \mathbb{R}$ and $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ are determined with the condition that u satisfies the homogeneous system. Obviously, $V \neq 0$. By replacing in the homogeneous system, we get:

$$re^{rt}V = A(e^{rt}V) = e^{rt}AV.$$

By simplifying with e^{rt}, we obtain:

$$(A - rI)V = 0. \tag{1.7}$$

As $V \neq 0$, we conclude that r is an eagen value of matrix A, and V is an eagen vector associated to this value. Eagen values are determined with the condition that the algebraic homogeneous system (1.7) admits solutions unequal to 0, meaning from equation:

$$det(A - rI) = 0. \tag{1.8}$$

This equation is named the characteristic equation of the system and is a second grade polynomial. It can be written:

$$\begin{vmatrix} a_{11} - r & a_{12} \\ a_{21} & a_{22} - r \end{vmatrix} = 0, \tag{1.9}$$

where, by solving the determinant, we obtain:

$$r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{12}a_{21} = 0,$$

meaning

$$r^2 - (trA)r + detA = 0.$$

Here,

$$trA = \sum_{i=0}^{n} a_{ii}$$

is the trace of the matrix. Considering how the roots of the characteristic equations might be, we distinguish the following cases:

1.3.6.1 Case of real and distinct roots

Let r_1, r_2 be the roots of the characteristic equation, which are presumed to be real and distinct. For each one, we choose a non-null solution of system (1.7). Let them be V_1, V_2. Meaning that we obtained two solutions for the homogeneous system:

$$u_1 = e^{r_1 t} V_1, \quad u_2 = e^{r_2 t} V_2.$$

As $r_1 \neq r_2$, they are linear independent. Matrix $U(t)$, with columns u_1, u_2 thus determined, is a fundamental matrix.

1.3.6.2 Case of equal roots

Let us presume that the roots of the characteristic equation are equal. Then $r := r_1 = r_2 = \frac{trA}{2}$. A solution of the system can be determined similarly to the previous case, of form $u_1 = e^{rt} V$, where V is a non-null solution of the algebraic homogeneous system (1.7). A second solution u_2, linear independent of u_1, is searched by the form $u_2 = e^{rt}(tV + W)$, with V being the previous, and W being vector which is obtained with the condition that verifies the system. We have:

$$u_2' = re^{rt}(tV + W) + e^{rt} V$$

meaning that u_2 is a solution if

$$re^{rt}(tV + W) + e^{rt} V = e^{rt} A(tV + W).$$

By simplifying with e^{rt} and by reducing the equal terms, we obtain the following algebraic linear and non-homogeneous system:

$$(A - rI)W = V.$$

This system is compatible and so we can choose a solution W of it.

1.3.6.3 Case of complex roots

Let us presume that the roots of the characteristic equation are complex, being them $\alpha \pm \beta i$. Considering one of them, for example $\alpha + \beta i$ and by proceeding like in the case of real and distinct roots, let us determine a solution of the system, this time complex:

$$u = e^{rt} V = e^{(\alpha + \beta i)t}(V_1 + iV_2) = e^{\alpha t}(\cos \beta t + i \sin \beta t)(V_1 + iV_2) =$$

$$= e^{\alpha t}[V_1 \cos \beta t - V_2 \sin \beta t + i(V_1 \sin \beta t + V_2 \cos \beta t)].$$

where V_1, V_2 represents the column vector of the real parts, respectively the coefficients of the imaginary parts of V, meaning $V = V_1 + iV_2$ and

$V_1, V_2 \in \mathbb{R}^2$. With the system being linear, homogeneous and with real coefficients, at once with a complex solution u, admits as solutions the conjugate function $\bar{u}$, as well as any linear combination of theirs with complex coefficients, particularly the real functions:

$$u_1 := Re(u) = \frac{1}{2}(u + \bar{u}) = e^{\alpha t}(V_1 \cos \beta t - V_2 \sin \beta t),$$

$$u_2 := Im(u) = \frac{1}{2i}(u - \bar{u}) = e^{\alpha t}(V_1 \sin \beta t + V_2 \cos \beta t).$$