Mathematical Models applied in Economy

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Chapter 1

Ordinary Differential Equations and Linear Systems

This chapter presents the general notions of an equation with separable variables, the otion of a linear differential equations of first order, after which we will consider the linear differential systems. In this part, we will present some results regarding the existence and uniqueness of a solution, as well as its possible representations. In the end, we will present the linear systems with constant coefficients and its particular cases. The author's contribution is the selection and solving of the examples found at the end of each chapter.

1.1 Equations with separable variables

1.1.1 The general form of an equation with separable variables

Let us consider the equation:

$$x' = f(t)g(x) \tag{SE}$$

where $f : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ and $g : (x_1, x_2) \subset \mathbb{R} \to \mathbb{R}$ are continuous functions and g does not cancel on (x_1, x_2). Before showing how to solve (SE), we recall the following definition:

Definition 1.1.1.1. *We call a solution on the interval $I \subset \mathbb{R}$ for the differential equation $F(t, x, x') = 0$ (where F is a real function defined on an open set from $\mathbb{R}^3$) a function $x : I \to \mathbb{R}$, differentiable on I and which verifies the equation on I, meaning:*

$$F(t, x(t), x'(t)) = 0, \forall t \in I$$

It is understood that x is such as $(t, x(t), x'(t))$ is in the domain of function F for $t \in I$. When reffering to a solution, we will usually point the interval on which it is defined (even the maximal interval if possible).

We return now to (SE). We assume that $x = x(t)$,$t \in (t_1, t_2)$ is a solution for (SE). Then

$$\int_{x_0}^{x(t)} \frac{d\xi}{g(\xi)} = \int_{t_0}^{t} f(s)ds, t \in (t_1, t_2) \tag{1.1}$$

where t_0 is a random point from the interval (t_1, t_2) and $x_0 = x(t_0)$. We define

$$G(y) = \int_{x_0}^{y} \frac{d\xi}{g(\xi)}, y \in (x_1, x_2) \tag{1.2}$$

knowing that G is a differentiable function (with continuous derivative) on (x_1, x_2) and strictly monotone. Therefore we can talk about G^{-1}, defined on the set $G((x_1, x_2))$ which has the same properties as function G. Since relation (1.1) can be written:

$$G(x(t)) = \int_{t_0}^{t} f(s)ds, t \in (t_1, t_2) \tag{1.3}$$

results that solution x has the following expression:

$$x(t) = G^{-1}\left(\int_{t_0}^{t} f(s)ds\right), t \in (t_1, t_2) \tag{1.4}$$

Mutually, a function $x = x(t)$ defined by relation (1.4) (where x_0 is arbitrary in (x_1, x_2) and t goes through a neighbourhood of point t_0 such that $\int_{t_0}^{t} f(s)ds$ is in the domain of function G^{-1}) is a solution for (SE), also checking the Cauchy condition $x(t_0) = x_0$.

1.1.2 Examples

1) $\begin{cases} xx' = e^{-t} \\ x(0) = e \end{cases}$

Solution: Let us consider first the equation $xx' = e^{-t}$. Then we have

$$x\frac{dx}{dx} = e^{-t}.$$

Separating the variables, we obtain

$$xdx = e^{-t}dt.$$

By integration, we get:

$$\int x dx = \int e^{-t} dt \Leftrightarrow$$

$$\frac{x^2}{2} = -e^{-t} + c_1, c_1 \in \mathbb{R}.$$

$$x^2 = -2e^{-t} + c, c \in \mathbb{R}.$$

This form represents the general solution in implicit form. We impose the Cauchy condition $x(0) = e$. Then

$$x^2 = -2e^0 + c.$$

Hence $e^2 = -2 + c$. Thus $c = e^2 + 2$. Hence the solution in implicit form is

$$x^2 = -2e^{-t} + e^2 + 2.$$

Hence

$$x = \pm\sqrt{-2e^{-t} + e^2 + 2}.$$

We choose $x = \sqrt{-2e^{-t} + e^2 + 2}$. Thus, for $t > 0$ we have the solution:

$$x = \sqrt{2(1 - e^{-t}) + e^2}.$$

2) $x' = x^2 - x$, $t \in \mathbb{R}$. We consider the equation:

$$x' = x(x-1).$$

For $x \neq 0$ and $x \neq 1$ we can separate the variables:

$$\frac{1}{x(x-1)} dx = dt.$$

By integrating, we obtain:

$$\int \frac{1}{x(x-1)} dx = \int dt.$$

$\Leftrightarrow$

$$\int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx = t + c_1, c_1 \in \mathbb{R}.$$

Next, resolving the integrals, we get:

$$ln|x-1| - ln|x| = t + c_1,$$

$$ln|\frac{x-1}{x}| = t + c_1$$

$\Leftrightarrow$

$$|\frac{x-1}{x}| = c_2 e^t, c_2 \in \mathbb{R}.$$

$$\frac{x-1}{x} = \pm c_2 e^t.$$

We can consider $\pm c_2$ a constant $c \in \mathbb{R}^*$. Then, the implicit form of the solution is:

$$\frac{x-1}{x} = ce^t, c \in \mathbb{R}^*.$$

Next, $x - 1 = xce^t$, resulting that

$$x = \frac{1}{1 - ce^t}, c \in \mathbb{R}^*. \tag{1.1}$$

We also notice that $x(t) = 0$ and $x(t) = 1$, where $t \in \mathbb{R}$ are solution for our equation. For $c = 0$ in (1.1) we get $x(t) = 1$. Hence the solutions are

$$x(t) = 0, t \in \mathbb{R}$$

$$x(t) = \frac{1}{1 - ce^t}, c \in \mathbb{R}, t \in J$$

where J is defined by the restriction $1 - ce^t = 0$.

(i) $c \in R \Rightarrow e^t = \frac{1}{c} \Rightarrow$

  (a) $c > 0 \Rightarrow e^t = \frac{1}{c} \Leftrightarrow t = \ln\frac{1}{c} = -\ln c \Rightarrow x(t) = \frac{1}{1-ce^t}$, $t \in (-\infty, -\ln c)$ or $t \in (-\ln c, +\infty)$.

  (b) $c < 0 \Rightarrow e^t = \frac{1}{c} \Leftrightarrow t \in \emptyset \Rightarrow x(t) = \frac{1}{1-ce^t}, t \in \mathbb{R}$

(ii) $c = 0 \Rightarrow x(t) = 1, t \in \mathbb{R}$.

For example, if we have a Cauchy problem of the following form: $\begin{cases} x' = x^2 - x \\ x(0) = 2 \end{cases}$

then $x(t) = 0$ and $x(t) = 1$, $t \in \mathbb{R}$ are not solutions.
For $x(t) = \frac{1}{1-ce^t}$:

$$x(0) = 2 \Leftrightarrow \frac{1}{1-c} = 2 \Leftrightarrow 1 - c = \frac{1}{2} \Leftrightarrow c = \frac{1}{2}.$$

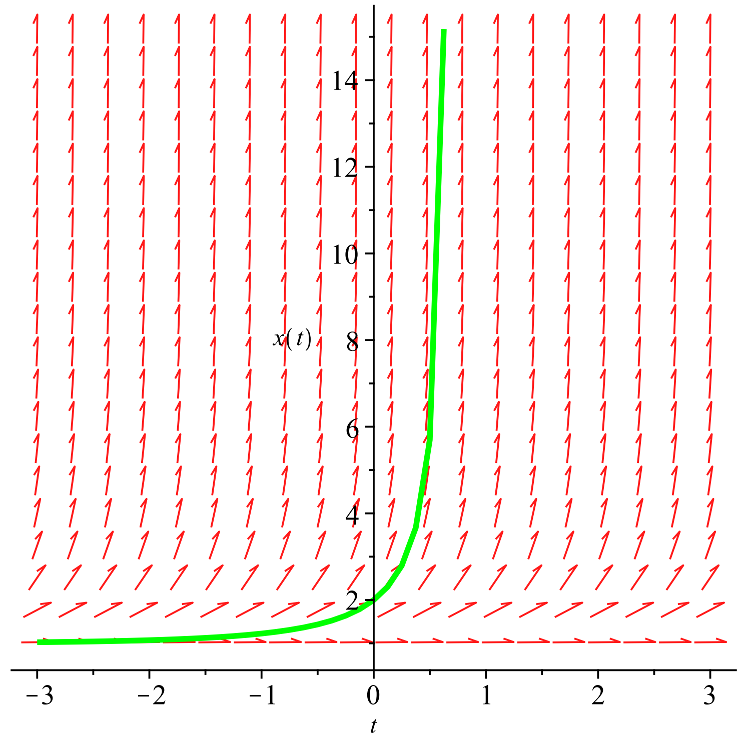
In conclusion, the solution for the given Cauchy problem is:

$$x(t) = \frac{1}{1 - \frac{1}{2}e^t}, t \in (-\infty, -\ln 2).$$

For the phase portrait, we have $x' = f(x)$. $f(x) = 0 \Leftrightarrow x = 0$ or $x = 1$. The function $f(x) = x^2 - x$ has the following signs on $\mathbb{R}$:

x	$-\infty$		0		1		∞
$x^2 - x$		$+$	0	$-$	0	$+$	

Figure 1.1: The graphic of the solution from exercise 2).



1.2 Linear Differential Equations of First Order

The following section introduces the reader to the general method of solving Linear Differential Equations of First Order, the solving method used in practice and a few examples.

1.2.1 The General Form of a linear differential equation of first order

A Linear differential equation has the following expression:

$$x' = a(t)x + b(t) \tag{LE}$$

where $a, b : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ are continous on (t_1, t_2) (bounded or not). If $x = x(t)$, $t_1 < t < t_2$ is a solutions for (LE), then multiplying with $exp(-\int_{t_0}^{t} a(s)ds)$, where t_0 is arbitrary chosen from (t_1, t_2), the following equation is obtained

$$\frac{d}{dt}\left[e^{-\int_{t_0}^{t} a(s)ds} x(t)\right] = b(t)e^{-\int_{t_0}^{t} a(s)ds}, t \in (t_1, t_2)$$

So

$$x(t) = e^{\int_{t_0}^{t} a(s)ds} x_0 + \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a(\sigma)d\sigma} ds, t \in (t_1, t_2) \qquad \text{(SOL)}$$

where x_0 is an arbitrary real number. Reciprocally, we can easily agree that any function $x = x(t), t \in (t_1, t_2)$ given by the formula (SOL) is a solution for (LE). Actually, (SOL) asserts the solution of (LE) with the Cauchy condition $x(t_0) = x_0$.

Sometimes, it is more convienent to use the following form of (SOL):

$$x(t) = e^{\int a(s)ds} \cdot \int b(t) e^{-\int a(t)dt} dt \qquad \text{(SOL*)}$$

with the convention that $\int a(t)dt$ is a fixed primitive of $a = a(t)$ (the same in (SOL) and (SOL*)).

In practice, a method that does not require the use of (SOL) formula is based on the algebraic link which exists between the set of the (LE) solutions and the set of the associated homogeneous equation solutions:

$$x' = a(t)x.$$

This link is contained in the following theorem:

Theorem 1.2.1.1. *If x_p is a particular solution of* (LE)*, than any other solution x of the equation is the sum between a certain solution x_o of the homogeneous equation and x_p, meaning:*

$$x = x_o + x_p.$$

Reciprocically, any solution x_o of the homogeneous equation is the difference between a certain solution x of (LE) *and x_p.*

The theorem sustains that for solving a (LE), we have to go through two stages:

1. to solve the associated homogeneous equation;
2. to determine a particular solution of (LE).

1.2.2 Examples

1) $x' + x\tan t = \frac{1}{\cos t}$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Solution: For solving this equation, we will use the result from 1.2.1.1. The associated homogeneous equation is:

$$xo' = -x_o \tan t.$$

Separating the variables, we get:

$$\frac{dx_o}{x_o} = -\tan t dt.$$

By integration, we obtain:

$$\ln x_o = \ln \cos t + c_1, c_1 \in \mathbb{R} \Leftrightarrow$$

$$x_o = c \cos t, c \in \mathbb{R}.$$

The particular solution for the given linear differential equation of first order is:

$$x_p = c(t) \cos t.$$

By replacing in the given equation, we obtain:

$$c'(t) \cos t - c(t) \sin t + c(t) \tan t \cos t = \frac{1}{\cos t} \Leftrightarrow$$

$$c'(t) \cos t = \frac{1}{\cos t} \Leftrightarrow$$

$$c'(t) = \frac{1}{\cos^2 t} \Leftrightarrow$$

$$c(t) = \tan t.$$

Meaning that the particular solution is:

$$x_p = \tan t \cos t \Leftrightarrow x_p = \sin t.$$

Hence, the solution is:

$$x = x_p + x_o \Leftrightarrow x = \sin t + c \cos t.$$

2) $\begin{cases} x' + x = e^{2t} \\ x(0) = 1 \end{cases}$

Solution: The associated homogeneous equation is:

$$x_o' = -x_o.$$

Separating the variables, we get:

$$\frac{dx_o}{x_o} = -dt.$$

By integration, we obtain:

$$\ln x_o = -t + c_1, c_1 \in \mathbb{R} \Leftrightarrow$$

$$x_o = ce^{-t}, c \in \mathbb{R}.$$

Hence, the particular solution for the given linear differential quation of first order has the following form:

$$x_p = c(t)e^{-t}.$$

Further, replacing in the given equation we get:

$$c'(t)e^{-t} - e^{-t}c(t) + e^{-t}c(t) = e^{2t} \Leftrightarrow$$

$$c'(t) = e^{3t} \Leftrightarrow c(t) = \frac{1}{3}e^{3t}.$$

Thus, the general solution has the following form:

$$x = ce^{-t} + \frac{1}{3}e^{2t}, c \in \mathbb{R}.$$

Then, by attaching the Cauchy condition, we get that the constant $c = \frac{2}{3}$, which means that the solution for the given Cauchy problem is:

$$x = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.$$

1.3 Linear Differential Systems

1.3.1 The general form of a linear differential system

Many evolutive processes from the real world can not be described by only one variable. Therefore, for two or more variables, we must consider a system of two or more differential equations. In the following chapter, we will consider differential systems of the first order:

$$\begin{cases} x' = f(t, x, y) \\ y' = g(t, x, y) \end{cases} \tag{DS}$$

where f,g are given functions and the unknowns are the functions x, y of variable t.

Definition 1.3.1.1. *A solution for* (DS) *is a pair* $(x, y) \in C^1(J)$*, where* $J \subseteq \mathbb{R}$*, which satisfy on* J *the two equations from* (DS)*, for any* $t \in J$*.*

(DS) is linear if functions f and g depend affinely on x and y. Then (DS) has the following form:

$$\begin{cases} x' = a_{11}(t)x + a_{12}(t)y + b_1(t) \\ y' = a_{21}(t)x + a_{22}(t)y + b_2(t) \end{cases} \tag{DS*}$$

Coefficients a_{ij} and free terms b_i are alleged to be continuous functions on an interval J. If, in particular, coefficients a_{ij} are constants, we say that the linear system is with constant coefficients. If $b_i = 0$, we say that the system is homogeneous. If both situations take place, the system is linear and homogeneous, with constant coefficients.

1.3.2 Matrix Analysis Theory

Let us define $\mathcal{M}_{nm}(\mathbb{K})$ the set of matrices with n rows and m columns, with elements from the field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$. It is known that $(\mathcal{M}_{nm}(\mathbb{K}), +, \cdot, \mathbb{R})$ is a linear space of dimension $n \cdot m$. It can be organized as a normalized space using the following norm:

$$\|A\| = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}$$

Then $(\mathcal{M}_{nm}(\mathbb{K}), \|\cdot\|)$ is a Banach space. The defined norm has the following properties on the Banach space presented earlier:

1. $\|A + B\| \leq \|A\| + \|B\|$
2. $\|\lambda A\| = |\lambda| \|A\|$
3. $\|Ax\|_{\mathbb{R}^n} = \|A\| \|x\|_{\mathbb{R}^n}$
4. $\|A \cdot B\| \leq \|A\| \cdot \|B\|$

Next, let $M \in \mathcal{M}_{nn}(\mathbb{K})$. Then:

$$e^M = \sum_{k \geq 0} \frac{1}{k!} M^k \tag{3.1}$$

Proof. Let $S_n = \sum_{k=0}^{n} \frac{1}{k!} M^k$ the partial sum of serie (3.1). Next, we prove that S_n is a Cauchy sequence in the Banach space $(\mathcal{M}_{nm}(\mathbb{K}), \|\cdot\|)$.

$$\|S_{n+p} - S_n\| = \| \frac{1}{(n+1!)} A^{n+1} + ... + \frac{1}{(n+p!)} A^{n+p} \|$$

$$\leq \frac{1}{(n+1)!} \|A^{n+1}\| + ... + \frac{1}{(n+p)!} \|A^{n+p}\| = |a_{n+p} - a_n|$$

where

$$a_n = \sum_{k=0}^{n} \frac{1}{k!} \|M\|^k.$$

Due to the fact that

$$\sum_{k \geq 0} \frac{1}{k!} x^k$$

is a convergent series and the sum of it is equal to e^x, $\forall x \in \mathbb{R} \Rightarrow$

$$\sum_{k \geq 0} \frac{1}{k!} \|M\|^k$$

is convergent, therefore a_n is convergent, which means that a_n is Cauchy $\Rightarrow$ $\forall \varepsilon > 0$, $\exists n(\varepsilon) \in \mathbb{N}$ so that

$$\|S_{n+p} - S_n\| \leq \varepsilon$$

$\forall n \geq n(\varepsilon)$, $\forall p \in \mathbb{N} \Rightarrow (S_n)$ is a Cauchy sequence $\Rightarrow$ the existence of (3.1) is proved.

$\square$

1.3.3 Existence and uniqueness theorems

Let us consider the system:

$$\begin{cases} u' = A(t)u + B(t) \\ u(t_0) = u_0 \end{cases} \tag{Sys1}$$

where $t \in J = [t_0 - a, t_0 + a], a > 0$ $t_0 \in J$ and $u_0 \in \mathbb{R}^2$.
Regarding the existence and uniqueness of the solution of the previous system, we present the following result:

Theorem 1.3.3.1. *Considering* (Sys1)*, we assume that* $A \in C(J, \mathcal{M}_2(\mathbb{R}))$*,* $B \in C(J, \mathbb{R}^2)$*. Then, there exists a unique solution* $u^* \in C^1(J, \mathbb{R}^2)$*.*

Proof. We will use the following Lemma:

Lemma 1.3.3.2. *The solution* u^* *is equivalent to the following system of integral equations:*

$$u(t) = \int_{t_0}^{t} \Big[A(s)u(s) + B(s)\Big] ds + u_0 \tag{IntSol}$$

Proof. "$\Rightarrow$ "If u satisfies (Sys1), then by integrating from t_0 to $t \in J$, we get:

$$u(t) - u(t_0) = \int_{t_0}^{t} \Big[A(s)u(s) + B(s)\Big] ds$$

$u(t_0) = u_0 \Rightarrow$ (IntSol).
"$\Rightarrow$"Let u be a solution of (IntSol). Then $u'(t) = A(t)u + B(t)$. Since the right side is continuous, we obtain that $u \in C^1(J, \mathbb{R})$. Moreover, $u(t_0) = u_0$. The lemma is proved. $\square$

Resuming the proof of the theorem, let us denote $A : C(J, \mathbb{R}^2) \to C(J, \mathbb{R}^2)$, $u \longmapsto Au$, where: $Au(t) := \int_{t_0}^{t} \Big[A(s)u(s) + B(s)\Big] ds + u_0$

"(IntSol) $\Leftrightarrow u = Au$"

We consider on $C(J, \mathbb{R}^2)$ the Bielecki norm:

$$\|u\|_B = max_{t \in J}(\|u\|_{\mathbb{R}^2} e^{-\tau|t-t_0|}), \tau > 0$$

$(C(J,\mathbb{R}^2), \|.\|)$ is a Banach space.
Then we have that $\|Au - Av\|_B \leq L_A \|u-v\|_B, \forall u, v \in C(J, \mathbb{R}^2, L_A \in (0,1))$.
For $t \geq t_0$:

$$\|Au(t) - Av(t)\|_{\mathbb{R}^2} = \|\int_{t_0}^{t} A(s)(u(s) - v(s))ds\|$$
$$\leq \int_{t_0}^{t} \|A(s)(u(s) - v(s))\| ds$$
$$\leq \int_{t_0}^{t} \|A(s)\|_{\mathcal{M}_2(\mathbb{R}^2)} \|u(s) - v(s)\|_{\mathbb{R}^2} ds$$
$$\leq M_A \int_{t_0}^{t} \|u(s) - v(s)\| e^{-\tau(s-t_0)} e^{\tau(s-t_0)} ds$$
$$\leq \int_{t_0}^{t} \max_{(s\in J)} (\|u(s) - v(s)\| e^{-\tau(s-t_0)}) e^{\tau(s-t_0)} ds$$
$$\leq M_A \|u - v\|_B \frac{1}{\tau} (e^{\tau(t-t_0)} - 1)$$
$$\leq \frac{M_A}{\tau} \|u - v\|_B e^{\tau(t-t_0)}$$
$$\Rightarrow \|Au - Av\| \leq \frac{M_A}{\tau} \|u - v\|_B.$$

Let $L_A = \frac{M_A}{\tau}$ such as $\tau > M_A$. Then $L_A < 1$.

□

1.3.4 Representations of the solution

Let us remark that (DS) can be written as a single vectorial equation:

$$u' = F(t, u) \tag{VDS}$$

where u and F are vectorials, with two real components, more exactly column matrix:

$$u = \begin{bmatrix} x \\ y \end{bmatrix}, \quad F(t, u) = \begin{bmatrix} f(t, x, y) \\ g(t, x, y) \end{bmatrix}$$

The condition for the Cauchy Problem of the system can be written:

$$u(t_0) = u_0, u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Also, (DS*) can be written:

$$u' = A(t)u + B(t) \tag{MDS}$$

where

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \quad B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

Theorem 1.3.4.1 (of representing the solutions of linear systems)**.** *Let* $A \in C(J, \mathcal{M}_\in))$ *and* $B \in (J, R^2)$*. Then the solutions of the linear system* (MDS) *are defined by the formula*

$$u = e^{\int_{t_0}^t A(\sigma)d\sigma} C + \int_{t_0}^t e^{\int_s^t A(\sigma)d\sigma} B(s)ds \tag{1.2}$$

where $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ *and* $C_1, C_2 \in \mathbb{R}$.

Theorem 1.3.4.2 (of existence, uniqueness and representation of the solution)**.** *Let* $A \in C(J, M_2(\mathbb{R})), B \in C(J, \mathbb{R}), t_0 \in Jandu_0 \in \mathbb{R}^2$*. Then the Cauchy Problem has a unique solution defined on* J*, given by the formula:*

$$u = e^{\int_{t_0}^t A(\sigma)d\sigma} u_0 + \int_{t_0}^t e^{\int_s^t A(\sigma)d\sigma} B(s)ds \tag{1.3}$$

Theorem 1.3.4.3 (the structure of the set of solutions)**.** *(a) The set of solutions of a bidimensional linear and homogeneous system is a linear bidimensional space.*
(b) If u_p *is a particular solution of a linear non-homogeneous system* (MDS)*, then any other solution* u *of the system* (MDS) *is the sum between the solution of the homogeneous system* (u_o) *with the particular solution* u_p*:*

$$u = u_o + u_p$$

Definition 1.3.4.1. *A matrix in which the columns are the linearly independent solutions of the homogeneous system is called fundamental matrix of the system.*

Lemma 1.3.4.4. *(a) Any fundamental matrix is unsingular for any* $t \in J$*.*
(b) Any fundamental matrix $U(t)$ *satisfies the differential matricial equation:*

$$U'(t) = A(t)U(t)$$

1.3.5 Representation with fundamental matrix

In this paragraph, let us submit the possibility of representing the solutions of the non-homogeneous system in terms of the fundamental matrix $U(t)$, of which purpose is to replace the matrix $e^{\int_{t_0}^{t} A(\sigma)d\sigma}$ from (1.2).
Firstly, let

$$u_p = U(t)C(t)$$

be a particular solution of the non-homogeneous system, where the vectorial function

$$C(t) = \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix}$$

is to be determined with the condition that u_p satisfies the given system. Hence

$$u_p' = U'(t)C(t)U(t)C'(t)$$

and by replacing we find that

$$U'(t)C(t) + U(t)C'(t) = A(t)U(t)C(t) + B(t).$$

Point b) from lemma 1.3.4.4 guarantees that

$$U'(t) = A(t)U(t).$$

Thus $U(t)C'(t) = B(t)$, which means that $C'(t) = U^{-1}(t)B(t)$. Then, according to point a) from 1.3.4.4, matrix $U(t)$ is invertible. Let us choose

$$C(t) = \int_{t_0}^{t} U^{-1}(s)B(s)ds$$

and so we get a particular solution of the non-homogeneous system:

$$u_p = \int_{t_0}^{t} U(t)U^{-1}(s)B(s)ds.$$

Thus, an equivalent of (1.2) is:

$$u = U(t)C + \int_{t_0}^{t} U(t)U^{-1}(s)B(s)ds. \tag{1.4}$$

The analogue of (1.3) results immediately:

$$u = U(t)U^{-1}(t_0)u_0 + \int_{t_0}^{t} U(t)U^{-1}(s)B(s)ds. \tag{1.5}$$

Hence, we can present the following theorem:

Theorem 1.3.5.1 (of representation with fundamental matrix)**.** *If* $U(t)$ *is a fundamental matrix of* (MDS)*, then:*

(i) The solutions of the homogeneous system are the functions $u_o = U(t)C$, where C is a random vector from $\mathbb{R}^2$;

(ii) The solutions of the non-homogeneous system are the functions defined by (1.4), *where $C \in \mathbb{R}^2$;*

(iii) The solution of the Cauchy problem for (Sys1) *is the function defined by formula* (1.5).

According to 1.3.5.1, solving a linear system depends on finding a fundamental matrix. Generally, this is not possible, but for the systems with constant coefficients is possible.

1.3.6 Linear systems with constant coefficients

In this paragraph, we consider the system (DS*), where the coefficients $a_{ij}(i, j = 1, 2)$ are constant, and $b_1, b_2 \in C(J)$. By using matrices, we can rewrite the system as:

$$u' = Au + B(t),$$

where

$$u = \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}.$$

Due to the fact that all the previous results stay valid, the solutions of the system have the following form:

$$u = u_o + u_p.$$

The explicit form of the solution is:

$$u = e^{(t-t_0)\cdot A} \cdot C + \int_{t_0}^{t} e^{(t-s)\cdot A} B(s) ds,$$

where the constant vector C is random. Also, the Cauchy problem, with $u(t_0) = u_0$ as the initial condition admits only one solution, which is

$$u = e^{(t-t_0)\cdot A} \cdot u_0 + \int_{t_0}^{t} e^{(t-s)\cdot A} B(s) ds \tag{1.6}$$

If by the replacement of $e^{(t-t_0)\cdot A}$ it is considered another fundamental matrix $U(t)$, then the solutions of the system and also for the Cauchy problem are given by formulas (1.4) and (1.5). The question of how we can determinate the fundamental matrix and how we can directly solve the system still remains.
For this, let us search for solutions for the associated homogeneous system based on the following form:

$$u = e^{rt} V,$$

where $r \in \mathbb{R}$ and $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ are determined with the condition that u satisfies the homogeneous system. Obviously, $V \neq 0$. By replacing in the homogeneous system, we get:

$$re^{rt}V = A(e^{rt}V) = e^{rt}AV.$$

By simplifying with e^{rt}, we obtain:

$$(A - rI)V = 0. \tag{1.7}$$

As $V \neq 0$, we conclude that r is an eagen value of matrix A, and V is an eagen vector associated to this value. Eagen values are determined with the condition that the algebraic homogeneous system (1.7) admits solutions unequal to 0, meaning from equation:

$$det(A - rI) = 0. \tag{1.8}$$

This equation is named the characteristic equation of the system and is a second grade polynomial. It can be written:

$$\begin{vmatrix} a_{11} - r & a_{12} \\ a_{21} & a_{22} - r \end{vmatrix} = 0, \tag{1.9}$$

where, by solving the determinant, we obtain:

$$r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{12}a_{21} = 0,$$

meaning

$$r^2 - (trA)r + detA = 0.$$

Here,

$$trA = \sum_{i=0}^{n} a_{ii}$$

is the trace of the matrix. Considering how the roots of the characteristic equations might be, we distinguish the following cases:

1.3.6.1 Case of real and distinct roots

Let r_1, r_2 be the roots of the characteristic equation, which are presumed to be real and distinct. For each one, we choose a non-null solution of system (1.7). Let them be V_1, V_2. Meaning that we obtained two solutions for the homogeneous system:

$$u_1 = e^{r_1 t}V_1, \quad u_2 = e^{r_2 t}V_2.$$

As $r_1 \neq r_2$, they are linear independent. Matrix $U(t)$, with columns u_1, u_2 thus determined, is a fundamental matrix.

1.3.6.2 Case of equal roots

Let us presume that the roots of the characteristic equation are equal. Then $r := r_1 = r_2 = \frac{trA}{2}$. A solution of the system can be determined similarly to the previous case, of form $u_1 = e^{rt}V$, where V is a non-null solution of the algebraic homogeneous system (1.7). A second solution u_2, linear independent of u_1, is searched by the form $u_2 = e^{rt}(tV + W)$, with V being the previous, and W being vector which is obtained with the condition that verifies the system. We have:

$$u_2' = re^{rt}(tV + W) + e^{rt}V$$

meaning that u_2 is a solution if

$$re^{rt}(tV + W) + e^{rt}V = e^{rt}A(tV + W).$$

By simplifying with e^{rt} and by reducing the equal terms, we obtain the following algebraic linear and non-homogeneous system:

$$(A - rI)W = V.$$

This system is compatible and so we can choose a solution W of it.

1.3.6.3 Case of complex roots

Let us presume that the roots of the characteristic equation are complex, being them $\alpha \pm \beta i$. Considering one of them, for example $\alpha + \beta i$ and by proceeding like in the case of real and distinct roots, let us determine a solution of the system, this time complex:

$$u = e^{rt}V = e^{(\alpha+\beta i)t}(V_1 + iV_2) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(V_1 + iV_2) =$$

$$= e^{\alpha t}[V_1\cos\beta t - V_2\sin\beta t + i(V_1\sin\beta t + V_2\cos\beta t)].$$

where V_1, V_2 represents the column vector of the real parts, respectively the coefficients of the imaginary parts of V, meaning $V = V_1 + iV_2$ and $V_1, V_2 \in \mathbb{R}^2$. With the system being linear, homogeneous and with real coefficients, at once with a complex solution u, admits as solutions the conjugate function $\bar{u}$, as well as any linear combination of theirs with complex coefficients, particularly the real functions:

$$u_1 := Re(u) = \frac{1}{2}(u + \bar{u}) = e^{\alpha t}(V_1\cos\beta t - V_2\sin\beta t),$$

$$u_2 := Im(u) = \frac{1}{2i}(u - \bar{u}) = e^{\alpha t}(V_1\sin\beta t + V_2\cos\beta t).$$

1.3.7 Examples

1) $U' = AU$, where $A = \begin{bmatrix} 0 & 4 \\ 5 & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

<u>Solution:</u> First, let us attach the characteristic equation: $r^2 - (tr(A))r + det(A) = 0$. In our case, $tr(A) = 1$ and $det(A) = -20$, meaning that the equation becomes: $r^2 - r - 20 = 0$. By solving the equation, we get the following roots: $r = 5$ and $r = -4$. Let us consider the first case, where $r = 5$. Next, we need to determine an eigenvector $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$,by solving $(A - rI)V = 0$. A convienent eigenvector would be $V = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ Hence, the solution is:

$$U_1 = e^{rt}V = e^{5t} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 4e^{5t} \\ -5e^{5t} \end{bmatrix}.$$

Next, we consider the case where $r = -4$. Thus, an eigenvector would be $V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence, the solution is:

$$U_2 = e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{-4t} \\ -e^{-4t} \end{bmatrix}.$$

Then, the fundamental matrix of the solutions of the system is:

$$U = c_1U_1 + c_2U_2,$$

where $c_1, c_2 \in \mathbb{R}$.

2) $U' = AU$, where $A = \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}$, $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

<u>Solution:</u> By attaching the characteristic equation, we get that $r = 1$ is a solution of the equation of multiplicity 2. We are in the case of equal roots. Next, by solving the equation $(A - rI)V = 0$, we get that $V = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of the given equation. By solving $(A - rI)W = V$, we get $W = \begin{bmatrix} 1 \\ -\frac{2}{8} \end{bmatrix}$. Then:

$$U_1 = e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

and

$$U_2 = e^t \begin{bmatrix} -2t + 1 \\ t - \frac{2}{8} \end{bmatrix}.$$

Hence, the fundamental matrix of the solutions has is $U = c_1U_1 + c_2U_2$, where $c_1, c_2 \in \mathbb{R}$.

3) $U' = AU$, where $A = \begin{bmatrix} 5 & 5 \\ -4 & 3 \end{bmatrix}$, $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

<u>Solution:</u> By solving the characteristic equation, we denote that we are in the

case of complex roots, where $r = 1 \pm 2i$. Next, we find that $V = \begin{bmatrix} 1 \\ -\frac{4}{5} + i\frac{2}{5} \end{bmatrix}$ is a convienient eigenvector for the case. The solution is

$$U = e^{(1+2i)t} \begin{bmatrix} 1 \\ -\frac{4}{5} + i\frac{2}{5} \end{bmatrix} = e^t(\cos 2t + i \sin 2t) \left(\begin{bmatrix} 1 \\ -\frac{4}{5} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{5} \end{bmatrix} i \right).$$

By separating the real part and the imaginary part, we get two real solutions:

$$U_1 = e^t \left(\cos 2t \begin{bmatrix} 1 \\ -\frac{4}{5} \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ \frac{2}{5} \end{bmatrix} \right)$$

and

$$U_2 = e^t \left(\cos 2t \begin{bmatrix} 0 \\ \frac{2}{5} \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ -\frac{4}{5} \end{bmatrix} \right).$$

The chapter introduced the reader to the general notions of an equation with separable variables, of linear differential equations of first order and the linear differential systems, needed for presenting the main part of the paper. The notions have been extracted from the classic books mentioned in the bibliography of the paper. The author's contribution was the selection and solving of the exercises found at the end of each section.

Chapter 2

Difference Equations

In this chapter, we will introduce the general form of a linear difference equation and its particular cases, followed by the generated dynamical systems. After which, we will present the notion of equilibrium point, the stability criteria and a few important results. The author's contribution are the selection and solving of the examples from the end of each section.

2.1 Linear difference equations of first order

The general form of a linear difference equation of first order is:

$$x_{n+1} = a_n x_n + b_n \tag{2.1}$$

for any $n \geq 0$, where $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are given series of real numbers. (2.1) is obtained starting from an initial value $x_0 \in \mathbb{R}$. Hence:

$$\begin{aligned}
x_1 &= a_0 x_0 + b_0 \\
x_2 &= a_1 x_1 + b_1 = a_1(a_0 x_0 + b_0) + b_1 \\
&= a_1 a_0 x_0 + a_1 + b_0 + b_1 \\
&\dots \\
x_n &= a_{n-1} x_{n-1} + b_{n-1} = a_{n-1} \cdot \dots \cdot a_0 x_0 + a_{n-1} \cdot \dots \cdot a_1 b_0 + \dots + a_{n-1} \cdot b_{n-2} + b_{n-1}.
\end{aligned}$$

Through mathematical induction, it is proved that the solution of the linear difference equation of first order has the following form:

$$x_n = \left(\prod_{j=0}^{n-1} a_j\right) \cdot x_0 + \sum_{j=0}^{n-1} \left(\prod_{k=j+1}^{n-1} a_k\right) \cdot b_j. \tag{2.2}$$

If (2.1) is valid for n starting from an $n_0 \geq 0$, meaning that:

$$x_{n+1} = a_n \cdot x_n + b_n, n \geq n_0 \tag{2.3}$$

Then (2.2) becomes:

$$x_n = \left(\prod_{j=n_0}^{n-1} a_j\right) \cdot x_{n_0} + \sum_{j=n_0}^{n-1} \left(\prod_{k=j+1}^{n-1} a_k\right) \cdot b_j. \tag{2.4}$$

Ordinarilly, there are multiple particular cases of (2.1) that can appear in practice, which ease the use of formula (2.2).

2.1.1 Particular cases of (2.1)

2.1.1.1 Case $a_n \equiv a$

In this case, (2.1) becomes:

$$x_{n+1} = a \cdot x_n + b_n, n \geq 0, \tag{2.5}$$

where $a \in \mathbb{R}$ and $(b_n)_{n\in\mathbb{N}}$ is a given series of real numbers. From (2.1) we have that:

$$\prod_{j=0}^{n-1} a_j = \prod_{j=0}^{n-1} a,$$
$$\prod_{k=j+1}^{n-1} a_k = \prod_{k=j+1}^{n-1} a = a^{n-1-(j+1)+1} = a^{n-j-1}$$

and the solution is:

$$(2.5) x_n = a^n \cdot x_0 + \sum_{j=0}^{n-1} a^{n-j-1} \cdot b_j. \tag{2.6}$$

2.1.1.2 Case $b_n \equiv b$

In this case, (2.1) becomes:

$$x_{n+1} = a_n \cdot x_n + b, n \geq 0, \tag{2.7}$$

where $b \in \mathbb{R}$ and $(a_n)_{n\in\mathbb{N}}$ is a given series of real numbers. From (2.2), we obtain:

$$x_n = \left(\prod_{j=0}^{n-1} a_j\right) \cdot x_0 + \sum_{j=0}^{n-1} \left(\prod_{k=j+1}^{n-1} a_k\right) \cdot b_j =$$
$$= \left(\prod_{j=0}^{n-1} a_j\right) \cdot x_0 + \sum_{j=0}^{n-1} \left(\prod_{k=j+1}^{n-1} a_k\right) \cdot b$$

meaning that the solution is of form:

$$x_n = \left(\prod_{j=0}^{n-1} a_j\right) \cdot x_0 + b \cdot \sum_{j=0}^{n-1} \left(\prod_{k=j+1}^{n-1} a_k\right). \tag{2.8}$$

2.1.1.3 Case $a_n \equiv a$ and $b_n \equiv b$

In this case, (2.1) becomes:

$$x_{n+1} = a \cdot x_n + b, n \geq 0, \tag{2.9}$$

where $a, b \in \mathbb{R}$. From (2.2) we obtain:

$$\begin{aligned} x_n &= \left(\prod_{j=0}^{n-1} a\right) \cdot x_0 + \sum_{j=0}^{n-1} \left(\prod_{k=j+1}^{n-1} a\right) \cdot b = \\ &= a^n \cdot x_0 + b \cdot \sum_{j=0}^{n-1} a^{n-j-1} = \\ &= a^n \cdot x_0 + b \cdot (a^{n-1} + a^{n-2} + ... + a^0). \end{aligned}$$

If $a \neq 1$, we have:

$$\begin{aligned} x_n &= a^n \cdot x_0 + b \cdot \frac{a^n - 1}{a - 1} = \\ &= a^n \cdot \left(x_0 + \frac{b}{a-1}\right) - \frac{b}{a-1}. \end{aligned}$$

And if $a = 1$, we have:

$$\begin{aligned} x_n &= a^n \cdot x_0 + b(a^{n-1} + a^{n-2} + ... + a^0) = \\ &= x_0 + n \cdot b. \end{aligned}$$

Thus the solution is:

$$x_n = \begin{cases} a^n \cdot \left(x_0 + \frac{b}{a-1}\right) - \frac{b}{a-1}, a \neq 1 \\ x_0 + n \cdot b, a = 1. \end{cases}$$

2.1.2 Examples

1) $x_{n+1} = -x_n + n$
Solution: In this example, $a_n = a = -1 \in \mathbb{R}$, meaning we are in case 2.1.1.1. Then, the products become:

$$\prod_{j=0}^{n-1} a = a^n = (-1)^n$$

and

$$\prod_{k=j+1}^{n-1} a = (-1)^{n-j-1}.$$

Hence, by applying (2.5), the solution is:

$$x_n = (-1)^n \cdot x_0 + \sum_{j=0}^{n-1} (-1)^{n-j-1} \cdot j.$$

Let us choose $x_0 = 1$, then the solution is:

$$x_n = (-1)^n + \sum_{j=0}^{n-1} (-1)^{n-j-1} \cdot j.$$

2) $x_{n+1} = \frac{n+2}{n+1} \cdot x_n$.
Solution:

In this example, $a_n = \frac{n+2}{n+1}$ and $b_n = 0 \in \mathbb{R}$, meaning we are in case 2.1.1.2. Then, the solution is:

$$x_n = \prod_{j=0}^{n-1} \frac{j+2}{j+1} \cdot x_0 = \frac{2}{1} \cdot ... \cdot \frac{n+2}{n+1} \cdot x_0 = (n+2) \cdot x_0.$$

Let us choose $x_0 = 1$. Then the solution is: $x_n = n + 2$.

2.1.3 Conclusions

2.2 Dynamical systems

2.2.1 General notions

$(G, +, \tau)$ is a topological semigroup if:

i) $(G, +)$ is a semigroup;

ii) (G, τ) is a separate topological space;

iii) operator
$$+ : GxG \to G, (t_1, t_2) \longmapsto t_1 + t_2$$
is continuous.

Let (X, d) be a metric space, $(G, +, \tau)$ a topological semigroup and $\varphi : GxX \to X$.

Definition 2.2.1.1. *The triplet (X, G, φ) is called a dynamical system if:*

1. *$\varphi(0, x) = x, \forall x \in X$;*

2. *$\varphi(t, \varphi(s, x)) = \varphi(t + s, x), \forall t, s \in G, \forall x \in X$;*

3. *operator φ is continuous.*

Space X is called phase space or states space. Operator $\varphi(+,\cdot) : X \to X$, $t \in G$ is called the movement of x in relation to the dynamical system (X, G, φ). The set $\varphi(G, x)$ is called the trajectory of x or the trajectory of the dynamical system passing through x. If $(G, +)$ is a group, then the dynamical system is a discrete dynamical system, and if $G = \mathbb{R}$ then the dynamical system is continuous.

2.2.2 Dynamical systems defined by a linear difference equation of first order

2.2.2.1 Example

Let us consider the difference equation:

$$x_{n+1} = ax_n, a \in \mathbb{R}^* \tag{2.10}$$

We know that $x_n = a^n x_0, x_0 \in \mathbb{R}$ represents the solution of (2.10) with start value $x_0 \in \mathbb{R}$. Let us define $\varphi : \mathbb{Z}x\mathbb{R} \to X$ through:

$$\varphi(k, x) = a^k x.$$

then $(\mathbb{R}, \mathbb{Z}, \varphi)$ is an inversable discrete dynamical system generated by (2.10).

2.2.2.2 Example

Let $f : \mathbb{R} \to \mathbb{R}$ and the difference equation:

$$x_{n+1} = f(x_n). \tag{2.11}$$

We note through

$$x_n(x_0) = \underbrace{(f \circ ... \circ f)}_{\text{of n times}}(x) = f^n(x_0), x_0 \in \mathbb{R} \tag{2.12}$$

the solution of the equation with start value $x_0 \in \mathbb{R}$. We define $\varphi : \mathbb{N}x\mathbb{R} \to \mathbb{R}$ through:

$$\varphi(n, x) = f^n(x),$$

then $(\mathbb{R}, \mathbb{N}, \varphi)$ is a discrete dynamical system generated by (2.11). The trajectory of the dynamical system that passes through $x_0 \in \mathbb{R}$ is given by the set:

$$\varphi(\mathbb{N}, x_0) = O_f(x_0) = \{x_0, x_1, ..., x_n, ...\} \tag{2.13}$$

named also the orbit generated by x through f.

Definition 2.2.2.1. *Let (X, G, φ) be a dynamical system. We say that $x^* \in X$ is a fixed point of the dynamical system if $\varphi(t, x^*) = x^*, \forall t \in G$.*

Let us consider the dynamical system generated by (2.11). If $x^* \in \mathbb{R}$ is a fixed point for this dynamical system, then:

$$\varphi(n, x^*) = x^*, \forall n \in \mathbb{N}$$
$$f^n(x^*) = x^*, \forall n \in \mathbb{N}$$

from where we deduce that $x^* \in \mathbb{R}$ is a solution for the equation $f(x) = x$. So the orbit generated by the fixed point x^* is reduced to a point

$$\varphi(\mathbb{N}, x^*) = O_f(x^*) = \{x^*\}.$$

2.3 Dynamics generated by difference equations of first order

2.3.1 Equilibrium point. Stability criteria

As we have seen in 2.2.2.2, the difference equation of first order:

$$x_{n+1} = f(x_n) \tag{2.14}$$

generates the dynamical system $(\mathbb{R}, \mathbb{N}, \varphi)$, where $\varphi(n, x) = f^n(x)$. The fixed points of this dynamical system are given by the solutions of the equation

$$x = f(x) \tag{2.15}$$

From the perspective of the difference equation (2.14), the fixed points of the generated dynamical system are obtained from the constant solutions:

$$(x_n) = (x^*, x^*, ...x^*) \tag{2.16}$$

where $x^* \in \mathbb{R}$ is the solution of (2.15).

Definition 2.3.1.1. *A point $x^* \in \mathbb{R}$ is called an equilibrium point (stationary) for equation* (2.14) *if it is a fixed point for the generated dynamical system* $(\mathbb{R}, \mathbb{N}, \varphi)$*. The constant solution defined by* (2.16) *is called equilibrium solution (stationary).*

One of the main objectives in the study of dynamics generated by difference equations is the study of the behaviour of their solutions to the equilibrium solutions. We will introduce the stability notions that describe this behaviour:

Definition 2.3.1.2. *We say that an equilibrium point $x^* \in \mathbb{R}$ for* (2.14) *is:*

1. ***global atractor** relative to set $D \subseteq \mathbb{R}$ if:*

$$x_n(x_0) = \varphi(x, x_0) \to x^*, n \to \infty, \forall x_0 \in D$$

2. ***locally stable*** *if for any $\epsilon > 0$ exists $\delta > 0$ such that if $|x_0 - x^*| < \delta$ then:*

$$|x_n(x_0) - x^*| < \epsilon, \forall n \in \mathbb{N}$$

3. ***locally asymptotic stable*** *if it is locally stable and exists $\eta > 0$ such that if $|x_0 - x^*| < \eta$ then:*

$$|x_n(x_0) - x^*| \to x^*, n \to \infty$$

4. ***globally asymptotic stable*** *relative to set $D \subseteq \mathbb{R}$ if it is locally asymptotic stable and is global atractor relative to set D;*

5. ***exponentially stable*** *relative to set $D \subseteq \mathbb{R}$ if exists $\alpha \in [0,1)$ and $c(x_0) > 0$ such that:*

$$|x_n(x_0) - x^*| \leq c(x_0) \cdot \alpha^n, \forall x_0 \in D$$

6. ***unstable*** *if it is not locally stable, meaning that exists $\epsilon > 0$ such that for any series $(x_l^o)_{l\in\mathbb{N}}$ which satisfy $x_l^o \to x^*$ exists $N_l \in \mathbb{N}$ such that:*

$$|x_{N_l}(x_l^o) - x^*| > \epsilon$$

Remark. *Any exponentially stable equilibrium point relative to set $D \subseteq \mathbb{R}$ is globally asymptotic stable relative to set $D \subseteq \mathbb{R}$.*

2.3.1.1 Cobweb Diagram

The Cobweb diagram represents the graphic visualization of a dynamic generated by a difference equation of form (2.14). It is built in the following way:

1. We calculate $x_1 = f(x_0)$, for a given x_0;

2. We draw the vertical from x_0 until it intersects the graphic of f in $(x_0, f(x_0)) = (x_0, x_1)$;

3. We draw the horizontal line from (x_0, x_1) until it intersects the diagonal $y = x$, thus obtaining the point (x_1, x_2).

The intersection of the vertical from (x_1, x_1) with the graphic of f represents the point $(x_1, f(x_1)) = x_1, x_2$. The process continues until we determine x_n relative to a fixed N. Thus, the Cobweb diagram is obtained and it provides us information regarding the stability of the equilibrium points.

2.3.2 Theorems with regard to the stability criteria of the equilibrium points

Theorem 2.3.2.1. *Let us consider* (2.14) *for which* $f : I \to I, I \subseteq \mathbb{R}$ *is a closed interval. If f is an* α*-contraction, meaning that there exists* $\alpha \in [0, 1)$ *such that*

$$|f(x) - f(y)| \leq \alpha|x - y|, \forall x, y \in I$$

then:

1. (2.14) *has an unique equilibrium point* $x^* \in I$*;*
2. *the equilibrium point* x^* *is exponentially stable:*

$$|x_n(x_0) - x^*| \leq \frac{\alpha^n}{1-\alpha} \cdot |f(x_0) - x_0|, \forall x_0 \in I \tag{2.17}$$

Proof. Let $x_0 \in I$ and let us consider $x_n(x_0) = f^n(x_0)$ the solution of quation $x_{n+1} = f(x_n)$. From the contraction condition we denote that

$$|x_{n+1}(x_0 - x_n(x_0)| \leq \alpha|x_n(x_0) - x_{n-1}(x_0)| \leq ... \leq \alpha^n|fx_0 - x_0|,$$

from where we obtain that

$$\begin{aligned}|x_{n+p}(x_0) - x_n(x_0)| &\leq \sum_{j=0}^{p-1} |x_{n+j+1}(x_0) - x_{n+j}(x_0)| \leq \\ &\leq \left(\sum_{j=0}^{p-1} \alpha^{n+j}\right)|f(x_0) - x_0| = \\ &= \alpha^n \cdot \frac{1-\alpha^p}{1-\alpha}|f(x_0) - x_0| \leq \frac{\alpha^n}{1-\alpha}|f(x_0) - x_0|.\end{aligned}$$

We can observe that for $n \to \infty$ and for all $p \in \mathbb{N}$ we have $|x_{n+p}(x_0) - x_n(x_0)| \to 0$, meaning that the series $(x_n(x_0))_{n\in\mathbb{N}}$ is fundamental. With $I \subseteq \mathbb{R}$ being a closed interval, we have that the series $(x_n(x_0))_{n\in\mathbb{N}}$ is convergent. Let $x^* = \lim_{n\to\infty} x_n(x_0)$. I is a closed interval, so $x_* \in I$. From

$$x_{n+1}(x_0) = f(x_n(x_0))$$

and f being continuous, applying the limit in the relation above we obtain:

$$x^* = f(x^*)$$

meaning that x^* is an equilibrium point for (2.14). If x^* and y^* are two equilibrium points for (2.14), then:

$$|x^* - y^*| \leq |f(x^*) - f(y^*)| \leq \alpha|x^* - y^*|,$$

hence

$$(1-\alpha)|x^*-y^*| \le 0,$$

which implies that $|x^*-y^*|=0$, so $x^*=y^*$, meaning that (2.14) admits an unique equilibrium point. Estimation (2.17) is given by the inequality:

$$|x_{n+p}(x_0)-x_n(x_0)| \le \alpha^n \frac{1-\alpha^p}{1-\alpha}|f(x_0)-x_0|$$

By making $p \to \infty$, the exponential stability of the equilibrium point x^* is proven.

□

Theorem 2.3.2.2. *Considering* (2.14) *for which* $f: I \to I, I \subseteq \mathbb{R}$ *is an interval, we presume that:*

1. (2.14) *has an unique equilibrium point* $x^* \in I$*;*

2. f *is an* α*-cvasicontraction, meaning that there exists* $\alpha \in [0,1)$ *such that:*

$$|f(x)-x^*| \le \alpha|x-x^*|, \forall x \in I.$$

Then the equilibrium point x^*

$$|x_n(x_0)-x^*| \le \alpha^n|x_0-x^*|, \forall x_0 \in I. \tag{2.18}$$

Proof. Let $x_0 \in I$ and let us consider $x_n(x_0)=f^n(x_0)$ the solution of (2.14), with x_0 as the starting value. Estimation (2.18), which demonstrates the exponential stability of the equilibrium point x^*, can be proved through mathematical induction by using the α-cvasicontraction condition. Hence:

$$\begin{aligned}
&|x_1(x_0)-x^*| \le \alpha|x_0-x^*|,\\
&|x_2(x_0)-x^*| \le |x_1(x_0)-x^*| \le \alpha|x_0-x^*|,\\
&\qquad\qquad\qquad .\\
&\qquad\qquad\qquad .\\
&\qquad\qquad\qquad .\\
&|x_n(x_0)-x^*| \le \alpha^n|x_0-x^*|.
\end{aligned}$$

□

Theorem 2.3.2.3 (The stability criteria in first aproximation)**.** *Let* $x^* \in I$ *be an equilibrium point for* (2.14)*,* $f: I \to \mathbb{R}$ *such that f is differentiable in* x^**. Then:*

1. *if* x^* *is locally stable, then* $|f'(x^*)| \le 1$*;*

2. *if* $|f'(x^*)| < 1$ *then* x^* *is locally asymptotic stable;*

3. if $|f'(x^*)| > 1$ then x^* is unstable.

Proof. (a) How f is differentiable in x^* then:

$$|f'(x^*)| = \lim_{x \to x^*} \frac{|f(x) - f(x^*)|}{|x - x^*|}$$

If the equilibrium point x^* is locally stable, then for all $\epsilon > 0$ exists $\delta > 0$ such that if $|x_0 - x^*| < \delta$ we have:

$$|x_n(x_0) - x^*| < \epsilon, \forall n \in \mathbb{N}$$
$$|f^n(x_0) - x^*| < \epsilon, \forall n \in \mathbb{N}$$

For a fixed $x \in I$ we choose $\epsilon = |x - x^*|$ and from x^*'s property of local stability we obtain:

$$|f^n(x) - x^*| < \epsilon = |x - x^*|, \forall n \in \mathbb{N},$$

so for $n = 1$ we have:

$$|f(x) - x^*| < |x - x^*|,$$
$$\frac{|f(x) - x^*|}{|x - x^*|} < 1,$$

and by applying the limit $x \to x^*$, results:

$$|f'(x^*)| \leq 1.$$

(b) For $\epsilon > 0$ exists δ with the property as $0 < \delta < \epsilon$ such that for $|x - x^*|$ we have:

$$|f(x) - x^*| = |f(x) - f(x^*)| \leq |f'(x^*)| \cdot |x - x^*|,$$

How $|f'(x^*)| \leq 1$ then f is a cvasicontraction on $(x^* - \delta, x^* + \delta)$ and applying 2.3.2.2 we deduce that:

$$|x_n(x_0) - x^*| \leq |f'(x^*)|^n \cdot |x_0 - x^*|, \forall x_0 \in (x^* - \delta, x^* + \delta),$$

meaning

$$|x_n(x_0) - x^*| \leq |f'(x^*)|^n \cdot \delta < \epsilon,$$

which proves that x^* is locally stable. In addition

$$|x_n(x_0) - x^*| \leq |f'(x^*)|^n \cdot \delta \to 0, n \to \infty,$$

so x^* is locally asimptotic stable. (c) is an immediat consequence of point (a).

□

Theorem 2.3.2.4. *Let $x^* \in I$ be an equilibrium point for* (2.14)*,$f : I \to \mathbb{R}$ such that $f'(x^*)$ exists and $f'(x^*) = 1$. Then:*

1. *if $f''(x^*)$ exists and $f''(x^*) \neq 0$ then x^* is unstable;*
2. *if $f''(x^*)$ exists and $f'''(x^*)$ such that $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* is unstable;*
3. *if $f''(x^*)$ exists and $f'''(x^*)$ such that $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* is locally asymptotic stable.*

Proof. Let us consider the difference equation:

$$y_{n+1} = g(y_n), \tag{2.19}$$

where

$$g(x) = f^2(x) = (f \circ f)(x).$$

Obviously,if x^* is an equilibrium point for (2.14), then x^* is also an equilibrium point for (2.19). We will demonstrate that if x^* is a locally asimptotic stable point for (2.19), then it is also a locally asimptotic stable point for (2.14). For a fixed $\epsilon > 0$ exists $\delta > 0$ such that for all x that satisfy $|x - x^*| < \delta$ takes place:

$$|f(x) - x^*| \leq |f(x^*)| \cdot |x - x^*| = |x - x^*|,$$
$$|g^n(x) - x^*| < \epsilon, |g^n(x) - x^*| \to 0, n \to \infty.$$

So, for a chosenx_0, such that $|x_0 - x^*| < \delta$, we have that also

$$|f(x_0) - x^*| \leq |x_0 - x^*| < \delta,$$

resulting that

$$|g^n(x_0) - x^*| < \epsilon, |g^n(f(x_0)) - x^*| < \epsilon$$

and

$$|g^n(x_0) - x^*| \to 0, |g^n(f(x_0)) - x^*| \to 0, n \to \infty,$$

but

$$y_n(x_0) = g^n(x_0) = x_{2n}(x_0), y_n(f(x_0)) = g^n(f(x_0)) = x_{2n+1}(x_0),$$

so

$$|x_n(x_0) - x^*| < \epsilon, |x_n(x_0) - x^*| \to 0, n \to \infty,$$

meaning that x^* is locally asimptotic stable also for (2.14). □

Theorem 2.3.2.5. *Let x^* be an equilibrium point of* (2.14)*. If $f \in C^3(\mathbb{R})$ and $f'(x^*) = -1$, then:*

1. *if $-3[f''(x^*)]^2 - 2f'''(x^*) < 0$, then x^* is locally asymptotic stable(?);*

2. *if* $-3[f''(x^*)]^2 - 2f'''(x^*) > 0$*, then* x^* *is unstable;*

Theorem 2.3.2.6. *Let* x^* *to be an equilibrium point of* (2.14)*. If* $f \in C^k(\mathbb{R}), k \geq 2$ *and* $f'(x^*) = 1$*,* $f^{(j)}(x^*) = 0, j = \overline{2, k-1}$*,* $f^{(k)}(x^*) \neq 0$*, then:*

1. *if k is even, then* x^* *is unstable;*

2. *if k is odd and* $f^{(k)}(x^*) > 0$*, then* x^* *is unstable;*

3. *if k is odd and* $f^{(k)}(x^*) < 0$*, then* x^* *is asymptotically stable(??).*

Lemma 2.3.2.7. *Suppose that* $f(x) \in \mathbf{C}(\mathbb{R})$ *and* $f'(x^*) = -1$*.*

1. *If* x^* *is an equilibrium point for* (2.14)*, then it is an equilibrium point for* (2.19)*;*

2. *If the equilibrium point* x^* *is locally asymptotic stable with respect to* (2.19)*, then it is also locally asymptotic stable for* (2.14)*;*

3. *If the equilibrium point* x^* *of* (2.14) *is unstable with respect to* (2.19)*, then it is unstable with respectto* (2.14)*.*

Theorem 2.3.2.8. *Let* x^* *to be an equilibrium point of* (2.14)*. Suppose that* $f \in \mathbf{C}^{2k-1}(\mathbb{R})$ *and* $f'(x^*) = -1, f^{(j)}(x^*) = 0(j = \overline{2, k-1}), f^{(k)}(x^*) \neq 0$*.*

i) *If k is odd and* $f^{(k)}(x^*) > 0$*, then* x^* *is asymptotically stable.*

ii) *If k is odd and* $f^{(k)}(x^*) < 0$*, then* x^* *is unstable.*

iii) *Assume that k is even, and there exists an integer* $l < k$ *such that:*

$$f^{(j)}(x^*) = 0(j = \overline{(k+1, 2l-3)}), f^{(2l-1)}(x^*) \neq 0$$

  a) *If* $f^{(2l-1)}(x^*) > 0$*, then* x^* *is asymptotically stable.*

  b) *If* $f^{(2l-1)}(x^*) < 0$*, then* x^* *is unstable.*

iv) *Assume that k is even, and:*

$$f^j(x^*) = 0(j = \overline{k+1, 2k-3}).$$

  a) *If* $\frac{k}{2}(\frac{f^{(k)}(x^*)}{k!})^2 + \frac{f^{(2k-1)}(x^*)}{(2k-1)!} > 0$*, then* x^* *is asymptotically stable.*

  b) *If* $\frac{k}{2}(\frac{f^{(k)}(x^*)}{k!})^2 + \frac{f^{(2k-1)}(x^*)}{(2k-1)!} < 0$*, then* x^* *is unstable.*

Application We consider te stability of the zero solution of

$$x_{n+1} = x_n e^{-x_n^k}, n \in \mathbb{N}, \tag{2.20}$$

where $x_n \in \mathbb{R}$ and k is a positive integer.

Theorem 2.3.2.9. *If k is even, then the zero solution of* (2.20) *is asymptotically stable. If k is odd, then the zero solution of* (2.20) *is unstable.*

Proof. Let $f(x) = xe^{-x^k}$. Then we have:

$$\begin{aligned} f(x) &= x\left(1 + (-x^k) + \frac{1}{2!}(-x^k)^2 + \frac{1}{3!}(-x^k)^3 + ...\right) \\ &= x = x^{k+1} + \frac{1}{2!}x^{2k+1} - \frac{1}{3!}x^{3k+1}.... \end{aligned}$$

Thus we have

$$f'(0) = 1, f^{(j)}(0) = 0, (j = \overline{2,k}), f^{(k+1)}(0) = -(k+1)! < 0$$

Using Theorem 2.3.2.6, we complete the proof. □

2.3.3 Examples

1) Depreciation Depreciation refers to the process in which a credit is paid through a series of periodic payments. Each payment covers both the primary credit and the interest rates.
Let $p(n)$ be the remaining credit of the n payment and $g(n)$ the percentage of the interest rate equal to "r".
The model is based on the idea that the primary credit $p(n+1)$ is equal to the $p(n)$ credit plus the interest rate $rp(n)$ corresponding to the n periods and minus the n payment $g(n)$. So:

$$\begin{aligned} p(n+1) &= p(n) + rp(n) - g(n) \\ p(n+1) &= (1+r)p(n) - g(n). \end{aligned}$$

Usually in practice the rate $g(n) = T$, which is constant. Resulting:

$$p(n) = (1+r)^n p(0) - \frac{T}{r}[(1+r)^n - 1]. \tag{2.21}$$

If we wish to pay the credit in exactly "n" payments, then T should be (if $p(n) = 0$):

$$T = p_0 \cdot \frac{r}{1-(1+r)^{-n}}$$

2) If exists, find the equilibruim points and specify their type for the following equations:

a) $x(n+1) = \frac{1}{3}x^3(n) + x(n)$

Solution: In this case, $f(x) = \frac{1}{3}x^3 + x$. To find the equilibruim point, we must solve $f(x) = x$. Meaning that we have:

$$\frac{1}{3}x^3 + x = x \Rightarrow x^3 = 0 \Rightarrow x = 0.$$

So the only equilibrium point for the given equation is $x = 0$. In order to apply Theorem 2.3.2.4, we need to find the value of $f'(0), f''(0)$ and $f'''(0)$.

$$f'(x) = x^2 + 1;$$

$$f''(x) = 2x$$

and

$$f'''(x) = 2.$$

Resulting that $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = 2$. Hence, $x = 0$ is an unstable equilibrium point.

b) $x(n+1) = \tan^{-1} x(n)$

Solution:

In this case, $f(x) = \tan^{-1} x$. If $f(x) = x$, we obtain that $x \tan x = 1$, resulting that the difference equation does not have equilibrium points.

c) $x(n+1) = -x^3(n) - x(n)$
In this case, $f(x) = -x^3 - x$. Solving $f(x) = x$, we obtain the equilibrium point $x = 0$. $f'(x) = -3x^2 - 1$, meaning that $f'(0) = 1$. $f''(x) = -6x$ resulting $f''(0) = 0$ and $f'''(x) = -6$ then $f'''(0) = -6$. Hence, by applying Theorem 2.3.2.5, we get that $x = 0$ is an unstable equilibrium point.

3) Let $x(n+1) = x^2(n) + 3x(n)$. Find the equilibrium point and specify its stability.

Solution:

By applying the formula, we denote that $f(x) = x^2 + 3x$. To find the equilibrium point, we need to solve $f(x) = x$. We get $x^2 + 2x = 0$, meaning that we have two equilibrium points $x_1^o = 0$ and $x_2^o = -2$. Knowing that $f'(x) = 2x + 3$, we get that $f'(0) = 3 > 0$, resulting that $x_1^o = 0$ is unstable according to Theorem 2.3.2.6. For $x_2^o = -2$, we have $f'(-2) = -1 < 0$, resulting that we can apply Theorem 2.3.2.5. Thus we have $f''(x) = 2$ and $f'''(x) = 0$ resulting that $-3 \cdot [f''(-2)]^2 - 2 \cdot f'''(-2) = -3 \cdot 2^2 - 0 = -3 \cdot 4 = -12 < 0 \Rightarrow x_2^o$ is locally asimptotic stable.

4) An application from economics Let $S(n)$ be the number of units from a merchandise supplied on the market in period "n". Let $D(n)$ be the number of units from the same commodity demanded in period "n". Let $p(n)$ be the price/unit in period "n". We presume that:

$$\begin{cases} D(n) = -m_d \cdot p(n) + b_d \\ m_d, b_d > 0. \end{cases} \tag{2.22}$$

Assumption: $D(n)$ depends linearly on $p(n)$. Equation (2.22) is also called the price-demand curve. m_d represents the consumers' sensitivity constant at price. It is certain that m_d should be positive as the growth of $p(n)$ implies the declining of the demand.
We presume:

$$\begin{cases} S(n+1) = -m_s \cdot p(n) + b_s \\ m_s, b_s > 0. \end{cases} \tag{2.23}$$

Equation (2.23) represents the price-offer curve. Assumption: The offer in period $n+1$ depends linearly on the price from the previous period. Assumption: The price from the market is the price to which the supplied quantity equals the demanded quantity, for example: $D(n+1) = S(n+1) \Rightarrow$

$$p(n+1) = A \cdot p(n) + B, \tag{2.24}$$

where $A = -\frac{m_s}{m_d}$ and $B = \frac{b_d - b_s}{m_d}$. We obtained a first order difference equation. The equilibrium price results from the intersection between the price-offer curve and the price-demand curve. If we note $f(p(n)) := Ap(n) + B$, then p^* (equilibrium price) is the only fixed point of $f(p)$. For example:

$$p^* = f(p^*) \Leftrightarrow p^* = Ap^* + B \Rightarrow p^* = \frac{B}{1-A}.$$

We denote three important cases:

i $A \in (-1, 0) \Rightarrow p^*$ is asymptotically stable;

ii $A = -1 \Rightarrow$ The prices are oscilating between two values. For example, if $p(0) = p_0 \Rightarrow p(1) = -p_0 + B.p(2) = p_0 \Rightarrow p^*$ is stable;

iii $A < -1 \Rightarrow$ the prices are oscilating towards infinity $\Rightarrow p^*$ is unstable.

The explicit solution of equation (2.24) is:

$$p(n) = \left(p_0 - \frac{B}{1-A}\right) \cdot A^n + \frac{B}{1-A}.$$

Conclusions: If the bidders are less sensitive to the prices, than the consumers ($m_s < m_d \Rightarrow A \in (-1, 0)$) then the market is stable. Otherwise, it becomes unstable.

Dynamic systems generated by difference equation are encountered in many mathematical models used in fields such as chemistry, biology, physics, economy and so on. They represent a useful tool in describing real behaviors and finding the equilibruim state of the models. In the next chapter, we will witness how the notions described above helped in solving real economic matters which appeared in specific times with the evolution of the civilization.