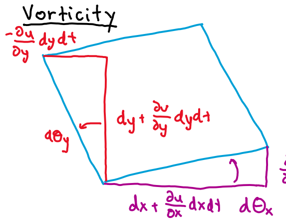
FLOW BASICS

Delia Stephens
Fluids S19

COMPRESSIBLE FLOW DIFF EQ

Vorticity



rotation rate $=\Omega$

$\Omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$

$\Omega_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$

$\Omega_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$

$\Omega=\frac{1}{2}\nabla\times\vec{V}=\frac{1}{2}\vec{\omega}$

$\omega=\nabla\times\vec{V}$, $\nabla=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$

$\text{curl}=\vec{\omega}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\hat{x}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\hat{y}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\hat{z}$

Normal Strain

$\epsilon_{xx}=\frac{\partial u}{\partial x}$ $\epsilon_{yy}=\frac{\partial v}{\partial y}$ $\epsilon_{zz}=\frac{\partial w}{\partial z}$

$\epsilon_{xx}=\frac{1}{\ell_x}\frac{d\ell_x}{dt}$, $\epsilon_{yy}=\frac{1}{\ell_y}\frac{d\ell_y}{dt}$, $\epsilon_{zz}=\frac{1}{\ell_z}\frac{d\ell_z}{dt}$

Shear Strain

$\epsilon_{xy}=\frac{1}{2}\left(\frac{d\theta_x}{dt}-\frac{d\theta_y}{dt}\right)=\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$

$\epsilon_{yz}=\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)$, $\epsilon_{xz}=\frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$

$\epsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$

Divergence

$\nabla\cdot\vec{V}=\frac{\partial u_i}{\partial x_i}=\epsilon_{ii}$
$=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$

$\lim_{\delta v\to 0}\frac{1}{\delta v}\frac{d}{dt}(\delta v)=\nabla\cdot\vec{V}$

Conservation of Mass

$\iiint_S \rho\vec{V}\cdot\hat{n}\,dS=\iiint_V \nabla\cdot(\rho v)\,dV$

$\iiint\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{V})\,dV\right]=0$

Conservation of Momentum

$\frac{\partial}{\partial t}(\rho u_j)+\nabla\cdot(\rho u_j\vec{V})=-\frac{\partial p}{\partial x_j}+\vec{f}_j^{\tau}$

$\left(\iint_S \tau_j\,dS=\iiint_V f_j^{\tau}\,dV\right)$

viscous forces pressure forces

Conservation of Energy

rate of work of viscous stresses per unit volume
$\iint_S \tau\cdot\vec{V}\,dS=\iiint_V \dot{w}^{\tau}dV$

$\frac{\partial}{\partial t}(\rho e_o)+\frac{\partial}{\partial x_i}(\rho e_o u_i)=-\frac{\partial}{\partial x_i}(pu_i)+\dot{w}^{\tau}-\frac{\partial q_i}{\partial x_i}$

$\frac{\partial}{\partial t}(\rho e_o)+\nabla\cdot(\rho e_o\vec{V})=-\nabla\cdot(p\vec{V})+\dot{w}^{\tau}-\nabla\cdot\dot{q}$

Substantial Derivative

- time rate of change of quantity following along w/ flow, pos. given by $X(t)$

$\frac{\partial\rho}{\partial t}+\frac{\partial p}{\partial x_i}\frac{dx_i}{dt}$] position

$\frac{D}{Dt}()=\frac{\partial}{\partial t}()+u_i\frac{\partial}{\partial x_i}()$
$=\frac{\partial}{\partial t}()+\vec{V}\cdot\nabla()$

$\frac{DT}{Dt}=\frac{\partial T}{\partial t}+10\cdot\frac{\partial T}{\partial x}$

Convective Governing Eq'ns

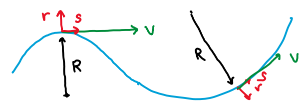
$\frac{D\rho}{Dt}=-\rho\frac{\partial u_i}{\partial x_i}=-\rho\nabla\cdot\vec{V}$

$\rho\frac{Du_j}{Dt}=-\frac{\partial p}{\partial x_j}+f_j^{\tau}$

$\rho\frac{De_o}{Dt}=-\frac{\partial}{\partial x_i}(pu_i)+\dot{w}^{\tau}-\frac{\partial q_i}{\partial x_i}$

$\frac{\partial}{\partial t}(\rho u_j)+\frac{\partial}{\partial x_i}(\rho u_j u_i)=u_j\left[\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i)\right]+\rho\frac{Du_j}{Dt}=\rho\frac{Du_j}{Dt}$

STREAMLINE CURVATURE



$\frac{\partial p}{\partial r}=\rho\frac{V^2}{R}$

$-\frac{\partial p}{\partial s}=\rho V\frac{\partial V}{\partial s}$

Camber

a larger camber (curvier) gives you a greater pressure gradient.

$P<P_\infty$, $C_p<0$ $P>P_\infty$, $C_p>0$

Thickness

thickness ↑ $R_u\downarrow$ $\frac{\partial p}{\partial r}=\rho\frac{V^2}{R}\uparrow$ $\rho_\infty-p_u\uparrow$ $p_u\downarrow$
$R_\ell\uparrow$ $\frac{\partial p}{\partial r}=\rho\frac{V^2}{R}\downarrow$ $\rho_\infty-p_\ell\downarrow$ $p_\ell\downarrow$

($p_u<p_\infty$, $p_\ell>p_\infty$)

Leading Edge

$C_p<-2$: suction, detrimental

$\lim_{R\to 0}\frac{\partial p}{\partial r}=\rho\frac{V^2}{R}\to\infty$

INCOMPRESSIBLE FLOW

$\nabla\cdot\vec{V}=0$

$\rho\frac{Du_j}{Dt}=-\frac{\partial p}{\partial x_j}+f_j^{\tau}$, $\rho\frac{D\vec{V}}{Dt}=-\nabla p+\vec{f}^{\tau}$

$\rho\left[\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla(\vec{V}^2)-\vec{V}\times\omega\right]=-\nabla p+\vec{f}^{\tau}$

Vorticity

$\rho\frac{D\omega}{Dt}=\rho(\omega\cdot\nabla)\vec{V}+\underbrace{\nabla\times\vec{f}^{\tau}}_{\text{viscous torque}}$

-if $\omega=0$, then $\rho(\omega\cdot\nabla)\vec{V}=0$ (and 2D)

Flow Modelling

assume:

- steady - uniform freestream
- inviscid - irrotational
- incompressible

$\vec{V}_\infty=V_\infty\cos\alpha\,\hat{i}+V_\infty\sin\alpha\,\hat{k}$

$\nabla\cdot\vec{V}=0$, $\vec{V}=\nabla\phi$ ② (grad. of vel. pot.)

$\nabla^2\phi=0$ ① $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

$p+\frac{1}{2}\rho V^2=p_\infty+\frac{1}{2}\rho V_\infty^2$ ③

① Solve for ϕ.
② Find $\vec{V}$
③ Find pressures.

Boundary conditions: flow tangency

$\vec{V}\cdot\hat{n}=0$

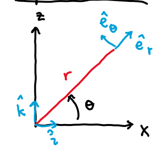
$\nabla\phi\cdot\hat{n}=\partial\phi/\partial n=0$

$\left.\begin{array}{l}\vec{V}\cdot\vec{t}_\infty=V_\infty\\ p=p_\infty\\ \nabla\phi\cdot\vec{t}_\infty=V_\infty\end{array}\right\}\vec{x}\cdot\vec{t}_\infty\to-\infty$

$\left(\vec{t}_\infty=\frac{\vec{V}_\infty}{V_\infty}\right)$

COORDINATE SYSTEMS

Cylindrical



$x=r\cos\theta$, $y=r\sin\theta$

$\hat{e}_r=\cos\theta\,\hat{i}+\sin\theta\,\hat{k}$

$\hat{e}_\theta=-\sin\theta\,\hat{i}+\cos\theta\,\hat{k}$

$u_r=u\cos\theta+w\sin\theta=\frac{\partial\phi}{\partial r}$

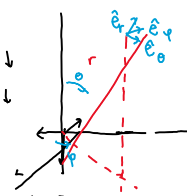
$u_\theta=-u\sin\theta+w\cos\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}$

$\nabla\cdot\vec{V}=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$

$\nabla\times\vec{V}=\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right]\hat{j}$

$\nabla^2\phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0$

Spherical



$x=r\cos\theta$

$y=r\sin\theta\cos\varphi$

$z=r\sin\theta\sin\varphi$

$\hat{e}_r=\cos\theta\,\hat{i}+\sin\theta\cos\varphi\,\hat{j}+\sin\theta\sin\varphi\,\hat{k}$

$\hat{e}_\theta=-\sin\theta\,\hat{i}+\cos\theta\cos\varphi\,\hat{j}+\cos\theta\sin\varphi\,\hat{k}$

$\hat{e}_\varphi=-\sin\varphi\,\hat{j}+\cos\varphi\,\hat{k}$

$u_r=u\cos\theta+v\sin\theta\cos\varphi+w\sin\theta\sin\varphi=\frac{\partial\phi}{\partial r}$

$u_\theta=-u\sin\theta+v\cos\theta\cos\varphi+w\cos\theta\sin\varphi=\frac{1}{r}\frac{\partial\phi}{\partial\theta}$

$u_\varphi=-v\sin\varphi+w\cos\varphi=\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}$

$\nabla\cdot\vec{V}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}$

$\nabla\times\vec{V}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & (r\sin\theta)\hat{e}_\varphi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ u_r & ru_\theta & (r\sin\theta)u_\varphi\end{vmatrix}$

$\nabla^2\phi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}=0$

CIRCULATION

$\Gamma=-\oint_c\vec{V}\cdot d\vec{\ell}$

KUTTA-JOUKOWSKY

$L'=\rho V_\infty\Gamma$

D'ALEMBERT'S PARADOX

$D'=0$

SHOCKWAVES

Assumptions:

* adiabatic (NOT reversible)
* steady: $\frac{\partial}{\partial t}=0$ $h=C_pT$
* body forces negligible
* ideal gas, calorically perfect (yay!)

Stagnation Properties

$\frac{\rho_o}{\rho}=\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}$ $\frac{P_o}{P}=\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$

$\frac{T_o}{T}=1+\frac{\gamma-1}{2}M^2=\frac{h_o}{h}$ Losses: $1-\frac{P_{o2}}{P_{o1}}$

$\rho_1u_1=\rho_2u_2$

$\rho_1u_1^2+p_1=\rho_2u_2^2+p_2$

$h_1+\frac{1}{2}u_1^2=h_2+\frac{1}{2}u_2^2$

$p_2=\frac{\gamma-1}{\gamma}\rho_2h_2$

Mach Jump Rel'n

$M_2^2=\frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2-\frac{\gamma-1}{2}}$

Static Jump Relations

$\frac{\rho_2}{\rho_1}=\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}$

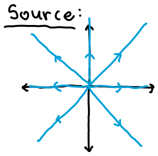
$\frac{P_2}{P_1}=1+\frac{2\gamma}{\gamma+1}(M_1^2-1)$

$\frac{T_2}{T_1}=\frac{h_2}{h_1}=\frac{P_2}{P_1}\frac{\rho_2}{\rho_1}$

FLOW MODELS

2D NONLIFTING FLOWS

Source:



$\phi = \frac{\Lambda}{2\pi}\ln(r)$

$u_r = \frac{\Lambda}{2\pi r}$

$u_\theta = 0$

- emits mass at rate of $\rho\Lambda$/span
- $\nabla\cdot\vec{V} = 0$ except for origin

$\phi = \frac{\Lambda}{2\pi}\ln\sqrt{x^2+z^2}$

$u = \frac{\Lambda}{2\pi}\ \frac{x}{x^2+z^2}$

$w = \frac{\Lambda}{2\pi}\ \frac{z}{x^2+z^2}$

Rankine Oval

- source with Λ and sink, $-\Lambda$ @ $(\ell,0)$ $(-\ell,0)$

$r_i = \sqrt{(x-x_i)^2+(z-z_i)^2}$

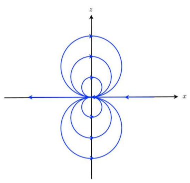
$\theta_i = \arctan\left(\frac{z-z_i}{x-x_i}\right)$

$\phi_i = \frac{\Lambda_i}{2\pi}\ln(r_i) = \frac{\Lambda_i}{2\pi}\ln\sqrt{(x-x_i)^2+(z-z_i)^2}$

$u_{ri} = \frac{\Lambda_i}{2\pi r_i}$, $u = \frac{\Lambda_i}{2\pi}\ \frac{x-x_i}{(x-x_i)^2+(z-z_i)^2}$

$u_{\theta i} = 0$, $w = \frac{\Lambda_i}{2\pi}\ \frac{z-z_i}{(x-x_i)^2+(z-z_i)^2}$

Doublet:



$\phi = \frac{K}{2\pi}\ \frac{\cos\theta}{r} = \frac{K}{2\pi}\ \frac{x}{x^2+z^2}$

$u_r = -\frac{K}{2\pi}\ \frac{\cos\theta}{r^2}$

$u_\theta = -\frac{K}{2\pi}\ \frac{\sin\theta}{r^2}$

$u = \frac{K}{2\pi}\ \frac{z^2-x^2}{(x^2+z^2)^2}$

$w = \frac{K}{2\pi}\ \frac{-2xz}{(x^2+z^2)^2}$

(derivative of Rankine Oval w/ respect to x

Flow Over Nonlifting Cylinder:

- freestream flow in x-direction + double

$\phi = V_\infty r\cos\theta + \frac{K}{2\pi}\ \frac{\cos\theta}{r}$ $\quad u_r(R,\theta) = 0$

$u_r = V_\infty\cos\theta - \frac{K}{2\pi}\ \frac{\cos\theta}{r^2}$ $\quad K = 2\pi R^2 V_\infty$

$u_\theta = -V_\infty\sin\theta - \frac{K}{2\pi}\ \frac{\sin\theta}{r^2}$

$r=R$, $u_r=0$, $u_\theta = -2V_\infty\sin\theta$, $V = 2V_\infty|\sin\theta|$

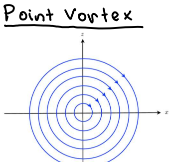
$p(R,\theta) = p_\infty + \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V^2$

$= p_\infty + \frac{1}{2}\rho V_\infty^2(1-4\sin^2\theta)$

$C_p(R,\theta) = \frac{p(R,\theta)-p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1-4\sin^2\theta$

2D LIFTING FLOWS

Point Vortex



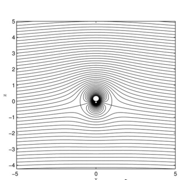
$\phi = -\frac{\Gamma}{2\pi}\theta$

$u_r = 0$

$u_\theta = -\frac{\Gamma}{2\pi r}$

$\oint_c \vec{V}\cdot d\vec{\ell} = -\iint_s (\nabla\times\vec{V})\cdot\hat{j}\,ds = 0$ if not origin, ∞ at origin

Lifting Flow over Rotating Cylinder



$\phi = V_\infty R\cos\theta\left(\frac{r}{R}+\frac{R}{r}\right) - \frac{\Gamma}{2\pi}\theta$

$u_r = V_\infty\cos\theta\left(1-\frac{R^2}{r^2}\right)$

$u_\theta = -V_\infty\sin\theta\left(1+\frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi r}$

at surface,

$u_r = 0$, $u_\theta = -2V\sin\theta - \frac{\Gamma}{2\pi R}$

$V = \left|2V\sin\theta + \frac{\Gamma}{2\pi R}\right|$

$2V_\infty\sin\theta_{stag} + \frac{\Gamma}{2\pi R} = 0$

$\sin\theta_{stag} = -\frac{\Gamma}{4\pi V_\infty R}$

$C_p(R,\theta) = \frac{p(R,\theta)-p_\infty}{\frac{1}{2}\rho V_\infty^2} =$

AIRFOIL FLOWS

Kutta condition: look at trailing edge and determine if flow smooth

Flat Plate

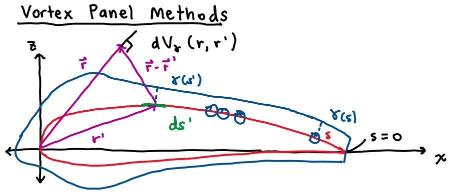
$\Gamma = \pi V_\infty c\sin\alpha$ $\quad$ (c = chord)

$L' = \rho V_\infty\Gamma = \rho V_\infty^2\pi c\sin\alpha$

$C_L = 2\pi\sin\alpha$, $\alpha\approx 0 \rightarrow C_L \approx 2\pi\alpha$

$D' = 0$, $C_D = 0$

Vortex Panel Methods



place point vortices with strength $\gamma(s)ds$ at pt. r', $d\vec{V}_\gamma(\vec{r},\vec{r}') = -\frac{\gamma(s')ds'}{2\pi|\vec{r}-\vec{r}'|}\hat{e}_\theta$

$\hat{e}_\theta = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}\times\hat{j}$

$\vec{V}_\gamma(\vec{r}) = \frac{1}{2\pi}\int \gamma(s')\frac{\hat{j}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^2}ds'$

$\vec{V}(r) = \vec{V}_\infty + \vec{V}_\infty(\vec{r})$

$-\vec{V}_\infty\cdot\hat{n}(\vec{r}) = \frac{1}{2\pi}\int \gamma(s')\frac{[\hat{j}\times(\vec{r}-\vec{r}')]\cdot\hat{n}(\vec{r})}{|\vec{r}-\vec{r}'|^2}ds'$

$\gamma(0)+\gamma(s_{te}) = 0$

$\Gamma = \int_{s=0}^{s_{te}}\gamma(s')ds'$

$$\begin{bmatrix} K_{1,1} & K_{1,2} & \dots & K_{1,N} & K_{1,N+1} \\ K_{2,1} & K_{2,2} & \dots & K_{2,N} & K_{2,N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{N-1,1} & K_{N-1,2} & \dots & K_{N-1,N} & K_{N-1,N+1} \\ K_{N,1} & K_{N,2} & \dots & K_{N,N} & K_{N,N+1} \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{N-1} \\ \gamma_N \\ \gamma_{N+1} \end{pmatrix} = \begin{pmatrix} -\mathbf{V}_\infty\cdot\hat{\mathbf{n}}_1 \\ -\mathbf{V}_\infty\cdot\hat{\mathbf{n}}_2 \\ \vdots \\ -\mathbf{V}_\infty\cdot\hat{\mathbf{n}}_{N-1} \\ -\mathbf{V}_\infty\cdot\hat{\mathbf{n}}_N \\ 0 \end{pmatrix}$$

$$K_{i,j} = \begin{cases} K_{i,1}^{(1)} & \text{if } j=1 \\ K_{i,j}^{(j-1)} + K_{i,j}^{(j)} & \text{if } 1<j<N+1 \\ K_{i,N+1}^{(N)} & \text{if } j=N+1 \end{cases}$$

RANKINE OVAL PROBLEM

$\vec{V}_\infty = V_\infty\hat{i}$, $\Lambda = \mp 4V_\infty\ell$, $x = \pm\ell$

a) along $x=0$, determine $(u-V_\infty)/V_\infty$

$u = V_\infty + \frac{\Lambda}{2\pi}\ \frac{x+\ell}{(x+\ell)^2+z^2} - \frac{\Lambda}{2\pi}\ \frac{x-\ell}{(x-\ell)^2+z^2}$

$\frac{u-V_\infty}{V_\infty} = \frac{2}{\pi}\ \frac{\frac{x}{\ell}+1}{(\frac{x}{\ell}+1)^2+(\frac{z}{\ell})^2} - \frac{2}{\pi}\ \frac{\frac{x}{\ell}-1}{(\frac{x}{\ell}-1)^2+(\frac{z}{\ell})^2}$

along $x=0$, $r = \sqrt{x^2+z^2} = |z|$

$\frac{u-V_\infty}{V_\infty} = \frac{4}{\pi}\ \frac{1}{1+(\frac{r}{\ell})^2}$

b) Along $z=0$, $r = \sqrt{x^2+z^2} = |x|$

$\frac{u-V_\infty}{V_\infty} = -\frac{4}{\pi}\ \frac{1}{(\frac{r}{\ell})^2-1}$

3D NONLIFTING FLOWS

Source

$\phi = -\frac{\lambda}{4\pi r}$, $u_r = \frac{\lambda}{4\pi r^2}$, $u_\theta = 0$, $u_\varphi = 0$

mass emitted at rate of $\rho\lambda$

at origin, $\nabla\cdot\vec{V} = \infty$. elsewhere, it is 0

Doublet

$\phi = \frac{\mu}{4\pi}\ \frac{\cos\theta}{r^2}$, $u_r = -\frac{\mu}{2\pi}\ \frac{\cos\theta}{r^3}$, $u_\theta = -\frac{\mu}{4\pi}\ \frac{\sin\theta}{r^3}$, $u_\varphi = 0$

Flow Over Nonlifting Sphere

$u_r = V_\infty\cos\theta - \frac{\mu}{2\pi}\ \frac{\cos\theta}{r^3}$

$u_\theta = -V_\infty\sin\theta - \frac{\mu}{4\pi}\ \frac{\sin\theta}{r^3}$

$u_\varphi = 0$

$u_r(R,\theta) = 0 \rightarrow \mu = 2\pi R^3 V_\infty$

$u_r = V_\infty\cos\theta\left(1-\frac{R^3}{r^3}\right)$

$u_\theta = -V_\infty\sin\theta\left(1+\frac{1}{2}\frac{R^3}{r^3}\right)$

$u_\varphi = 0$, at $r=R$

$u_r = 0$

$u_\theta = -\frac{3}{2}V_\infty\sin\theta$

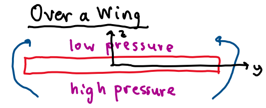
$u_\varphi = 0$

$V = \frac{3}{2}V_\infty|\sin\theta|$

$p(R,\theta) = p_\infty + \frac{1}{2}\rho V_\infty^2\left(1-\frac{9}{4}\sin^2\theta\right)$

$C_p(R,\theta) = \frac{p(R,\theta)-p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1-\frac{9}{4}\sin^2\theta$

Over a Wing

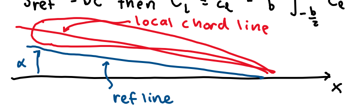


(front view)

$L'(y=\pm\frac{b}{2}) = 0$, $AR\uparrow$ works like 2D

$L = \int_{-\frac{b}{2}}^{\frac{b}{2}} L'(y)dy$, $c_\ell(y) = L'(y)/q_\infty c$

$S_{ref} = bc$ then $C_L = \bar{c}_\ell = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} c_\ell(y)dy$



$M_{bend} = \int_0^{\frac{b}{2}} y\,L'(y)dy$ $\quad$ places where $c_\ell(y)$ is high more likely to stall

$\frac{L'(y)}{q_\infty c_{ref}} = \frac{c_\ell(y)c(y)}{c_{ref}}$

TWO PT. SOURCE PROBLEM

$u = \frac{\Lambda}{2\pi}\ \frac{x-x_s}{(x-x_s)^2+(z-z_s)^2}$

$w = \frac{\Lambda}{2\pi}\ \frac{z-z_s}{(x-x_s)^2+(z-z_s)^2}$

$C_p = 1-\left(\frac{|V|}{V_\infty}\right)^2 = 1-\left(\frac{u}{V_\infty}\right)^2-\left(\frac{w}{V_\infty}\right)^2$

2D POINT VORTICES

$\vec{V}_{vortex} = -\frac{\Gamma}{2\pi r}\hat{e}_\theta$ $\quad$ $(0,1)$: Γ, $(0,-1)$: $-\Gamma$

$r_1 = \sqrt{x^2+1} = r_2$

$\hat{e}_{r_1} = \left(\frac{x}{r_1}, -\frac{1}{r_1}\right) = \frac{\vec{r}_1}{r_1} \leftarrow (x,-1)$

$\hat{e}_{\theta_1} = \left(\frac{1}{r_1}, \frac{x}{r_1}\right)$

$\hat{e}_{r_2} = \left(\frac{x}{r_2}, \frac{1}{r_2}\right)$

$\hat{e}_{\theta_2} = \left(-\frac{1}{r_2}, \frac{x}{r_2}\right)$

$\vec{V}(x,0) = V_\infty\hat{i} - \frac{\Gamma}{2\pi r_1}\left(\frac{1}{r_1}\hat{i}+\frac{x}{r_1}\hat{k}\right) + \frac{\Gamma}{2\pi r_2}\left(-\frac{1}{r_2}\hat{i}+\frac{x}{r_2}\hat{k}\right)$

$\vec{V}(x,0) = \left[V_\infty - \frac{\Gamma}{\pi(x^2+1)}\right]\hat{i}$

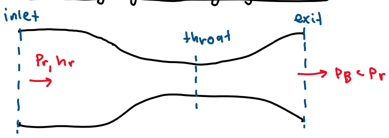
$\vec{V}\cdot\hat{n} = 0$, so

$V_\infty - \frac{\Gamma}{\pi(x^2+1)} = 0$ at $x = -0.8$

SHOCKWAVES

MORE SHOCKWAVES

Converging / Diverging Nozzle



Subsonic Case:

① $P_e = P_B$, isentropic relationship to find me

$$M_e^2 = \frac{2}{\gamma-1}\left[\left(\frac{P_r}{P_B}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

② use Me to determine A_e/A^*

③ Find M(x) everywhere

$$\frac{A(x)}{A^*} = \frac{A(x)}{A_e}\frac{A_e}{A^*}$$

④ Use M(x) to find p(x)

$$p(x) = p_r\left(1+\frac{\gamma-1}{2}(M(x))^2\right)^{-\frac{\gamma}{\gamma-1}}$$

Choked Flow:

$\dot m$ maximized when M=1 at throat (choked)

$$\dot m = \rho V A = \frac{\gamma p_0}{\sqrt{(\gamma-1)h_0}}\, M \left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A$$

$$\dot m_{choked} = \frac{\gamma p_r}{\sqrt{(\gamma-1)h_r}}\left(1+\frac{\gamma-1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A_t$$

$p_B < (p_B)_{choked}$ → supersonic ~~downstr~~

Choked Flow with Normal Shock:

- flow upstream doesn't change as P_B lowered
- supersonic: M↑ P↓
- normal shock occurs and then flow returns to subsonic, so $p_e = p_B$
- shock incurs total pressure loss: $P_0 < P_r$ after shock
- as P_B lowered, shock goes to exit
  $(P_B)_{exit\ shock}$ → M_1 just upstream the supersonic $A/A^* = A_e/A_t$.
  $P_B < (P_B)_{exit\ shock}$ → Mach diamonds

Determining Losses of Choked Flow:

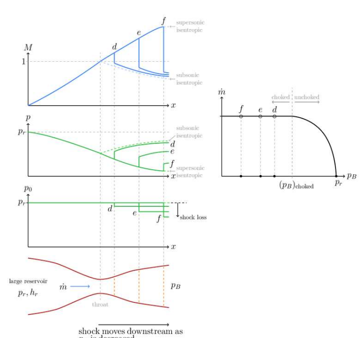
$$\dot m = \frac{\gamma p_e}{\sqrt{(\gamma-1)h_r}}\, M_e\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{1}{2}} A_e$$

$$M_e^2\left(1+\frac{\gamma-1}{2}M_e^2\right) = \left(\frac{P_r}{P_B}\frac{A_t}{A_e}\right)^2\left(1+\frac{\gamma-1}{2}\right)^{-\frac{\gamma+1}{\gamma-1}}$$

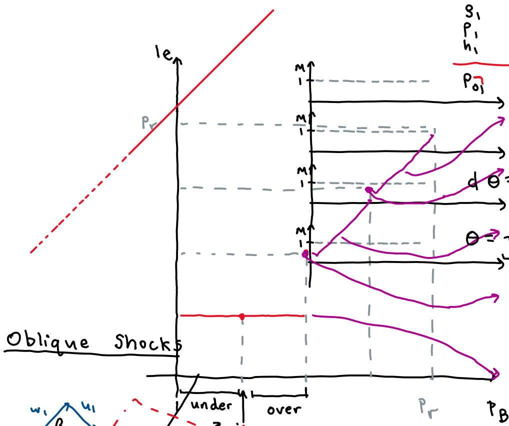
$$P_{0e} = P_e\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P_{0e}}{P_r} = \left(\frac{P_{02}}{P_{01}}\right)_{shock}, \quad \dot m = \frac{\gamma p_0}{\sqrt{(\gamma-1)h_0}}\left(1+\frac{\gamma-1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A^*$$

$$(P_0)_1 (A^*)_1 = (p_0)_2 (A^*)_2$$

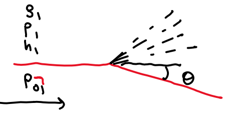


Supersonic Exit Flows

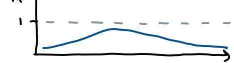


Expansion Waves

M_1
ρ_1
P_1
h_1
P_{01}



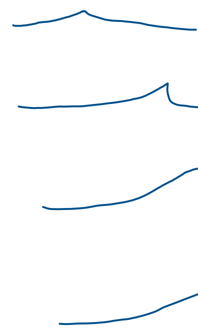
$M_2 > M_1$
$\rho_2 < \rho_1$
$P_2 < P_1$
$h_2 < h_1$
$P_{02} = P_{01}$



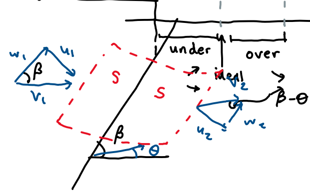
$$d\theta = \sqrt{M^2-1}\,\frac{dV}{V}$$

$$\theta = \nu(M_2) - \nu(M_1)$$

use Prandtl-Meyer



Oblique Shocks



$u_1 = V_1 \sin\beta \quad w_1 = V_1\cos\beta$

$\rho_1 u_1 = \rho_2 u_2$

$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$

$w_1 = w_2$

$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$

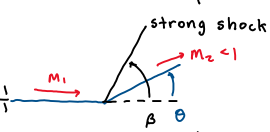
$P_2 = \frac{\gamma-1}{\gamma}\rho_2 h_2$

$$M_{n1} = \frac{u_1}{a_1} = \frac{V_1 \sin\beta}{a_1} = M_1 \sin\beta$$

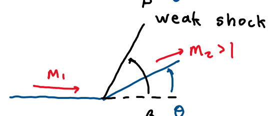
$$M_{n2} = \frac{u_2}{a_2} = \frac{V_2\sin(\beta-\theta)}{a_2} = M_2\sin(\beta-\theta)$$

$$M_{n2}^2 = \frac{1+\frac{\gamma-1}{2}M_{n1}^2}{\gamma M_{n1}^2 - \frac{\gamma-1}{2}}, \quad M_2 = \frac{M_{n2}}{\sin(\beta-\theta)}$$

$$\tan\theta = \frac{2}{\tan\beta}\,\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos 2\beta)+2}$$



bowed shock: $\theta > \theta_{max}$



Static Jump Relations

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{n1}^2}{2+(\gamma-1)M_{n1}^2}$$

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(M_{n1}^2 - 1)$$

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \left[1+\frac{2\gamma}{\gamma+1}(M_{n1}^2-1)\right]\frac{2+(\gamma-1)M_{n1}^2}{(\gamma+1)M_{n1}^2}$$

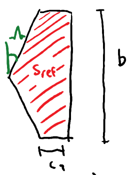
① Find angle β from upstream and deflection angle θ
② Find upstream normal mach M_{n1}
③ Calculate M_{n2} and M_2
④ Find ratios of static quantities using M_{n1} shock tables
⑤ Find downstream static conditions

UNIT 1

AERODYNAMIC FORCES

$\vec{A} = \iint(-p\hat{n}+\vec{\tau})\,dS$, $\vec{A}' = \int_S(-p\hat{n}+\vec{\tau})\,dS$

GEOMETRY



λ = taper ratio = $\frac{C_T}{C_R}$

Λ = sweep angle

AR = aspect ratio = $\frac{b^2}{S_{ref}}$

S_{ref} = ref, $\bar{c} = \frac{S_{plan}}{b}$

S_{ref} usually = S_{plan}

$\vec{D} = \vec{A}\cdot\frac{\vec{V}_\infty}{|\vec{V}_\infty|}$ $\qquad C_D = \frac{D}{q_\infty S_{ref}}$

$\qquad C_L = \frac{L}{q_\infty S_{ref}}$

$q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$

$D = D_{friction} + D_{induced} + D_{form} + D_{wave}$

$C_{Di} = \frac{C_L^2}{\pi AR e}$ (e = span efficiency)

$C_{Dp} = C_{Df} + C_{Dform}$

$C_{Dw} = D_{wave}/q_\infty S_{ref}$

$(CDA)_k = C_{Dother,k}\, A_{other\,k}$

STALLS, REYNOLDS #, SIM.

$L = mg = q_\infty S_{ref} C_L = \frac{1}{2}\rho_\infty V_\infty^2 S_{ref} C_L$

$V_\infty = \sqrt{\frac{2mg}{\rho_\infty S_{ref} C_L}}$ $\qquad$ min V_∞, max C_L

$\vec{\tau}_{wall} = \mu\frac{\partial u_s}{\partial n}\Big|_{wall}$ dynamic vis. coeff.

$Re_\infty^A = \frac{\rho_\infty V_\infty^A \ell_{ref}}{\mu_\infty}$ $\qquad M = \frac{V}{\sqrt{\gamma RT}} = \frac{V_1}{a_1}$

for C_L to be $= C_{LT}$, $M_\infty = M_T$, $a_1 = \alpha_2$

$Re_\infty = Re_T$

CONSERVATION OF MASS

$\iint_S \rho V\,dS = 0$ for closed surface S

practically, mass in = mass out

CONSERVATION MOMENTUM

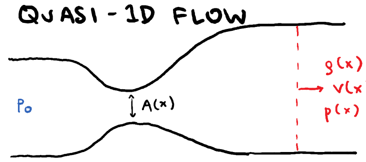
$$\frac{d}{dt}\iiint_V \rho\vec{V}\,dS + \underbrace{\iint_S \rho\vec{V}\,\vec{V}\cdot\hat{n}\,dS}_{\text{flow out}} = \underbrace{\iint_S(-p\hat{n}+\vec{\tau})\,dS}_{\text{aero forces}} + \underbrace{\iiint_V \rho\vec{g}\,dV}_{\text{momentum}} + \sum\vec{F}_{ext}$$

often easier to break into parts

y-mass : $\iint_S \rho v\,\vec{V}\cdot\hat{n}\,dS = \hat{y}\cdot\iint(-p\hat{n}+\vec{\tau})\,dS + \hat{y}\cdot\iiint_V \rho\vec{g}\,dV$

$\iint_S \rho_\infty\,\hat{n}\,dS = 0$

QUASI-1D FLOW



← location max camber

M P T T

↑ max camber $\qquad$ ↑ thickness

adiabatic + steady flow → h_t is const.

+ perfect → T_0 const.

+ reversible → P_0 const.

incompressible : ρ constant, VA constant

compressible : ρVA constant = $\dot{m}$

Incompressible

VA = constant

P_0 = total pressure = $P + \frac{1}{2}\rho v^2$ $\qquad T_0 = T + \frac{v^2}{2C_p}$

$P(x) = P_1 + \frac{1}{2}\rho v_1^2\left[1-\left(\frac{A_1}{A}\right)^2\right]$ $\qquad h_0 = h + \frac{v^2}{2}$

Compressible

ρVA = constant

$\rho = P/RT$

$V = Ma$

$a = \sqrt{\gamma RT}$

$T = T_0\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}$

$P = P_0\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma}{\gamma-1}}$

$\dot{m} = \rho VA$ $\quad P_0\sqrt{\frac{\gamma}{RT_0}}\,M\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}A$

$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1+\frac{\gamma-1}{2}M^2}{(\gamma+1)/2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$, A^* = area where the Mach number is 1 (adiabatic, reversible, etc.)

$M_e = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_0}{P_e}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}$

if P_0, T_0 constant then A^* const

if subsonic, $P_e = P_B$ $\qquad$ if we have shaft work, $P_0/\sqrt{T_0}$ will change

$F_{ext} = \rho_e u_e^2 A_e = \gamma p_\infty M_e^2 A_e$

CONSERVATION OF ENERGY

$$\underbrace{\iiint\frac{\partial}{\partial t}(\rho e_0)\,dV}_{\text{steady flow, 0}} + \underbrace{\iint_S \rho h_0\vec{V}\cdot\hat{n}\,dS}_{\text{enthalpy flux}} = \underbrace{\iint\vec{\tau}\cdot\vec{V}\,dS}_{\text{viscous work}} - \underbrace{\iint\vec{q}\cdot\hat{n}\,dS}_{\text{h t}} + \iiint\rho\vec{g}\cdot\vec{V}\,dV + \sum\vec{F}_{ext}\cdot\vec{V}_{ext}$$