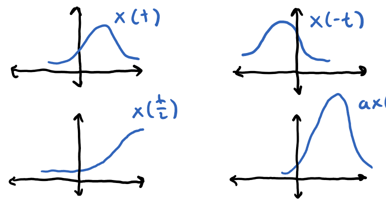
EXAM 1 CHEAT SHEET

Transformations and Characteristics



$x(-t) = x(t)$ even

$x(-t) = -x(t)$ odd

$x(t+T) = x(t)$

Exponential Signals

$x(t) = Ce^{at}$ (if C and a are real, not periodic)

$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$ $\quad T_0 = \frac{2\pi}{\omega_0}$

$x(t) = A\cos(\omega t + \phi) = A \cdot Re[e^{j(\omega t+\phi)}]$

$A\sin(\omega t + \phi) = A \cdot Im[e^{j(\omega t+\phi)}]$

$x(t) = Ce^{at}; \quad C = |C|e^{j\theta}, \; a = \sigma + j\omega_0$

$= |C|e^{j\theta} e^{(\sigma + j\omega_0)t}$

$= |C|e^{\sigma t} e^{j(\omega_0 t + \theta)}$

$= |C|\underbrace{e^{\sigma t}}_{\text{envelope function}}[\cos(\omega_0 t+\theta) + j\sin(\omega_0 t + \theta)]$

$x[n] = C\alpha^n$ if $|\alpha| > 1$, exp. growth; < 1, exponential decay

$C = |C|e^{j\theta}, \; \alpha = |\alpha|e^{j\omega_0}$

$x[n] = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$

Useful Signals

1. $x(t) = c$ (static signal)
2. $\sigma(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$ (unit step)
3. $x(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} = t\sigma(t)$
4. $\delta_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & 0 \leq t \leq \epsilon \\ 0, & \text{otherwise} \end{cases}$

$\sigma(t) = \int_{-\infty}^{\infty} \delta(\tau)\, d\tau$

Power/Energy

$P(t) = |x(t)|^2$

$P_{avg} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 \, dt$

$E_{Avg} = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt$

Convolution

$u(t) \longrightarrow \boxed{G(t)} \longrightarrow y(t) = g(t) * u(t)$

$\int_{-\infty}^{\infty} \underbrace{g(t-\tau)}_{\text{impulse response}} \underbrace{u(\tau)}_{\text{input}} d\tau = (g * u)(t)$

$\sum_{k=-\infty}^{\infty} u[k]\, g[n-k] = y[n] = u * g$

1. $f(t)\,\delta(t) = f(0)\,\delta(t)$ ("sifting")
   $f(t) * \delta(t) = f(t) = \delta(t) * f(t)$
2. $f(t) * g(t) = g(t) * f(t)$
3. $f * (g + h) = f*g + f*h$
4. $f * (g * h) = (f * g) * h$

$y[n] = g[0]\,u[n]$

$b = 2\sqrt{ac} \rightarrow$ critically damped $\rightarrow$ decay rate: $-\sqrt{\frac{c}{a}}$

LaPlace Transform

$e^{st} \longrightarrow \boxed{G} \longrightarrow G(s)e^{st}$

$G(s) = \int_{-\infty}^{\infty} e^{-s\tau} g(\tau)\, d\tau = \mathcal{L}[g(t)]$

if g causal,

$G(s) = \int_0^{\infty} e^{-s\tau} g(\tau)\, d\tau$

$G(s) = \frac{N(s) \leftarrow \text{zeros}}{D(s) \leftarrow \text{poles}}$, $\quad \underset{\text{output}}{Y(s)} = \underset{\text{transfer}}{G(s)}\underset{\text{input}}{U(s)}$

Properties of Laplace

1. $g(t) = \delta(t)$ $\quad e^{st} \rightarrow \boxed{G} \rightarrow e^{st}$
   $G(s) = 1$
2. $g(t) = \sigma(t)$ $\quad e^{st} \rightarrow \boxed{G} \rightarrow \frac{1}{s}e^{st}$
   $G(s) = \frac{1}{s}$
3. $g(t) = e^{at}\sigma(t)$
   $G(s) = \frac{1}{s-a}$ $\quad e^{st} \rightarrow \boxed{G} \rightarrow \frac{1}{s-a}e^{at}$ (?)
   $(Re[s] > Re[a])$
4. $\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha F(s) + \beta G(s)$
5. $\mathcal{L}[f * g] = F(s) * G(s)$
6. $\mathcal{L}[\dot{g}(t)] = sG(s) - g(0^-)$
7. $f(t) = \int_0^t g(\tau)\, d\tau$
   $\mathcal{L}[F(t)] = \frac{1}{s}\mathcal{L}[g(t)]$
8. $\mathcal{L}[e^{at}g(t)] = G(s-a)$
9. $\mathcal{L}[g(t-t_0)] = e^{-st_0}G(s)$
10. $\mathcal{L}[t g(t)] = -dG(s)/ds$

First Order Systems

$T\dot{x} - x = 0 \qquad \dot{x} = \frac{1}{T}x$

$x(t) = e^{t/T}x(0)$

$\dot{x} = bx - dx$

$x(t) = e^{(b-d)t}x(0)$

$c\dot{x} = -ax$

$x(t) = e^{-at}x(0)$

Inverse Laplace Transforms

$G(s) = \frac{a_m s^m + a_{m-1}s^{m-1} + \dots + a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + \dots + b_1 s + b_0}$

$= \frac{N(s)}{D(s)}$

1. $n > m$, $G(s)$ strictly proper
   $D(s) = (s-p_1)(s-p_2)\dots(s-p_n)$
   $G(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \dots + \frac{c_n}{s-p_n}$
   $c_i = \lim_{s\to p_i}(s-p_i)G(s)$
2. $m = n$, no repeated poles
   $G(s) = C_0 + \frac{C_1}{s-p_1} + \dots + \frac{C_n}{s-p_n}$
   $C_0 = \lim_{s\to\infty} G(s)$
3. $m > n$, no repeated poles
   a) Polynomial long division to find higher order coefficients
   b) heaviside

   $G(s) = \frac{2s^2 + 5s + 5}{s+1} = C_0 s + C_1 + \frac{C_2}{s+1}$

   $$\begin{array}{r} 2s + 3 + \frac{2}{s+1} \\ s+1 \,\big)\, 2s^2 + 5s + 5 \\ -2s^2 - 2s \\ \hline 3s + 5 \\ -3s + 3 \\ \hline 2 \end{array}$$

   $G(s) = 2s + 3 + \frac{2}{s+1}$
   $= 2\dot{\delta}(t) + 3\delta(t) + 2e^{-t}\sigma(t)$
4. repeated poles
   can't use coverup method
   $c_j = \frac{1}{(j-1)!}\lim_{s\to p_i}\left[\frac{d^{r-1}}{ds^{r-1}}\left[(s-p_i)^r G(s)\right]\right]$

System Properties

1. Linearity (superposition)
2. Time Invariant $\left(\begin{array}{l} u(t) \rightarrow y(t) \\ u(t-t_0) \rightarrow y(t-t_0) \end{array}\right)$
3. Causal ($u(t-20)$ ✓, $u(t+5)$ ✗)

Second Order Systems

$a\ddot{x} + b\dot{x} + cx = 0$

$a[s^2X(s) - sx(0) - \dot{x}(0)] + b[sX(s) - x(0)] + cX(s) =$

$X(s) = \frac{asx(0) + a\dot{x}(0) + bx(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$

$x(t) = \mathcal{L}^{-1}[X(s)]$

$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

1. $b^2 - 4ac > 0$
   $X(s) = \frac{\alpha s + \beta}{as^2 + bs + c} = \frac{c_1}{s - \lambda_1} + \frac{c_2}{s - \lambda_2}$
   $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$, $\quad c_1 = \frac{\lambda_1 \alpha + \beta}{\sqrt{b^2 - 4ac}}$
   overdamped $\quad c_2 = \frac{-\lambda_2\alpha - \beta}{\sqrt{b^2 - 4ac}}$
2. $b^2 - 4ac = 0$
   $X(s) = \frac{\alpha s + \beta}{a(s-\lambda)^2} = \frac{c_1}{s-\lambda} + \frac{c_2}{(s-\lambda)^2}$
   $x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$ $\quad \lambda = \frac{-b}{a}$
   critically damped $\quad c_1 = \alpha/a$
   $c_2 = \frac{\beta + \lambda\alpha}{a}$
3. $b^2 < 4ac$
   $X(s) = \frac{\alpha s + \beta}{a(s-\lambda)(s-\lambda^*)} = \frac{c_1}{s-\lambda} + \frac{c_1^*}{s-\lambda^*}$
   $\lambda = \frac{-b + j\sqrt{4ac - b^2}}{2a} = \sigma + j\omega$
   $x(t) = c_1 e^{\lambda t} + c_1^* e^{\lambda^* t}$
   $c_1 = \frac{\alpha}{a} + j\frac{ab - 2a\beta}{4a^2\omega}$
   $x(t) = \underbrace{Ae^{\sigma t}}_{\text{envelope}}\cos(\omega t + \phi)$, $A = 2|c_1|$
   $\phi = \tan^{-1}\left(\frac{Im[c_1]}{Re[c_1]}\right)$

EXAM 1 CHEAT SHEET

BIBO Stability

$$\int_0^\infty |g(t)|\,dt = M < \infty$$

- if and only if ROC of $G(s)$ contains $j\omega$-axis

Fourier Series

If $x(t)$ periodic, then:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{-j(\frac{2\pi}{T_0})t}$$

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(\frac{2\pi}{T_0})kt}$$

real and even? c_k's also real and even

real and odd? c_k's imaginary

Parceval's Theorem

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)^2\,dt = \int_{-\infty}^{\infty} |Y(f)|^2 df$$

Fourier Transform

$$\mathcal{F}[\Pi(t)] = \int_{-\infty}^{\infty} \Pi(t) e^{-j\omega t} dt$$

Properties of Fourier

① Unit Impulse

$\mathcal{F}[\delta(t)] = 1$

② One-sided exp. decay

$$x(t) = \begin{cases} 0, & t < 0 \\ e^{-at}, & t \geq 0 \end{cases}$$

$\mathcal{F}[x(t)] = \frac{1}{a+2\pi j f}$

③ Linearity

$\mathcal{F}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{F}(t) + \beta\, \mathcal{F}[g(t)]$

④ Time-Shifted

$x(t - t_0) \Leftrightarrow e^{-2\pi j f t_0} X(f)$

⑤ Time-Scaled

$x(at) = \frac{1}{|a|} X\left(\frac{f}{a}\right)$

⑦ Duality

$x(t) \Leftrightarrow X(f)$

$X(f) \Leftrightarrow x(-t)$

⑧ Conjugate Duality

$x^*(t) \Leftrightarrow X^*(-f)$

⑨ Odd/Even

same as FS

⑩ Derivative

$\dot{x}(t) \Leftrightarrow 2\pi j f X(f)$

⑪ Integral

$$\int_{-\infty}^{t} x(\tau)d\tau \Leftrightarrow \frac{X(f)}{2\pi j f} + \frac{X(0)\delta(f)}{2}$$

⑫ Of periodic signals

???

⑬ Convolution

$\mathcal{F}[g * h] = G(f)H(f)$

⑭ Multiplication

$g(t) * h(t) \Leftrightarrow G(f)H(f)$

$G(t) * H(t) \Leftrightarrow g(-f)^* h(-f)$

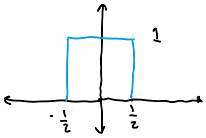
$g(t)\,h(t) \Leftrightarrow G(f) * H(f)$

Euler

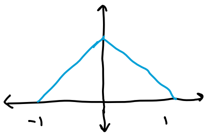
$e^{j\theta} = \cos\theta + j\sin\theta$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$



$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Known Fourier

$\cos(2\pi f_0 t)$

$c_1 = c_{-1} = \frac{1}{2}$, $c_k = 0$

$\mathcal{F}[\Lambda(t)] = \mathrm{sinc}^2(f)$

$\mathcal{F}[\cos(2\pi f_0 t)] = \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$

Routh Stability

If one of coefficients of $D(s)$ in $G(s) = \frac{N(s)}{D(s)}$ is not postive (≤ 0), then system not stable.

$$\begin{array}{llll}
s^n & 1 & a_2 & a_4 \cdots \\
s^{n-1} & a_1 & a_3 & a_5 \cdots \\
s^{n-2} & b_1 = \frac{a_1 a_2 - a_3}{a_1} & b_2 = \frac{a_1 a_4 - a_5}{a_1} & b_3 = \frac{a_1 a_6 - a_7}{a_1} \\
s^{n-3} & c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} & \frac{b_1 a_5 - a_1 b_3}{b_1} &
\end{array}$$

all elements of first column of Routh array positive

① If first el 0, replace row with small ϵ, take limit as $\epsilon \to 0$

② If all zero, poly with elements of previous row

$d(s) = \beta_1 s^{i+1} + \beta_2 s^{i-1} + \beta_3 s^{i-3}$

roots must be checked separately

(complex conjugate pairs of roots, mirrored across imaginary axis)

$$\mathrm{sinc}(2k) = \frac{\sin(2\pi k)}{2\pi k}$$

$$\mathrm{sinc}(k) = \frac{\sin(\pi k)}{\pi k}$$

$\omega_0 T_s < \pi$

$\frac{\omega_0}{\pi} < \frac{1}{T_s}$

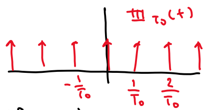
$\frac{1}{\pi} = f_0$

UNIT 2 CHEAT SHEET

SAMPLING

Impulse Train:

$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$



Fourier Series:

$\sum_{k=-\infty} \delta(t - kT_0) \Leftrightarrow f_0 \sum_{k=-\infty}^{\infty} \delta(f - kf_0)$

$Ш_p(t) \Leftrightarrow \frac{1}{p} Ш_{\frac{1}{p}}(f)$

$Ш_{T_0}(t) \Leftrightarrow f_0 Ш_{f_0}(f)$

multiply by impulse train → sampling

$x(t) Ш_{T_s}(t) = \sum x(kT_s)\delta(t - kT_s)$

convolve w/ impulse train → periodize

$x(t) * Ш_p(t) = \sum_{k=-\infty}^{\infty} x(t - kp)$

$X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(f - \frac{k}{T_s})$

Recover original signal?

- must be sampled at $2W$

$H(f) = T_s \Pi(\frac{f}{2W})$

$\frac{1}{T_s} = 2W, \quad \tilde{x}(t) = \sum x(kT_s) \text{sinc}(\frac{t}{T_s} - k)$

DISCRETE TIME

$\sigma[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

$x[n] = c$

$x[n] = z^n = c\alpha^n$

$\left[G(z) = \sum_{k=0}^{\infty} g[k] z^{-k} \right]$

Convolution

$u[n] = \sum u[k]\delta[n-k]$

$y[n] = \sum u[k] g[n-k]$

$y[n] = u[n] * g[n]$

Z-TRANSFORM

transfer function DT LTI: z-Transform of impulse

$X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$

DON'T FORGET TO DIVIDE BY Z LCCODE

Known Z-Transforms

$\mathcal{Z}[\delta[n]] = 1$

$\mathcal{Z}[\sigma[n]] = \frac{z}{z-1}, \quad |z| > 1$

$\mathcal{Z}[\alpha^n \sigma[n]] = \frac{z}{z-\alpha}, \quad |z| > |\alpha|$

$z^{-1}\left[\frac{1}{(z-\alpha)^2}\right] = \alpha^k k\sigma[k]$

ROC unit circle if BIBO stable

Inverse Z-Transform

① Factorize denominator

② Compare to known transforms

Properties

① Linearity: $\mathcal{Z}[\alpha f[n] + \beta g[n]] = \alpha F(z) + \beta G(z)$

② $Z[x[n - n_0]] = z^{-n_0} X(z)$

③ $Z^+[x[n] - x[n-1]] = (1 - z^{-1}) X(z) - x[-1]$

④ $f[n] = \sum_{k=0}^{n} g(k)$

$Z[f[n]] = \frac{z}{z-1} G(z)$

⑤ $Z[a^n g[n]] = G(z/a)$

⑥ $Z[g[n] * u[n]] = G(z) U(z)$

⑦ $Z[n g[n]] = -z \frac{dG(z)}{dz}$

LCCODE Sol'n

$y[k] - 3y[k-1] + 2y[k-2] = 2u[k-1] - 2u[k-2]$

$u[k] = \begin{cases} k, & k \geq 0 \\ 0, & k < 0 \end{cases} \quad k\sigma[k]$

① Take z-transforms:

$Y(z) - 3z^{-1}Y(z) + 2z^{-2}Y(z) = 2z^{-1}U(z) - 2z^{-2}U(z)$

$U(z) = -z \frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$

INFORMATION

associated w/ event w/ probability?

$I = \log(\frac{1}{p}) = -\log(P)$

ENTROPY

N mutually exclusive, each w prob p_i

expected info: entropy $= -\sum_{i=1}^{N} p_i \log(p_i)$

two mutually exclusive events w/ prob p and $1-p$:

$H(p) = -p\log(p) - (1-p)\log(1-p)$

CODE

ideal → no more bits than expected info

instantaneous → decodable as soon as codeword received

$L = \sum_{i=1}^{N} p_i l_i$ (expected length)

Huffman:

① Set S of all pairs of symbols and probabilities

② Two pairs, lowest prob as the nodes of the tree

③ Repeat until only one.

④ Every split: branch 0, other 1

STATE SPACE

state variables: variables that make up state of system

$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t)$ $\quad A: n \times n \quad C: m \times n$

$y(t) = C\bar{x}(t) + D\bar{u}(t)$ $\quad B: n \times r \quad D: m \times r$

number of state variables = "order" (number of integrat

Linearization

$\dot{\bar{x}}: A\bar{x} + B\bar{u}$

$\bar{y}: C\bar{x} + D\bar{u}$

equilibrium pt.

$\dot{\bar{x}} = \bar{0}$

$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial u_1} & \cdots & & \frac{\partial f_n}{\partial u_n} \end{bmatrix}$

Transfer Func.

$\dot{\bar{x}}(t) = A\bar{x}(t) + Bu(t)$ $\quad$ (SISO) $A: n \times n \quad C: 1 \times n$

$y(t) = C\bar{x}(t) + Du(t)$ $\quad B: n \times 1 \quad D: 1 \times 1$

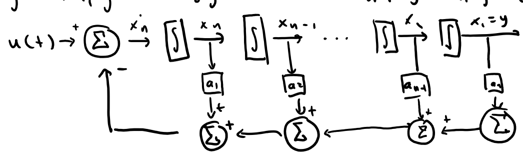
$\bar{X}(s) = (sI - A^{-1}) B U(s)$

$G(s) = C(sI - A)^{-1}B + D$

poles roots of $\det(sI - A) = 0$ (eigenvalues)

Block Diagrams

$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1}\dot{y} + a_n y = u$



NATURAL RESPONSE

$\dot{\bar{x}} = A\bar{x}$

General solution: $\bar{x}(t) = \sum_{i=1}^{n} a_i \bar{v}_i e^{\lambda_i t}$

λ_i = eigenvalues

$\bar{v}_i$ = eigenvectors

a_i = initial conditions

$\bar{x}(t) = A\Phi(t)\bar{x}(0)$

$\Phi(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]$

Properties

① $\Phi(0) = 1$

② $\Phi^{-1}(t) = \Phi(-t)$

③ $\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0)$

④ $[\Phi(t)]^k = \Phi(kt)$

Forced

FORCED RESPONSE

$\dot{\bar{x}} = A\bar{x} + B\bar{u}$ $\quad \bar{x} = n \times 1, \ A = n \times n,$

zero init. conds $\quad B = n \times r, \ u = r \times 1$

COMPLETE SOL'N

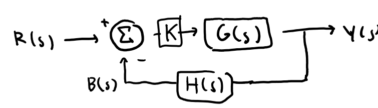
$s\bar{X}(s) - \bar{x}(0) = A\bar{X}(s) + BU(s)$

$\bar{x}(t) = \Phi(t)\bar{x}(0) + \int_0^t \Phi(t - \tau) B\bar{u}(\tau) d\tau$

$\bar{y}(t) = C(\bar{x}(t)) + D\bar{u}$

the step response is the integral of the impulse response

BLOCK DIAGRAMS



$\frac{Y(s)}{R(s)} = \frac{K\,G(s)}{1 + K\,G(s)\,H(s)}$

② Combine, partial frac expansion