# Unit 2 Review

Thursday, December 13, 2018 11:53 AM

Delia Stephens

This Unit Contains:
1. Buckling Theory ✓
2. Torsion ✓
3. Beam Theory ✓
4. Rod Theory ✓
5. Elasticity Theory ✓
6. Strain ✓
7. Stress ✓

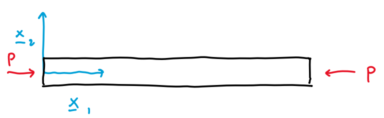
## BUCKLING

- arises when we think about equilibrium in the deformed condition (before, focused on the fact the equilibrium in undeformed held in deformed)

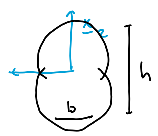
- compressive loads don't drive deformation; rather, they decrease the stiffness of the system
  $P = P_{CR} \rightarrow$ system has no stiffness (equilibrium for any lateral load)

there's a lot of stuff abt. this single-degree-of freedom example, but IDK how relevant this is.

### COLUMNS




General symmetric cross-section

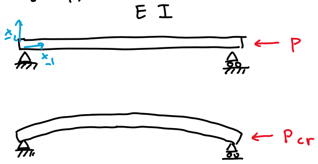



(estimated by bh)

$L >> b, h$ (long and slender)
EI is constant

- only compressive loads along long direction
- shortening $\bar{u}_1$ and buckling $\bar{u}_2$

$$\bar{u}_2''(x_1) = \frac{M_3(x_1)}{EI}$$




### EQUATIONS

Equilibrium

$$\frac{dN}{dx_1} = 0 \qquad \frac{dS_2}{dx_1} = -p_2 \qquad \frac{dM_3}{dx_1} + S_2 - \underbrace{\boxed{N_1 \frac{d\bar{u}_2}{dx_1}}}_{\text{new}} = 0$$

the compressive load; shear; distributed load

Compatibility + Constitutive

$$M_3 = EI \frac{d^2\bar{u}_2}{dx_1^2}$$

### $4^{th}$-Order Summary

$$EI \frac{d^4 \bar{u}_2}{dx_1^4} - \underbrace{N_1 \frac{d^2 \bar{u}_2}{dx_1^2}}_{\text{"moderately large" assumption}} = p_2$$

### The Questions we Need to Ask

- what is the value of $P_{cr}$ (at what load does buckling occur?)
- What shape does the buckled column assume?
- Will buckling always occur provided compressive load large enough?

## SOLUTION METHOD (HOMOGENEOUS)

1. $N_1 = -P$, assume $p_2(x_1) = 0$

$$EI\, u_2^{iv}(x_1) + P u''(x_1) = 0$$

2. Hey! This is a differential equation!

$$\lambda^4 + \frac{P}{EI}\lambda^2 = 0$$

3. $\bar{u}_2(x_1) = A \sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) + B\cos\left(\sqrt{\frac{P}{EI}}\, x_1\right) + C + Dx_1$

## BOUNDARY CONDITIONS

(Homogeneous, same as for beam-bending)

| | |
|---|---|
| | $\bar{u}_2 = 0$, $\bar{u}_2' = 0$ |
| | $\bar{u}_2 = 0$, $M_3 = 0$ |
| | $\bar{u}_2' = 0$, $S_3 = 0$ |
| — | $M_3 = 0$, $S_3 = 0$ |

$S_3 = -EI\bar{u}_2'''$

$M_3 = EI\bar{u}_2''$

## GENERAL SOL'N (WEIRD BOUNDARIES)

1. Start with the general solution to the fourth-order ODE.

$$\bar{u}_2(x_1) = A\sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) + B\cos\left(\sqrt{\frac{P}{EI}}\, x_1\right) + C + Dx_1$$

2. Apply the boundary conditions (two at each end)

$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0\end{bmatrix}$$

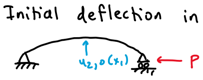
3. To have a non-trivial solution, the determinant of M must be 0. Set $\det(M) = 0$ and solve for roots $P_{cr}$ ($\lambda$) of eq'n.

4. Homogeneous boundary conditions:

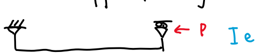
$$P_{cr} = \frac{c\pi^2 EI}{L^2}$$ c: coefficient of edge fixity (?)

## INITIAL IMPERFECTIONS

1. Initial deflection in column.




2. Load not applied along the centerline of the column.




### Solution Approach

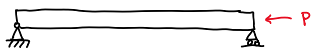
1. $\frac{d^4\bar{u}_2}{dx_1^4} + \frac{P}{EI}\frac{d^2\bar{u}_2}{dx_1^2} = 0$

2. General sol'n from before:

$$\bar{u}_2(x_1) = A\sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) + B\cos\left(\sqrt{\frac{P}{EI}}\, x_1\right) + C + Dx_1$$

3. Boundary conditions for type 2 (to solve A, B, C, D)

## EXAMPLE 1 (SIMPLY SUPPORTED)




$u_2(0) = 0 \qquad M(0) = 0$

$u_2(L) = 0 \qquad M(L) = 0$

$u_2(0) = A\sin(0) + B\cos(0) + C + D(0) = 0$

$B + C = 0$

$u_2(L) = A\sin\left(\sqrt{\frac{P}{EI}}L\right) + \cancel{B\cos\left(\sqrt{\frac{P}{EI}}L\right) + C} + DL = 0$

$u_2''(x_1) = -A\left(\frac{P}{EI}\right)\sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) - B\frac{P}{EI}\cos\left(\sqrt{\frac{P}{EI}}\, x_1\right)$

$M(x_1) = -AP\sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) - BP\cos\left(\sqrt{P/EI}\, x_1\right)$

$M(0) = 0 = -B\cancel{P}$

$B = 0,\ C = 0$

$M(L) = -A\cancel{P}\sin\left(\sqrt{\frac{P}{EI}}L\right) = \emptyset$

$A\sin\left(\sqrt{P/EI}\, L\right) = 0, \quad A\sin\left(\sqrt{\frac{P}{EI}}L\right) + DL = 0$

$\downarrow$

$A\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0 \qquad D = 0$

$A = 0$ (a stupid result) OR

$\sin\left(\sqrt{\frac{P}{EI}}\, L\right) = 0$

$\sqrt{\frac{P}{EI}}\, L = n\pi \quad n \in \mathbb{Z}$

Solve for P to see when buckling occurs.

$$\left(\sqrt{\frac{P}{EI}}\, L\right)^2 = (n\pi)^2$$

$$\frac{PL^2}{EI} = n^2\pi^2$$

$$P = \frac{n^2\pi^2 EI}{L^2}$$

plug buckling P into the deflection formula to see what it looks like.

$$\bar{u}_2(x_1) = A\sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) = A\sin\left(\frac{n\pi x_1}{L}\right)$$

- the lowest buckling load is the one for which $n = 1$

$$\bar{u}_2(x_1) = A\sin\left(\sqrt{\frac{P}{EI}}\,x_1\right) + B\cos\left(\sqrt{\frac{P}{EI}}\,x_1\right) + C + Dx_1$$

③ Boundary conditions for type 2 (to solve A, B, C, D)

$u_2(0) = 0 \qquad u_2(L) = 0$

$M_3(0) = Pe \qquad M_3(L) = 0$

$u_2(0) = 0 = 0A + B + C + 0D = 0$

$u_2''(0) = \frac{Pe}{EI} \rightarrow B = -e,\; C = e$

$u_2(L) = 0 \quad A\sin\left(\sqrt{\frac{P}{EI}}\,L\right) + DL = -e\left[1 - \cos\left(\sqrt{\frac{P}{EI}}\,L\right)\right]$

$u_2''(L) = \frac{Pe}{EI} \quad A\sin\left(\sqrt{\frac{P}{EI}}\,L\right) = -e\left[1 - \cos\left(\sqrt{\frac{P}{EI}}\,L\right)\right]$

$$A = -e\,\frac{1 - \cos\left(\sqrt{\frac{P}{EI}}\,L\right)}{\sin\left(\sqrt{\frac{P}{EI}}\,L\right)}, \quad D = 0$$

④ $$\bar{u}_2(x_1) = -e\left[\frac{1-\cos\left(\sqrt{\frac{P}{EI}}\,L\right)}{\sin\left(\sqrt{\frac{P}{EI}}\,L\right)}\sin\left(\sqrt{\frac{P}{EI}}\,x_1\right) + \cos\left(\sqrt{\frac{P}{EI}}\,x_1\right) - 1\right]$$

as long as $P \rightarrow \frac{n^2\pi^2 EI}{L^2}$, then we're all good. $\quad \sin\left(\sqrt{\frac{P}{EI}}\,L\right) =$

$\sqrt{\frac{P}{EI}}\,L = n\pi$

$\frac{P}{EI}L^2 = n^2\pi^2$

$P = \frac{n^2\pi^2 EI}{L^2}$ (coincidentally, $P_{cr}$)

## FAILURE OF COLUMNS

In the "perfect" case, a column will fail if it buckles

$\bar{u}_2 \rightarrow \infty$, so $M_3 \rightarrow \infty$, so $\sigma_{11} \rightarrow \infty$

Long and Slender

$P_{cr} = \frac{c\pi^2 EI}{L^2}$

$\sigma_{11} = -P/A$, so $\sigma_{cr} = \frac{c\pi^2 EI}{L^2 A}$ where did the negative go?

Short Columns

reach "ultimate compressive stress" ⟶ "failure by squishing"

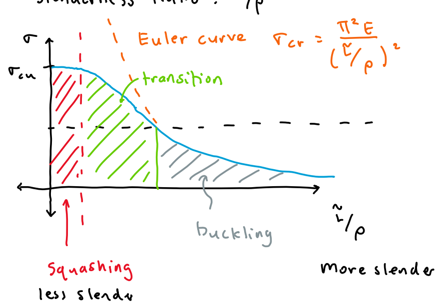
$\sigma_{11} = \frac{-P}{A} = -\sigma_{cu}$

Effective Length, $\tilde{L} = L/\sqrt{c}$

Radius of gyration, $\rho = \sqrt{I/A}$

$\sigma_{cr} = \frac{c\pi^2 EI}{L^2 A} \Rightarrow \sigma_{cr} = \frac{c\pi^2 E}{(\tilde{L}/\rho)^2}$

Slenderness Ratio : $\tilde{L}/\rho$




$\tilde{L}/\rho$ big: fail by buckling

$\tilde{L}/\rho$ small: squish

transition region: plastic deformation

$\sigma_{cy} < |\sigma_{11}| < \sigma_{cu}$

$\sigma_{cy}$ = compressive yield stress

$\sigma_{cu}$ = ultimate yield stress

# TORSION (CIRCULAR SHAFTS)

- only external loads concentrated T or distributed torques $t_3$

Assumptions

① [illegible]

- only external loads concentrated $T$ or distributed torques $t_3$

Assumptions
1. Planar cross-sections $x_3$ = constant
2. Angle of twist $(\phi(x_3))$ function of $x_3$
3. No other deformation allowed.

$$u_1(x_1, x_2, x_3) = -\phi(x_3)\, x_2$$
$$u_2(x_1, x_2, x_3) = \phi(x_3)\, x_1$$
$$u_3(x_1, x_2, x_3) = 0$$

Strains

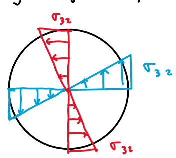
$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$
$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$
$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0$$
$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\Big(-\phi(x_3) + \phi(x_3)\Big) = 0$$
$$\epsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = -\frac{1}{2}\left(\frac{d\phi(x_3)}{dx_3} x_2 + 0\right) = \boxed{-\frac{1}{2} x_2 \frac{d\phi(x_3)}{dx_3}}$$
$$\epsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(\frac{d\phi(x_3)}{dx_3} x_1 + 0\right) = \frac{1}{2} x_1 \frac{d\phi(x_3)}{dx_3}$$

Stresses

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = 0 \Rightarrow \sigma_{11} = \sigma_{22} = \sigma_{33} = 0$$
$$\sigma_{12} = 2G\epsilon_{12} = 0$$
$$\sigma_{13} = 2G\epsilon_{13} = -\frac{2}{2} G x_2 \frac{d\phi(x_3)}{dx_3}$$
$$\sigma_{23} = 2G\epsilon_{23} = \frac{2}{2} G x_1 \frac{d\phi(x_3)}{dx_3}$$

only stresses shear stresses acting on plane and pointing tangentially

linear radially




Because only stresses in $\sigma_{31}, \sigma_{23}$ → only contributes to shears in $x_1, x_2$ and moment in $x_3$

$$S_1(x_1) = \int_A \sigma_{31}(x_1, x_2, x_3)\, dA$$
$$S_2(x_3) = \int_A \sigma_{23}(x_1, x_2, x_3)\, dA$$
$$M_3(x_1) = \int_A \left(x_1 \sigma_{23}(x_1, x_2, x_3) - x_2 \sigma_{31}(x_1, x_2, x_3)\right) dA$$

$$\boxed{\begin{aligned} T(x_3) &= M_3(x_3) = GJ\phi'(x_3) \\ J &= \int_A r^2\, dA = \frac{\pi R^4}{2} \end{aligned}}$$

GJ = torsional stiffness (depends on material and geometry)

Equilibrium

$$T'(x_3) + t(x_3) = 0$$

Compatibility

$$T(x_3) = GJ\phi'(x_3)$$

$$(GJ(\phi'(x_3)))'(x_3) + t(x_3) = 0$$

$$\boxed{\begin{aligned} \sigma_{13} &= \frac{-Tx_2}{J} \\ \sigma_{23} &= \frac{-Tx_1}{J} \end{aligned}}$$

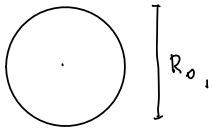
statically determinate if we know a value of $T$ somewhere in the bar. Use the equilibrium equation and THEN sub into CC.
Statically indeterminate → use two kinematic boundary conditions.

$\sigma_{3\theta}$ shear stress: $\tau$

$$\tau = \sqrt{\sigma_{13}^2 + \sigma_{23}^2}$$

$$\boxed{\tau = \frac{Tr}{J}}$$

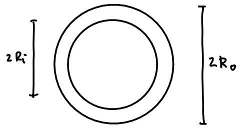
CALCULATING J



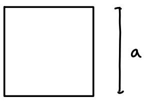

$J = \frac{\pi R_0^4}{2}$

$\int_0^{2\pi}\int_0^R r^2\, r\,dr\,d\theta$

$\frac{1}{4}R^4 \cdot 2\pi = \frac{\pi R^4}{2}$




$J = \frac{\pi R_o^4}{2} - \frac{\pi R_i^4}{2}$




$J = .141\, a^4$

# BEAM THEORY

goal: compute internal resultant forces and moments, stresses and deformations of beam subjected to distributed load.

## DERIVATION

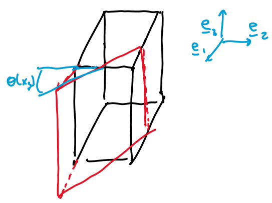
Assumptions

1. Cross-sections are rigid
2. Cross-sections are planar
3. Only deformations in $\underline{e}_2$

$u_1(x_1, x_2, x_3) = -x_2 \frac{d\bar{u}_2}{dx_1}$

$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1)$

what is a stress?




Strain Field

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = -x_2\, \bar{u}_2''(x_1)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -\frac{d\bar{u}_2}{dx_1} + \frac{d\bar{u}_2}{dx_1} = 0$$

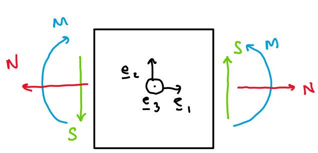
- axial fibers stretch/contract proportionally to $x_1, x_3$ plane.
- no shear strains (b/c cross sections remain planar)
- no strains in plane (cross-sections rigid)

Constitutive Law for Cross-Section

$\sigma_{22}, \sigma_{33} = 0$

$\sigma_{11}(x_1, x_2, x_3) = E\,\varepsilon_{11}(x_1, x_2) = -E x_2\, \bar{u}_2''(x_1)$

Stress resultants: integral effects of internal stresses on cross-section




$N(x_1) = \int_A \sigma_{11}(x_1, x_2, x_3)\,dA$

$M(x_1) = -\int_A x_2\,\sigma_{11}(x_1, x_2)\,dA$

$S(x_1) = \int_A \sigma_{12}(x_1, x_2)\,dA$

$$M(x_1) = -\int_A x_2\left(-E x_2\, \bar{u}_2''(x_1)\right) dA$$

$$= \left[\int_A E x_2^2\, dA\right] \bar{u}_2''(x_1)$$

$$M_3(x_1) = H(x_1)\, \bar{u}_2''(x_1)$$

$$= E \int_A x_2^2\, dA\; \bar{u}_2''(x_1)$$

$$\boxed{M_3(x_1) = E I\, \bar{u}_2''(x_1)}$$

important: $\sigma_{11}(x_1, x_2) = \frac{-M(x_1)}{\phantom{I}}\, x_2$

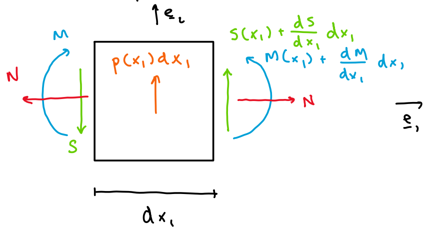
$$\boxed{M_3(x_1) = EI\,\bar{u}_2''(x_1)}$$

important: $\sigma_{11}(x_1, x_2) = \dfrac{-M(x_1)}{I}\, x_2$

Equilibrium

$$\cancel{-S(x_1)} + p_2(x_1)\,dx_1 + \cancel{S(x_1)} + S'(x_1)\,dx_1 = 0$$

$$S'(x_1) + p_2(x_1) = 0$$




$$-M(x_1) + S(x_1)dx_1 - p_2(x_1)dx_1 \frac{dx_1}{2} + M(x_1) + M'(x_1)\,dx_1 = 0$$

$$M'(x_1) + S(x_1) = 0$$

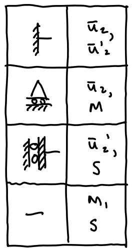
Equilibrium

$$\boxed{\begin{aligned} &S'(x_1) + p_2(x_1) = 0 \\ &M'(x_1) + S(x_1) = 0 \\ &M''(x_1) = p_2(x_1) \end{aligned}}$$

CC

$$(EI\,\bar{u}_2''(x_1))'' = p(x_1)$$

## BOUNDARY CONDITIONS




## SHEAR STRESSES IN BENT BEAMS

stress equilibrium

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\sigma_{11}(x_1, x_2) = \frac{-M(x_1)}{I} x_2$$

$$\frac{\partial \sigma_{12}}{\partial x_2} = \frac{M'(x_1)}{I} x_2$$

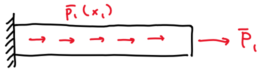
$$\sigma_{12} = \frac{-S(x_1)}{I} \frac{x_2^2}{2} + C$$

shear stress $(\sigma_{12}) = 0$ at top and bottom surfaces of the beam

$$C = \frac{S(x_1)}{I} \frac{\left(\frac{h}{2}\right)^2}{2}$$

$$\boxed{\sigma_{12}(x_1, x_2) = \frac{-S x_1}{I}\left[\left(\frac{h}{2}\right)^2 - x_2^2\right]}$$

# ROD THEORY




$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1)$$

$$\left.\begin{aligned} &u_2(x_1, x_2, x_3) = 0 \\ &u_3(x_1, x_2, x_3) = 0 \end{aligned}\right\} \text{cross-sections deform rigidly}$$

Strain Field

$\varepsilon_{11}(x_1, x_2, x_3) = \bar{u}_1'(x_1)$, all other 0

$$\sigma_{11}(x_1, x_2, x_3) = E\,\varepsilon_{11}$$

$$\sigma_{11}(x_1, x_2, x_3) = E\,\bar{u}_1'(x_1) \longrightarrow \text{uniaxial stress}$$

Stress Resultant

$N_1(x_1) = \int_A \sigma_{11}(x_1, x_2, x_3)\, dA$

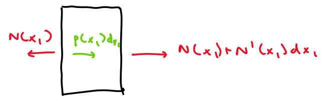
$= \int_A E u_1'(x_1)\, dA$

$= \int_A E\, dA\, \bar{u}_1'(x_1)$

$S = \int_A E(x_1, x_2, x_3)\, dA$

$N(x_1) = EA(x_1)\, \bar{u}_1'(x_1)$

Equilibrium




$-N(x_1) + p_1(x_1)dx_1 + N(x_1) + N'(x_1)dx_1 = 0$

$N'(x_1) + p_1 = 0$

$$\boxed{\frac{d(EA(x_1)\bar{u}_1'(x_1))}{dx_1} + p_1 = 0}$$

2 boundary conditions

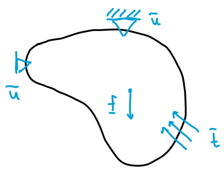
$(\bar{u}_1 = 0,\ N_1 = 0 \text{ or } P_1)$

Adding Thermal Changes

$N(x_1) = EA(x_1)\left[\bar{u}_1'(x_1) - \alpha\Delta\theta(x_1)\right]$

and $N'(x_1) + p(x_1) = 0$ + BC

# ELASTICITY




$\sigma_{ji,j} + f_i = 0$ in B

$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ in B

$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$ in B

$\varepsilon_{ij} = S_{ijkl}\,\sigma_{kl}$ in B

For linear isotropic material:

$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$

$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right]$, similar for $\varepsilon_{22}$

$\varepsilon_{23} = \frac{1}{E}\left[(1+\nu)\sigma_{23} - \nu\sigma_{kk}\delta_{23}\right]$

$\varepsilon_{23} = \frac{1+\nu}{E}\sigma_{23}$

$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right]$

$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu(\sigma_{33} + \sigma_{11})\right]$

$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right]$

$2\varepsilon_{23} = \frac{1}{G}\sigma_{23}$

$2\varepsilon_{31} = \frac{1}{G}\sigma_{31}$

$2\varepsilon_{12} = \frac{1}{G}\sigma_{12}$

$G = \frac{E}{2(1+\nu)}$

$n_i\sigma_{ij} = t_j = \bar{t}_j$

$u_i = \bar{u}_i$

# STRESS TENSOR

## MATHEMATICAL PRELIM

free indices → not repeated in same additive term; goes through all values in range

$a_{i1} \rightarrow a_{11}, a_{21}, a_{31}$

$t_i = \sigma_{ij} n_j$ implies summation:

$t_1 = \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3$

$t_2 = \sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3$

$t_3 = \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3$

Basis: any set of linearly independent

$t_i = \sigma_{ij} n_j$ implies summation:

$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$
$t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$
$t_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$

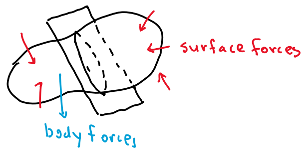
Basis: any set of linearly independent

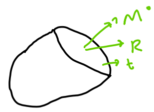
$\delta_{ij} = \underline{e}_i \cdot \underline{e}_j \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$

$\varepsilon_{ij} = \begin{cases} 0 & \text{when anything repeated} \\ 1 & \text{clockwise} \\ -1 & \text{counterclockwise} \end{cases}$


$\underline{e}_i \times \underline{e}_j = \varepsilon_{ijk} \underline{e}_k$

$\underline{e}_1 \times \underline{e}_2 = \varepsilon_{121} \underline{e}_1 + \varepsilon_{122} \underline{e}_2 + \varepsilon_{123} \underline{e}_3$
$= 0 + 0 + \underline{e}_3$



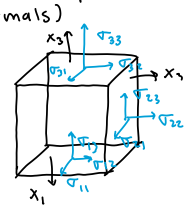

stress vector = force/unit area

$\underline{t} = \lim_{\Delta S \to 0} \underline{f}/\Delta S$

resultant stress vector $\int_S \underline{t}\, dS = \underline{R}$

resultant moment vector $\int_S \underline{r} \times \underline{t}\, dS = \underline{M}^o$

can decompose stress vectors (imagine they go through the same point, but have different normals)




$\underline{t}^{(i)} = \sigma_{ij} \underline{e}_j$

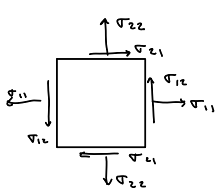
$\underline{t}^{(n)} = \underline{n}\,(e_1 \underline{t}^{(1)} + e_2 \underline{t}^{(2)} + \underline{e}_3 \underline{t}^{(3)}) = \underline{n} \cdot (\underline{e}_1 \otimes \underline{t}^{(1)} + \underline{e}_2 \otimes \underline{t}^{(2)} + \underline{e}_3 \otimes \underline{t}^{(3)})$

$\underline{\sigma} = \underline{e}_i \sigma_{ij} \underline{e}_j = \sigma_{ij} \underline{e}_i \underline{e}_i$

$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \longrightarrow$ components of stress tensor (acting on planes w/ normal $\underline{e}_i$)

$t_j^{(\underline{n})} = t_j(\underline{n}) = \sigma_{ij} n_i$

# TRANSFORMATIONS




$\tilde{\sigma}_{ij} = \sigma_{k\ell} (\tilde{\underline{e}}_i \cdot \underline{e}_k)(e_\ell \cdot \tilde{\underline{e}}_j)$

$\tilde{\sigma}_{11} = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos(2\alpha) + \sigma_{12} \sin(2\alpha)$
$\tilde{\sigma}_{22} = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2} \cos(2\alpha) - \sigma_{12} \sin(2\alpha)$
$\tilde{\sigma}_{12} = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin(2\alpha) + \sigma_{12} \cos(2\alpha)$

Maximum Normal Stress

- zero shear stress

$\tan(2\alpha_p) = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}$

- principal stresses are eigenvalues of matrix of stress tensor components
- principal directions corresponding eigenvectors
- $\sigma$ symmetric, eigenvalues real and eigenvectors orthogonal

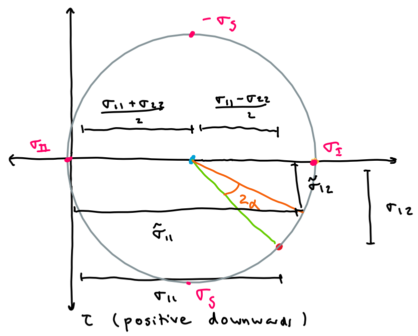
$$\sigma_{I,II} = \frac{\sigma_{11}+\sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

$$\tan(2\alpha_s) = -\frac{\sigma_{11}-\sigma_{22}}{2\sigma_{12}}$$

$$\sigma_s = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

here, normal stresses are average stress: $\frac{\sigma_{11}+\sigma_{22}}{2}$

## MOHR'S CIRCLE




① Plot reference pt. $A = (\sigma_{11}, \sigma_{12})$
stress components on plane w/ normal $\underline{e}_1$ ($\alpha = 0$)
② Plot circle's center, C, at avg. stress $\left(\frac{\sigma_{11}+\sigma_{22}}{2}, 0\right)$
③ The distance between the two is the radius, R.
④ Sketch the circle centered at C w/ radius R.
⑤ $\sigma_I$, $\sigma_{II}$, $\sigma_s$, and $-\sigma_s$ are labelled appropriately.

# STRAIN

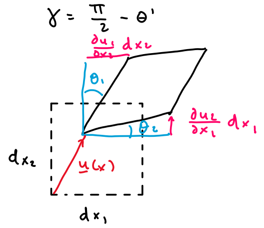
## DEFINITIONS

normal strain = elongation or contraction of a line segment per unit length

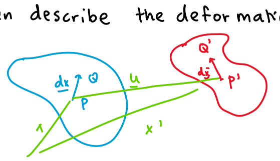
$$\varepsilon = \lim_{\Delta S \to 0} \frac{\Delta S' - \Delta S}{\Delta S} \quad (\Delta S = \text{line segment undeformed})$$

shear strain = change in angle between two perpendicular line segments

$$\gamma = \frac{\pi}{2} - \theta'$$




can describe the deformation a body undergoes with deformation mapping.




$$x' = \varphi(\underline{x}) = \underline{x} + \underline{u}$$

Lots of math happens. It doesn't seem very important.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \underbrace{\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}}_{\text{ignore HOT}}\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- strain tensor symmetric

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right)$$

- strain tensor symmetric

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right)$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial x_2}{\partial u_1}\right) = \varepsilon_{21}$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial x_3}{\partial u_1}\right) = \varepsilon_{31}$$

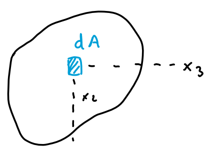
$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial x_2}{\partial u_3} + \frac{\partial x_3}{\partial u_2}\right) = \varepsilon_{32}$$

and the transformations

$$\tilde{\varepsilon}_{ij} = \varepsilon_{lk}(\tilde{\underline{e}}_l \cdot \underline{e}_i)(\tilde{\underline{e}}_j \cdot \underline{e}_k)$$

so everything above works.
(and formulas given).

# BEAM INERTIA




① First Moment of Area = Center of Area

$$Q_2 = \int_A x_3 \, dA, \quad Q_3 = \int_A x_2 \, dA$$

Geometric center:

$$x_3^G = \frac{Q_2}{A} = \frac{\int_A x_3 \, dA}{A}, \quad x_2^G = \frac{Q_3}{A} = \frac{\int_A x_2 \, dA}{A}$$

② Second Moments of Area

$$I_{22} = \int_A x_3^2 \, dA, \quad I_{33} = \int_A x_2^2 \, dA$$

$$I = I_{33} = \int_A x_2^2 \, dA$$

$x_2$ = distance to center of area

$$I = I^G + A d_2^2$$ distance to neutral axis

neutral axis