Unit 1 Review

Wednesday, February 21, 2018 8:43 PM

INTRO TO DIFFERENTIAL EQUATIONS

- equation that relates an unknown function and its derivatives

Ordinary Diff Eq: only one variable
Partial Diff Eq: multivariable functions

To build a diff eq,

1. Start with
   $\ddot{y}$ $\dot{y}$ y
2. Multiply each term by a function of t
   $e^t\ddot{y}$ $5\dot{y}$ t^2y
3. Add, set equal to 0 (homogeneous)
   $e^t\ddot{y}+5\dot{y}+t^2y=0$
   or set equal to $q(t)$
   $e^t\ddot{y}+5\dot{y}+t^2y=q(t)$

SEPARATION OF VARIABLES

$\frac{dy}{dt}=f(y,t)$

1. Write $f(y,t)=g(y)\cdot h(t)$
2. Get LHS in terms of y and RHS in terms of t.
   $\frac{dy}{g(y)}=h(t)\,dt$ → check for $g(y)=0$ (may be extra sol'ns!)
3. a) Integrate both sides
   $G(y)=H(t)+C$
   b) Solve for y.
4. Check for the extra solutions that result from dividing by 0.

Integrating Factors

Given $\dot{y}+p(x)y=g(x)$, $P(x)$ is the antiderivative of $p(x)$.
Multiply both sides by $e^{P(x)}$ to get something that looks like a product rule: $y'e^{P(x)}+p(x)e^{P(x)}y=g(x)e^{P(x)}$
Differentiate:
$\frac{d}{dx}\left[ye^{P(x)}\right]=q(x)e^{P(x)}$
$ye^{P(x)}=\int q(x)e^{P(x)}dx$
$y=e^{-P(x)}\int q(x)e^{P(x)}dx$

VARIATION OF PARAMETERS

- method for solving inhomogeneous linear equations
  1. Get the equation into standard form ($\dot{y}+p_1(t)y=q(t)$)
  2. Solve the associated homogeneous equation to get a general solution.
  3. Substitute the general solution with the constant as u, a function of t for y in the original differential equation and solve for.
  4. Combine the answer from pt. 3 with pt. 2 and add to general solution.
- uses the concept of "linear combinations" → the solution to inhomogeneous eq'n is the linear combination of the particular solution and the general solution to homogeneous equation

MODELING

- honestly, modeling is a little bit wishy washy but it's always helpful to take it back to reality
  1. Identify the relevant quantities (known and unknown) → identify their units.
     Key words: rate of x means $\frac{dx}{dt}$
  2. Identify the independent variable (typically time). The other variables are either functions of this variable or constants.
  3. Write equations expressing how things change.
  4. Think about the sanity checks: when is nothing changing? what happens when independent variable is 0? Really big? Really small? Draw pictures if helpful.
  5. Solve your differential equations (if that's what the question is).

INTRODUCTION TO COMPLEX NUMBERS

- in the form $a+bi$
  Real ↗ ↖ Imaginary part

$\mathrm{Im}(a+bi)=b$ (NOT bi) $i^2=-1$
$\mathrm{Re}(a+bi)=a$

- complex numbers are a field, whatever that means

not sure abt section on complex roots of

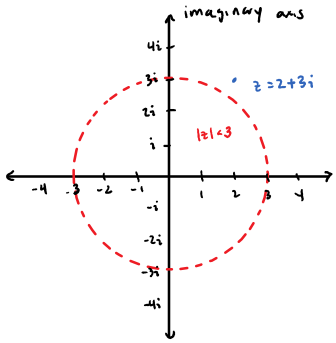
imaginary axis

ui

$z=a+bi$

- complex numbers are a field, whatever that means

not sure abt section on complex roots of



$z = a+bi$

$|z|$ = distance from origin

$\sqrt{a^2+b^2}$

$\bar{z} = a-bi$

$Re(z) = \frac{z+\bar{z}}{2}$, $Im(z) = \frac{z-\bar{z}}{2}$

- can be added, subtracted, multiplied, and divided
  * for division, it's a good idea to multiply be the complex conjugate

POLAR FORMS OF COMPLEX NUMBERS

- Euler's Formula says that
  $e^{it} = \cos t + i \sin t$

- Given $z = a+bi$, you can express z as

  $\underbrace{r}_{|z|} e^{i\theta}$ ← arg $(\tan^{-1}(\frac{b}{a}))$

- this makes operations with complex numbers much easier:
  * $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$
  * $\frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$
  * $(re^{i\theta})^n = r^n e^{i\theta n}$
  * $\overline{re^{i\theta}} = re^{-i\theta}$
- Find $z^n = x$ (complex roots problem)
  1. convert z to complex number $(z = re^{i\theta})$
     and x to complex number $(xe^{i\pi}$ or $xe^{-i\pi})$
  2. $n\theta = \pi + 2\pi k$
     $\theta = \frac{\pi}{n} + \frac{2\pi}{n} k$
- n^{th} roots of unity the complex solutions to $z^n = 1 \longrightarrow e^{i(\frac{2\pi}{n})}$

SINUSOIDAL FUNCTIONS

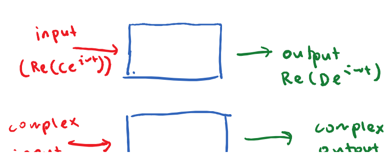
- every second order linear ODE requires two initial conditions to fully solve
  $\ddot{x} + x = 0$
  $x(t) = c_1 \cos(t) + c_2 \sin(t)$
- important identities below:
  $e^{it} = \cos t + i \sin t$ $\quad \overline{(e^{-it})} = \overline{(\cos t - i \sin t)}$
  $\cos t = \frac{e^{it}+e^{-it}}{2}$ $\quad \sin t = \frac{e^{it}-e^{-it}}{2i}$

- there are three ways to write a sinusoidal function
  1. Amplitude-phase form: $A\cos(\omega t - \phi)$
  2. Complex Form: $Re(ce^{i\omega t}) \longrightarrow c$ = complex number
  3. Linear Combination: $a\cos\omega t + b\sin\omega t \longrightarrow a$ and b = real numbers

  $c = Ae^{-i\phi} = a-bi$
  $\bar{c} = Ae^{i\phi} = a+bi$
- physical significance of sinusoids:
  A = amplitude $= |c| = \sqrt{a^2+b^2}$
  t_0 = time lag (time for max value) $= \phi/\omega$
  P = period $= 2\pi/\omega$
  ω = angular frequency
  ν = frequency $= \frac{\omega}{2\pi}$
  ϕ = phase lag

COMPLEX GAIN

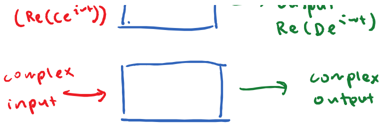
$G = \frac{\text{complex output}}{\text{complex input}} = \frac{Ce^{i\omega t}}{ce^{i\omega t}} = \frac{C}{c}$



- multiplying by the complex gain amounts to
  * multiplying A by $|G|$

complex input $ce^{i\omega t}$ c

- multiplying by the complex gain amounts to
  * multiplying A by $|G|$
  * increasing ϕ by $-\text{ang } G$



SECOND ORDER HOMOGENEOUS LINEAR ODEs

$a_2\ddot{y} + a_1\dot{y} + ay = 0$

① Write down the characteristic equation:

$a_2 r^2 + a_1 r + a = 0$

② Solve the characteristic equation to list complex roots with multiplicity.

③ a) If roots are distinct $r_1 \neq r_2$, then

Basis: $e^{r_1 t}, e^{r_2 t}$

General solution: $c_1 e^{r_1 t} + c_2 e^{r_2 t}$

b) If roots are equal $(r_1 = r_2)$, then

Basis: e^{rt}, te^{rt}

General Solution: $c_1 e^{rt} + c_2 te^{rt}$

HARMONIC OSCILLATORS AND DAMPED FREQUENCIES

(does damped frequencies work if you're overdamped or critically damped?)

$m\ddot{x} + b\dot{x} + k = 0$

① $mr^2 + br + k = 0$

② Roots: $\dfrac{-b \pm \sqrt{b^2 - 4mk}}{2}$

③ Case 1: $b^2 < 4mk$ ("underdamped")

Complex roots: $-\frac{b}{2m} \pm i\frac{\sqrt{4mk - b^2}}{2m}$

$\omega_D = \dfrac{\sqrt{4mk - b^2}}{2m}$ = damped frequency

$s = \frac{b}{2m}$

Basis: $(e^{(-s + i\omega_d)t}, e^{(-s - i\omega_d)t})$

Real-Valued Basis: $e^{-st}\cos(\omega_d t), e^{-st}\sin(\omega_d t)$

General Solution: $e^{-st}(A\cos(\omega_d t - \phi))$ $T = \frac{2\pi}{\omega_d}$

Case 2: $b^2 = 4mk$ ("critically damped")

Roots: $-\frac{b}{2m}, -\frac{b}{2m}$

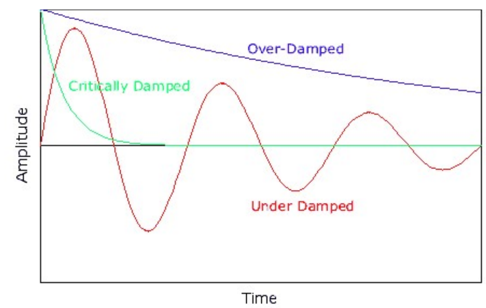
Basis: $e^{-\frac{b}{2m}t}, te^{-\frac{b}{2m}t}$

General Real Sol'n: $e^{-\frac{b}{2m}t}(c_1 + c_2 t)$

Case 3: $b^2 > 4mk$ ("overdamped")

Roots (both real): $\dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$ $(s_1$ and $s_2)$

General Sol'n: $ae^{-s_1 t} + be^{-s_2 t}$



Unit 2 Review

Wednesday, March 14, 2018 8:40 PM

OLD NEWS

$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

period: $2\pi/\omega$

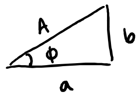
$\nu = \omega/2\pi$

time lag $= \phi/\omega$

① $a\cos(\omega t) + b\sin(\omega t)$

② $\mathrm{Re}(ce^{i\omega t})$; $c = Ae^{-i\phi}$

③ $A\cos(\omega t - \phi)$



$c = a - bi = Ae^{-i\phi}$

WEEK 3

SOLVING HOMOGENEOUS LINEAR ODEs

$a_n y^{(n)} + \ldots + a_1 y^{(1)} + a_0 y = 0$

① Characteristic Polynomial =
$a_n r^n + \ldots + a_1 r + a_0$

② Factor characteristic polynomial

③ a) If distinct, sol'ns are Span $(e^{r_1 t}, \ldots, e^{r_n t})$
b) If not distinct, multiply by t for every time a root appears

VECTOR SPACE

A set of functions is a vector space if:

① The zero function 0 is in S

② multiplying any function by a scalar gives another function in S

③ Adding any two functions gives another function in S

the Span of homogeneous linear ODEs = a vector space

- the "dimension" of a basis is the number of solutions in that basis
- dimension of n^{th} order homogeneous ODE = n

- two functions are "linearly dependent" if they are scalar multiples or linear combinations

WEEK 4

OPERATORS

- take an input function and return another function
- can add, multiply, etc. operators
- can rewrite any homogeneous linear ODE in operator form
  * looks like the char poly

ERF

$z_p = \frac{1}{p(r)} e^{rt}$ = particular solution

① Find the general solution to associated homogeneous equation

② Linear combination of particular solution from ERF and general sol'n

Sometimes, $p(r) = 0$, so we need to use the GERF.

$$y_p = \frac{1}{p^{(m)}(r_0)} t^m e^{r_0 t}$$

original r value

the multiplicity of the polynomial

Complex Gain $= \frac{1}{p(i\omega)} = G$

Gain $= \frac{1}{|p(i\omega)|} = g$

Phase lag $= -\arg(G)$

STABILITY

- system in which changes in the initial conditions have vanishing effect on the long-term behavior of the function

	General Solution	Condition
$a+bi$	$e^{at}(C_1\cos(bt)+C_2\sin(b_2t))$	$a<0$
real s, s	$e^{st}(C_1+C_2)$	$s<0$
real $r_1 \neq r_2$	$c_1e^{r_1t}+c_2e^{r_2t}$	$r_1, r_2 < 0$

- roots of char poly have negative real part

WEEK 5

RESONANCE *

- harmonic oscillator driven at a frequency at or near its natural frequency
- observe what happens to the gain as the solution to char poly approaches 0
- with a damped system, the gain is large but bounded

RLC CIRCUITS

$I = \dot{Q}$

$V_R = RI = R\dot{Q}$

$V_L = L\dot{I} = L\ddot{Q}$

$V_C = \frac{1}{C}Q$

$V = V_R + V_L + V_C$

input

$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t) \dashrightarrow L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}(t)$

Kirchoff: at each junction, $I_{in} = I_{out}$
: around each loop, sum of $V = 0$

FINDING EIGENVALUES

$\dot{\vec{x}} = A\vec{x}$, where A = a vector

$\lambda^2 - tr(A)\lambda + \det(A) = 0$

FINDING EIGENVECTORS

Given eigenvalues, $\vec{v} = \begin{pmatrix} v \\ w \end{pmatrix}$, pick a value

$(A - \lambda I)\vec{v} = 0$

elements multiples of each other

CONVERTING N^{TH} ORDER → 1^{ST} ORDER ODE

$y := \dot{x}$

- express $\dot{x}, \dot{y}$ in terms of x, y

$\dot{x} = y$

$\dot{y} = \ddot{x} = -5\dot{x} - 6x = -6x - 5y$

Ex.: $\left.\begin{array}{l} \dot{x} = 2x - y \\ \dot{y} = 5x + 7y \end{array}\right\}$ 2nd order ODE only x

$y = 2x - \dot{x}$

$\dot{y} = 5x + 7(2x - \dot{x})$

$\dot{y} = 5x + 14x - 7\dot{x}$

$\dot{y} = 19x - 7\dot{x} = 2\dot{x} - \ddot{x}$

$\ddot{x} - 9\dot{x} + 19x = 0$

COMPLEX EIGENVALUES

- if λ is a nonreal eigenvalue, the other is $\bar{\lambda}$
- if $\vec{v}$ is a nonzero vector associated with λ, the other vector $\bar{\vec{v}}$ is associated with $\bar{\lambda}$
- basis: $(e^{\lambda t}\vec{v}, e^{\bar{\lambda} t}\bar{\vec{v}})$
- real-valued basis: $(Re(e^{\lambda t}\vec{v}), Im(e^{\lambda t}\vec{v}))$

VISUALIZING SOLUTIONS

trajectories?

PHASE PORTRAITS

- diagram showing trajectories in a phase plane
  1. Find the eigenvalues of A.
  2. If the eigenvalues are distinct real numbers and nonzero, find and draw eigenlines.
     * if opposite signs, saddle (asymptotic to both)

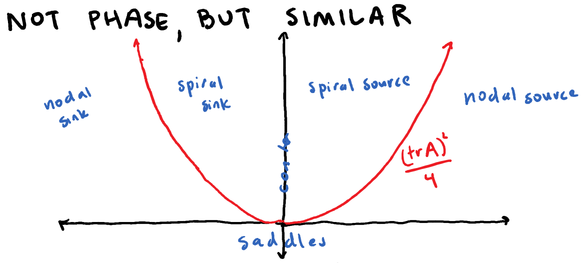
smallest
) |eigenvalue|

* if same signs, repelling / attracting node (trajectories tangent to "slow")

⑤ If eigenvalues complex ($a \pm bi$), check a:
* if $+$, repelling spiral
* if $-$, attracting spiral
* if 0, center

} direction determined by evaluating a few pts.

NOT PHASE, BUT SIMILAR



$\det A < 0$, $\lambda_1 \lambda_2$ opposite values : saddle
$\det A > 0$, $trA < 0$: nodal sink
$trA > 0$: nodal source

$\det A > \frac{(trA)^2}{4}$, $trA > 0$: spiral source
$trA < 0$: spiral sink
$trA = 0$: center

degenerate node: $\lambda \neq 0$, $A = \lambda I$
star node: $\lambda \neq 0$, $A \neq \lambda I$

- can't change phase portrait? structurally stable!

ENERGY

$m\ddot{x} + kx = 0$
$KE = \frac{1}{2}m\dot{x}^2$
$PE = \frac{1}{2}kx^2$

$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$
$\dot{E} = m\dot{x}\ddot{x} + k\dot{x}x$ parameterized by ellipse
$\dot{E} = \dot{x}(m\ddot{x} + kx) = 0$ because $m\ddot{x} + kx = 0$ (undamped case)

in a damped oscillator,
$m\ddot{x} + b\dot{x} + kx = 0$
$m\ddot{x} + kx = -b\dot{x}$
$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$
$\dot{E} = m\dot{x}\ddot{x} + k\dot{x}x$
$= \dot{x}(m\ddot{x} + kx)$
$= -b\dot{x}^2$ spirals inwards :)

Unit 3 Review

Tuesday, May 22, 2018 11:43 AM

For Unit 3, you used the flashcards extensively. These were pretty helpful.

Unit 4 Review

Tuesday, May 22, 2018 11:44 AM

HEAT EQUATION

Homogeneous Boundary Conditions

$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ $u(0,t)=0$ $u(x,0)=f(x)$

$u(L,t)=0$

① Find the general solution (ignore $u(0,t)=0, u(L,t)=0$)

a) separate variables

$u(x,t) = v(x)\,w(t)$

$\frac{\partial u}{\partial x} = v'(x)w(t)$ $\frac{\partial u}{\partial t} = \dot{w}(t)v(x)$

$\frac{\partial^2 u}{\partial x^2} = \ddot{v}(x)w(t)$

$\dot{w}(t)v(x) = \alpha^2 \ddot{v}(x)w(t)$

b) get variables to same side, move coefficient to t-side

$\frac{1}{\alpha^2}\frac{\dot{w}(t)}{w(t)} = \frac{\ddot{v}(x)}{v(x)}$ (assume that neither is 0 function)

c) equate to some $-\lambda$

$\frac{1}{\alpha^2}\frac{\dot{w}(t)}{w(t)} = \frac{\ddot{v}(x)}{v(x)} = -\lambda$

$\ddot{v}(x) = -\lambda v(x)$, $\dot{w}(t) = -\lambda\alpha^2 w(t)$

$\ddot{v}(x) + \lambda v(x) = 0$, $\dot{w}(t) + \lambda\alpha^2 w(t) = 0$

d) solve using old techniques

$r^2 + \lambda = 0$

$c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x} = 0$

$c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) = 0$

$\frac{dw}{dt} = -\lambda\alpha^2 w(t)$

$\int \frac{1}{w}\dot{w} = \int -\lambda\alpha^2$

$\ln(w) = -\lambda\alpha^2 t$

$w = e^{-\lambda\alpha^2 t}$

② plug in boundary conditions

$u(0,t) = v(0)w(t) = 0$

$v(0) = c_1 = 0$

$v(x) = c_2 \sin\sqrt{\lambda}x$

dropped something in here → is incorrect!

③ sub into OG eq'n

$u(x,t) = c_2 e^{-\lambda\alpha^2 t} \sin\sqrt{\lambda}x$ = general sol'n $\lambda = +n^2$

④ init cond: $u(x,0) = c_2 e^{-\lambda\alpha^2 t} \sin\sqrt{\lambda}x = f(x)$ (if $f(x)=1$, then square wave)

$$u(x,t) = \sum_{n\geq 1,\text{odd}} \underbrace{\frac{4}{n\pi}}_{b_n=\frac{4}{n\pi}} e^{-n^2\alpha^2 t} \sin(nx) \qquad x \in [0,L],\ t \geq 0$$

Homogeneous Boundary (Alternative)

$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ $u(0,t)=0$ $u(x,0)=f(x)$

$u(L,t)=0$

① Make a substitution:

$x_1 = \frac{\pi}{L}x$ $t_1 = bt$

$u_1(x_1,t_1) = u\left(\frac{L}{\pi}x_1, \frac{t_1}{b}\right)$

$\frac{\partial u_1}{\partial t_1} = \frac{1}{b}\frac{\partial u}{\partial t}$ $\frac{\partial u_1}{\partial x} = \frac{L}{\pi}\frac{\partial u}{\partial x}$

$\frac{\partial^2 u_1}{\partial x_1^2} = \left(\frac{L}{\pi}\right)^2 \frac{\partial^2 u}{\partial x^2}$

② Sub in initial value for $\frac{\partial u}{\partial t}$, solve for $\frac{\partial^2 u}{\partial x^2}$

$\frac{\partial u_1}{\partial t_1} = \frac{1}{b}\cdot\alpha\frac{\partial^2 u}{\partial x^2}$ $\frac{\partial^2 u}{\partial x^2} = \left(\frac{\pi}{L}\right)^2 \frac{\partial^2 u_1}{\partial x_1^2}$

$\frac{\partial u_1}{\partial t} = \frac{\alpha}{b}\cdot\left(\frac{\pi}{L}\right)^2 \frac{\partial^2 u_1}{\partial x_1^2}$

③ $\frac{\alpha}{b} = \left(\frac{\pi}{L}\right)^2$, so $b = \alpha\left(\frac{L}{\pi}\right)^2$

④ $\frac{\partial u_1}{\partial t_1} = \frac{\partial^2 u}{\partial x_1^2}$ $\longrightarrow$ know from previous efforts that gen sol'n $u_1(x_1, t_1) = \sum b_n e^{-n^2 t_1} \sin(n x_1)$

⑤ sub in $t_1 = bt$.

$u_1(0, t_1/b) = 0$

$u_1(L, t_1/b) = 0$

⑥ $u(x,t) = \sum b_n e^{-n^2 bt} \sin\left(n \frac{\pi}{L} x\right)$

$$u(x,t) = \sum b_n e^{-n^2\left(\alpha\left(\frac{L}{\pi}\right)^2\right)t} \sin\left(n \frac{\pi}{L} x\right)$$

⑦ solve for b_n

$$b_n = \frac{2}{\pi}\int_0^{\pi} u_1(x_1, 0) \sin(n x_1)\, dx_1$$

$$= \frac{2}{\pi} \int_0^L u_1\left(\frac{\pi}{L}x, 0\right) \sin\left(n \frac{\pi}{L} x\right) dx$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{L} \int_0^L u(x,0) \sin\left(n \frac{\pi}{L} x\right) dx$$

$$= \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx$$