a) 6 iteration for newton's method and secant method can be found in pictures below. First guess for newton method was $x_0 = 1$. First 2 guess for secant method was $\underbrace{x_0 = 2}_{\text{previous}}$, $\underbrace{x_1 = 1}_{\text{current}}$ **Q1**

b) Because we do not know the root, our best bet for the error estimation is $|x_{current} - x_{previous}|$. We can see the error estimations for our results in the picture below.

6 iterations for Newton's method

x_current	f(x_current)	f_prime(x_current)	|x_cu
1.000000000000000	-1.000000000000000	3.000000000000000	0.
1.333333333333333	-0.182605043526210	2.000000000000000	0.
1.424635855096438	-0.008970891970714	1.807735033265204	0.
1.429598358895743	-0.000024211231888	1.797988662416834	0.
1.429611824626865	-0.000000000177443	1.797962307736241	0.
1.429611824725556	0.000000000000000	1.797962307543088	0.

6 iterations for secant method

iteration	x_current	f(x_current)	f_prime(x_current)	|x_current - x_next|
1	1.000000000000000	-1.000000000000000	3.000000000000000	0.564146655935188
2	1.564146655935188	0.225214974090973	1.557304959111037	0.103699576961356
3	1.460447078973832	0.054523350306825	1.738887329495403	0.033124345741087
4	1.427322733232745	-0.004120833372396	1.802449584012716	0.002327595011217
5	1.429650328243963	0.000069226424073	1.797886952478229	0.000038455556041
6	1.429611872687922	0.000000086234524	1.797962213673629	0.000000047963371

c) By Taylor's approximation we can write any function like this:

(1) $f(x) = f(x_n) + f'(x_n)(x - x_n) + \underbrace{\frac{f''(\xi)(x - x_n)^2}{2!}}_{R_2(x)}$

This is a remainder, when we approximate the function with its first derivative.

(2) We know that $f(x^*) = 0$ so x^* is a root of $f(x)$.

Let's insert x^* to Taylor's approximation above.

$$0 = f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(\xi)(x^* - x_n)^2}{2!}$$

$$\frac{f(x_n)}{f'(x_n)} + (x^* - x_n) + \frac{f''(\xi)(x^* - x_n)^2}{2f'(x_n)} = 0$$

(3) $$x^* - \underbrace{\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}_{x_{n+1}} = -\frac{f''(\xi)}{2f'(x_n)}(x^* - x_n)^2$$

$$\underbrace{|x^* - x_{n+1}|}_{\text{This is error } e_{n+1}} = \underbrace{\left|-\frac{f''(\xi)}{2f'(x_n)}\right|}_{\text{Let's call this } M} \underbrace{(x^* - x_n)^2}_{\text{This is error } e_n}$$

$e_{n+1} = M e_n^2$

$\boxed{\text{Answer}}$

$|x^* - x_{n+1}| = M(x^* - x_n)^2$

Thus, we have found the equation that relates the error between the n. and (n+1). steps.

and we have showed speed of convergence of Newton's method is quadratically convergent.

$\boxed{Q2}$

$$A_1 = \begin{bmatrix} -1 & 3 & 0 \\ -4 & -1 & 3 \\ 0 & -4 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & -1 & 3 \\ -1 & 3 & 0 \\ 0 & -4 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -4 & -1 & 3 \\ 0 & 13/4 & -3/4 \\ 0 & -4 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & -1 & 3 \\ 0 & -4 & -1 \\ 0 & 13/4 & -3/4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -4 & -1 & 3 \\ 0 & -4 & -1 \\ 0 & 0 & -25/16 \end{bmatrix} \Rightarrow U$$

$$\frac{13}{16} - \frac{12}{16} = \frac{-25}{16}$$

L

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/4 & -13/16 & 1 \end{bmatrix}$$

P

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

So $\underbrace{PA_1 = LU}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ -2 & -1 & 3 \\ 0 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{4} & \frac{-13}{16} & 1 \end{bmatrix} \begin{bmatrix} -4 & -1 & 3 \\ 0 & -4 & -1 \\ 0 & 0 & \frac{-25}{16} \end{bmatrix}$$

$PA = LU$ form for matrix A_1

A2

$$\begin{bmatrix} 1 & 1 & 2 \\ 4 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 4 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

U

$$\rightarrow \begin{bmatrix} 4 & 4 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

L

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

P

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$PA_2 = LU$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 4 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 4 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

PA=LU form of matrix A_2

We will solve the linear system $A_2 x = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}$

$$PA_2 x = P \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix}$$

Because $PA_2 = LU$ we can write this as

$$L \underbrace{Ux}_{y} = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix}$$

First let's call $Ux = y$ and then solve this system for y by forward substitution.

$\Leftrightarrow$

$$Ly = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$$

$$y_1 = -4$$

$$y_2 = -\frac{1}{2} y_1 - 2 = 0$$

$$y_3 = -\frac{1}{4} y_1 + 3 = 4$$

So we found the y.
We said $Ux = y$
So let's solve for x
this time by backward substitution.

$$\begin{bmatrix} 4 & 4 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}}$$

we found the x we seek.

$2x_3 = 4 \Rightarrow x_3 = 2$

$-2x_2 = -x_3 = -2 \Rightarrow x_2 = 1$

$4x_1 = -4x_2 - 4 = -8 \Rightarrow x_1 = -2$

Answer

$$x = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

GIPTA

$AG = I$; We are asked to find matrix G. Q3

Let's call ith column of G as x_i. Then we can write G like this: $[x_1 \; x_2 \; x_3]$

$$AG = I \iff A[x_1 \; x_2 \; x_3] = \underbrace{[e_1 \; e_2 \; e_3]}_{\text{columns of } I.}$$

$$\iff [Ax_1 \; Ax_2 \; Ax_3] = [e_1 \; e_2 \; e_3]$$

$$\iff \begin{array}{c} Ax_1 = e_1 \\ \wedge \\ Ax_2 = e_2 \\ \wedge \\ Ax_3 = e_3 \end{array}$$

We have 3 linear systems to find G meaning A^{-1}.

Let's, first, find LU decomposition of A.

$$\underset{A}{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix}} \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\Rightarrow \underset{U}{\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & -8/3 \end{bmatrix}}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$$

We have found the $A=LU$ form.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & -8/3 \end{bmatrix}$$

$$L(\underbrace{U x_1}_{y}) = e_1 \iff \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$y_1 = 1$

$y_2 = \frac{1}{2}$

$y_3 = \frac{2}{3} y_2 = \frac{1}{3}$

$$y = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & -8/3 \end{bmatrix} \begin{bmatrix} x_{1(1)} \\ x_{1(2)} \\ x_{1(3)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 5/8 \\ 1/4 \\ -1/8 \end{bmatrix}$$

$-\frac{8}{\cancel{3}} x_{1(3)} = \frac{1}{\cancel{3}} \Rightarrow x_{1(3)} = -\frac{1}{8}$

$\frac{3}{2} x_{1(2)} = x_{1(3)} + \frac{1}{2} = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8} \Rightarrow x_{1(2)} = \frac{1}{4}$

$2 x_{1(1)} = x_{1(2)} + 1 = \frac{5}{4} \Rightarrow x_{1(1)} = \frac{5}{8}$

(3)

$$L(\underbrace{Ux_2}_{y}) = e_2 \iff \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$y_1 = 0$

$y_2 = 1$

$y_3 = \frac{2}{3} y_2 = \frac{2}{3}$

$$y = \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix}$$

$$Ux_2 = \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix} \iff \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & -8/3 \end{bmatrix} \overbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}^{x_2} = \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix}$$

$(-8/3)c = 2/3 \Rightarrow c = -\frac{1}{4}$

$\frac{3}{2} b = c + 1 = \frac{3}{4} \Rightarrow b = \frac{1}{2}$

$2a = b = \frac{1}{2} \Rightarrow a = \frac{1}{4}$

$$x_2 = \begin{bmatrix} 1/4 \\ 1/2 \\ -1/4 \end{bmatrix}$$

$$\underbrace{L(Ux_3)}_{y} = e_3 \iff \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} y_1 &= 0 \\ y_2 &= 0 \\ y_3 &= 1 \end{aligned} \implies y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Ux_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \iff \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$c = -\frac{3}{8}$$

$$\frac{3}{2}b = c = -\frac{3}{8} \implies b = -\frac{1}{4}$$

$$2a = b = -\frac{1}{4} \implies a = -\frac{1}{8}$$

$$x_3 = \begin{bmatrix} -\frac{1}{8} \\ -\frac{1}{4} \\ -\frac{3}{8} \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 5/8 \\ 1/4 \\ -1/8 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1/4 \\ 1/2 \\ -1/4 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -1/8 \\ -1/4 \\ -3/8 \end{bmatrix}$$

$$A^{-1} = G = [x_1 \ \ x_2 \ \ x_3]$$

Answer

$$= A^{-1} = \begin{bmatrix} 5/8 & 1/4 & -1/8 \\ 1/4 & 1/2 & -1/4 \\ -1/8 & -1/4 & -3/8 \end{bmatrix}$$

Thus we have found the inverse of the matrix A using LU decomposition.

Q4

$(A^T A)\underbrace{w}_{\text{Here are the weights}} = A^T \underbrace{b}_{\text{Here are the outputs.}}$

Here are the inputs in the form of this:

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & & & & \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix}$$

- m is the number of inputs.
- We are trying to fit this data to:

$w_1 x^0 + w_2 x^1 + \dots + w_n x^{n-1}$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \qquad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

- Aw contains the predictions.

(a) For this section our A matrix should be like this

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ & \vdots \\ 1 & x_{20} \end{bmatrix} = A \quad , \quad w = \begin{bmatrix} a \\ b \end{bmatrix} \quad , \quad \ell(x) = a + bx$$

After solving the linear equations $(A^T A) w = A^T y$ we find our weights like this:

$$w = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1.923 \\ 0.500285 \end{bmatrix}$$

Thus we found the weights for our function $\ell(x) = a + bx$

(b) This time our matrix A is in the form:

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ & \vdots & \\ 1 & x_{20} & x_{20}^2 \end{bmatrix}, \quad w = \begin{bmatrix} e \\ d \\ c \end{bmatrix}$$

$$, \; p(x) = e + dx + cx^2$$

After solving $(A^T A)w = A^T y$, we found our weight vector:

$$w = \begin{bmatrix} e \\ d \\ c \end{bmatrix} = \begin{bmatrix} 0.60295 \\ 1.22309 \\ -0.06857 \end{bmatrix}$$

(C) In section C, our matrix A should be like this:

$$\begin{bmatrix} 1 & \ln(x_1) \\ 1 & \ln(x_2) \\ \vdots & \\ 1 & \ln(x_{20}) \end{bmatrix} = A, \quad w = \begin{bmatrix} k \\ m \end{bmatrix}, \quad f(x) = k + m\ln(x)$$

After solving $(A^TA)w = A^Ty$ for w, we get this:

$$w = \begin{bmatrix} k \\ m \end{bmatrix} = \begin{bmatrix} 1.78006 \\ 1.94533 \end{bmatrix}$$

(d) You can find all matrices A, weight vectors w, of section a, b and c, printed in the terminal after you run the program.

Likewise you can find the graphs of 3 functions $l(x)$, $p(x)$ and $f(x)$ in separated windows and you can see the training points around them.

Please do not forget reading comments between the codes and printings in the terminal. They are vital to understand what I did.

I especially did not write $(A^TA)w = A^Ty$ like $w = (A^TA)^{-1}A^Ty$ because it is much more healthy and robust to solve linear equations this way in numpy. Using inverses can cause unwanted things.