



Introduction to Quantum Computation

Artur Miroszewski

Quantum Cosmos Lab, Jagiellonian University

High Performance and Disruptive Computing in Remote
Sensing School 2025



JAGIELLONIAN UNIVERSITY
IN KRAKÓW



Quantum Cosmos Lab



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09:30 - 11:00 Introduction to Quantum Computation

11:00 - 11:30 Break

11:30 - 12:00 Quantum Algorithms

12:00 - 13:00 Hands-on Session 1

13:00 - 14:30 Lunch

14:30 - 15:30 Quantum machine learning

15:30 - 16:00 Break

16:00 - 17:00 Hands-on Session 2



09:30 - 11:00 **Introduction to Quantum Computation**

11:00 - 11:30 Break

11:30 - 12:00 Quantum Algorithms

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13:00 - 14:30 Lunch

14:30 - 15:30 Quantum machine learning

15:30 - 16:00 Break

16:00 - 17:00 Hands-on Session 2

Introduction to Quantum Computation

- Basic notions and postulates of quantum mechanics
- Measurement
- Introduction of quantum gates and circuits
- Computational paradigms and quantum computing hardware technologies

Quantum Algorithms

- Logic behind quantum algorithms
- Advanced circuits
- Review of main algorithms

Quantum Field Theory

Quantum Mechanics

Quantum Computing

*This
course*

Error correction

Noise

*Quantum
channels*

...

*Infinite-
dimensional
Hilbert spaces*

*"Physical
systems"*

...

*Relativistic
physics*

Fields

...



Basic definition

$$z = \underbrace{a}_{\text{real part}} + i \underbrace{b}_{\text{imaginary part}}$$

$$\operatorname{Re}[z] = a, \quad \operatorname{Im}[z] = b$$
$$i^2 = -1 \rightarrow \sqrt{-1} = i$$

Complex conjugation

$$z^* = a - ib$$

$$\operatorname{Re}[z] = \frac{1}{2}(z + z^*)$$

$$\operatorname{Im}[z] = \frac{1}{2}(z - z^*)$$

Basic operations

Addition

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

Multiplication

$$z_1 \cdot z_2 = (a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Modulus

$$|z| = \sqrt{z \cdot z^*} = \sqrt{a^2 + b^2}$$

Example

$$z = 2 - 3i$$

$$\operatorname{Re}[z] = 2, \quad \operatorname{Im}[z] = -3$$

$$z^* = 2 + 3i, \quad |z| = \sqrt{4 + 9} = \sqrt{13}$$

Math reminder - Complex numbers

Basic definition

$$z = \underbrace{r}_{\text{modulus}} \cdot e^{i \overbrace{\varphi}^{\text{argument}}}$$

$$\operatorname{Re}[z] = r \cdot \cos \varphi, \quad \operatorname{Im}[z] = r \cdot \sin \varphi$$

$$z^* = r \cdot e^{-i\varphi}$$

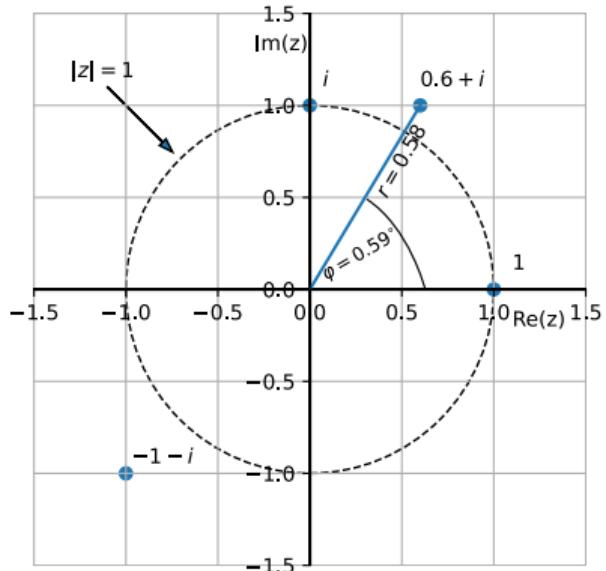
Basic operations

Multiplication

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$$

Modulus

$$|z| = r$$





Basic definitions

$$a, b, c, d \in \mathbb{C}, M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$M^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, M^* = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix},$$

$$M^\dagger = (M^T)^* = (M^*)^T = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix},$$

$$(M_1 \cdot M_2)_{ij} = \sum_k (M_1)_{ik} \cdot (M_1)_{kj}$$

In general

$$M_1 \cdot M_2 \neq M_2 \cdot M_1$$

$$[M_1, M_2] = M_1 \cdot M_2 - M_2 \cdot M_1$$

Hermitian matrix

- $H = H^\dagger$
- Generalization of symmetric matrices
- All eigenvalues real

Unitary matrix

- $U \cdot U^\dagger = \mathbb{I}$
- $U^{-1} = U^\dagger$
- Generalization of orthogonal matrices
- $|\det(U)| = 1, \det(U) = e^{i\varphi}$



Postulate I

In quantum mechanics the state of a physical system is represented by a vector in a Hilbert space \mathcal{H} : a complex vector space with an inner product.

(This course - only finite dimensional Hilbert spaces)

Dirac notation

Vector: 'ket' $|\psi\rangle$ Dual vector: 'bra' $\langle\varphi|$

Customarily, for $d = \dim(\mathcal{H})$:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad d, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, |d-1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\langle 0| = [1 \ 0 \ \dots \ 0], \quad \langle\psi| = (|\psi\rangle)^\dagger$$



Hilbert space

A Hilbert space \mathcal{H} is a complex inner product space.

- Linear combination (superposition):

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle, \quad |\varphi\rangle = \sum_{i=0}^{d-1} \beta_i |i\rangle,$$

- Inner product (bra-ket):

$$\langle i|j\rangle = \delta_{ij}$$

$$\langle\psi|\varphi\rangle = \sum_{ij} \alpha_i^* \beta_j \langle i|j\rangle = \sum_i \alpha_i^* \beta_i$$

- Assumption of normalization:

$$\|\psi\|^2 = \langle\psi|\psi\rangle = 1$$

State examples

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{d-1} \end{bmatrix}$$

$$|\varphi_1\rangle = \frac{1}{\sqrt{30}} (|0\rangle + 2i|1\rangle + 3|2\rangle + 4i|3\rangle)$$

$$|\varphi_2\rangle = \frac{1}{2} ((1+i)|0\rangle + i|1\rangle + |2\rangle)$$

$$|\varphi_1\rangle = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2i \\ 3 \\ 4i \end{bmatrix}, \quad |\varphi_2\rangle = \frac{1}{2} \begin{bmatrix} 1+i \\ i \\ 1 \\ 0 \end{bmatrix}$$

$$\langle\varphi_1|\varphi_2\rangle = \frac{1}{2\sqrt{30}} ((1-i) + 2 + 3 + 0) = \frac{6-i}{2\sqrt{30}}$$



'Matrix' multiplication

$$\begin{aligned}\langle \varphi_1 | \varphi_2 \rangle &= (|\varphi_1\rangle)^\dagger |\varphi_2\rangle = \\ &= \frac{1}{\sqrt{30}} [1, -2i, 3, -4i] \cdot \frac{1}{2} \begin{bmatrix} 1+i \\ i \\ 1 \\ 0 \end{bmatrix} = \frac{6+i}{2\sqrt{30}}\end{aligned}$$

Exercise 1

1. Find N , such that the state

$$|\varphi\rangle = \frac{1}{N} (|0\rangle + 5i|1\rangle)$$

is normalized, $\|\varphi\|^2 = \langle \varphi | \varphi \rangle = 1$.

2. Is N unique?



Exercise 1 part 1

1. Find N , such that the state

$$|\varphi\rangle = \frac{1}{N} (|0\rangle + 5i|1\rangle)$$

is normalized, $\|\varphi\|^2 = \langle \varphi | \varphi \rangle = 1$.

Answers: a) $\sqrt{26}$, b) $17i$, c) 6 , d) 36

$$\langle \varphi | \varphi \rangle = \frac{1}{N^*} (\langle 0 | - 5i \langle 1 |) \cdot \frac{1}{N} (|0\rangle + 5i|1\rangle) = \frac{26}{|N|^2} = 1 \Rightarrow |N| = \sqrt{26}$$

Exercise 1 part 2

2. Is N unique?

Answers: a) Yes, b) No

No, we only know the modulus of N , but the phase is arbitrary, $N = \sqrt{26}e^{i\alpha}$.
Customarily, we choose N real, $N = \sqrt{26}$.



Qubit

Two state system:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

From normalization condition $|\alpha|^2 + |\beta|^2 = 1$

Examples:

$$|0\rangle, \alpha = 1, \beta = 0$$

$$|1\rangle, \alpha = 0, \beta = 1$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

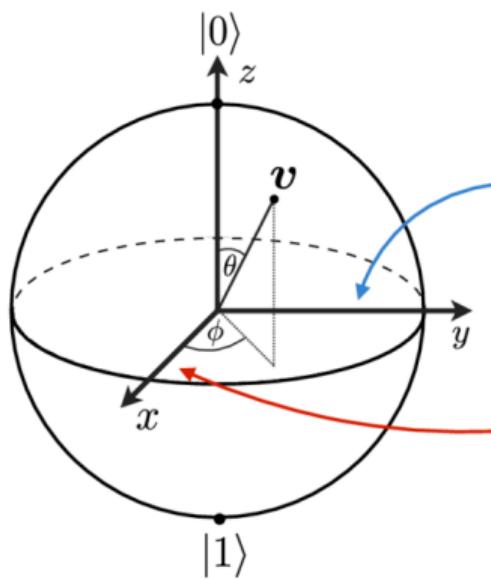
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|i+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|i-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

State on the Bloch sphere

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$$



Pole states:

$$|i+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|i-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$



Tensor product

Hilbert space of a composite system is the tensor product of the Hilbert spaces for the subsystems.

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \dim(\mathcal{H}) = \dim(\mathcal{H}_1) \cdot \dim(\mathcal{H}_2)$$

Customarily:

$$|i\rangle \otimes |j\rangle = |i\rangle |j\rangle = |i\ j\rangle$$

For qubits:

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle, |\varphi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$$

$$|\psi\rangle \otimes |\varphi\rangle = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

$$|\psi\rangle \otimes |\varphi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$$



Entangled states

States of the form $|\psi\rangle \otimes |\varphi\rangle$ are product states, all others are entangled states.

Consider:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Interpretation:

A product state $|\psi\rangle \otimes |\varphi\rangle$ has the meaning that system ψ has the property $|\psi\rangle$ and system φ the property $|\varphi\rangle$. For an entangled state one typically cannot assign definite properties to the individual systems ψ and φ .



Exercise 2

Which of the following states is entangled?

$$|\psi\rangle = |00\rangle - |01\rangle - |10\rangle + |11\rangle, \quad |\phi\rangle = |00\rangle + |01\rangle - |10\rangle + |11\rangle$$

- a) Both are entangled,
- b) Only $|\psi\rangle$,
- c) Only $|\phi\rangle$,
- d) None.



Exercise 2

Which of the following states is entangled?

$$|\psi\rangle = |00\rangle - |01\rangle - |10\rangle + |11\rangle, \quad |\phi\rangle = |00\rangle + |01\rangle - |10\rangle + |11\rangle$$

- a) Both are entangled,
- b) Only $|\psi\rangle$,
- c) Only $|\phi\rangle$,
- d) None.

Solution

Remember the formula $(a - b)^2 = a^2 - ab - ba + b^2$? Mixed terms are negative, hence
 $|\psi\rangle = (|0\rangle - |1\rangle) \otimes ((|0\rangle - |1\rangle))$ - not entangled.

We would expect that there will be even negative terms if all basis states are present.

$|\phi\rangle = |00\rangle + |01\rangle - |10\rangle + |11\rangle$ - entangled.

Changing the relative phase leads to different states. Only global phase does not matter!

**Postulate IIa**

Every measurable physical quantity \mathcal{A} is described by a Hermitian matrix A acting in the state space \mathcal{H} .

The result of measuring a physical quantity \mathcal{A} must be one of the eigenvalues of the corresponding matrix A .

Postulate IIb

When the physical quantity \mathcal{A} is measured on a system in a normalized state $|\psi\rangle$, the probability of obtaining an eigenvalue (denoted a_n) of the corresponding observable A is given by the amplitude squared of the appropriate state (projection onto corresponding eigenvector).

$$Pr[a_n] = |\langle a_n | \psi \rangle|^2,$$

where $A|a_n\rangle = a_n|a_n\rangle$.



Mathematical modeling $\xrightarrow{\text{measurement}}$ Real world



Postulate II in quantum computation

In quantum computing, we typically work in the computational basis, which consists of the orthonormal states:

$$|\underbrace{0 \dots 00}_n\rangle, |0 \dots 01\rangle, |0 \dots 10\rangle, \dots, |1 \dots 11\rangle$$

for an n -qubit system.

A measurement in the computational basis is a projective measurement where the projectors are:

$$P_k = |k\rangle\langle k|, \quad \text{for } k \in \{0 \dots 00, 0 \dots 01, \dots, 1 \dots 11\}$$

Immediately after the measurement, the system collapses (with probability $\langle\psi|P_k|\psi\rangle = |\langle k|\psi\rangle|^2$) into the (normalized) post-measurement state:

$$|\psi_k\rangle = \frac{P_k|\psi\rangle}{\sqrt{\langle\psi|P_k|\psi\rangle}}.$$



Postulates of quantum mechanics: Postulate II

Hermitian matrices as observables

Hermitian matrices have real eigenvalues.
Although we work with complex numbers,
the measurement outcome is real, like in
classical physics.

Pauli matrices and the computational basis

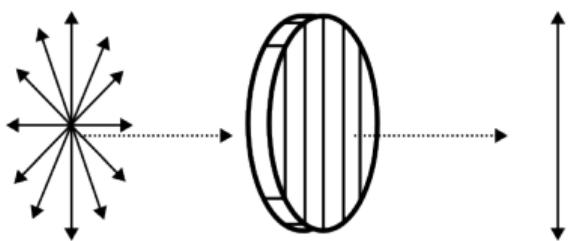
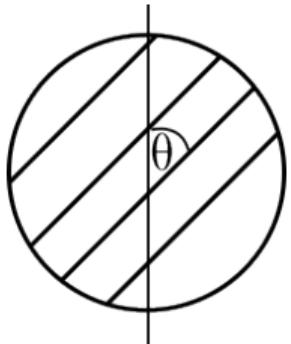
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Z|0\rangle = 1|0\rangle, Z|1\rangle = -1|1\rangle$$

$$X|0\rangle = |1\rangle, X|1\rangle = |0\rangle$$

$$X|+\rangle = 1|+\rangle, X|-\rangle = -1|-\rangle$$

$$Y|i+\rangle = 1|i+\rangle, Y|i-\rangle = -1|i-\rangle$$

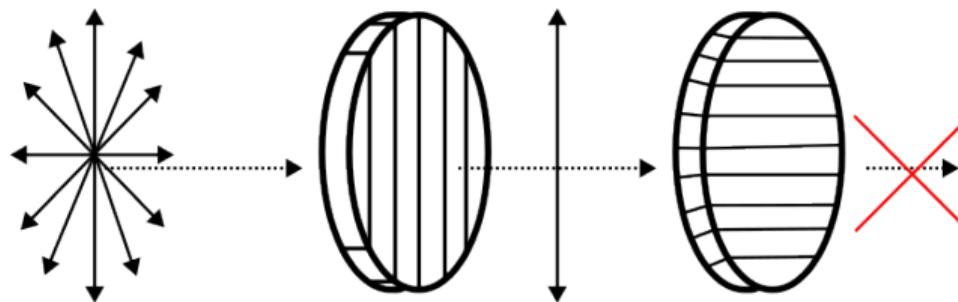


- Two orthogonal polarizations of light, vertical and horizontal, introduce a basis: $|v\rangle, |h\rangle$
- Two dimensional Hilbert space: $\mathcal{H} = \text{span}(|v\rangle, |h\rangle)$, $\langle v|h \rangle = 0$
- Any polarization photon state: $|\theta\rangle = \cos(\theta)|v\rangle + \sin(\theta)|h\rangle$
- The action of the polarizer:
 - ◊ The intensity of light after a polarizer is proportional to the probability that the photon had a correct polarization

$$Pr[v] = |\langle v|\theta \rangle|^2, Pr[h] = |\langle h|\theta \rangle|^2$$

For unpolarized light (Malus' law 1809)

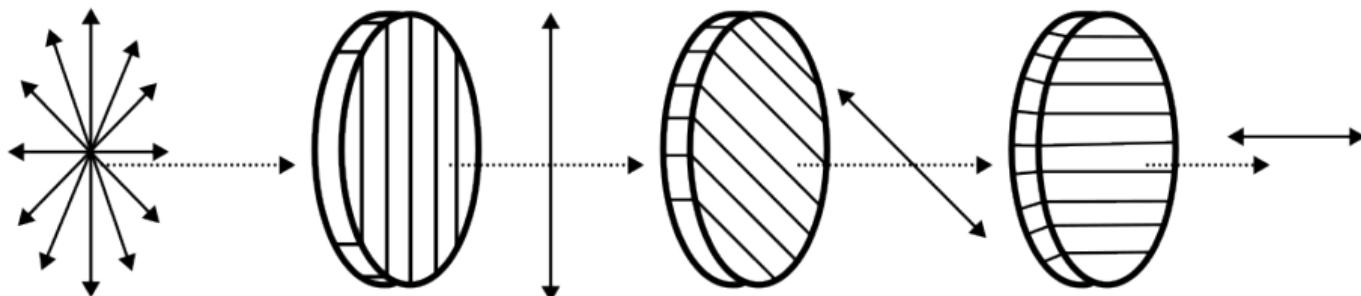
$$\diamond Pr[v|unpolarized] = \int |\langle v|\theta \rangle|^2 d\theta = \int \cos^2(\theta) d\theta = \frac{1}{2}$$



1. Checking the polarization filters: apply two filters orthogonally.
2. After the first polarizer the light is in state $|v\rangle$
3. The second polarizer projects the state on the horizontal orientation eigenvector $|h\rangle$,

$$Pr[h|v] = |\langle h|v \rangle|^2 = 0$$

The light vanishes in this setup.



1. Insert tilted polarizer ($\theta = 45^\circ$),

$$|45^\circ\rangle = \frac{1}{\sqrt{2}}|\nu\rangle + \frac{1}{\sqrt{2}}|h\rangle$$

$$+ | - 45^\circ\rangle = \frac{1}{\sqrt{2}}|\nu\rangle - \frac{1}{\sqrt{2}}|h\rangle$$

$$|\nu\rangle = \frac{1}{\sqrt{2}}|45^\circ\rangle + \frac{1}{\sqrt{2}}| - 45^\circ\rangle$$

2. After the first polarizer we have state $|\nu\rangle$, after the second

$$Pr[45^\circ|\nu] = |\langle 45^\circ|\nu\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2$$

3. After the final polarizer

$$Pr[h|45^\circ] = |\langle h|45^\circ\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2$$

4. The intensity in the whole process

$$Pr[v|unpolarized]Pr[45^\circ|\nu]Pr[h|45^\circ] = \frac{1}{8}$$



Postulates of quantum mechanics: Postulate III

Postulate III

The time evolution of a closed system is described by a unitary transformation on the initial state.

$$|\psi(t)\rangle = U(t; t_0)|\psi(t_0)\rangle$$

High-level evolution in QC

In quantum computing, unitary operations will be called gates.

Time evolution is discretized - single time step is the execution of a single (layer) of gate(s).



Exercise 3

Assume that we have a unitary U_{cl} such that it is able to copy an arbitrary one-qubit state to the second register

$$U_{cl}(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$

What is the result of the action of U_{cl} on the state $(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle$?

Answers:

- a) $\alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$
- b) $\alpha|00\rangle + \beta|11\rangle$



Solution?

Both answers seem to be correct . . .

- First apply U_{cl} , then multiply

$$U_{cl}((\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle) = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) = \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$$

- First multiply, then apply U_{cl}

$$U_{cl}((\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle) = U_{cl}(\alpha|00\rangle + \beta|10\rangle) = U_{cl}(\alpha|00\rangle) + U_{cl}(\beta|01\rangle) = \alpha|00\rangle + \beta|11\rangle$$



No-cloning theorem

Solution?

Both answers seem to be correct . . .

- First apply U_{cl} , then multiply

$$U_{cl}((\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle) = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) = \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$$

- First multiply, then apply U_{cl}

$$U_{cl}((\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle) = U_{cl}(\alpha|00\rangle + \beta|10\rangle) = U_{cl}(\alpha|00\rangle) + U_{cl}(\beta|01\rangle) = \alpha|00\rangle + \beta|11\rangle$$

No-cloning theorem

There is no unitary operator U on $\mathcal{H} \otimes \mathcal{H}$ such that for all normalized states $|\phi\rangle_A$ and $|e\rangle_B$ in \mathcal{H}

$$U(|\phi\rangle_A|e\rangle_B) = e^{i\alpha(\phi,e)}|\phi\rangle_A|\phi\rangle_B$$

for some real number α depending on ϕ and e .

Matrix representation in a basis

Operation from $|i\rangle$ to $|j\rangle$

$$U_{ji} = |j\rangle\langle i|$$

Identity gate

Before	After
$ 0\rangle$	$ 0\rangle$
$ 1\rangle$	$ 1\rangle$

I

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$I = |0\rangle\langle 0| + |1\rangle\langle 1| =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix representation in a basisOperation from $|i\rangle$ to $|j\rangle$

$$U_{ji} = |j\rangle\langle i|$$

NOT gate

Before	After
$ 0\rangle$	$ 1\rangle$
$ 1\rangle$	$ 0\rangle$



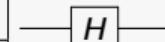
$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X = |1\rangle\langle 0| + |0\rangle\langle 1| =$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hadamard gate

Before	After
$ 0\rangle$	$\frac{1}{\sqrt{2}}(0\rangle + 1\rangle)$
$ 1\rangle$	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$



$$H = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Hermitian and Unitary

$$H \cdot H^\dagger = H \cdot H = \mathbb{I}$$

- Changes basis

$$HXH = Z$$

- Hadamard transform

$$H^{\otimes n}(|0\rangle)^{\otimes n} = H \otimes \cdots \otimes H |0 \dots 0\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle$$



Matrix representation in a basis

Operation from $|i\rangle$ to $|j\rangle$

$$U_{ji} = |j\rangle\langle i|$$

Phase gate - Exercise 4

Before	After
$ 0\rangle$	$ 0\rangle$
$ 1\rangle$	$i 1\rangle$



What is the matrix representation of S ?

- a) $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, c) $\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$, d) $\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$



Matrix representation in a basis

Operation from $|i\rangle$ to $|j\rangle$

$$U_{ji} = |j\rangle\langle i|$$

Phase gate - Exercise

Before	After
$ 0\rangle$	$ 0\rangle$
$ 1\rangle$	$i 1\rangle$



What is the matrix representation of S ?

$$b) \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Gates as an exponentiation of gates

We (theoretically) can perform arbitrary continuously parameterized gates

$$R_X(\theta) = e^{-iX \frac{\theta}{2}}$$

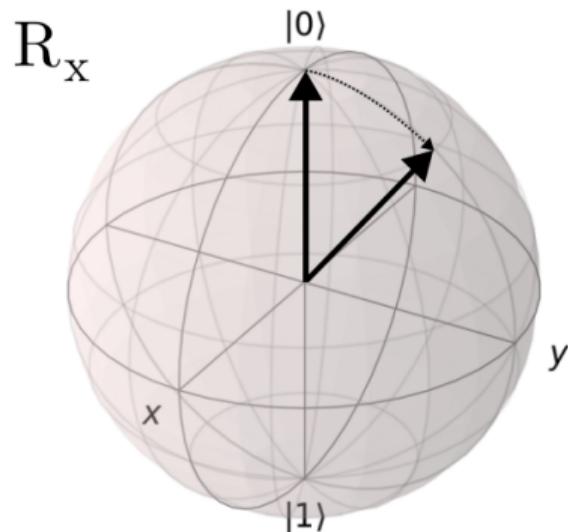
$$R_X(\theta) = \begin{bmatrix} \cos(\theta/2) & -i \sin(\theta/2) \\ -i \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

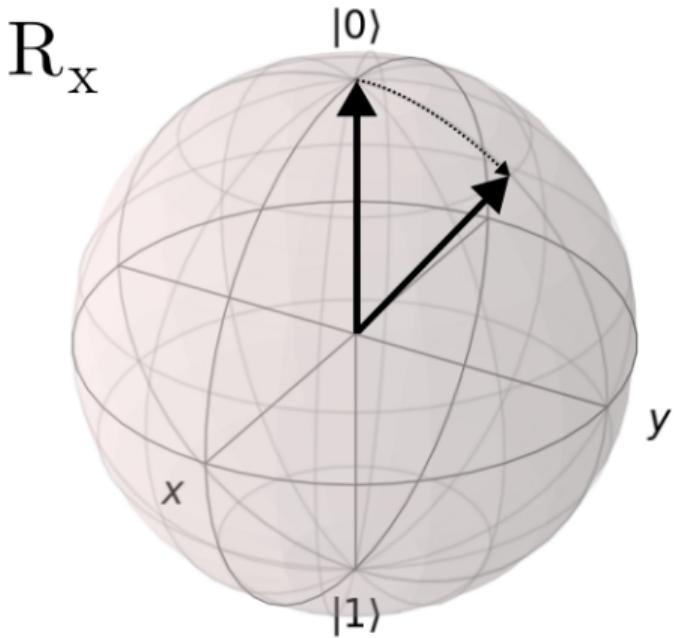
$$R_Y(\theta) = e^{-iY \frac{\theta}{2}}$$

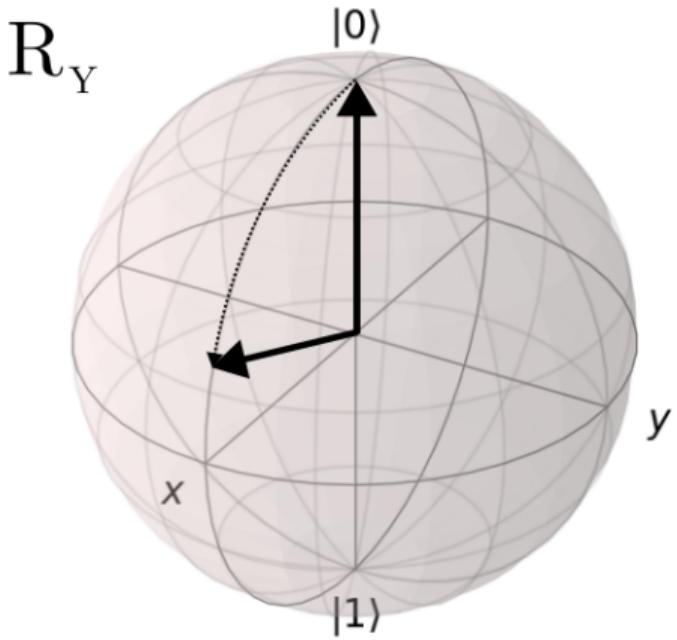
$$R_Y(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

$$R_Z(\theta) = e^{-iZ \frac{\theta}{2}}$$

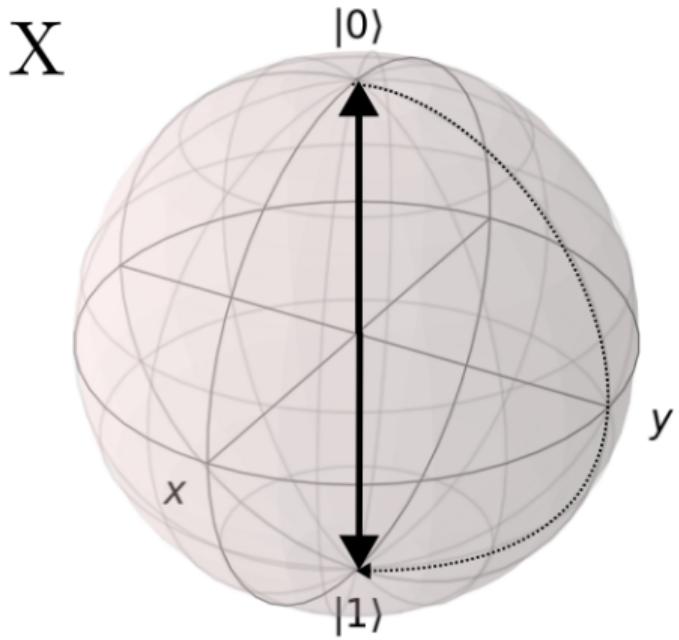
$$R_Z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$



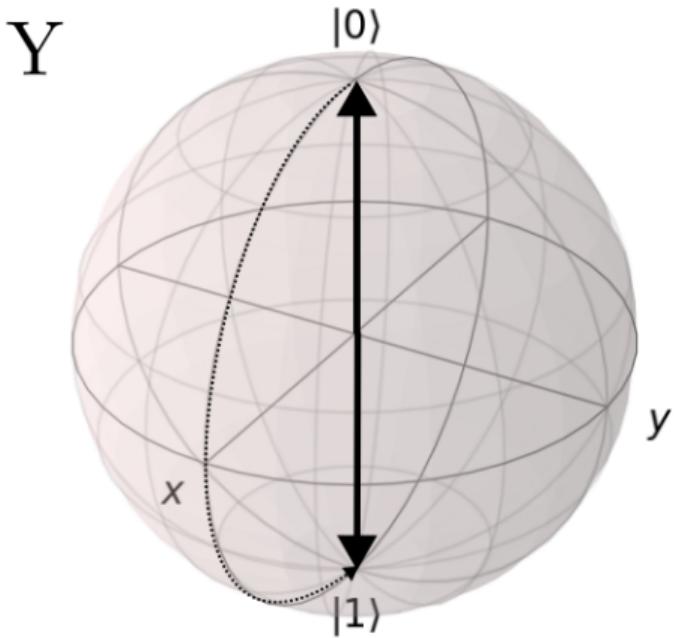




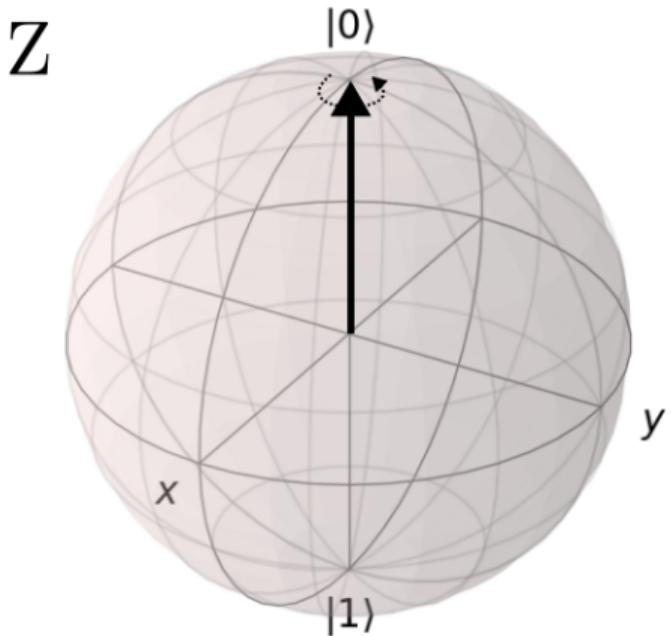
Quantum computing as unitary rotations

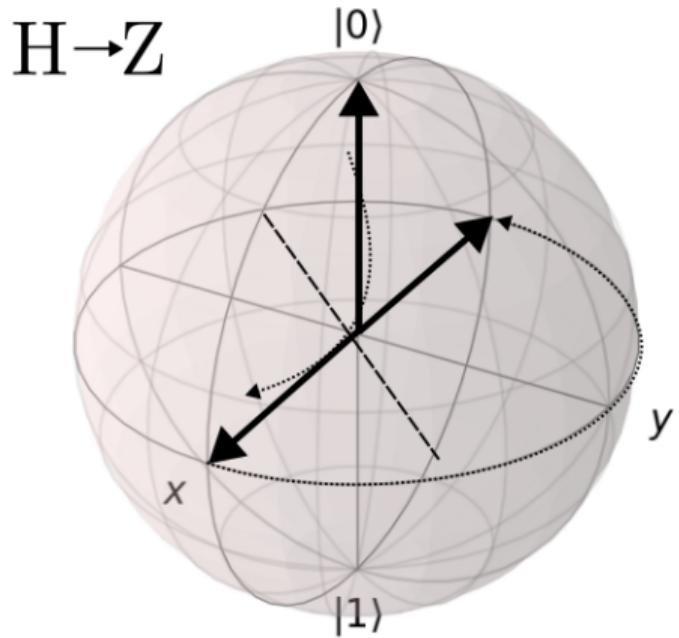


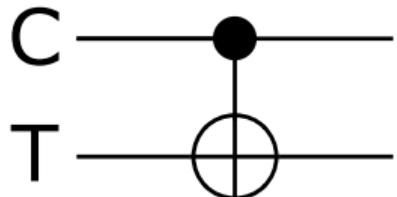
Quantum computing as unitary rotations



Quantum computing as unitary rotations







CNOT truth table

Before		After	
C	T	C	T
$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$
$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$
$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$
$ 1\rangle$	$ 1\rangle$	$ 1\rangle$	$ 0\rangle$

Matrix representation

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

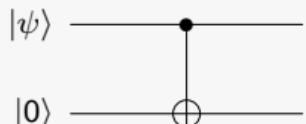
$$CNOT = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11| =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

CNOT gate - action on an arbitrary quantum state

Arbitrary state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



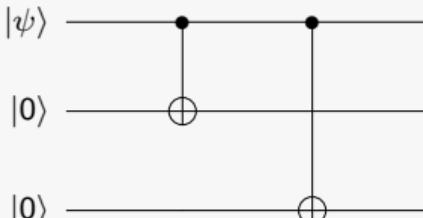
$$\begin{aligned} CNOT [(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle] &= \\ &= CNOT(\alpha|00\rangle + \beta|10\rangle) = \\ &= \alpha|00\rangle + \beta|11\rangle \end{aligned}$$

For $\alpha = \frac{1}{\sqrt{2}}$, $\beta = \frac{1}{\sqrt{2}}$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Even more qubits

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle$$

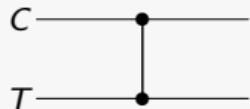


$$\alpha|000\rangle + \beta|111\rangle$$

Error correction - repetition codes

Controlled Z gate

diagram



CNOT truth table

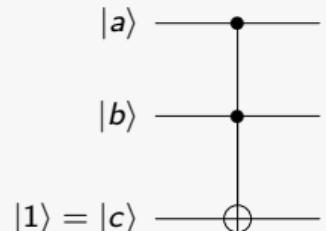
Before		After	
C	T	C	T
$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$
$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$
$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$
$ 1\rangle$	$ 1\rangle$	$ 1\rangle$	$- 1\rangle$

Matrix representation

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

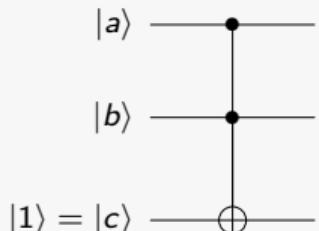
$$CZ = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 11| - |11\rangle\langle 11| =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

**Toffoli gate and ancilla****Truth table**

Before			After		
$ a\rangle$	$ b\rangle$	$ c\rangle$	$ a\rangle$	$ b\rangle$	$ c\rangle$
$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 0\rangle$	$ 1\rangle$
$ 0\rangle$	$ 1\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$
$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$
$ 1\rangle$	$ 0\rangle$				

Toffoli gate and ancilla



Truth table

Before		After	
$ a\rangle$	$ b\rangle$		$ c\rangle$
$ 0\rangle$	$ 0\rangle$		$ 1\rangle$
$ 0\rangle$	$ 1\rangle$		$ 1\rangle$
$ 1\rangle$	$ 0\rangle$		$ 1\rangle$
$ 1\rangle$	$ 1\rangle$		$ 0\rangle$

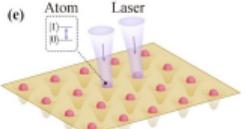
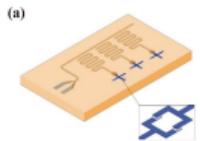
Universality argument $Q \rightarrow C$

- Quantum computations are inherently reversible.
- The classical NAND gate is irreversible but universal for classical computation.
- To simulate NAND, we introduce additional qubit(s) called *ancilla*.
- We apply a reversible quantum circuit (e.g., the Toffoli gate) to perform the computation.
- The overall quantum operation is reversible.
- By discarding (ignoring) some qubits, we obtain an effective irreversible operation on a subsystem.
- This allows us to simulate a NAND gate within a reversible quantum system.
- Since NAND is universal, quantum computers can simulate any classical computation.

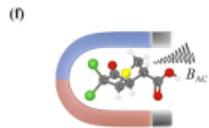
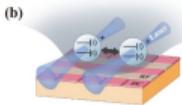


- Quantum computation is described using the formalism of quantum mechanics: states are vectors in a Hilbert space and transformations are unitary operators.
- The outcome of a quantum computation is obtained by measurement; intermediate quantum states are not directly observable.
- Measurement collapses a quantum state probabilistically according to the Born rule, giving classical outcomes based on the quantum amplitudes.
- Quantum systems evolve in exponentially large Hilbert spaces due to the tensor product structure of qubits, enabling potentially more expressive computation.
- Quantum computations proceed via unitary transformations (quantum gates) applied to qubits.
- Unitarity of quantum gates implies that all quantum computations are inherently reversible.
- Any classical (irreversible) computation can be implemented reversibly on a quantum computer by using additional ancilla qubits and reversible gate constructions.

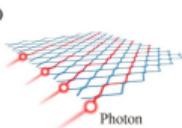
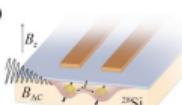
Quantum technologies



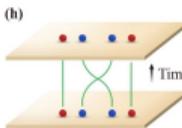
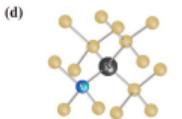
(a) Superconducting quantum processor:
An array of artificial atoms made of Josephson junctions and capacitors, fabricated with lithography, controlled with microwave electronics
Number of qubits with individual control:
~100
Gate fidelities:
Systems with 53 qubits: $F_1 > 99\%$; $F_2 > 99\%$
Pros:
Fast gate speed, tailored qubits, high controllability and scalability
Cons:
Crosstalk between qubits, low temperature, supporting technology for scaling up



(b) Trapped ion quantum computing:
Laser-cooled atomic ions held by radio-frequency electric fields in ultra-high vacuum environment
Number of qubits with individual control:
20–30
Gate fidelities:
Systems with dozens of ions: $F_1 = 99.9\%$; $F_2 > 99\%$
Pros:
Extremely long coherence time and excellent gate operations on few ions;
Reconfigurable connectivity between ion qubits
Cons:
Technically challenging in integration



(c) Semiconductor spin-based quantum computing:
Electron or hole spins in semiconductor (silicon) quantum dot
Number of qubits with individual control:
6
Gate fidelities:
Donor spin qubit: $F_1 = 99.99\%$; $F_2 = 99.5\%$
Gate-defined qubit: $F_1 = 99.9\%$; $F_2 = 99.51\%$
Pros:
Semiconductor fabrication
Long coherence and fast high-fidelity gate
Work at temperature > 1 K
Small footprint
Cons:
Challenge of nanoscale fabrication



(d) NV center quantum computing:
Point defects in diamond; electron and nuclear spins with long coherence time; atom-like properties and solid-state host environment
Number of qubits with individual control:
10 (one electron + nine ^{13}C nuclear spins)
Gate fidelities:
 $F_1 = 99.995\%$; $F_2 = 99.2\%$
Pros:
Work at room temperature
Excellent quantum sensor
Handy in quantum network
Cons:
Hard to scale up

(e) Neutral atom array quantum computing:
Neutral atom arrays trapped in optical tweezers with controlled interactions based on the Rydberg interactions
Number of qubits in digital quantum processors: 24
Number of qubits in analog quantum simulators: 289
Pros:
Both digital and analog quantum simulations
Scaling up beyond 100 qubits in programmable geometries
Cons:
Improving fidelities of two-qubit gates

(f) NMR quantum computing:
Nuclear spins in molecule
Number of qubits with individual control:
12
Gate fidelities:
 $F_1 = 99.98\%$; $F_2 \sim 99.3\%$
Pros:
Work at room temperature
Long coherence time
Digital quantum simulator
Cons:
Hard to scale up

(g) Photonic quantum computing:
Coherently manipulating a large number of single photons or canonically conjugate pair of variables for electromagnetic modes to process quantum information
Number of qubits with individual control:
18
Number of photons with coherent control:
255 (Jinzheng 3.0 to be published)
Gate fidelities:
 $F_1 = 99.84\%$; $F_2 = 99.69\%$
Pros:
Robust against decoherence, working at room temperature, compatible with CMOS fabrications, natural interface for distributed quantum computing
Cons:
Big challenge to realize deterministic photon-photon gate

(h) Topological quantum computing:
Fault-tolerant quantum computation based on non-abelian braiding of anyons
Number of qubits with individual control:
N/A
Gate fidelities:
N/A
Pros:
Intrinsic topological protection
Few physical qubits to construct a logic qubit
Promising to achieve large-scale, error-corrected computation
Cons:
The ideal materials or systems not found yet.
Zero topological qubit so far



Technical details of different technologies

Parameters	SC	T.ions	Photonic	N.atoms	S.spin	NV	CPUs
Clock cycle	1MHz	1KHz	10Hz	1MHz	0.76MHz	1MHz	3GHz
Measurement	660ns	300μs	x	200ms	1.3μs	x	x
2-qubit gate	34ns	200μs	x	< 100μs	x	700ns	x
1-qubit gate	25ns	15μs	x	x	x	9ns	x
Readout fidelity	99.4%	97.3%	50.0%	99.1%	99%	98%	x
1Q fidelity	99.99%	99.99%	99.84%	99.83%	99.99%	99.99%	x
2Q fidelity	99.97%	99.9%	99.69%	99.4%	99.5%	99.2%	x

Adiabatic quantum computing

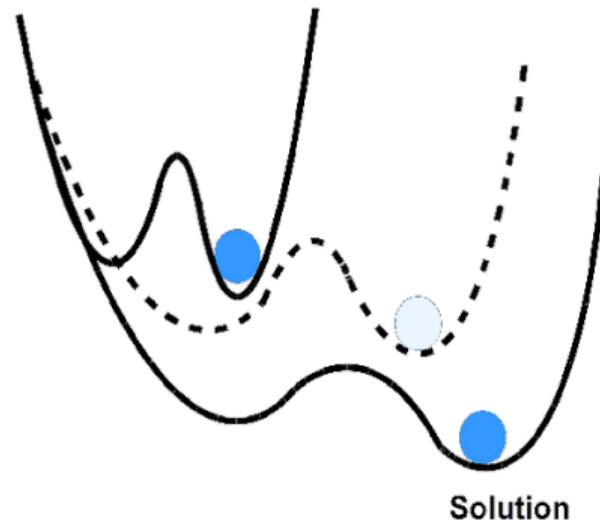
Adiabatic theorem

A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum.

$$H(t) = \left(1 - \frac{t}{T}\right) H_0 + \frac{t}{T} H_1, T = \mathcal{O}\left(\frac{1}{\Delta_{min}^2}\right)$$

$$H(t) = - \sum_{i,j} J_{ij} \sigma_i \sigma_j - \mu \sum_i h_i \sigma_i, \sigma_i = \{-1, 1\}$$

Easy implementation for Quadratic unconstrained binary optimization (QUBO) problems.



Adiabatic evolution

How to access quantum computing devices?

