Issues with sequence correction for Gaussian probability

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Let K^{-1} be the prior precision, m be the prior mean, c_i is the ith standard basis vector. We approximate the truncated normal $p(x) = N(m, K)1\{a \leq Ax \leq b\}$ with $q(x) = N(x; \mu, \Sigma)$. We work in the natural parameterization where $\eta = \Sigma^{-1}\mu$ and $\Lambda = \Sigma^{-1}$ because we have the prior precision when we use the sequence to convert polygonal problems to axis aligned problems.

Directly converting the cavity distribution of Cunningham et al 2011 to natural parameterization is problematic because

$$\frac{1}{\sigma^2_{\setminus i}} = \tau_{\setminus i} = \frac{1}{c_i^T \Sigma c_i} - \frac{1}{\tilde{\sigma}_i^2} \neq c_i^T \Lambda c_i - \frac{1}{\tilde{\sigma}_i^2}$$

That is, we can't just access the diagonal of the approximate precision Λ to compute the precision of the ith cavity distribution, we need to get the diagonal of its inverse, Σ.

Cunningham et al denote the cavity distribution from removing the ith factor as $q^{\setminus i}(x) = N(x, u^{\setminus i}, V^{\setminus i})$, and gives their forms in equations (30) and (31):

$$V^{\setminus i} = \left(\Sigma^{-1} - \frac{1}{\tilde{\sigma}_i^2} c_i c_i^T\right)^{-1}$$

$$u^{\setminus i} = V^{\setminus i}\left(\Sigma^{-1}\mu - \frac{\tilde{\mu}_i}{\tilde{\sigma}_i^2} c_i\right)$$

Therefore the natural parameters are

$$R^{\setminus i} = V^{\setminus i^{-1}} = \Sigma^{-1} - \frac{1}{\tilde{\sigma}_i^2} c_i c_i^T$$

$$e^{\setminus i} = (V^{\setminus i})^{-1} u^{\setminus i} = \Sigma^{-1}\mu - \frac{\tilde{\mu}_i}{\tilde{\sigma}_i^2} c_i$$

As in Cunningham et al (33), define $A \in \mathbb{R}^{n \times n}$ as

$$A = \left[c_i, a_2, \ldots, a_n\right]$$

Then we marginalize over the

$$
\begin{aligned}
q_{\backslash i}(c_i^T x) = q_{\backslash}(x_i) &= \int_{\backslash c_i; x} q^{\backslash i}(x') dx' \\
&= \int_{\backslash c_i; x} Z^{\backslash i} N(x'; e^{\backslash i}, R^{\backslash i}) \\
&= \int_{\backslash c_i; x} Z^{\backslash i} \exp\left[{e^{\backslash i}}^T x' - \frac{1}{2} {x'}^T R^{\backslash i} x' \right] dx' \\
&= \int_{\backslash e_1; (c^T x) e_1} Z^{\backslash i} \exp\left[{e^{\backslash i}}^T A y - \frac{1}{2} y^T A^T R^{\backslash i} A y \right] dy, \quad \text{since } y = A^T x' \text{ and } \mid dy/dx \mid = \mid A \mid = 1 \\
&= \int_{\backslash e_1; (c^T x) e_1} Z^{\backslash i} N(y; A^T e^{\backslash i}, A^T R^{\backslash i} A) dy \\
&= Z^{\backslash i} N(c_i^T x; c_i^T e^{\backslash i}, c_i^T R^{\backslash i} c_i)
\end{aligned}
$$

These cavity parameters are

$$
\begin{aligned}
c_i^T R^{\backslash i} c_i &= c_i^T \left(\Sigma^{-1} - \frac{1}{\tilde{\sigma}_i^2} c_i c_i^T \right) c_i \\
&= c_i^T \Sigma^{-1} c_i - \frac{1}{\tilde{\sigma}_i^2} c_i^T c_i c_i^T c_i \\
&= \Lambda_{ii} - \frac{1}{\tilde{\sigma}_i^2} \\
c_i^T e^{\backslash i} &= c_i^T \left(\Sigma^{-1} \mu - \frac{\tilde{\mu}_i}{\tilde{\sigma}_i^2} c_i \right) \\
&= c_i^T \Sigma^{-1} \mu - \frac{\tilde{\mu}_i}{\tilde{\sigma}_i^2} c_i^T c_i \\
&= \eta_i - \frac{\tilde{\mu}_i}{\tilde{\sigma}_i^2}
\end{aligned}
$$

But this does not match the moment parameterization of the cavity updates. Something has gone wrong in this derivation.