Statistical Inference with Simulation

STAT 211 - 509

2018-10-08

Statistical Inference

- There is a population we wish to study. We take a sample of data, and we want to use it learn about the population.
- We model the observed sample data as outcomes of random variables that represent the population. We want to know the parameters of these random variables, the population parameters.
- **Population**: the entire group of interest.
- **Sample**: a part of the population selected to draw conclusions about the population.

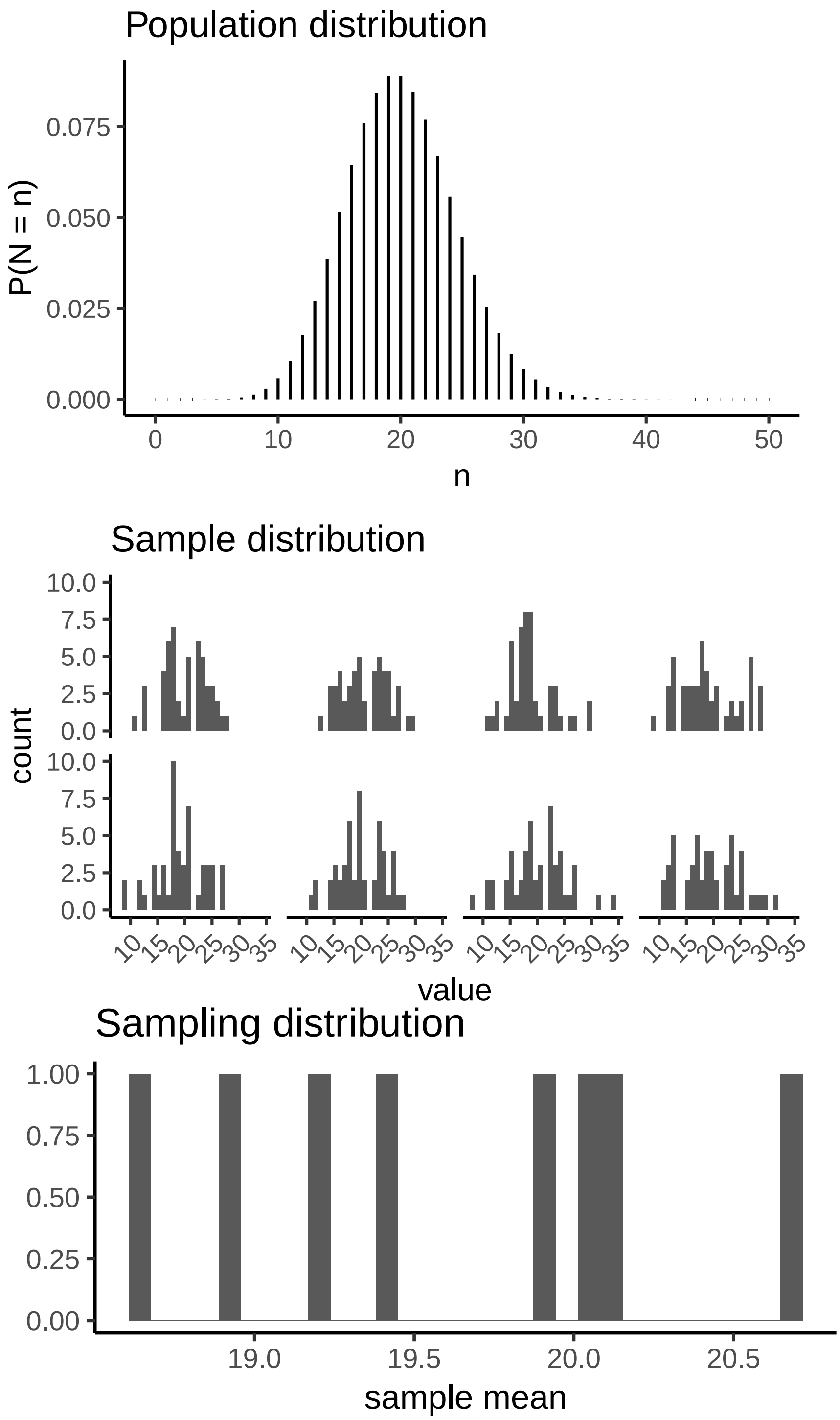
3 problems in statistical inference:

- **Point estimation** : single estimate of the parameter of interest
  - Maximum likelihood estimation
- **Confidence interval** : a range of "plausible" values for the parameter of interest, at a stated level of "confidence"
- **Hypothesis test** : a formal decision about the value of the parameter of interest, again at a stated level of "confidence"

Sampling Distribution

- We get model the data, $X_1, \ldots, X_n$, as independently following some distribution.
- Compute a sample **statistic** to estimate the parameter of interest.
  - Mean: $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$
  - Standard deviation: $\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu})^2}$
- In general, a statistic is a function $g(X_1, \ldots, X_n)$. This is a function of random variables, so it too is a random variable.
  - So the statistic has a distribution, called its **sampling distribution**

Three distributions



Using the sampling distribution

- Confidence interval : use the sampling distribution to obtain an interval of statistic values with a specified probability, e.g. 0.95, of being observed.
- Hypothesis test : use the sampling distribution that would apply for a particular hypothesized parameter value to compute the probability of seeing data like those you saw; reject the hypothesized value if this probability, called a **p-value** , is sufficiently small, e.g. less than 0.05.

Computing / Approximating Sampling Distributions

- If we fully specify a probability model, we can often derive a sampling distribution exactly.
- Other times, we can approximate it using asymptotic (limiting) results, which requires that we have a "large" sample size n .
- Alternatively, if we can somehow *draw random samples from the sampling distribution*, we can use them to approximate the sampling distribution.
- Recall how we have used R, e.g. via the 'rbinom' function, to simulate from a distribution, enabling us to approximate probabilities as simple proportions.

The Bootstrap

- The **bootstrap** involves sampling with replacement from the observed data:

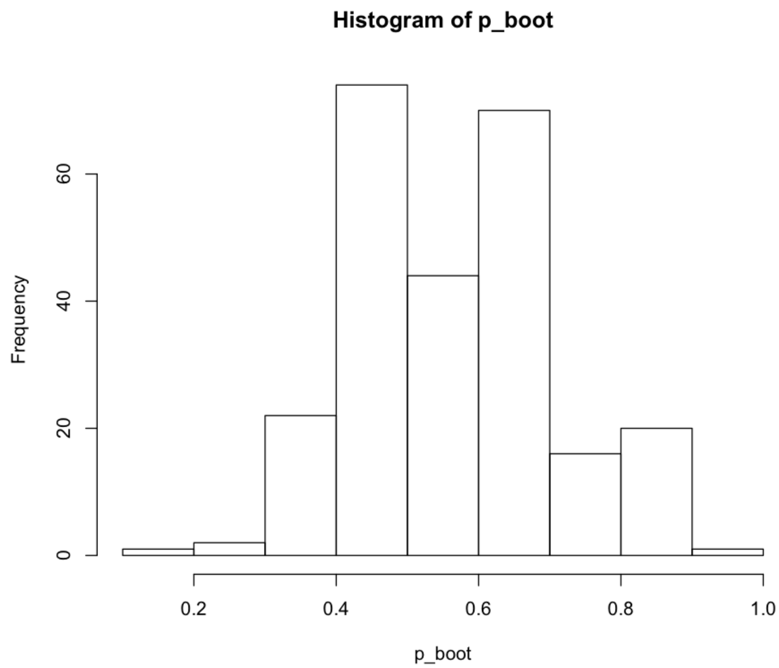
  Repeat B times:
  data_b = sample with replacement from data
  statistic_b = value of statistic for 'data_b'

- Now you have B simulated values of the statistic, and you can use them to approximate the sampling distribution, enabling both confidence intervals and hypothesis tests.

Example: NFL

Confidence Intervals

- A procedure for generating intervals such that if you repeated your experiment and constructed intervals many times, about $100(1-\alpha)\%$ of them will cover the unknown parameter value.



- Any one confidence interval either contains the parameter with probability 1 or otherwise it doesn't, contains with probability 0.
- α is the confidence level.

Confidence Intervals with the Bootstrap

- One way to construct an approximate $100(1-\alpha)\%$ confidence interval for a parameter (e.g., p in the Binomial case) is to:
  - Use the bootstrap to approximate the sampling distribution of $\hat{p}$.
  - Obtain the $\alpha/2$ and $1-\alpha/2$ percentiles of the bootstrapped sampling distribution.
  - For example, the interval from the 2.5th percentile to the 97.5th percentile of the bootstrapped sampling distribution of is an approximate 95% confidence interval for p.

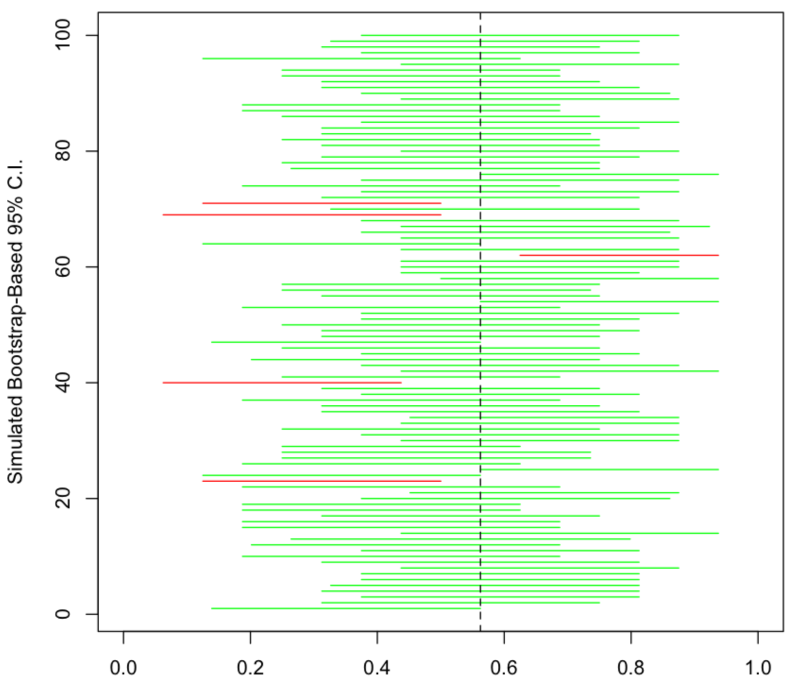
Interpretation of Confidence Intervals

- We expect of $100(1-\alpha)\%$ of all such confidence intervals to contain the parameter being
  estimated.

- That means we expect α% of such intervals to *not* contain the parameter being estimated.
- Once we observe data and use it to compute an actual confidence interval:
  - We can say we obtained our interval by using a technique that can be expected to 'cover' the true parameter value with probability .
  - We cannot say whether our particular interval covers the true parameter value or not.

Example: NFL

- See " nfl.r " and 'ch_5_1' lecture video.



Hypothesis Tests

- The **null hypothesis** (H_o) is what we choose to believe until shown sufficient evidence to the contrary.
- The **alternative hypothesis** (H_a) is what we will conclude if the null hypothesis is
  rejected.

- A **p-value** is the probability of seeing a statistic value like ours, or even more 'extreme', if *the null hypothesis is true*.
- To compute a p-value, we require the sampling distribution of our statistic when *the null hypothesis is true*.
- Potential errors:
  - Type I: Reject H_o when it is true.
  - Type II: Fail to reject H_o when it is false.

Hypothesis Tests with the Bootstrap

- One way to compute a p-value based on a statistic T, for which we observe the value T_o with our data:
  - Transform the data to force the null hypothesis to be true.
  - Use the bootstrap on the transformed data to approximate the sampling distribution of T under the null hypothesis.
  - Compute the proportion of simulated values of T that are 'as or more extreme' relative to T_o.
  - For example, in the Binomial case, 'as or more extreme' is as follows:
    * $H_o: \ p = p_o$ vs. $H_a: \ p > p_o$: 'as or more extreme' means $\geq T_o$.
    * $H_o: \ p = p_o$ vs. $H_a: \ p < p_o$: 'as or more extreme' means $\leq T_o$.
    * $H_o: \ p = p_o$ vs. $H_a: \ p \neq p_o$: : 'as or more extreme' means $\geq |T_o|$ in absolute value

Interpretation of Hypothesis Tests

- Based on our p-value and desired confidence level , we make a decision as follows:
  - If p-value $\leq \alpha$, reject H_o in favor of H_a.
  - Otherwise, "fail to reject" H_o.
- If H_o is true but we observe p-value $\leq \alpha$, we reject H_o, committing a **Type I error**.
- If H_a is true but we observe p-value $> \alpha$, we fail to reject H_o, committing a **Type II error**.
- If H_o is true, we expect to commit a Type I error no more than α% of the time.

- Once we observe data and use it to compute an actual p-value, making a decision based on whether it is :
  - We can say we made our decision based on a technique that can be expected to commit a Type I error no more than of α% of the time.
  - We cannot say whether we have committed a Type I error in this instance.