Modelling Floods With Differential Equations

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Abstract

The project is about studying how different forms of riverbeds can affect the formation of a flash flood. When we talk about the forms of a riverbed, we will consider various cross-sectional areas of the riverbeds, the incline of the riverbed and the textures of the riverbed. Such factors will enable us to establish a relationship between the variables and the formations of flash floods. The Mathematical models produced come from applying the theory behind conservation laws; The project uses analytical and numerical techniques to provide solutions for the models. In particular, the Method Of Characteristics and the Godunov Scheme. In conclusion, we find an established relationship between independent and dependent parameters, which influences the outcome of the models produced.

Keywords: Flash Floods, Riverbed, Volumetric Flow, Conservation Laws

1 Introduction

Flash floods are essential to study because of the mere danger it presents to us. In particular, areas with soil degradation, areas lacking rainfall, and areas with poor flood defence systems have to deal with regular flooding. We define a *flash flood* as a flood that occurs with 6hours of heavy rainfall (or any other cause) [1]. The cause of flash floods can be for many reasons (manufactured or environmental). Contributing factors include hefty rainfall from thunderstorms, Dam or Levee breaks, Debris Flow or even tsunamis.

Flash floods can occur anywhere on the Earth's surface, provided that critical conditions are satisfied. However, the problem arises because areas in the world are more likely to be hit by a flash flood. These tend to be areas near volcanoes, mountain ranges, rivers, fault lines and low lying areas (such as coastal areas and river flood plains) [3] . The human population tends to live in these areas as 70% of the population is about 5km from a water source.[7]Hence, they are more likely to experience a flash flood at some point.

Flash floods are expensive as they cause much damage to businesses, properties, and people; a flash flood could have substantial social, economic, and environmental negative impacts on a region. For example, People will have no choice but to leave their homes because their family home is damaged. Diseases may spread because of poor sanitation systems. Businesses may not return as they fear that there could be a flash flood in the near future. These problems will lead to more significant problems, which will force the government to spend more on the region for relief efforts.

An example of this is Hurricane Katrina in 2005. Where the total damage amounted to \$125bn (2005), 1800 lives were lost, and hundreds of thousands of people were displaced.[4]

The point emphasised here is that we need to study flash floods more to understand how to combat them in the future. If we do not find ways to put defences against them, we will face a problem that will escalate to an even bigger problem. Modelling Floods may not be the leading solution to the problem, but it helps contribute to understanding it. Hence, we as a society know how to address the problem and move forward to decrease the impact of a flash flood.

2 Methodology

In the report, we will apply the theory of conservation laws to derive an equation that will eventually get us to a point where we have an autonomous scalar conservation law - a partial differential equation. There are multiple ways to solve the equation, but we will focus on two methods. One is the method of characteristics (a numerical approach), and the other is the Godunov scheme (an analytical technique). We will also need to consider the shape of the river's cross-sectional areas and study the relationship between the CSA and the riverbed's perimeter; we do this via algebraic manipulation.In addition, we infer what the model means when applying it to the context when a flash flood happens. We want to show which conditions cause minor damage hence less expensive results and which ones do the converse. We show this in the analytics.

2.1 Conservation Laws

The following sections 2.1 and 2.1.1 is a summary from [5]

We start with the concept of a scalar conservation law in one space dimension, which we will be working on for our models. The general form of a conservation law comes as:

$$u_t + [f(u,x,t)]_x = 0 \tag{2.1}$$

This is the diffrential form/ strong form of the conservation law. In our case, our $f(u,x,t)$ is usually independent of x ant t. Hence, we have an autonomous scalar conservation law. This reduces (2.1) to:

$$u_t + [f(u)]_x = 0 \tag{2.2}$$

$f(u)$ is defined as our flux function and the PDE is hyperbolic. Using the chain rule we could express our PDE in explicit quasilinar form.

$$u_t + f'(u)u_x = 0 \tag{2.3}$$

$f'(u) = \frac{\partial f}{\partial u}$ which is defined as the characteristic velocity. (2.3) is the final form of our conservation law we want to get through the final derivation to either use the method of characteristics or the Godunov scheme along with the initial data to produce a model.

2.1.1 Method Of Characteristics

We now have the (2.2), and we want to solve this via the method of characteristics. Assume u is constant on the (x,t) plane and $x = \zeta(x_0,t)$ which starts at the point x_0 at $t = 0$.

$$u[\zeta(x_0,t),t] = u(x_0,0) = g(x_0) \tag{2.4}$$

For all $t \geq 0$ we take the partial derivative $\frac{\partial}{\partial t}$ and applying the chain rule. This produces:

$$0 = \frac{\partial}{\partial t}\bigg|_{x_0} u[\zeta(x_0,t),t] = u_t + u_x\frac{\partial \zeta}{\partial t}$$
$$= -f'(u)u_x + u_x\frac{\partial \zeta}{\partial t} = u_x(\frac{\partial \zeta}{\partial t} - f'(u)) \tag{2.5}$$

Assuming x_0 is ignored we can write (2.5) as:

$$\frac{d}{dt}u[\zeta(t),t] = u_t + u_x\zeta'(t) = 0 \tag{2.6}$$

Here we have $u(x,t)$, $x = \zeta(t)$. We further expand this by taking rate of change of x. Using the fact that $u_x \neq 0$; the initial condition $\zeta(x_0,0) = x_0$ we have:

$$\frac{\partial}{\partial t}\zeta(x_0,t) = f'[u(x_0,0)] = f'[g(x_0)] \tag{2.7}$$

Here, we have that ζ depends on x_0 and t and there's no derivation w.r.t x_0 on the LHS of (2.7); The RHS of (2.7) has that there is no dependence on ζ or t. Hence, we can integrate both sides to get:

$$\zeta(x_0,t) = x_0 + f'[g(x_0)]t \tag{2.8}$$

The integration constant is fixed by the intial condition. Hence the characteristic curves for (2.2) are the straight lines:

$$x = x_0 + f'[g(x_0)]t \tag{2.9}$$

We now have the solution $u(x,t)$ in implicit form. We can express this in closed form and write the solution in explicit form as:

$$u(x,t) = g[x_0(x,t)] \tag{2.10}$$

We define $f'(u)$ as the characteristic velocity; this is the solution to the conservation law (2.2). The Shock Speed is given as:

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R} \tag{2.11}$$

2.1.2 The Godunov Scheme for Hyperbolic Systems

The following is a summary from [12]

Another approach to solving our hyperbolic PDE (2.2) is the Godunov scheme; we discretise our conservation law (2.2) following the integral form (eqn).

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{u_{i+1/2}^{n+1/2} - u_{i-1/2}^{n+1/2}}{\Delta x} \tag{2.12}$$

Using the self-similar solution to the Riemann problem (2.2). We compute the time average flux function as:

$$\begin{aligned} u^*_{i+1/2}(x/t) &= \mathscr{R}\mathscr{P}[u_i^n, u_{i+1}^n] \\ f_{i+1/2}^{n+1/2} &= f(u^*_{i+1/2}(0)) \end{aligned} \tag{2.13}$$

This leads us to define the Godunov Flux as:

$$F_{i+1/2}^{n+1/2} = F^*(u_i^n, u_{i+1}^n \tag{2.14}$$

3 Theory

3.1 Waves, Rivers and Shocks

External forces form waves; they can come in many forms. Mainly with offshore winds interacting with the surface water - there is an energy transfer. The local winds act like a counterbalance against it. Another force is the friction created as a counter to the forces acting. The combination of forces causes the waves to form.[8] As the waves are propagating and are full of energy, there will be a time when the wave's amplitude will reach a critical level. A large amount of energy is transferred into kinetic energy, defined as a wave breaking. This can cause some violent, turbulent behaviour where its hard to predict what occurs due to the wave breaking.[9] The project's goal is to study and model what happens at this point and how quickly it occurs. We define a wave breaking as a point of discontinuity, a point where a shock has formed.

3.2 Derivation of The Consevation Law

This section is a summary from [10]

The theory behind the mathematical modelling of a flash flood comes from the concept of conservation laws for water on a river bed. The equations (2.2) can be applied and derived from the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0 \tag{3.1}$$

We want to study the dynamics of a fluid; we want to talk about floods. Consider a fluid with a closed surface S and normal vector $\hat{n}$. The volume enclosed by the surface is V_0 and fluid density $\rho(\mathbf{r},t)$.The velocity field of the particles $\mathbf{q} = (u,v,w)$, where $\mathbf{q} = \mathbf{q}(\mathbf{r},t)$ and $\mathbf{r}(t)$ represents the path taken by the fluid particles. Therefore, the mass within V_0 at fixed t is:

$$M = \int_{V_0} \rho \, dV \tag{3.2}$$

The change in mass:

$$\frac{dM}{dt} = \frac{d}{dt}\int_{V_0} \rho \, dV = \int_{V_0} \frac{d\rho}{dt} \, dV \tag{3.3}$$

We have that the volume enclosed the surface S by V_0 and V_0 is fixed. Hence by using Gauss Theorem,

$$\int_{\partial V_0} \rho \mathbf{q} \, dS = \int_{V_0} \nabla \cdot (\rho \mathbf{q}) \, dV \tag{3.4}$$

This is the mass flow per unit time where $\rho\mathbf{q}$ represents the mass flux density. Provided that there are no more external mechanisms affecting the system. We get that the change of mass and the mass flow out must sum up to zero. hence the conservation law:

$$\int_{V_0} \frac{d\rho}{dt} \, dV = -\int_{V_0} \nabla \cdot (\rho \mathbf{q}) \, dV \tag{3.5}$$

Reformulating we get:

$$\int_{V_0} [\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{q})] \, dV = 0 \tag{3.6}$$

As V_0 is arbitrary, we arrive (3.1)

Our goal now is to use the continuity equations to express in the form we want. We are using the following independent variables, dependent variables and parameters, and we seek to manipulate the continuity equation to the form we want.

3.3 The Variables Considered

1. Time t (Independent Variable)
2. Area of bed filled with water $A(s,t)$ (Independent Variable)
3. Flux of water down river bed $Q(s,t)$ (Dependent Variable)
4. Area averaged fluid velocity $\hat{u}(s,t)$ (Dependent Variable)
5. Perimeter of bed wetted by river $l(A(s,t))$ (Dependent Variable)
6. Shear stress per unit area river bed $\tau(s,t)$ (Dependent Variable)
7. Density of water ρ (Parameter)
8. Angle of inclination of river bed α (Parameter)
9. Friction factor f (Parameter)

3.4 Application Of Variables to The Continuity Equations

The continuity equations for water conservations in river beds leads to the PDE in Q and A:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} = 0 \tag{3.7}$$

This comes from considering a small section of the riverbed between the points s and $s + ds$ (where ds is neglible) which leads to the volume of water in the section being $dV = A(s,t)ds$. $Q(s,t)$ represents the volumetric flow of water. Reconsidering the point $d + ds$ we get that $\frac{d}{dt}(A(s,t)ds) = Q(s,t) \approx -ds\frac{\partial Q(s,t)}{dt}$. With $Q(s,t) = A(s,t)\bar{u}(s,t)$ and reformulation we end up with (3.7).

We now want to consider the other variables and parameters. The two forces that will cause the motion of the riverbed - one being the force of friction caused by the water particles acting against the riverbed walls; another is the gravity caused by the water flowing down the stream at an angle. The Frictional Force exerted $dF_{fric} = dsl(A)\tau$. The gravitational force exerted $dF_{grav} = g\sin(\alpha)$. Expanding upon the shear stress $\tau = f\rho\bar{u}^2$. With the assumption of neglection of inertia, we will assume that the forces are balanced, leading to the relation $\tau = \frac{g\sin(\alpha)\rho A}{l(A)}$. Then, substituting into the $\tau = f\rho\bar{u}^2$ and rearranging for $\hat{u}$. We end up with:

$$\frac{\partial A}{\partial t} + \sqrt{\frac{g}{f}} \frac{\partial}{\partial s} \left(A^{\frac{3}{2}} \sqrt{\frac{\sin\alpha}{l(A)}} \right) = 0 \tag{3.8}$$

We now have the equations that we can use for our conservation system; we now look to studying how manipulating these factors contribute to the modelling of a flash flood.

3.5 Assumptions

This section will discuss the assumptions that we will use to carry out the project. In previous sections, we carried out a lot of mathematics and applied many physics without considering their limitations.

First, we'll consider the environment and base the model on it. In this model, we are assuming that the water. Travelled in a distance ds will follow travel through a consistent riverbed in terms of space and area; this means that the cross-sectional area will resemble a regular shape, and we can easily calculate the volume by using the equation *Volume* =*CSA* $\times$ *breadth*. We also assume that the breadth travelled is a straight Euclidean line. Hence, the riverbed must be consistent in shape and length.

Second, We'll assume that the water contains no impurities. As previously proven, the density of the water is not a factor in the master equation. However, we consider this because we will have to consider the forces acting between the dirt particles and the water particles. This project only considers the friction coming from the water particles coming into contact with the riverbed. But, a more realistic model will more likely factor this in as most of the world's rivers are impure.

Thirdly, We consider that there are no external mechanisms affecting the flow system. External mechanisms can range from heavy rain, heavy drought or extra water coming in from sewage. If we were to consider this will lead to the Master Equation changing as we will need to consider these mechanisms, and our equation will not be homogenous. Considering these factors could make the model more realistic, but we will ignore the factors for this model.

Fourthly, we'll say that the shape of the cross-sectional area is symmetric. Hence, we have evenly distributed areas. The reason is to facilitate simplistic models in which we can algebraically manipulate the constants easily.

Lastly, we consider the type of fluids in the model. Water is an incompressible fluid; hence, we have incompressible flow; this has vast implications for the master equations as it means that we must consider all terms. Whereas if we have a steady flow, the continuity equation will reduce as $\nabla \cdot (\rho \mathbf{q}) = 0$.

3.6 Initial Conditions

We want to have a wave as an initial condition and study the wave as time passes. This can be achieved by an exponential function or a power function. We also want the function to be symmetric to resemble a wave. This comes by squaring the input - to achieve symmetry, reflecting the function on the minima point where $s = 0$. This leads to our initial condition being: $A(s,0) = \exp\{-s^2\}$.

We now have everything we need for the model and now look to understand different variables and parameters that affect a flash flood's modelling.

4 The Models

There are two significant components of our Master Equation we will consider. One is the $A(s,t)$ the cross-sectional area of the riverbed and $l(A(s,t))$ the perimeter of the riverbed. We want to study the relationship between the length and the cross-sectional area. We want to understand how that relationship impacts the formation of a shock and the break of a wave. Due to the nature of waves, we want a breaking wave or shock to delay as much as possible over a period of time. but we want it to occur at a small distance to minimise the impacts of the wave break.

We will focus on three different types of river beds. A Rectangular Riverbed, A Triangular Riverbed and A Semi-Circular riverbed. Once we explore the relationship between the perimeter and the cross-sectional area,

4.1 Model I: The Rectangular Riverbed

The Area of the rectangular riverbed is defined as: $A(s,t) = wh$ - where w is the width and h is the height of the riverbed. $w, h \in \mathbb{R}$ and are constants.The perimeter of the riverbed is $L = 2h + w$. via algebraic manipulation, we get to the linear relation:

$$l(A) = \frac{1}{h}A + 2h \tag{4.1}$$

for $l > 0, A > 0, h > 0$. $l(A)$ is lower bounded by $\frac{1}{h}A$. Essentially, this shows that the perimeter of the riverbed is directly proportional to the cross-sectional area of the riverbed. Hence, the bigger the perimeter the, the bigger the CSA.(4.1) leads to our (3.8) reformulated to its new form.

$$\frac{\partial A}{\partial t} + \sqrt{\frac{g}{f}}\frac{\partial}{\partial s}\left(A^{\frac{3}{2}}\sqrt{\frac{h\sin\alpha}{A+2h^2}}\right) = 0 \quad (4.2)$$

The Quasi-linear form:

$$\frac{\partial A}{\partial t} + \sqrt{\frac{g}{f}}\left(\frac{Ah\sin\alpha}{A+2h^2}\right)^{\frac{1}{2}}\left(A+3h^2\right)\frac{\partial A}{\partial s} = 0 \quad (4.3)$$

The general solution of the system (solved by the method of characteristics) is given as:

$$A = \exp\left\{-\left(s - \sqrt{\frac{g}{f}}\left(\frac{Ah\sin\alpha}{A+2h^2}\right)^{\frac{1}{2}}\left(A+3h^2\right)t\right)^2\right\} \quad (4.4)$$

The Shock Speed is:

$$s = \sqrt{\frac{g}{f}}\sqrt{h\sin\alpha}\frac{\sqrt{\left(\frac{A_L^3 h\sin\alpha}{A_L+2h^2}\right)} - \sqrt{\left(\frac{A_R^3 h\sin\alpha}{A_R+2h^2}\right)}}{A_L - A_R} \quad (4.5)$$

4.2 Model II: The Triangular Riverbed

The Area of a triangular riverbed is defined as $A(s,t) = \frac{1}{2}ab\sin\beta$ where a,b represent the sides that are adjacent to each other with β being the angle between them. Given that the shapes are symmetric, we have that both $a = b$. We get $A(s,t) = \frac{1}{2}a^2\sin\beta$; the perimeter of the riverbed is $L = 2a$. Via algebraic manipulation, we get the relation:

$$l(A) = \sqrt{\frac{8A}{sin\beta}} \quad (4.6)$$

Given that this is a power relationship, it's hard to tell the relationship between $l(A)$ and $A(s,t)$. We use the principle of linear laws to explore this. It suffices that:

$$\ln l = \frac{1}{2}\left(\ln A + \ln\frac{8}{\sin\beta}\right) \quad (4.7)$$

for $l > 0, A > 0, 0 < \beta < \frac{\pi}{2}$. $\frac{1}{2}\left(\ln A + \ln\frac{8}{\sin\beta}\right)$ is lower bounded by $\frac{1}{2}\ln A$. The shows that $\ln l$ is directly proportional to $\frac{1}{2}\ln A$.Therefore, $\frac{1}{2}\left(\ln A + \ln\frac{8}{\sin\beta}\right)$ increases linearly in relation to $\ln l$; this shows that the increase of the perimeter leads to the increase of the area. (4.5) leads to the reformulation of (3.8) in its new form.

$$\frac{\partial A}{\partial t} + \sqrt{\frac{g}{f}}\frac{\partial}{\partial s}\left(A^{\frac{3}{2}}\sqrt{\frac{\sin\alpha}{\sqrt{\frac{8A}{\sin\beta}}}}\right) = 0 \quad (4.8)$$

The Quasi-linear form:

$$\frac{\partial A}{\partial t} + \sqrt{\frac{g}{f}}\left(\frac{5}{4}\left(\frac{A\sin\beta}{8}\right)^{\frac{1}{4}}(\sin\alpha)^{\frac{1}{2}}\right)\frac{\partial A}{\partial s} = 0 \quad (4.9)$$

The general solution of the system (solved by the method of characteristics) is given as:

$$A = \exp\left\{-\left(s - \frac{5}{4}\sqrt{\frac{g}{f}}\left(\frac{A\sin\beta}{8}\right)^{\frac{1}{4}}(\sin\alpha)^{\frac{1}{2}}t\right)^2\right\} \quad (4.10)$$

The Shock Speed is:

$$s = \sqrt{\frac{g}{f}}\sqrt{\sin\alpha}\sin\beta^{\frac{1}{4}}\frac{\sqrt{\left(\frac{A_L^3}{\sqrt{8A_L}}\right)} - \sqrt{\left(\frac{A_R^3}{\sqrt{8A_R}}\right)}}{A_L - A_R} \quad (4.11)$$

4.3 Model III: The Semi-Circular Riverbed

The Area of a semi-circular riverbed is defined as $A(s,t) = \frac{1}{2}^2$ where r represents the radius of the semi-circle. The radius also represents the depth of the riverbed. The perimeter of the riverbed is $L =$. Via algebraic manipulation, we get the relation:

$$l(A) = \sqrt{2A\pi} \tag{4.12}$$

Given that this is a power relationship, it's hard to tell the relationship between $l(A)$ and $A(s,t)$. We use the principle of linear laws to explore this. It suffices that:

$$\ln l = \frac{1}{2}(\ln A + \ln 2\pi) \tag{4.13}$$

for $l > 0$, $A > 0$. $\frac{1}{2}(\ln A + \ln 2\pi)$ is lower bounded by $\frac{1}{2}\ln A$. $\ln l$ is directly proportional to $\frac{1}{2}\ln A$, therefore $\frac{1}{2}(\ln A + \ln 2\pi)$ increases positively in relation to $\ln l$; this shows that increase of the perimeter leads to the increase of the area. (4.5) leads to the reformulation of (3.8) in its new form.

$$\frac{\partial A}{\partial t} + \sqrt{\frac{g}{f}}\frac{\partial}{\partial s}\left(A^{\frac{3}{2}}\sqrt{\frac{\sin\alpha}{2A\pi}}\right) = 0 \tag{4.14}$$

The Quasi-linear form:

$$\frac{\partial A}{\partial t} + \sqrt{\frac{g}{f}}\left(\frac{5}{4}\left(\frac{A}{2\pi}\right)^{\frac{1}{4}}(\sin\alpha)^{\frac{1}{2}}\right)\frac{\partial A}{\partial s} = 0 \tag{4.15}$$

The general solution of the system (solved by the method of characteristics) is given as:

$$A = \exp\left\{-\left(s - \frac{5}{4}\sqrt{\frac{g}{f}}\left(\frac{A}{2\pi}\right)^{\frac{1}{4}}(\sin\alpha)^{\frac{1}{2}}\right)^2\right\} \tag{4.16}$$

The Shock Speed is:

$$s = \sqrt{\frac{g}{f}}\sqrt{\frac{\sin\alpha}{\sqrt{2\pi}}}\frac{\left(A_L^{\frac{5}{4}} - A_R^{\frac{5}{4}}\right)}{A_L - A_R} \tag{4.17}$$

5 Discussion Of Figures and Results

In this section, 14 figures focus on the variations of the factors we consider when it comes to modelling a flash flood. We begin with constant variables as we study the best shapes that handle flash floods better. Afterwards, We look further on to study how the friction factor and the steepness of the riverbed impact the model's results. In those cases, we will consider one shape for each varying factor. The rectangular riverbed for friction and the triangular riverbed for steepness.

5.1 Shapes with Constant Variables

First, we focus on the cross-sectional areas of the riverbeds with constant variables. In this case, we keep the depth of the riverbed as $1m$, the frictional coefficient as 0.2, and the steepness angle as 3°. Another thing to note is that we keep the angle of the triangular riverbed as 45°. *Figures 1 - 6 depict the models for this case.*

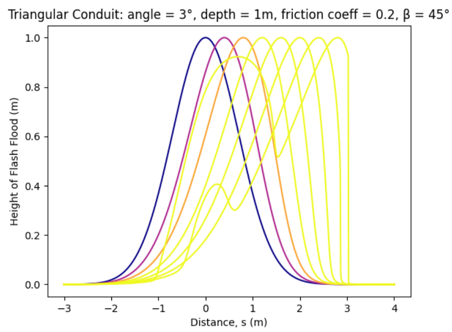

Figure 1: The graph for a Triangular Riverbed depicting how the wave moves through time at different distances. The shock starts to occur just past $s = 3$. This was calculated with the use of the Godunov Scheme.

Here we have that three shapes have shocks propagate at similar points. We want the shock to occur upstream but take longer as one; this means that populations will not feel the full brunt of the wave break downstream. Also, there is enough time to evacuate those in the local area to safety. In the case where we have constant conditions, we find that riverbeds with a rectangular cross-sectional area optimise these requirements to the maximum as the shock propagates close to the source point

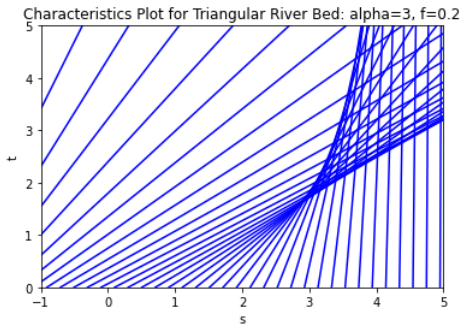

Figure 2: The graph for a Triangular Riverbed depicting the characteristics of wave moves in accordance to time and distance travelled. The shock starts to occur just past $s = 3, t = 2$. This was calculated via the method of characteristics.

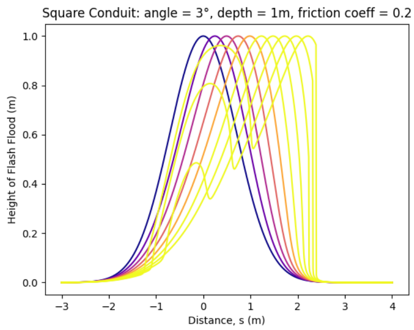

Figure 3: The graph for a Rectangular Riverbed depicting how the wave moves through time at different distances. The shock starts to occur just around $s = 2.5$. This was calculated with the use of the Godunov Scheme..

and occurs quickest compared to the other two riverbeds.

5.2 The Variation of Friction

Second, we want to understand how the frictional coefficient factors when modelling the flash flood. The Master Equation (3.8) shows that f has an inverse proportional relationship with $f(A)$. Hence, we should find that as the friction factor increases, the shock should propergate at a closer location. *Figures 7- 10 depict the models for this case.*

The results confirm our hypothesis as we see

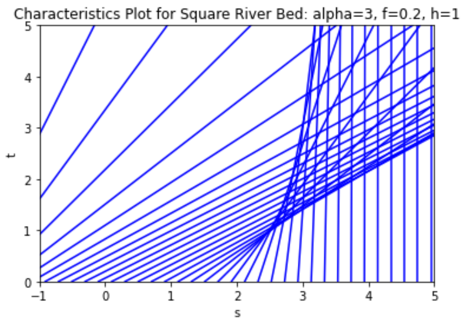

Figure 4: The graph for a Rectangular Riverbed depicting the characteristics of wave moves in accordance to time and distance travelled. The shock starts to occur just about $s = 2.5$, $t = 1$. This was calculated via the method of characteristics.

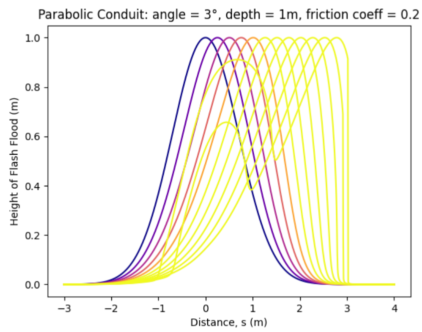

Figure 5: The graph for a Semi-Circular(Parabolic) Riverbed depicting how the wave moves through time at different distances. The shock starts to occur just past $s = 3$. This was calculated with the use of the Godunov Scheme.

that the higher the value of friction, the closer the distance from the source the shock begins to propagate. However, a varying friction value seems to affect the timing of where the shock propagates. We infer that a riverbed with much friction plays a significant role in the location of the wave breaking. Hence, riverbeds that tend to be smoother, such as the LA River, will be worse due to their smooth concrete layout [6] than the Congo River, where the riverbed comprises soil and mud [11] with a higher frictional factor.

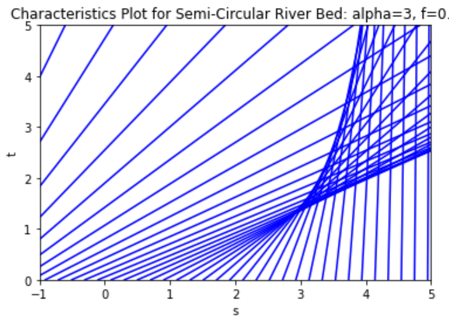

Figure 6: The graph for a Semi-Circular(Parabolic) Riverbed depicting the characteristics of wave moves in accordance to time and distance travelled. The shock starts to occur just before $s = 3$, $t = 2$. This was calculated via the method of characteristics.

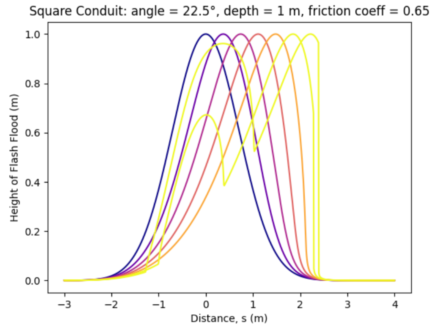

Figure 7: The graph for a Rectangular Riverbed with a friction coefficient of 0.65 depicting how the wave moves through time at different distances. The shock starts to occur just about $s = 2.65$. This was calculated with the use of the Godunov Scheme

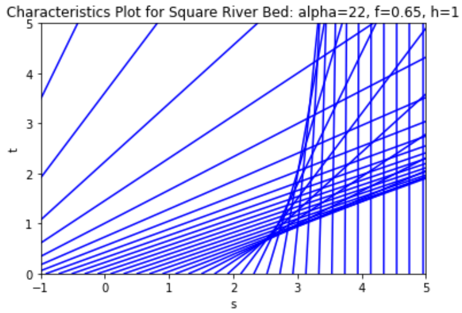

Figure 8: The graph for a Rectangular Riverbed a friction coefficient of 0.65 depicting the characteristics of wave moves in accordance to time and distance travelled. The shock starts to occur just before $s = 2.65$, $t = 0.6$. This was calculated via the method of characteristics.

5.3 The Variation of Steepness

Lastly, we want to focus on how the steepness of the riverbed has an impact on the model. As previously mentioned, flash floods can occur anywhere within touching distance of water. The areas could range from coastal areas to mountain ranges. We have that mountain ranges tend to be steeper than coastal areas; we hypothesise that the steeper the riverbed, the quicker and closer the wave breaks. *Figures 11- 14 depict the models for this case.*

$\sin\alpha$ has a directly proportional relationship with $f(A)$ for all riverbeds. Hence we have that hypothesis comes true; the results show this as we find that the riverbed with more steepness has a higher effect on the timing of the shock propagation as a larger angle increases the timing. However, the larger angle decreases the location of the shock propagation, which infers that the wave breaks close to the source but takes longer to occur.

6 Conclusion

The project aimed to establish how contributing factors can impact the mathematical modelling of a flash flood. We focused on the type of cross-sectional shape areas, the effect of friction and the impact of the steepness of a riverbed. We want to find the best scenario that would cause the minor damage and the scenario that causes the most damage. The earlier downstream a shock occurs, the more damage will be as floods tend to start at distances closer to the source and far from civilian populations. Hence, civilian populations will suffer severe impacts when a wave break occurs.

The results show that the best-case scenario when a riverbed has a CSA shape of a rectangle will have minor damage regardless of the variation of variables placed upon it. As

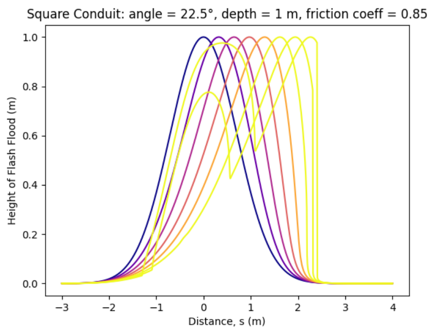

Figure 9: The graph for a Rectangular Riverbed with a friction coefficient of 0.85 depicting how the wave moves through time at different distances. The shock starts to occur just about $s = 2.5$. This was calculated with the use of the Godunov Scheme

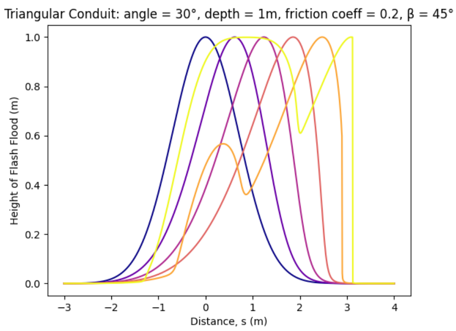

Figure 11: The graph for a Triangular Riverbed with a steepness of 30° depicting how the wave moves through time at different distances. The shock starts to occur just past $s = 3$. This was calculated with the use of the Godunov Scheme

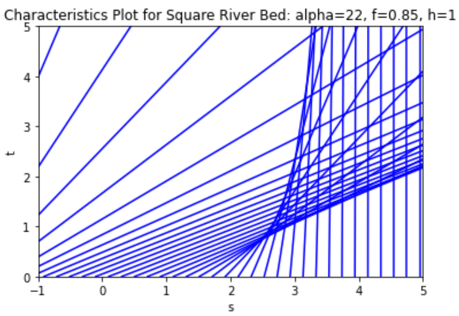

Figure 10: The graph for a Rectangular Riverbed a friction coefficient of 0.85 depicting the characteristics of wave moves in accordance to time and distance travelled. The shock starts to occur just before $s = 2.5$, $t = 0.8$. This was calculated via the method of characteristics.

we see, wave shock propagates from $(s,t) = (2.5,1)$ whereas the other riverbeds happen from $(s,t) = (3,2)$; this means the wave breaks quickly and at a shorter distance from the source.

When it comes to talking about friction, we state that the shape is independent of the effect of friction, so we imply that the relationship between friction and a CSA riverbed should be the same for all. Using the rectangular riverbed as a footprint, we find that the increase of the friction factor from $0.65 \to 0.85$ takes our propagating shock point (s,t) from $(2.65, 0.6) \to (2.5, 0.8)$ this shows that the friction does affect the timing and location of the breaking wave.

Thirdly, we find that the steeper the riverbed, the quicker the wave breaks; this means that areas in mountainous regions tend to be more likely not to feel the worse effects of flooding than near-flat coastal areas. This gives us the idea of the worse areas that could be affected: near-flat smooth areas that do not have a rectangular riverbed. The good thing is that there are not many areas with an oddly shaped riverbed. Most of the world's riverbed resembles a parabola or a rectangular CSA shape.

Finally, we shall consider the validity of the models. We have used many assumptions to make the models more streamlined in this project. The problem with this is that it encouraged us to take shortcuts when producing the model. In this project, we did not consider factoring in the possibility of rain and exterior factors that could influence the outcome of the model, which makes it unrealistic as usually other factors contribute to the formation and the impacts of a flash flood. Another thing to note is that most of the world's rivers are not straight and somewhat meandering. Hence, the assumption that a river is a straight line in the Euclidean sense does not make the models feasible in most contexts. This only applies to

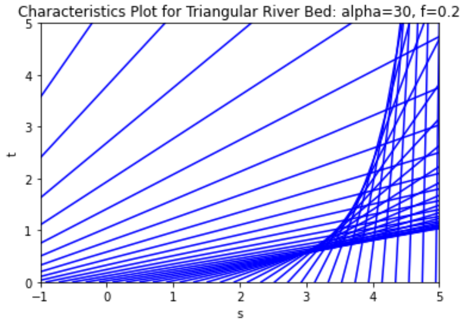

Figure 12: The graph for a Triangular Riverbed with a steepness of 30° depicting the characteristics of wave moves in accordance to time and distance travelled. The shock starts to occur just past $s = 3$, before $t = 1$. This was calculated via the method of characteristics.

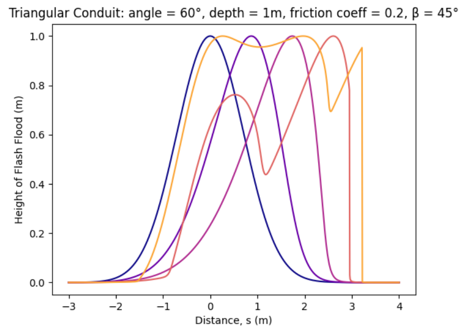

Figure 13: The graph for a Triangular Riverbed with a steepness of 60° depicting how the wave moves through time at different distances. The shock starts to occur just about $s = 3.25$. This was calculated with the use of the Godunov Scheme.

an artificial river such as a canal, as they tend to lie on straight lines. The most appropriate application of these models is the Suez Canal in Eygpt - where we have a nearly straight line waterway with a rectangular river basin made with smooth concrete.[2]

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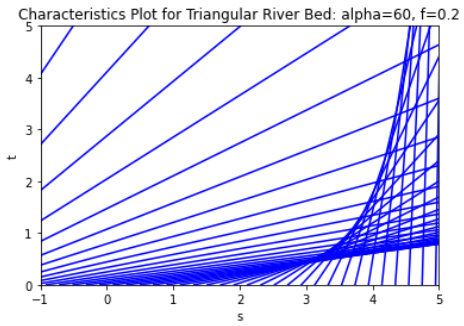

Figure 14: The graph for a Triangular Riverbed with a steepness of 60° depicting the characteristics of wave moves in accordance to time and distance travelled. The shock starts to occur just about $s = 3.25$, before $t = 0.5$. This was calculated via the method of characteristics.

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7 Appendix

7.1 Linear Law Relationships between The perimeter and The Cross-sectional Area

8 Individual Contribution

I contributed to the numerical analysis of the project by providing solutions to the PDE and the derivation of the conservation law, method of characteristics and lastly contributed to the derivations of the Cross-Sectional Areas.

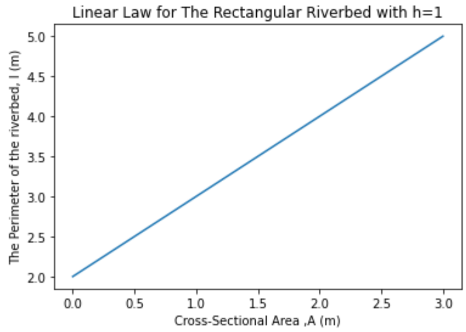

Figure 15: The linear relationship between the cross sectional area and perimeter of the rectangular riverbed of height 1m. The inference is explained in **Section 4.1**

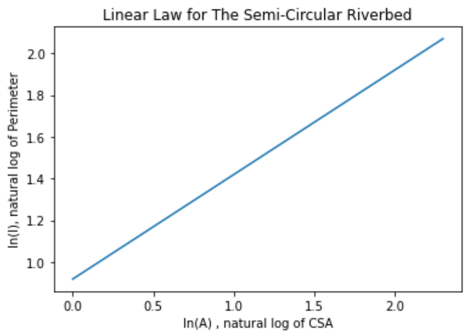

Figure 16: The Linear law relationship depicting how $\ln l$ depends on lnA for a semi circular Riverbed. The inference is explained in **Section 4.3**

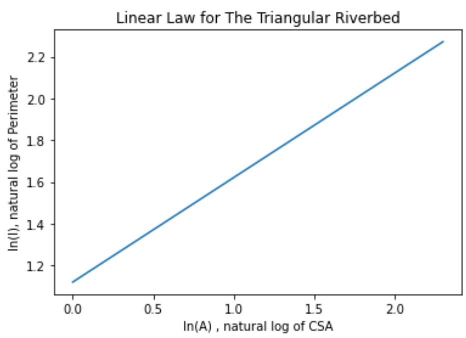

Figure 17: The Linear law relationship depicting how $\ln l$ depends on lnA for a Triangular Riverbed. The inference is explained in **Section 4.2**