MATH6144: The Mapping and Modelling Of Metro Systems

The mathematics behind cartography and modelling of transportation systems.

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Abstract

In the modern area, public transportation has always been one area of interest that arises when trying to understand how a system operates. One key area continuously researched and reviewed in public transportation logistics is the mapping and modelling a system because public transportation systems are constantly changing. The way maps are modelled and drawn needs to update at every iteration. In this project, the focus will be on a particular form of public transportation. The metro - Metro Trains run on lines that compile together to make a complex system of metro lines composed of stations, interchanges, and inks. The project's objective is to explore the various ways to map and model a metro system by looking at mathematics and cartography. Cartography is the study of map design; notions will be brought forward from the subject areas of Elementary Geometry, Differential Geometry, Map Projections, and Graph Theory.

What will be revealed is that there will be multiple ways to map and model a metro system – most notably the projection of a map by preserving a metric property and modelled as a graph/network. The problem is that each possible way has its advantages and disadvantages, but the Transport Authorities must decide which is the best way to represent a metro system. Transport Authorities make decisions on a case-by-case basis. However, it is essential to understand each case and its usefulness to the navigator. These arguments will be covered in the discussion of cartography; therefore, this project highlights that there will be several possible results. The discussion proves that cartography and mathematics have a large intersection and can be valuable for future research.

Keywords: Metro System, Map Projection, Earth's Sphere, Cartography

Contents

1 Introduction

Our project seeks to combine mathematics and cartography with helping understand the critical areas of the logistics of public transportation systems. The importance of the project is to show the complexity of public transportation systems and provide solutions for how to simplify systems for multiple motives. The focus will be on mainly metro systems, a subset of the more comprehensive public transportation network. Considering that metro systems are complex, the project will explore several ways to represent metro systems. This representation can be for many reasons but in several ways. The main two ways this project will represent a given metro system are by mapping and mathematical modelling – There are many advantages and disadvantages to following specific methods.

First, Metro systems will be defined, and we will show examples of Metro Systems to dictate what qualifies and what does not. Then we will review the general theory of cartography, where we will explore particular aspects such as the History of Cartography, the Cartographic Process and how it applies to metro systems. Then, we will seek to combine the Mathematical and Geographical notions when constructing a generic map using the Mercator projection. We look to start with the basic concepts in analysis and elementary geometry to work our way up to learning about concepts in differential geometry to understand the theory behind map projections.

Afterwards, we will use what we have learned to construct a map and a model for our Metro System. We will combine elements from the General Theory of Cartography and The Mathematics behind Map Making to construct a map for our Metro Systems. The Metro System we will use to construct a map and a model is the London Underground and the Glasgow Subway.

2 Metro Systems

The project aims to find the best way to produce maps of a metro system and discover ways to construct mathematical models of a metro system for exploitation. We begin by defining what a metro system is and look into examples of metro systems; this will aid us in understanding the structure of a metro system which will be helpful later as we discover that the form of the system is fundamental in producing a map and modelling.

2.1 Definitions and Remarks

Definition 2.1.1. *[27] A **Transit system** is a system used for the transportation of people and goods utilizing, without limitation, a street railway, an elevated railway having a fixed guideway, a commuter railroad, a subway, and motor vehicles or motor buses. It includes a complete system of tracks, stations, and rolling stock necessary to effectuate passenger service to or from the surrounding regional municipalities.*

Definition 2.1.2. *[33] A **Metro System** is an urban passenger transportation system using elevated or underground trains or a combination of both.*

Remark 2.1.3. *From the two definitions provided, we can make a few remarks about metro systems:*

1. *Metro Systems use electrified railways (trains) with high capacity to carry passengers from one location to another.*
2. *Metro Systems must operate over a specific locale (i.e. a metropolitan area, a regional municipality, a city).*
3. *Metro Systems must be connected or have robust connectivity - Generally, a passenger must get from one point to another in the network. (I.e., the majority of the time, a passenger should get to a target station [in the network] without having the system)*
4. *Metro systems must have a finite number of stations and connections/links; stations in the metro system connect to other stations in the system via such connections (i.e. the station is either an intermediate station or a terminus)*
5. *Metro systems are closed and discrete.*
6. *Metro Systems has trains that operate at a high frequency in a reasonable amount of time (Northern Line Train of the London Underground operates at 34 trains per hour during peak hours)[23]*

Examples of Metro Systems

The London Underground The London Underground can be a Metro System because it operates in a specific locale – London. It can connect passengers in short distances in said locale – Bank to Waterloo in London and operates at a high frequency in a reasonable amount of time – Victoria Line with 36 trains per hour during peak hours. [24] All stations in the tube network are connected and have robust connectivity. It uses trains and has a finite number of stations and links.

The Glasgow Subway The Glasgow Subway can be considered a Metro System because it operates over a certain locale – Glasgow. As it operates over the city centre region of the city, we can say it connects passengers in short distances in the region. Trains run "The Subway runs from 06:30 to 23:40 Monday to Saturday and 10:00 to 18:12 on Sunday. There are trains every four minutes at peak times and every six-to-eight minutes during off-peak times. A complete circuit takes 24 minutes." [44]. Hence, it operates at a high frequency over a reasonable amount of time.

Lastly, it uses trains and has a finite number of stations and links. The Glasgow Subway shape is a loop, which means that It has no terminus station, and the loop is bidirectional. The Glasgow subway represents this in two lines the inner loop and the outer loop.

2.2 The Basis For The Mapping and Modelling of Metro Systems

We revealed that the critical component of a metro system is that we have a finite number of stations connected via a link. We define a link as a connection between two stations such that at least one metro line runs through the link. We know that it is impossible to get off at any point other than a station due to how metro lines work. Hence, we can say that a metro system is discrete as all stations are objects whose values are countable, separate and distinct.

We define a station that has only one link as a terminus. A station with two links is an intermittent station, and lastly, a station with $n > 2$ is known as a junction; a station with multiple lines running through it is an interchange. Note that there are cases where there a station could be multiple things. An example of this is Uxbridge station on the London Underground, an interchange and a terminus.

Metro Systems has the purpose of helping many passengers travel via train at any given time in a locale. The passenger needs to be able to navigate the system to get from point a to point b. Hence, the need for a map as it helps the passenger complete the task efficiently and effectively. However, the Metro System cannot help the passenger reach their final destination or cannot use the system as a starting point. Usually, a passenger will have to travel from a location outside the system to presumably the closest station to the passenger, then use the system to get to the station closest to the passenger's final destination. We will talk about this when we talk about creating an embedded metric for a passenger's journey using the metro system.

But how do we go about doing this? We have a set of stations that we can see as points on a globe (sphere) that cover a surface area (operating locale) and an amalgamation of links that connect the points. We need to find a way to project this onto a 2D plane that is consumable for the navigator. However, there are other factors we need to consider, such as scale, distortion, symbolism, and typography. We will now use the general theory of cartography to understand the method of creating a map and then look into the process of modelling our network.

3 General Theory Of Cartography

We now look to learn about the general theory behind cartography. The idea is to use the concepts in cartography - such as the historical aspects, the types of maps, the purpose of a map, the cartographic process and map design features and apply this in designing a map for our metro system. After that, we will seek to look at the mathematical notions and principles behind cartography for metro systems, where we will try to find the best way to produce a map for our system.

3.1 Basic Definitions

Definition 3.1.1. *[25] A **Geographical Map** is a symbolic representation of selected characteristics of a place, usually drawn on a flat surface. Maps present information about the world in a simple, visual way. They teach about the world by showing sizes and shapes of countries, locations of features, and distances between places.*

Definition 3.1.2. *[19] **Cartography**is the study of being able graphically represent a geographical area. This can be done on curved surface or flat surface such as a map or chart.*

3.2 History of Cartography

Maps and Cartography have always been important to humanity throughout history. The oldest maps produced by humanity date back millennia: cave wall paintings. However, it took plenty of centuries to produce a world map (Although it was a poorly produced distorted projection of the world).

Ancient maps were the oldest known maps in the world which dates back *c.17,000BC*. The map was the Lascaux Cave Star Map[4], and it was discovered during *Fall 1940* by a group of boys around South West France. The ancient map was on a cave wall, and it depicted the area around Abauntz Lamizulo cave and animals such as red deer and ibex. However, the map goes against multiple cartographic principles. Even though it is a map, the map is very inaccurate.

Figure 1: Lascaux Cave Star Map [4]

In Contrast, the maps produced by the Greeks had better precision due to the progress and innovation made by Greek academics and philosophers. A critical philosopher was Anaximander, who created the first world map[4]. Although the accuracy of the map was limited, it was the first attempt at accurately depicting the world. Ptolemy was known as the man who pioneered the field of geography. He also contributed to the fields of astrology, astronomy, and mathematics. His works laid the groundwork for modern cartography.

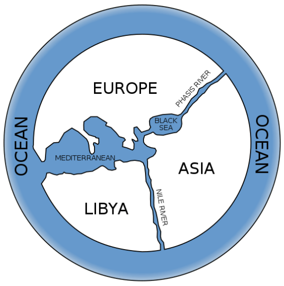


Figure 2: Map Produced By Anaximander [4]

Furthermore, the age of exploration began in the 15th Century; this came from innovations in technology like the telescope, compass, and the sextant. This led to the increased desire to explore worldwide and the demand for increasingly detailed and accurate world maps. European Cartographers conducted land surveys and explored uncharted areas to produce the most detailed maps. The map produced was the Mercator projection[25].

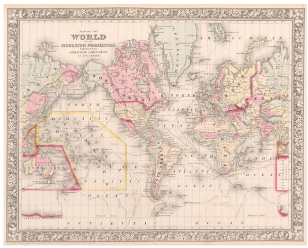


Figure 3: Mercator Projection Map used during the Age Of Exploration [25]

Now, the modern-day maps have evolved significantly with the rise of geospatial technology, which led to more accurate 2d and 3d maps. Such as those produced by google maps and google earth. Almost all modern-day maps use all cartographic principles, and with high accuracy. .

3.3 Types of Maps

There are three types of maps: Reference Maps, Thermatic Maps and Special Purpose Maps.[38]

- Reference Maps are maps that show the reader several spatial data types in an area. It is used to show the reader what is where and provide the reader with information about the area. The map includes the area's geographical features but can illustrate qualities in the map like political boundaries, cities, transportation routes, and topographic features. [6]
- Thematic Maps are maps that focus on a specific theme, point or multiple ideas that help us do quantitative or qualitative analysis. Such as the amount of precipitation weather patterns. Thematic maps can represent data with points, lines, areas or volumes. Thematic Maps use visible variables and symbols to represent the data in thematic maps. [6]
- Special Purpose Maps – represent the cross-section between Reference Maps and Thermatic Maps. Generally, they are designed for a specific type of user or to show a specific type of data. Maps like Navigational Map, Aeronautic Maps, Sanborn Maps, Soil Maps, and Municipal Utility Maps are considered Special Purpose Maps.[38] Maps of Metro Systems would be considered unique purpose maps because they are navigation maps that serve passengers. These is the type maps we'll be focusing on.

Figure 4: US General Refrence Map

Figure 5: US Time Zone Map

3.4 Map Purpose

A map can have many purposes. The Publisher of a map has to decide the purpose or intention and disclose this to the cartographer.[37]Cartographic Theory The cartographer must consider that for a map to be published to an audience. The purpose can range from teaching about political divisions and geographical features to a site map of a building. The audience who would read the map would need to be factored into the map's purpose. This could be dangerous as the way data is presented on a map can influence how we think about the data.

An example of this Is political thematic maps such as the U.S. Presidential Election Map. Given how the electoral system works, state electors will go to the candidate who receives the most votes in said state or 'Winner Takes All'. A state that consistently votes one party can enforce stereotypes of the votes in the state, but the electoral map does not show the proportion of the voters that voted for either candidate.

3.5 Cartographic Process

This section was written as a summary from source [37]

1 The Cartographic Process begins with the demand to express spatial data on a map to be read by an assumed audience. The spatial data can come from real-life or imaginary environments.

2 Once the demand is high enough, the cartographers would look for more information regarding the spatial data and the concepts.

3 Then, The Cartographer will consider the way the data is structured and how to represent this data structure upon a map.

4 The Cartographer now begins with the experimentation of the generalisation, symbolisation, typography, scale, insets, graphic primitives (margins and logos) and other essential map elements to find the best way for the map to be digestible and interpreted as intended to the reader.

5 The Cartographer will then settle on the final form of the map – whether it would be in physical form (book, large sheet of paper) or electronic form (on a website).

6 The Cartographer shows the map solution to the publisher. Once the publisher agrees with the map design, The Publisher will distribute the map to the intended audience.

3.6 Aspects Of Map Design

This section was written as a summary from source [19]

1. **Map Projections**: We need to present spatial data on a map. The map will be shown to the reader on a plane. Whether the map is physical or virtual, the represented data would need to be projected to the plane, especially if the map represents the data from Earth's surface. We know that it would be virtually impossible to accurately map the Earth's Surface onto a plane without the use of distortion. However, we can decide how we go about the Earth's surface distortion.

2. **Generalisation**: Given the nature of maps and the map purpose. This means that much information about the area being mapped may have to be abandoned in favour of valuable information about the area being mapped. Hence the process of generalisation comes in where we adjust the level of detail in geographic data to fit the purpose and constraints of the map.

3. **Symbology**: Map Symbols are used in maps to represent a geographic area's location, properties, and features. Symbols can come In visual forms such as colours, shapes, size and patterns.

4. **Composition**: The map needs to have an element of visual hierarchy. The Visual hierarchy ensures the process of map-reading is straightforward and clear.

5. **Labelling**: Text helps communicate aspects of the map to the reader. The labels are designed and positioned for effective communication.

6. **Layout**: The map must be set up with other elements such as title, legend, additional maps, text, images. Combined, this helps to facilitate visual communication of the map towards the reader.

3.7 Application To Metro Systems

We can now use the general theory of cartography and apply this to the process of creating a map of a metro system. Here, we have the purpose of wanting a metro system map varying from navigation to planning and modelling systems for specific situations (i.e. Engineering Works, Improving travel time). The intended audience of the map could be passengers, train drivers or executives. For the cartographic process to begin, we need there to be a demand for a new map. In the case of metro maps, there will always be a need for a new map for a multitude of reasons, most likely for clarity; as metro systems become more complex and denser, the need for easier to read and navigable maps will increase as well.

An example of this is the 2015 London Tube Map in comparison to the 2014 London Tube Map. The expansion of the London Overground from Liverpool Street to Enfield Town / Cheshunt / Chingford and the Shuttle service from Romford to Upminister line (Both Acquired from Greater Anglia in 2015) [34] made the London Tube Map denser, and innovative methods had to be used to be able to fit on to the map. We must also consider other aspects of map design - mainly map projections, generalisation and the composition of the map. We will look more into this in the next section.

Figure 6: 2014 Tube Map: Before London Overground Expansion [13]

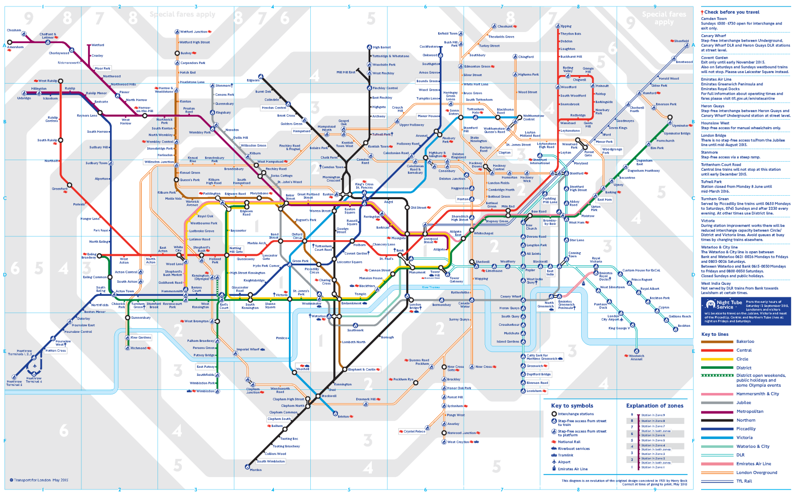


Figure 7: 2015 Tube Map: Post London Overground Expansion and the addition of TFL Rail [12]

4 General Construction of a Map

We now have the tools to complete the process of creating a map for our metro systems. We will focus on creating a map for passengers for navigation. The maps produced will use aspects of Thematic Maps, where we use variables and symbols to represent the spatial data and Reference Maps, where stations on a map are placed on a plane to express what is in the area. Hence, the maps produced will be Special Purpose Maps.

The maps we will produce can come in several forms. The goal is to find the most suitable map for our audience, being passengers. For instance, we will assume that the Earth is a perfect sphere and have a set of points and links on said sphere. A way to represent the spatial data as a map is to use software where we can scale up on the globe and look at the network to match our needs. An example of this is Google Earth, which uses this process of representation. Google Maps uses an alternate way of representation where we use the method of a map projection (Mainly, the Mercator Projection) where we project the globe onto a 2-dimensional Euclidean plane $\mathbb{E}^2$.

First, we will learn the mathematics behind creating a generic map for a specific area of our planet. We detail the steps that we will take to create a map. We will look at the assumptions taken, the justification for the methods used, the theorems and notions applied and the implications of the steps taken. Furthermore, we will then use this to create a map and a model for our metro system. The example used for the application will be the London Underground and the Glasgow Subway.

4.1 Basic Notions

We introduce basic concepts that are fundamental to constructing a map. In addition, we will use these definitions later on to provide an argument for the implication of other results.

Definition 4.1.1. *[2] A **Set** is a collection of objects called elements of the set. Elements can be any mathematical object ranging from numbers, points in space, symbols, lines, variables or any other set or subset of a set. A **set** with one element is defined as a **singleton**. A Set could be a finite set with a finite number of elements or an infinite set. Examples of sets are* $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}$ *and* $S = \{a, b, c\}$.

Definition 4.1.2. *[35] A **Mathematical Map (Function)** is a map* $f : X \to Y$ *between sets* X *and* Y *with the property that* f *is able to send a element from* X *to a specific element in* Y*. The result is that we are able to establish a relationship between the two corresponding sets. Examples of functions are:*

- $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = x$
- $g : \mathbb{R} \to \mathbb{R}^2$ *with* $g(x) = x^2$
- $h : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ *with* $h(x) = 1/x$

4.1.1 Continuous and Smooth Functions

Definition 4.1.3. *[29, Definition 6.5] Let* $f : X \to Y$ *be a function. The function* f *is **continuous at the point*** α *if:* $\alpha \in X$ *and* $\forall \epsilon > 0 \quad \exists \delta > 0$ *such that* $\forall x \in X \quad |x-\alpha| < \delta \implies |f(x)-f(\alpha)| < \epsilon$

The implication here is that a limit exists at α and that the function is defined at a.

Definition 4.1.4. *[29, Definition 6.6] Let* $f : U \to V$ *be a function with domain* U*. The function* f *is **continuous for the domain*** U *if it is continuous at each point* α *in* U *as:* $\forall \alpha \in U$ *and* $\forall \epsilon > 0 \quad \exists \delta > 0$ *such that* $\forall x \in U \quad |x - \alpha| < \sigma \implies |f(x) - f(\alpha)| < \epsilon$

Definition 4.1.5. *[31, Section 1.4] Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets. A mathematical map is called **smooth** if and only if it is infinitely differentiable. i.e if and only if all of its partial derivatives are continuous such that:*

$$\partial^\alpha f = \frac{\partial^{\alpha_1 + ... + \alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

$$\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$$

*Exist. We denote $C^\infty(U, V)$ for the set of smooth maps. In addition, for a **Smooth Map** $f = (f_1, ..., f_m) : U \to V$ and a point $x \in U$ the derivative of f at x is the linear map*

$$df(x)\zeta := \left.\frac{d}{dt}\right|_{t=0} f(x + t\zeta) = \lim_{t \to 0} \frac{f(x + t\zeta) - f(x)}{t}$$

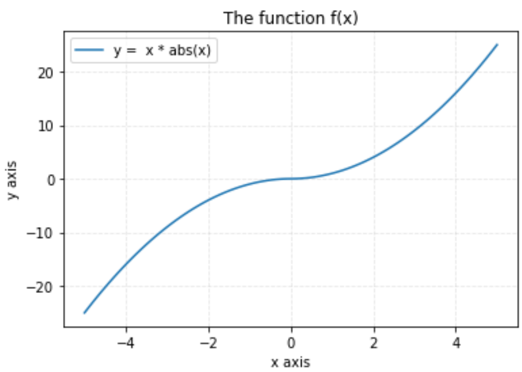
*Where $\zeta \in \mathbb{R}^n$. The linear map is represented by the **Jacobian matrix** of f at x, which for the linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$*

$$df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix} \in \mathbb{R}^{m \times n}$$

So, We want any function that will be applied to the project to be continuous for the domain and co-domain that will be applied for carrying out our map projection $f : \mathbb{R}^3 \to \mathbb{R}^2$. The continuous function ensures that every point in the set can is mapped onto the plane and nothing 'blows up'. The smoothness of the map is implied as a more robust form of continuity coming from the definition of being differentiable. However, this is not always the case; a function can be continuous but not smooth.

Example: Analysing Smoothness of Continuous Functions

Consider the two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ where $f(x) = x|x|$ and $g(x) = x^3$. By inspection both functions are continuous functions. But, we want to study the smoothness of the functions. If we differentiate twice $f(x)$ we get $f''(x) = \frac{2x}{|x|}$. Quite clearly, there will be an discontinuity at 0 - hence the function is not smooth as the function is not infinitely differentiable but it is a continuous function. Whereas, $g(x) = x^3$ will always be infinitely differentiable given its a polynomial. In fact, every polynomial belongs to C^∞ - the group of functions that are continuous and infinitely times differentiable [31]. The following figures details the smoothness of the functions through each derivative.



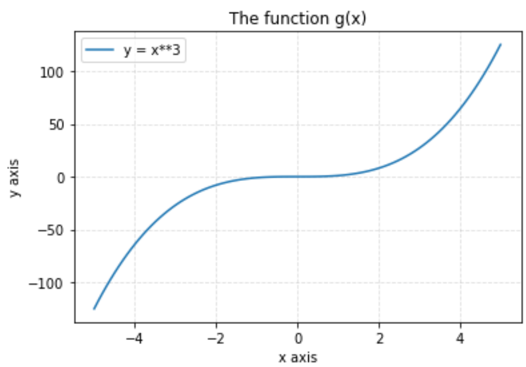


Figure 8: The Given Functions $f(x)$ and $g(x)$

An example of a consequence of using a non-smooth function is the modelling of a particles path for a given time t for $t \geq 0$ if we were to use the function $f(t) = t|t|$. When it comes to

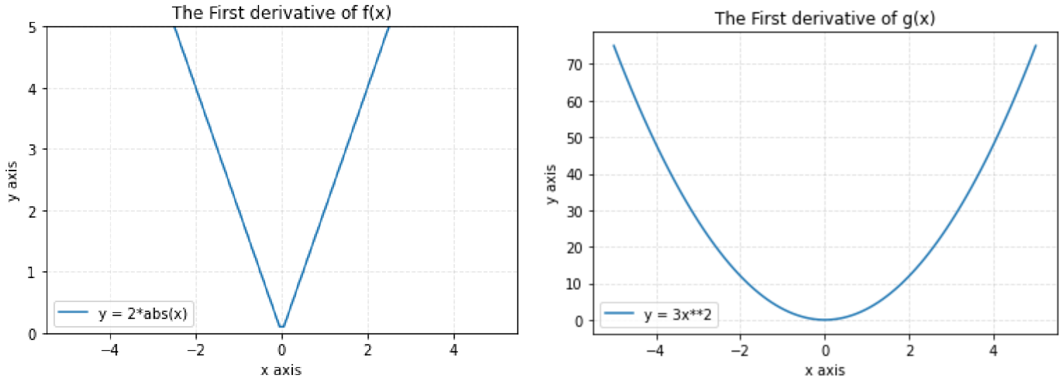


Figure 9: The First Derivative of the Given Functions $f(x)$ and $g(x)$

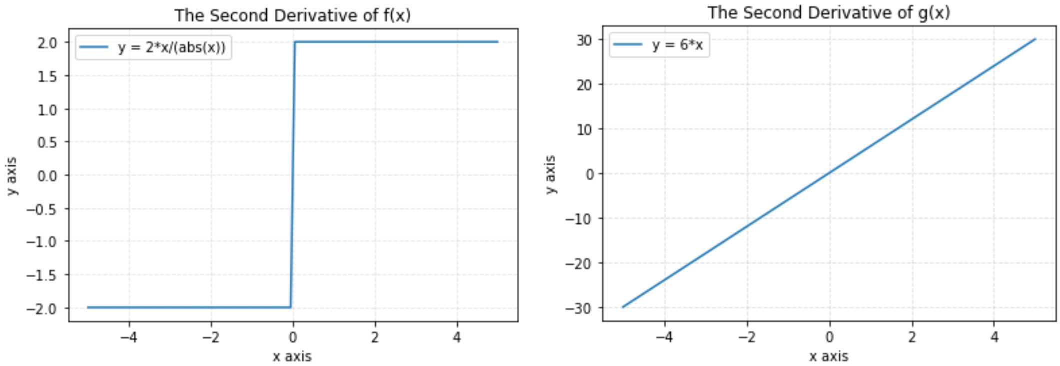


Figure 10: The First Derivative of the Given Functions $f(x)$ and $g(x)$

describing the particles acceleration $\ddot{f}(t) = \frac{2t}{|t|}$, we will have an infinite acceleration at $t = 0$ as $\ddot{f}(t)$ is discontinuous at $t = 0$; this is clearly unlikely hence using $f(t)$ as a function to describe the particles position at a time t is not a feasible model for the particle. Whereas, $g(t)$ would be , as it is a smooth function,

4.1.2 Curves and Arc-Length : A Particle's Path

Definition 4.1.6. *A* ***Curve*** *[40] is a continuous function from 1-D space to an n-dimensional space. In this project, we will refer to curves as the graphed functions of two or three-dimensional curves. We can parametrise the simplistic curves and represent them parametrically in n-dimensional space and system:*

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} \tag{1}$$

go While, more complex curves can be only be defined implcity - in the form :

$$f(x_1, x_2, ..., x_m) = 0$$

Example: The Circle

One of the most common examples of a parameterised curve is a circle. Take the equation:

$$x^2 + y^2 = 9$$

where r the radius $= 9$ and origin $(0,0)$. Then, using t as our parameter (where $t \in [0, 2\pi]$) we have the system of equations:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3\cos t \\ 3\sin t \end{bmatrix} \tag{2}$$

The parametrisation of curves will become notable as we define the environment we will be projecting from and justify it. We now look at the distances between points on a curve known as the Arc Length of a Function.

Definition 4.1.7. *[39] The Arc Length s is defined as the length along a curve,*

$$s = \int_{\gamma} |d\boldsymbol{\mathcal{L}}| dx$$

Where $d\boldsymbol{\mathcal{L}}$ represents the differential displacement vector along a curve γ. By defining the line segment $ds^2 = |d\boldsymbol{\mathcal{L}}|$. Parametising in the curve in t and given that ds/dt is the velocity at the end of the radius vector $\boldsymbol{r}$ gives us:

$$s = \int_a^b ds = \int_a^b \frac{ds}{dt} dt = \int_a^b |\boldsymbol{r}'(t)| dt$$

The equation expressed in 2-D Cartesian coordinates is given as with orientation $y = f(x)$. i.e. :

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

[11] with orientation $x = f(y)$

$$s = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

[11] If the curve is parametised in t and the curve is in the form $\boldsymbol{r} = x(t)\hat{\boldsymbol{x}} + y(t)\hat{\boldsymbol{y}}$

$$s = \int_a^b \sqrt{x'^2(t) + y'^2(t)} dx$$

We understand the path of a particle and how to calculate the path of a curve from points a to b. Knowing this is essential when talking about distances between points on a given surface or plane. Later on, we will seek to learn and understand how much a curve deviates from being a straight line. We will study its broader implications when it comes to the projection of our surface in $\mathbb{R}^3$ to plane $\mathbb{R}^2$; how we can project curves on a surface to a plane, and how to use the notion of curves to study the distance between points on a surface.

4.2 Metric Spaces

We now look to focus on the spaces in which we will project our spatial data. What we want to do is project our Earth Sphere in $\mathbb{E}^3$ to the Euclidean space $\mathbb{E}^{\not\vDash}$. From those spaces, we can apply distance functions (Metrics) such that our map infers the Scale and the distance between points in a given locale. We will build on these ideas when applying when constructing a map for a metro system.

Definition 4.2.1. *A **Metric Space**[14, Definition 1.1] (X, d) is a non empty set X equipped with a function d (known as the distance function). Such that X is mapped by $d : X \times Z \to \mathbb{R}$ and it obeys the following axioms.*

1. *$d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$. The distance function if positive definite and if we have it equal to 0 then the point x is the same as point y.*
2. *$d(x, y) = d(y, x)$ The distance function is symmetric $\forall x, y \in X$, $d(x, y) = d(y, x)$.*
3. *$d(x, y) + d(y, z) \geq d(x, z)$. The distance function satisfies the inequality equation $\forall x, y, z \in X$.*

4.2.1 The Euclidean Space

We want to learn about the Euclidean Space as this is what we want to be the basis for the mapping process. Eventually, we want to produce a map projection from a perfect sphere to a Euclidean plane. Hence, let us re-familiarise ourselves with traditional notation.

Definition 4.2.2. *[9, 1.2 Euclidean Space] The $\mathbb{R}^n$ space (where $n \in \mathbb{N}$) consists of points $\boldsymbol{x}, \boldsymbol{y}$ such that they are n-tuples of $\mathbb{R}$ Numbers. Therefore, we have that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ if and only if:*

$$\boldsymbol{x} = (x_1, ..., x_n)$$

$$\boldsymbol{y} = (y_1, ..., y_n)$$

We combine the space with the Euclidean Norm for $\boldsymbol{x}$ where we have:

$$\|\boldsymbol{x}\| = (\sum_{n=1}^{n} |x_n|^2)^{\frac{1}{2}}$$

*This is defined as the **Euclidean Metric** between the origin $\underline{\boldsymbol{0}}$ and the position vector at $\boldsymbol{x} \in \mathbb{R}^n$.*

$$\|\boldsymbol{x}\text{-}\boldsymbol{y}\| = (\sum_{n=1}^{n} |x_n - y_n|^2)^{\frac{1}{2}}$$

*This is defined as the **Euclidean Metric** between the position vector at $\boldsymbol{y}$ and the position vector at $\boldsymbol{x}$ (Where $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$). Finally, we have the n-dimensional Euclidean space $\mathbb{E}^n$ is defined as the $\mathbb{R}^n$ equipped with the Euclidean Metric.*

Examples of Metric Spaces

The Metrics Were provided by the sources [41], and [9]. I conducted the proof of satisfying Definition 4.3.1

The Usual Metric on $\mathbb{R}$ This is defined as:

$$d(x, y) = \|x - y\|$$

We check that this is a metric by proving that it follows the three axioms from above.

1. For the first axiom: We have that for any $t \in \mathbb{R}$ we have $|t| \geq 0$ with $|t| = 0 \iff t = 0$. Hence, $\forall x, y \in \mathbb{R}$ we have $|x-y| \geq 0$ with $|x-y| = 0 \iff x-y = 0 \iff x = y$.

2. For the second axiom: We have that $d(x,y) = d(y,x)$ hence $|x-y| = |y-x| = |-(x-y)| = |-1||x-y| = |x-y|$ $\forall x, y \in \mathbb{R}$ as required.

3. For the third axiom: We have if $\forall x, y, z \in \mathbb{R}$ if $x \leq y \leq z$ or $z \leq y \leq x$ then,

$$|x-z| = |x-y| + |y-z|$$

or

$$|x-z| < |x-y| + |y-z|$$

. Hence implying the triangle inequality is satisfied $\forall x, y, x \in \mathbb{R}$

The Euclidean Metric on $\mathbb{R}^2$ This is defined as:

$$d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Where $\underline{\mathbf{x}} = (x_1, x_2)^T$ and $\underline{\mathbf{y}} = (y_1, y_2)^T$ $\forall \underline{\mathbf{x}}, \underline{\mathbf{y}} \in \mathbb{R}^2$

We check that this is a metric by proving that it follows the three axioms from above.

1. For the first axiom: We have that in this metric, the square root of a positive real number is taken as the positive square root. Hence, this shows that the metric is positive definite where $d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \geq 0$. Where we have $d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = 0 \iff (x_1 - y_1)^2 = (x_2 - y_2)^2 = 0$ this happens when $x_1 = y_1$ and $x_2 = y_2$. Hence $\underline{\mathbf{x}} = \underline{\mathbf{y}}$.

2. .For the second axiom: As
$$(y_1 - x_1)^2 = (x_1 - y_1)^2$$
and
$$(y_2 - x_2)^2 = (x_2 - y_2)^2$$
It follows that $d(\underline{\mathbf{y}}, \underline{\mathbf{x}}) = d(\underline{\mathbf{x}}, \underline{\mathbf{y}})$

3. For the third axiom: Let's say we have three points $\underline{\mathbf{x}}, \underline{\mathbf{y}}, \underline{\mathbf{z}} \in \mathbb{R}^2$. Where $\underline{\mathbf{x}} = (x_1, x_2)$, $\underline{\mathbf{y}} = (y_1, y_2)$ and $\underline{\mathbf{z}} = (z_1, z_2)$/ Let $\alpha = x_1 - y_1$, $\beta = y_1 - z_1$, $\gamma = x_2 - y_2$ and $\delta = y_2 - z_2$. Taking the squares of both sides

$$d(\underline{\mathbf{x}}, \underline{\mathbf{z}})^2 = (x_1 - z_1)^2 + (x_2 - z_2)^2 = (\alpha + \beta)^2 + (\gamma + \delta)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\alpha\beta + \gamma\delta)$$

Meanwhile on the R.H.S we have

$$(d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + d(\underline{\mathbf{y}}, \underline{\mathbf{z}}))^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2\sqrt{(\alpha^2 + \beta^2)(\gamma^2 + \delta^2)}$$

For the triangle equality to follow we must show that:

$$(\alpha\beta + \gamma\delta) \leq (\alpha^2 + \gamma^2)(\beta^2 + \delta^2)$$

However, we know that:

$$0 \leq (\alpha\delta - \beta\gamma)^2 = \alpha^2\delta^2 + \beta^2\gamma^2 - 2\alpha\beta\gamma\delta = (\alpha^2 + \gamma^2)(\beta^2 + \delta^2) - (\alpha\beta + \gamma\delta)^2$$

. This satisfies the third axiom and d is an metric on $\mathbb{R}^2$.

4.2.2 Structure Of Metric Spaces

We want to consider the basic structure of Metric Spaces.

Definition 4.2.3. *X as a set containing the points x and y and r as the radius. The structure is given as:*

1. *The Open Ball (or neighbourhood) around $x \in X$ with radius r. With $x \in X$ and $r > 0$*
$$B(x) = \{y \in X | d(x, y) < r\}$$
2. *The Closed Ball (or neighbourhood) around $x \in X$ with radius r. With $x \in X$ and $r > 0$*
$$\hat{B}(x) = \{y \in X | d(x, y) \leq r\}$$
3. *The Sphere around $x \in X$ with radius r. With $x \in X$ and $r > 0$*
$$S(x) = \{y \in X | d(x, y) = r\}$$

Example: Balls on $\mathbb{R}^2$

We can apply Definition 4.2.3 to any metric space. Take the Euclidean Metric on $\mathbb{R}^2$. Then, the open ball, the closed ball and the spheres will correspond to the equation of a circle of radius r centered at $(0, 0)$ where the region indicated would be:

1. The Open ball is defined as:
$$x^2 + y^2 < r^2$$
2. The Closed Ball is defined as:
$$x^2 + y^2 \leq r^2$$
3. The Sphere is defined as:
$$x^2 + y^2 = r^2$$

Definition 4.2.4. *[9] Let (X, d) be a metric space. A **Metric Subspace** of X is a subset Y of X equipped with the metric d restricted to Y. The resulting metric on Y is called the **induced Metric**.*

An example of an induced metric is the absolute value metric on $\mathbb{R}$. $\mathbb{Q}$ and $\mathbb{N}$ are all subspaces of the mentioned metric. The use of subsets and subspaces unlocks a new definition:

Definition 4.2.5. *Let $\hat{X}$ be a non-empty subset of the metric space (X, d). The **diameter** of $\hat{X}$, denoted $diam(\hat{X})$, is defined as :*

$$diam(\hat{X}) = sup\{d(x, y) | x, y \in \hat{X}\}$$

if $diam(\hat{X})$ is finite. $\hat{X} \subseteq X$ is Bounded.

From the definition it suffices that:

Proposition 4.2.6. *If $\bar{X} \subseteq \hat{X}$ then $diam(\bar{X}) \leq diam(\hat{X})$*

Studying the diameter is interesting as it enables us to evaluate the sheer Scale of a metric space. The diameter for multiple spaces could be different. The reason for this is because of how they are defined. For example, the diameter of a Graph will be different compared to the diameter of the Euclidean Space. The diameter of a graph is defined as the supremum amount of edges that need to be transverse to get from one vertex to another vertex in the Graph (via a sequence of existing paths of edges whose weight of edges are defined).[17], and the diameter of the euclidean space is the norm between the two furthest points. The structure of Graphs is much different as the points are discrete, which means that there will be no other element in the neighbourhood of a point in the Graph for any given ball. In comparison, Euclidean Space is not discrete. There will always be another element of the Euclidean Space regardless of what ball we evaluate.

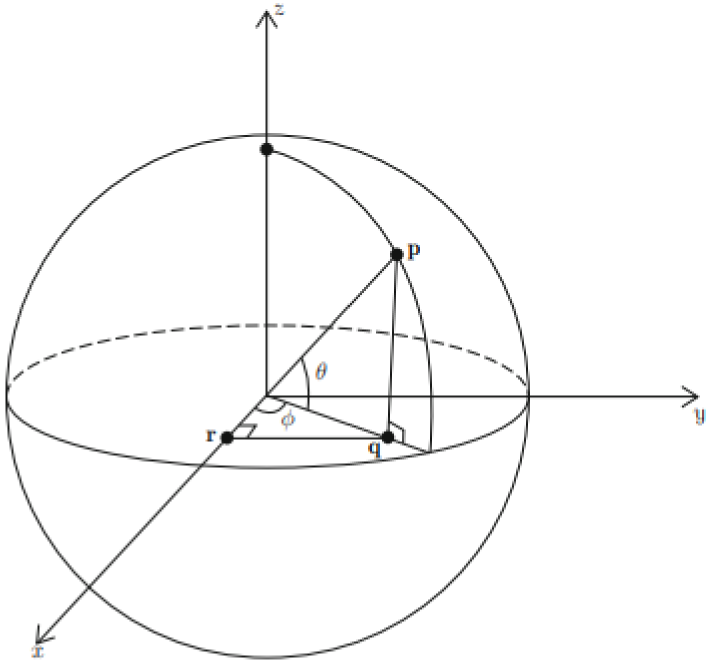


Figure 11: The Sphere with coordinates at **p** [30]

4.3 The Foundation for our Mathematical Map

A significant component of this project is understanding the environment, influencing our choices in producing a map. Our goal is to create a map representing an area (a surface patch) on Earth (a surface); this will involve performing a map projection from a 3-dimensional Euclidean plane to a 2-dimensional euclidean plane. However, what should our 3-dimensional surface be? Given that we live on a round ball, we have two options. We could either decide that the Earth is an Ellipsoid or a Sphere. In reality, the shape of the Earth is an ellipsoid; as the Earth rotates, the forces that act upon the north and south poles cause it to become near straightly flat as opposed to a curved and an imperfect sphere. [1] In this project, we do not factor this in, and we assume the shape of the Earth is a sphere, and this allows us to use conformal map projections to be able to carry out our task effectively. However, this will lead to some issues in the long run. There will be a loss of information, which will lead to some inaccuracy, and we will find some distortion in the area mapped. Furthermore, we seek to study the characteristics, results and models that could help us map the Earth.

4.3.1 Surfaces and Surface Patches

Definition 4.3.1. *[30, Definition 4.1.1] A subset S of $\mathbb{R}^3$ is a surface if, for every point $\boldsymbol{p} \in S$, there is an open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing $\boldsymbol{p}$ such that $S \cap W$ is homeomorphic to U. A subset of surface S of the form $S \cap W$, where W is an open subset of $\mathbb{R}^3$ is called an open subset of S.*

A homeomorphism $\boldsymbol{\sigma} : U \to S \cap W$ is defined as a ***surface patch*** *or, the parmatisation of the open subset $S \cap W$ of S.*

A collection of surface patches - where the images cover the entirety of S is defined as the ***Atlas*** *of S*

We now look at the types of surfaces and surface patches that will be used in this project.

Definition 4.3.2. *The Unit Sphere $\mathbb{S}^n$ in the Euclidean Space $\mathbb{E}^{n+1}$ is the set of points such that $\forall \boldsymbol{p} \in \mathbb{E}^{n+1}$. The Euclidean Distance to the origin $\boldsymbol{0}$ is 1. Hence,*

$$\mathbb{S}^n = \{\boldsymbol{p} \in \mathbb{E}^{n+1} | d(\boldsymbol{p}, \boldsymbol{0}) = ||\boldsymbol{p}|| = 1\}$$

We can further expand on the definition from [9, Section 2.1] by formulating it in the form of the sphere of Earth. Take the Equatorial radius of the planet to be $6,370,000m^2$ to 3 significant figures[16]. It suffices that, for our sphere $\mathbb{S}^2$ in the 3-dimensional Euclidean Plane $\mathbb{E}^2$ we have:

$$\mathbb{S}^2 = \{\boldsymbol{p} \in \mathbb{E}^3 | d(\boldsymbol{p}, \boldsymbol{0}) = ||\boldsymbol{p}|| = 6,380,000\}$$

Application 1: The Ellipsoid

We define the standard form equation (using a Cartesian coordinate system with the origin at centre) of the ellipsoid of the earth as using the fact that the polar radius is $6,370,000m^2$ to 3 significant figures:

$$\frac{x^2}{6,380,000^2} + \frac{y^2}{6,380,000^2} + \frac{z^2}{6,370,000^2} = 1$$

The parameterisation of the Cartesian coordinates $x, y, z \in \mathbf{p}$ and $\mathbf{x} \ \in \mathbb{E}^3$ is given as:

$$\boldsymbol{\sigma}(\theta, \phi) = \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6,380,000\cos\theta\sin\phi \\ 6,380,000\sin\theta\sin\phi \\ 6,370,000\cos\phi \end{bmatrix} \tag{3}$$

Where $\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi < 2\pi$.

Application 2: The Sphere

We define the standard form equation (using a Cartesian coordinate system with the origin at the centre) sphere of the Earth as:

$$x^2 + y^2 + z^2 = 6,380,000^2$$

The parameterisation of the coordinates $x, y, z \in \mathbf{p}$ and $\mathbf{p} \ \in \mathbb{E}^3$ is given as:

$$\boldsymbol{\sigma}(\theta, \phi) = \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6,380,000\cos\theta\sin\phi \\ 6,380,000\sin\theta\sin\phi \\ 6,380,000\cos\phi \end{bmatrix} \tag{4}$$

Where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi < 2\pi$.

In Polar coordinates, for $\mathbf{p} \in \mathbb{S}^2$ we say that our sphere with radius $6,380,000m$ is defined as:

$$(6,380,000, \theta, \phi)$$

Through studying the ellipsoid of the Earth and the Sphere of the Earth, we encounter our first issue. As we assume the Earth is a sphere of equal radius, we do not consider how the Earth is an oblate spheroid [31]. If we take that the Equatorial and Polar Radii are approximately equal, we find that we have an oblate spheroid as for:

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\zeta^2} = 1$$

We have $\alpha = \beta > \zeta$. Hence, the z-axis has less Curvature than the x,y axis. Therefore, by assuming that Earth is a sphere (where $\alpha = \beta = \zeta$), we overestimate the values of the z-axis, which shows that our projection will not be as accurate.

4.3.2 Spherical Distances

Now, we understand the formulation of points of our sphere. We now want to understand the distances between points. As this will be important as we look to conduct a map projection $f : \mathbb{E}^3 \to \mathbb{E}^2$. The hope is to maintain the magnitude of the distances between points through the map projection. We begin with the notion of a Great Circle and a Small Circle.

Definition 4.3.3. *We define a **Great Circle** as the intersection of a plane and the sphere that contains the sphere's diameter. [18, page. 87] A **Small Circle** is the intersection of the sphere and the plane such that it does not contain the sphere's diameter. [32, pages. 220-221].*

Using definition 4.3.3, we get that the spherical distance comes from the length of the shorter arc of the partition of the great circle. This produces the new definition:

Definition 4.3.4. *[9, Section 2.1] We define the distance between two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^2$ (a unit sphere as:*

$$d_{\mathbb{S}^2}(\boldsymbol{x}, \boldsymbol{y}) = \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}) \tag{5}$$

Expanding upon this we arrive at the spherical distance for our Sphere of the Earth being:

$$d_{\mathbb{S}^2}(\boldsymbol{x}, \boldsymbol{y}) = 6,380,000 \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}) \tag{6}$$

The distance function is analogous to the arc length (as expected). We have the radius of the Earth multiplied by the angle of separation between the two points. We now prove that the arc length is a part of the great circle and that the shorter segment is the spherical distance for a unit sphere.

We apply what we have learnt to metro systems. Take any two points in a locale of a metro system; then we can use Definition 4.3.4 to obtain the distances between points. We will find that the distances between points in cities compared to the great circles of the globe are much smaller.

Example: Distance Between Two Points in London

We know that for Earth the circumference of every Great Circle will be $400,000,000m$ to 3 significant figures. If we take two points in the city of London, assume that the Earth is a sphere and that the longitude angle $\hat{\theta}$ as $\hat{\theta} = \theta$ for angles east of the Greenwich Meridian and $\hat{\theta} - \frac{\pi}{2} = \theta$ for angles west of the Greenwich Meridian, and the latitude angle $\hat{\phi}$ is $\frac{\pi}{2} - \hat{\phi} = \phi$ for angles in the North Hemisphere and $\hat{\phi} - \frac{\pi}{2} = \phi$ For angles in the Southern Hemisphere [9, Page 12].

Two of those points are Stockwell with (Longitude, Latitude) $= (51.5°N, 0.13°W)$, angles (θ, ϕ) are denoted $(359.87°, 38.5°)$[26] and the Cartesian Coordinates:

$$(6380000 \cos 359.87° \sin 38.5°, 6380000 \sin 359.87° \sin 38.5°, 6380000 \cos 38.5°)$$

Finsbury Park (Longitude, Latitude) $= (51.6°N, 0.10°W)$, angles (θ, ϕ) are denoted $(359.9°, 38.4°)$ [26] and the Cartesian Coordinates:

$$(6380000 \cos 359.9° \sin 38.4°, 6380000 \sin 359.9° \sin 38.4°, 6380000 \cos 38.4°)$$

We use the formula in Definition 4.3.4 to arrive at the Earth's spherical distance between Stockwell and Finsbury Park being $11300m$ or $11.3km$. If we compare the Earth's spherical distance between the two points to the circumference of the great circle of the Earth, we find that the ratio is approximately 1 : 40000. We can see that as the distances are much smaller. We can infer that the local area will be much smaller than the Earth's surface area. As the relationships are similar, the smaller area could be a good thing as this means that a map projection of the area covered will be prone to more minor errors and more accurate.

We have just explored an example of the spherical distance between points in a certain locale area. What we want to do is preserve this distance as we project the points from the sphere onto a Euclidean plane $\mathbb{E}^2$. The function's name is an isometry, and we'll look into this in the next section.

4.4 Isometrics

Definition 4.4.1. *[9] Let X and Y be two non-empty sets. For two metric spaces (X, d_X) and (X, d_Y). A map between the two metric spaces is defined as **Distance Preserving** or an **Isometry** if $d_Y(f(x), f(y)) = d_X(x, y)\ \forall x, y$.*

Theorem 4.4.2. *[10] Every Isometry is Injective.*

Proof. *Let's say that there exists a non-injective isometry such that $\phi : S \to T$. As ϕ is not injective. There exists a $x, y \in S$, where $x \neq y$ and, $\phi(x) = \phi(y)$. But, $x \neq y \implies d_S(x, y) > 0$ While $\phi(x) = \phi(y) \implies d_T(x, y) = 0$. This is a contradiction; ϕ is a isometry.*

Whilst we get that a isometry is injective. This does not imply that they are automatically bijections. An example of this is the discrete metric on the integers $(\mathbb{Z}, d)$ as $d(x, x) = 0$; $x \neq y \implies d(x, y) = 1$. We can define the map $\phi : \mathbb{Z} \to \mathbb{Z}$ as $\phi(n) = 2n$ whilst this is injective and an isometry this is not surjective hence not a bijection.

Definition 4.4.3. *[36] [10]A **Global Isometry** is a isometry that is bijective. A **Local Isometry** is a bijective isometry between the subsets of two surfaces.*

Definition 4.4.4. *A **Euclidean Isometry** [10, pages 39 - 46] $\phi : \mathbb{E}^n \to \mathbb{E}^n$ is a geometric transformation of the euclidean space that preserves the Euclidean distance for every pair of points. There are four types of Euclidean Isometries: Translation, Reflection, Rotation or any sequence of the previously mentioned (i.e. a reflection and a translation).*

Definition 4.4.5. *[9, Definition 1.14] A Euclidean Transformation of the Euclidean Space $\mathbb{E}^n$ is a map $f : \mathbb{E}^n \to \mathbb{E}^n$ of the form.*

$$f = f_{A,\boldsymbol{t}} : \boldsymbol{x} \to A\boldsymbol{x} + t$$

Where A represents an $n \times n$ orthogonal matrix and $\boldsymbol{t} \in \mathbb{E}^n$ is any vector. f represents a composition of transformations. With $\boldsymbol{x} \to A\boldsymbol{x}$ represents an orthogonal transformation and $\boldsymbol{t} : \boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{t}$ a translation.

We expand on the definition by focusing on the orthogonal matrix component. We begin with a proposition.

Proposition 4.4.6. *[9, Proposition 1.12] Let A be an orthogonal matrix of order n be the linear transformation $f_A : \mathbb{E}^n \to \mathbb{E}^n$. Then, A preserves the scalar product and the norm on $\mathbb{E}^n$. Hence, $\forall \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n$*

$$(A\boldsymbol{x}) \cdot (A\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y} \quad \textit{and} \quad \|A\boldsymbol{x}\| = \|\boldsymbol{x}\|$$

Proof: *$\forall \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n$, using the associativity axiom of matrix multiplication and the fact that $ATA = I_n$, we have*

$$(A\boldsymbol{x}) \cdot (A\boldsymbol{y}) = (A\boldsymbol{x})^T(A\boldsymbol{y}) = (\boldsymbol{x}^T A^T)(A\boldsymbol{y}) = \boldsymbol{x}^T(A^T A)\boldsymbol{y} = \boldsymbol{x}^T I_n \boldsymbol{y}$$
$$= \boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y} \qquad \square$$

Theorem 4.4.7. *[9, Theorem 1.14] Let A be an $n \times n$ orthogonal matrix and let $f_A : \mathbb{E}^n \to \mathbb{E}^n$ be the linear transformation given by*

$$f_A(\boldsymbol{x}) = A\boldsymbol{x}$$

$\forall \boldsymbol{x} \in \mathbb{E}^n$. Then f_A is a Euclidean Isometry.

Proof: *$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n$ we have*

$$d(f_A(x), f_A(y)) = \|A\boldsymbol{x} - A\boldsymbol{y}\| = \|A(\boldsymbol{x} - \boldsymbol{y})\|$$

By the reverse distributivity axiom . Using the result in proposition 4.4.6. We have,

$$\|A(\boldsymbol{x} - \boldsymbol{y})\| = \|\boldsymbol{x} - \boldsymbol{y}\| = d(\boldsymbol{x}, \boldsymbol{y}) \qquad \square$$

Now, for a translation.

Theorem 4.4.8. *[9, Theorem 1.17] Every Euclidean transformation $f_{A,\boldsymbol{t}} : \mathbb{E}^n \to \mathbb{E}^n$ is an Euclidean isometry*

Proof: *We've seen that in Theorem 4.4.7 that $f_{A,\mathbf{0}} = f_A$ is an isometry. We now seek to prove that translations are Euclidean isometries. $\forall \boldsymbol{t} \in \mathbb{E}^n$. We have,*

$$d(f_A(x), f_A(y)) = d(\boldsymbol{x}+\boldsymbol{t}, \boldsymbol{y}+\boldsymbol{t}) = \|(\boldsymbol{x}+\boldsymbol{t}) - (\boldsymbol{y}+\boldsymbol{t})\| = \|\boldsymbol{x}-\boldsymbol{y}\| = d(\boldsymbol{x}, \boldsymbol{y}) \qquad \square$$

Using theorems 4.4.7 and 4.4.8 we can infer and prove the following theorem:

Theorem 4.4.9. *The composition of two isometries $f_{A,\boldsymbol{t}} = f_{\boldsymbol{t}} \circ f_A$ is also an isometry.*

Proof: *Considering in theories 4.4.7 and 4.4.8 have two isometric functions, we should expect a similar result. However, $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n$ and using proposition 4.4.6. We have*

$$d(f_{A,\boldsymbol{t}}(x), f_{A,\boldsymbol{t}}(y)) = \|(A\boldsymbol{x}+\boldsymbol{t}) - (A\boldsymbol{y}+\boldsymbol{t})\| = \|A\boldsymbol{x} - A\boldsymbol{y}\| = \|A(\boldsymbol{x}-\boldsymbol{y})\| = \|\boldsymbol{x}-\boldsymbol{y}\| = d(\boldsymbol{x}, \boldsymbol{y}) \qquad \square$$

4.4.1 Application: Isometries In $\mathbb{R}^2$

We now look at an application of applying euclidean isometries in the $\mathbb{R}^2$ plane. We will show and derive the proofs for each euclidean transformation and isometry formula. Then, we will look at an application of applying the transformations to a point in $\mathbb{R}^2$.

The proof and derivation for this section are provided by source [9, Example 1.11]. We extended it with proof that the proof that A_θ is orthogonal where we conduct the following example

in the $\mathbb{R}^2$ space, orthogonal transformations are rotations about the origin $\mathbf{0}$ or a reflection through a line or an axis through $\mathbf{0}$.

Rotation: A rotation through the angle θ in the anti-clockwise direction transforms the basis unit vector $\boldsymbol{e}_1 = (1,0) \to \boldsymbol{e}_1^{'} = (\cos\theta, \sin\theta)$. Whilst, the other standard basis vector $\boldsymbol{e}_2 = (1,0) \to \boldsymbol{e}_2^{'} = (-\sin\theta, \cos\theta)$. The vectors $\boldsymbol{e}_1^{'}$ and $\boldsymbol{e}_2^{'}$ are orthonormal. The corresponding 2×2 matrix is

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{7}$$

We'll prove that A_θ is orthogonal. We know that for a matrix to be orthogonal, the determinant must be equal to 1 or -1.

$$det(A_\theta) = (\cos\theta)\cdot(\cos\theta) - (-\sin\theta)\cdot(\sin\theta) = \cos^2\theta + \sin^2\theta = 1$$

Hence, the transformation is orthogonal.

Reflections: A reflection in (x, y) coordinates about the x axis is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This is is clearly a orthogonal matrix as $det(B_\theta) = -1$. The reflection formula can be infered from the last result. Hence the formula for the reflection in an axis through $\mathbf{0}$ making the angle θ with the positive x-axis is given by $\boldsymbol{x} \to B_\theta \boldsymbol{x}$, where

$$B_0 = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \tag{8}$$

The matrix is an orthogonal matrix as

$$det(B_\theta) = (\cos 2\theta)\cdot(-\cos 2\theta) - (\sin 2\theta)\cdot(\sin 2\theta) = -\cos^2 2\theta - \sin^2 2\theta = -1$$

Application: Examples of the application

Let $\boldsymbol{x} = (3,4)^T$ where $\boldsymbol{x} \in \mathbb{E}^2$ and $\|\boldsymbol{x}\| = 5$. Then for the following examples of Euclidean transformations we have:

Example 1: Rotation of $45°$ about the origin $\boldsymbol{x} \to A\boldsymbol{x}$ Hence,

$$\begin{pmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{7\sqrt{2}}{2} \end{pmatrix}$$

Example 2: Reflection on the line $y = x$ where $\theta = 45°$ $\boldsymbol{x} \to B_\theta \boldsymbol{x}$ Hence,

$$\begin{pmatrix} \cos 2 \cdot 45 & \sin 2 \cdot 45 \\ \sin 2 \cdot 45 & -\cos 2 \cdot 45 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Example 3: Translation by a vector $(9,9)^T$ $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{t}$ Hence,

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 9 \\ 9 \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \end{pmatrix}$$

Example 4: A glide reflection - Reflection on line $y = x$ where $\theta = 45$ and a translation by a vector $(9,9)^T$ $\boldsymbol{x} \to B_\theta \boldsymbol{x} + \boldsymbol{t}$

$$\begin{pmatrix} \cos 2 \cdot 45 & \sin 2 \cdot 45 \\ \sin 2 \cdot 45 & -\cos 2 \cdot 45 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 9 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 9 \\ 9 \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \end{pmatrix}$$

4.5 Geodesics and Basic Properties

Definition 4.5.1. *[9] Let C be a curve in any metric space X. Then, X is a* ***geodesic*** *in if for any $\boldsymbol{a}, \boldsymbol{b} \in C$, the shortest path in X from $\boldsymbol{a}$ to $\boldsymbol{b}$ is the line segment of C between $\boldsymbol{a}$ and $\boldsymbol{b}$.*

Definition 4.5.2. *[30, Deinition 9.1.1] Let γ be a curve on a surface S. Then, the curve is called a* ***geodesic*** *is $\ddot{\boldsymbol{\gamma}}(t)$ is zero or perpendicular to the tangent plane of the surface at the point $\boldsymbol{\gamma}(t)$. i.e, parallel to its unit normal, for all values of the parameter t.*

From the given definitions, we can unlock the following properties of the geodesic:

Proposition 4.5.3. *[30, Proposition 9.1.2] Any Geodesic has constant speed.*

Proof: *Let $\boldsymbol{\gamma}(t)$ be a geodesic on a surface S. Then, denoting $\frac{d}{dt}$ by a dot $\cdot$ and using properties of the dot product and norms:*

$$\frac{d}{dt}\|\dot{\boldsymbol{\gamma}}\|^2 = \frac{d}{dt}(\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}) = 2\ddot{\boldsymbol{\gamma}}\dot{\boldsymbol{\gamma}} = 0$$

Since $\boldsymbol{\gamma}$ is a geodesic, $\ddot{\boldsymbol{\gamma}}$ is perpendicular to the tangent plane hence perpendicular to the tangent vector $\dot{\boldsymbol{\gamma}}$ so $\ddot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} = 0$ and the magnitude of $\dot{\boldsymbol{\gamma}}$ is a constant. $\square$

Proposition 4.5.4. *[30, Proposition 9.1.4] Any straight line or a part of a straight line on a surface is a geodesic.*

Proof: *A general parametisation of any straight line comes in the form:*

$$\boldsymbol{\gamma}(t) = \boldsymbol{a} + \boldsymbol{b}t$$

Where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors and t is a parameter. Clearly $\dot{\boldsymbol{\gamma}} = 0$. Hence, any straight line has constant speed and therefore by proposition 4.5.3. Any straight line on a surface is a geodesic. $\square$

4.6 Curvature

Definition 4.6.1. *[42] We define **Curvature** as the amount a curve deviates from a straight line.*

We want to look at the mathematical aspects of the map projection we will perform on our sphere's surface patch. We need to look at aspects of Curvature, such as the Gaussian curvature of a surface. Once we understand this, Theorema Egresium will show that there is no way to project a sphere to a plane accurately without any distortion. We begin with the concept of elementary Curvature.

4.6.1 Elementary Curvature

We have already seen that the length of a curve is given by,

$$s = \int_{\gamma} \|\dot{\gamma}(t)\| dt \tag{9}$$

From this we can infer the definition:

Definition 4.6.2. *[30, Deinition 2.1.1] For γ such that it is a unit-speed curve with parameter t, the curvature $\kappa(t)$ at the point $\gamma(t)$ is defined to be $\|\ddot{\gamma}(t)\|$*

From the definition we should have that the curvature of a straight line is 0. Take a straight line equation $\gamma = mt + c$ and differentiate one we have $\dot{\gamma} = m$ and twice $\ddot{\gamma} = 0$. As expected.

Application: The Curvature Of A Circle

We now consider the curvature of circles in $\mathbb{R}^2$ centered at the origin with radius R and unit speed , The paramisation of the circle is given by:

$$\gamma(t) = (R \cos \frac{t}{R}, R \sin \frac{t}{R})$$

We compute $\dot{\gamma}$ to get:

$$\dot{\gamma}(t) = (-\sin \frac{t}{R}, \cos \frac{t}{R})$$

We compute $|\dot{\gamma}(t)|$

$$|\dot{\gamma}(t)| = \sqrt{\left(-\sin \frac{t}{R}\right)^2 + \left(\cos \frac{t}{R}\right)^2} = 1$$

As expected. Then we differentiate again to get:

$$\dot{\gamma}(t) = (-\frac{1}{R} \cos \frac{t}{R}, -\frac{1}{R} \sin \frac{t}{R})$$

We compute the magnitude to arrive at $\frac{1}{R}$ Hence the curvature of a circle with radius R and unit speed is defined as:

$$\kappa = \frac{1}{R}$$

The result makes sense as it shows that a circle with a smaller radius will have more Curvature than a larger circle with smaller Curvature. Nevertheless, what about curves $\gamma(t)$ that cannot be written explicitly. In this case, we consider a formula that calculates the Curvature of a curve considering $\gamma(t)$.

Proposition 4.6.3. *[46, Proposition 1.4.2] Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ then the curvature is given by the formula:*

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3} \tag{10}$$

Proof. *Let s be a unit speed parameter for γ then by the chain rule*

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt}$$

Therefore,

$$\kappa = \left\| \frac{d^2\gamma}{ds^2} \right\| = \left\| \frac{d}{ds}\left(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}} \right) \right\| = \left\| \frac{\frac{d}{dt}\left(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}} \right)}{\frac{ds}{dt}} \right\| = \left\| \frac{\frac{ds}{dt}\frac{d^2\gamma}{dt^2} - \frac{d^2s}{dt^2}\frac{d\gamma}{dt}}{\left(\frac{ds}{dt}\right)^3} \right\| \tag{11}$$

We have that,

$$\left(\frac{ds}{dt}\right) = \|\dot{\gamma}\|^2 = \dot{\gamma} \cdot \dot{\gamma}$$

and differentiating w.r.t t gives

$$\frac{ds}{dt}\frac{d^{2s}}{dt^2} = \dot{\gamma} \cdot \ddot{\gamma}$$

Apply the two equations to 11

$$\kappa = \left\| \frac{\left(\frac{ds}{dt}\right)^2 \ddot{\gamma} - \frac{d^2s}{dt^2}\frac{ds}{dt}\dot{\gamma}}{\left(\frac{ds}{dt}\right)^4} \right\| = \frac{\|(\dot{\gamma} \cdot \dot{\gamma}\ddot{\gamma} - (\dot{\gamma} \cdot \ddot{\gamma})\ddot{\gamma}\|}{\|\dot{\gamma}\|^4}$$

Via the use of the vector triple identity (where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$)

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

We have

$$\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma}) = (\dot{\gamma} \cdot \dot{\gamma})\ddot{\gamma} - (\dot{\gamma} \cdot \ddot{\gamma})\dot{\gamma}$$

and by using the fact that $\dot{\gamma}$ $\ddot{\gamma} \times \dot{\gamma}$ are perpendicular vectors, we have

$$\|\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma})\| = \|\dot{\gamma}\|\|\dot{\gamma} \times \ddot{\gamma}\|$$

Therefore,

$$\frac{\|(\dot{\gamma} \cdot \dot{\gamma})\ddot{\gamma} - (\dot{\gamma} \cdot \ddot{\gamma})\dot{\gamma}\|}{\|\dot{\gamma}\|^4} = \frac{\|\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma})\|}{\|\dot{\gamma}\|^4} = \frac{\|\dot{\gamma}\|\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^4} = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3} \quad \square$$

Example. Calculating the curvature of a parametised curve.

Take the parametised curve $\gamma(t) = (a\cos t, b\sin t, t)$ we seek to use proposition 4.6.3 to calculate the curvature of the curve. We differentiate once and twice to obtain the values of $\dot{\gamma}$ and $\ddot{\gamma}$.

$$\dot{\gamma(t)} = (-a\sin t, a\cos t, 1) \quad \text{and} \quad \ddot{\gamma(t)} = (-a\cos t, a\sin t, 0)$$

Computing $\ddot{\gamma(t)} \times \dot{\gamma(t)}$, we have

$$\ddot{\gamma(t)} \times \dot{\gamma(t)} = (-a\sin t, a\cos t, -a^2)$$

We now have the ingredients to compute the curvature

$$\kappa = \frac{\|(-a\sin t, a\cos t, -a^2)\|}{\|(-a\sin t, a\cos t, 1)\|^3} = \frac{(a^2 + a^4)^{\frac{1}{2}}}{(a^2+1)^{\frac{3}{2}}}$$

We see that there are no parameters t in this equation. The Curvature for this curve will be a constant.

4.6.2 The First Fundamental Form

We now seek to study the curvature of surfaces and surface patches. We follow 9 and look to compute the length of a curve on a surface S. We need to find the tangent vectors to the surface to compute this. A vector tangent to the surface $\dot{\gamma}$. We compute such from the following definition.

Definition 4.6.4. *[30, Definition 6.15] let $\boldsymbol{p}$ be a point on the surface S the **first fundamental form** of S at $\boldsymbol{p}$ is provided by associating the tangent vectors $\boldsymbol{v}, \boldsymbol{w} \in T_{\boldsymbol{p}}S$. It suffices that*

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{\boldsymbol{p}.S} = \boldsymbol{v} \cdot \boldsymbol{w}$$

Where $\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{\boldsymbol{p}.S}$ represents the dot product of the tangent vectors restricted to S at $\boldsymbol{p}$

Let us consider familiarly writing the formula. Let's sat that $\boldsymbol{\sigma}(u, v)$ is a surface patch of S. Then, any tangent vector to S at the point $\boldsymbol{p}$ in the image $\boldsymbol{\sigma}$ can be composed as a linear combination of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_v$. We define the maps:

Proof of the following is provided by sources [30] and [20]

$$du := T_{\mathbf{p}}S \to \mathbb{R} \quad \text{and} \quad dv := T_{\mathbf{p}}S \to \mathbb{R}$$

such that the linear maps:

$$du(u) = \lambda \quad dv(v) = \mu$$

Provided that $\mathbf{v} = \lambda \underline{\sigma_u} + \mu \underline{\sigma_v}$ For some $\lambda, \mu \in \mathbb{R}$. Using the fact that $\langle , \rangle$ is symmetric by-linear form we have

$$\langle \mathbf{v}, \mathbf{v} \rangle = \lambda^2 \langle \ \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_u \rangle + 2\lambda\mu \langle \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_u \rangle + \mu \langle \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_u \rangle \tag{12}$$

Let

$$E = \|\boldsymbol{\sigma}_u\|^2 \qquad F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v \qquad G = \|\boldsymbol{\sigma}_v\|^2$$

Then equation 12 becomes

$$\langle \mathbf{v}, \mathbf{v} \rangle = E\lambda^2 + 2F\lambda\mu + 6\mu^2 = Edu(\mathbf{u})^2 + 2Fdu(\mathbf{u})dv(\mathbf{v}) + Gdv(\mathbf{v})^2$$

In traditional form, this gives:

$$Edu^2 + 2Fdudv + Gdv^2 \tag{13}$$

This is the first fundamental form for $\boldsymbol{\sigma}(u, v)$. Where E, F, G are linear maps and du, dv is provided by the surface patch we choose. Hence, can derive $\dot{\gamma}$. if we have $\dot{\gamma} = \boldsymbol{\sigma}(u(t), v(t))$ for some smooth functions of $u(t)$ and $v(t)$ using equations 9 and 13. It suffices that the length of the curve is

$$s = \int E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 dt \tag{14}$$

Application. First Fundamental Form for our sphere

Lets calculate the FFF for our Earth Sphere. We have that

$$\boldsymbol{\sigma}(\theta, \phi) = \mathbf{p} = \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 6,380,000 \cos\theta \sin\phi \\ 6,380,000 \sin\theta \sin\phi \\ 6,380,000 \cos\phi \end{bmatrix} \tag{15}$$

Where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi < 2\pi$. We calculate $\underline{\sigma_u}$ and $\underline{\sigma_v}$

$$\boldsymbol{\sigma}_\theta = (-6,380,000 \sin\theta \sin\phi, 6,380,000 \cos\theta \sin\phi, 0)$$

and

$$\boldsymbol{\sigma}_\phi = (6,380,000 \cos\theta \cos\phi, 6,380,000 \sin\theta \cos\phi, -6,380,000 \sin\phi)$$

Therefore,

$$E = \|\boldsymbol{\sigma}_\theta\|^2 = 6,380,000^2 \sin^2\phi$$

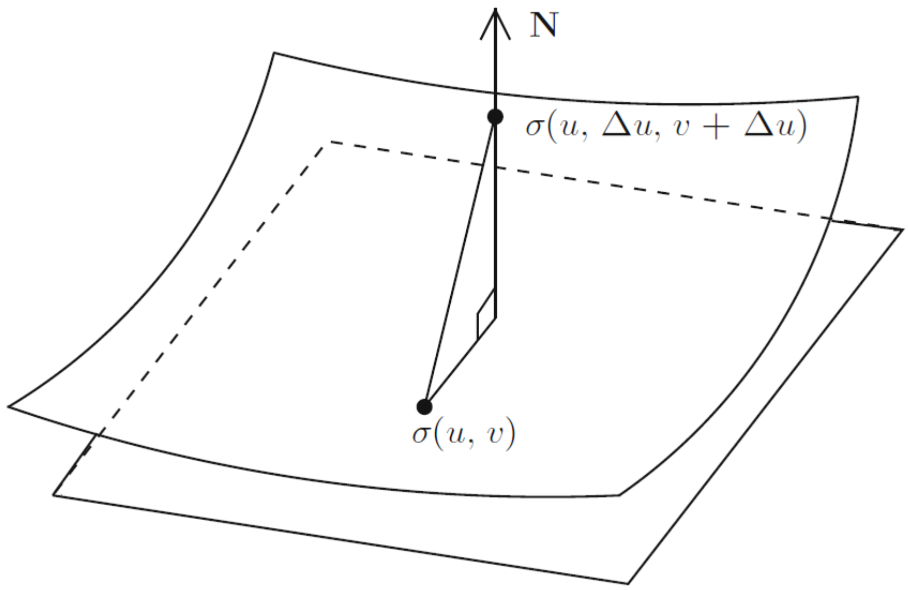


Figure 12: The plane showing the deviation of a curve from a plane. [30]

and,

$$F = \boldsymbol{\sigma}_\theta \cdot \boldsymbol{\sigma}_\phi = 0$$

lastly,

$$G = \|\boldsymbol{\sigma}_\phi\|^2 = 6,380,000^2$$

So, we have our FFF for our Earth Sphere is given as

$$6,380,000^2 \sin^2 \phi d\theta^2 + 6,380,000^2 d\phi^2 \tag{16}$$

Result 16 will be of great importance when we talk about the Gaussian Curvature of the sphere of our planet.

4.6.3 The Second Fundamental Form

Proof of the following derivation is provided by sources [30] and [20]

We continue with the discussion of the curvature of the sphere by considering how a surface deviates from a plane. Let $\boldsymbol{\sigma}$ be a surface patch in $\mathbb{R}^3$. With a standard normal $\boldsymbol{N}$. We evaluate what happens when $\boldsymbol{\sigma}(u,v)$ becomes $\boldsymbol{\sigma}(u+\Delta, v+\Delta)$. Whilst this happens we have that the plane deviates from the tangent plane of $\boldsymbol{\sigma}(u,v)$ by

$$(\boldsymbol{\sigma}(u+\Delta, v+\Delta) - \boldsymbol{\sigma}(u,v)) \cdot \mathbf{N}$$

Figure 12 shows the relationship described. We apply the 2-dimensional taylor expansion to arrive at:

$$\boldsymbol{\sigma}_u \Delta u + \boldsymbol{\sigma}_v \Delta v + \frac{1}{2}(\boldsymbol{\sigma}_{uu}(\Delta u)^2 + 2\boldsymbol{\sigma}_{uv}\Delta u \Delta v + \boldsymbol{\sigma}_v(\Delta v)^2) + remainder$$

We find that the values of the $remainder/((\Delta u)^2 + (\Delta v)^2) \to 0$ as $((\Delta u)^2 + (\Delta v)^2)$. Given that, $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are tangent vectors of the surface and perpendicular to the normal vector $\mathbf{N}$. It suffices that the deviation of $\boldsymbol{\sigma}$ is

$$\frac{1}{2}(L(\Delta u)^2 + 2M\Delta u \Delta v + N(\Delta v)^2) \tag{17}$$

Where

$$L = \boldsymbol{\sigma}_{uu} \cdot \boldsymbol{N} \qquad M = \boldsymbol{\sigma}_{uv} \cdot \boldsymbol{N} \qquad N = \boldsymbol{\sigma}_{vv} \cdot \boldsymbol{N} \tag{18}$$

With the results of 17 and 18. We can define the second fundamental form as:

Definition 4.6.5. *[30, page 160][20, page 119] The **second fundamental form** (denoted SFF for brevity) of the surface patch $\boldsymbol{\sigma}$ is provided by the expression.*

$$Ldu^2 + 2Mdudv + Ndv^2 \tag{19}$$

We'll now look at a example and a application of applying the SFF.

Example. The second fundamental form of a plane

Consider a plane $\boldsymbol{\sigma}(u,v) = \mathbf{i} + (u+1)\mathbf{j} + (v-1)\mathbf{k}$ We have that

$$\boldsymbol{\sigma}_u = \mathbf{j} \qquad \text{and} \qquad \boldsymbol{\sigma}_v = \mathbf{k}$$

Which are constant vectors. Hence we have $\boldsymbol{\sigma}_{uu} = \boldsymbol{\sigma}_{uv} = \boldsymbol{\sigma}_{vv} = 0$. The second fundamental form of the plane is 0.

Application. The second fundamental form of our Earth's Sphere

Using equation 15. we find the following relations.

$$\boldsymbol{\sigma}_\theta = (-6,380,000 \sin\phi \sin\theta, 6,380,000 \sin\phi\cos\theta, 0)$$

$$\boldsymbol{\sigma}_\phi = (6,380,000 \cos\phi\cos\theta, 6,380,000\cos\phi\sin\theta, -6,380,000\sin\phi)$$

Therefore,

$$\boldsymbol{\sigma}_\phi \times \boldsymbol{\sigma}_\theta = (6,380,000^2 \sin^2\phi\cos\theta, 6,380,000^2\sin^2\phi\sin\theta, 6,380,000^2\sin\phi\cos\phi)$$

$$\|\boldsymbol{\sigma}_\phi \times \boldsymbol{\sigma}_\theta\| = 6,380,000^2 \sin\phi$$

Therefore,

$$\boldsymbol{N} = \frac{\boldsymbol{\sigma}_\phi \times \boldsymbol{\sigma}_\theta}{\|\boldsymbol{\sigma}_\phi \times \boldsymbol{\sigma}_\theta\|} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$\boldsymbol{\sigma}_{\phi\phi} = (-6,380,000\sin\phi\cos\theta, -6,380,000\sin\phi\sin\theta, -6,380,000\cos\phi)$$

$$\boldsymbol{\sigma}_{\phi\theta} = (-6,380,000\cos\phi\sin\theta, 6,380,000\cos\phi\cos\theta, 0)$$

Hence,

$$\boldsymbol{\sigma}_{\theta\theta} = (-6,380,000\sin\phi\cos\theta, -6,380,000\sin\phi\sin\theta, 0)$$

$$L = \boldsymbol{\sigma}_{\phi\phi}\cdot\boldsymbol{N} = -6,380,000 \quad M = \boldsymbol{\sigma}_{\phi\theta}\cdot\boldsymbol{N} = 0 \quad N = \boldsymbol{\sigma}_{\theta\theta}\cdot\boldsymbol{N} = -6,380,000\sin^2\phi$$

and our Second Fundamental Form is

$$-6,380,000\sin^2\phi d\theta^2 - 6,380,000 d\phi^2 \tag{20}$$

Now we understand the two concepts of the first fundamental form and the second fundamental form. We will now combine the two concepts to introduce the concept of Gaussian Curvature.

4.6.4 Gaussian Curvature

Definition 4.6.6. *The **Gaussian Curvature** K of a Surface Patch $\boldsymbol{\sigma}$ is defined as the formula combining results in 13 and 18. The formula is:*

$$K = \frac{LN - M^2}{EG - F^2} \tag{21}$$

Where L,M and N represent the coefficients of the second fundamental form and E, G and F represent the coefficients of the first fundamental form.

Application: The Gaussian Curvature of the Earths Sphere

Using the relations:

$$E = 6,380,000^2 \sin^2 \phi \qquad F = 0 \qquad G = 6,380,000^2$$

and

$$L = -6,380,000 \qquad M = 0 \qquad N = -6,380,000 \sin^2 \phi$$

We place the given values into 21 to get the Gaussian Curvature for our Earth's Sphere.

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(-6,380,000) \cdot (-6,380,000 \sin^2 \phi) - 0}{(6,380,000^2 \sin^2 \phi) \cdot (6,380,000^2) - 0} = \frac{1}{6,380,000^2} = 2.456 \times 10^{-14} \tag{22}$$

We can learn that the Gaussian Curvature for our Earth's Sphere is a positive constant. We'll talk about the implications of this in the next section.

From this result, we can infer that any sphere will have a positive Gaussian Constant if we substitute $6,380,000$ for r (as r is a constant representing the radius of the sphere). We'll find we have the following paramatisation of a sphere:

$$\boldsymbol{\sigma}(\theta, \phi) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \sin\phi \\ r \sin\theta \sin\phi \\ r\cos\phi \end{bmatrix} \tag{23}$$

We find that the relationship between the Gaussian Curvature and the radius r is: $K = \frac{1}{r^2}$

Application: The Gaussian Curvature of a Generic Plane

We now calculate the Gaussian Curvature of a Generic Plane: The Generic Plane Equation is given as:

$$ax + by + cz = 0$$

The parameterisation of a Genric Sphere is:

$$\boldsymbol{\sigma}(u, v) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ \alpha u + \beta v + \gamma \end{bmatrix} \tag{24}$$

Where $a, b, c, d, \alpha, \beta, \gamma \in \mathbb{R}$. We calculate our tangent vectors $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$

$$\boldsymbol{\sigma}_u = (0, 1, \beta) \qquad \boldsymbol{\sigma}_v = (1, 0, \alpha)$$

The tangent vectors are vectors of constants. Hence we have that

$$\boldsymbol{\sigma}_{uu} = \boldsymbol{\sigma}_{uv} = \boldsymbol{\sigma}_{vv} = 0$$

and the following relations

$$E = 1 + \beta^2 \qquad F = \alpha\beta \qquad G = 1 + \alpha^2$$

and

$$L = M = N = 0$$

The Gaussian Curvature for a generic plane is given as:

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(0) \cdot (0) - 0}{(1 + \beta^2) \cdot (1 + \alpha^2) - \alpha\beta^2} = \frac{0}{(1 + \beta^2) \cdot (1 + \alpha^2) - \alpha\beta^2} = 0 \tag{25}$$

The results shows that the Gaussian Curvature of any generic plane is zero. We also find that the Gaussian Curvature for a generic plane is a constant.

Application: The Gaussian Curvature of a Cylinder of unit radius

We now calculate the Gaussian Curvature for a cylinder of unit radius. Consider the parameterisation of the sphere:

$$\boldsymbol{\sigma}(u,v) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos u \\ \sin u \\ v \end{bmatrix} \tag{26}$$

We caluate the following :

$$\begin{aligned} \boldsymbol{\sigma}_u &= (-\sin u, \cos u, 0) \\ \boldsymbol{\sigma}_{uu} &= (-\cos u, -\sin u, 0) \\ \boldsymbol{\sigma}_v &= (0,0,1) \\ \boldsymbol{\sigma}_{vv} &= (0,0,0) \\ \boldsymbol{\sigma}_{uv} &= (0,0,0) \\ \boldsymbol{N} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} &= (\cos u, -\sin u, 0) \end{aligned}$$

and we obtain the following relations

$$E = 1 \qquad F = 0 \qquad G = 1$$

and

$$L = -\cos^2 u + \sin^2 u \qquad M = 0 \qquad N = 0$$

Hence, the Gaussian Curvature for a Cylinder of unit radius is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(-\cos^2 u + \sin^2 u) \cdot (0) - 0}{(1) \cdot (1) - 0} = \frac{0}{1} = 0 \tag{27}$$

The Gaussian Curvature for a Cylinder with a unit radius is zero. We can follow the constant analogy for a sphere and extend the notion of the Gaussian Curvature for any cylinder of radius r where $r > 0$ has a Gaussian Curvature of zero. We also find that the Gaussian Curvature for a cylinder is a constant.

4.6.5 Theorema Egregium and its implications

So, what do the results 22 ,25 and 27 imply. Essentially, we want to be able to perform a map projection $\mathbb{R}^3 \to \mathbb{R}^2$. But, we find that the Gaussian Curvatures for our three surfaces have different values whilst being a constant. The Gaussian Curvature for our Earth's Sphere has positive constant Gaussian Curvature, whilst the other two surfaces have zero Gaussian Curvature. We have reached a roadblock this is because of the **Theorema Egregium** which states:

Theorem 4.6.7. *[21, Theorem 4.1.5] The Gaussian Curvature of Surfaces is invariant under local isometries. i.e. two isometric surfaces always have the same Gaussian Curvature.*

Given that our three surfaces have different Gaussian Curvatures of different values. What can be implied by the theorem is that there exists no isometry between any sphere, any cylinder, and any plane as a Sphere cannot be unfolded to a flat plane without any form of distortion. Conversely, a plane cannot be folded to become a sphere without distortion.[21] The impact of this on our map means that we will not be able to provide an accurate view of the environment. Distances on our plane will not accurately reflect the distances on the planet. In addition, this means the information about the environment will be lost. However, we do find that we can take liberties when producing our map and defining aspects of the map. For example, the scale factor, the placement of objects, and the map's generalisation and visualisation.

Furthermore, we do have that the relationship between the Gaussian Curvature and the radius of the sphere is given as $K = \frac{1}{r^2}$, which shows that the larger the radius of the sphere, the smaller the Gaussian Curvature. The Gaussian Curvature will always be a positive constant if we have a large r (like in our case). $K \to 0$ as $r \to \infty$ the impact of distortion on the distances on the sphere to a plane can be minimised by having a large sphere radius. The distances on the sphere will still be distorted, but the amount of distortion needed lowers as the radius increases. Thereby increasing accuracy.

Finally, what we can infer from this is that every map projection will distort some distances between points on our sphere. We'll now look at different types of map projections we can use to fulfil the process of creating a map.

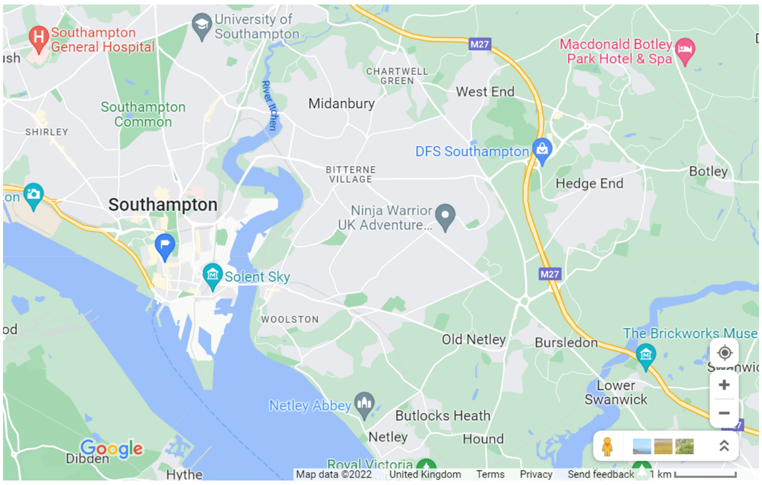


Figure 13: A Google Map showing the area of Southampton with the Geometric scale of 1:1km shown on the bottom right. [26]

4.7 Conducting Our Map Projection

This section is based on source [28], source [15], source [45], source [22] and section 65 of source [20]. With further details and discussion of the implications was provided by myself .

We now look to conduct a map projection for our Earth's sphere. The map projection we will be performing is a central cylindrical map projection called the *Mercator Projection*. The Mercator projection is a conformal map projection meaning that the angles between points on the surface are preserved [30, Section 6.3]. However, in section 4.6.5, we see that any map projection will not preserve the distances on the sphere on the plane, which means that our cylindrical projections will also have some distortion between points on a plane. We will begin with the concept of homothety and Scale, talking about deriving a general formula for any cylindrical map projection. Furthermore, we will discuss applying the general formula for the Mercator projection.

4.7.1 Scale Of the Map

Definition 4.7.1. *[22] [28] There are three ways to define the Scale of a map. Either Arithmetically, Geometrically, or through verbiage. They are respectively defined as:*

Arithmetical *- The scale can represented in the form of a ratio such as 1:3000m - where in this case one unit represents 3000m or 3km. The ratio of distances will be given as* $s = \frac{1}{3000}$ *or in a generic case* $s = \frac{1}{n}$ *for the ratio* $1 : n$.

Geometrical *- A visual representation of distances. An example is a map from Google Maps. Like, figure 13 details the ratio of distances via visual aid.*

Verbal *- Verbiage indicating the approximate ratio of distances on a map.*

When we apply a scale to a map. The function undergoes a Homothety

Definition 4.7.2. *[8] A* ***homothety*** *is defined as a transformation of space which dilates distances for a fixed point. A homothety could be an enlargement (where we zoom in on a figure/ figure is larger), identity transformation (result in no change in figure/figure is congruent), otherwise a contraction (resulting in a smaller figure).*

An application of a homothety can be applied to metric spaces. Take an isomemtry in between Euclidean n-spaces $f : \mathbb{E}^n \to \mathbb{E}^n$ where $d(f(x), f(y)) = d(x, y)$. f can be a simple scaling that would be considered a homothety.

4.7.2 Extent and Resolution of the Map

When constructing a map, it is important to consider the extent and the resolution of the map. The map's extent describes the visible area of representation of the map [22]. For example, in figure 13 the extent the is the Central to Eastern portions of the Southampton Area and the suburbs of Hedge End and Netley. The map's resolution is the small estimate unit that is mapped [22]; in the case mentioned earlier, the most satisfactory level of spatial data we can see is the Roads, rivers, and towns; hence, our resolution for this map.

4.7.3 Cylindrical Projection

We are focusing on creating a cylindrical projection. Three key aspects need to be considered before undergoing a map projection.

Definition 4.7.3. *[28] We define the Equator, Meridians and Parallels of Latitude, respectively, as the following:*

1. *The **Equator** is the great circle equidistant from the North and South Poles.*
2. *The **Meridians** are the circular arcs joining the North or South Poles. The arcs are perpendicular to the Equator and are the curves one transverses when travelling due north or south from any point.*
3. *The **Parallels of Latitude (or Parallels)** are the circles perpendicular to the meridians. These are also circles of fixed distance from the North or South Poles and are curves one transverse when travelling east or west from any point.*

We undergo the process of a cylindrical projection to construct a map via the following process [28, Page 3 or 235]: [45, Similar Method on Page 6-7]

1. We represent the Equator as a horizontal line segment. The length of the line segment is used to determine the scale factor of the map produced along the Equator.
2. We denote L as the length of the Equator; w as the width of the map - the length of the horizontal segment representing the Equator. The distances along the Equator are represented by the fixed factor $\frac{w}{L}$,
3. Then, the meridians will be represented as vertical lines whose length could be finite or infinite. The parallels are represented by horizontal line segments of the Equator's same fixed length w.
4. The final step involves cutting along a meridian and unrolling the cylinder to a flat plane to produce our map of the Earth.

The result of the cylindrical projection allows us to access two properties [28]:

1. The map produced will be in the shape of a rectangle or an infinite vertical strip representing all of the Earth except the poles with two vertical sides corresponding to a unique vertical line.
2. The map produced will have a constant scale factor along every parallel of latitude. Namely, the quarter circle along a meridian from the equator to the North or South Pole is divided into $90°$, and the latitude of any point is the number of degrees along the meridian north or south of the equator. It follows that at latitude β north or south, the parallel is a circle of radius $R\cos\beta$ where R is the radius of the Earth and the length of the equator is $L = 2\pi R$.

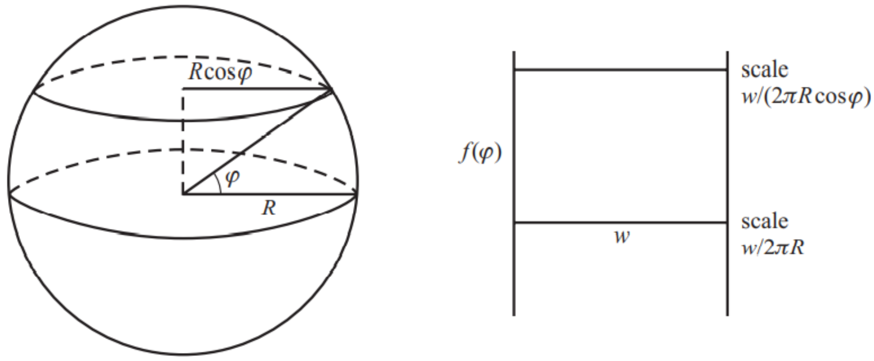


Figure 14: figure demonstrating different scales along parallels [28]

Therefore, at latitude β (for $-90^\circ < \beta \leq 90^\circ$, $-\frac{\pi}{2} \leq \beta < \frac{\pi}{2}$ in radians), the parallel is mapped with fixed scale

$$s_\beta = \frac{w}{2\pi R\cos\beta} = \frac{w}{L\cos\beta} = s\sec\beta \tag{28}$$

Where $s = w/L$ represents the scale of the map along the equator.

The given properties are the expected result out of discussion in 4.6.5 we see that along every parallel, the distances between parallels will be distorted as a result of having different scale factors. Hence, the map projection would not perform an isometry on the entire Earth.

We define the longitude α of a point(where $-180^\circ \leq \alpha < 180^\circ$, $-\pi \leq \alpha < \pi$ in radians) is the value of θ where the meridian through the point hits the equator.It is defined in the domain wher $-180^\circ \leq \alpha < 180^\circ$ ($-\pi \leq \alpha < \pi$ in radians) as the equator is divided up by 360° and that assigns to each point on the equator at an angle θ in the domain. [28] We also

With the latitude β and longitude α defined, we get the following definition.

Definition 4.7.4. *[28] The formula for a generic cylindrical projection is defined as follows:*

$$x = \frac{w\alpha}{360} \qquad y = f(\beta) \tag{29}$$

if α, β are in radians

$$x = \frac{w\alpha}{2\pi} \qquad y = f(\beta) \tag{30}$$

Where, $f(\beta)$ is a monotonically increasing function. where $f(0) = 0$ and $f(90^\circ) = H$.The reason for this is to ensure that if $x \leq y$ then $f(x) \leq f(y)$; $x \geq y$ then $f(x) \geq f(y)$ and $x = y$ then $f(x) = f(y)$. The set of points will be an ordered set and in a case where $f(\beta)$ is a monotonic function on interval $[-\frac{\pi}{2}, \frac{pi}{2}]$ means that f is differentiable and hence continuous [29].

4.7.4 Application: The Mercator Projection

We now want to take the equations and apply them in the case of conducting a Mercator projection. The Mercator projection is an example of a map projection used for producing maps of the world and local areas. For example, Google Maps [26] uses the Mercator projection for local areas whilst using other projections for Global mapping.

This is a summary from sources [28] and [45]

The Mercator projection does the Scale of the parallels and equates that to the Scale of the Meridian. We've seen that the Scale of the parallels s_β is constant and uniform at a given latitude. Whereas the the scale of the meridians s_α is not constant and is dependent on $f(\beta)$. The arc of a meridian is defined by an interval of latitude $a < \beta < b$, while the image of the arc on the map will be a vertical line segment $f(a) < y < f(b)$. Given that we assumed that the Earth is a sphere, the length of the arc will be $\frac{L(b-a)}{360}$ and the image on the map will have length $f(b) - f(a)$. Hence, the overall scale factor of the arc of the meridian is

$$\frac{360}{L}\frac{f(b) - f(a)}{b - a}$$

Hence, the new definition

Definition 4.7.5. *[28] The longitudinal scale of a cylindrical projection s_α given in 29 at a latitude $\beta = a$ is defined as*

$$s_\alpha = \lim_{b \to a} = \frac{360}{L}\frac{f(b) - f(a)}{b - a} = \frac{360}{L} \cdot f'(a)$$

The $\frac{360}{L}$ is factored in to account for the fact that we measured the distance of the meridian in degrees rather than the actual arc length in radians. We will now use this definition to derive a formula for $f(\beta)$.

So by using the values

$$s = w/L \qquad s_\alpha = \frac{2\pi}{L} \cdot f'(\beta) \quad \text{(in radians)} \qquad s_\beta = s \sec\beta$$

We use the fact, that Mercator was trying to have $s_\alpha(\beta) = s_\beta(\beta)\ \forall\beta$. Hence,

$$\frac{2\pi}{L} \cdot f'(\beta) = \frac{w}{L}\sec\beta$$

which reduces the equations to

$$f'(\beta) = \frac{df}{d\beta} = \frac{w}{2\pi}\sec\beta \tag{31}$$

The equation 31 is a first order differential equation that can be solved by the separation of variables. Therefore, by using boundary conditions $f(0) = 0$. The result is:

$$f(\beta) = \frac{w}{2\pi} \cdot \log(\sec\beta + \tan\beta) \tag{32}$$

The results provide us the definition

Definition 4.7.6. *[28, page 7 / 239] The Mercator projection for our Earth's sphere $\boldsymbol{\sigma}(u, v) \to \boldsymbol{X}(x, y)$ is defined as in radians:*

$$x = \frac{w\theta}{2\pi} \qquad y = \frac{w}{2\pi} \cdot \log(\sec\phi + \tan\phi) \tag{33}$$

Where,

$$0 \leq \theta \leq 2\pi \qquad -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

4.7.5 Distortion Of Distances

We now build on example 4.3.2 and section 4.6.5. We look to study how much they calculate the distortion in the map projection. We've already seen in 4.3.2 that the spherical distance between Finsbury Park and Stockwell towns is 11.3km to 3 significant figures; We calculate the distance between the two points after the Mercator projection is applied. In this case, we will just use the values of θ and β given as $(38.5°, 359.87°)$ for Stockwell and $(38.4°, 359.9°)$ for Finsbury Park. Here we have under map projections our points with a width of 420mm (A2 - Paper):

$$(x,y)_S = \left(0.4198, 3 \times 10^{-4}\right) \qquad (x,y)_F = \left(0.4200, 3.68 \times 10^{-4}\right)$$

The Euclidean distance for the two new projected points is $3.5 \times 10^{-5}m$

Another point on the Earth that has the same spherical distance of $11.3km$ is Notre Dame and Le Perreux in Paris. But are at different altitudes and latitudes. The (θ, ϕ) coordinate for Notre Dame is $(2.4°, 41.1°)$,and for La Perreux $(2.4°, 41.2°)$ Here we have under map projections our points with a width $420mm(A2 - Paper)$:

$$(x,y)_{ND} = \left(2.8 \times 10^{-3}, 3.99 \times 10^{-4}\right) \qquad (x,y)_{LP} = \left(2.92 \times 10^{-3}, 4.01 \times 10^{-4}\right)$$

The Euclidean distance for the two new projected points is $2.51 \times 10^{-3}m$. As the distances are not equal, we see how having different latitudes and longitudes can affect the amount a distance is distorted through a map projection.

We now have all the ingredients to perform a Map Projection and Mathematical Basis to create a generic map. We will now look at applying concepts and ideas to creating a map for a Metro System.

5 Mapping and Modelling of a Metro System

We now want to construct a map and a model for our metro system. We will be using the case studies of the London Underground and the Glasgow Subway. We will begin with a discussion on describing our situation and how we can apply the mathematics in chapter 4 and the geography/ cartography elements in chapter 3 in hopes of combining the two for the application of constructing a map and modelling a metro system.

5.1 Situational Considerations

First, we need to consider our situation when constructing the map. As previously defined, a metro system is an urban passenger transportation system that uses elevated and underground trains or a combination of both. We remarked that systems operate over a specific locale, are generally connected, serve only a finite number of stations, and only run trains through a given number of links. We also remarked that a metro system is discrete as the train only stops at stations served by train lines. We must factor these facts in when producing the map, and we must understand the map's purpose and the map's intended audience. These factors influence the way we will go about making our map.

So, what is the map's purpose, and who would be the intended audience? If our intended audience is the typical passenger, the map's purpose would be for navigation. The map we produce would only need to reflect the passenger's path to get to their destination. If our map's intended audience is engineers and mathematicians, then the purpose of the map could range from data analytics to future planning. We will consider making a map for the typical passenger in our case. Generally, the demand for a passenger map would be higher than the need for a map showing all the train tracks (i.e. including routes not open to the public) metro trains could take.

5.2 Map Projection

First, we consider the how-to map projection aspect of map design. In the project, we have learnt about surfaces and surface patches, applying this concept here. We define the operating locale of the metro system to be the area of coverage the metro system operates. Suppose we assume for a moment that all stations and links are considered points on a curve on a surface. Then the operating locale can be seen as the surface patch, a subset of the sphere's surface. Usually, metro systems operate over cities; hence we expect the surface area of the surface patch to be projected to be much much smaller than the area of the surface.

As a result, the map produced by the map projection will be more accurate compared to the nap produced globally. In the case of our map, we will apply a cylindrical map projection called the Mercator Projection. Given that, the Mercator projection will take every point on the surface of the Earth's Sphere onto the Flat plane (once the cylinder is divided). We find that we have a map of the whole world produced.

The result is problematic as we want the extent of our map to show the operating locale. The London Underground serves North London, Inner South London, Buckinghamshire and Hertfordshire and Essex in our two application cases. [43] Whereas the Glasgow subway only serves the inner city. We also would like the map's resolution to be the distances between train stations, as the purpose of the map is to give the passenger an indication of how far certain places metro systems are on a map.

5.3 Scale, Extent and Resolution

So, how do we achieve the Extent and Resolution of our Metro map? We have now conducted the map projection that takes us from $\mathbb{E}^3 \to \mathbb{E}^2$. We find that the problem with our map is the map's

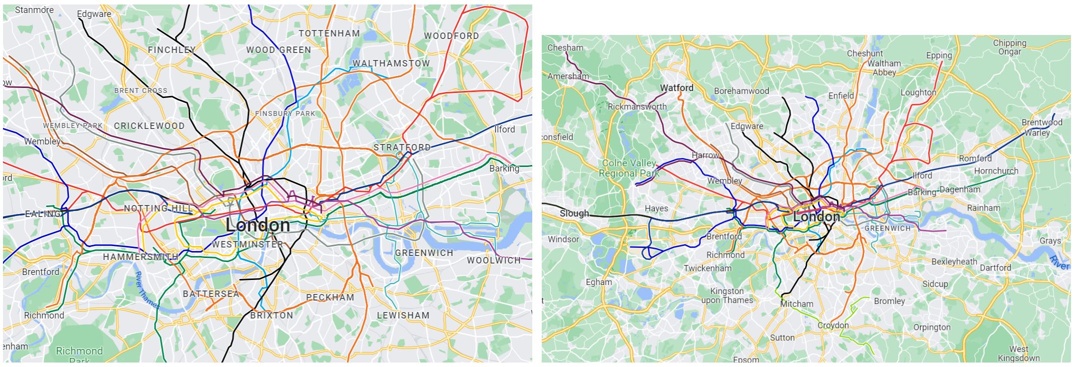


Figure 15: The maps showing how the curves of the train lanes are on in $\mathbb{R}^2$ and, how they look at a distance with the left figure showing the metro lines within the North and South Circlular Road's and, the figure on the right showing the entire map of the London Underground Netwoek with all metro lines considered [26]

scale; we want to make sure that our map shows the metro system and only the metro system. For this to be done, we consider a euclidean transformation which is a euclidean isometry, and we apply a scale factor to each of the points in $\mathbb{E}^2$ an appropriate scaling can be formulated as:

$$T(\boldsymbol{x}) = \begin{pmatrix} c_x & 0 \\ 0 & c_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For any $\boldsymbol{x} \in \mathbb{E}^2$, $\boldsymbol{x} = (x, y)^T$ and c_y, c_x represent scale factors in the y axis and x axis respectively. What we have seen is an homothety the euclidean transformations are an euclidean isometry in the $\mathbb{E}^2$ space.

5.4 Distances between stations

We detour to talk about the distances between stations in a metro system. Therefore, defining a metric for the system. We know that trains move on train tracks to get from one station to another station on the system. We can model the train's path as a path, and we model the train as a particle. We can then use the theory we've learned to form a model to calculate the length of a curve and, thereby the exact distance between stations.

We know from definition 4.1.7 the arc length is defined as

$$s = \int_{\gamma} |d\boldsymbol{\mathcal{L}}| dx$$

Where $d\boldsymbol{\mathcal{L}}$ represents the differential displacement vector along a curve γ. By defining the line segment $ds^2 = |d\boldsymbol{\mathcal{L}}|$ If we are able to derive a parametisation of our displacement vector representing the path taken by the particle. We find that it should be easy to calculate the distances between stations - provided that they are linked to one another. However, in reality its pretty much near difficult to write a entire curve as a function of parameter explicitly. Hence, we must consider other options. A more realistic approach is to focus splitting the curve up to n number of segments in hopes of being able to parameterise the curve, The total distance from one station to another is the total sum of the arc length of all the split segments. Hence,

$$s = \int_{\gamma} |d\boldsymbol{\mathcal{L}}| dx = \int_{\gamma_1} |d\boldsymbol{\mathcal{L}_1}| dx + \int_{\gamma_2} |d\boldsymbol{\mathcal{L}_2}| dx + ... + \int_{\gamma_n} |d\boldsymbol{\mathcal{L}_n}| dx$$

Another much easier approach is treating the curves between stations as geodesics. We know that a geodesic on a surface is a straight line, and a geodesic is the shortest path between points. Hence we can use these facts to measure the distance between two stations as the infimum euclidean metric between the two points.

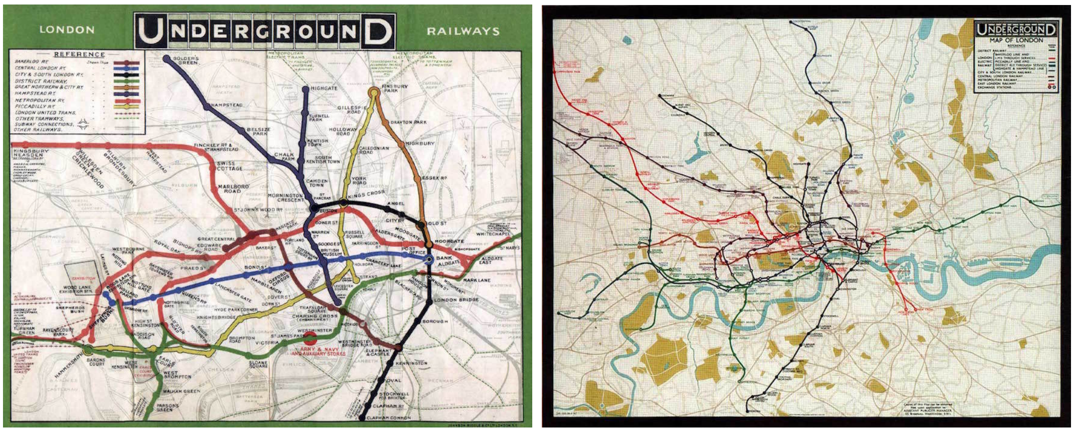


Figure 16: The maps detailing the initial efforts taken to create a reasonable map of the London Underground in Early 20th century [3]

5.5 Generalisations

We now have the curves projected on the plane and the distances between any pair of stations in the metro system. One problem that arises is how to create a map that gives a good representation of the metro system and how to make the passenger understand the notion of distance between points on a metro system.

5.5.1 Generalising the Map

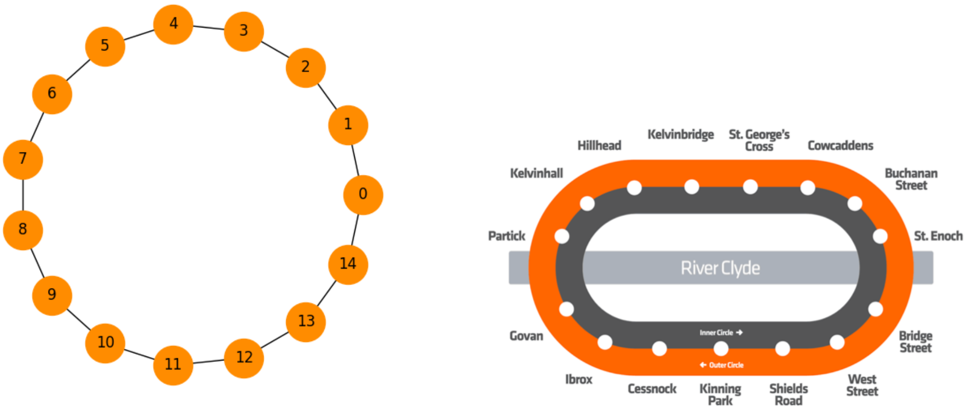


Figure 17: An example of a topological map (right) that can be made into a graph (left) [44]

One of the problems cartographers faced when trying to construct a map for the London Underground is that they want to make the London Underground system a reflection of the city of London [3]. As they would usually draw the lines on top of London's busiest roads - the idea was that the passenger would know where they were relative to the Main road they were closest to. The figures in 17 shows the attempts made by the cartographers to create the map. A problem with this is that London is a large area and the London Underground today covers about $402km$ of the track with the vast majority of stations in zones 1-2 as of 2015.[12] Zones 1-2 cove up the Central London. Hence, another problem is that we cannot fit all stations and lines into a plane that can accurately represent the distances between points.

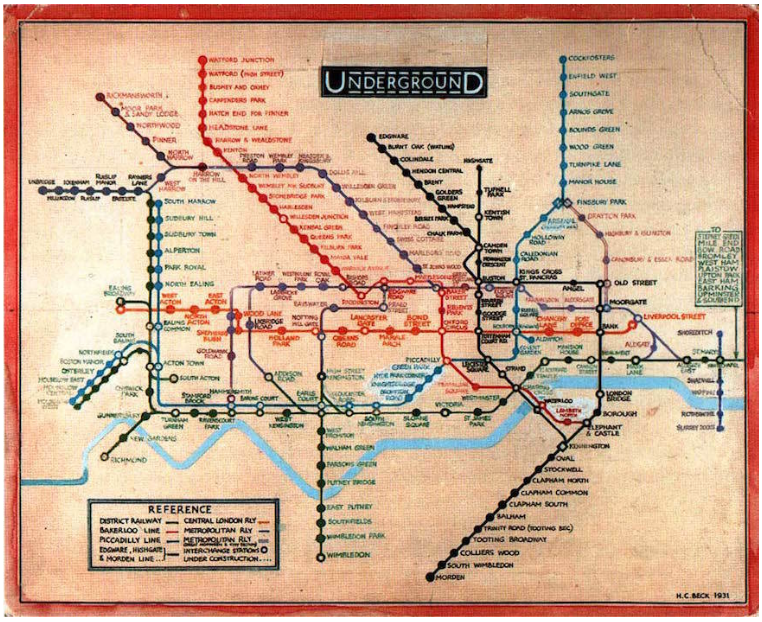


Figure 18: The map produced by Harry Beck in 1933 [3]

In 1933, Henry Beck (a cartographer) came up with a map which took a different approach. He realised that passengers that used the London Underground didn't need to understand aspects of the outside of the Metro System, such as main roads or significant tourist attractions, to be able to know how to get from one point in the system to another, He believed that all passengers need to know was how to use the metro system. How to get from one point on the map to another. The map he produced is a topological map. [7]

Definition 5.5.1. *A* ***Topological Map*** *is a diagram that has been simplified so that only the vital information remains and the unnecessary detail has been removed.*

In the case of metro systems, we represent the diagram as a graph showing the relationship between discrete points via their adjacent edges. [17] We represent the vertices as the stations and the links between connected stations as the edges in the set. A major piece of information lost is that the scale, distance and direction are subject to change and variation.[5] Figure 17 how a metro system can be modelled as a graph hence a topological map. In the case of the London Underground, the major piece of information kept is the River Thames' location; the distance between two stations represents one stop. [3] [7]

5.5.2 The Issue with Topological Maps and Solutions

A significant issue of the topological map is that it does not provide the map reader with an accurate representation of distances between stations in the metro system. An example of this is the two stations that have a large distance between and a small distance between the two. Take Chesham to Chalfont And Latimer on the Metropolitan line. The distance between the two stations is 6.31km [7]. Whereas the distance between Leicester Square and Covent Garden is 300m [7]. Now one might think that the distances are the same due to the length of each edge/ link being limited to one - which is an inaccurate representation of distances in the Metro System.

So, how do we deal with the issue? For starters, we can change the way we look at distance as people are more likely to be concerned with the amount of time it takes to get from one station to another, how long it takes to change metro lines and how long it takes to get from a station to a station. As we've calculated the distance of the curve between stations. We can run experiments or models to calculate the average time for someone to get from one station to another. We can assign this value as the weight of each edge in the graph and define the metric d as the infimum of the sum of the sequence of weights of edges that have to be traversed to get from station A to station B. We want the infimum to get to a point in the network as quickly as possible.

5.5.3 Other Aspects of Map Design

Lastly, we focus on other aspects of map design such as the composition, the symbology, the labelling and the layout.

1. The Composition of the map for a metro system is as follows, The top of the visual hierarchy is the metro lines, as it shows the relationship between lines and stations on the system. Then, is the stations related to the points in the system. Last in the visual hierarchy is geographical features such as rivers or coastlines; this is to give the map reader a perspective of where a station is relative to the geographical feature.

2. Metro Systems use a variety of symbols to help communicate to the map reader. For example, stations may be represented as a blob, whereas interchange stations may be represented as a different coloured blob. The colour distinguishes Metro line routes - for example, the Bakerloo line on the London Underground is brown whilst other lines are of a different colour to communicate to the map reader which line is which.

3. Every station, line and other feature of a Metro System map feature clear text to communicate information to the reader.

4. Using the London Underground [12] as an example, a Metro Map has the main map taking centre stage with information to communicate to the map reader towards the sides of the map.

We have now completed creating a map for a Metro System.

6 Conclusion

The project is about the Mapping and Modelling of Metro Systems, where we detailed the steps and the aspects that contribute to the cartographic process. First, we defined what a Metro System is and argued why we should construct a map for a Metro System. The main reason is that the system is constantly changing and adapting. Hence, we need to create a map that reflects the changes to the environment of a Metro System.

We then had a look at the General Theory of Cartography as we studied essential aspects of the process of map-making, the history of cartography, and the different types of maps. Eventually, we discussed how we could apply the theory to create a map and model for our Metro System.

Then, we discussed the Mathematics behind the mapping and modelling of Metro Systems, where we revisited elementary geometry and explored the new concepts such as surfaces in $\mathbb{R}^3$, Spherical Geometry, Curvature and Theorema Egresium, where we learn that there isometry between a sphere and a plane. These are aspects of differential geometry which lead up to deriving the formula for a cylindrical map projection - the Mercator Projection.

Lastly, We applied the concepts we learned throughout the project and created a Map and a Model for our Metro System. We create a map by conducting a map projection and performing a homothety via a Euclidean Isometry to get the correct scale, Extent and Resolution of the map. We also learned how to apply the theory behind curves to calculate the distance between stations and how we can use the notion of distance by referring to it as the average time to travel between stations to communicate to the reader the scale of distances between stations in a Metro System. We also discussed the issues of using a topological map compared to a topographical map. As we learn that, information about a metro system's environment is lost.

The hope for the project is to understand how a set of points in a system on the surface of the Earth can be mapped onto a plane for us (the map reader) to use for navigation or other motives. The limitation of the project is that the project did not look further at the aspects of modelling that could be implemented for analytical reasons. For example, we said we can model the Metro System as a graph, but we do not talk thoroughly about what the properties of the graph can tell us about the Metro System Network.

The recommendation for future research on the topic is to focus on producing models representing how a Metro System could be run. However, when it comes to the mapping aspect the mathematical elements are sufficient to describe how we can produce a map for the Metro System.

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