MATH6173: Statistical Computing

Supplementary Coursework Report

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1 Section 1, Question 1

1.1 Part C: Graph

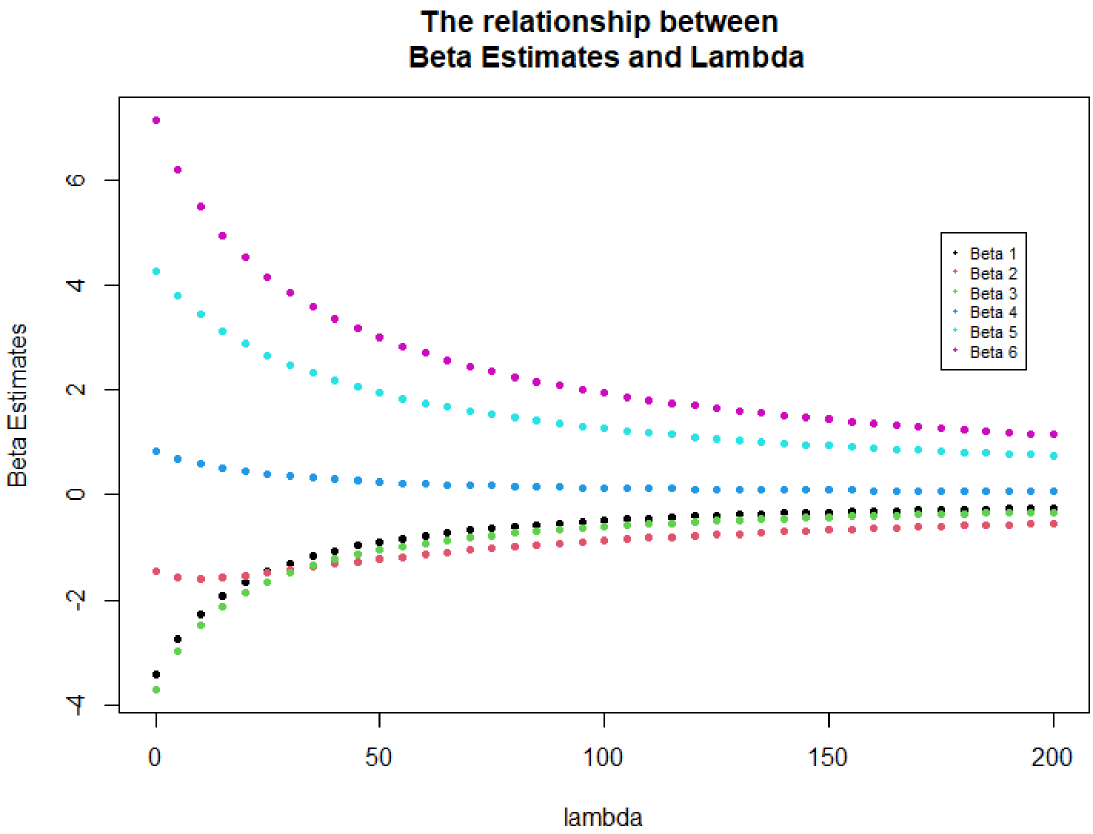

Figure 1: The Relationship Between Beta Estimates and Lambda

1.2 Part D: Graph

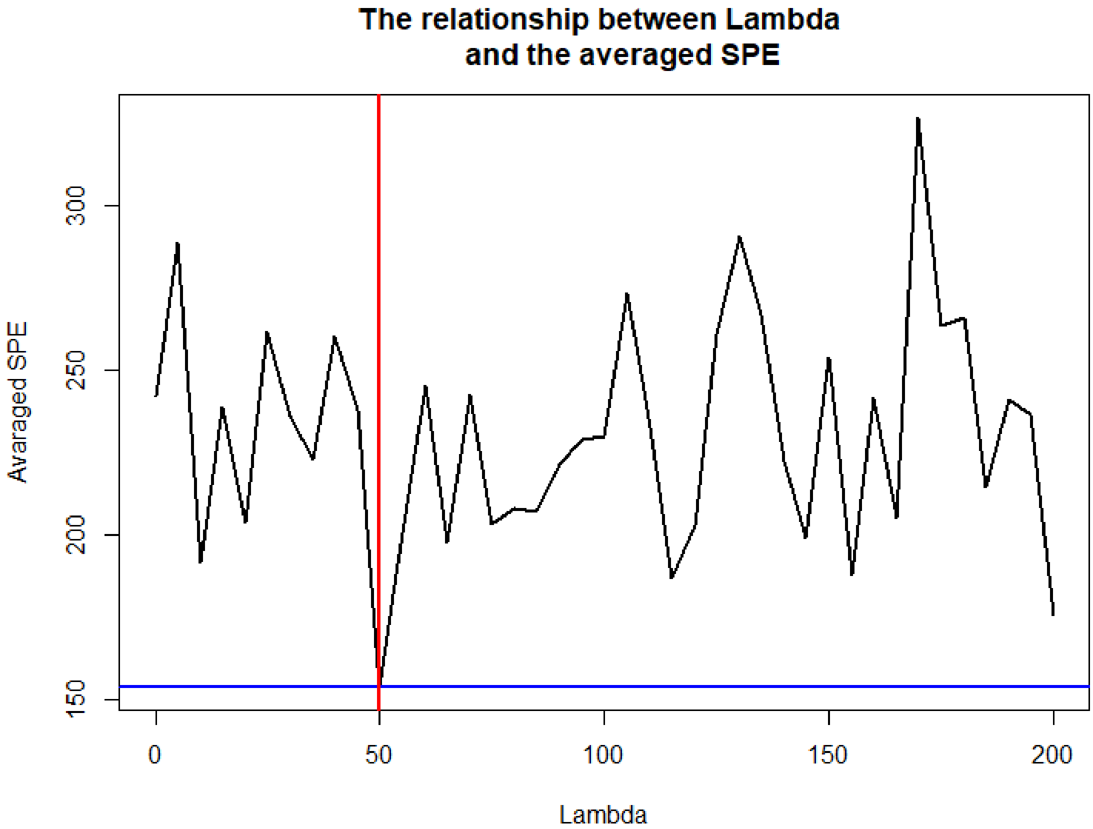

Figure 2: The Relationship Between Lambda and Averaged SPE

The optimal λ is 50 for this specific case and the average SPE 153.8

2 Section 1, Question 2

2.1 Part A: Algorithm

2.1.1 Prerequisites

1. $$M = 1 + a$$

2. $$\frac{\pi(x)}{Mp(x)} = \frac{1}{1+a}(1 + a\sin(2\pi \log x))$$

Algorithm 1 Accept - Reject Sampling Algorithm

1. Draw $X = x$ from $LN(0, 1)$
2. Draw $Y = y$ from $Unif(0, 1)$

if

$$y \leq \frac{1}{1+a}(1 + a\sin(2\pi \log x))$$

then

accept $X = x$ as a a sample of $\pi(x)$;

else

reject $X = x$

end if

2.2 Part B: Algorithm

2.2.1 Prerequisites

1. $$w(x) = (1 + a\sin(2\pi \log x))\mathbb{1}(x > 0)$$

Algorithm 2 Importance Sampling Algorithm

1. Sample i.i,d random variables $X_1, ..., X_n$ from $LN(0, 1)$
2. Compute importance weights:
$$w(X_1), ..., w(X_n)$$
3. Estimate the tail probability as:
$$I_n = \frac{1}{n}\sum_{i=1}^{n} w(X_i) \cdot \mathbb{1}(X_i > 4)$$

2.3 Part C: Algorithm

2.3.1 Prerequisites

1. $$\frac{\pi(x)}{\pi(x^{(j-1)})}\frac{f(x^{(j-1)})}{f(x)} = \frac{1 + a\sin(2\pi\log x)}{1 + a\sin(2\pi\log x^{(j-1)})}$$

2. $$f(x) = \frac{1}{x\sqrt{2\pi}}\exp^{-\frac{(\log x)^2}{2}}$$

Algorithm 3 Tne Metropolis Hastings Algorithhm

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1. **Initialisation:**
$$\boldsymbol{X}^{(0)} = \boldsymbol{x}^{(0)}$$
2. **For** $j = 1, 2$
    1. Draw $X = x \sim f(x)$
    2. Compute
    $$\alpha = \min\left\{\frac{\pi(y)}{\pi(x^{(j-1)})}\frac{f(x^{(j-1)})}{f(y)}, 1\right\}$$
    3. Draw $U = u \sim Unif(0, 1)$
    4. Take
    $$X^{(j)} = x^{(j)} = \begin{cases} x & \text{if } u \leq \alpha \\ x^{(j-1)} & \text{if } u > \alpha \end{cases} \tag{1}$$

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3 Section 1, Question 3

3.1 Part A: Derivation

The Derivation is on the NEXT PAGE.

S1 Q3a I)

$$\pi(x,y) = \frac{1}{2}(x+y+1) \quad 0 < x, y < 1$$

$$\pi(x|y) = \frac{\pi(x,y)}{\pi(y)}$$

$$\pi(y) = \int_0^1 \pi(x,y)\,dx = \int_0^1 \frac{1}{2}(x+y+1)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + xy + x\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} + y + 1\right]$$

$$= \frac{1}{4}(1 + 2y + 2)$$

$$= \frac{1}{4}(2y+3)$$

$$\frac{\pi(x,y)}{\pi(y)} = \frac{\frac{1}{2}(x+y+1)}{\frac{1}{4}(2y+3)}$$

$$= \frac{2(x+y+1)}{2y+3}$$

$$0 < x, y < 1$$

S1 Q3a II)

$$\pi(x,y) = \frac{1}{2}(x+y+1) \qquad 0 < x, y < 1$$

$$\pi(y|x) = \frac{\pi(x,y)}{\pi(x)}$$

$$\pi(x) = \int_0^1 \pi(x,y)\,dy = \int_0^1 \frac{1}{2}(x+y+1)\,dx$$

$$= \frac{1}{2}\left[xy + \frac{y^2}{2} + y\right]_0^1$$

$$= \cancel{\frac{1}{2}} \quad \frac{1}{2}\left[x + \frac{1}{2} + 1\right]$$

$$= \frac{1}{4}\left[2x + 1 + 2\right]$$

$$= \frac{1}{4}(2x+3)$$

$$\frac{\pi(x,y)}{\pi(x)} = \frac{\frac{1}{2}(x+y+1)}{\frac{1}{4}(2x+3)}$$

$$= \frac{2(x+y+1)}{2x+3}$$

$$0 < x, y < 1$$

3.2 Part B1: Algorithm to generate X given Y = y

Algorithm 4 Inverse Transformation Algorithm 1

1. draw U = u from $Unif(0,1)$
2. compute

$$x = F^{-1}(u) = -(y+2) + \sqrt{((y+2)^2 + u(2y+3)}$$

Given Y = y

3. deliver $X = x$

3.3 Part B2: Algorithm to generate Y given X = x

Algorithm 5 Inverse Transformation Algorithm 2

1. draw U = u from $Unif(0,1)$
2. compute

$$y = F^{-1}(u) = -(x+2) + \sqrt{((x+2)^2 + u(2x+3)}$$

Given X = x

3. deliver $Y = y$

3.4 Part D: Algorithm

Algorithm 6 Gibbs Sampler Algorithm

1. Set value of initial vector $(x^{(0)}, y^{(0)})$
2. **For** $j = 1, 2, ...$
3. Draw $X^{(j)} = x^{(j)}$

$$\pi(x|y^{(j-1)}) = \frac{2(x + y^{(j-1)} + 1)}{2y^{(j-1)} + 3}$$

▷ This is done via the Inverse transformation Method

4. Draw $Y^{(j)} = y^{(j)}$

$$\pi(y|x^{(j)}) = \frac{2(y + x^{(j-1)} + 1)}{2x^{(j)} + 3}$$

▷ This is done via the Inverse transformation Method

3.5 Part E: Graph

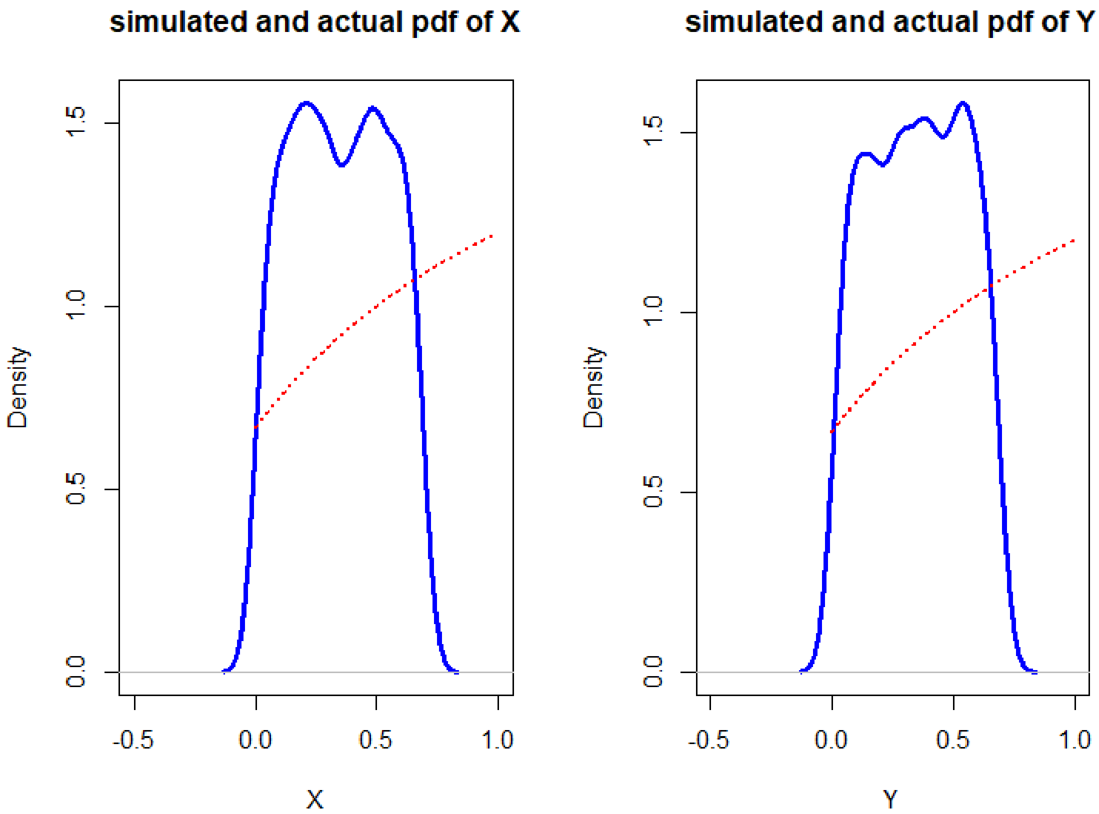

Figure 3: The Relationship Between simulated and actual pdf's of X and Y

4 Section 1, Question 4

4.1 Part A: Algorithm

Algorithm 7 Sampling from N(0,1)

1. Initialise: seed Y_0, and parameters a, b, M ▷ This is the LCG for the uniform distribution $U(0, 2\pi)$
2. **for** $i = 1, 2, ..., n$ **do**
   1. $$Y_i = (aY_{i-1} + b) mod \quad M$$
   2. Compute :
   $$\theta_i = \frac{Y_i}{M \cdot 2\pi}$$
   3. Draw $R^2 = r^2 \sim \chi^2(2)$ ▷ Here we aim to get our radius values to calculate X1 and X2 for a given radius
   4. Compute $u = \sqrt{r^2}$
   5. Compute
   $$X_{1,i} = u \cos\theta_i \qquad X_{2,i} = u \sin\theta_i$$
3. Combine X_1 and X_2 to form vector X
4. Deliver vector X

4.2 Part E: Graph

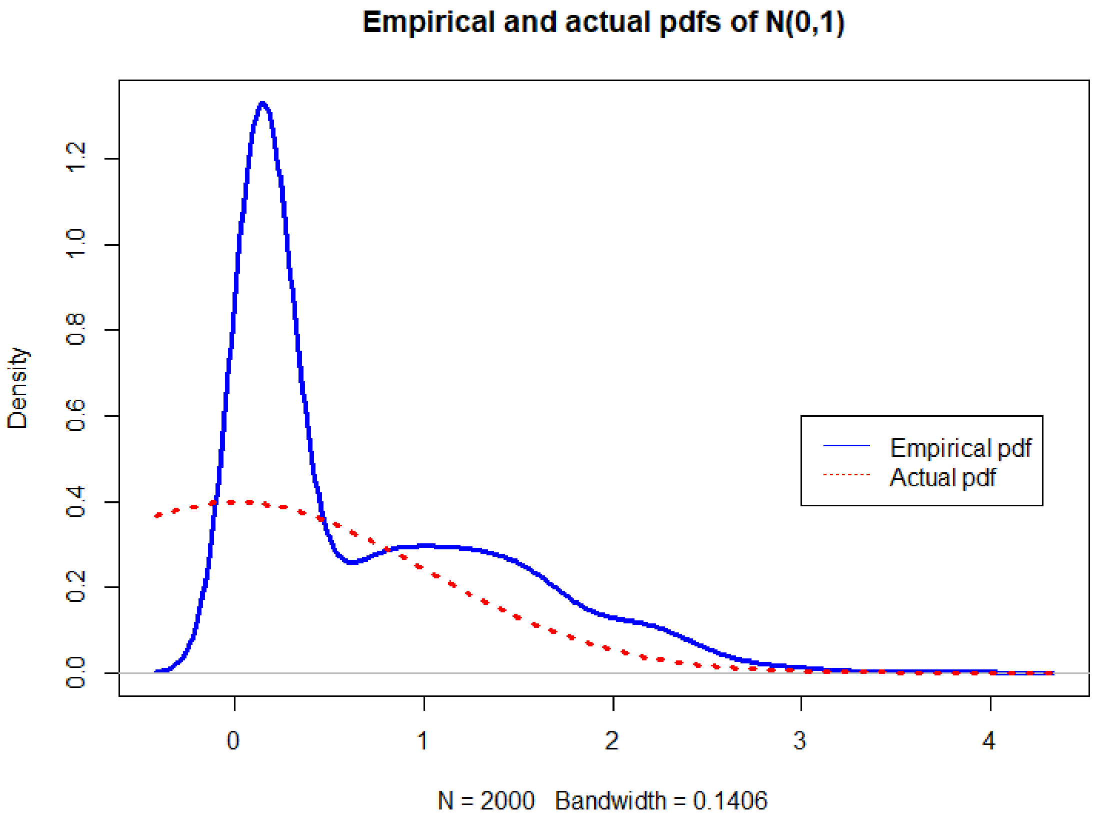

Figure 4: Empirical and Actual pdfs of N(0,1)

5 Section 2, Question 2

$$f_X(x) = \pi_1 \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} \exp(-\beta_1 x) + \pi_2 \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} x^{\alpha_2 - 1} \exp(-\beta_2 x) + \pi_3 \frac{\beta_3^{\alpha_3}}{\Gamma(\alpha_3)} x^{\alpha_3 - 1} \exp(-\beta_3 x)$$

For this problem, we seek to find the maximum likelihood estimates of the parameters $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$ and $\pi = (\pi_1, \pi_2, \pi_3)$. From a sample of X with N observations. We do this by implementing the EM - Algorithm to find the estimates of the parameters $\theta = \{\alpha, \beta, \pi\}$. With $j \in \{1, 2, 3\}$. The four steps :

Algorithm 8 Expectation-Maximisation Algorithm

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1. Initialisation
2. Expectation (E-step)
3. Maximisation (M-step)
4. **If** the Convergence criteria are satisfied. **Then**, the Algorithm terminates. **Otherwise**, repeat 2nd step (onwards) until the condition is reached.

Implementing, this algorithm in R involves creating four functions **gammaMixEstep**, **gammaMixMstep**, **gammaMixCalcLogLik**, and **gammaMixEM**.

5.1 gammaMixEstep

This function seeks to calculate the conditional distribution $p(Z_i = j|X_i, \theta^{cur})$ for all j and for all $i \in \{1, ...N\}$. We apply the formula:

$$p(Z_i = j|X_i, \theta^{cur}) = \frac{p(Z_i = j)p(X_i|Z_i = j, \theta^{cur})}{\sum_{n=1}^{3} p(Z_i = n)p(X_i|Z_i = n, \theta^{cur})}$$

The arguments of the function are

1. *piCur* a numeric vector of length 3 containing the current values of π.
2. *alphaCur* a numeric vector of length 3 containing the current values of α.
3. *betaCur* a numeric vector of length 3 containing the current values of β.
4. X a numeric vector containing all the values of $X = (X_1, ..., X_N)$ - this is the observed data mentioned previously.

The result of this function is a numeric matrix with N rows and 3 columns . Each entry of the matrix is a value for $p(Z_i = j|X_i, \theta^{cur})$.

5.2 gammaMixMstep

The second function created is to carry out the Maximisation step in the algorithm. We take the matrix created in **gammaMixEstep** and we use this to find the new values of parameters $\theta^{new} = (\alpha^{new}, \beta^{new}, \pi^{new})$. The new parameters will optimise the function $Q(\theta, \theta^{cur})$.

The arguments of the function are:

1. W the matrix produced by **gammaMixEstep**
2. X a numeric vector containing all the values of $X = (X_1, ..., X_N)$ - this is the observed data mentioned previously.

The result is a list with three elements where

1. the first object - a vector containing $\alpha^{new} = (\alpha_1^{new}, \alpha_2^{new}, \alpha_3^{new})$
2. the second object - a vector containing $\beta^{new} = (\beta_1^{new}, \beta_2^{new}, \beta_3^{new})$
3. the third object - a vector containing $\pi^{new} = (\pi_1^{new}, \pi_2^{new}, \pi_3^{new})$

5.3 gammaMixCalcLogLik

The third function created will calculate the log-likelihoods of the three Gamma mixture model for θ^{cur} and θ^{new} respectively.

The arguments of the function:

1. X the same vector we've been using the entire time
2. *alphaCur* same vector used in **gammaMixEstep**
3. *alphaNew* α^{new} computed by **gammaMixMstep**
4. *betaCur* same vector used in **gammaMixEstep**
5. *betaNew* β^{new} computed by **gammaMixMstep**
6. *piCur* same vector used in **gammaMixEstep**
7. *piNew* π^{new} computed by **gammaMixMstep**

The result of this function is that we return a numeric vector with two log-likelihood values evaluated in the computation. The first element represents the log-likelihood parameter values θ^{cur}. The second element represents the log-likelihood parameter values θ^{new}.

5.4 gammaMixEM

Lastly, we would like to bring all these functions together by being able to call them via this function. **gammaMixEM** contains the following arguments:

1. *alphaInt* contains the initial values for the vector α
2. *betaInt* contains the initial values for the vector β
3. *piInt* contains the initial values for the vector π
4. X a numeric vector containing all the values of $X = (X_1, ..., X_N)$ - this is the observed data mentioned previously.
5. *convergeEps* a positive numeric value that specifies the convergence criteria

This function implements the EM algorithm to calculate the MLEs of π, β and α of the mixed gamma model. The flow of the algorithm will follow the algorithm at the start but with further modifications in this function. The function flow is as follows:

1. **gammaMixEStep**
2. **gammaMixMstep**
3. **gammaMixCalcLogLik**

In this function, a while loop will be used to ensure that the convergence criteria is satisfied, If the convergence criteria are not satisfied the θ^{cur} becomes θ^{new} and we repeat the algorithm until the condition is satisfied.

We also add a condition in the algorithm to check if the π values sum up to 1. If the values originally provided do not satisfy this condition the algorithm will stop and an error message will show up. Lastly, the result of the matrix will be a numeric vector with two log-likelihood values evaluated in the computation. The first element represents the log-likelihood parameter values θ^{cur}. The second element represents the log-likelihood parameter values θ^{new}.