Minitest 1

Important notes:

- This exam is CLOSED book.
- The use of phones is not permitted. Please leave your phone by the examiner if you must go to the bathroom during the exam.
- Please clearly write your name and fill in your student ID by shading the appropriate entries on the grid above.
- Please write your answers on the white space below the question.
- Please show as much of your work as possible; this includes explaining the reasoning behind your calculations, we really like to give partial credit.
- All problems can be solved without lengthy computations. We advise you to look for a simple solution if you can.
- This exam has **2 problems** (100 points total) on **XX pages**, including this one.

Good luck!

Problem 1: *Measuring information* (50 points total)

(a) (5 points) Let X be an ensemble with the uniform distribution over the set $\{-2, -1, 0, 1, 2\}$. What is the value of $p_X(-2)$?

Answer 1a
Since there are five elements in the set we have that $p_X(-2) = 1/5$.

(b) (5 points) Let $f = x^2$, and $Y = f(X)$. What is the value of $p_Y(1)$?

Answer 1b
By definition

$$p_Y(1) = \sum_{x:f(x)=1} p_X(x) = p_X(-1) + p_X(1) = 2/5$$

Let us imagine that you want to transmit a word from the foreign language Icish to a friend. In this language words have always only two letters, first a consonant then a vowel. The consonants in the language are $\mathcal{C} = \{b, c\}$ and they occur respectively with probabilities $\{1/2, 1/2\}$. The vowels in the language are $\mathcal{V} = \{a, e\}$. The probability of having a vowel in a word depends on the consonant as follows:

	a	e
b	1/2	1/2
c	1/4	3/4

Let CV be the joint ensemble that represents the occurrence of the different words and C, V be the ensembles representing the occurrence of consonants and vowels respectively.

(c) What is the entropy of C? (10 points)

Answer 1a

$$H(C) = -\sum_c p_C(c) \log p_C(c) = -(1/2 \log(1/2) + 1/2 \log(1/2)) = 1$$

(c) What is the entropy of V? (10 points)

Answer 1c

Let us first find the probabilities of the two vowels.

$$p_V(a) = \sum_c p_{VC}(a,c) = \sum_c p_{VC}(a|c)p(c) = \frac{1}{2}(\frac{1}{2} + \frac{1}{4}) = \frac{3}{8}$$

$$p_V(e) = \sum_c p_{VC}(a,c) = \sum_c p_{VC}(a|c)p(c) = \frac{1}{2}(\frac{1}{2} + \frac{3}{4}) = \frac{5}{8}$$

Now we can find the entropy:

$$H(V) = -\sum_v p_V(v)\log p_V(v) = -(3/8\log(3/8) + 5/8\log(5/8))$$

(d) What is the entropy of a vowel given that the consonant was b? (10 points)

Answer 1d

From the definition:

$$H(V|b) = -\sum_v p_{VC}(v|b)\log p_{VC}(v|b) = -(1/2\log(1/2) + 1/2\log(1/2)) = 1$$

We have focused our study on information measures on the entropy function. However, there is a whole zoo of entropic measures with different properties and operational interpretations. Given an ensemble X, we define its collision entropy as follows:

$$H_c(X) = -\log \sum_{x\in\mathcal{X}} (p_X(x))^2 \ .$$

This is also a very useful quantity with applications in cryptography (you will learn about it if you take the master's course Quantum communications and cryptography).

(e) (10 points) Prove that the collision entropy can not be greater than entropy. That is prove: $H(X) \geq H_c(X)$. (Hint: Jensen's inequality can make this proof very simple)

Answer 1e

We have by convexity of the log and direct application of Jensen's inequality that:

$$\sum_x p_x \log\frac{1}{p_x} \leq \log\left(\sum_x (p_x)^2\right),$$

that is: $-H(X) \leq -H_c(X)$, from which follows the desired statement.

Problem 2: *Coding information* (40 points total)
This assembly language instruction *add* $\$t0, \$t3, \$t5$ performs an addition between two operands. Its corresponding R-type instruction has the following field values:

0	11	13	8	0	32
6 bits	5 bits	5 bits	5 bits	5 bits	6 bits

(a) Convert this instruction into machine code (hexadecimal). (10 points)

Answer 2a

(b) Take the four most significant digits of the previous hexadecimal number and convert the resulting 4-digit hexadecimal number to decimal. (5 points) (Use $015F_{16}$ in case you did not obtain the hexadecimal number in question 2a)

Answer 2b

(c) Let's assume now that the values stored in registers $\$t3$ and $\$t5$ are the following 6-bit two's complement numbers: 110110 and 100011. What is the decimal number they represent? (5 points)

Answer 2c

(d) Convert the previous 6-bit numbers to decimal assuming now they are 6-bit sign/magnitude numbers (5 points).

Answer 2d

(e) Perform the addition of the two 6-bit two's complement numbers of question 2c (assume the result is also a 6-bit two's complement number). Is the result correct? If not, explain why. (5 points)

Answer 2e

(f) Convert the previous 6-bit two's complement numbers to 8-bit two's complement numbers. (10 points)

Answer 2f

Problem 3: *Representation of switching functions* (10 points total)

Consider the the following switching expression: $f(x_1, x_2, x_3) = x_1'x_2 + x_2'x_3 + x_1x_2x_3'$

(a) Determine its truth table (8 points)

Answer 3a

(b) What is its 1-set? (2 points)

Answer 2b