Prepared for submission to JHEP

Boundary terms in the action for Causal Sets

M. Buck[a] , **F. Dowker**[a,b] **and I. Jubb**[a]

[a] *Theoretical Physics Group, Blackett Laboratory, Imperial College, London, SW7 2AZ, U.K.*

[b] *Institute for Quantum Computing, University of Waterloo, ON, N2L 2Y5, Canada*

Abstract: We propose a formula for the boundary term in the action of a causal set that is well-approximated by a continuum manifold with spacelike boundary. The boundary term is proportional to the difference in the number of elements immediately to future and the number of elements immediately to the past of the surface. We show that in the continuum limit one recovers the Gibbons-Hawking-York boundary term in the mean.

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NOTES:
1. define s as the RV, not as its mean

1 Introduction

In furthering causal set theory it is crucial that we understand the kinematics of the theory. The action of a given causal set is a crucial piece of the kinematics that would be extremely useful to know and understand. Proposals for the action of a causal set are available [**?**] and these hold analytically in some cases and numerically in many more. These cases being when the causal set is embeddable in some existing spacetime. `MB: rephrase` One can show that the action of the causal set then agrees with the Einstein-Hilbert action of the spacetime in some continuum limit. How this limiting procedure is carried out will be described in more detail below.

It is well known that the Einstein-Hilbert action is not the full story in the continuum. In the presence of spacetime boundaries the gravitational action must include a boundary term S_{GHY}, the Gibbons-Hawking-York action, in order to yield a well-defined variational principle [1]. The contributions of this term play an essential role in particular in the quantum theory. For instance, in the calculation of the black hole entropy via the Euclidean path-integral, it is the boundary terms that produce the answer $A/4l_p^2$ necessary for the unification of black hole mechanics with thermodynamics. To give another example, in Regge calculus boundary terms are necessary in order that the quantum mechanical amplitudes satisfy the correct composition law and that they have the correct classical limit[**? ?**].

MB1: paragraph about boundary term in causets. In this paper we discuss a proposal for the causal set analogue of the boundary action for *spacelike* hypersurfaces.

We define a *causal set* (or *causet*) as a locally finite partial order. This means it is a pair $(\mathcal{C}, \preceq)$ where $\mathcal{C}$ is the set of points and $\preceq$ is a partial order relation on $\mathcal{C}$ that has the following properties. It is (i) reflexive: $x \preceq x$, (ii) acyclic: $x \preceq y \preceq x \Rightarrow x = y$, and (iii) transitive: $x \preceq y \preceq z \Rightarrow x \preceq z$ for all points $x, y, z \in \mathcal{C}$. We define an inclusive order interval as the set $I(x, y) \equiv \{z \in \mathcal{C} | x \preceq z \preceq y\}$ for any $x, y \in \mathcal{C}$. The *locally finite* condition is simply that the cardinality of any order interval is finite, that is $|I(x, y)| < \infty$. This condition ensures we are dealing with a discrete structure, as all the other properties of the relation would apply to an order relation between points on a continuous manifold.

In order to say we have a causal set analogue for a continuous expression we need a well defined procedure for relating the discrete theory to the continuum. This procedure, or tool, is called a *Poisson sprinkling*, or just a *sprinkling*. It is a Poisson process which provides a way to generate a causet from a d-dimensional Lorentzian manifold $(\mathcal{M}, g)$ by selecting points in $\mathcal{M}$ to be the elements of $\mathcal{C}$, with an order relation given by the causal order of the manifold. The number of points chosen in a region of spacetime volume, V, is a Poisson random variable. This means that the expected number of points in some region will be ρV, where ρ is the density of the sprinkling. The density is related to the discreteness scale, l, by $\rho = l^{-d}$ in d spacetime dimensions. It is called a sprinkling as one can envisage the point selection process as a 'sprinkling' of points into the manifold. If a causet, $\mathcal{C}$, can be generated with relatively high probability by a sprinkling into the manifold, $(\mathcal{M}, g)$, then we say that the manifold is a good approximation for the causal set.

2 The Claims

Consider a sufficiently well-behaved d-dimensional spacetime (M, g) and Cauchy surface Σ in M. The causal past and future sets $M^{\pm} = J^{\pm}(\Sigma)$ form a partition of M and $\partial M^{\pm} = \Sigma$. The Gibbons-Hawking-York boundary term for $M^{\pm}$ is in this case given by

$$S_{GHY} = \pm \frac{1}{l_p^{d-2}} \int_\Sigma K d\Sigma \tag{2.1}$$ MB: lose ±?

where l_p is the rationalised Planck length and K is the trace of the extrinsic curvature $K_{\mu\nu} = h^\rho_\mu h^\sigma_\nu \nabla_\rho n_\sigma$ of Σ defined with future-pointing timelike unit normal $n_\mu = \partial_\mu S / \sqrt{g^{\mu\nu} \partial_\mu S \partial_\nu S}$. This translates into a past-pointing normal vector n^μ.

Now observe that the integral in (2.1) can be thought of as the "volume gradient" across the surface Σ. More formally

$$\int K d\Sigma = \frac{\partial}{\partial n} \int d\Sigma, \tag{2.2}$$

where the right hand side is the derivative of the area $\int d\Sigma$ as each point of Σ is moved an equal distance along the unit normal vector, n^μ, which corresponds to a rate of change of the surface volume backwards in time, as n^μ is past-pointing. In the causal set, the analogue of a spacelike hypersurface is an anti-chain, i.e. a subset $\mathcal{A} \subseteq \mathcal{C}$ in which all elements are unrelated. The intuitive analogue of the boundary term would then be the rate of change of the number of causal set elements below and above the antichain. We shall see that this intuitive idea indeed bears out.

MB: does the antichain defn really work?

MB: Write some stuff about antichains being a bit pathological and all that.

Consider a causal set $\mathcal{C}$ obtained by a Poisson sprinkling at density $\rho = 1/l^d$ into a d-dimensional spacetime (M, g). A Cauchy surface $\Sigma \subset M$ induces a partition $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$ of the sprinkling, where $\mathcal{C}^+$ and $\mathcal{C}^-$ denote the restrictions of $\mathcal{C}$ to the points sprinkled to the future and past of Σ, respectively. Spacetime volume in the causal set is obtained simply by counting the number of elements, and hence the volume gradient corresponds to the difference in the number of "immediate neighbours" to the future and to the past of the surface. The most natural definition for the nearest neighbours to the future/past of Σ in the sprinkling is to take the minimal/maximal elements in $\mathcal{C}^\pm$, respectively. Thus $x \in \mathcal{C}^-$ is a maximal element in $\mathcal{C}^-$ if it is maximal in the precedence relation of the causal set, i.e. if there exists no element $y \in \mathcal{C}^-$ such that $x \preceq y$. Figure 1 shows an illustration of the idea. The minimal and maximal elements have been highlighted in COLOUR.

We will also show that it is possible to restrict oneself to either C^+ or C^- and still calculate the GHY term. To do this requires us to define the k-next-to-maximal/minimal elements in C^- or C^+ respectively. These are the elements of the causal set that have k elements to their future or past. This means that the 0-to-maximal/minimal elements are the maximal/minimal elements. Elements with 1 element to their past/future have been highlighted in COLOUR in Figure 1, and those with 2 have been highlighted in COLOUR.

Let us denote the number of k-next-to-maximal elements in $\mathcal{C}^-$ by $N^{(k)}_{max}$ and the number of k-next-to-minimal elements in $\mathcal{C}^+$ by $N^{(k)}_{min}$.

We propose the following definitions for the discrete Gibbons-Hawking-York boundary term (one can adopt the appropriate definition to best suit their calculation needs):

MB: $S[\mathcal{C}, \Sigma]$?

$$
\mathcal{S}_{GHY}^{(d)}[\mathcal{C}] = (l/l_p)^{d-2}\,\frac{c_d}{2\,\Gamma\left(\frac{2}{d}\right)} \times \begin{cases} N_{max}^{(0)} - N_{min}^{(0)} \\ d\,N_{max}^{(1)} - N_{max}^{(0)} \\ N_{min}^{(0)} - d\,N_{min}^{(1)} \end{cases} \tag{2.3}
$$

where the constant c_d only depends on the spacetime dimension and is given by

$$
c_d = \frac{d(d+1)}{(d+2)}\left[\frac{A_{d-2}}{d(d-1)}\right]^{\frac{2}{d}}, \tag{2.4}
$$

$A_d = 2\pi^{\frac{d+1}{2}}/\Gamma\left(\frac{d+1}{2}\right)$ and denotes the volume of the unit d-sphere. In fact, one can write down a general expression for the boundary term as a sum over k of $N_{max/min}^{(k)}$ terms with particular coefficients. This can be written as follows:

$$
\begin{aligned}
\mathcal{S}_{GHY}^{(d)}[\mathcal{C}] = (l/l_p)^{d-2} c_d &\left(\sum_m p_m \frac{\Gamma\left(\frac{2}{d}+m\right)}{\Gamma\left(\frac{1}{d}+m\right)} - \sum_n q_n \frac{\Gamma\left(\frac{2}{d}+n\right)}{\Gamma\left(\frac{1}{d}+n\right)}\right)^{-1} \\
&\times \left(\sum_m p_m \frac{m!}{\Gamma\left(\frac{1}{d}+m\right)} N_{max}^{(m)} + \sum_n q_n \frac{n!}{\Gamma\left(\frac{1}{d}+n\right)} N_{min}^{(n)}\right)
\end{aligned} \tag{2.5}
$$

The coefficients, $p_m, q_n \in \mathbb{R}$, must satisfy the following relation:

$$
\sum_m p_m + \sum_n q_n = 0 \tag{2.6}
$$

The different definitions of the boundary term in (2.3) are just simple cases of this general formula, and the objects that one would preferably use in a calculation. From above one can see that at least two terms are needed in the sum to find the boundary term. The condition (2.6) means that there will be one free parameter if two terms are present[1]. This free parameter will be cancelled by the term after c_d in (2.5). Thus, with only two terms, the form of the discrete boundary term is fixed. If one chooses to include more than two terms the coefficients of the $N_{max/min}^{\,(k)}$ terms are no longer unique.

To support this proposal we show that in the continuum limit of $l \to 0$ (i.e. $\rho \to \infty$) we obtain

$$
\lim_{l\to 0}\left\langle S_{GHY}^{(d)}[\mathcal{C}]\right\rangle = \frac{1}{l_p^{d-2}}\int_\Sigma d^{d-1}x\,\sqrt{h}\,K \tag{2.7}
$$

[1]For example you could include only the maximal and minimal terms (like the first case in (2.3)) with $p_0 = p$ and $q_0 = -p$ where $p \in \mathbb{R}$. This would satisfy (2.6) and would leave p as a free parameter. In this case you would get a factor of p^{-1} from the term after c_d in (2.5) which will cancel the p from the last term.

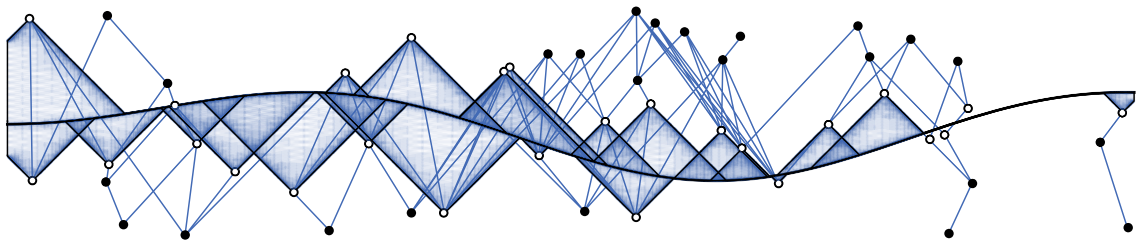

Figure 1: An illustration of the idea. Pictured here is a causal set obtained by a sprinkling into a spacetime partitioned by a spacelike hypersurface. The minimal and maximal points about the surface are highlighted and the shaded regions illustrate the regions whose volumes $V_\blacktriangle$ and $V_\blacktriangledown$ are needed in the proof.

for the most general definition above. Here $\langle\cdot\rangle$ denotes the mean over sprinklings.

We also propose a discrete definition for the spatial volume of the surface:

$$\mathcal{A}^{(d)}[\mathcal{C}] = (l/l_p)^{d-1} \frac{b_d}{\Gamma\left(\frac{1}{d}\right)} N^{(0)}_{max/min} \tag{2.8}$$

where the constant b_d is given by

$$b_d = d\left[\frac{A_{d-2}}{d(d-1)}\right]^{\frac{1}{d}} \tag{2.9}$$

Again on can find a general expression written as a sum of different terms.

$$\mathcal{A}^{(d)}[\mathcal{C}] = (l/l_p)^{d-1} b_d \sum_m p_m \frac{m!}{\Gamma\left(\frac{1}{d}+m\right)} N^{(m)}_{max/min} \tag{2.10}$$

where the coefficients, p_m, must satisfy $\sum_m p_m = 1$. This time we see that only one term is needed to find the surface volume, and that when more than one term is added the coefficients will not be unique. We support this claim in the same way as the boundary term, and show that

$$\lim_{l\to 0}\left\langle \mathcal{A}^{(d)}[\mathcal{C}]\right\rangle = \frac{1}{l_p^{d-1}} \int_\Sigma d^{d-1}x\,\sqrt{h} \tag{2.11}$$

where the right hand side is the spatial volume of the surface Σ.

3 The Proof

3.1 Poisson Sprinklings for $\langle N^{(m)}_{max}\rangle$ and $\langle N^{(m)}_{min}\rangle$

In order to prove (2.7) and (2.11) let us first derive expressions for the mean values of $N^{(m)}_{max}$ and $N^{(m)}_{min}$. For any instance of the sprinkling, the probability that a sprinkled point $p \in M$ below the surface Σ is *k-next-to-maximal* is given by the probability that k points of the sprinkling lie in the region $J^+(p) \cap J^-(\Sigma)$, the intersection of the causal future of p with the causal past of the surface Σ.[2] This region will in general be the interior of some curvy d-dimensional cone truncated by the surface Σ, as illustrated in Figure 1. We will refer to these regions as *truncated cones* below. The Poisson process assigns a probability

$$\mathbb{P}\left(\text{k points in } J^+(p) \cap J^-(\Sigma)\right) = \frac{(\rho\, V_{\blacktriangledown}(p))^k}{k!} e^{-\rho V_{\blacktriangledown}(p)} \tag{3.1}$$

to this event, where $V_{\blacktriangledown}(p) \equiv V(J^+(p) \cap J^-(\Sigma))$ is the spacetime volume of the region $J^+(p) \cap J^-(\Sigma)$. The probability of sprinkling an element into an infinitesimal four-volume dV_p at p is ρdV_p where $\rho = l^{-d}$ is the sprinkling density, and so the total expected number of k-next-to-maximal elements below Σ is

$$\left\langle N^{(k)}_{max}\right\rangle = \rho \int_{J^-(\Sigma)} dV_p\, \frac{(\rho\, V_{\blacktriangledown}(p))^k}{k!} e^{-\rho V_{\blacktriangledown}(p)} \tag{3.2}$$

Similarly the expected number of k-next-to-minimal elements above Σ is

$$\left\langle N^{(k)}_{min}\right\rangle = \rho \int_{J^+(\Sigma)} dV_p\, \frac{(\rho\, V_{\blacktriangle}(p))^k}{k!} e^{-\rho V_{\blacktriangle}(p)} \tag{3.3}$$

where $V_{\blacktriangle}(p) \equiv V(J^+(\Sigma) \cap J^-(p))$.

In the limit $\rho \to \infty$, both quantities will diverge, but if some combination of maximal and minimal terms grows slower than or at order $\rho^{1-\frac{2}{d}}$, the proposed action (2.5) will tend to a finite value in the continuum limit.

Consider a set of synchronous or Gaussian Normal Coordinates (GNC) $x^\mu = (t, \mathbf{x})$ adapted to Σ such that in a neighbourhood U_Σ of Σ the line element is

$$ds^2 = -dt^2 + h_{ij}(t, \mathbf{x})dx^i dx^j. \tag{3.4}$$

In these coordinates, the surface Σ corresponds to $t = 0$, and the coordinate t measures the proper time elapsed along geodesics whose tangent vector is proportional to the

[2] For notational convenience we use the symbol x to refer to the causal element $x \in \mathcal{C}$, to its embedding in the manifold M, and to its coordinates in some chart on M.

surface normal on Σ. The integrals (3.2) and (3.3) seem intractable as they stand, since the integration is over the entire causal past/future of the surface and the volume expressions will be complicated in the presence of curvature. However, for large ρ (small l), the integrands $\exp(-\rho V)$ will be exponentially suppressed unless the volumes are small. Consider the spacetime region $U_\epsilon = \{|t| < \epsilon\}$ around the surface Σ for some $\epsilon > 0$ (with ϵ small enough such that the GNC system is valid throughout U_ϵ). We will assume that for any such ϵ, the contribution from points with $|t| > \epsilon$ to the integrals (3.2) and (3.3) can be made arbitrarily small by setting ρ large enough, since the volumes of the truncated past/future lightcones for such points will be too large. Hence, as $\rho \to \infty$, the integration ranges in (3.2) and (3.3) can be cut off at finite time $t = \pm\epsilon$ with ϵ arbitrarily small, which allows us to expand the time integration-variable about zero. These assumptions of course impose certain regularity conditions on the surface Σ, but we will assume that they are reasonable conditions. The integrals then simplify to

MB: say more about this? we can't have an asymptotically null surface i suppose

$$\begin{aligned}
\left\langle N^{(k)}_{max} \right\rangle &= \rho \int_\Sigma d^{d-1}x \int_{-\varepsilon}^{0} dt\, h^{\frac{1}{2}} \left(1 + \frac{1}{2}\frac{\dot{h}}{h} t + O(t^2)\right) \frac{(\rho\, V_\blacktriangledown(p))^k}{k!} e^{-\rho V_\blacktriangledown(t,\mathbf{x})} \\
\left\langle N^{(k)}_{min} \right\rangle &= \rho \int_\Sigma d^{d-1}x \int_{0}^{\varepsilon} dt\, h^{\frac{1}{2}} \left(1 + \frac{1}{2}\frac{\dot{h}}{h} t + O(t^2)\right) \frac{(\rho\, V_\blacktriangle(p))^k}{k!} e^{-\rho V_\blacktriangle(t,\mathbf{x})}
\end{aligned} \tag{3.5}$$

where $h \equiv det\,(h_{ij}(0, \mathbf{x}))$ and $\dot{} \equiv \frac{\partial}{\partial t}$ and the metric determinant has been in expanded in small t.

The only truncated cones that contribute will have small volumes, as we are restricting ourselves close to the surface. This means that the volumes can be approximated by the volumes of less curvy cones, and it can be shown that higher order corrections, in the approximation scheme, vanish in the limit of $\rho \to \infty$. This will all be made more precise below.

MB: this sounds circular w.r.t. last paragraph

3.2 Lightcone Volumes

In order to evaluate the volume $V_\blacktriangledown(p)$ or $V_\blacktriangle(p)$ of a truncated cone we will perform a coordinate transform to Riemann Normal Coordinates (RNCs) in a neighbourhood containing the cone. The discussion for the two volume integrals is identical so we will outline that for $V_\blacktriangle(p)$ (i.e. for points to the future of Σ).

Fix $p \in M$ and denote its coordinate values in GNCs by $x^\mu_p = (t_p, \mathbf{x}_p)$. It will be convenient to use RNCs centered not at the tip p of the cone but instead at the point p_0 where the unique geodesic through p whose tangent is normal to Σ intersects Σ. In GNCs this simply corresponds to the point p_0 with coordinates $x^\mu_0 = x^\mu(p_0) = (0, \mathbf{x}_p)$. RNCs centered at p_0 will be given the symbol $y^{\overline{\mu}}$. We need to assume that for any

MB: let's put all facts about x^μ and $y^{\overline{\mu}}$ for p and p_0 here and expansion of det in

p in U_ϵ, the Riemann normal neighbourhood $U_p \subset M$ (throughout which the RNC system centered at p_0 is well-defined) contains the truncated cone $J^-(p) \cap J^+(\Sigma)$. This assumption seems reasonable given the that U_ϵ can be made arbitrarily "thin" as $\rho \to \infty$. In RNCs the metric and the Christoffel symbols at p_0 are those of flat space:

$$g_{\overline{\mu\nu}}(p_0) = \eta_{\overline{\mu\nu}} = A^\mu_{\overline{\mu}} A^\nu_{\overline{\nu}} g_{\mu\nu}(p_0) \ , \quad \Gamma^{\overline{\mu}}_{\overline{\nu\rho}}(p_0) = 0 \tag{3.6}$$

The $A^\mu_{\overline{\mu}}$ govern the coordinate transformation from GNCs to RNCs to linear order. To second order the coordinate transformation is given by

$$y^{\overline{\mu}} = A^{\overline{\mu}}_{\ \nu} x^\nu + \frac{1}{2} A^{\overline{\mu}}_{\ \mu} \Gamma^\mu_{\ \nu\rho}(p_0) x^\nu x^\rho + O((x-x_0)^3). \tag{3.7}$$

The inverse relation to first order is

$$x^\mu = A^\mu_{\overline{\nu}} y^{\overline{\nu}} + O(y^2) \tag{3.8}$$

and one finds that the $A^\mu_{\overline{\mu}}$ satisfy

$$A^{\overline{\mu}}_{\ \mu} A^\mu_{\overline{\nu}} = \delta^{\overline{\mu}}_{\overline{\nu}} \ , \quad A^\mu_{\overline{\mu}} A^{\overline{\mu}}_{\ \nu} = \delta^\mu_\nu \tag{3.9}$$

These relations for RNC will cover all that will be needed in this discussion.

Now the volume of the truncated cone can be found as follows. The volume of the cone is given by

$$V_\blacktriangle(p) = \int_{\mathcal{X}_p} d^d x \, \sqrt{-g} \tag{3.10}$$

where the integration region, $\mathcal{X}_p \equiv J^-(p) \cap J^+(\Sigma)$ will be a complicated expression. As we are dealing with points close to the surface we can use the RNCs $y^{\overline{\mu}}$, defined as above, about the point p_0. Let us denote the time-coordinate of p in GNCs by $T = x^0_p = t_p$. For the transformation from GNCs to RNCs at p_0 one finds that $A^{\overline{0}}_{\ 0} = 1$, $A^{\overline{0}}_{\ i} = 0$ and $\delta_{\overline{ij}} = A^i_{\overline{i}} A^j_{\overline{j}} h_{ij}(p_0)$, which in particular implies $y^{\overline{0}}_p = \overline{t}_p = t_p = T$. In RNC one can expand the metric determinant in (3.10) to find

MB: more detail needed i think

$$V_\blacktriangle(p) = \int_{\mathcal{X}_p} d^d y + \int_{\mathcal{X}_p} d^d y \left(-\frac{1}{6} R_{\overline{\mu\nu}}(p_0) y^{\overline{\mu}} y^{\overline{\nu}} \right) + O(T^{d+3}) \tag{3.11}$$

where $R_{\overline{\mu\nu}}(p_0)$ is the Ricci tensor in RNC evaluated at p_0. The second term comes in at $O(T^{d+2})$ so the volume we have to calculate has reduced to

MB: justify beforehand which orders to keep?

$$V_\blacktriangle(p) = \int_{\mathcal{X}_p} d^d y + O(T^{d+2}) \tag{3.12}$$

Terms of $O(T^{d+2})$ can be retained till the end, but in the limit, $\rho \to \infty$, they vanish. This will be proved below.

This is now a simple volume integral but we need to find the boundaries of $\mathcal{X}_p$ to write down the integration limits. There are two boundaries to the (solid) truncated cone: the lightcone emanating at p and the base, given by the intersection of Σ with the interior of the lightcone. First we look at the lightcone. Following [2] it can be shown that the first curvature correction to the lightcone comes in at $O(T^{d+2})$ and so can be ignored for our purposes. This means that the lightcone can be treated as effectively flat in RNCs and thus corresponding to the set $(y^{\bar{1}})^2 + (y^{\bar{2}})^2 + ... + (y^{\overline{d-1}})^2 = T - \bar{t}$. MB: what limit on $\bar{t}$?

The base of the cone in GNCs is simply a part of the surface $t = 0$, so we can use (3.7) to find the equation for the surface in RNCs. Equation (3.7) gives

$$\bar{t} = \frac{1}{2}\Gamma^0_{\ ij}(p_0)x^i x^j + O(x^3) \tag{3.13}$$

The linear part on the right of (3.7) vanishes as $A^{\bar{0}}_{\ \mu}x^\mu = x^0$ (as $A^{\bar{0}}_{\ i} = 0$ and $A^{\bar{0}}_{\ 0} = 1$) and $x^0 = t = 0$ for the bottom surface. Using the inverse RNC relation (3.8) one can find the equation for the bottom surface in RNCs:

$$\bar{t} = \frac{1}{2}\Gamma^0_{\ ij}(p_0)A^i_{\ \bar{\imath}}A^j_{\ \bar{\jmath}}y^{\bar{\imath}}y^{\bar{\jmath}} + O(y^3) \tag{3.14}$$

Let us rewrite this equation in spherically symmetric coordinates, i.e. define $r = \sqrt{\delta_{\bar{\imath}\bar{\jmath}}y^{\bar{\imath}}y^{\bar{\jmath}}}$ and the usual angular coordinates $\phi_1, .., \phi_{d-2}$ in terms of the spatial coordinates $y^{\bar{1}} = r\cos(\phi_1), \dots, y^{\overline{d-1}} = r\sin(\phi_1)\cdots\sin(\phi_{d-3})\sin(\phi_{d-2})$. Then

$$\bar{t} = \frac{1}{2}\left(\Gamma^0_{\ ij}(p_0)A^i_{\ \bar{\imath}}A^j_{\ \bar{\jmath}}\frac{y^{\bar{\imath}}y^{\bar{\jmath}}}{r^2}\right)r^2 + O(y^3) = \frac{1}{2}f(\mathbf{x}_p, \boldsymbol{\phi})r^2 + O(y^3) \tag{3.15}$$

where $\boldsymbol{\phi}$ stands collectively for all the angular coordinates $\phi_1, .., \phi_{d-2}$. The function $f(\mathbf{x}_p, \boldsymbol{\phi})$ depends on $\mathbf{x}_p$ since $\Gamma^0_{\ ij}$ and $A^i_{\ \bar{\imath}}$ depend on p_0.

With the boundaries of the integration region in place, we can now write down the integral explicitly in spherical coordinates:

$$V_{\blacktriangle}(p) = \int_{S^{d-2}} d\Omega_{d-2}\int_0^{r_{max}(\phi)} r^{d-2}dr\int_{\frac{1}{2}f(\mathbf{x}_p,\phi)r^2}^{-r+T} d\bar{t} + O(T^{d+2}) \tag{3.16}$$

where $r_{max}(\boldsymbol{\phi})$ is the value of the radial coordinate for which the surface Σ intersects the lightcone at angle $\boldsymbol{\phi}$, as shown in Figure 2. To find this value one needs to solve

$$\frac{1}{2}f(\mathbf{x}_p, \boldsymbol{\phi}){r_{max}}^2(\boldsymbol{\phi}) = -r_{max}(\boldsymbol{\phi}) + T \tag{3.17}$$

for $r_{max}(\boldsymbol{\phi})$ and take the positive solution. The solution can be expanded in T and is simply $r_{max} = T + O(T^2)$, with angular dependent terms contributing at $O(T^2)$. The

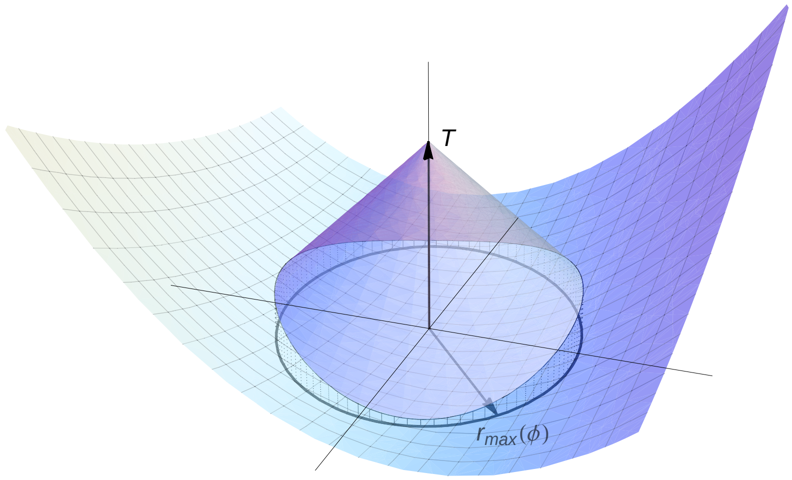

Figure 2: The size of the region inside the top and bottom bounding surfaces is the volume we want to calculate.

$O(T^2)$ term will contribute at $O(T^{d+2})$ in the volume integral and so can be ignored. Substituting $r_{max} = T$ into (3.16) lets us evaluate the integral and we find

$$V_\blacktriangle(p) = \frac{A_{d-2}}{d(d-1)} T^d \left(1 - \frac{d}{2(d+1)} \Gamma^0_{\ ij}(p_0) A^i_{\ \bar\imath} A^j_{\ \bar\jmath} \delta^{\bar\imath\bar\jmath} T\right) + O(T^{d+2}) \tag{3.18}$$

where A_{d-2} is the volume of the unit $(d-2)$-sphere and the $\delta^{\bar\imath\bar\jmath}$ comes from the fact that cross terms $(\bar\imath \neq \bar\jmath)$ vanish under the angular integration. The defining relations for $A^i_{\ \bar\imath}$ can now be rearranged to give $A^i_{\ \bar\imath} A^j_{\ \bar\jmath} \delta^{\bar\imath\bar\jmath} = h^{ij}(p_0)$. Now in GNCs the extrinsic curvature on the surface is given by

$$K = g^{\mu\nu} \nabla_\mu n_\nu = -\Gamma^0_{\ ij} h^{ij} = -\frac{1}{2}\frac{\dot h}{h}. \tag{3.19}$$

Substituting this into (3.18) with we obtain

$$V_\blacktriangle(T, \mathbf{x}) = \frac{A_{d-2}}{d(d-1)} T^d \left(1 + \frac{d}{2(d+1)} K(0, \mathbf{x}) T\right) + O(T^{d+2}) \tag{3.20}$$

$$V_\blacktriangledown(-T, \mathbf{x}) = \frac{A_{d-2}}{d(d-1)} T^d \left(1 - \frac{d}{2(d+1)} K(0, \mathbf{x}) T\right) + O(T^{d+2}), \tag{3.21}$$

having dropped the subscript p. Given the volume expressions in GNCs we now proceed to evaluate the integrals for $\langle N^{(k)}_{min} \rangle$ and $\langle N^{(k)}_{max} \rangle$.

3.3 The mean of $S^{(d)}_{GHY}[\mathcal{C}]$

Using the previous equations for the volumes of the truncated cones we find that (3.5) reduces to

$$
\begin{aligned}
\left\langle N^{(k)}_{max}\right\rangle &= \frac{\rho^{k+1}}{k!}\int_\Sigma d^{d-1}x\int_{-\varepsilon}^{0} dt\, h^{\frac{1}{2}}\left(1+\frac{1}{2}\frac{\dot{h}}{h}t\right)\\
&\qquad\times\left(A(-t)^d\right)^k\left(1-B(-t)\right)^k e^{-\rho A(-t)^d[1-B(-t)]}+\dots\\
\left\langle N^{(k)}_{min}\right\rangle &= \frac{\rho^{k+1}}{k!}\int_\Sigma d^{d-1}x\int_{0}^{\varepsilon} dt\, h^{\frac{1}{2}}\left(1+\frac{1}{2}\frac{\dot{h}}{h}t\right)\\
&\qquad\times\left(A\,t^d\right)^k\left(1+B\,t\right)^k e^{-\rho A\,t^d[1+B\,t]}+\dots
\end{aligned}
\tag{3.22}
$$

where we have defined

$$
\begin{aligned}
A &:= \frac{A_{d-2}}{d(d-1)}\\
B &:= \frac{A_{d-2}}{2(d-1)(d+1)}K(0,\mathbf{x})
\end{aligned}
\tag{3.23}
$$

Once again the higher order terms that have been ignored in both equations will be shown to vanish in the limit. From (3.19) we can substitute K in for $\frac{1}{2}\frac{\dot{h}}{h}t$. We swap the integration variable to $t\to -t$ in the first equation and expand the $O(t^{d+1})$ part of the exponentials to give

$$
\begin{aligned}
\left\langle N^{(k)}_{max}\right\rangle &= \frac{\rho^{k+1}}{k!}\int_\Sigma d^{d-1}x\int_{0}^{\varepsilon} dt\, h^{\frac{1}{2}}\left(1+K\,t\right)\\
&\qquad\times\left(A\,t^d\right)^k\left(1-B\,t\right)^k\left(1+\rho AB\,t^{d+1}\right)e^{-\rho A\,t^d}+\dots\\
\left\langle N^{(k)}_{min}\right\rangle &= \frac{\rho^{k+1}}{k!}\int_\Sigma d^{d-1}x\int_{0}^{\varepsilon} dt\, h^{\frac{1}{2}}\left(1-Kt\right)\\
&\qquad\times\left(A\,t^d\right)^k\left(1+B\,t\right)^k\left(1-\rho AB\,t^{d+1}\right)e^{-\rho A\,t^d}+\dots
\end{aligned}
\tag{3.24}
$$

The reason for expanding the exponentials is that the integrals will have to be put into the form of Gaussian integrals in order to evaluate them in the limit $\rho\to\infty$. The above expressions are almost the same apart from a few sign differences, so from now on we will just work with the expression for $\left\langle N^{(k)}_{min}\right\rangle$. By expanding the brackets, and retaining only necessary orders, the integral can be split into two parts of differing powers of ρ.

$$\left\langle N^{(k)}_{min} \right\rangle = \frac{\rho^{k+1} A^k}{k!} \int_\Sigma d^{d-1}x\, h^{\frac{1}{2}} \int_0^\varepsilon dt\, \left\{ \left(t^{dk} + (kB - K)\, t^{dk+1} \right) + O\left(t^{dk+2} \right) \right\} e^{-\rho A\, t^d}$$
$$- \frac{\rho^{k+2} A^{k+1}}{k!} \int_\Sigma d^{d-1}x\, h^{\frac{1}{2}} \int_0^\varepsilon dt\, \left\{ B t^{dk+d+1} + O\left(t^{dk+d+2} \right) \right\} e^{-\rho A\, t^d} \tag{3.25}$$

To find out how this diverges in the limit of $\rho \to \infty$, it suffices to find the divergence of the following general expression

$$\lim_{\rho \to \infty} \rho^p \int_0^\varepsilon dt\, t^q e^{-\rho A t^d} \tag{3.26}$$

where $p, q \in \mathbb{R}$. We make the substitution $z = \rho A t^d$ to put (3.26) into the form of an incomplete gamma function.

$$\lim_{\rho \to \infty} \frac{A^{-\left(\frac{q+1}{d}\right)}}{d} \rho^{p - \left(\frac{q+1}{d}\right)} \int_0^{\rho A \varepsilon^d} dz\, z^{\left(\frac{q+1}{d}\right) - 1} e^{-z} \tag{3.27}$$

The pre-factor has come from the substitution. In the limit we can retain the ρ outside the integral while taking the integration limit to ∞. In doing this the integral becomes a gamma function.

$$\lim_{\rho \to \infty} \frac{A^{-\left(\frac{q+1}{d}\right)}}{d} \rho^{p - \left(\frac{q+1}{d}\right)} \int_0^\infty dz\, z^{\left(\frac{q+1}{d}\right) - 1} e^{-z} = \lim_{\rho \to \infty} \frac{A^{-\left(\frac{q+1}{d}\right)}}{d} \rho^{p - \left(\frac{q+1}{d}\right)} \Gamma\left(\frac{q+1}{d} \right) \tag{3.28}$$

We can use this procedure to take the limit of (3.25) and formulate the answer in terms of gamma functions. The limits of both $\left\langle N^{(k)}_{min} \right\rangle$ and $\left\langle N^{(k)}_{min} \right\rangle$ are then given by

$$\lim_{\rho \to \infty} \langle N^{(k)}_{max} \rangle = \lim_{\rho \to \infty} \left\{ \rho^{1 - \frac{1}{d}} \left(b_d \right)^{-1} \frac{\Gamma\left(\frac{1}{d} + k \right)}{k!} \int_\Sigma d^{d-1}x\, \sqrt{h} \right.$$
$$\left. + \rho^{1 - \frac{2}{d}} \left(c_d \right)^{-1} \frac{\Gamma\left(\frac{2}{d} + k \right)}{k!} \int_\Sigma d^{d-1}x\, \sqrt{h} K + O\left(\rho^{1 - \frac{3}{d}} \right) \right\}$$
$$\lim_{\rho \to \infty} \left\langle N^{(k)}_{min} \right\rangle = \lim_{\rho \to \infty} \left\{ \rho^{1 - \frac{1}{d}} \left(b_d \right)^{-1} \frac{\Gamma\left(\frac{1}{d} + k \right)}{k!} \int_\Sigma d^{d-1}x\, \sqrt{h} \right.$$
$$\left. - \rho^{1 - \frac{2}{d}} \left(c_d \right)^{-1} \frac{\Gamma\left(\frac{2}{d} + k \right)}{k!} \int_\Sigma d^{d-1}x\, \sqrt{h} K + O\left(\rho^{1 - \frac{3}{d}} \right) \right\} \tag{3.29}$$

These object diverge as $\rho^{1-\frac{1}{d}}$ in the limit. The factor of ρ that must be included in the formula for the surface volume is exactly the inverse of this ($\rho^{\frac{1}{d}-1}$), as can be seen from (2.8) if one substitutes in $l = \rho^{-\frac{1}{d}}$. If we include this factor above and then take the limit we see that the only remaining terms are proportional to the surface volume. The other terms all have negative powers of ρ and therefore vanish as $\rho \to \infty$. From this it follows that the mean of any linear combination of terms like $\rho^{\frac{1}{d}-1}\left\langle N_{max}^{(k)}\right\rangle$ or $\rho^{\frac{1}{d}-1}\left\langle N_{min}^{(k)}\right\rangle$ will give something proportional to the surface volume, so long as the terms containing the surface volume don't exactly cancel. To simplify the coefficients in this linear sum we take out a factor of $\frac{k!}{\Gamma\left(\frac{1}{d}+k\right)}$. The last step in arriving at (2.10) is to enforce the condition that the coefficients sum to unity, in order for the linear sum to equal the surface volume and not just be proportional. This concludes the proof of the surface volume conjecture.

If we multiply (3.29) by the appropriate factor of ρ from the boundary term ($\rho^{\frac{2}{d}-1}$) we find that the above equation still diverges as $\rho^{\frac{1}{d}}$ in the limit. This divergence comes from the surface volume term, while the term proportional to the GHY boundary term is finite. In order to recover the boundary term we must take a linear combination of these objects such that the term proportional to the surface are vanishes, while the term proportional to the boundary term does not. This is where the condition (2.6) comes in. This condition for the coefficients is only so simple because we have taken out a factor of $\frac{k!}{\Gamma\left(\frac{1}{d}+k\right)}$ from each coefficient. The remaining sum is proportional to the boundary term, and in order to get it exactly we simply have to divide out by the proportionality constant. At this point one arrives at (2.5) and the proof is concluded.

4 Fluctuations

So far we have only talked about the mean of the causal set boundary action. Let us now turn to its fluctuations (the standard deviation) $\sigma[S_{GHY}^{(d)}(\mathcal{C})] = \mathrm{Var}[S_{GHY}^{(d)}(\mathcal{C})]^{\frac{1}{2}}$.[3] We can make a heuristic argument to estimate the dependence of fluctuations on $\rho = l^{-d}$. In any spacetime region of fixed volume V the number of causal set elements N experiences Poisson fluctuations of order $\sqrt{N}$. We will use the first case of (2.3) as it is the simplest case in which to form a heuristic argument for the fluctuations. Since $N_{max}^{(0)}$ and $N_{min}^{(0)}$ are quantities associated with a codimension-1 surface (they count elements "near" Σ), we may expect them to inherit fluctuations of order $N^{\frac{d-1}{2d}}$. This comes from the fact that any one dimension will attribute $N^{\frac{1}{d}}$ points to the total volume N (as $(N^{\frac{1}{d}})^d = N$).

[3] Whenever we say fluctuations we refer to the standard deviation of the random variable, not to its variance.

Thus, a codimension-1 surface will have some power of N such that multiplying it by an additional dimensional factor of $N^{\frac{1}{d}}$ will give back N. This gives a surface volume of $N^{\frac{d-1}{d}}$, and with Poisson fluctuations being the square root of the mean we end up with fluctuations that go like $N^{\frac{d-1}{2d}}$. The action $S^{(d)}_{GHY}$, being proportional to $\rho^{\frac{2-d}{d}}$ times the difference of two independent[4] random variables with standard deviation $N^{\frac{d-1}{2d}} = (\rho V)^{\frac{d-1}{2d}}$ should see fluctuations of order $\rho^{\frac{2-d}{d}}\rho^{\frac{d-1}{2d}} = \rho^{\frac{3-d}{2d}}$. This suggests that for $d = 2$ these fluctuations should grow like $\rho^{\frac{1}{4}}$ as $\rho \to \infty$, for $d = 3$ they should be constant, and for $d > 3$ they should be damped.

In order to test this heuristic argument one may like to write down the integral expression for $\mathrm{Var}[S^{(d)}_{GHY}(\mathcal{C})]^{\frac{1}{2}}$ but it is complicated enough even in finite regions of flat space for which it is not very illuminating to reproduce here. It also seems far less tractable than the expression for the mean because it involves terms of the type $\int d^d x \int d^d y \exp\left[-\rho V(J^+(x) \cap J^+(y))\right]$ for which the above methods of cutting off the time-integration range and Taylor expanding will not follow through. Instead we will show the results of computer simulations that support results of the heuristic argument.

The simulations were carried out as follows. Denote the discreteness scale by l. Take a d-cube $[0, L]^d$ in d-dimensional Minkowski space with metric $ds^2 = -dt^2 + d\mathbf{x}^2$ and define the hypersurface $\Sigma : t = L/2$, which partitions the cube into two halves. Sprinkle at density $\rho = l^{-d}$ into the cube, which in any given run will place N points inside the cube where N is a Poisson random number with mean $\langle N \rangle = \rho V = (L/l)^d$. Evaluate $S^{(d)}_{GHY} = \rho^{\frac{2-d}{d}} \frac{c_d}{2\Gamma\left(\frac{2}{d}\right)}(N_{min} - N_{max})$ for Σ by counting the minimal/maximal elements in the upper/lower half of the cube, and repeat for R runs.

[4] The independence is as a feature of the Poisson process in M: the number of elements $N(R)$ in any subregion $R \subset M$ is a Poisson variable itself, and for any two disjoint regions R_1, R_2 the numbers $N(R_1)$ and $N(R_2)$ are independent random variables.

References

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