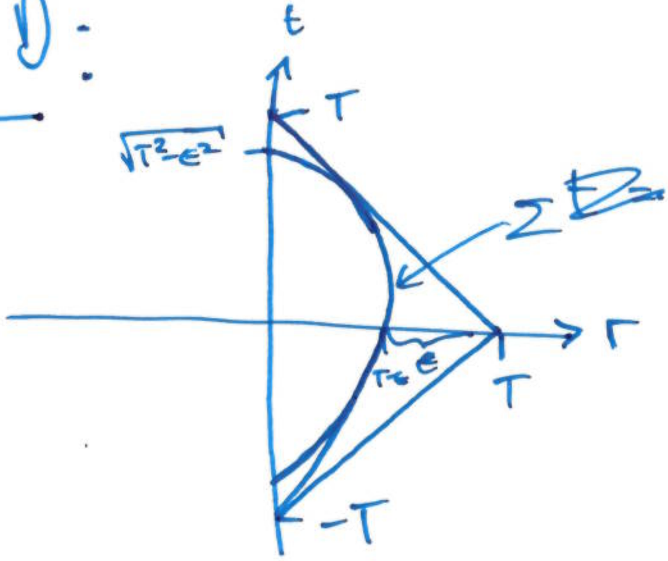
General D:



$$\Sigma: \quad S(t,r) = t^2 - (T-r)^2 + \epsilon^2 = 0.$$

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-1}^2$$

- Will be easier to use Rindler-type coordinates.
- Let
$$t = e^{\xi} \sinh\eta \qquad \xi \in (-\infty, \infty)$$
$$r = T - e^{\xi}\cosh\eta \qquad \eta \in (-\infty, \infty).$$
- Then
$$g_{\eta\eta} = -\left(\frac{\partial t}{\partial\eta}\right)^2 + \left(\frac{\partial r}{\partial\eta}\right)^2$$
$$= -e^{2\xi}\cosh^2\eta + e^{2\xi}\sinh^2\eta$$
$$= -e^{2\xi}$$

and
$$g_{\xi\xi} = -e^{2\xi}\sinh^2\eta + e^{2\xi}\cosh^2\eta = e^{2\xi}$$
$$\Rightarrow ds^2 = e^{2\xi}(-d\eta^2 + d\xi^2) + (T - e^{\xi}\cosh\eta)^2 d\Omega_{d-1}^2$$

In $(\eta, \zeta, \underline{\theta})$-coords the surface Σ (2)
is just

$$e^{2\zeta}\sinh^2\eta - e^{2\zeta}\cosh^2\eta + \epsilon^2 = 0$$

$$\Rightarrow \quad e^{2\zeta} = \epsilon^2$$

$$\Rightarrow \quad \zeta = \ln\epsilon \equiv \zeta_\epsilon .$$

So Σ: $S(\zeta) = \zeta - \zeta_0 = 0$.

Then $n_\mu \propto D_\mu S = \partial_\mu S = (0, 1, \underline{0})$.

$n^2 = $ $\Rightarrow n_\mu = \alpha(0, 1, \underline{0})$.

Normalise: $n^2 = +1$

$$\Rightarrow g^{\mu\nu} n_\mu n_\nu = \alpha^2 e^{-2\zeta} = 1$$

$$\Rightarrow |\alpha| = e^{\zeta}.$$

What about the sign? ~~As~~ n as a covector should be "outward pointing". Now $d\zeta$ points into the surface. So $\alpha = -e^{\zeta}$.

Let's use

$$t = \zeta \sinh\eta$$
$$r = T - \zeta\cosh\eta$$

instead with $\eta \in (-\infty,\infty)$, $\zeta \in [0,\infty)$.

Then $g_{\eta\eta} = -\zeta^2$, $g_{\zeta\zeta} = +1$.

$$\Rightarrow \quad ds^2 = -\zeta^2 d\eta^2 + d\zeta^2 + (T-\zeta\cosh\eta)^2 d\Omega_{d-1}^2 .$$

Also Σ: $S(\zeta) = \zeta - \epsilon = 0$.

$$\Rightarrow \quad n_\mu = \alpha\,(0,1,\underline{0})$$

$$n^2 = +1 \quad \Rightarrow \quad |\alpha| = 1$$

and for n to be outward pointing, we take $\alpha = -1$.

Then $$K = g^{\mu\nu}\nabla_\mu n_\nu = g^{\mu\nu}\cancel{\partial_\mu n_\nu}^{\,0} - g^{\mu\nu}\Gamma^\kappa_{\mu\nu} n_\kappa$$
$$= + g^{\mu\nu}\Gamma^\zeta_{\mu\nu} .$$

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So we just need the $\Gamma^{\zeta}_{\phi\phi}$ components (since g is diagonal).

$$\Gamma^{\zeta}_{\eta\eta} = \cancel{\tfrac{1}{2}(-\zeta^{-2})} = \frac{1}{2} g^{\zeta\zeta}(-g_{\eta\eta,\zeta})$$

$$= \frac{1}{2} \times 2\zeta = \zeta .$$

$$\Gamma^{\zeta}_{\zeta\zeta} = \frac{1}{2} g^{\zeta\zeta}(g_{\zeta\zeta,\zeta}) = \frac{1}{2} \times 0 = 0 .$$

$$\Gamma^{\zeta}_{\phi\phi} = \frac{1}{2}(-g_{\phi\phi,\zeta})$$

$$= -\frac{1}{2}\omega_i \left(-2(T-\zeta\cosh\eta)\cosh\eta\right)$$

$$= (T-\zeta\cosh\eta)\cosh\eta \, \omega_i$$

where ω_i is the spherical metric element, i.e. $ds^2 = -\zeta^2 d\eta^2 + d\zeta^2 + (T-\zeta\cosh\eta)^2(\omega_1 d\theta_1^2 + \dots \omega_{d-1} d\theta_{d-1}^2)$

$$\text{So } K = (-\zeta^{-2})\zeta + 0 + \sum_i \frac{\cancel{(T-\zeta\cosh\eta)}\cosh\eta\,\cancel{\omega_i}}{(T-\zeta\cosh\eta)^{\cancel{2}}\,\cancel{\omega_i}}$$

$$= -\zeta^{-1} + (d-1)\frac{\cosh\eta}{T-\zeta\cosh\eta}$$

$$= \zeta^{-1}\left[\frac{\zeta(d-\cancel{1})\cosh\eta - (T-\cancel{\zeta\cosh\eta})}{T-\zeta\cosh\eta}\right]$$

$$\ldots K = \frac{1}{\xi}\left[\frac{d\,\xi\cosh\eta - T}{T - \xi\cosh\eta}\right].$$

Note that in $(t, r, \underline{\underline{\epsilon}})$-coordinate this is, on Σ:

$$K\Big|_{\Sigma} = \frac{1}{\epsilon}\left[\frac{d(T-r) - T}{r}\right]$$

$$= \frac{1}{\epsilon}\left[(d-1)\frac{T}{r} - d\right],$$

which provides a consistency check with my previous calculations.

- The induced metric h on Σ is just the restriction (pull-back) onto $\xi = \epsilon$.

$$\Rightarrow \cancel{h_{\mu\nu} =}\ ds^2\Big|_{\Sigma} = -\epsilon^2 d\eta^2 + (T - \epsilon\cosh\eta)^2\, d\Omega^2_{d-1}$$

$$\Rightarrow \sqrt{|h|} = \epsilon\,(T - \epsilon\cosh\eta)^{d-1}$$

To get S_{GHY}, need to integrate $K\sqrt{|h|}\,dV$ over boundary.

The time (η-) limits are correspond to $r=0$

$$\Rightarrow \quad T - \epsilon\cosh\eta = 0$$

$$\Rightarrow \quad \cosh\eta = \frac{T}{\epsilon}$$

$$\Rightarrow \quad \eta_\pm = \pm \cosh^{-1}\left(\frac{T}{\epsilon}\right).$$

So

$$S_{GHY} = \int \sqrt{|h|}\, K \, d\eta \, d\Omega_{d-1}$$

$$= V_{d-1} \int_{\eta_-}^{\eta_+} d\eta \, \cancel{\epsilon}\,(T - \epsilon\cosh\eta)^{d-1} \frac{1}{\cancel{\epsilon}} \left(\frac{d\epsilon\cosh\eta - T}{T - \epsilon\cosh\eta} \right)$$

$$(\dagger) = V_{d-1} \int_{\eta_-}^{\eta_+} d\eta \, (T - \epsilon\cosh\eta)^{d-2} (d\epsilon\cosh\eta - T).$$

Let $u = \epsilon\cosh\eta$. Then $d\eta = \frac{du}{\epsilon\sinh\eta} = \epsilon^{-1}\left(u^2-1\right)^{-\frac{1}{2}} du$

& $u_\pm = \pm\frac{T}{\epsilon}$

$$= V_{d-1} \int_{-\frac{T}{\epsilon}}^{\frac{T}{\epsilon}} \epsilon^{-1}(u^2-1)^{-\frac{1}{2}} (T-u)^{d-2} (du - T)$$

Doesn't simplify the integral.
But note that the integrand in (+) is even, so

$$S_{GHY} = 2V_{d-1}\int_0^{\eta_+} (T-\epsilon\cosh\eta)^{d-2}(d\,\epsilon\cosh\eta - T)\,d\eta$$

Mathematica can do this integral!