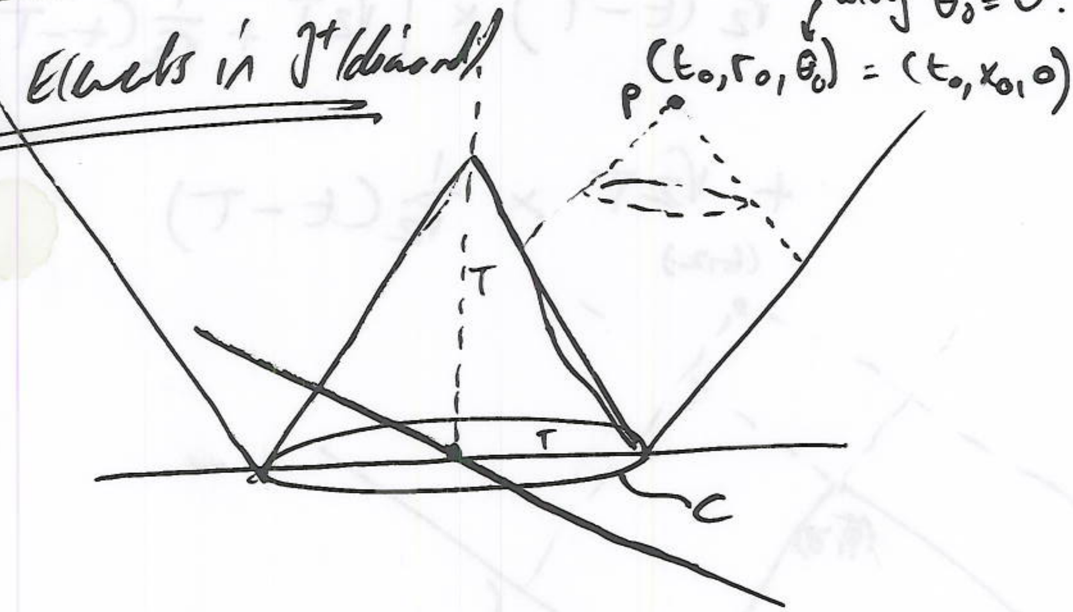
$d=2+1$ FLAT DIAMOND

Minimal Elements in J^+(diamond)



Circle C of radius T at $t=0$.

$$J^+(C) = \{(t,r,\theta) : |r-T| < t > 0\}$$

$$J^-(p) = \{(t,r,\theta) : t < t_0 \ \& \ \sqrt{r^2+r_0^2-2rr_0\cos\theta} < t_0 - t\}$$

In Cartesian coordinates:

$$J^+(C) = \{(t,x,y) : \left|\sqrt{x^2+y^2} - T\right| < t \ \& \ t>0\}$$

$$J^-(p) = \{(t,x,y) : \sqrt{(x-x_0)^2+y^2} < t_0 - t \ \& \ t < t_0\}$$

$$R = J^+(C) \cap J^-(p)$$

Case 1 : $p \notin J^+((T,0,0))$

This condion implies that $t=$ const surfaces of $J^+(c) \cap J^-(p)$ are singly connected.

C_0 : $0 < t < t_0$, $x_0 > 0$, $T > 0$

C_1 : $-(T-t_0)^2 + (x_0)^2 + (y_0)^2 > 0$

or equivalently

$$\boxed{x_0^2 + y_0^2 > (T-t_0)^2}$$

Eq:

C_2 : $\left|\sqrt{x^2+y^2} - T\right| < t$

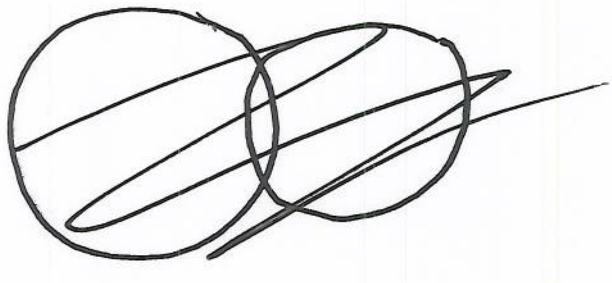
C_3 : $(x-x_0)^2 + y^2 < (t_0-t)^2$

Integration limits: A for $t<$

$$y_\pm = \pm \frac{\sqrt{(T+t_0)^2 - x_0^2}\,\sqrt{x_0^2 - (2t+T-t_0)^2}}{2x_0}$$

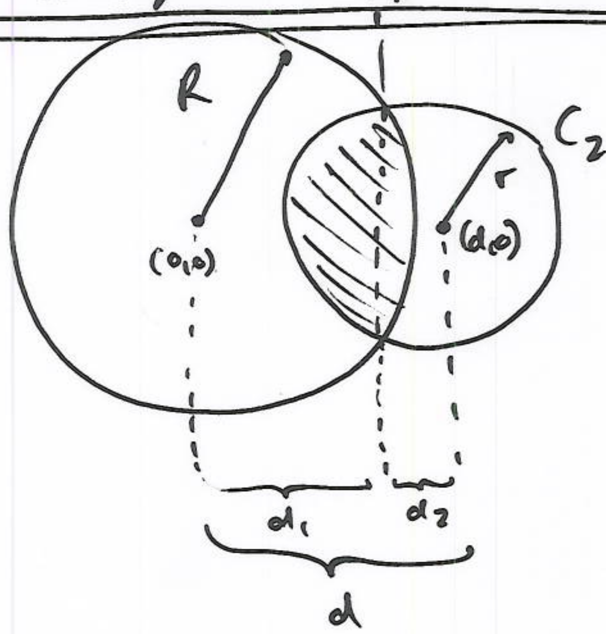
When $p \notin J^+((T,0,0))$, the $t = \text{const}$ slices Σ_t in R take two forms. Near enough to p they're just circles since $J^-(p) \cap \Sigma_t \subset J^+(c)$. Further down they're both restricted by the backwardlightcone from p and the edges of $J^+(c)$.

Maybe we can do this using the formula for the area of two intersecting circles.



Area of overlap of ~~two~~ x_* circles

[1]



Assume:

- $d>0, r>0, R>0$
- $d+r>R$
-

Facts:

- ~~$d-r<x_*$~~
- ~~$x_* \leq R$~~

$$A = r^2 \cos^{-1}\left[\frac{d^2+r^2-R^2}{2dr}\right] + R^2 \cos^{-1}\left[\frac{d^2+R^2-r^2}{2dR}\right]$$

$$-\frac{1}{2}\sqrt{(-d+r+R)(d+r-R)(d-r+R)(d+r+R)}$$

~~In our case, for]~~

Let's derive this formula.

$$C_1: \quad x^2+y^2=R^2$$

$$C_2: \quad (x-d)^2+y^2=r^2$$

Intersection: $x_*^2-(x_*-d)^2 = R^2-r^2$

$$2x_*d-d^2 = R^2-r^2$$

$$x_* = \frac{R^2-r^2+d^2}{2d}$$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \begin{array}{l} |x_*| < R \\ |x_*-d| < r \end{array}$$

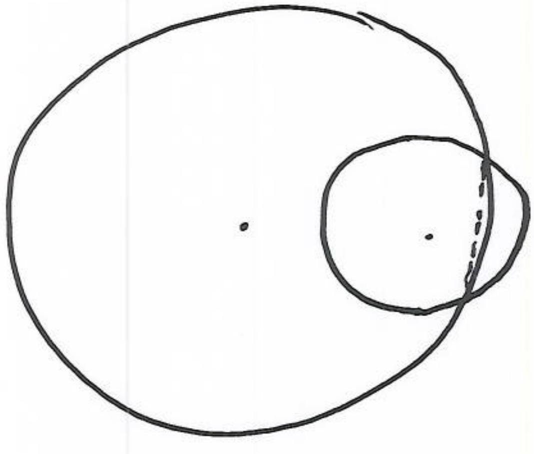
$$y_* = \pm\sqrt{R^2-x_*^2} = \pm\frac{1}{2d}\sqrt{4R^2d^2-(R^2-r^2+d^2)^2}$$

When $d+r > R$ ($C_2 \not\subset C_1$) then

$$A = \int_{-y_*}^{y_*} dy \int_{d-\sqrt{r^2-y^2}}^{x_*} dx + \int_{-y_*}^{y_*} dy \int_{x_*}^{\sqrt{R^2-y^2}} dx$$

$$= \int_{-y^*}^{y^*} dy \int_{d-\sqrt{r^2-y^2}}^{\sqrt{R^2-y^2}} dx$$

$= \ldots$ doesn't work because



would be missing out some bits.

$\Rightarrow$ This calc is only valid if $d > R$.

$\boxed{3'}$

$A_t =\ -R < x_* < R$ and $x_* < r+d$

$x_* > -r+d$

So

$$A = \int_{d-r}^{x_*} dx \int_{-\sqrt{r^2-(x-d)^2}}^{+\sqrt{r^2-(x-d)^2}} dy + \int_{x_*}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy$$

$$= 2\int_{d-r}^{x_*} dx \sqrt{r^2-(x-d)^2} + 2\int_{x_*}^{R} dx \sqrt{R^2-x^2}$$

$x-d = r\sin\theta$ $\qquad$ $x = R\sin\theta$

$$= 2r^2\int_{-\frac{\pi}{2}}^{\sin^{-1}\left(\frac{x_*-d}{r}\right)} d\theta \cos^2\theta + 2R^2\int_{\sin^{-1}\left(\frac{x_*}{R}\right)}^{\frac{\pi}{2}} d\theta \cos^2\theta$$

$$= \cancel{2}r^2\left[\frac{x_*-d}{r}\cos\left[\sin^{-1}\left(\frac{x_*-d}{r}\right)\right] + \sin^{-1}\left(\frac{x_*-d}{r}\right) - 0 - \left(-\frac{\pi}{2}\right)\right]$$

$$+ R^2\left[0 + \frac{\pi}{2} - \frac{x_*}{R}\cos\left[\sin^{-1}\left(\frac{x_*}{R}\right)\right] - \sin^{-1}\left(\frac{x_*}{R}\right)\right.$$

Using $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$ we have

$$A = r^2\left[\frac{x_*-d}{r}\sqrt{1-\frac{(x_*-d)^2}{r^2}} + \sin^{-1}\left(\frac{x_*-d}{r}\right) + \frac{\pi}{2}\right]$$

$$+ R^2\left[-\frac{x_*}{R}\sqrt{1-\frac{x_*^2}{R^2}} - \sin^{-1}\left(\frac{x_*}{R}\right) + \frac{\pi}{2}\right]$$

$$= (x_*-d)\sqrt{r^2-(x_*-d)^2} - x_*\sqrt{r^2-x_*^2}$$

$$+ r^2\sin^{-1}\left(\frac{x_*-d}{r}\right) - R^2\sin^{-1}\left(\frac{x_*}{R}\right)$$

$$+ \frac{\pi}{2}(r^2+R^2)$$ ✓ mathematica

$\cancel{= r^2 \sec^{-1}[}$ Can use $\arcsin^{-1}(z) = \frac{\pi}{2} - \cos^{-1}(z)$

$$= -d\sqrt{R^2-x_*^2} + R^2\cos^{-1}\left(\frac{x_*}{R}\right) + r^2\cos^{-1}\left(\frac{x_*-d}{r}\right)$$

Checked with Mathematica.

out: $x^2 + y^2 < (t+T)^2$

back: $(x-x_0)^2 + y^2 < (t_0 - t)^2$

in: $x^2 + y^2 > (T-t)^2$

~~back~~

At what time t do in & back stop overlapping? It's when their intersection is at $y=0$

$$\Rightarrow \cancel{x^2 - (x-x_0)^2} \; \cancel{x} = \cancel{t+T} \; T-t$$

$$\Rightarrow (\underbrace{T-t-x_0}_{-ve})^2 = (t_0-t)^2$$

$$\Rightarrow T - t - x_0 = -(t_0 - t)$$

$$\Rightarrow t_{in} = \frac{T - x_0 + t_0}{2} = \frac{t_0 - x_0 + T}{2}$$

Out & back: $y=0 \Rightarrow x = t+T$

$$x - x_0 = t_0 - t$$

$$\Rightarrow t + T - x_0 = t_0 - t$$

$$\Rightarrow t_{out} = \frac{t_0 + x_0 - T}{2}.$$

Can we say anything about t_{in} vs. t_{out}?

Depends on sign of $x_0 - T$:

$$t_{in} = t_{out} - (x_0 - T)$$

So $x_0 > T \Rightarrow t_{in} < t_{out}$

$x_0 < T \rightarrow t_{in} > t_{out}$.

· ~~So from $t=0 \rightarrow t=$~~

· So there are two cases:

$$x_0 > T \Rightarrow \int_0^{t_{in}} dt \left[A(\text{out} \cap \text{back}) - A(\text{out in} \cap \text{back})\right)$$

$$+ \int_{t_{in}}^{t_{out}} dt \left[A(\text{out} \cap \text{back})\right]$$

$$+ \int_{t_{out}}^{t_1} dt \; A(\text{back})$$