FRW $\rightarrow$ RNC at $(0,0)$

- $ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2)$

- Non-zero $\Gamma^{\mu}_{\rho\sigma}$:

$$\Gamma^{0}_{ij} = -\frac{1}{2}\left(-g_{ij,0}\right) = \dot{a}a\,\delta_{ij}$$

$$\Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \frac{1}{2} g^{ik}\left(g_{kj,0}\right) = \frac{1}{2} a^{-2}\delta^{ik}\delta_{kj}\, 2\dot{a}a$$

$$= \delta^{i}_{j}\frac{\dot{a}}{a} = \delta^{i}_{j} H.$$

- New coords $\bar{x}^{\bar{\mu}} = A^{\bar{\mu}}{}_{\nu} x^{\nu} + \frac{1}{2} B^{\bar{\mu}}{}_{\rho\sigma} x^{\rho} x^{\sigma}$

s.t. $\cancel{\eta_{\bar{\mu}\bar{\nu}} =}\ g_{\mu\nu}\big|_p = \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}} \frac{\partial x^{\bar{\nu}}}{\partial x^{\nu}} \eta_{\bar{\mu}\bar{\nu}}\big|_p$ at $\cancel{t}\ x^{\mu}(p) = (0,0)$

$$\Rightarrow A^{\bar{\mu}}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a(0) & 0 \\ 0 & 0 & a(0) \end{pmatrix}$$

- $B^{\bar{\mu}}{}_{\rho\sigma} = A^{\bar{\mu}}{}_{\mu} \Gamma^{\mu}_{\rho\sigma}\big|_p$

- Non-zero components of $\Gamma^{\mu}_{\rho\sigma}$ give

$$B^{0}_{ij} = \Gamma^{0}_{ij}\big|_p = \dot{a} a \delta_{ij}$$

$$B^{\bar{i}}_{0j} = A^{\bar{i}}_{i} \Gamma^{i}_{0j}\big|_p = a(0)\delta^{\bar{i}}_{i} \Gamma^{i}_{0j}$$

$$= a(0)\delta^{\bar{i}}_{i} \delta^{i}_{j} H = \delta^{\bar{i}}_{j} \dot{a}(0)$$

$$\Rightarrow \cancel{\dot{x}}\ \hat{t} = t + \frac{1}{2} \dot{a}(0) a(0)(x^2+y^2)$$

$$\hat{x} = a(0)x + \cancel{\tfrac{1}{2}} \dot{a}(0) t x$$

$$\hat{y} = a(0)y + \cancel{\tfrac{1}{2}} \dot{a}(0) t y$$

$(+)$

$$g^{\hat{\mu}\hat{\nu}} = \frac{\partial x^{\hat{\mu}}}{\partial x^{\mu}} \frac{\partial x^{\hat{\nu}}}{\partial x^{\nu}} g^{\mu\nu}$$

$$g^{\hat{0}\hat{0}} = -1 + \frac{\dot{a}(0)^2 a(0)^2}{a(t)^2}(x^2+y^2)$$

$$g^{\hat{1}\hat{1}} = -\cancel{\tfrac{1}{4}} \dot{a}(0)^2 x^2 + \frac{1}{a(t)^2}\left(a(0) + \cancel{\tfrac{1}{2}} \dot{a}(0) t\right)^2$$

$$= \frac{a(0)^2}{a(t)^2}\left(1 + \frac{H(0)}{\cancel{4}} t\right)^2 - \frac{\dot{a}(0)^2}{\cancel{4}} x^2$$

$$g^{\hat{1}\hat{1}} = \frac{a_0^2}{a_t^2}(1+H_0 t)^2 - \dot{a}_0^2 x^2$$ where we've defined $a_0 \equiv a(0)$, $a_t \equiv a(t)$ etc.

Check that this has zero first-order contributions in t, x, y:

$$g^{\hat{1}\hat{1}} = \frac{a_0^2}{a_0^2(1+H_0 t)^2}(1+H_0 t)^2 - \dot{a}_0^2 x^2 + \text{h.o.}$$

$$= \frac{a_0^2}{a_0^2(1+H_0 t+\frac{1}{2}\frac{\ddot{a}_0}{a_0}t^2)^2}(1+H_0 t)^2 - \dot{a}_0^2 x^2 + \text{h.o.}$$

$$= (1-2H_0 t - \cancel{\tfrac{1}{2}}\frac{\ddot{a}_0}{a_0}t^2)(1+2H_0 t + H_0^2 t^2) - \dot{a}_0^2 x^2 + \text{h.o.}$$

$$= (1 - \frac{\ddot{a}_0}{a_0}t^2 + H_0^2 t^2) - \dot{a}_0^2 x^2 + \text{h.o.}$$

$$= 1 + \left(H_0^2 - \frac{\ddot{a}_0}{a_0}\right)t^2 - \dot{a}_0^2 x^2 \quad \checkmark$$

$$g^{\hat{2}\hat{2}} = \frac{a_0^2}{a_t^2}(1+H_0 t)^2 - \dot{a}_0^2 y^2$$

$$g^{\hat{0}\hat{1}} = -\frac{\partial \hat{t}}{\partial t}\frac{\partial \hat{x}}{\partial t} + \frac{1}{a_t^2}\frac{\partial \hat{t}}{\partial x}\frac{\partial \hat{x}}{\partial x} + \frac{1}{a_t^2}\frac{\partial \hat{t}}{\partial y}\frac{\partial \hat{x}}{\partial y}$$

$$= -\dot{a}_0 x + \frac{1}{a_t^2}\dot{a}_0 a_0 x a_0(1+H_0 t) + \frac{1}{a_t^2}\dot{a}_0 a_0 y \times 0$$

$$= -\dot{a}_0 x + \frac{a_0^2}{a_t^2}(\cancel{\dot{a}_0 x}1+H_0 t)\dot{a}_0 x$$

$$= \dot{a}_0\left(\frac{a_0^2}{a_t^2} - 1\right)x + \frac{\dot{a}_0 H_0 a_0^2}{a_t^2} t x$$

$$g^{\hat{0}\hat{2}} = \dot{a}_0\left(\frac{a_0^2}{a_t^2}-1\right)y + \frac{a_0^2\dot{a}_0 H_0}{a_t^2}ty$$

$$g^{\hat{1}\hat{2}} = -\frac{\partial\hat{x}}{\partial t}\frac{\partial\hat{y}}{\partial t} + \cancel{\frac{\partial\hat{x}}{\partial t}}\,0 + 0$$

$$= \cancel{-(1+H_0t)(1+H_0t)} - \dot{a}_0 x\,\dot{a}_0 y$$

$$= -\dot{a}_0^2 xy$$

Let's compute the Riemann tensor in (t,x,y) coordinates.

$$R^\mu{}_{\nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^k_{\nu\sigma}\Gamma^\mu_{k\rho} - \Gamma^k_{\nu\rho}\Gamma^\mu_{k\sigma}$$

Non-zero derivatives of Γ:

$$\Gamma^0_{ij,0} = \delta_{ij}(\ddot{a}a + \dot{a}^2)$$

$$\Gamma^i_{0j,0} = \delta^i_j\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = \delta^i_j\left(\frac{\ddot{a}}{a} - H^2\right)$$

Then

$$\cancel{R^0{}_{\nu\rho\sigma} = \Gamma^0_{\nu\sigma,\rho} - \Gamma^0_{\nu\rho,\sigma} + \Gamma^k_{\nu\sigma}\Gamma^0_{k\rho} - \Gamma^k_{\nu\rho}\Gamma^0_{k\sigma}}$$

$$\cancel{= \delta^i_\nu\delta^j_\sigma\delta^0_\rho\,\delta_{ij}(\ddot{a}a + \dot{a}^2)}$$

Check the expansion:

$$\underline{\hat{t} = \hat{x}^0 =} \quad g^{\hat{0}\hat{0}} = 1^{00} + \alpha R^0{}_{\nu}{}^0{}_{\sigma} x^\nu x^\sigma$$

$$R^0{}_{\nu}{}^0{}_{\sigma} = g^{0\rho} R^0{}_{\nu\rho\sigma}$$

Note this is the Riemann tensor in the original coordinates.

$$= - R^0{}_{\nu 0 \sigma}$$

$$= -\left[\Gamma^0_{\nu\sigma,0} - \Gamma^0_{\nu 0,\sigma} + \cancel{\Gamma^k_{\nu\sigma} \Gamma^0_{k0}} - \Gamma^k_{\nu 0} \Gamma^0_{k\sigma} \right]$$

$$\Rightarrow R^0{}_i{}^0{}_j = - R^0{}_{i0j} = -\left[\Gamma^0_{ij,0} + \cancel{\Gamma^0_{ij} \Gamma^0_{00}} - \Gamma^k_{i0} \Gamma^0_{kj} \right]$$

$$= -\left[\delta_{ij} (\ddot{a} a + \dot{a}^2) - \delta^k_i H \delta_{kj} \dot{a} a \right]$$

$$= -\left(\delta_{ij} (\ddot{a} a + \dot{a}^2) - \delta_{ij} \dot{a}^2 \right)$$

$$\underline{= - \delta_{ij} \ddot{a} a} \, .$$

$$R^0{}_0{}^0{}_0 = 0 \, , \quad R^0{}_i{}^0{}_0 = 0 \, , \quad R^0{}_0{}^0{}_i = 0 \, .$$

To second order, let's invert (+) from page 2.

$$t = \tilde{t} - \frac{1}{2} H(0) \frac{\tilde{x}^2+\tilde{y}^2}{(1+H(0)t)^2}$$

$$\cancel{\hat{t} = \left(t + \frac{1}{2}H(0)(\tilde{x}^2+\tilde{y}^2)\right)(1-2H(0)t+3H(0)^2t^2)+O(...)}$$

$$\cancel{\Rightarrow \frac{3}{2}H_0^3(\tilde{x}^2+\tilde{y}^2)t^2 + (1-H(0)^2(\tilde{x}^2+\tilde{y}^2))t} + \frac{1}{2}H_0(\tilde{x}^2+\tilde{y}^2)-\hat{t}=0$$

$$t(1+H(0)t)^2 - \tilde{t}(1+H(0)t)^2 = -\frac{1}{2}H(0)(\tilde{x}^2+\tilde{y}^2)$$

· ~~Figure out~~ More general attempt: find RNC in terms of GNC

$\cancel{\underline{GNC}: d\tilde{s}^2 = -d\tilde{t}^2 + h_{ij}(t,\underline{x})\, d\tilde{x}^i d\tilde{x}^j}$

$\Downarrow$

$\cancel{\underline{RNC}: \text{need } g_{\bar{\mu}\bar{\nu}} \text{ at } (\tilde{t}_0, \tilde{x}_0).}$

$\cancel{ds^2 = -dt}$

RNCs will be denoted by y or overlined x

Do inversion systematically.

We have $x^{\bar{\mu}} = A^{\bar{\mu}}{}_{\nu} x^{\nu} - \frac{1}{2} A^{\bar{\mu}}{}_{\rho} \Gamma^{\rho}_{\alpha\beta} x^{\alpha} x^{\beta}$.

~~$x^\mu = A^\mu_\nu x^\nu +$~~ RNC inverse relations: 9

$$y^\mu = A^\mu_\nu x^\nu + B^\mu_{\rho\sigma} x^\rho x^\sigma.$$ Expand to 2nd order.

$$x^\mu = (A^{-1})^\mu_\nu y^\nu + C^\mu_{\rho\sigma} y^\rho y^\sigma$$

$$y^\mu = A^\mu_\nu (A^{-1})^\nu_\rho y^\rho + A^\mu_\nu C^\nu_{\rho\sigma} y^\rho y^\sigma$$

$$+ B^\mu_{\rho\sigma} \left((A^{-1})^\rho_\alpha y^\alpha + C^\rho_{\alpha\beta} y^\alpha y^\beta\right)$$

$$\left((A^{-1})^\sigma_\alpha y^\alpha + O(y^2)\right)$$

$$y^\mu = y^\mu + A^\mu_\nu C^\nu_{\rho\sigma} y^\rho y^\sigma$$

$$+ B^\mu_{\alpha\beta} (A^{-1})^\alpha_\rho (A^{-1})^\beta_\sigma y^\rho y^\sigma$$

$$\Rightarrow A^\mu_\nu C^\nu_{\rho\sigma} + B^\mu_{\alpha\beta} (A^{-1})^\alpha_\rho (A^{-1})^\beta_\sigma = 0$$

$$\Rightarrow C^\mu_{\rho\sigma} = (A^{-1})^\mu_\nu B^\nu_{\alpha\beta} (A^{-1})^\alpha_\rho (A^{-1})^\beta_\sigma$$

Now $B^{\mu}_{\rho\sigma} = -\frac{1}{2} A^{\mu}{}_{\nu} \Gamma^{\nu}_{\rho\sigma}$ so

$$C^{\mu}_{\rho\sigma} = -\frac{1}{2}(A^{-1})^{\mu}{}_{\nu} A^{\nu}{}_{\gamma} \Gamma^{\gamma}_{\alpha\beta} (A^{-1})^{\alpha}{}_{\rho} (A^{-1})^{\beta}{}_{\sigma}$$

$$= -\frac{1}{2} \Gamma^{\mu}_{\alpha\beta} (A^{-1})^{\alpha}{}_{\rho} (A^{-1})^{\beta}{}_{\sigma}$$

So to second order we have

$$x^{\mu} = (A^{-1})^{\mu}{}_{\nu} y^{\nu} - \frac{1}{2} \Gamma^{\mu}_{\alpha\beta} (A^{-1})^{\alpha}{}_{\rho} (A^{-1})^{\beta}{}_{\sigma} y^{\rho} y^{\sigma}$$

If we denote y^{μ} by $x^{\bar\mu}$ and "$(A^{-1})^{\mu}{}_{\nu} = A^{\mu}{}_{\bar\nu}$", i.e. $A^{\mu}{}_{\bar\nu} A^{\bar\nu}{}_{\rho} = \delta^{\mu}_{\rho}$ we have

$$\cancel{x^{\mu} = A^{\mu}{}_{\bar\nu} x^{\bar\nu} - \frac{1}{2} \Gamma^{\mu}_{\alpha\beta} A^{\alpha}{}_{\bar\rho} A^{\beta}{}_{\bar\sigma} x^{\bar\rho} x^{\bar\sigma}}$$

$$x^{\mu} = A^{\mu}{}_{\bar\nu} x^{\bar\nu} - \frac{1}{2} \Gamma^{\mu}_{\alpha\beta} A^{\alpha}{}_{\bar\rho} A^{\beta}{}_{\bar\sigma} x^{\bar\rho} x^{\bar\sigma}.$$

Where $A^{\mu}{}_{\bar\nu}$ and $\Gamma^{\mu}_{\alpha\beta}$ denote the constant matrices that satisfy:

$$\eta_{\bar\mu\bar\nu} = A^{\mu}{}_{\bar\mu} A^{\nu}{}_{\bar\nu}\, g_{\mu\nu}(\overbrace{0, \underline{x}_0}^{\text{GNCs on } \Sigma})$$

$$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta}(0, \underline{x}_0).$$

What does the $\hat{t}=0$ surface look like in RNCs? ~~Also~~ Since $g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & h \end{pmatrix}$

~~$X^\mu = 2A^\mu_\alpha$~~ we have $\eta_{\bar\mu\bar\nu} = A^\mu_{\bar\mu} A^\nu_{\bar\nu} g_{\mu\nu}$

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & \underline{\underline{a}} \end{pmatrix} \quad \text{and } a \text{ satisfies}$$

$$\cancel{1 = 2aga} \quad \cancel{\text{where } g_{\bar\mu\bar\nu}}$$

$$\underline{\underline{1}} = \underline{\underline{a}}^{-1} \underline{\underline{h}}\, \underline{\underline{a}}^{-1} \quad \text{where } h_{ij} \equiv h_{ij}(a, \underline{x}_0).$$

$$\underline{\underline{a}}^{+1} = \underline{\underline{h}}\, \underline{\underline{a}}^{-1}$$

$$\underline{\underline{1}} = \underline{\underline{h}}\, \underline{\underline{a}}^{-2} \Rightarrow \underline{\underline{a}}^2 = \underline{\underline{h}} \Rightarrow \underline{\underline{a}} = \sqrt{\underline{\underline{h}}}$$

- Since $\underline{\underline{h}}$ is positive-definite it has a unique positive-definite square root.

- ~~Then for $\hat{t}=0$~~

~~$X^\mu =$~~

- Recall that $\underline{\underline{A}}$ represents the matrix $A^{\bar\mu}_{\mu}$ in

$$X^{\bar\mu} = A^{\bar\mu}_{\nu} X^\nu - \frac{1}{2} A^{\bar\mu}_{\rho} \Gamma^{\rho}_{\mu\nu} X^\mu X^\nu$$

where $X^{\bar\mu} = \text{RNC}$, $X^\mu = \text{GNC}$.

⌜Clarify relation: as at the beginning $\tilde{x}^\mu$ denotes RNCs, $x^{\bar{\mu}}$ denotes GNCs.⌟

- We want to find the equation for the $t=0$ surface in RNCs. For that we need the inverse relation derived on the last pages. We have:

$$0 = t = x^0 = A^0{}_{\bar{\nu}} x^{\bar{\nu}} - \frac{1}{2} \Gamma^0_{\alpha\beta} A^\alpha{}_{\bar{\mu}} A^\beta{}_{\bar{\nu}} x^{\bar{\mu}} x^{\bar{\nu}} + \mathcal{O}(\bar{x}^3)$$

where $A^\alpha{}_{\bar{\mu}}$ are the components of A^{-1}.

Now $A^0{}_{\bar{i}} = 0$ and

$$\Gamma^0_{\alpha\beta} = \frac{1}{2} g^{00} \left(\cancel{g_{0\alpha,\beta}}^{\,0} + \cancel{g_{0\beta,\alpha}}^{\,0} - g_{\alpha\beta,0} \right)$$

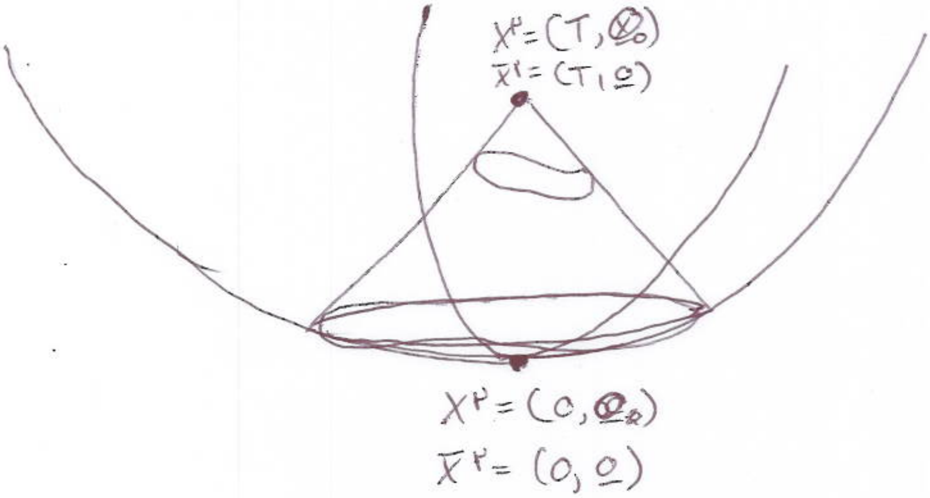
$$= + \frac{1}{2} g_{\alpha\beta,0} \quad (= 0 \text{ unless } \alpha = i, \beta = j).$$

$\Rightarrow$ only non-zero components in $\Gamma^0_{\alpha\beta}$ are Γ^0_{ij}.

So
$$0 = \bar{t} - \frac{1}{2} \Gamma^0_{ij} A^i{}_{\bar{i}} A^j{}_{\bar{j}} x^{\bar{i}} x^{\bar{j}} + \mathcal{O}(\bar{x}^3)$$

$$\Rightarrow \boxed{\bar{t} = \frac{1}{2} \Gamma^0_{ij} A^i{}_{\bar{i}} A^j{}_{\bar{j}} x^{\bar{i}} x^{\bar{j}}}$$

· So the $t=0$ surface is a geodesic surface in RNCs.



· Note that we've rescaled the GNCS ~~~~ to the particular point $(0, x_0)$ by defining $\underline{x} \to \underline{x}' = \underline{x} - \underline{x}_0$ so that $(0, \underline{x}_0) \to (0, \underline{0})$.
~~Note~~ Then $ds^2 = -dt^2 + k_{ij}(t, \underline{x}' + \underline{x}_0)\, dx'^i dx'^j$
so the k_{ij} that enters in A & Γ is that evaluated at $(0, \underline{x}_0)$ in the original GNC.

· Also note that $X^\mu = (T, \underline{0})$ corresponds to $\bar{X}^\mu(T, \underline{0})$
because
$$\bar{X}^{\bar{\mu}} = A^{\bar{\mu}}{}_\nu X^\nu - \frac{1}{2} A^{\bar{\mu}}{}_\nu \Gamma^\nu_{\rho\sigma} X^\rho X^\sigma$$
$$X^{\bar{0}} = X^0 - \frac{1}{2} \Gamma^0_{\rho\sigma} X^\rho X^\sigma$$
$$= X^0 - \frac{1}{2} \Gamma^0_{ij} X^i X^j = T.$$
$$X^{\bar{i}} = 0 - A^{\bar{i}}_i \Gamma^i_{\rho\sigma} X^\rho X^\sigma \quad \text{but } \Gamma^i_{00} = 0 \text{ so}$$
$$= 0 \ \forall\, \bar{i}.$$

- We need to integrate from the bottom geodesic to the backward lightcone emanating at $(\bar{T},\underline{0})$. What's the equation for this lightcone?

- Simone's analysis $\Rightarrow$ only need to find lightcone in tangent space at $(\bar{T},\underline{0})$; then treat light-rays as straight as any corrections due to geodesic acceleration occur at ~~sub~~ orders higher than T^2.

- To find the light-cone note that at $(T,\underline{0})$:

$$g_{\bar{\mu}\bar{\nu}}(T,\underline{0}) = 1_{\bar{\mu}\bar{\nu}} - \frac{1}{3}T^2 R_{\bar{\mu}0\bar{\nu}0} + O(T^3).$$

so the null geodesic vector $\xi^{\bar{\mu}}$ satisfies

$$-(\xi^0)^2 + \underline{\xi}^2 = \frac{1}{3} R_{\bar{\mu}0\bar{\nu}0}\, \xi^{\bar{\mu}} \xi^{\bar{\nu}}.$$

- Recall $R^{\bar{\mu}}{}_{\bar{\rho}\bar{\nu}\bar{\sigma}} = \Gamma^{\bar{\mu}}_{\bar{\rho}\bar{\sigma},\bar{\nu}} - \Gamma^{\bar{\mu}}_{\bar{\rho}\bar{\nu},\bar{\sigma}} + \Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\kappa}}\Gamma^{\bar{\kappa}}_{\bar{\rho}\bar{\sigma}} - \Gamma^{\bar{\mu}}_{\bar{\sigma}\bar{\kappa}}\Gamma^{\bar{\kappa}}_{\bar{\rho}\bar{\nu}}$ so

$$R^{\bar{\mu}}{}_{0\bar{\nu}0} = \Gamma^{\bar{\mu}}_{00,\bar{\nu}} - \Gamma^{\bar{\mu}}_{0\bar{\nu},0} + \Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\kappa}}\Gamma^{\bar{\kappa}}_{00} - \Gamma^{\bar{\mu}}_{0\bar{\kappa}}\Gamma^{\bar{\kappa}}_{0\bar{\nu}}$$