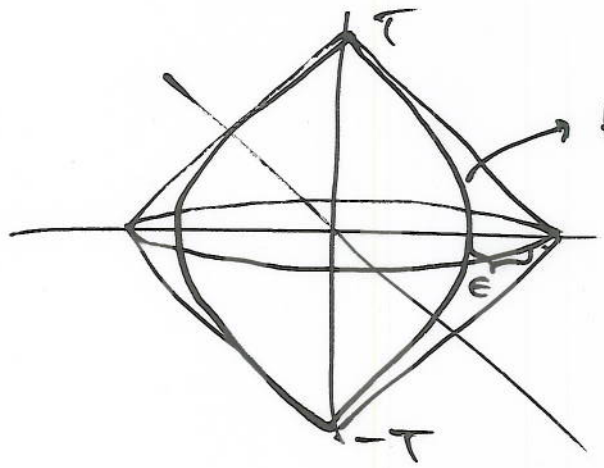
# GHY-BT from diamond in $M^2$ (CONTINUUM) ①




$t^2 = \pm\sqrt{(T-r)^2-\epsilon^2}$

$$\Sigma : \; S(t,r) = t^2 - (T-r)^2 + \epsilon^2 = 0$$

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2$$

$$n_\mu \propto \nabla_\mu S \propto (t, T-r, 0)_\mu$$

$$n^2 = +1 \;\Rightarrow\; n_\mu = \frac{1}{\epsilon}(t, T-r, 0)_\mu$$

$$K = g^{\mu\nu}\nabla_\mu n_\nu = g^{\mu\nu}\partial_\mu n_\nu - g^{\mu\nu}\Gamma^\rho_{\mu\nu} n_\rho \to \text{only } \Gamma^r_{\mu\nu}, \Gamma^t_{\mu\nu}$$

$$\Gamma^t_{\mu\nu} = 0, \quad \Gamma^r_{\theta r} = \Gamma^r_{rr} = 0, \quad \Gamma^r_{\theta\theta} = -r$$

$$K = \frac{1}{\epsilon}(-1-1) - g^{\theta\theta}\Gamma^r_{\theta\theta} n_r = \frac{1}{\epsilon}\left[-2 + \frac{1}{r}(T-r)\right]$$

$$= \underline{\frac{1}{\epsilon}\left[\frac{T}{r} - 3\right]}.$$

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu = \begin{pmatrix} -1 - t^2/\epsilon^2 & -t(T-r)/\epsilon^2 & 0 \\ -t(T-r)/\epsilon^2 & 1-(T-r)^2/\epsilon^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

$$h = \left(-1 - t^2/\epsilon^2\right)\left(1 - \frac{(T-r)^2}{\epsilon^2}\right) r^2$$

$$+ \; t(T-r)/\epsilon^2 \times \left[-t(T-r)/\epsilon^2 \times r^2\right]$$

$$= -\left[\left(1 + \frac{t^2}{\epsilon^2}\right)\left(1 - \frac{(T-r)^2}{\epsilon^2}\right) r^2 + \frac{t^2 (T-r)^2 r^2}{\epsilon^2}\right]$$

$$= -r^2\left[1 + \frac{t^2}{\epsilon^2} - \frac{(T-r)^2}{\epsilon^2}\right]$$

$$= -\frac{r^2}{\epsilon^2}\left[\epsilon^2 + t^2 - (T-r)^2\right] = 0$$

of course ... Actually need to remove the $r$-cols/rows since $h_{ij} = \frac{\partial x^\mu}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^j} g_{\mu\nu}$ and for $\bar{x}^i = (t, \theta)$ we get the determinant

Now $x^\mu = (t, \pm(T \mp \sqrt{t^2 + \epsilon^2}), \theta)$.

Actually, using Rindler coordinates centred on $r = T$ seems more convenient.

$$X^\mu = \left(e^{\xi}\sinh\eta,\ T-e^{\xi}\cosh\eta,\ \theta\right).$$ Then $s=0$ $\Leftrightarrow$ $\underline{\xi = \ln\epsilon}$.

$$h_{ij} = \frac{\partial X^\mu}{\partial x^i}\frac{\partial X^\nu}{\partial x^j} g_{\mu\nu} \qquad x^i = (\eta,\theta)$$

$$h_{\eta\eta} = \frac{\partial X^\mu}{\partial \eta}\frac{\partial X^\nu}{\partial \eta} g_{\mu\nu}$$

$$= -e^{2\xi}\cosh^2\eta + e^{2\xi}\sinh^2\eta = -e^{2\xi}$$

$$h_{\eta\theta} = 0$$

$$h_{\theta\theta} = r^2 = \cancel{e^{2\xi}\cosh^2\eta}\,(T-e^{\xi}\cosh\eta)^2$$

$$h_{ij} = \begin{pmatrix} -e^{2\xi} & 0 \\ 0 & (T-\cosh\eta\, e^{\xi})^2 \end{pmatrix}$$

$$|h| = \cancel{e^{4\xi}\cosh^2\eta}\; e^{2\xi}(T-\cosh\eta\, e^{\xi})^2$$

$$\sqrt{-h} = \cancel{e^{2\xi}\cosh\eta}\; e^{\xi}(T-\cosh\eta\, e^{\xi})$$

$K = \frac{1}{\epsilon}\left(\frac{T}{r} - 3\right)$ $\sqrt{-h} = e^3 r$ where $r = T - e^3\cosh\eta$.
$= \underline{\epsilon r}$.

Now we evaluate

$$\int \sqrt{-h}K = \int_0^{2\pi} d\theta \int_{\eta_{min}}^{\eta_{max}} d\eta \sqrt{-h}\, K$$

Now $t_{max} = |t(0)| = \sqrt{T^2 - \epsilon^2}$

$t_{min} = -|t(0)| = -\sqrt{T^2 - \epsilon^2}$

$\eta_{max} = \sinh^{-1}\left(\frac{t_{max}}{\epsilon}\right) = \sinh^{-1}\left(\frac{\sqrt{T^2-\epsilon^2}}{\epsilon}\right)$

$= \underline{\cosh^{-1}(T/\epsilon)}$.

$$\int \sqrt{-h}\, K = 2\pi \int_{-\cosh^{-1}(T/\epsilon)}^{\cosh(T/\epsilon)} d\eta\; \epsilon r \times \frac{1}{\epsilon}\left(\frac{T}{r} - 3\right)$$

$$= 2\pi \int d\eta\; (T - 3r) = 2\pi \int d\eta\; (T - 3(T - \epsilon\cosh\eta))$$

$$= 2\pi \left[ 8\epsilon \sinh(\cosh^{-1}(T/\epsilon)) - 4T\cosh^{-1}(T/\epsilon)\right]$$

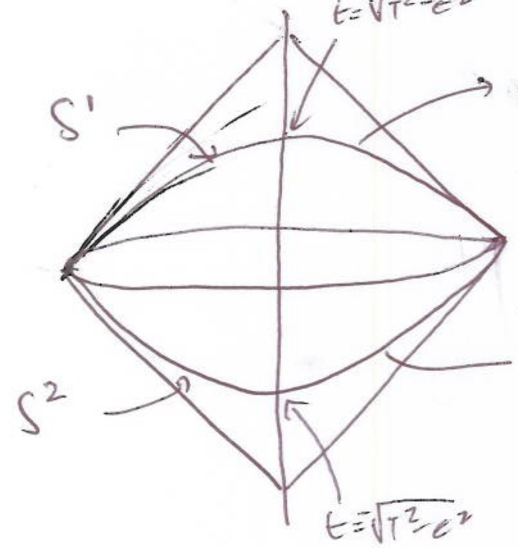
$$= 2\pi \left[ 6\epsilon \sqrt{\frac{T^2}{\epsilon^2} - 1} - 4T\cosh^{-1}(T/\epsilon)\right]$$

(5)

$$\int \sqrt{-4}k = 2\pi \left[ 6\sqrt{T^2-\epsilon^2} - 4T\cosh^{-1}\left(\frac{T}{\epsilon}\right) \right]$$

$$\lim_{\epsilon\to 0} \int \sqrt{-4}k = 2\pi \left[ 6T - 4T\left( \ln(2T) - \ln\epsilon + \mathcal{O}(\epsilon^2)\right)\right]$$

$\longrightarrow$ logarithmic divergence.




$$r=\sqrt{(T-t)^2-\epsilon^2}$$

$$t = T-\sqrt{r^2+\epsilon^2}$$

$$t=-T+\sqrt{r^2+\epsilon^2}$$

$$S^1(t,x) = r^2+\epsilon^2-(T-t)^2 = 0$$

$$S^2(t,x) = r^2+\epsilon^2-(T+t)^2 = 0$$

$$n^1_\mu \propto (T-t, r, 0) \;;\; n^2_\mu \propto (-t-T, r, 0)$$

$$(n^1)^2 = -1 \Rightarrow -\alpha_1^2(T-t)^2 + \alpha_1^2 r^2 = -1$$

$$\Rightarrow \alpha_1 = \frac{1}{\epsilon}$$

$$\boxed{\begin{aligned} n^1_\mu &= \frac{1}{\epsilon}(T-t, r, 0) \\ n^2_\mu &= \frac{1}{\epsilon}(-T-t, r, 0)\end{aligned}}$$

$$(n^2)^2 = -1 \Rightarrow \alpha_2 = 1/\epsilon .$$

$$K = g^{\mu\nu}\nabla_\mu n_\nu = -\partial_0 n_0 + \partial_1 n_1 - \cancel{g^{\mu\nu}\Gamma^0_{\mu\nu} n_0}$$

$$- g^{\mu\nu}\Gamma^1_{\mu\nu} n_1 \quad \text{only } \Gamma^r_{\theta\theta} = -r$$

$$K^1 = -\frac{1}{\epsilon}(-1) + \frac{1}{\epsilon} - \frac{1}{r^2}\times(-r)\times\frac{r}{\epsilon} = \frac{1}{\epsilon}(2+1) = \frac{3}{\epsilon}$$

$$K^2 = -\frac{1}{\epsilon}(-1) + \frac{1}{\epsilon} - \frac{1}{r^2}(-r)\frac{r}{\epsilon} = \frac{3}{\epsilon}$$

Calculate $I_1 = \int_{S_1} \sqrt{h}\, k_1$ :

Let $x^\mu = (T - e^{\zeta}\cosh\eta,\ e^{\zeta}\sinh\eta,\ \theta)$

where $S^1 = 0 \iff \zeta = \ln\epsilon$. So $S^1$

is parametrised by $\eta$ & $\theta$, both

now being spatial coordinates.

$$h_{\eta\eta} = -1 \times \sinh^2\eta\, e^{2\zeta} + 1 \times e^{2\zeta}\cosh^2\eta = e^{2\zeta} = \epsilon^2$$

$$h_{\eta\theta} = 0$$

$$h_{\theta\theta} = r^2 \quad (= e^{2\zeta}\sinh^2\eta = \epsilon^2\sinh^2\eta).$$

$$\sqrt{+h} = r\epsilon.$$

$$\cancel{\int_{S_1} \sqrt{h}\, k_1 = 2\pi \int_{-\cosh^{-1}(}}$$

Integration limits : $t = 0 \Rightarrow \eta = \operatorname{arccosh}\left(\frac{T}{\epsilon}\right)$

so ~~$\eta_{min} = 0$, $\eta_{max} = \cosh^{-1}\left(\frac{T}{\epsilon}\right)$.~~

$\cancel{t = \sqrt{T^2 - \epsilon^2}}$ $\rightarrow \eta = 0$ (since $r = 0 \Rightarrow \eta = 0$).

$t = T - \epsilon$

So $\eta_{min} = 0$, $\eta_{max} = \cosh^{-1}(T/\epsilon)$.

$$\int_{S_1} \sqrt{h}\, K_1 = 2\pi \int_0^{\cosh^{-1}(T/\epsilon)} d\eta \; r\cancel{\epsilon} \times \frac{3}{\cancel{\epsilon}}$$

$$= 6\pi\epsilon \int_0^{\cosh^{-1}(T/\epsilon)} d\eta \, \sinh\eta$$

$$= 6\pi\epsilon \left[\cosh\left(\cosh^{-1}(T/\epsilon)\right) - 1\right]$$

$$= 6\pi\epsilon\left[\frac{T}{\epsilon} - 1\right] = \underline{6\pi\left[T - \epsilon\right]}.$$

→ the $O(T)$ term doesn't match the $\Theta$ calculation - double check this.
(i.e. the previous calc could give the same result "up to renormalisation")