Derivation of GHY-BT variations

$$I_{EH} + I_{GHY} = \frac{1}{2\kappa^2}\int_M d^{d+1}x\sqrt{-g}\,R + \frac{1}{\kappa^2}\int_{\partial M} d^d x\sqrt{-h}\,K$$

Assume ∂M is timelike. Then $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$.

Now we vary $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$.

Variation of $\sqrt{-g}$:

$g = g_{ab} A^{ab}$ where A^{ab} are the (signed) subdeterminants.
Note that $A^{(ab)}$ is independent of $g_{(ab)}$.

$\delta g = \delta g_{ab} A^{ab}$. Now $g^{ab} = \frac{1}{g} A^{ab}$ so

$$\delta g = \delta g_{ab}\, g\, g^{ab}$$

$$\Rightarrow \delta\sqrt{-g} = -\frac{1}{2}\delta g\,(-g)^{-\frac{1}{2}} = -\frac{1}{2}(-g)^{-\frac{1}{2}}\, g\, g^{ab}\,\delta g_{ab}$$

$$= \frac{1}{2}\sqrt{-g}\, g^{ab}\,\delta g_{ab}.$$

$$R^{\mu}{}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} + \Gamma^{\mu}_{\rho\kappa}\Gamma^{\kappa}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\kappa}\Gamma^{\kappa}_{\nu\rho}$$

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\alpha\kappa}\Gamma^{\kappa}_{\mu\nu} - \Gamma^{\alpha}_{\nu\kappa}\Gamma^{\kappa}_{\mu\alpha}$$

$$\cancel{R = g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu,\alpha} - g^{\mu\nu} \dots}$$

$$\cancel{\delta R_{\mu\nu} = \delta\Gamma}$$

tensor!

$$\nabla_\lambda\left(\delta\Gamma^{\mu}_{\rho\sigma}\right) = \delta\Gamma^{\mu}_{\rho\sigma,\lambda} + \Gamma^{\mu}_{\lambda\kappa}\delta\Gamma^{\kappa}_{\rho\sigma} - \Gamma^{\kappa}_{\lambda\rho}\delta\Gamma^{\mu}_{\kappa\sigma} - \Gamma^{\kappa}_{\lambda\sigma}\delta\Gamma^{\mu}_{\kappa\rho}$$

$$\begin{aligned}
\delta R^{\mu}{}_{\nu\rho\sigma} &= \delta\Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\rho\kappa}\delta\Gamma^{\kappa}_{\nu\sigma} - \Gamma^{\kappa}_{\rho\nu}\delta\Gamma^{\mu}_{\sigma\kappa} \\
&\quad - \left(\delta\Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\kappa}_{\nu\sigma}\delta\Gamma^{\mu}_{\rho\kappa} + \Gamma^{\mu}_{\sigma\kappa}\delta\Gamma^{\kappa}_{\nu\rho}\right) \\
&= \nabla_\rho\delta\Gamma^{\mu}_{\nu\sigma} + \Gamma^{\kappa}_{\rho\sigma}\Gamma^{\mu}_{\kappa\nu} \\
&\quad - \left(\nabla_\sigma\Gamma^{\mu}_{\nu\rho} + \Gamma^{\kappa}_{\rho\sigma}\Gamma^{\mu}_{\kappa\nu}\right) \\
&= \nabla_\rho\delta\Gamma^{\mu}_{\nu\sigma} - \nabla_\sigma\Gamma^{\mu}_{\nu\rho}
\end{aligned}$$

$\delta R_{\mu\nu} = \cancel{\delta g^{\rho\sigma} R}$ $\quad R_{\mu\nu} = \delta^{\rho}_{\sigma} R^{\rho}{}_{1}$ $\quad R_{\nu\sigma} = \delta^{\rho}_{\mu} R^{\mu}{}_{\nu\rho\sigma}$

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\lambda}\left(g_{\lambda\rho,\sigma} + g_{\lambda\sigma,\rho} - g_{\rho\sigma,\lambda}\right)$$

$$\delta\Gamma^\mu_{\rho\sigma} = \frac{1}{2}\delta g^{\mu\lambda}\left(g \dots\right) + \frac{1}{2} g^{\mu\lambda}\left(\left(\delta g_{\lambda\rho,\sigma}\right) + \dots\right)$$

Now $\delta(g^{\mu\nu} g_{\nu\sigma}) = \delta g^{\mu\nu} g_{\nu\sigma} + g^{\mu\nu}\delta g_{\nu\sigma}$

$$\Rightarrow \delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma}\delta g_{\rho\sigma}$$

$$\cancel{\delta\Gamma^\mu_{\rho\sigma} = g^{\mu\lambda} g^{\nu\sigma} g_{\lambda\sigma}}$$

$$\cancel{= g^{\mu\alpha} g^{\lambda\beta}\delta g_{\alpha\beta}\left(g_{\lambda\rho,\alpha} + g_{\lambda\sigma,\rho}\right.} - g_{\rho\sigma,\lambda})$$

$$\cancel{\nabla_\lambda \delta g_{\mu\nu} = \delta g_{\mu\nu}}$$

$$= -g^{\mu\alpha}\delta g_{\alpha\beta}\,\Gamma^\beta_{\rho\sigma} + g^{\mu\lambda}\left(+\dots\right)$$

Now $\delta g_{\lambda\rho,\sigma} = \nabla_\sigma \delta g_{\lambda\rho} + \Gamma^\alpha_{\sigma\lambda} g_{\alpha\rho} + \Gamma^\alpha_{\sigma\rho} g_{\lambda\alpha}$

etc. so

$$(\delta g_{\lambda\rho,\sigma} + \delta g_{\lambda\sigma,\rho} - \delta g_{\rho\sigma,\lambda})$$

$$= \Big(\nabla_\sigma \delta g_{\lambda\rho} + \cancel{\Gamma^{\alpha}_{\sigma\lambda}\delta g_{\alpha\rho}} + \Gamma^{\alpha}_{\sigma\rho}\delta g_{\lambda\alpha}$$

$$+ \nabla_\rho \delta g_{\lambda\sigma} + \cancel{\Gamma^{\alpha}_{\rho\lambda}\delta g_{\alpha\sigma}} + \Gamma^{\alpha}_{\rho\sigma}\delta g_{\alpha\lambda}$$

$$- \nabla_\lambda \delta g_{\rho\sigma} - \cancel{\Gamma^{\alpha}_{\lambda\rho}\delta g_{\alpha\sigma}} - \cancel{\Gamma^{\alpha}_{\lambda\sigma}\delta g_{\alpha\rho}} \Big)$$

$$= \big(\nabla + \nabla - \nabla + 2\,\Gamma^{\alpha}_{\rho\sigma}\delta g_{\alpha\lambda} \big)$$

$$\delta\Gamma^{\mu}_{\rho\sigma} = -g^{\mu\alpha}\cancel{\Gamma^{\beta}_{\rho\sigma}\delta g_{\alpha\beta}} + \frac{1}{2} g^{\mu\lambda}(\nabla_\sigma g_{\lambda\rho} + \nabla_\rho g_{\lambda\sigma} - \nabla_\lambda g_{\rho\sigma})$$

$$+ g^{\mu\lambda}\cancel{\Gamma^{\alpha}_{\rho\sigma}\delta g_{\alpha\lambda}}$$

$$= \underline{\frac{1}{2} g^{\mu\lambda}(\nabla_\sigma g_{\lambda\rho} + \nabla_\rho g_{\lambda\sigma} - \nabla_\lambda g_{\rho\sigma})}$$

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\rho_{\rho\mu}$$

$$= \frac{1}{2}\nabla_\rho \left(g^{\rho\lambda}\left(\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\lambda\mu} - \nabla_\lambda \delta g_{\mu\nu}\right)\right)$$

$$- \frac{1}{2}\nabla_\nu \left(g^{\rho\lambda}\left(\nabla_\mu \delta g_{\lambda\rho} + \nabla_\rho \delta g_{\lambda\mu} - \nabla_\lambda g_{\mu\rho}\right)\right)$$

$$\cancel{= - g^{\rho\alpha} g^{\lambda\beta}}$$

$$\cancel{\text{Use } \nabla_\rho(\delta^\mu_\nu) = \nabla_\rho(g^{\mu\alpha} g_{\alpha\nu}) = \nabla_\rho g^{\mu\alpha} g_{\alpha\nu} + g^{\mu\alpha} \nabla_\rho g_{\alpha\nu}}$$

$$\cancel{\Rightarrow \nabla_\rho g^{\alpha\beta} = - g^{\alpha\mu} g^{\beta\nu} \nabla_\rho g_{\mu\nu}}$$

$$\cancel{= - g^{\rho\alpha} g^{\lambda\beta} \nabla_\rho \delta g}$$

Note $\nabla g = 0$.

$$= \frac{1}{2} g^{\rho\lambda} \nabla_\rho \nabla_\mu \delta g_{\lambda\nu} + \cancel{\frac{1}{2} g^{\rho\lambda} \nabla_\rho \nabla_\nu \delta g_{\lambda\mu}} - \frac{1}{2}\nabla^2 \delta g_{\mu\nu}$$

$$- \frac{1}{2} g^{\rho\lambda} \nabla_\nu \nabla_\mu \delta g_{\lambda\rho} - \cancel{\frac{1}{2} g^{\rho\lambda} \nabla_\rho \nabla_\nu \delta g_{\lambda\mu}} + \frac{1}{2} g^{\rho\lambda} \nabla_\nu \nabla_\lambda \delta g_{\mu\rho}$$

$$= \frac{1}{2}\left(\nabla^\lambda \nabla_\mu \delta g_{\lambda\nu} + \nabla^\lambda \nabla_\nu \delta g_{\lambda\mu} - g^{\rho\lambda} \nabla_\nu \nabla_\mu \delta g_{\lambda\rho} - \nabla^2 \delta g_{\mu\nu}\right)$$

$$\nabla_\nu \nabla_\mu \delta g_{\rho\sigma} = \nabla_\nu (\delta g_{\rho\sigma;\mu} -$$

$$\nabla_\mu \nabla_\nu \delta g_{\rho\sigma} - \nabla_\nu \nabla_\mu \delta g_{\rho\sigma}$$

$$= \ldots +$$

$$\cancel{\delta R = \frac{1}{2}\left(\nabla^\lambda \nabla^\nu \delta g_{\lambda\nu} + \nabla^\lambda \nabla^\nu \delta g_{\lambda\nu} - g^{\rho\lambda} \nabla^2 \delta g_{\rho\lambda} - g^{\mu\nu} \nabla^2 \delta g_{\mu\nu} \right)}$$

$$\cancel{= \nabla^\mu \nabla^\nu \delta g_{\mu\nu} -}$$

$$\delta R = \delta(g^{\mu\nu} R_{\mu\nu})$$

$$= - g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma} R_{\mu\nu} + \nabla^\mu \nabla^\nu \delta g_{\mu\nu} - g^{\mu\nu} \nabla^2 \delta g_{\mu\nu}$$

$$= - R^{\mu\nu} \delta g_{\mu\nu} + \nabla^\mu (\nabla^\nu \delta g_{\mu\nu} - g^{\rho\sigma} \nabla_\mu \delta g_{\rho\sigma})$$

So

$$\delta I_{EH} \propto \int_M d^{d+1}x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} R \,\delta g_{\mu\nu} - R^{\mu\nu} \delta g_{\mu\nu} \right\}$$

$$+ \underbrace{\int_M d^{d+1}x \sqrt{-g}\, \nabla^{\mu}\left(\nabla^{\nu} \delta g_{\mu\nu} - g^{\rho\sigma} \nabla_{\mu} \delta g_{\rho\sigma} \right)}_{\text{surface term.}}$$

First term gives $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Surface term needs to be cancelled if action is to be truly stationary.