Black Holes

MSc in Quantum Fields and Fundamental Forces
Imperial College London

Fay Dowker

Blackett Laboratory,
Imperial College,
London, SW7 2AZ,
U.K.

Contents

Recommended Reading:

Popular/Historical:

- K.S. Thorne, *Black Holes and Time Warps*, Picador, 1994.
- W. Israel, *Dark Stars: The Evolution of an Idea*, in *Three Hundred Years of Gravitation* (S.W. Hawking & W. Israel, eds.), Cambridge University Press, 1987.

Textbooks:

- S.W. Hawking & G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, 1973.
- R.M. Wald, *General Relativity*, Chicago University Press, 1984.
- S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Addison-Wesley, 2004.

Lecture Notes:

- P. Townsend, *Black Holes*. Available online at arXiv:gr-qc/9707012.

General:

- S. W. Hawking & R. Penrose, *The Nature of Space and Time*, Princeton University Press, 1996.

  This is a series of lectures by Hawking and Penrose concluded by a final debate. Hawking's part of the lectures is available online at arXiv:hep-th/9409195.

1 The Schwarzschild Black Hole

The Schwarzschild metric (1916) is a solution to the vacuum Einstein equations $R_{\mu\nu} = 0$. It is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2, \tag{1.1}$$

where $0 < r < \infty$ is a radial coordinate and $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the round metric on the two-sphere.

The line-element (1.1) is the *unique* spherically symmetric solution to the vacuum Einstein equations. This result is known as Birkhoff's theorem and it has strong implications. For example, since (1.1) is independent of t, Birkhoff's theorem implies that *any* spherically symmetric vacuum solution must be time-independent (*static*).

To study the physics of black holes, we will start with a simple model of spherically symmetric gravitational collapse: a ball of pressure-free dust that collapses under its own gravity. Since the dust ball is spherically symmetric, Birkhoff's theorem tells us that outside of it (where the energy-momentum tensor vanishes), the metric must be Schwarzschild. Since the metric is continuous, it must be Schwarzschild *on* the surface, too. Let us follow the path of a massive particle on the surface of the dust ball. Assume it follows a radial geodesic, $d\theta = d\phi = 0$. Since the pull of gravity in the r-direction is the same for all particles at a given radius, the particle will remain at the surface throughout its trajectory.

The action of a massive relativistic particle moving in a spacetime (M, g) along a trajectory $x^\mu(\tau)$ parametrised by τ is given by

$$S = \int \mathcal{L}\, d\tau = \frac{1}{2m}\int \left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - m^2\right)d\tau. \tag{1.2}$$

An implicit assumption in this form of the action is that the parameter τ is equal to the proper time measured along the particle's worldline (more on this in section 3.2). For the radial trajectory of the surface particle we have $x^\mu(\tau) = (t(\tau), r(\tau), \theta_0, \phi_0)$. Define

$$R(t) \equiv r(\tau(t)), \tag{1.3}$$

the radius of the particle's position as a function of coordinate-time t. We want to find the equation of motion for $R(t)$. A simple way to get the first equation is from $d\tau^2 = -ds^2$, which implies

$$1 = \left[\left(1 - \frac{2M}{R}\right) - \left(1 - \frac{2M}{R}\right)^{-1}\left(\frac{dR}{dt}\right)^2\right]\left(\frac{dt}{d\tau}\right)^2. \tag{1.4}$$

The factor $dt/d\tau$ can be found from the Euler-Lagrange equations of the action:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(dx^\mu/d\tau)}\right) = 0. \tag{1.5}$$

The action of the dust particle in the Schwarzschild background takes the form

$$S = \frac{1}{2m}\int\left(-\left(1-\frac{2M}{R}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{R}\right)^{-1}\left(\frac{dR}{d\tau}\right)^2 - m^2\right) d\tau. \tag{1.6}$$

Since it has no explicit dependence on the t-coordiante (it only depends on $dt/d\tau$), the Euler-Lagrange equation for the t-coordinate

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(dt/d\tau)}\right) = \frac{d}{d\tau}\left[-2\left(1-\frac{2M}{R}\right)\frac{dt}{d\tau}\right] = 0 \tag{1.7}$$

reveals a constant of motion along the geodesic $x^\mu(\tau)$:

$$\epsilon = \left(1-\frac{2M}{R}\right)\frac{dt}{d\tau} = const. \tag{1.8}$$

The constant ϵ is equal to the energy per unit rest mass of the particle, as measured by an inertial observer at $r=\infty$. To see this, recall that the energy of a relativistic particle is given by the t-component of the momentum (co-)vector $p_\mu = (-E, \mathbf{p}) = m g_{\mu\nu}\, dx^\mu/d\tau$. Then

$$E/m = -p_0/m = g_{00}\frac{dt}{d\tau} = \left(1-\frac{2M}{R}\right)\frac{dt}{d\tau} = \epsilon > 0. \tag{1.9}$$

A particle starting at rest at infinity has $\epsilon = 1$ at $R=\infty$ and therefore $\epsilon = 1$ everywhere along its geodesic. Consequently, a particle starting at rest at some finite radius $R = R_{\rm max}$ must have $\epsilon < 1$. We will therefore assume $0<\epsilon<1$ from now on.

Plugging (1.8) into (1.4) we obtain

$$1 = \left[1-\frac{2M}{R} - \left(1-\frac{2M}{R}\right)^{-1}\dot{R}^2\right]\left(1-\frac{2M}{R}\right)^{-2}\epsilon^2, \tag{1.10}$$

where we have defined $\dot{R} \equiv dR/dt$. This gives the following equation for $\dot{R}$:

$$\dot{R}^2 = \epsilon^{-2}\left(1-\frac{2M}{R}\right)^2\left(\frac{2M}{R} - 1 + \epsilon^2\right). \tag{1.11}$$

We immediately see that $\dot{R}$ vanishes at

$$R = R_{\rm max} \equiv 2M/(1-\epsilon^2) > 2M \tag{1.12}$$

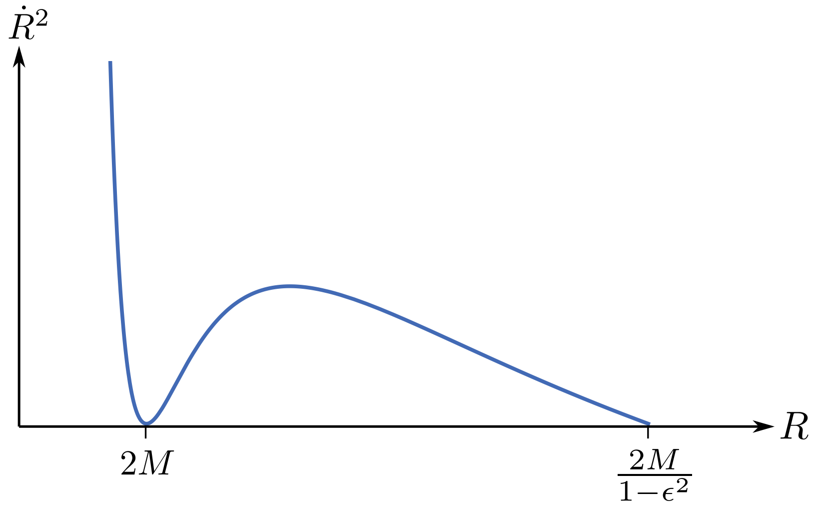

Figure 1. $\dot{R}^2$ as a function of R for the massive surface particle.

and at $R = 2M$, as illustrated in figure 1. The zero at $R_{\rm max}$ simply corresponds to the initial condition that the particle starts off at rest at $R_{\rm max}$. However, $\dot{R}$ approaches zero again as $R \to 2M^+$: the particle appears to *slow down* as it approaches the Schwarzschild radius.

How much time t_{2M} does it take the particle to reach $R = 2M$ from $R = R_{\rm max}$? Integrating (1.11), we find

$$t_{2M} = \int_0^{t_{2M}} dt = -\epsilon \int_{R_{\rm max}}^{2M} dR \left(1 - \frac{2M}{R}\right)^{-1} \left(\frac{2M}{R} - 1 + \epsilon^2\right)^{-\frac{1}{2}} = \infty\,! \tag{1.13}$$

To convince yourself that this integral is infinite, look at the contribution to the integral from the region near $R = 2M$. Let $R = 2M + \rho$ with $0 < \rho \ll 2M$ and Taylor expand in ρ:

$$\begin{aligned} t_{2M} &= \text{finite number} - \epsilon \int_\rho^0 d\rho \left(1 - \frac{2M}{2M+\rho}\right)^{-1} \left(\frac{2M}{2M+\rho} - 1 + \epsilon^2\right)^{-\frac{1}{2}} \\ &= \text{finite number} - \epsilon \int_\rho^0 d\rho \left[\frac{2M}{\rho} + \mathcal{O}(\rho^0)\right]. \end{aligned} \tag{1.14}$$

This is infinite since the $1/\rho$ term in the integral diverges logarithmically at 0. Hence, as far as the coordinate time t is concerned, the particle never reaches the Schwarzschid radius.

This result seems physically rather strange. The surface of a collapsing ball of dust mysteriously slows down as it approaches $R = 2M$ and never actually reaches $R = 2M$. However, notice that while t may be a meaningful measure of time for

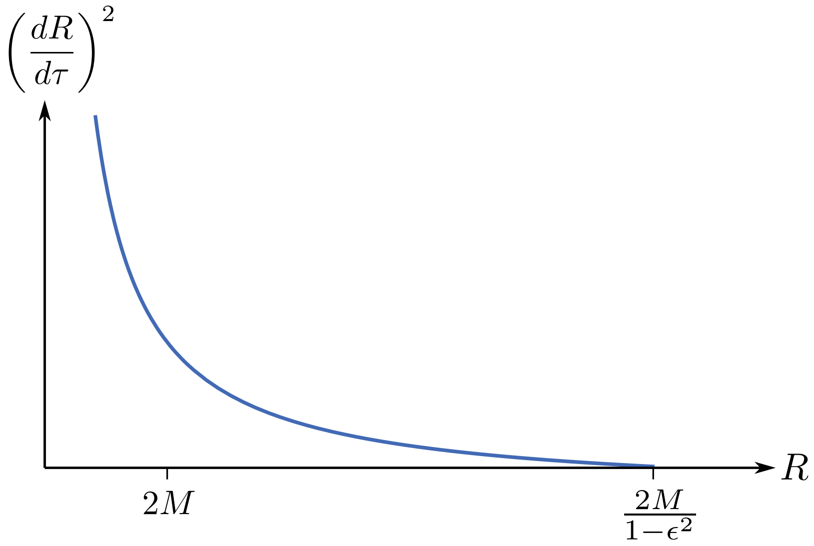

Figure 2. $(dR/d\tau)^2$ as a function of R for the massive surface particle.

inertial observers at $r = \infty$ (t is proportional to their proper time since the metric is flat at infinity), it has no physical meaning to the infalling particle itself (or to an observer falling along with it). This should not be surprising. After all, t is just a coordinate, and **coordinates in general relativity have no physical meaning.**

So let us instead work with the proper time τ of the infalling particle. How does R change as a function of τ? Using (1.8) and (1.4) we find

$$\left(\frac{dR}{d\tau}\right)^2 = (1-\epsilon)^2 \left(\frac{R_{\max}}{R} - 1\right). \tag{1.15}$$

We see that nothing curious at all happens as $R \to 2M^+$. The particle (and therefore the surface of the dust ball) passes smoothly through $R = 2M$. This is confirmed by the plot of $(dR/d\tau)^2$ against R in figure 2.

Let us calculate the proper time τ_{2M} that elapses for the particle between $R = R_{\max}$ and $R = 2M$. Using (1.15), we find (exercise)

$$\tau_{2M} = \int_0^{\tau_{2M}} d\tau = \int_{R_{\max}}^{2M} \left(\frac{d\tau}{dR}\right) dR = \frac{2M}{(1-\epsilon^2)^{\frac{3}{2}}} \left(\epsilon\sqrt{1-\epsilon^2} + \arcsin\epsilon\right), \tag{1.16}$$

which is indeed finite. The strange behaviour near $R = 2M$ has disappeared — a first indication that the singularity in the metric at $R = 2M$ is due to a pathology in the time coordinate t.

The proper time taken to reach $r=0$ is also finite:

$$\tau_0 = \int_{R_{\max}}^{0} \left(\frac{d\tau}{dR}\right) d\tau = \frac{\pi M}{(1-\epsilon^2)^{\frac{3}{2}}}. \tag{1.17}$$

This means that the entire dust star collapses to the single point $r=0$ in a finite proper time, a first sign that there is a true singularity at $r=0$ — a *curvature singularity* — at which physical (coordinate-independent) quantities such as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverge.

While the calculations above have given us some useful insights, we have been a bit careless. We worked out the geodesics using the Schwarzschild metric, and traced them through $r=2M$, even though the metric (1.1) breaks down at $r=2M$. Furthermore, for $r<2M$, the factor $\left(1-\frac{2M}{r}\right)$ becomes negative, which makes t appear like a space-coordinate and r like a time-coordinate. Statements such as "the star collapses to $r=0$ in finite time" then become somewhat suspect. To make our calculations sound, let us replace the coordinate system (t,r,θ,ϕ) by a more suitable set of coordinates.

1.1 Eddington-Finkelstein Coordinates

If we want to overcome the problems encountered in the Schwarzschild coordinate system, an obvious place to start is the time-coordinate t. We will replace t by a coordinate that is "adapted to null geodesics". For a null geodesic we have $ds^2=0$, which implies

$$dt^2 = \left(1-\frac{2M}{r}\right)^{-2} dr^2. \tag{1.18}$$

First, we define the radial (Regge-Wheeler) coordinate r_* via

$$dr_*^2 \equiv \left(1-\frac{2M}{r}\right)^{-2} dr^2, \tag{1.19}$$

so that null geodesics obey the simple equation $dt^2 = dr_*^2$. Solving (1.19) and requiring r_* to be real and increasing with r, we obtain

$$r_* = r + 2M\ln\left(\frac{r-2M}{2M}\right). \tag{1.20}$$

The range $2M < r < \infty$ corresponds to $-\infty < r_* < \infty$, since $r_* \to -\infty$ as $r \to 2M^+$. For this reason, r_* is also known as a *tortoise coordinate*: as we approach the Schwarzschild radius, r changes more and more slowly with r_* since $dr/dr_* \to 0$.

The solutions $t \pm r_* = \mathit{const}.$ correspond to ingoing and outgoing null geodesics. Let us define a new pair of *lightcone coordinates*:

$$\begin{aligned} u &= t - r_* \\ v &= t + r_*, \end{aligned} \tag{1.21}$$

such that lines of constant v correspond to ingoing null geodesics and lines of constant u to outgoing null geodesics.

1.1.1 Ingoing Eddington-Finkelstein coordinates

We now make a coordinate transformation from (t, r, θ, ϕ) to (v, r, θ, ϕ), i.e. we replace t by the coordinate v that labels ingoing radial null lines. The coordinates (v, r, θ, ϕ) are called *ingoing Eddington-Finkelstein (IEF) coordinates* and the line-element in IEF coordinates reads (exercise)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega_2^2. \tag{1.22}$$

This metric is perfectly regular at $r = 2M$ (i.e. it is non-degenerate: $\det g_{\mu\nu} \neq 0$). In fact, it is fine *everywhere* except at $r = 0$. We may therefore extend the range of r across $r = 2M$ to all $r > 0$ (you can easily check that (1.22) is a solution to the vacuum Einstein equations $R_{\mu\nu} = 0$ for all $r > 0$). We can see clearly now that there is nothing peculiar about $r = 2M$; there is nothing going on in the local physics that would tell you that you are approaching or passing through the surface $r = 2M$. The original coordinates (t, r, θ, ϕ) were simply not suitable to describe the spacetime region beyond $r = 2M$ due to their bad behaviour there.

Figure 3 shows radial ingoing ($v = const.$) and outgoing ($u = const.$) null geodesics in the IEF metric. Notice how the lightcones tip over as we cross the Schwarzschild radius: while "outgoing" photons that start off oustide the horizon eventually escape to infinity, "outgoing" photons emitted at $r = 2M$ *remain* at $r = 2M$ forever.[1] "Outgoing" photons emitted at $r < 2M$ stay within $r < 2M$ and fall towards $r = 0$. As they reach $r = 2M$, their geodesics become parallel to ingoing null geodesics.

We call the surface $r = 2M$ (a two-sphere in $3 + 1$ dimensions) the *event horizon* and the region $r < 2M$ the *black hole*. The surface dust particle is massive, so its worldline must lie between ingoing and outgoing null geodesics (inside the lightcone). As you can see in figure 3, this means that the particle must eventually hit $r = 0$. Furthermore, no signal from an event inside the event horizon can ever escape the black hole to reach an observer at $r > 2M$.

From figure 3 we can also get an idea how someone hovering at a fixed radius $r > 2M$ outside the black hole will perceive the infalling matter as it falls across the

[1]This reveals one at first perhaps counter-intuitive aspect of the horizon, which we will see in more detail in later sections: the surface $r = 2M$ is a *null* surface (null geodesics travel along it), even though it is a surface of constant r.

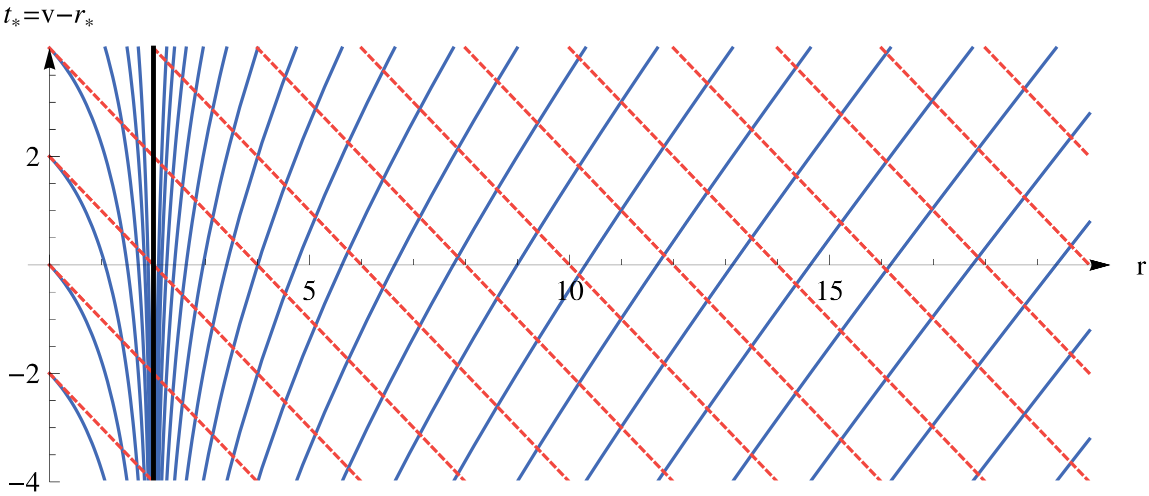

Figure 3. Ingoing (dashed red) and outgoing (solid blue) null geodesics in IEF coordinates for $M = 1$. The horizon at $r = 2M$ is highlighted in black.

horizon. Imagine a signal being transmitted from the surface of the dust ball at a constant rate. Due to the bending of the lightcones in the vicinity of the horizon, the signals reaching the observer will become sparser and sparser as the surface approaches $r = 2M$. If we think of these signals as electromagnetic radiation, i.e. light, then the light is shifted toward the lower (i.e. red) end of the spectrum, and the signal received by the distant observer becomes redder and redder until it eventually disappears.

1.1.2 Outgoing Eddington-Finkelstein coordinates

In the previous section, we made a coordinate transformation by replacing t by v. Another obvious possibility is to use the outgoing EF coordinate $u = t - r_*$ instead. This leads to *outgoing Eddington-Finkelstein (OEF)* coordinates (u, r, θ, ϕ).

The metric in OEF coordinates is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 d\Omega_2^2. \tag{1.23}$$

Note the similarity between (1.22) and (1.23): the only difference is the sign of the cross-term. However, the physical picture of the spacetime described by the line-element (1.23) turns out to be very different. To see this, let us consider again "ingoing" ($v = const.$) and "outgoing" ($u = const.$) null geodesics. They are depicted in figure 4, where we have defined the OEF time $t_* = u + r_*$. "Ingoing" photons emitted at $r > 2M$ or $r < 2M$ never cross $r = 2M$: they approach $r = 2M$

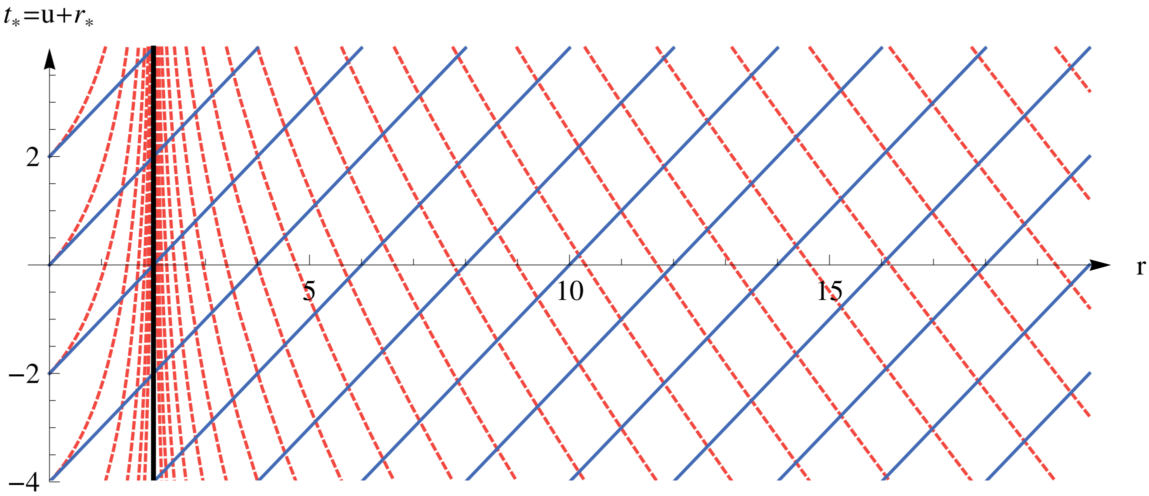

Figure 4. Ingoing (dashed red) and outgoing (solid blue) null geodesics in the OEF metric for $M = 1$. The horizon at $r = 2M$ is highlighted in black.

and hover the horizon forever. Conversely, all outgoing null geodesics escape to infinity. Looking at the lightcones, we see that everything inside $r = 2M$ is ejected. The interior region $r < 2M$ is therefore called a *white hole* and $r = 2M$ the white hole horizon. It is the exact time reversal of a black hole (you can see this by turning figure 3 upside down and comparing it with figure 4).

What happens in the spacetime described by OEF coordinates is clearly very different from what happens in the spacetime described by IEF coordinates. This may seem contradictory, since both are obtained by coordinate transformations from the original Schwarzschild spacetime. We will learn how to make sense of this apparent contradiction by introducing yet another coordinate system.

1.2 Kruskal-Szekeres Coordinates

First, change coordinates to (u, v, θ, ϕ), replacing both Schwarzschild coordinates r and t by $u = t - r_*$ and $v = t + r_*$ (where r_* is the function of r defined in (1.20)). The metric then reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du dv + r^2 d\Omega_2^2, \tag{1.24}$$

where $r = r(u, v)$ can be given in terms of u and v via (1.20) and $r_* = (v - u)/2$. This metric has a coordinate singularity at $r = 2M$, so it is only defined for $r \neq 2M$. To remove the singularity, we define a new pair of coordinates U and V in the region $r > 2M$ by

$$U = -\exp\left(-\frac{u}{4M}\right) \qquad \text{and} \qquad V = \exp\left(\frac{v}{4M}\right). \tag{1.25}$$

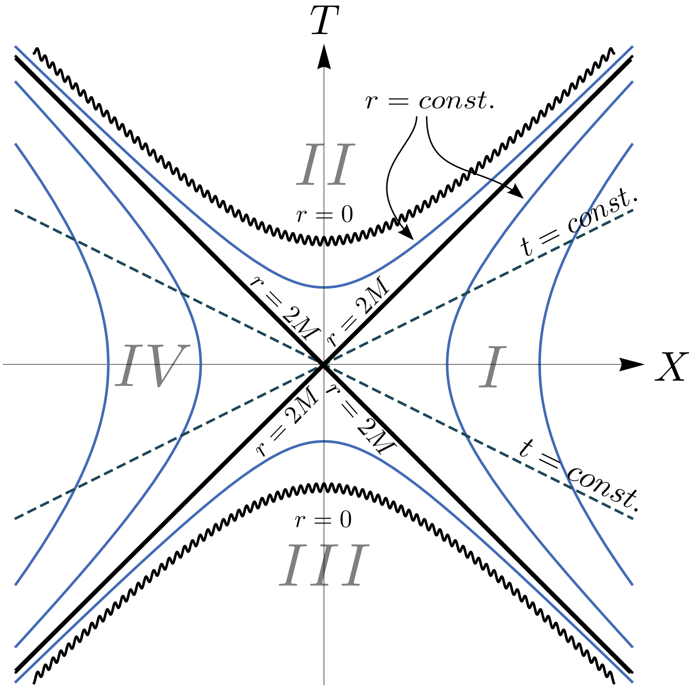

Figure 5. Kruskal spacetime. Each point represents a 2−sphere of radius $r(T, X)$. The black lines at 45° correspond to $r = 2M$. The singularity $r = 0$ is represented by a wavy line. Solid blue hyperbolae correspond to $r = const.$ surfaces and straight dashed lines correspond to $t = const.$ surfaces.

Note that $U < 0$ and $V > 0$ for all values of r. The coordinates (U, V, θ, ϕ) are called *Kruskal-Szekeres (KS) coordinates*. The Schwarzschild metric in KS coordinates is (exercise):

$$ds^2 = -\frac{32M^3}{r} \exp\left(-\frac{r}{2M}\right) dU dV + r^2 d\Omega_2^2, \tag{1.26}$$

where $r = r(U, V)$ is defined in terms of U and V by the implicit equation $UV = -\left(\frac{r-2M}{2M}\right) \exp\left(\frac{r}{2M}\right)$.[2]

This metric is well-defined for $r = 2M$ and indeed for all $r > 0$. Notice that for $r < 2M$, we have $UV > 0$, which is incompatible with (1.25). Let us therefore *forget* the initial definitions of U and V in terms of the Schwarzschild coordinates for now (much like we did for u and v when we went from Schwarzschild to Eddington-Finkelstein coordinates), and instead investigate the new spacetime with metric (1.26) an extended coordinate ranges $-\infty < U, V < \infty$.

[2]We can write r explicitly as $r(U, V) = 2M\left[1 + W\left(-UV/e\right)\right]$ where e is Euler's constant and W stands for the Lambert W-function.

Figure 5 is a picture of the spacetime described by the KS metric. Since null lines are conventionally plotted at 45°, we define time and space coordinates $T = U + V$ and $X = T - V$, which label the vertical and horizontal axes in the figure. The U and V axes are then at 45° to the T and X axes. At the horizon $r = 2M$, we have $UV = 0$, which means either $U = 0$ or $V = 0$. This corresponds to the solid diagonals. The singularity $r = 0$ corresponds to the (two branches of the) hyperbola described by $UV = 1$, which is represented by a wavy line (singularities will always be represented by wavy lines). In general, surfaces of $r = const.$ correspond to hyperbolae $UV = const.$ with $UV < 1$, as shown in blue on the diagram. Spatial sections with $t = const.$ have $U/V = const.$ and $|U/V| < 1$, which corresponds to straight lines through the origin with gradient between -1 and 1. Finally, ingoing and outgoing null geodesics are respectively given by $U = const.$ and $V = const.$

We can now see he relation between the different coordinate systems. The original Schwarzschild coordinates (t, r, θ, ϕ) only cover region I. When we changed to IEF and OEF coordinates, we extended our spacetime to regions II and III, respectively. There is a new region that we only see in KS coordinates, region IV. This region is a mirror image of region I, which can be seen by defining

$$U = \exp\left(-\frac{u}{4M}\right) \qquad \text{and} \qquad V = -\exp\left(\frac{v}{4M}\right) \tag{1.27}$$

on region IV and noting that the metric (1.26) will be of the same form as the Schwarzschild metric. (The proper statement is that region IV is *isometric* to region I. In general relativity, two spacetimes or regions of spacetime that are isometric are physically identical.)

The geometry of spatial sections is also worth a closer look. Consider a slice with $t = const. \iff T/X = const.$ On figure 5, it corresponds to a straight line through the point $(T, X) = (0, 0) \iff r = 2M$, where regions I and IV attach to each other. In both regions the metric on the slice is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{1.28}$$

and in both regions r increases from $2M$ to ∞. For large r, we have $\left(1 - \frac{2M}{r}\right) \to 1$ and so the geometry becomes Euclidean. As we approach $r = 2M$, i.e. the centre in figure 5, the metric (1.28) starts to deviate from the Euclidean metric. What is the geometry near $r = 2M$? Since we cannot draw a three-dimensional surface, let us suppress one angular coordinate by setting θ to its equatorial value $\theta = \frac{\pi}{2}$. Then the metric is $ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2$. It is not hard to show (exercise) that this is just the metric on a quartic surface $x^2 + y^2 = (z^2/8M + 2M)^2$ embed-

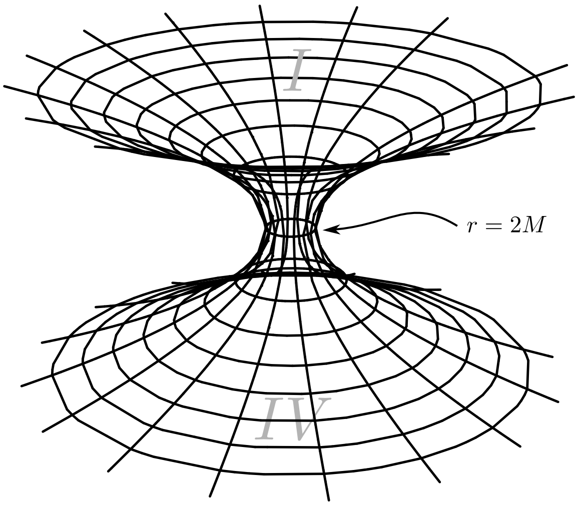

Figure 6. The Einstein-Rosen bridge with one spatial dimension suppressed. Each circle represents a two-sphere in the three-dimensional analogue.

ded in three-dimensional Euclidean space $\mathbb{E}^3$ with cartesian coordinates x, y, z.[3] The geometry is shown in figure 6, where two identical copies of the surface have been attached to each other at the circle $r = 2M$. One side corresponds to region I and the other side to region IV. This is known as the *Einstein-Rosen bridge*, which is one example of a *wormhole*. Note that no observer can ever cross the wormhole, as you can see clearly in the Kruskal diagram (and, further along in the lectures, in the Penrose diagram for Kruskal space): in order to cross the wormhole from region I to IV or vice-versa, the trajectory of the observer would have to be spacelike somewhere.

We can also get an intuitive sense for the strange sign change that appears in the original Schwarzschild metric (1.1) between $r > 2M$ and $r < 2M$, which makes r appear like a time coordinate when $r < 2M$. Indeed, if $r = 0$ were just a "position in space", as one might naively think of it, it would seem that one could simply avoid it by navigating around it. In figure 5, we see that for anyone who has fallen across the horizon, the singularity $r = 0$ is *not* a position in space — it becomes a *moment of time*, as unavoidable as 9am tomorrow morning.

Finally, we can justify in hindsight some of the coordinate extensions (analytic continuations) that we performed rather ad-hoc in the last sections. For example, consider the extension from region I covered by Schwarzschild coordinates to regions $I + II$ covered by IEF coordinates. The original coordinate system with time co-

[3]Hint: Recall that the metric on $\mathbb{E}^3$ is $ds^2 = dx^2 + dy^2 + dz^2$. Consider the surface parametrised by $x = r\cos\phi$, $y = r\sin\phi$ and $z = \sqrt{8M(r-2M)}$, show that it satisfies the quartic equation above and find the metric induced on it. For an illustration see http://wolfr.am/WIRjK3.

ordinate $-\infty < t < \infty$ covers region I only. If we think of region I as a physical spacetime in its own right, then a particle will hit a "boundary" ($r = 2M$) in finite proper time τ_{2M}. In light of this, it seems physically quite reasonable to work instead with the spacetime $I + II$ obtained by extending the original coordinate system, where this artificial boundary disappears.

1.3 Penrose Diagrams

In this section we introduce a useful way of representing the causal structure of an infinite spacetime on a finite piece of paper. This involves performing a *conformal transformation* on the metric:

Definition. A *conformal transformation* is a map from a spacetime (M, g) to a spacetime $(M, \tilde{g})$ such that

$$\tilde{g}_{\mu\nu}(x) = \Lambda(x)^2 g_{\mu\nu}(x)$$

where $\Lambda(x)$ is a smooth function of the spacetime coordinates and $\Lambda(x) \neq 0 \; \forall \; x$.

Conformal transformations preserve the causal structure of a spacetime. To see this, consider a vector V^μ on M. Then it follows from $\Lambda(x)^2 > 0$ that

$$\begin{aligned} g_{\mu\nu}V^\mu V^\nu > 0 &\iff \tilde{g}_{\mu\nu}V^\mu V^\nu > 0 \\ g_{\mu\nu}V^\mu V^\nu = 0 &\iff \tilde{g}_{\mu\nu}V^\mu V^\nu = 0 \\ g_{\mu\nu}V^\mu V^\nu < 0 &\iff \tilde{g}_{\mu\nu}V^\mu V^\nu < 0. \end{aligned} \tag{1.29}$$

Hence, curves that are timelike/null/spacelike with respect to g are timelike/null/spacelike with respect to $\tilde{g}$. Furthermore, by consequence of the second line, null geodesics in g correspond to null geodesics in $\tilde{g}$ (whereas timelike/spacelike geodesics in g are *not* necessarily geodesics in $\tilde{g}$).

The idea of a Penrose diagram is this. First, we use a coordinate transformation on the spacetime (M, g) to bring "infinity" to a finite coordinate distance, so that we can draw the entire spacetime on a sheet of paper. The metric will typically diverge as we approach the "points at infinity", i.e. the edges of the finite diagram. To remedy this, we perform a conformal transformation on g to obtain a new metric $\tilde{g}$ that is regular on the edges. Then $(M, \tilde{g})$ is a good representation of the original spacetime (M, g) insofar that it has exactly the same causal structure. It is customary to add the points at infinity to M to form a new manifold $\tilde{M}$. The resulting spacetime $(\tilde{M}, \tilde{g})$ is what is called the *conformal compactification* of (M, g).

Note that the curvature tensors are in general not preserved under conformal transformations, e.g. $\tilde{R}^{\mu}_{\nu\rho\sigma} \neq R^{\mu}_{\nu\rho\sigma}$, $\tilde{R} \neq R$ etc. In that sense, the conformally compactified spacetime $(\tilde{M}, \tilde{g})$ is unphysical: it provides a good representation of the causal structure of the physical spacetime (M, g), but it should otherwise not be viewed as a picture of what is going on (for example, as noted above, the geodesics of massive test particles in $(\tilde{M}, \tilde{g})$ do not correspond to geodesics of massive test particles in (M, g)).

1.3.1 Example 1: Minkowski space in $1+1$ dimensions

The two-dimensional Minkowski metric is given by

$$ds^2 = -dt^2 + dx^2 \tag{1.30}$$

where $-\infty < t, x < \infty$. We introduce light-cone coordinates $u = t - x$ and $v = t + x$, in which the metric $g_{\mu\nu}$ takes the simple form

$$ds^2 = -du dv. \tag{1.31}$$

Now, to shrink "infinity" down to a finite coordinate distance, we define a new set of coordinates via

$$u = \tan \tilde{u} \qquad \text{and} \qquad v = \tan \tilde{v}. \tag{1.32}$$

These coordinates indeed have a finite range $-\frac{\pi}{2} < \tilde{u}, \tilde{v} < \frac{\pi}{2}$ (the range is open since points with $u, v = \tan(\pm\frac{\pi}{2}) = \pm\infty$ are not part of the spacetime). The line-element in terms of $\tilde{u}$ and $\tilde{v}$ is

$$ds^2 = -\frac{1}{(\cos \tilde{u} \cos \tilde{v})^2} d\tilde{u} d\tilde{v}. \tag{1.33}$$

It diverges as $u, v \to \pm\infty \iff \tilde{u}, \tilde{v} \to \pm\frac{\pi}{2}$. Define a new metric $\tilde{g}_{\mu\nu}$ through a conformal transformation on $g_{\mu\nu}$:

$$d\tilde{s}^2 = (\cos \tilde{u} \cos \tilde{v})^2 \, ds^2 = -d\tilde{u} d\tilde{v}. \tag{1.34}$$

This metric is regular at the "points at infinity" where either $\tilde{u}$ or $\tilde{v}$ are equal to $\pm\frac{\pi}{2}$ and we can now add them to the spacetime. The resulting spacetime $(\tilde{M}, \tilde{g})$ is the conformal compactification of (M, g). Both spacetimes are shown in $(\tilde{u}, \tilde{v})$ coordinates in figure 7.

The two points $(\tilde{u}, \tilde{v}) = (\frac{\pi}{2}, \frac{\pi}{2})$ and $(-\frac{\pi}{2}, -\frac{\pi}{2})$ are denoted by $i^{\pm}$. All future (past) directed timelike curves end up at i^+ (i^-), so we refer to i^+ (i^-) as *future (past) timelike infinity*. Future directed null geodesics either end up at $\tilde{v} = \frac{\pi}{2}$ with constant $|\tilde{u}| < \frac{\pi}{2}$ or at $\tilde{u} = \frac{\pi}{2}$ with constant $|\tilde{v}| < \frac{\pi}{2}$. This set of points is denoted by $\mathcal{J}^+$

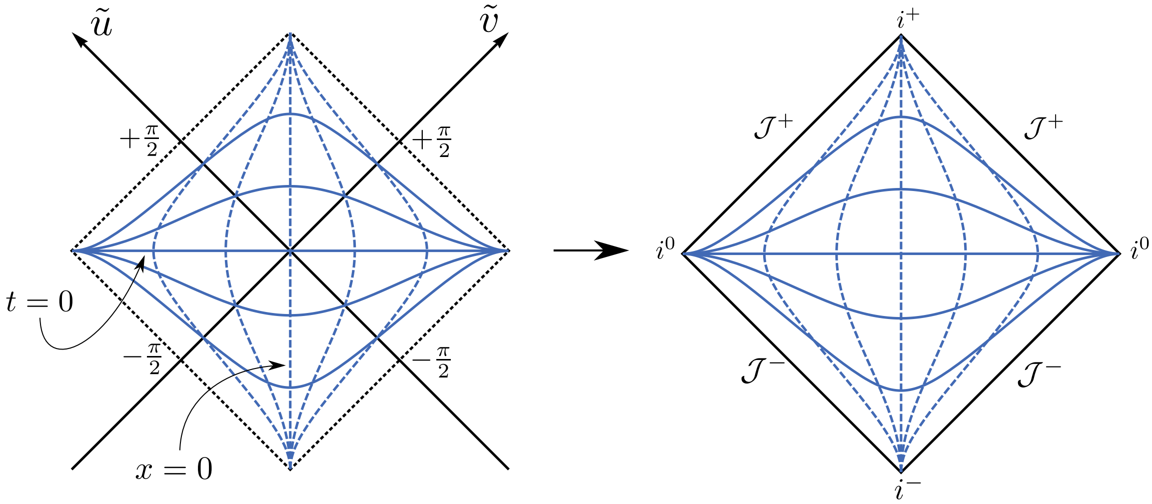

Figure 7. **Left:** Minkowski space (M, g) in $(\tilde{u}, \tilde{v})$ coordinates. The boundaries $\tilde{u}, \tilde{v} = \pm\frac{\pi}{2}$ are not part of M and g diverges there. Some timelike and spacelike geodesics of g have been included: lines with $r = const.$ are represented by dashed curves and lines with $t = const.$ are represented by solid curves. **Right:** The Penrose diagram of conformally compactified Minkowski space $(\tilde{M}, \tilde{g})$, with future/past timelike infinity $i^{\pm}$, future/past null infinity $\mathscr{J}^{\pm}$ and spacelike infinity i^0. The timelike and spacelike geodesics of Minkowski space are clearly not all geodesics for $(\tilde{M}, \tilde{g})$, since $\tilde{g}$ is flat in $(\tilde{u}, \tilde{v})$ coordinates.

("scri-plus") and referred to as *future null infinity*. An analogous definition holds for past null infinity $\mathscr{J}^-$ ("scri-minus"). Finally, *spacelike infinity* i^0 denotes the set of endpoints of spacelike geodesics, which corresponds here to $(\tilde{u}, \tilde{v}) = (\frac{\pi}{2}, -\frac{\pi}{2})$ and $(\tilde{u}, \tilde{v}) = (-\frac{\pi}{2}, \frac{\pi}{2})$.

1.3.2 Example 2: Minkowski space in $3+1$ dimensions

The four-dimensional Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{1.35}$$

can be written in terms of spherical polar coordinates (t, r, θ, ϕ) by choosing an arbitrary point as the origin, say $x = y = z = 0$. We then have

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \tag{1.36}$$

where $d\Omega_2^2$ is the round metric on the 2-sphere. Define light-cone coordinates $u = t - r$ and $v = t + r$ and perform the same transformation (1.32) as above to bring infinity to finite coordinate distance. The only difference to the previous analysis is that since $r \geq 0$, we have the additional constraint $u \leq v \iff \tilde{u} \leq \tilde{v}$. The metric in

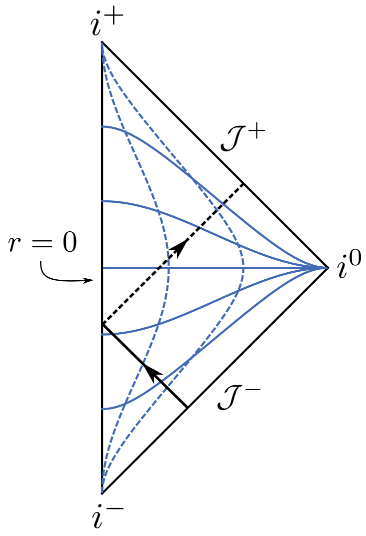

Figure 8. The Penrose diagram of $3+1$ dimensional Minkowski space with some lines of constant r and t. Each point represents a $2-$sphere. As the null geodesic shown passes through $r = 0$, it emerges on another copy of the Penrose diagram whose points represent the antipodes on the two-spheres.

$(\tilde{u}, \tilde{v}, \phi, \theta)$ coordinates reads

$$ds^2 = -\frac{1}{(2\cos\tilde{u}\cos\tilde{v})^2}\left[-4d\tilde{u}d\tilde{v} + \sin^2(\tilde{v} - \tilde{u})\, d\Omega_2^2\right] \tag{1.37}$$

and its conformal compactification is obtained via

$$d\tilde{s}^2 = (\cos\tilde{u}\cos\tilde{v})^2\, ds^2 = -4d\tilde{u}d\tilde{v} + \sin^2(\tilde{v} - \tilde{u})\, d\Omega_2^2. \tag{1.38}$$

On the Penrose diagram of $1+1$ dimensional Minkowski space in figure 7, $\tilde{u} \leq \tilde{v}$ implies that we only include the part that lies to the right of the vertical line $x = 0$. The resulting diagram for the $3+1$ space is shown in figure 8. Every point on the diagram represents a two-sphere of radius $\sin(\tilde{v} - \tilde{u})$.

There exists a somewhat more illustrative way to picture the conformal compactification. Define $\tilde{T} = \tilde{v} + \tilde{u}$ and $\chi = \tilde{v} - \tilde{u}$. The coordinate ranges are then (exercise) $-\pi \leq \tilde{T} \leq \pi$ and $0 < \chi < \pi$ and the metric reads

$$d\tilde{s}^2 = -d\tilde{T}^2 + d\chi^2 + \sin^2\chi d\Omega_2^2. \tag{1.39}$$

The spatial part $d\chi^2 + \sin^2\chi d\Omega_2^2$ of this metric is just the round metric of a three-sphere S^3 parametrised by polar coordinates (χ, θ, ϕ). The spacetime (1.39) therefore represents a static universe with spherical spatial slices, which corresponds to a finite slab of the *Einstein static universe (ESU)* whose topology is $\mathbb{R} \times S^3$. This is shown

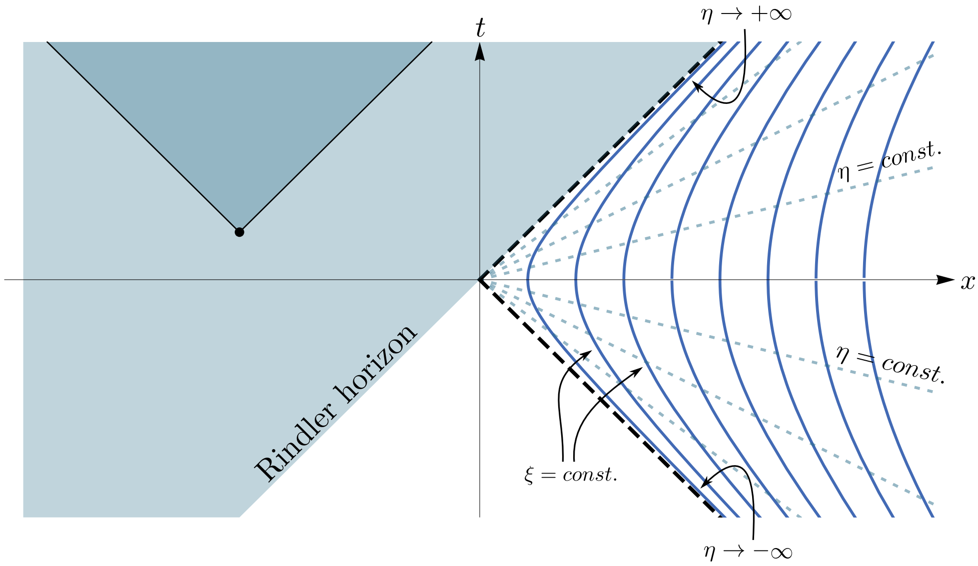

Figure 9. Eternally accelerating observers in Minkowski space. Their worldlines are shown in blue and labelled by ξ. Events in the shaded region such as the black dot are hidden to them. The Rindler horizon is the boundary between the shaded and unshaded regions. Rindler space corresponds to the right wedge outlined by the dashed black null lines. The straight dotted lines are lines of constant Rindler time η.

in figure (to come). The triangle in figure 8 then corresponds to that part of the region wrapped around the cylinder facing you.

1.3.3 Example 3: Rindler space in $1+1$ dimensions

Rindler space is a subregion of Minkowski space associated with observers eternally accelerated at a constant rate. The worldlines of these "Rindler observers" (accelerating in the positive x-direction) is given by the hyperbolae $x_\mu x^\mu = -t^2 + x^2 = \xi^2$. You can check (exercise) that the 4-acceleration $a^\mu = d^2x^\mu/d\tau^2$ along this worldline is indeed constant: $a_\mu a^\mu = 1/\xi^2$.

Consider the one-parameter family of Rindler observers depicted in figure 9. The region $x \leq t$ is forever hidden to them, which makes the line $x = t$ a horizon to these observers.[4] *Rindler space* corresponds to the right wedge $x > |t|$ foliated by the worldlines of the accelerated observers in figure 9.

[4]This horizon is different from the event horizon in the Schwarzschild black hole spacetime, since the Schwarzschild horizon is observer/frame-independent, while the horizon here is associated with a family of special observers/frames.

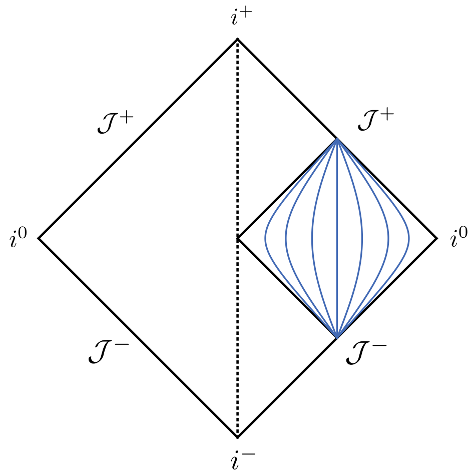

Figure 10. The Penrose diagram for $1+1$ dimensional Rindler space, seen as a portion of Minkowski space. Some accelerated worldlines (curves of constant ξ) have been drawn for $\xi = \frac{2}{9}, \frac{2}{5}, \frac{2}{3}, 1, \frac{3}{2}, \frac{5}{2}, \frac{9}{2}$. Note that all of the worldlines represent observers accelerating in the postitive $x-$direction, even though they appear to bend toward the right for $\xi > 1$. This distortion is a side effect of the coordinate transformation $(u, v) \to (\tilde{u}, \tilde{v})$.

To obtain the Rindler metric, we introduce a new set of space and time coordinates ξ and η in the subregion $x > |t|$ of Minkowski space. As space coordinate we use ξ, which labels the hyperbolic worldlines $x^2 - t^2 = \xi^2$. As time coordinate we use the proper time $\eta = \tanh^{-1}(t/x)$ measured by a Rindler observer, synchronised such that $\eta = 0$ when the observer passes the x-axis. The relation to Cartesian coordinates is

$$x = \xi \cosh \eta \qquad \text{and} \qquad t = \xi \sinh \eta \tag{1.40}$$

and the coordinate ranges are $0 < \xi < \infty$ and $-\infty < \eta < \infty$. The Minkowski metric in (η, ξ) coordinates is

$$ds^2 = -\xi^2 d\eta^2 + d\xi^2. \tag{1.41}$$

Since Rindler space is just a subregion of Minkowski space, the Penrose diagram is just a piece of figure 7, as shown in figure 10.

1.3.4 Example 4: Kruskal space in $3+1$ dimensions

Recall the Kruskal metric (1.26):

$$ds^2 = -\frac{32M^3}{r} \exp\left(-\frac{r}{2M}\right) dU dV + r^2 d\Omega_2^2.$$

Define a new set of null coordinates via $U = \tan\tilde{U}$ and $V = \tan\tilde{V}$, such that $-\frac{\pi}{2} < \tilde{U}, \tilde{V} < \frac{\pi}{2}$. Then the line-element becomes

$$ds^2 = (2\cos\tilde{U}\cos\tilde{V})^{-2}\left[-4\frac{32M^3}{r}\exp\left(-\frac{r}{2M}\right)d\tilde{U}d\tilde{V} + r^2\cos^2\tilde{U}\cos^2\tilde{V}d\Omega_2^2\right] \quad (1.42)$$

We perform the usual conformal transformation, which leaves us with

$$\begin{aligned} d\tilde{s}^2 &= (2\cos\tilde{U}\cos\tilde{V})^2 ds^2 \\ &= -4\frac{32M^3}{r}\exp\left(-\frac{r}{2M}\right)d\tilde{U}d\tilde{V} + r^2\cos^2\tilde{U}\cos^2\tilde{V}d\Omega_2^2, \end{aligned} \quad (1.43)$$

and we add the points at infinity. The curvature singularity $UV = 1$ now corresponds to

$$\tan\tilde{U}\tan\tilde{V} = 1 \iff \sin\tilde{U}\sin\tilde{V} = \cos\tilde{U}\cos\tilde{V} \iff \cos(\tilde{U}+\tilde{V}) = 0 \quad (1.44)$$

which implies $\tilde{U}+\tilde{V} = \pm\frac{\pi}{2}$, or $\tilde{T} = \pm\frac{\pi}{4}$ if we define $\tilde{T}$ and $\tilde{X}$ through $\tilde{U} = \tilde{T} - \tilde{X}$ and $\tilde{V} = \tilde{T} + \tilde{X}$ in analogy to section 1.2. The Penrose diagram including the points at infinity and the singularity is shown in figure 11. Also shown is the curve corresponding to the surface of a collapsing star. If we only keep the region exterior to the surface, we obtain the Penrose diagram for the collapsing star on the right. The interior of the star is excluded since the stress-energy tensor is non-zero there and spacetime is not described by the Schwarzschild metric. Therefore, regions I and III are the only regions of Kruskal spacetime relevant to the description of a black hole formed through stellar collapse.

1.3.5 Example 5: de Sitter space in $3+1$ dimensions

The de Sitter metric is a solution of the Einstein equations in the presence of a positive cosmological constant $\Lambda > 0$:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \quad (1.45)$$

It corresponds to a universe with uniform positive energy density and negative pressure. To see this, note that (1.45) can be interpreted as the Einstein equations in the presence of an energy momentum tensor

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \quad (1.46)$$

with $T_{\mu\nu} = -\Lambda g_{\mu\nu}$, which implies positive energy density $T_{00} = \Lambda$ and negative uniform pressure $T_{ii} = -\Lambda$ for $i = 1, 2, 3$.

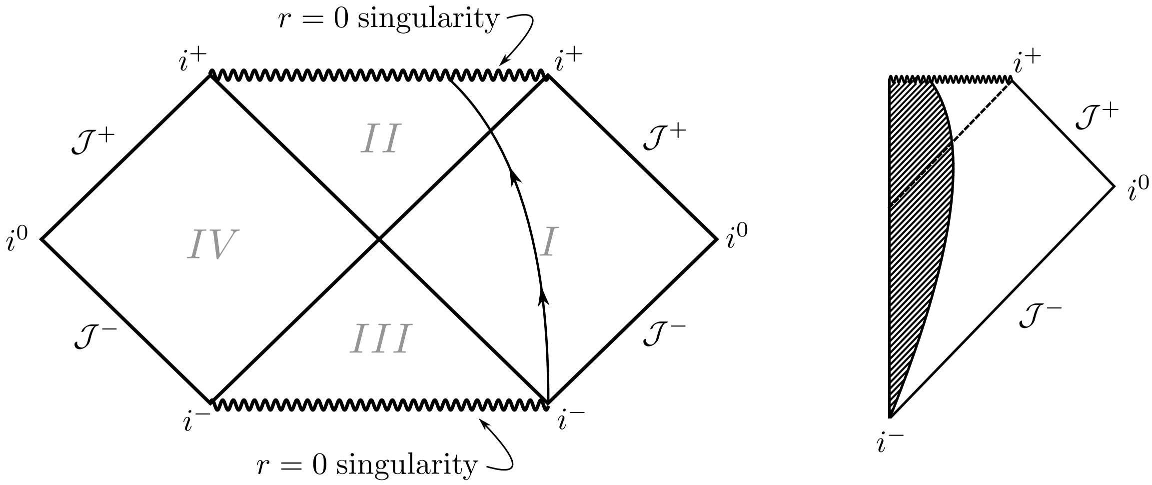

Figure 11. **Left:** The Penrose diagram for Kruskal space. The possible trajectory of the surface of a collapsing star is shown — the region to the left of the curve corresponds to the interior of the star, where spacetime is not described by the Kruskal metric. **Right:** The Penrose diagram for a collapsing star. The curved line represents the surface and the shaded region corresponds to the interior of the star. The horizon corresponds to the dashed line.

de Sitter space admits a number of different coordinate systems, some of which cover the whole space and some of which cover only part of it. The geometry of spatial sections in these coordinates can also vary (closed, open, flat). One convenient set of coordinates are closed global coordinates (t, χ, θ, ϕ) in which the line-element reads

$$ds^2 = -dt^2 + a^2 \cosh^2(t/a) d\Omega_3^2, \tag{1.47}$$

corresponding to a spacetime whose spatial sections are three-spheres with a time-dependent radius $a \cosh(t/a)$.

The best way to picture de Sitter space is as a hyperboloid embdedded in Minkowski space of one dimension higher (so $3+1$ dimensional de Sitter space is a hyperboloid in $4+1$ dimensional Minkowski space). Since our illustrations are restricted to three dimensions, let us suppress two of the angular coordinates by setting $\chi = \theta = \frac{\pi}{2}$. In that case, spatial cross sections are circles instead of three-spheres and $d\Omega_3^2 \to d\phi^2$. To see that this two-dimensional de Sitter space is a hyperboloid embedded in three-dimensional Minkowski space, consider the surface parametrised by $T = a \sinh(t/a)$, $X = a \cosh(t/a) \cos\phi$ and $Y = a \cosh(t/a) \sin\phi$, where T, X, Y are Cartesian coordinates in Minkowski space. Then it is easy to check that the embedding $2+1$ dimensional Minkowski metric $ds^2 = -dT^2 + dX^2 + dY^2$ reproduces (1.47) and furthermore that de Sitter space satisfies the equation of a hyperboloid: $-T^2 + X^2 + Y^2 = a^2$. This is shown in figure 12.

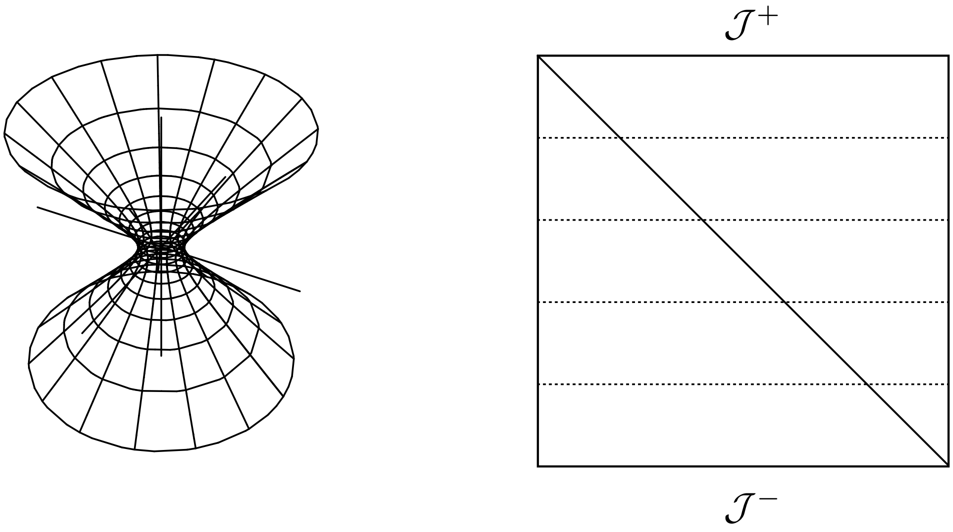

Figure 12. **Left:** The $1+1$ dimensional de Sitter hyperboloid embedded in $2+1$ dimensional Minkowski space. Some curves of constant θ and t are shown. **Right:** The Penrose diagram of de Sitter space. Dotted lines are lines of constant η. The diagonal line is the de Sitter horizon for a comoving observer at the north pole $\chi = 0$.

To construct the Penrose diagram of de Sitter space we pull out the factor of $a^2 \cosh^2(t/a)$ in the metric (1.47):

$$ds^2 = a^2 \cosh^2(t/a) \left[-d\eta^2 + d\Omega_3^2\right], \tag{1.48}$$

and define the "conformal time coordinate" η accordingly: $d\eta^2 = dt^2/(a^2 \cosh^2(t/a))$. Integrating, we obtain $\eta = \pm 2 \tan^{-1}\left(e^{t/a}\right) + c$. Choose the upper sign and fix $c = -\frac{\pi}{2}$ so that η is monotonically increasing with t and has the symmetric range $\eta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Substitution into (1.48) then yields

$$ds^2 = \frac{a^2}{\cos^2 \eta} \left[-d\eta^2 + d\Omega_3^2\right]. \tag{1.49}$$

This metric is conformal to the Einstein Static Universe, as shown in figure (to come) and the Penrose diagram is simply a square (corresponding to the half of the surface on the ESU cylinder facing you).

1.3.6 Example 6: Schwarzschild in $3+1$ dimensions with $M < 0$

In the Schwarzschild metric (1.1)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

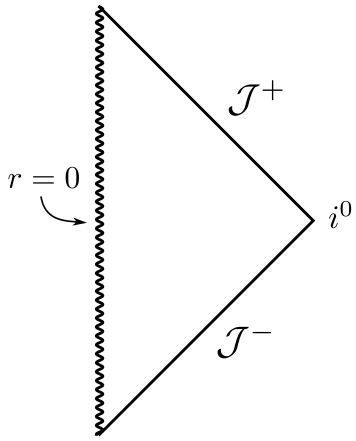

Figure 13. The Penrose diagram for the negative mass black hole. The singularity at $r = 0$ can be seen from $\mathcal{J}^+$.

there is no a priori restriction on the values of M. Let's consider the case where $M < 0$ (without going into the question what "negative mass" really means). The term $\left(1 - \frac{2M}{r}\right) = (1 + \frac{2|M|}{r})$ is then always positive and there are no singularities in the metric except the curvature singularity at $r = 0$.

We can construct the Penrose diagram in the same manner as in the positive mass case, by writing the metric in terms of $u = t - r_*$ and $v = t + r_*$ coordinates (with the restriction $r \geq 0 \iff v \geq u$) and proceeding with the conformal compactification as usual. The Penrose diagram is shown in figure 13.

Notice that this singularity is different from the black hole singularity: it can be seen from $\mathcal{J}^+$. Conversely, the singularity inside the black hole is hidden behind the horizon $r = 2M$. A singularity that can be seen from $\mathcal{J}^+$ is known as a *naked singularity*. The white hole has a naked singularity too, as it can be seen in figure 11. While naked singularities occur quite frequently in solutions to Einsteins equations, their physical status is debated. Roger Penrose has formulated the *cosmic censorship hypothesis: "Nature abhors a naked singularity"*, which encapsulates the expectation that naked singularities (except for the Big Bang) are unphysical and do not occur in the real world.

2 Charged & Rotating Black Holes

The Schwarzschild black hole is only the simplest among a number of black hole solutions to the Einstein equations. In fact, the astrophysical black holes for which we have observational evidence all appear to be rotating, while the Schwarzschild solution has zero angular momentum. In this section we review two further, more general, black hole solutions.

2.1 The Reissner-Nordström Solution

Gravity coupled to the electromagnetic field is described by the Einstein-Maxwell action

$$S = \frac{1}{16\pi G}\int \sqrt{-g}\,(R - F_{\mu\nu}F^{\mu\nu})\,d^4x \tag{2.1}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and A_μ is the electromagnetic (four-)potential. The normalisation of the Maxwell term in (2.1) is such that the Coulomb force between two charges Q_1 and Q_2 separated by a (large enough) distance r is $G|Q_1Q_2|/r^2$. This corresponds to *geometrised units of charge.*

The equations of motion derived from the variation of the Einstein-Maxwell action are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= 2\left(F_{\mu\lambda}F^{\lambda}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \\ \nabla_\mu F^{\mu\nu} &= 0. \end{aligned} \tag{2.2}$$

They admit the spherically symmetric solution

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_2^2, \tag{2.3}$$

which is known as the *Reissner-Nordström solution.* The electric potential is $A_0 = \frac{Q}{r}$ and the other components of A_μ vanish. We therefore interpret Q as the charge of the black hole (by analogy with the electric potential of a point charge) and M as its mass. Without loss of generality we will assume that $Q > 0$. By a theorem analogous to Birkhoff's theorem, the Reissner-Nordström solution is the *unique* spherically symmetric solution to the Einstein-Maxwell equations.

It is convenient to introduce the function

$$\Delta = Q^2 - 2Mr + r^2 = (r - r_+)(r - r_-) \tag{2.4}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The metric then reads

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2 d\Omega_2^2. \tag{2.5}$$

There are three separate cases to look at: $Q > M$, $Q < M$ and $Q = M$. Let's consider them in turn.

2.1.1 Super-Extremal RN: $Q > M$

If $Q > M$ then Δ has no real roots and the metric is regular for all $r > 0$. There is a curvature singularity at $r = 0$. This is the same situation as for the negative mass black hole, and the Penrose diagram looks exactly the same (figure 13).

Note that an electron has charge $e = 1.6 \times 10^{-19}$ C and mass $m_e = 9.1 \times 10^{-31}$ kg. In geometrised units this means $Q = e/\sqrt{4\pi\epsilon_0 G} = 1.4 \times 10^{-29}$ kg, so $Q \gg M$. Could it be that an electron is just a charged black hole? No. The electron is a quantum mechanical object, whose Compton wavelength $\lambda = h/mc = 2.4 \times 10^{-12}$ m is much larger than its Schwarzschild radius $r_s = 2Gm_e/c^2 = 1.4 \times 10^{-57}$m.

2.1.2 Sub-Extremal RN: $Q < M$

Now Δ has two real roots $r_+ > r_-$ and there are two coordinate singularities. As always, we can remove them if we find a suitable coordinate system. Recalling our strategy with the Schwarzschild metric, let us define a tortoise coordinate r_*via

$$\frac{\Delta}{r^2} dr_*^2 = \frac{r^2}{\Delta} dr^2, \tag{2.6}$$

in terms of which the metric takes the form

$$ds^2 = -\frac{\Delta}{r^2}(dt^2 - dr_*^2) + r^2 d\Omega_2^2. \tag{2.7}$$

Radial null geodesics are then given by the simple equation $t \pm r_* = const.$ (and $\theta = \phi = const.$). A solution of (2.6) with a convenient choice of sign and integration constant is

$$r_* = r + \frac{1}{2\kappa_+} \ln\left(\frac{r - r_+}{r}\right) + \frac{1}{2\kappa_-} \ln\left(\frac{r - r_-}{r}\right), \tag{2.8}$$

where

$$\kappa_+ = \frac{r_+ - r_-}{2r_+^2} > 0 \qquad \text{and} \qquad \kappa_- = \frac{r_- - r_+}{2r_-^2} < 0. \tag{2.9}$$

Define the null coordinates $u = t - r_*$ and $v = t + r_*$ and ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ). In terms of the latter, the metric becomes

$$ds^2 = -\frac{\Delta}{r^2} dr^2 + dr dv + r^2 d\Omega_2^2, \tag{2.10}$$

which is regular for all $r > 0$, including $r = r_+$ and $r = r_-$. To understand the spacetime structure close to $r = r_\pm$ we can use two different sets of Kruskal-type coordinates at each of the two radii:

$$U^\pm = -\exp\left(-\kappa_\pm u\right) \qquad \text{and} \qquad V^\pm = \exp\left(\kappa_\pm u\right). \tag{2.11}$$

This gives rise to the Penrose diagram shown in figure 14. Notice that a timelike trajectory can avoid $r = 0$, since the $r = 0$ singularity is timelike itself. In fact, to hit $r = 0$, one must accelerate toward it (this time it *is* like a position in space).

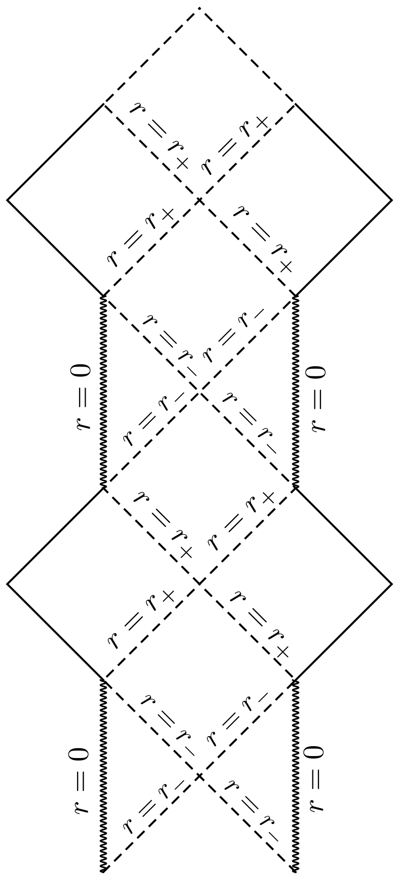

Figure 14. The Penrose diagram for the sub-extremal Reissner-Nordström solution.

2.1.3 Extremal RN: $Q = M$

The metric of the extremal Reissner-Nordström solution is

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2, \tag{2.12}$$

which has one coordinate singularity at $r = r_+ = r_- = M$. To get rid of it, define the tortoise coordinate $dr_* = (1 - \frac{M}{r})^{-2} dr$ such that

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 (dt^2 - dr_*^2) + r^2 d\Omega_2^2, \tag{2.13}$$

and change to ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ), where $v = t + r_*$ labels ingoing null geodesics, which should be clear from a look at (2.13). This leaves us with the improved line-element

$$ds^2 = -\left(1 - \frac{M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega_2^2 \tag{2.14}$$

which is regular at $r = M$. The inner and outer horizons have now coalesced. The result is the Penrose diagram shown in figure (to come).

2.2 Rotating Black Holes

So far we have only discussed solutions with spherical symmetry. We now introduce the *Kerr-Newman solution* (1965) to the Einstein-Maxwell equations, which describes a rotating charged black hole of mass M, charge Q and angular momentum $J \equiv Ma$ (so a is the angular momentum per unit mass). In Boyer-Lindquist coordinates (t, r, θ, ϕ), in which the black hole rotates about the polar axis, the metric part of the solution is given by

$$\begin{aligned} ds^2 = &- \left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma} \right) dt^2 + \frac{\Sigma}{\Delta} dr^2 - 2\frac{a \sin^2\theta}{\Sigma} \left(r^2 + a^2 - \Delta \right) dt d\phi \\ &+ \Sigma d\theta^2 + \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \right) \sin^2\theta d\phi^2 \end{aligned} \tag{2.15}$$

where where $\Sigma = r^2 + a^2 \cos^2\theta$ and $\Delta = r^2 - 2Mr + Q^2 + a^2$. The components of the electromagnetic potential are

$$A_t = \frac{Qr}{\Sigma}, \qquad A_\phi = -\frac{Qar\sin\theta}{\Sigma}, \qquad A_r = A_\theta = 0. \tag{2.16}$$

For $a = Q = 0$, we recover the Schwarzschild solution (1.1). For $a = 0$, we recover the Reissner-Nordström solution (2.3). Finally, the solution is symmetric under the simultaneous replacements $\phi \to -\phi$ and $a \to -a$, so we can set $a \geq 0$ without loss of generality.

When a black hole is rotating, there is no analogue of Birkhoff's theorem. This means that, during gravitational collapse with rotating matter, we cannot use the same reasoning as in the spherically symmetric case to argue that, on the surface of the collapsing matter, the metric should be of the form given above. All we can say is that, after enough time has passed and matter and spacetime have "settled down" to equilibrium, they will be described by the Kerr-Newman solution.

We will investigate the structure of the simple but illustrative special case of a rotating black hole with zero charge $Q = 0$. The metric (2.15) then reduces to the *Kerr solution* (1963):

$$ds^2 = \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + (r^2 + a^2) \sin^2\theta d\phi^2 + \frac{2Mr}{\Sigma} \left(a \sin^2\theta d\phi - dt \right)^2 - dt^2 \tag{2.17}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2 \cos^2\theta$. This metric is a solution to the vacuum Einstein equations. It has coordinate singularities at

$$\Delta = 0 \iff r = r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{2.18}$$

Below we will show a coordinate transformation that removes them. It also has a curvature singularity at

$$\Sigma = 0 \iff r = 0 \text{ and } \cos\theta = 0. \tag{2.19}$$

The latter condition implies that the curvature singularity is only there when $\theta = \frac{\pi}{2}$, i.e. when $r = 0$ is approached along the equator. When approached from any other angle, there is no singularity at $r = 0$.

There are again three cases to consider: $M < a$, $M = a$ and $M > a$. We will concentrate on the $M > a$ solution, for which there are two coordinate singularities at r_+ (the "outer" horizon) and $r_- < r_+$ (the "inner" horizon). To remove them, we do a coordinate transformation to ingoing Kerr coordinates (v, r, θ, χ), where $v = t + r_*$ and r_* and χ are defined by

$$dr_* = \frac{r^2 + a^2}{\Delta} dr \qquad \text{and} \qquad d\chi = d\phi + \frac{a}{\Delta} dr. \tag{2.20}$$

The definition of χ implies that $\phi = const.$ does not correspond to $\chi = const.$ For example, in order to stay at $\chi = const.$ as you fall in ($dr < 0$), you need to rotate too: $d\phi = -a/\Delta dr$. In terms of ingoing Kerr coordinates the metric becomes

$$\begin{aligned} ds^2 = &- \left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma} \right) dv^2 + 2dvdr - 2\frac{a \sin^2\theta}{\Sigma}(r^2 + a^2 - \Delta) dv d\chi \\ &- 2a \sin^2\theta d\chi dr + \left[\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \right] \sin^2\theta d\chi^2 + \Sigma d\theta^2. \end{aligned} \tag{2.21}$$

There are no more factors of Δ in the numerators and the metric is regular at r_+ and r_-. The only remaining singularity is the curvature singularity at $\Sigma = 0$.

To draw the Penrose diagram is more difficult because the metric is not spherically symmetric. Since the curvature singularity at $r = 0$ only appears when $\theta = \frac{\pi}{2}$, the Penrose diagram should look very different for $\theta \neq \frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$. In order to represent both cases, it is customary to draw a Penrose diagram that is an amalgam of the Penrose diagram for an observer falling in from the north pole ($\theta = 0$) and of that for an observer falling in in the equatorial plane ($\theta = \frac{\pi}{2}$) at fixed χ. Notice that $\chi = const.$ means that ϕ is not constant, so the observer falling in at $\theta = \frac{\pi}{2}$ rotates about the polar axis.

The procedure is very similar to that for the sub-extremal Reissner-Nordström solution in section 2.1.2. First, perform a coordinate transformation to coordinates (u, v, θ, ϕ) where $u = t + r_*$ and $v = t - r_*$ with r_* as defined in (2.20). Then, define Kruskal-type coordinates $U^\pm$ and $V^\pm$ close to $r = r_\pm$, respectively, and draw the Penrose diagram. This leads to the infinitely sequence of spacetime regions we saw in figure 14. Up to this point, the analysis is identical for $\theta = 0$ and $\theta = \frac{\pi}{2}$. The only difference is that the Penrose diagram for $\theta = 0$ has a curvature singularity at $r = 0$, whereas the Penrose diagram for $\theta = \frac{\pi}{2}$ has none. In the amalgam Penrose diagram for the Kerr spacetime, we indicate this by drawing an interrupted wavy

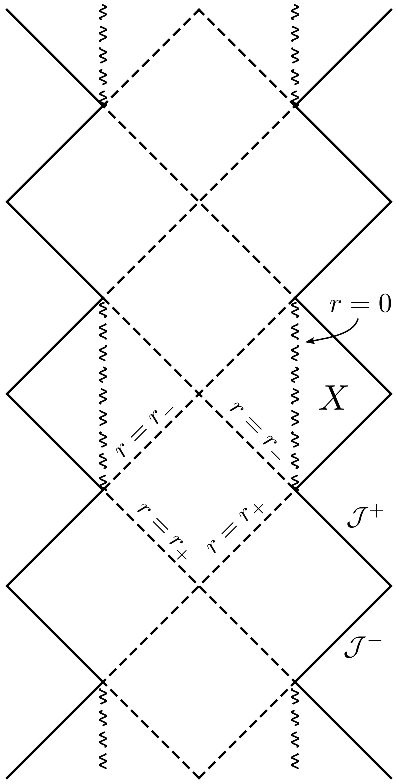

Figure 15. The Penrose diagram for the sub-extremal Kerr black hole.

line at $r = 0$. The result is shown in figure 15.

2.2.1 Ring singularities

How can it be that there is a singularity for $\theta = \frac{\pi}{2}$ but not otherwise? The simple explanation is that the singularity has the shape (topology) of a ring. Indeed, for fixed r, v and θ, the metric (2.17) becomes

$$ds^2 = \frac{(r^2 + a^2)^2 - \sin^2\theta(r^2 - 2Mr + a^2)a^2}{a^2\cos^2\theta}\sin^2\theta d\chi^2, \tag{2.22}$$

which tends to $ds^2 = a^2 \sin^2\theta d\chi^2$ as $r \to 0$. When $\theta = \frac{\pi}{2}$, we obtain

$$ds^2 = a^2 d\chi^2, \tag{2.23}$$

the metric of a ring with radius a. If you travel toward $r = 0$ from any other angle than $\theta = \frac{\pi}{2}$, you will not encounter the singularity. Instead, you will fall through the interior of the "ring" and emerge in a *new* region of spacetime.

2.2.2 Closed Timelike Curves

A *closed timelike curve (CTC)*, also called a time machine, is a curve that is everywhere timelike and that eventually returns to where it started from *in spacetime*. *CTC*s have been extensively explored in the context of general relativity. In fact, they are ubiquitous and appear in a number of spacetimes that are solutions to the Einstein equations.

Kerr spacetime is one such example: region X in figure 15 contains CTCs. To see this, consider a curve in region X at fixed t, $\theta = \frac{\pi}{2}$ and $r < 0$. Then

$$ds^2 = \left(r^2 + a^2 + \frac{2Ma}{r}\right) d\chi^2. \tag{2.24}$$

Close enough to the singularity, where r is small and negative, $r < 0$ and $|r| < 2Ma/(r^2 + a^2)$, (2.24) is negative and the curve is timelike. Since χ is a periodic coordinate with $\chi \equiv \chi + 2\pi$, the curve is also closed: it is a *CTC*.

However, it turns out that region X is unphysical. Much like in the case of the sub-extremal RN solution of section 2.1.2, the inner horizon at $r = r_-$ becomes a curvature singularity in the presence of the smallest perturbations to the Kerr metric: at the inner horizon, perturbations are infinitely blueshifted, which leads to divergences in the curvature scalars.

3 Killing Vectors & Killing Horizons

So far we have produced a number of different black hole solutions and looked at some of their features. In this section we introduce the concepts and machinery that we will need to really understand the structure of black hole spacetimes.

Notation: to denote partial derivatives we use shorthand notations such as $\partial_t = \frac{\partial}{\partial t}$ *and* $\partial_\mu = \frac{\partial}{\partial x^\mu}$ *as well as* $X_{,\mu} = \partial_\mu X$ *when applied to a tensor* X.

3.1 Symmetries & Killing Vectors

Let (M, g) be a Lorentzian manifold. Given a smooth vector field ξ on M, an *integral curve* of ξ is a curve $\gamma : \mathbb{R} \to M$ whose tangent vector is equal to ξ at every point $p \in \gamma$. This can be expressed as the demand that

$$\xi_p(f) = \frac{d}{d\lambda}(f \circ \gamma(\lambda))\bigg|_p \tag{3.1}$$

for all smooth functions $f : M \to \mathbb{R}$. Equivalently, given a coordinate system x^μ on M, the components of ξ in that coordinate system must satisfy

$$\xi^\mu_p = \frac{d}{d\lambda} x^\mu(\gamma(\lambda))\bigg|_p , \tag{3.2}$$

where λ parametrises γ.

When ξ is smooth and everywhere non-zero, the set of integral curves form a *congruence*: every point $p \in M$ lies on a unique integral curve. Given any congruence there is an associated one-parameter family of diffeomorphisms from M onto itself, defined as follows: for each $s \in \mathbb{R}$, define $h_s : M \to M$, where $h_s(p)$ is the point parameter distance s from p along ξ, i.e. if $p = \gamma(\lambda_0)$ then $h_s(p) = \gamma(\lambda_0 + s)$. These transformations form an abelian group: the composition law is $h_s \circ h_t = h_{s+t}$, the identity is h_0 and the inverse is $(h_s)^{-1} = h_{-s}$.

This leads to the concept of the *Lie derivative* $\mathcal{L}_\xi$ along the vector field ξ. When applied to a vector V at point p it is defined as

$$(\mathcal{L}_\xi V)_p = \lim_{\delta\lambda \to 0} \frac{V_p - (h_{\delta\lambda})_* V_{h_{-\delta\lambda}(p)}}{\delta\lambda}. \tag{3.3}$$

Here $(h_s)_*$ denotes the *push-forward* associated with the group element h_s, which maps a vector defined at p to a vector defined at $h_s(p)$. It can be shown (exercise) that the Lie derivative of a vector is equal to the bracket

$$(\mathcal{L}_\xi V)_p = [\xi, V]_p$$

where $[X, Y]^\mu = X^\nu Y^\mu_\nu - Y^\nu X^\mu_\nu$.

The Lie derivative can be applied to any tensor on M, with appropriate definitions analogous to (3.3). In particular, given a metric tensor g on M, we can take its Lie derivative. Since the components of g transform covariantly (whereas vector fields transform contravariantly), the Lie derivative of g involves the *pull-back* h^*_s, which maps a covector at $h_s(p)$ to a covector at p:

$$(\mathcal{L}_\xi g)_p = \lim_{\delta\lambda \to 0} \frac{g_p - (h_{\delta\lambda})^* g_{h_{\delta\lambda}(p)}}{\delta\lambda}. \tag{3.4}$$

It can then be shown that

$$(\mathcal{L}_\xi g)_{\mu\nu} = \nabla_\mu \xi_\nu - \nabla_\mu \xi_\nu. \tag{3.5}$$

If the metric does not change under the transformation h_s we say that the transformation is an *isometry* and that the metric has a *symmetry*. In this case $\mathcal{L}_\xi g = 0$, which implies

$$\nabla_\mu \xi_\nu + \nabla_\mu \xi_\nu = 0. \tag{3.6}$$

This is known as *Killing's equation* and a vector ξ that satisfies (3.6) is a *Killing vector*. Note that this equation implicitly involves the metric, which is hidden in ∇. Finding the symmetries of a spacetime amounts to finding the vectors that satisfy the Killing equation; this can be done either by inspection or by integrating (3.6).

A useful fact to know when looking for Killing vectors is the following. Given any vector field ξ we can (locally) find a coordinate system (x^1, x^2, x^3, x^4) in which ξ takes the form

$$\xi = \frac{\partial}{\partial x^1}, \tag{3.7}$$

i.e. $\xi^\mu = (1, 0, 0, 0)$. This implies that

$$(\mathcal{L}_\xi g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^1}. \tag{3.8}$$

Hence, if we find a coordinate system in which $g_{\mu\nu}$ is independent of one of the coordinates, say y, then we know that $\frac{\partial}{\partial y}$ must be a Killing vector since

$$(\mathcal{L}_{\frac{\partial}{\partial y}} g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial y} = 0. \tag{3.9}$$

The converse statement is not true: if $g_{\mu\nu}$ depends on all the coordinates, that does not mean that that g has no Killing vectors.

3.1.1 Example 1: Schwarzschild

Using the fact above, a quick look at the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

reveals the two Killing vectors $k = \partial_t$ and $m = \partial_\phi$. This means that Schwarzschild spacetime is *static* and *axisymmetric*:

Definition. An asymptotically flat spacetime is *stationary* if it admits a Killing vector k such that $k^2 \to -1$ asymptotically. If in addition the metric is invariant under $t \leftrightarrow -t$, the spacetime is *static*.

Definition. An asymptotically flat spacetime is *axisymmetric* if it admits a Killing vector ℓ that is spacelike asymptotically and whose integral curves are closed.

Schwarzschild spacetime is not just axisymmetric but spherically symmetric: it has two more spacelike Killing vectors, which together with ∂_ϕ generate the symmetries of a sphere — the $SO(3)$ transformations. The other two Killing vectors cannot be "read off" from the metric but they can be found by solving Killing's equation.

3.1.2 Example 2: Kerr-Newman

The Kerr-Newman solution (2.15)

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 - 2\frac{a\sin^2\theta}{\Sigma}\left(r^2+a^2-\Delta\right)dtd\phi$$
$$+\Sigma d\theta^2 + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2$$

is axisymmetric and stationary: since $g_{\mu\nu}$ is independent of t and ϕ, it has Killing vectors $k = \partial_t$ and $\ell = \partial_\phi$. However, the metric is not invariant under $t \leftrightarrow -t$ due to the $dtd\phi$ term, so it is not static.

A number of *uniqueness theorems*, proved between 1967 and 1975, have established the remarkable fact that the two-parameter Kerr family (parameters M and J) is the unique stationary asymptotically flat solution of the vacuum Einstein equations and that the three-parameter Kerr-Newman family (parameters M, J and Q) is the unique stationary asymptotically flat black hole solution of the electrovacuum Einstein-Maxwell equations.

3.2 Conservation Laws

In classical physics, the presence of symmetries is very closely tied to the existence of conservation laws. As we shall see in this section, this is also the case in general relativity. We first take a look at the geodesic motion of test particles.

Consider the action of a particle in a spacetime (M, g) moving on a curve γ with parameter λ and endpoints A and B. Pick a coordinate system x^μ and denote the coordinates of the curve by $x^\mu(\lambda)$. Then the action for γ is given by

$$I\left(x^\mu\right) = m\int d\tau = m\int_{\lambda_A}^{\lambda_B}\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}. \tag{3.10}$$

If we deform the curve by a small amount $\delta x^\mu(\lambda)$ and require the action to be stationary with respect to this variation, we obtain the Euler-Lagrange equations:

$$\frac{\delta I}{\delta x^\mu} = 0 \implies \frac{d}{d\lambda}\dot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma \equiv \nabla_{(\lambda)}\dot{x}^\mu \propto \dot{x}^\mu. \tag{3.11}$$

The dot denotes differentiation with respect to proper time, $\frac{d}{d\tau}$. The solutions to (3.11) are the geodesics in (M, g). If the right hand side vanishes, $\nabla_{(\lambda)}\dot{x}^\mu = 0$, the geodesic is said to be *affinely parametrised* and λ is an *affine parameter*. If we set $\lambda = \tau$ then (3.11) reduces to

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0. \tag{3.12}$$

For the purpose of finding geodesics we can cast (3.10) into a more useful form by introducing an independent function $e(\lambda)$ (called the *auxiliary field* or *einbein*):

$$I\left(x^{\mu}, e\right)=\frac{1}{2} \int_{\lambda_{A}}^{\lambda_{B}} d \lambda\left[e^{-1}(\lambda) g_{\mu \nu} \dot{x}^{\mu} \dot{x}^{\nu}-m^{2} e(\lambda)\right] . \tag{3.13}$$

To see that this is equivalent to (3.10), notice that from $\delta I / \delta e=0$ we get

$$-e^{-2} g_{\mu \nu} \dot{x}^{\mu} \dot{x}^{\nu}-m^{2}=0 \implies e=\frac{1}{m} \sqrt{-g_{\mu \nu} \dot{x}^{\mu} \dot{x}^{\nu}}=\frac{1}{m} \frac{d \tau}{d \lambda} \tag{3.14}$$

and from $\delta I / \delta x^{\mu}$ we get

$$\frac{d}{d \lambda} \dot{x}^{\mu}+\Gamma_{\rho \sigma}^{\mu} \dot{x}^{\rho} \dot{x}^{\sigma}=\left(e^{-1} \dot{e}\right) \dot{x}^{\mu}. \tag{3.15}$$

This equation is already the geodesic equation $\nabla_{(\lambda)} \dot{x}^{\mu}=\left(e^{-1} \dot{e}\right) \dot{x}^{\mu}$ and (3.14) relates e to the choice of parameter λ. To turn this into the equation for an affinely parametrised geodesic, we need to set $\dot{e}=0$. Then $d \tau / d \lambda=$ *const.*, which implies that $\lambda=a \tau+b$.

Consider now an infinitesimal translation of the curve γ along a Killing vector field k (leaving e unchanged). In the coordinate chart x^{μ} this corresponds to

$$x^{\mu} \rightarrow x^{\mu}+\alpha k^{\mu} \tag{3.16}$$

where α is an infinitesimal constant. Then the action will be changed by an amount

$$\begin{aligned}
\delta I & =I\left(x^{\mu}+\alpha k^{\mu}, e\right)-I\left(x^{\mu}, e\right) \\
& =\frac{\alpha}{2} \int d \lambda\left[e^{-1}\left(g_{\mu \nu} \dot{k}^{\mu} \dot{x}^{\nu}+g_{\mu \nu} \dot{x}^{\mu} \dot{k}^{\nu}+g_{\mu \nu, \sigma} \dot{x}^{\mu} \dot{x}^{\nu} k^{\sigma}\right)\right] \\
& =\frac{\alpha}{2} \int d \lambda\left[e^{-1}\left(g_{\mu \nu} \dot{x}^{\sigma} \dot{x}^{\nu} k^{\mu}{}_{, \sigma}+g_{\mu \nu} \dot{x}^{\mu} \dot{x}^{\sigma} k^{\nu}{}_{, \sigma}+g_{\mu \nu, \sigma} \dot{x}^{\mu} \dot{x}^{\nu} k^{\sigma}\right)\right] \\
& =\frac{\alpha}{2} \int d \lambda\left[e^{-1} \dot{x}^{\mu} \dot{x}^{\nu}\left(g_{\sigma \nu} k^{\sigma}{}_{, \mu}+g_{\mu \sigma} k^{\sigma}{}_{, \nu}+g_{\mu \nu, \sigma} k^{\sigma}\right)\right] \\
& =\frac{\alpha}{2} \int d \lambda\left[e^{-1} \dot{x}^{\mu} \dot{x}^{\nu}\left(\nabla_{\mu} k_{\nu}+\nabla_{\nu} k_{\mu}\right)\right] \\
& =0
\end{aligned} \tag{3.17}$$

where the last line follows from the fact that k satisfies Killing's equation. To get from the second to the third line we used the fact that $\dot{k}^{\mu}=\frac{d}{d \tau} k^{\mu}=\frac{d x^{\nu}}{d \tau} \partial_{\nu} k^{\mu}=\dot{x}^{\nu} k^{\mu}{}_{, \nu}$. We see that k being Killing leads to a symmetry of the particle action. Associated with this symmetry is a quantity (charge) that is conserved along geodesics:

Claim. Let k be a Killing vector. Then

$$Q=k^{\mu} p_{\mu} \tag{3.18}$$

is conserved along geodesics, where p_μ is the momentum of the particle defined by

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = e^{-1}\dot{g}_{\mu\nu}x^\nu = mg_{\mu\nu}\frac{dx^\nu}{d\tau}. \tag{3.19}$$

The last equality follows from equation (3.14) for e.

Proof. Consider a small variation $\delta x^\mu = \alpha k^\mu$ generated by the Killing vector k. Set $\lambda = \tau$ for simplicity (the proof for general λ is similar). Then

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\delta \dot{x}^\mu = 0 \tag{3.20}$$

for a geodesic. Using the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \dot{p}_\mu, \tag{3.21}$$

the previous equation can be written as

$$0 = \dot{p}_\mu \alpha k^\mu + p_\mu \alpha \dot{k}^\mu = \alpha\frac{d}{d\tau}(k^\mu p_\mu) = \alpha \frac{d}{d\tau}Q. \tag{3.22}$$

Therefore Q is constant along a geodesic.

3.2.1 Example 1: Schwarzschild

Recall the Killing vector $k = \partial_t$ in Schwarzschild spacetime. Then $k^\mu = (1, 0, 0, 0)$ and we have

$$Q = k^\mu p_\mu = k^\mu m g_{\mu\nu}\dot{x}^\nu = mg_{00}\dot{x}^0 = -m\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \equiv -E, \tag{3.23}$$

where we have identified Q with minus the energy of the particle in the rest frame of the black hole. This agrees with the the definition of the constant $\epsilon = E/m$ in equation (1.9) of section 1.

The conserved quantity associated with $\ell = \partial_\phi$ is

$$Q = mg_{33}\dot{x}^3 = mr^2 \sin^2\theta\dot{\phi} \equiv J, \tag{3.24}$$

the angular momentum of the particle along the axis of symmetry as measured by an observer at rest at infinity.

3.2.2 Example 2: Kerr

The vectors k and ℓ are also Killing vectors for the Kerr metric (2.15,2.17):

$$ds^2 = \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{2Mr}{\Sigma}\left(a\sin^2\theta d\phi - dt\right)^2 - dt^2.$$

In Schwarzschild spacetime, you (or a test particle) can travel along integral curves of $k = \partial_t$ anywhere outside the horizon — you will then appear stationary[5] to observers at infinity, since your position in space is not changing. This is possible because g_{00} is negative everywhere for $r > 2M$, so that $k^2 = g_{00}$ is negative and the integral curves of k are timelike. It turns out that this is not the case in Kerr spacetime: there is a region around the outer horizon, called the *ergosphere* or *ergoregion*, in which it is impossible for you (and any test particle) to remain stationary with respect to observers at infinity — everything rotates. This happens because

$$g_{00} = -\left(1 - \frac{2Mr}{\Sigma}\right) \tag{3.25}$$

becomes positive in the region

$$\frac{2Mr}{\Sigma} > 1 \implies \xi(r) \equiv \Sigma - 2Mr = r^2 + a^2\cos^2\theta - 2Mr < 0, \tag{3.26}$$

part of which lies *outside* the outer horizon $r = r_+$ when $a \neq 0$. This is easy to see by noting that the equation for $\xi(r)$ is a parabola with roots at $\tilde{r}_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta}$ and $\tilde{r}_+$ is bigger than $r_+ = M + \sqrt{M^2 - a^2}$ for $\theta \neq 0, \pi$. Hence g_{00} is positive in the ellipsoidal region $r_+ < r < \tilde{r}_+$, which has a maximum extent on the equator $\theta = \frac{\pi}{2}$ where $\tilde{r}_+ = M + \sqrt{M^2 + a^2}$.

In the ergoregion, orbits of ∂_t are not timelike, so you cannot travel along them and remain stationary with respect to observers at infinity. In order for a curve $x^\mu = (t, r, \theta, \phi)$ to be timelike, its tangent vector $u^\mu = dx^\mu/d\tau$ must satisfy $u^2 = -1$. But in the ergoregion, *every* term in $u^2 = g_{\mu\nu}u^\mu u^\nu$ is positive except for $g_{t\phi}u^t u^\phi$, which means that $u^\phi = d\phi/d\tau$ must be non-zero. Moreover, since $u^t > 0$ for a future-directed worldline and $g_{t\phi} < 0$, u^ϕ must be positive. Any timelike worldline is therefore dragged around in the direction of rotation of the black hole. This effect is an example of *frame dragging.*

3.2.3 Example 3: The Penrose Process

The Penrose process is a process that allows you to extract energy from a rotating black hole. Imagine sending a particle into the ergioregion. Prepare it carefully such

[5]Note that the use of the word "stationary" here has nothing to do with the definition of a stationary spacetime. When used to describe the relative motion of observers or test particles, stationary just means "not moving in space".

that, once in the ergoregion, it decays into two particles, one of which falls into the black hole and one of which escapes the ergoregion again. Denote the energy of the initial particle by $E = -k_\mu p^\mu$ and that of the final particles by $E_1 = -k_\mu p_1^\mu$ and $E_2 = -k_\mu p_2^\mu$, where k is the asymptotically timelike Killing vector. Conservation of four-momentum $p^\mu = p_1^\mu + p_2^\mu$ implies that $E = E_1 + E_2$. The fact that k is spacelike in the ergoregion allows you to arrange the decay such that the energy of the particle that falls into the black hole is negative with respect to you: $E_1 = k_\mu p_1^\mu < 0$. To see this, choose a coordinate system in which $k^\mu = (0, x, 0, 0)$. Prepare your decay such that $p_1^\mu = (1, y, 0, 0)$, where y is small enough for p_1 to be timelike and adjusted such that $xy > 0$. Then E_1 is negative and $E_2 > E$: the particle that reemerges from the ergoregion has more energy than the particle you sent in.

3.3 Hypersurfaces

The horizon is a central aspect of black holes. So far we have audio.

Let $S(x)$ be a smooth function of the spacetime coordinates. Consider the family of hypersurfaces $S(x) = const.$ The normal vector to $S(x) = const.$ is given by

$$n^\mu = f(x) g^{\mu\nu} \partial_\nu S, \tag{3.27}$$

where $f(x)$ is an arbitrary (smooth) normalisation function. A hypersurface is spacelike (timelike) when n is timelike (spacelike) everywhere on it. When the hypersurface is timelike (spacelike) we can always find an $f(x)$ such that $n^2 = -1$ ($n^2 = +1$). A hypersurface is null when n is null, $n^2 = 0$.

3.3.1 The induced metric

Consider a spacelike (timelike) hypersurface Σ in a spacetime (M, g). Normalise the normal n to Σ such that $n^2 = -1$ ($n^2 = +1$). Then the *induced metric* h on Σ is given by

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{3.28}$$

The metric h is the restriction of g onto Σ: while h is degenerate on the full tangent space at any point $p \in \Sigma$, it is positive definite for spacelike Σ (Lorentzian for timelike Σ) on the subspace of the tangent space spanned by vectors tangent to curves in Σ (i.e. vectors v that satisfy $v.n = 0$).

To give an example, consider first a spatial slice Σ defined by $S(x) = t = const.$ in 3+1 dimensional Minkowski space in Cartesian coordinates $(t, \mathbf{x})$. Then $n^\mu = (1, 0, 0, 0)$ and

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu = \mathrm{diag}(-1, 1, 1, 1) - \mathrm{diag}(1, 0, 0, 0) = \mathrm{diag}(0, 1, 1, 1). \tag{3.29}$$

On the spatial slice Σ, h is just the positive definite three-dimensional flat Euclidean metric. Next, consider a timelike hypersurface Σ defined by $S(x) = x^1 = const.$ with normal $n^\mu = (0, 1, 0, 0)$. The induced metric on it is $h_{\mu\nu} = \mathrm{diag}(-1, 0, 1, 1)$, which carries Lorentzian signature on the subspace of the tangent space spanned by vectors tangent to Σ.

3.3.2 Null hypersurfaces

For example, the surface $r = 2M$ in Schwarzschild spacetime is null. MB1: finish. And Minkowski examples. The normal vectors to null hypersurfaces have a peculiar property: they are also tangent to the hypersurface. Indeed if $n^\mu = g^{\mu\nu}\partial_\nu \Sigma$ is null then?

Claim: The integral curves of n^μ in Σ are null geodesics.

Proof: For $\ell^\mu = f g^{\mu\nu}\partial_\nu S$ we have

$$\begin{aligned} \ell.\nabla l^\nu &= \ell^\mu \nabla_\mu (f g^{\nu\rho}\partial_\rho S) \\ &= \ell^\mu (\nabla_\mu f) g^{\nu\rho}\partial_\rho S + f \ell^\mu g^{\nu\rho}\nabla_\mu \partial_\rho S \\ &= (\ell.\nabla f) f^{-1}\ell^\nu + f\ell^\mu g^{\nu\rho}\nabla_\mu\partial_\rho S. \end{aligned} \tag{3.30}$$

The second term vanishes:

$$\begin{aligned} \ell^\mu f g^{\nu\rho}\nabla_\mu\partial_\rho S &= f\ell^\mu g^{\nu\rho}\nabla_\rho\partial_\mu S \\ &= f\ell^\mu g^{\nu\rho}\nabla_\rho (f^{-1}\ell_\mu) \\ &= f g^{\nu\rho}(\nabla_\rho f^{-1})\ell^2 + \ell^\mu g^{\nu\rho}\nabla_\rho \ell_\mu \\ &= \frac{1}{2} g^{\nu\rho}\nabla_\rho(\ell^2) \end{aligned} \tag{3.31}$$

using the fact that $\ell^2 = 0$ on Σ in the third line. In the last line, note that $\nabla_\rho(\ell^2)$ is *not* evidently zero, because ℓ^2 can be non-zero outside of Σ and so its derivative can be non-zero. However, since ℓ^2 vanishes everywhere on Σ, any non-zero contribution to $\nabla_\rho(\ell^2)$ must be proportional to the normal to Σ: $\nabla_\rho(\ell^2) \propto \ell_\rho$. Together with (3.30) this implies that $\ell.\nabla l^\nu \propto \ell^\nu$ and therefore the integral curves of ℓ^μ are geodesics. □

If the integral curves of ℓ^μ are not affinely parametrised, then we can always find some function $h(x)$ such that $\tilde{\ell}^\mu = h(x)\ell^\mu$ has affinely parametrised integral curves, i.e. $\tilde{\ell}.\nabla\tilde{\ell}^\mu = 0$. These curves are called the *null geodesic generators* of the null hypersurface.

3.4 Killing Horizons

A null hypersurface Σ is a *Killing horizon* of a Killing vector ξ if ξ is normal to Σ on Σ. Let ℓ be normal to Σ and affinely parametrised such that $\ell.\nabla\ell = 0$. Then, for some function f, $\xi = f\ell$ on Σ. It follows that ξ satisfies

$$\begin{aligned}
\xi^\mu \nabla_\mu \xi^\nu &= f\ell^\mu \nabla_\mu (f\ell^\nu) \\
&= f\ell^\mu \ell^\nu \nabla_\mu f + f^2 \ell^\mu \nabla_\mu \ell^\nu \\
&= f\ell^\nu \ell^\mu \nabla_\mu f \\
&\equiv \kappa \xi^\nu
\end{aligned} \tag{3.32}$$

on Σ. In the last line we have defined κ, the *surface gravity*: $\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu$. It takes its name from the fact that κ is constant over the horizon and equals the force that an observer at infinity would have to exert in order to keep a unit mass at the horizon.

3.4.1 Example 1: Schwarzschild

In Schwarzschild spacetime, the surfaces of constant r are the "cylinders"of constant $S(r) = r$ and the black hole horizon corresponds to the particular surface $S(r) = 2M$. To find the normal vector $\ell^\mu = f g^{\mu\nu} \partial_\nu S$ to the horizon, let us work in IEF coordinates. We need the inverse of the IEF metric (1.22), which is given by

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \left(1 - \frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2 \sin^2\theta \end{pmatrix}. \tag{3.33}$$

The normal vector to the surface $S(r) = 2M$ is then

$$\ell^\mu = f(g^{vr}\partial_r S, g^{rr}\partial_r S, 0, 0) = f\left(1, \left(1 - \tfrac{2M}{r}\right), 0, 0\right) = f(1,0,0,0) = f\partial_v. \tag{3.34}$$

This vector is null at $r = 2M$ since $\ell^2 = f^2 g^{rr} = f^2\left(1 - \frac{2M}{r}\right) = 0$ at $r = 2M$. Hence $r = 2M$ is a null hypersurface. $\ell = f\partial_v$ is a Killing vector (since $g_{\mu\nu}$ is independent of v) and it is timelike everywhere outside the horizon.

Remember that we already encountered a timelike Killing vector in Schwarzschild spacetime: ∂_t in Schwarzschild coordinates. In fact ∂_t and ∂_v are the same vector field. To check this, denote Schwarzschild coordinates by (t, r, θ, ϕ) and IEF coordinates by $(v, \tilde{r}, \tilde{\theta}, \tilde{\phi})$. Then

$$\begin{aligned}
dv &= dt + \left(1 - \tfrac{2M}{r}\right)^{-1} dr \\
d\tilde{r} &= dr \\
d\tilde{\theta} &= d\theta \\
d\tilde{\phi} &= d\phi
\end{aligned} \tag{3.35}$$

and therefore

$$\partial_t = \frac{\partial v}{\partial t}\partial_v + \frac{\partial \tilde{r}}{\partial t}\partial_{\tilde{r}} + \frac{\partial \tilde{\theta}}{\partial t}\partial_{\tilde{\theta}} + \frac{\partial \tilde{\phi}}{\partial t}\partial_{\tilde{\phi}} = \partial_v. \tag{3.36}$$

The surface gravity κ is evaluated most easily in IEF coordinates. The equation (3.32) defining κ is a vector equation so we only need to evaluate one of its components. For $\xi = \partial_v$, the $\mu = v$ component of $\xi^\sigma \nabla_\sigma \xi^\mu$ is

$$\xi^\sigma \nabla_\sigma \xi^v = \xi^\sigma \xi^v{}_{,\sigma} + \xi^\sigma \Gamma^v_{\sigma\rho} \xi^\rho = \Gamma^v_{vv}. \tag{3.37}$$

Now

$$\Gamma^v_{vv} = \frac{1}{2} g^{v\sigma} \left(g_{v\sigma,v} + g_{\sigma v,v} - g_{vv,\sigma} \right) = \frac{1}{2} g^{vr} (-g_{vv,r}) = \frac{M}{r^2}, \tag{3.38}$$

which, on the horizon, reduces to $\Gamma^v_{vv} = 1/4M$. Substitution into (3.37) then gives $\kappa = 1/4M$.

Note that the particular normalisation of ξ is important. Had we used a vector with a different normalisation $\tilde{\xi} = c\xi$ we would have obtained $\tilde{\xi}.\nabla\tilde{\xi}^\nu = (c\kappa)\tilde{\xi}^\nu$, leading to a different value $\tilde{\kappa} = c\kappa$ for the surface gravity. In Schwarzschild spacetime, the normalisation that leads to $\kappa = 1/4M$ is fixed by requiring that $\xi^2 = -1$ asymptotically.

3.4.2 Example 2: Kerr

The Killing horizon of a Kerr black hole is slightly more complicated in an interesting way. Recall the Killing vectors $k = \partial_t$ and $m = \partial_\phi$ of the Kerr-Newman black hole in Boyer-Lindquist coordinates (t, r, θ, ϕ) of section 3.1.2. Let's consider the zero charge case of the Kerr black hole. In Kerr coordinates (v, r, θ, χ) — defined above (2.20) — the Killing vectors correspond to $k = \partial_v = \partial_t$ and $m = \partial_\chi = \partial_\phi$. The frame dragging effect translates to the fact that k is *not* normal to the horizon. Instead, for the rotating Kerr black hole, the normal to the horizon is a combination of k and m (exercise):

$$\xi = k + \Omega_H m, \tag{3.39}$$

where $\Omega_H = a/(r_+^2 + a^2)$ is interpreted as the angular velocity of the black hole. The intuition here is that as you move forward in time along a null generator of the horizon, you also have to rotate by a certain amount. Given ξ we can evaluate the surface gravity κ_+ on the outer horizon $r = r_+$ using $\xi.\nabla\xi^\nu = \kappa\xi^\nu$, which evaluates to (exercise)

$$\kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)}. \tag{3.40}$$

3.5 Black Hole Uniqueness

As already mentioned in section 3.1.2, the Kerr-Newman three-parameter family is the unique stationary black hole solution of the Einstein-Maxwell theory. An equilibrium black hole in the presence of the electromagnetic field is therefore fully characterised by the three numbers M, J and Q.

However, the electromagnetic field is only one among many matter fields in Nature. Through a series of theorems known as "no hair" theorems it has been established that, in general, black holes have no other properties besides mass, angular momentum and charge (of whichever fields are present). To illustrate the general spirit of these proofs, we look at a simple example here.

Claim: A static black hole cannot be the source of a real (minimally coupled) scalar field.

Proof: Let ϕ be a real scalar field that satisfies the Klein-Gordon equation

$$\nabla^2\phi - m^2\phi = 0. \tag{3.41}$$

By the word "static" in the claim we mean that scalar field and spacetime (i.e. the metric field) have settled down to equilibrium, which implies that the scalar field does not vary in time anymore: "$\dot{\phi} = 0$". In covariant language the statement is that there exists a timelike Killing vector k $(= \partial_t)$ of the metric such that $k^\mu \nabla_\mu \phi = 0$. Given k we can foliate the spacetime by $t = \mathit{const}.$ hypersurfaces whose unit normals n are proportional to k, $n \propto k$. The black hole horizon is a Killing horizon of k.

Consider the integral

$$I = \int_R \phi(\nabla^2\phi - m^2\phi)\sqrt{-g}d^4x, \tag{3.42}$$

where $R \subset M$ is the spacetime region *outside* of the horizon. From the Klein-Gordon equation (3.41) it follows immediately that $I = 0$. On the other hand, using Stokes' theorem to integrate by parts the first term, we get

$$I = \int_V (-g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2)\sqrt{-g}d^4x + \int_{\partial V} \phi\nabla_\mu\phi dS^\mu \tag{3.43}$$

where ∂V is the boundary of the region V and dS^μ is the normal surface element on ∂V.

We now show that the surface integral in (3.43) vanishes identically. There are four contributions to ∂V: the horizon $r = 2M$, the surfaces $t = \pm\infty$ and the surface $r \to \infty$ (the "sphere at infinity"). On the horizon and on the $t = \pm\infty$ surfaces, the normal surface element is proportional to k, so the integrand vanishes: $\nabla_\mu \phi dS^\mu \propto k^\mu \nabla_\mu \phi = 0$. We are left with the boundary term at $r \to \infty$. To see that this vanishes too, note that since the spacetime is static and asymptotically flat, the metric takes the form

$$ds^2 = f(r)dt^2 + g(r)dr^2 + r^2 d\Omega_2^2 \tag{3.44}$$

where $f(r) = -1 + f_1 r^{-1} + \mathcal{O}(r^{-2})$ and $g(r) = 1 + g_1 r^{-1} + \mathcal{O}(r^{-2})$. Using $\nabla^2 = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$, a quick expansion shows that the leading terms in the Klein-Gordon equation in the limit $r \to \infty$ are

$$\partial_r^2\phi + 2r^{-1}\partial_r\phi - m^2\phi = 0, \tag{3.45}$$

which implies that the field must fall off like $\phi \sim 1/r$ or faster. For a sphere of radius r the normal surface element takes the form $dS \propto rdr\partial_r$, so the leading contribution in the integrand is $\phi\nabla_\mu\phi dS^\mu \propto rdr\phi\partial_r\phi \sim dr/r^2$. The integral of this vanishes in the limit $r \to \infty$.

Hence, the first integral in (3.43) must vanish identically. In terms of the induced metric $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ on hypersurfaces of $t = const.$ the integrand reads

$$g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = h^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + (n.\nabla\phi)^2 = h^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \geq 0 \tag{3.46}$$

having dropped the second term due to the fact that $n \propto k$ and $k.\nabla\phi = 0$. The last inequality follows from the fact that h is positive definite (see section 3.3.1). Therefore, the integrand is always negative or zero and the integral is zero only if the integrand vanishes everywhere. For $m > 0$ this implies that $\phi = 0$ everywhere. For $m = 0$ it implies that $\nabla_\mu\phi = 0 \implies \phi = const.$ everywhere, which is physically indistinguishable from $\phi = 0$ everywhere. □

3.6 Komar Integrals

For a Killing vector k, the Killing equation $\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0$ is the statement that the symmetric part of $\nabla_\mu k_\nu$ vanishes: $\nabla_{(\mu}k_{\nu)} = 0$. It follows that $\nabla_\mu k_\nu$ is antisymmetric:

$$\nabla_\mu k_\nu = \nabla_{[\mu}k_{\nu]} = \frac{1}{2}\left(\nabla_\mu k_\nu - \nabla_\mu k_\nu\right) \equiv K_{\mu\nu}. \tag{3.47}$$

The tensor $K_{\mu\nu}$ looks similar to the electromagnetic tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and, indeed, just as Maxwell's equations lead to a conserved electric charge, Einstein's equations lead to a conserved quantity when the spacetime has a Killing vector. This is expressed most conveniently in the language of differential forms.

Lightening Review of Differential Forms

Given a d-dimensional spacetime (M, g), a *differential form* of order k (k-form for short) on M is a totally antisymmetric covariant k-index tensor. By definition, zero-forms are functions and one-forms are covectors. Given a (local) coordinate chart x^μ on M, the coordinate differentials $dx^1, dx^2, \ldots, dx^d$ are examples of 1-forms and together they form a complete basis. A general one-form can therefore be written as $F = F_\mu dx^\mu$, where F_μ are the components of F. A k-form F can be written as

$$F = \frac{1}{k!} F_{\mu_1\mu_2\ldots\mu_k}\, dx^{\mu_1} \wedge dx^{\mu_2} \ldots \wedge dx^{\mu_k}, \tag{3.48}$$

where $\wedge$ is the antisymmetric *wedge product*: $dx^\alpha \wedge dx^\beta = -dx^\beta \wedge dx^\alpha$. The *exterior derivative* of a k-form F is defined as the $(k+1)$-form dF whose components are

$$(dF)_{\mu_1\mu_2\ldots\mu_{k+1}} = (k+1)\partial_{[\mu_1}\omega_{\mu_2\mu_3\ldots\mu_{k+1}]}. \tag{3.49}$$

A consequence of the antisymmetrisation in this definition is that $d(dF) = 0$ for any form F. The Hodge star $\star$ is a linear map that sends k-forms to $(d-k)$-forms. If F is a k-form, then $\star F$ is defined as the $(d-k)$-form whose components are

$$(\star F)_{\mu_1\mu_2\ldots\mu_{d-k}} = \sqrt{-g}\, \varepsilon_{\mu_1\mu_2\ldots\mu_{d-k}}{}^{\sigma_1\sigma_2\ldots\sigma_k} F_{\sigma_1\sigma_2\ldots\sigma_k}, \tag{3.50}$$

where ε denotes is the Levi-Civita symbol. Stokes' theorem states that for any form F the integral of the exterior derivative dF over a region $R \subseteq M$ is equal to the integral of F over the boundary ∂R of R:

$$\int_R dF = \int_{\partial B} F. \tag{3.51}$$

The electromagnetic tensor defines the two-form $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$. In terms of the four-potential $A = A_\mu dx^\mu$ it is given by $F = dA$, which implies that $dF = d(dA) = 0$. Maxwell's equations are completely encapsulated in the two equations

$$\begin{aligned} d\star F &= 4\pi \star j \\ d\star j &= 0 \end{aligned} \tag{3.52}$$

where $j = j_\mu dx^\mu$. The second equation is the continuity equation. The Hodge star $\star$ exchanges magnetic and electric fields, i.e. the magnetic and electric components of F are interchanged in $\star F$. Consider the surface shown in figure X. The electric charge in region B satisfies

$$Q(B) = \int_B \star j = \frac{1}{4\pi}\int_B d\star F = \frac{1}{4\pi}\int_{\partial B} \star F \tag{3.53}$$

by Stoke's theorem. This charge is conserved, which can be shown as follows. Take any two spacelike surfaces B_1 and B_2. Consider the region V shown in figure X, which is bounded by B_1 and B_2 and chosen to be large enough that all the sources (i.e. the support of j) lie inside V, in other words $j = 0$ on the cylindrical boundaries and outside of V. Then by virtue of the continuity equation

$$\begin{aligned} 0 &= \int_V d \star j \\ &= \int_{\partial V} \star j \\ &= \int_{B_1} \star j - \int_{B_2} \star j \\ &= \frac{1}{4\pi} \int_{\partial B_1} \star F - \frac{1}{4\pi} \int_{\partial B_2} \star F \\ &= Q(B_1) - Q(B_2), \end{aligned} \tag{3.54}$$

which proves that Q is conserved.

Let's turn to the analogous situation for $K_{\mu\nu}$. The associated two-form is defined as $K = \frac{1}{2} K_{\mu\nu}\, dx^\mu \wedge dx^\nu$. How does K show up in Einstein's equations? First note that $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) k^\sigma = R^\sigma{}_{\rho\mu\nu} k^\rho$. For $\sigma = \mu$ we get $\nabla_\mu \nabla_\nu k^\mu = R_{\rho\nu} k^\rho$ or

$$\nabla_\mu K^\mu{}_\nu = R_{\rho\nu} k^\rho, \tag{3.55}$$

which looks just like $\nabla_\mu F^\mu{}_\nu = 4\pi j_\nu$. Einstein's equations $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$ can be written as

$$R_{\rho\nu} = 8\pi G \left(T_{\rho\nu} - \frac{1}{2} g_{\rho\nu} T \right). \tag{3.56}$$

Let

$$\zeta_\nu \equiv \frac{1}{8\pi G} R_{\rho\nu} k^\rho \tag{3.57}$$

and $\zeta = \zeta_\nu dx^\nu$. Then Einstein's equations take the form

$$\begin{aligned} d \star K &= 8\pi G \star \zeta \\ d \star \zeta &= 0. \end{aligned} \tag{3.58}$$

In analogy to (3.53) we can then define a *Komar integral*

$$Q_k(B) \equiv \frac{c}{8\pi G} \int_{\partial B} \star K \tag{3.59}$$

for any Killing vector k. It can be shown that, for B large enough such that all matter is inside of it and spacetime is vacuum outside of it, $Q_k(B)$ is conserved.

The physical interpretation of Q_k depends on the Killing vector k. Some examples (whose proof is left as an exercise) for appropriate choices of constants c and regions B are:

- In Schwarzschild spacetime, $Q_k(B) = M$ where $k = \partial_t$.
- In Kerr spacetime, $Q_m(B) = J$ where $m = \partial_\phi$.
- In Kerr-Newman spacetime, $Q_k(B) = M$ and $Q_m(B) = J$.

4 Black Hole Thermodynamics

4.1 Overview

In 1973, Bardeen, Carter and Hawking (BCH) wrote a paper titled "The Laws of Black Hole Thermodynamics", which summarises work on black holes as a series of laws analogous to the laws of thermodynamics. The main points of this analogy are shown in table 1. BCH emphasised that black holes have zero temperature (nothing can escape black holes, so they cannot radiate) and cannot therefore have a physical entropy — in their mind, the analogy between the laws of black hole mechanics and the laws of thermodynamics was purely formal.

A young graduate student named Jacob Bekenstein disagreed. He noted that the second law of thermodynamics would be violated if black holes had no entropy, since one could throw arbitrarily entropic objects into the black hole, thereby lowering the total entropy of the exterior universe. He claimed that black holes *must* have an entropy $S_{BH} \propto A$ to save the second law of thermodynamics. Bekenstein's *generalised second law* states that

$$dS_{total} \geq 0 \tag{4.1}$$

where $S_{total} = S_{external} + S_{BH}$. In 1974, it was Hawking who announced that black holes are hot and radiate just like any hot body with a temperature

$$T_H = \frac{\hbar \kappa}{2\pi k_B}, \tag{4.2}$$

Law	Thermodynamics	Black Holes
0^{th}	The temperature T is constant throughout a system in thermal equilibrium	The surface gravity κ is constant over the event horizon of a stationary black hole
1^{st}	$dE = TdS + \sum_i \mu_i dN_i$	$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ + \Psi_H dQ$
2^{nd}	$dS \geq 0$	$dA \geq 0$
3^{rd}	T cannot be reduced to zero by a finite number of operations	κ cannot be reduced to zero by a finite number of operations

Table 1. Analogy between the laws of thermodynamics and black holes.

from which it follows that a black hole has an entropy given by

$$S_{BH} = \frac{A}{4G\hbar} \tag{4.3}$$

the *Bekenstein-Hawking entropy.* Since $G\hbar$ has units of length squared, we may define the *Planck length* $l_P = \sqrt{G\hbar}$ so that the Bekenstein-Hawking entropy

$$S_{BH} = \frac{1}{4}\frac{A}{l_P^2} \tag{4.4}$$

can be interpreted as a quarter of the area of the black hole, counted in units of l_P^2.

So, after all, the analogy of table 1 between black hole mechancis and thermodynamics seems to be more than a formal peculiarity: if one identifies the temperature with $\hbar\kappa/2\pi k_B$ it becomes a physical unification. In the next sections we will discover the origin of the results discussed here.

4.2 The First Law of Black Hole Mechanics

The first law encapsulates conservation of energy: if a stationary black hole with parameters M, Q and J is perturbed to a new stationary state then the changes in M, Q and J satisfy

$$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ + \Phi_H dQ, \tag{4.5}$$

where Ω_H is the angular velocity of the black hole and Φ_H is the "electric surface potential". We will use the uniqueness theorem that the only stationary, asymptotically flat black hole solutions of the Einstein-Maxwell equations are give by the Kerr-Newman family (2.15)

$$\begin{aligned} ds^2 = &- \left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta}dr^2 - 2\frac{a\sin^2\theta}{\Sigma}\left(r^2 + a^2 - \Delta\right) dtd\phi \\ &+ \Sigma d\theta^2 + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2. \end{aligned}$$

In order to verify (4.5), let us express κ, Ω_H, Φ_H and A in terms of M, Q and J (or $a = J/M$, equivalently):

- The surface gravity κ on the outer horizon is given by (3.40) $\kappa = (r_+ - r_-)/2(r_+^2 + a^2)$.
- The surface area A of the horizon is defined at time t_0 as the area of the intersection of the hypersurface Σ of constant $t = t_0$ with the horizon H defined by $r = r_+$. For the Kerr-Newman metric, the induced line-element on the intersection of the two hypersurfaces $t = const.$ and $r = const.$ is

$$ds^2 = h_{\mu\nu}dx^\mu dx^\nu = \Sigma d\theta^2 + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2. \tag{4.6}$$

On $H \cap \Sigma$ we have $\Delta = 0$ and therefore $\sqrt{h} = (r_+^2 + a^2)\sin\theta$, so that

$$A = \int_{H\cap\Sigma} \sqrt{h} d\theta d\phi = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta (r_+^2 + a^2)\sin\theta = 4\pi(r_+^2 + a^2). \tag{4.7}$$

- The angular velocity $\Omega_H = a/(r_+^2 + a^2)$ was derived in (3.39) above: it is the non-zero constant in $\chi = \partial_t + \Omega_H \partial_\phi$ that makes χ a Killing vector tangent to the generators of the horizon.

- The electric potential Φ_H is defined as the potential difference between infinity and the horizon, i.e. the work done in bringing a unit charge from $r = \infty$ to $r = r_+$:

$$\Phi_H = (\chi^\mu A_\mu)|_{r=r_+} - (\chi^\mu A_\mu)|_{r=\infty} = (A_t + \Omega_H A_\phi)|_{r=r_+} = \frac{Q r_+}{r_+^2 + a^2}. \tag{4.8}$$

The second term in the first line was dropped because A_μ vanishes as $r \to \infty$.

Putting these together, we obtain

$$A = A(M, Q, J) = 4\pi \left(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - a^2}\right). \tag{4.9}$$

Since M, Q and J are independent parameters, this implies that

$$dA = \frac{\partial A}{\partial M} dM + \frac{\partial A}{\partial Q} dQ + \frac{\partial A}{\partial J} dJ, \tag{4.10}$$

which can be rearranged to (exercise)

$$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ + \Phi_H dQ. \tag{4.11}$$

The proof is deceptively simple. All the hard work goes into proving the uniqueness theorems: you need to know that the black hole settles down to another Kerr-Newman black hole and not some other spacetime. It is worth noting that there exist proofs, known as "physical process proofs", that do not assume this.

4.3 Working up to Hawking's Area Theorem

Recall that the integral curves of a smooth non-zero vector field form a congruence: a collection of curves through a region of spacetime such that every point in the region lies on exactly one of the curves. Every congruence naturally defines an associated coordinate system (λ, y^a) with $a = 1, 2, 3$, where λ is the parameter such that the vector $\partial_\lambda = t$ is tangent to the curve of the congruence at the point (λ, x^a). In other words, the indices y^a label the curves and λ is a parameter along the curves. The vectors $\eta_a = \partial_{y^a}$ are called the "connecting vectors".

We shall assume that the curves are geodesics and that λ is an affine parameter so that $t.\nabla t^\mu = 0$. For a null congruence $t^2 = 0$ and for a timelike congruences we can set $t^2 = -1$. The vectors t and η_a commute because they form a coordinate basis:

$$[t, \eta_a] = 0 \implies \eta_a^\mu \nabla_\mu t^\nu - \eta_a^\mu \nabla_\mu t^\nu = 0 \tag{4.12}$$

for $a = 1, 2, 3$. Since $t^\mu \nabla_\mu = \partial_\lambda$, this means that the "rate of change" of the connecting vectors along a curve of the congruence is given by

$$\frac{d}{d\lambda}\eta_a^\nu = t^\mu \nabla_\mu \eta_a^\nu = \eta_a^\mu \nabla_\mu t^\nu \equiv B^\nu_\mu \eta_a^\mu. \tag{4.13}$$

Then tensor B^ν_μ is said to measure the *geodesic deviation*, i.e. the extent to which the connecting vectors fail to be parallelly transported along curves of the congruence. When η_a is parallelly transported, B^ν_μ is zero.

Note that there is an ambiguity in the definition of the connecting vectors: if η_a is a connecting vector from γ to $\tilde{\gamma}$, then $\eta_a + \zeta t$ with $\zeta \in \mathbb{R}$ is also a connecting vector from γ to $\tilde{\gamma}$. We may use this freedom to choose some convenient set of connecting vectors. For example, when the congruence is timelike, it is always possible to choose all three connecting vectors η_a to be orthogonal to t everywhere: $t.\eta_a = 0$ for $a = 1, 2, 3$. This condition will be preserved along the congruence since

$$\begin{aligned}\frac{d}{d\lambda}(t.\eta_a) = t^\mu \nabla_\mu (t_\nu \eta_a^\nu) &= (t^\mu \nabla_\mu t^\nu)\eta_a^\nu + t^\mu t_\nu \nabla_\mu \eta_a^\nu \\ &= t_\nu \eta_a^\mu \nabla_\mu t^\nu = \frac{1}{2}\eta_a^\mu \eta_\mu (t^2) = 0.\end{aligned} \tag{4.14}$$