

Reference / Summary Sheet for Complex Analysis

I've attempted to list everything I learned in an upper division course on Complex Analysis here for quick reference:

Complex Identities for a $z = x + iy$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) = \frac{e^i - e^{-iz}}{2i}$$

and for $x \in \mathbb{R}$:

$$\sinh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

so we can derive the complex hyperbolic functions:

$$\sinh(z) = \sinh(x)\cos(y) + i\cosh(x)\sin(y)$$

$$\cosh(z) = \text{TODO}$$

$$\tanh(z) = \text{TODO}$$

more identities of hyperbolic trig functions

List of Important Theorems and Definitions

still in progress, unordered, expanded on below.

- Rouché's Theorem
- Fundamental Theorem of Algebra
- Type I/II Integrals

- singularities
- zeroes
- residues
- Residue Theorem
- $\text{ord}_w f(z)$
- meromorphic function
- extended complex plane
- linear fractional
- Cauchy's Residue Theorem
- winding numbers
- Laurent Series
- poles
- order of a pole
- non-isolated singularity
- isolated singularity
- simply connected topological space
- path connected topological space
- hessian matrix and critical points
- Homotopy
- analytic branch of complex log
- principal branch of complex log
- ended hw 5 continue after this

Singular Points

- a singularity of $f(z)$ is a point where it fails to be analytic. You can classify them:

Non-Isolated Singularities

1. Branch points: functions are not analytic here, so not analytic in a deleted neighborhood of a branch point
 - ex 1: $f(z) = \sqrt{z-3}$ has a branch point at $z = 3$
 - ex 2: $f(z) = \ln(z^2 + z - 2) = 0$ has branch pts where $f(z) = 0$
- 2.

Isolated Singularities

- a point z_0 is an isolated singularity if you can find a deleted delta neighborhood around it with no other singularity. If you can't find a deleted neighborhood s.t. f is analytic around z_0 , it is an essential / non isolated singularity
1. TYPE 1: **poles/non-essential singularities**: a type of non-essential, isolated singularity. classify its order by : $\lim_{z \rightarrow z_0} (z - z_0)^n f(z) \neq 0$ but

$\lim_{z \rightarrow z_0} (z - z_0)^{n+1} f(z) \neq 0$ then f has a zero of order n at the pole z_0 .
The limit of $f(z)$ at a pole is ∞ .

- pole classification techniques:
 - if $f(z)$ rational function: factor the denominator (here poles are where f is dividing by zero), assured all zeroes of the denominator exist in the complex plane by FTA, if you find simple roots (distinct and constant), poles (the zeroes) have order 1.
 - ex: $\frac{1}{z^4+16}$
 - factor the denominator. if you see a double factor, probably a pole of order 2 (double pole)
 - ex: $\frac{1}{x^2+4x+4}$
 - check your order guesses with limit definition
 - use quadratic equation when you can't easily factor (valid for complex numbers): $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$
 - ex: $\frac{1}{z^2 - z + 1}$
2. TYPE 2: **Removable Singularities:** isolated / non essential singularity
 z_0 is removable if $\lim_{z \rightarrow z_0}$ exists and is not ∞ . By this fact, f remains continuous and analytic at z_0
 3. TYPE 3: **Essential Singularities** Any singular point that is not a pole or a removable singularity is an essential singularity. The limits of the function at essential singularities truly do not exist. (they're not ∞). Also, we cannot find an n s.t. $\lim_{z \rightarrow z_0} (z - z_0)^n f(z) \neq 0$
 - ex: $f(z) = e^{\frac{1}{z-2}}$ has an essential singularity at $z = 2$.

Strategy for problems of classification:

Step 1: Visualize the singularity, if provided. If the behavior tends to positive infinity or negative infinity (or both), it's probably a pole. If the behavior at the singularity of the function seems continuous, it might be a removable singularity. An essential singularity looks like neither a pole (strictly tending to positive/negative infinity) or a removable singularity.

Note that a non isolated singularity will have other singularities “really close” to the proposed singularity. If the function behaves erratically (ex: $\tan(\frac{1}{z})$ at $z = 0$), it might be a non-isolated singularity. Always formally prove that it is non isolated by finding formulas for the other singularities. Once you have such a formula for “dividing by zero”, take the limit

Step 2: After you have a guess, check the limit theorems of each type. A removable singularity should have a limit that exists at that point. A pole will yield an infinite limit. An essential singularity will yield a truly nonexistent limit.

Step 3: If you have a pole, verify the order with the limit equation.

Laurent Series Expansion