10-16

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TODO ① log review of lecture on google sheet.
② start HW
③ Exam 1 Plan.
④ ask abt roots of unity + finish notes.
⑤ Note: only 4-5 p's graded on HW - select answer

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TOPICS INDEX

Lecture 1 Sep 1 2020

- Dr. Zhang is recent grad of U Illinois. Post Doc @ Math Dept.
- can collaborate with friends - no copying.
- TODO: form teammate on Piazza
- Zhang office hours: by poll (1hr a week) TA · 2hr/week.
- 1 chapter a week! next week: Chapter 2 (diff of complex analytic functions).

EXAM 1 { Exam 1 Ch.1 - Ch.5 or Ch.4)

- most useful part of class: Residue Theorem.

FOCUS of LEC
- cubic equation not that important
- focus on 1.1, 1.2

WHY i? (ANSWERED IN FOLLOWING EXS):

Quadratic Formula in $\mathbb{R}$:

QUAD FORM IN $\mathbb{R}$

$$ax^2 + bx + c = 0 \quad , a \neq 0 \quad , a, b, c \in \mathbb{R}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

discriminant

This eqn has $\begin{cases} 2 \text{ rts if } \Delta = b^2 - 4ac > 0 \\ 1 \text{ rt if } \Delta = 0 \\ 0 \text{ rts if } \Delta < 0 \end{cases}$

~~WHY~~

but solving cubics...

CUBIC PROBLEM need to take sqroots of neg #'s even when all 3 roots are real.

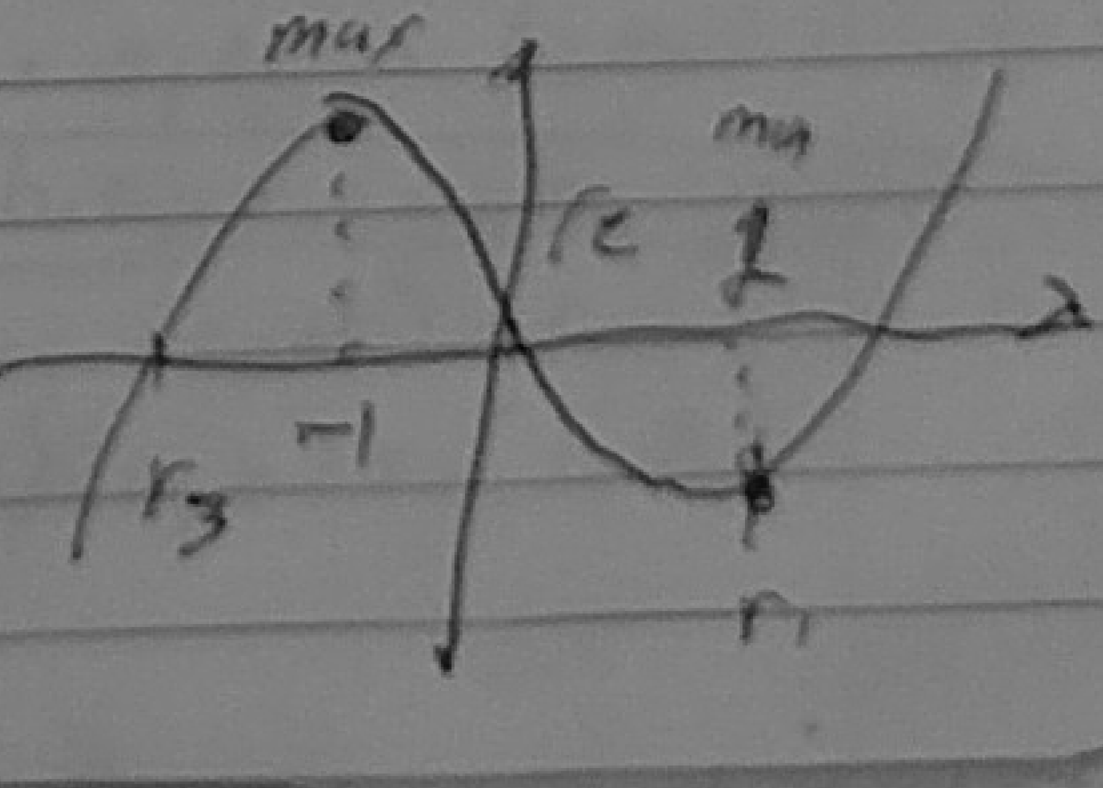
ex: $x^3 - 3x + 1 = 0$

Claim: The $f(x) = x^3 - 3x + 1$ has 3 real roots.

pf find local max/min + sketch graph

$$f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

$$f(1) = -1 \quad f(-1) = 3$$



Cardano's formula: CUBIC PROB cont...

for cubic, using Cardanos:

substitu

$x = u+v \Rightarrow (u+v)^3 - 3(u+v) + 1 = 0$

$u^3 + 3u^2v + 3uv^2 + v^3 - 3u - 3v + 1 = 0$

$u^3 + v^3 + 3(u+v)(uv-1) = 1$

we can solve ↑ by solving

$u^3 + v^3 = -1$ ①

$uv - 1 = 0$ ② $\Longrightarrow u^3v^3 = 1$

Now solve for u^3 and v^3. (easier than solve u and v, cause)

$\Rightarrow v^3 = \frac{1}{u^3}$

plug $v^3 = \frac{1}{u^3}$ into ① > $\left[u^3 + \frac{1}{u^3} = -1\right]$

→ no real solutions (why?)

$> u^6 + u^3 + 1 = 0$

$\Rightarrow u^3 = \frac{-1 \pm \sqrt{-3}}{2}$

~~> so at least one solution of~~

$\Rightarrow u = \sqrt[3]{\frac{-1+\sqrt{-3}}{2}} \Rightarrow v = \sqrt[3]{\frac{-1-\sqrt{-3}}{2}}$

$\Rightarrow$

$x = \sqrt[3]{\frac{-1+\sqrt{-3}}{2}} + \sqrt[3]{\frac{-1-\sqrt{-3}}{2}}$ (is a real solution!) to $x^3 - 3x + 1 = 0$

but problem is....

even for real sols, we need to use $\sqrt{\text{neg \#}}$ to write the solution using Cardanos! This is our problem ☺ ⇒ so $\sqrt{\text{neg \#}}$ must mean something.

DIFF EQS example:

Consider $[y'' + \lambda y = 0]$

when $\begin{cases} \lambda > 0 \Rightarrow y = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x} \\ \lambda < 0 \Rightarrow y = c_1 \cos(\sqrt{-\lambda} x) + c_2 \sin(\sqrt{-\lambda} x) \\ \lambda = 0 \Rightarrow \end{cases}$

$\Rightarrow$ indicates might be some relation b/w $e^{\sqrt{\lambda} x}$ $(\lambda < 0)$ and trig fns

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Eulers ID

$$e^{\sqrt{-1}\,\theta} = \cos\theta + \sqrt{-1}\sin\theta \quad \text{(oscillations)}$$

GAUSS

- When is an integer the sum of 2 perfect squares?
  (use $\sqrt{-1}$) for solution.

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end of Why i? / Historical background.
Now.... Complex Numbers:

COMPLEX NUMBERS DEF + OPS.

- $\mathbb{C} := \{(a,b) \mid a,b \in \mathbb{R}\}$
- As a set, $\mathbb{C}$ ($+/-$) is the same as $\mathbb{R}^2$.
- Same arithmetic operations as $\mathbb{R}^2$ (pairwise add/$-$):
  $(a_1, b_1) \pm (a_2, b_2) = (a_1 \pm a_2, b_1 \pm b_2)$
- multiplication operation
  $(a_1, b_1)(a_2 b_2) := (a_1 a_2 - b_1 b_2, \ a_1 b_2 + a_2 b_1)$:

DERIVATION:

$$\begin{aligned}(a_1 + b_1 i)(a_2 + b_2 i) &= a_1(a_2 + b_2 i) + b_1 i (a_2 + b_2 i) \\ &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i \cdot i \\ &= a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i\end{aligned}$$

MUL INV PRF:

Claim: when $(a,b) \neq (0,0)$, $\exists$ a uniq (c,d) s.t.
$(a,b)(c,d) = (c,d)(a,b) = (1,0)$

$\exists$ $(a+bi)(c+di) = 1$

$$c+di = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$

$$\Rightarrow c = \frac{a}{a^2+b^2} \qquad d = \frac{-b}{a^2+b^2}$$

$\Rightarrow$ COMPLEX $\mathbb{C}$ has division.

TODO: Check multiplication is associative, ~~cancel~~.

SOLVING QUAD EQUATIONS:

- solve quadratic equations ~~to~~ with complex coefficients:

> $ax^2+bx+c=0 \quad a \neq 0$

> $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

> now write $\sqrt{b^2-4ac}$ as $x+yi$

> set $b^2-4ac = u+vi$

> ~~(u+vi)^2 = x+yi~~ $u+vi = (x+yi)^2$

$$= x^2-y^2+2xyi$$

$$\Rightarrow \begin{cases} u = x^2-y^2 \\ v = 2xy \end{cases}$$ solve for x & y.

(TODO-TASK).

~~example:~~

What is $\sqrt{i}$? We can show this is $\left[\sqrt{i} = \pm\left(\frac{1-i}{\sqrt{2}}\right)\right]$ with work above.

for cubic eqs — Use Cardano's formula.

Questions

How abt eqs of higher degree? Do they have roots?

↳ P1: Given a polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 = 0 \quad a_i \in \mathbb{C}$$

is there a complex w that solves this?

↳ P2: Is there a formula for zeroes like in quad/cubic case?

(1) Was solved by Gauss: True.
Fundamental Theorem of Algebra (provable by C. Analysis) or Topology

(2) answered by diff ~~m~~ - Galois
when degree $n \geq 5 \Rightarrow$ no formula (insolvability of quintics)
(no formula using $+, -, \div, \times, \sqrt{\ }$)
(Abstract Algebra)

COMPLEX PLANES

- as a vector space, $\mathbb{C} = \mathbb{R}^2$
- $z = x + yi \longleftrightarrow (x, y) \in \mathbb{R}^2$

$\operatorname{Re} z := x$-coord

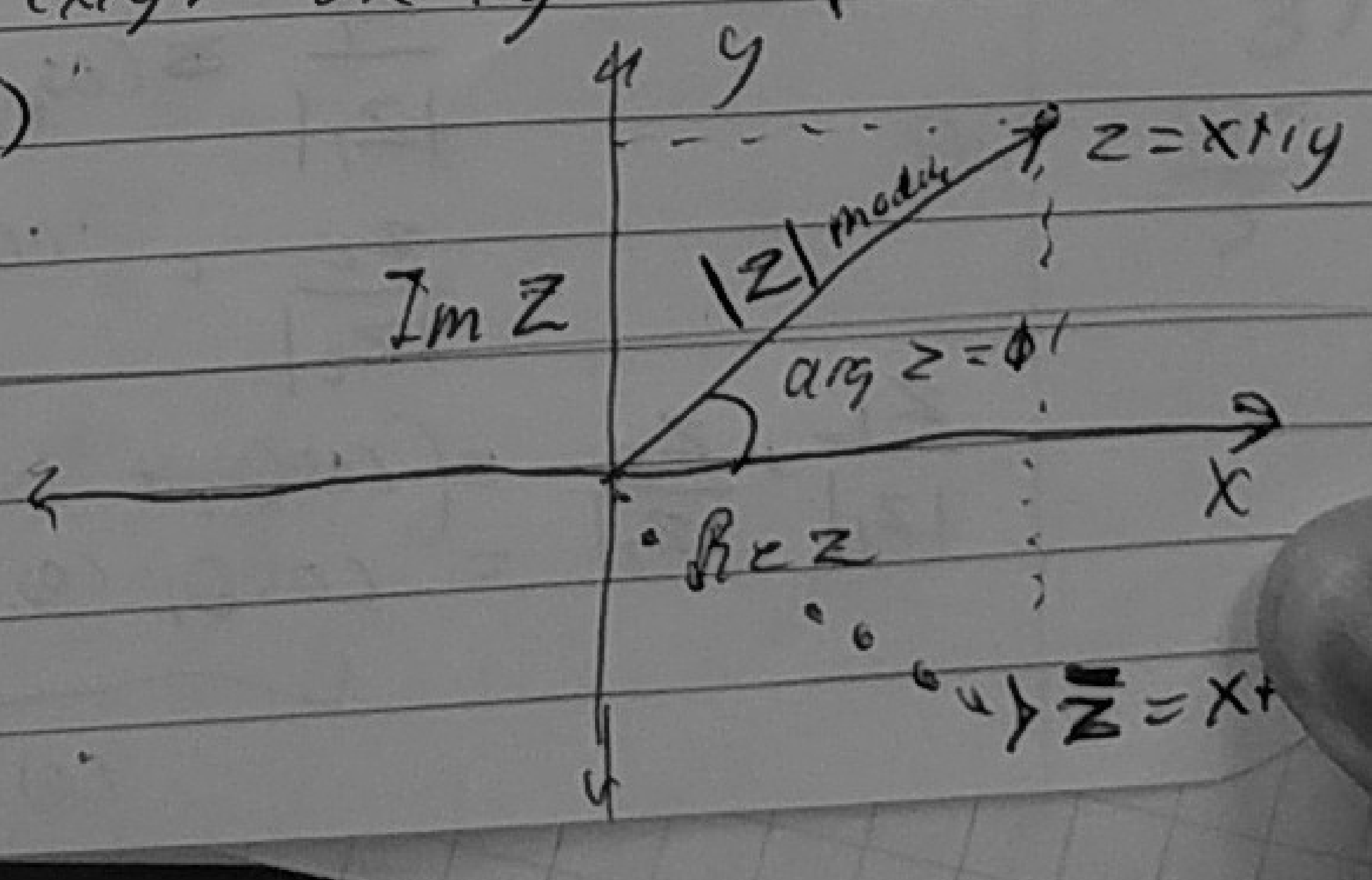
$\operatorname{Im} z :=$ y coord

modulus $|z| :=$ length of $(x,y) = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}}$

argument ~~arg~~ $z = \phi$ (mod 2π)

conjugate $\bar{z} = x - iy$

pictures of complex plane:



example:

Sketch some subsets of $\mathbb{C}$:

$$\{ z \mid \operatorname{Re} z = |z-1| \}$$

$$x = |x-1+iy| = \sqrt{(x-1)^2+y^2}$$

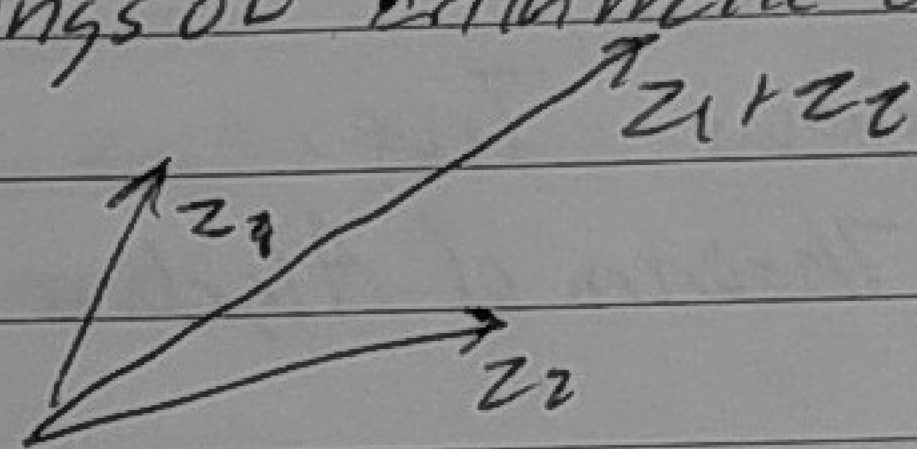
$$x^2 = (x-1)^2 + y^2 = x^2 - 2x + 1 + y^2$$

$$\Rightarrow y^2 = 2x - 1$$

pf: ~~sketch~~ work it in x, y if you can't sketch

Geometric meanings of arithmetic expr

• vector +



• mult (intuitively):

$\mathbb{C}$ as vector is length + dir — mod + arg.

$$z_1 z_2 = \begin{cases} \text{length } |z_1 z_2| \\ \text{direc/arg: } \arg z = \theta \pmod{2\pi} \end{cases}$$

• don't use i as index lol.

$z_j = x_j + iy_j \quad j = 1, 2$ TODO:

$|z_j| = \sqrt{x_j^2 + y_j^2}$ (check $|z_1 z_2| = |z_1| \cdot |z_2|$)

(stopped understanding)

→ argument (intuitively part)

for $\arg z_1 z_2 \longrightarrow z_1 = |z_1| \cdot \dfrac{z_1}{|z_1|}$, ($\dfrac{z_1}{|z_1|}$ is unit vector complex #)

$$\frac{z_1}{|z_1|} = \cos_{\theta}\theta_1 + i\sin_{\theta}\theta_1 \quad (\theta_1 = \arg z_1)$$

$$\frac{z_2}{|z_2|} = \cos\theta_2 + i\sin\theta_2$$

$$\frac{z_1}{|z_1|} \cdot \frac{z_2}{|z_2|} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= \underbrace{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2}_{\cos(\theta_1+\theta_2)} + \underbrace{(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)}_{\sin(\theta_1+\theta_2)} i$$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

8 $\Rightarrow \arg(z_1 z_2) = \theta_1+\theta_2 = \arg(z_1) + \arg(z_2)$

$\Rightarrow z_1 z_2 = \begin{cases} \text{length } |z_1 z_2| = |z_1|\,|z_2| \\ \text{arg } \arg(z_1 z_2) = \arg z_1 + \arg z_2 \end{cases}$

DeMoivre's ~~Theorem~~ Formula:

$$(\cos\theta + i\sin\theta)^n = \begin{cases} \text{length } 1 \\ \text{arg: } n\theta \end{cases}$$

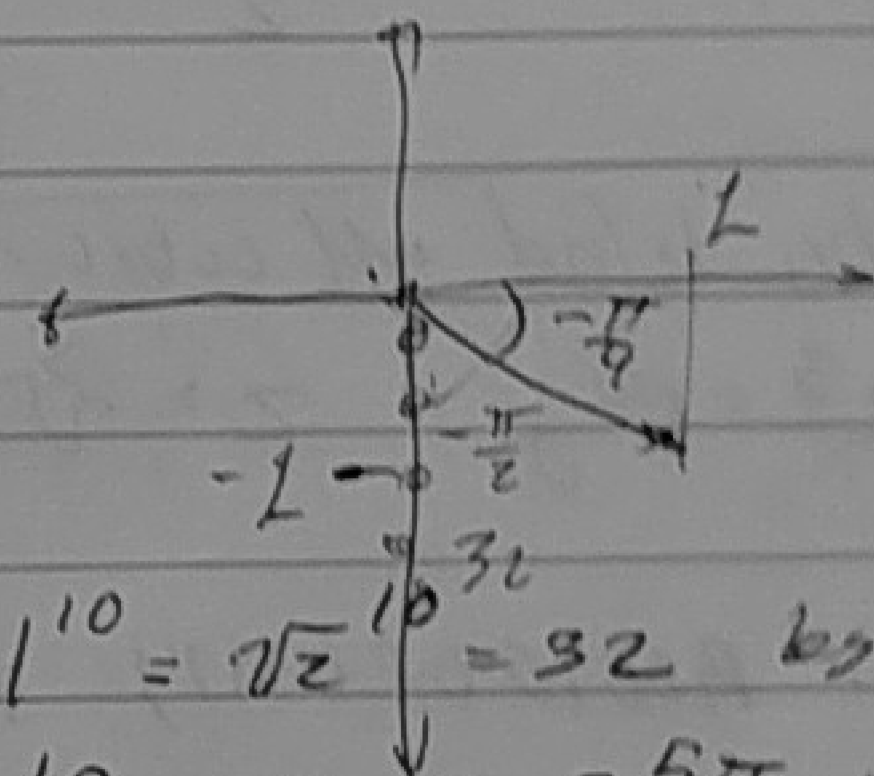
$$= \cos n\theta + i\sin n\theta$$

ex: $(1-i)^{10}$:

$z = 1-i$

$\rightarrow |z| = \sqrt{2}$

$\rightarrow \arg z = \frac{-\pi}{4}$



now, $z^{10} = \begin{cases} |z^{10}| = |z|^{10} = \sqrt{2}^{10} = 32 \\ \arg z^{10} = 10 \arg z = \frac{-5\pi}{2} = \frac{-\pi}{2} \end{cases}$

$= z^{10} = \langle l = 32, \text{ arg of } -\frac{\pi}{2} \rangle$

$\Rightarrow z^{10} = (-32i)$ - second grp'

ROOTS of UNITY: find all cubic roots of unity $z^3 = 1$

$|1| = 1$

$\arg(1) = 0$

$\rightarrow |z|^3 = 1 \rightarrow |z| = 1$

$\rightarrow \arg z^3 = 0 = 3\arg z = 0 \Rightarrow \arg z = 2n\pi,\ 2\pi n + \frac{2\pi}{3},\ 2n\pi + \frac{4\pi}{3}$

$\Rightarrow z = 1,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$\Rightarrow 1,\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

~~A~~lternately, to find roots of unity just solve:

$$z^3 = 1 \Rightarrow z^3 - 1 = 0$$
$$\Rightarrow (z-1)\left[(z^2+z+1)\right] = 0$$
$$\Rightarrow (z-1)$$

quad solve

$z^2 + z + 1 = 0$

using quad formula:

$$z = \frac{-1 \pm \sqrt{-3}}{2}$$

agrees with computation above!! w roots of unity.

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more generally, to find all cubic roots of unity,

$z^3 = r$, $z = \sqrt[3]{r}$, ... confused here?

TODO: ROOTS OF UNITY formula