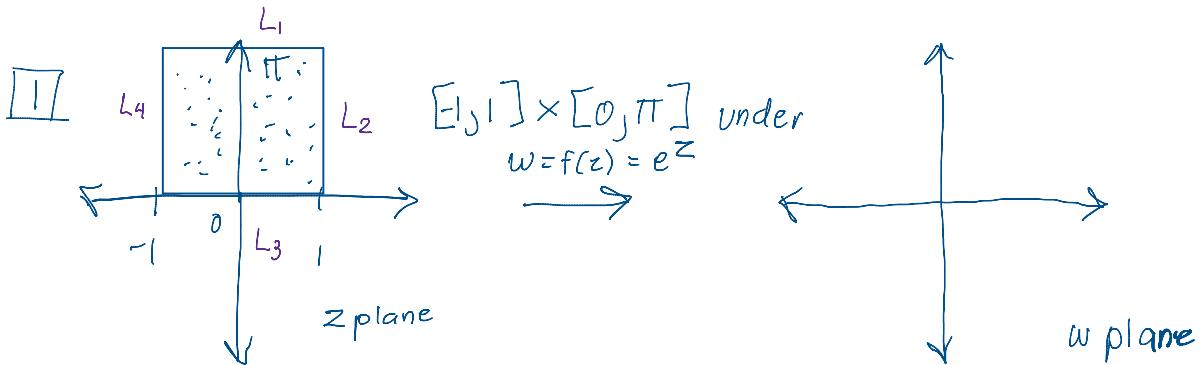


# Homework 9

Friday, November 27, 2020 3:44 PM

**Exercise 1.** Find the images of

- (1) the rectangle  $[-1, 1] \times [0, \pi]$  under the map  $f(z) = e^z$ .
- (2) the circle  $|z| = R$  under  $f(z) = \frac{1}{z-R}$ .
- (3) the half disk  $\{z \in \mathbb{C} \mid |z| < 1, \operatorname{Im} z > 0\}$  under  $f(z) = -\frac{1}{2}(z + \frac{1}{z})$ . (Hint: Set  $f(z) = w$  and solve for  $z$ . For which values of  $w$  does this equation have a solution in the half disk?)



Step 1:  $w = f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) = e^x \cos y + i e^x \sin y$

$$\Rightarrow w = u + iv \quad \begin{cases} u(x, y) = e^x \cos y \\ v(x, y) = e^x \sin y \end{cases}$$

$u, v \in \mathbb{R}$  - val functions

Step 2: see where  $w$  maps the boundary.  $\{L_1, L_2, L_3, L_4\}$  case by case.

$L_1$ : line going from  $(-1, \pi)$  to  $(1, \pi)$ ,  $y \equiv \pi$ ,  $x \in [-1, 1]$  varies within the interval.  
then

$$u := e^x \cos y = e^x \cos \pi = e^x (-1) = -e^x, \quad x \text{ travels in } [-1, 1].$$

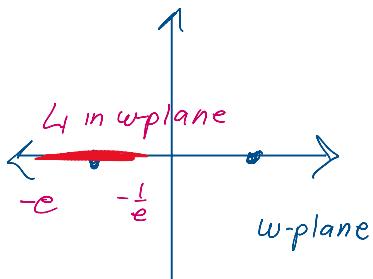
$$v := e^x \sin y = e^x \sin(\pi) = e^x \cdot 0 = 0$$

$L_1$  mapped to a line where  $v \equiv 0$ ,  $-e^{-1} \leq u \leq -e$

$L_2$ : line going from  $(1, 0) \rightarrow (1, \pi)$

$$u := e^x \cos y = e^x \cos(y), \quad y \in [0, \pi]$$

$$v := e^x \sin y = e^x \sin(y), \quad y \in [0, \pi]$$



$-c$     $\frac{-z}{e}$     $\downarrow$     $w\text{-plane}$

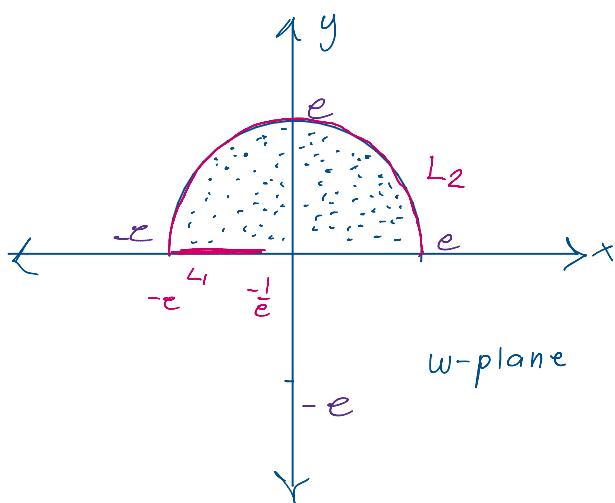
from  $w = u + iv$ ,  $w = e \cos y + i e \sin y$

$$= e(\cos y + i \sin y)$$

$$\text{where } |w| = \sqrt{u^2 + v^2} = \sqrt{e^2 \cos^2 y + e^2 \sin^2 y} \\ = \sqrt{e^2(1)} \\ = \pm e,$$

$\Rightarrow w$  is circle with  $u^2 + v^2 = e^2$ , so radius =  $e$ .

- but it's only a half circle because  $y$  stays in  $[0, \pi]$ .



$L_3$ : line going from  $(-1, 0) \rightarrow (1, 0)$

$$u := e^x \cos y = e^x \cos(0) = e^x(1) = e^x, \quad -1 \leq x \leq 1.$$

$$v := e^x \sin y = e^x \sin 0 = 0 \quad \forall x \in [-1, 1]$$

so  $L_3$  gets mapped to a line  $e^{-1} \leq u \leq e^1, v=0$ .

$L_4$ : line going from  $(-1, 0)$  to  $(-1, \pi)$

$$u := e^x \cos y = e^{-1} \cos y \quad 0 \leq y \leq \pi$$

$$U := e^x \cos y = e^{-1} \cos y \quad 0 \leq y \leq \pi$$

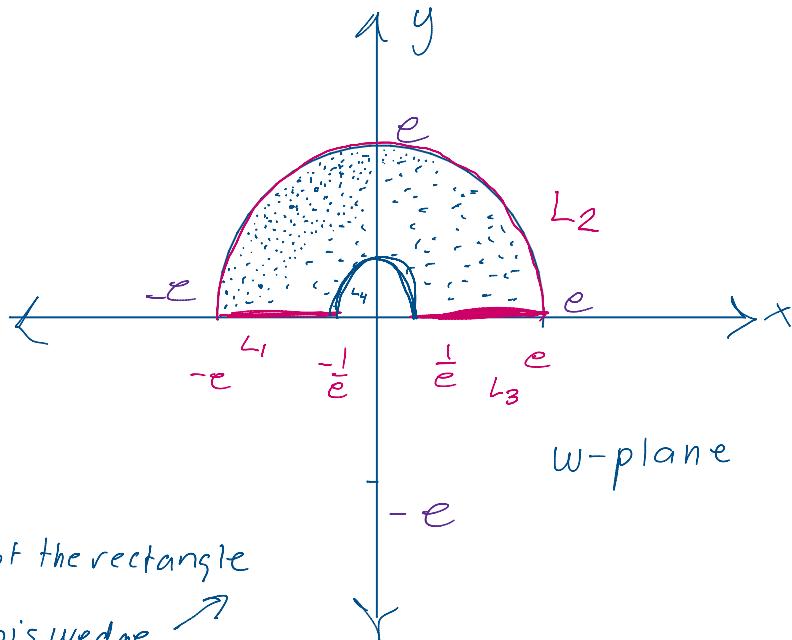
$$V := e^x \sin y = e^{-1} \sin y \quad 0 \leq y \leq \pi$$

then  $w = U + iV$

$$= e^{-1} \cos y + i e^{-1} \sin y$$

$$\text{where } |w| = \sqrt{U^2 + V^2} = \sqrt{e^{-2} \cos^2 y + e^{-2} \sin^2 y} = \sqrt{e^{-2}(1)} = \frac{\sqrt{1}}{\sqrt{e^2}} = \frac{1}{\sqrt{e^2}} = \frac{1}{e^{\pm}}$$

so  $w$  is a circle centered at 0 with radius  $= e^{-1}$



So the image of the rectangle

under  $w$  is this wedge.

(region inside rectangle became region inside wedge).

$$[2] \quad |z| = R \quad f(z) = \frac{1}{z - R}$$

$$z = \{re^{i\theta} \mid r=R\} \text{ and also } z = |z|e^{i\theta} = Re^{i\theta}$$

$$\text{so } f(z) = w = \frac{1}{Re^{i\theta} - R} = (Re^{i\theta} - R)^{-1} = \frac{1}{R \cos \theta + i \sin \theta - R}$$

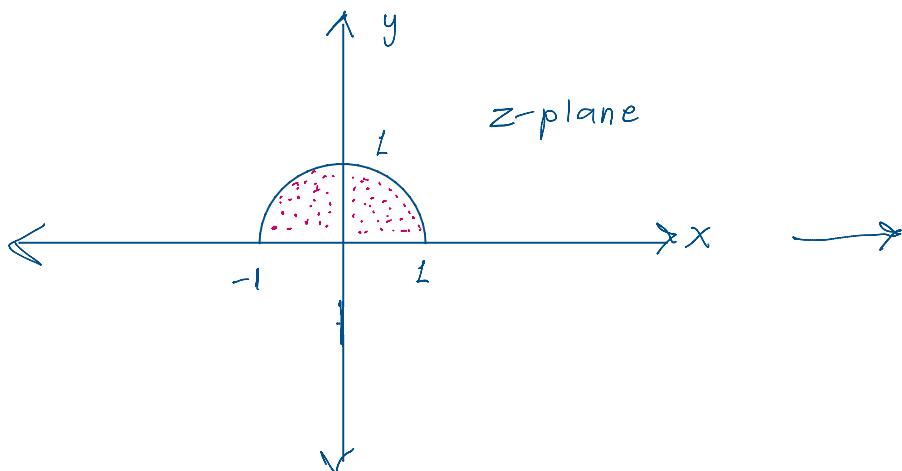
so  $z$  becomes the set  $\{ \text{TODO 3} \}$ , a circle with radius  $\underline{\quad}$  and

center \_\_\_\_\_.

Since  $f(z) = \frac{1}{z-R}$  is a fractional linear map, with  $a=0, b=1, c=1$ ,  
and we showed F.L maps (in Lecture) map  $d = -R$   
circles/lines to circles/lines, this also makes sense.

[3] half disk  $\{z \in \mathbb{C} \mid |z|=1, \operatorname{Im}(z)>0\}$

under map  $w = \varphi(z) = -\frac{1}{2} \left( z - \frac{1}{z} \right)$



Strategy 1: take  $\frac{dw}{dz} = \frac{1}{2} \left( 1 - \frac{1}{z^2} \right)$ ,  $\frac{dw}{dz} = 0$  when  $z=1, -1$   
these are critical points.

Taking the Taylor expansion

Strategy 2: can parametrize the boundary?

$$w = f(z) = -\frac{1}{2} \left( z \right) + \frac{1}{2z} = -\frac{z^2}{2z} + \frac{1}{2z} = \frac{1-z^2}{2z} = \frac{(z-1)(z+1)}{2z} = w$$

$$\begin{aligned} z &= \pm \sqrt{w^2 + 1} - w, \quad w := u + iv \\ &= \pm \sqrt{(u+iv)^2 + 1} - u - iv \end{aligned}$$

$$= \pm \sqrt{U^2 + 2iUV - V^2 + 1 - U - iV}$$

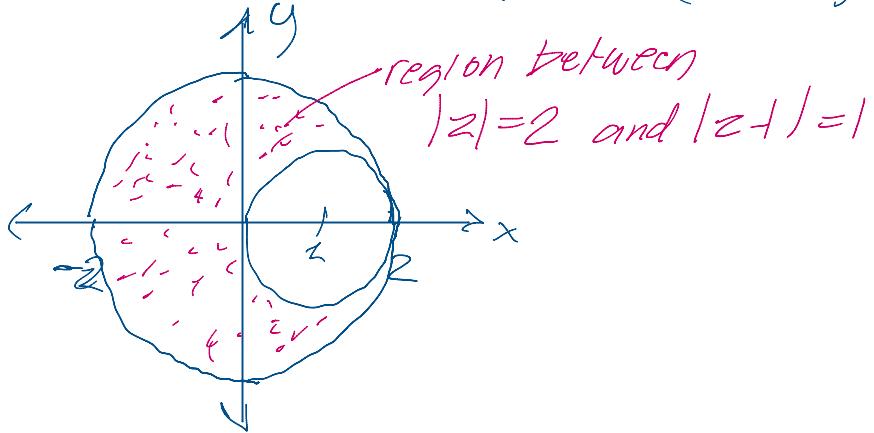
$$|z| = 1 \Rightarrow \text{???$$

4.\* Find a conformal mapping of the region "between" the circles:  $|z| = 2$  and  $|z - 1| = 1$  onto the unit disc.

5.\* Find a conformal mapping of the semi-infinite strip:  $x > 0, 0 < y < 1$  onto the unit disc.

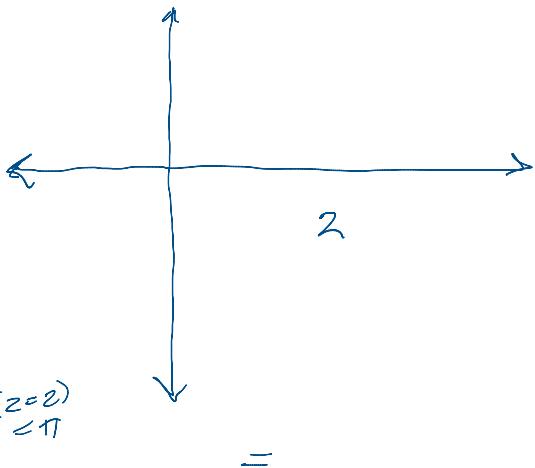
6.\* Find a conformal mapping of the semi-disc  $S = \{z : |z| < 1, \operatorname{Im} z > 0\}$  onto the unit disc.

13.4 first we sketch:  $|z| = 2 \rightarrow \sqrt{x^2 + y^2} = 2$   
 $x^2 + y^2 = 4$   
 $|z - 1| = 1 \rightarrow \sqrt{(x-1)^2 + y^2} = 1$



(S1) since the two circles are tangent at  $z=2$ ,  
try transformation  $f(z) = \frac{1}{z-2} = w$

this maps 2 circles to:



$$w = \frac{1}{z-2} = \frac{1}{|z|e^{i\theta}-2} = \frac{1}{2e^{i\theta}-2}$$

$$|z|=2$$

$$f(z)=\infty$$

$$f(2+0i)=\infty$$

$$\arg(z=2) < \pi$$

$$\frac{(2\cos\theta - i2\sin\theta - 2)(2\cos\theta + i2\sin\theta - 2)}{4\cos^2\theta} =$$

$$-4i\sin\theta\cos\theta$$

$$-4\cos^2\theta$$

$$+4i\sin\theta\cos\theta$$

$$-(2i)^2\sin^2\theta = 4\sin^2\theta + 4\cos^2\theta - 8\cos\theta + 4$$

$$-4i\sin\theta = 4(1) - 8\cos\theta + 4$$

$$-2(2\cos\theta) = \underline{-8\cos\theta + 8}$$

$$+2(2i)\sin\theta$$

$$+2(2)$$

separate real/img parts:

$$\Rightarrow \frac{2\cos\theta - 2i\sin\theta - 2}{8 + 8\cos\theta} = \frac{\cancel{2\cos\theta - 2}}{8 + 8\cos\theta} - \frac{2i\sin\theta}{8 + 8\cos\theta}$$

$$= \underbrace{\frac{1}{4} \frac{\cancel{8(\cos\theta + 1)}}{8(\cos\theta + 1)}}_{\text{Real}} - \frac{2i\sin\theta}{8 + 8\cos\theta}$$

$$\approx -\frac{1}{4}$$

$$|z-1|=1 \rightarrow \sqrt{(x-1)^2 + y^2} = 1 \quad r=1, \alpha=1$$

$$(x-1)^2 + y^2 = 1$$

$$(x-1)^2 + y^2 = 1$$

$\hookrightarrow (1+e^{i\theta})$  is form of  $z$

$$\begin{aligned} f(w) &= \frac{1}{z-2} = \frac{1}{(1+e^{i\theta})-2} = \frac{1}{e^{i\theta}-1} = \frac{1}{\cos\theta+i\sin\theta-1} (\cos\theta-i\sin\theta-1) \\ f(+1) &= \infty \\ &= \frac{\cos\theta-1-i\sin\theta}{2-2\cos\theta} \\ &= \frac{\cos\theta-1}{2(1-\cos\theta)} + \frac{-i\sin\theta}{2(1-\cos\theta)} \\ &\xrightarrow{\text{sum}} \frac{1}{2} [\text{real part}] \end{aligned}$$

now what's the image of these circles?

$$f(z) \text{ for } |z|=2 = \{ z \mid \operatorname{Re}(z) = -\frac{1}{2} \}$$

$$f(z) \text{ for } |z|=1 = \{ z \mid \operatorname{Re}(z) = -\frac{1}{2} \}$$

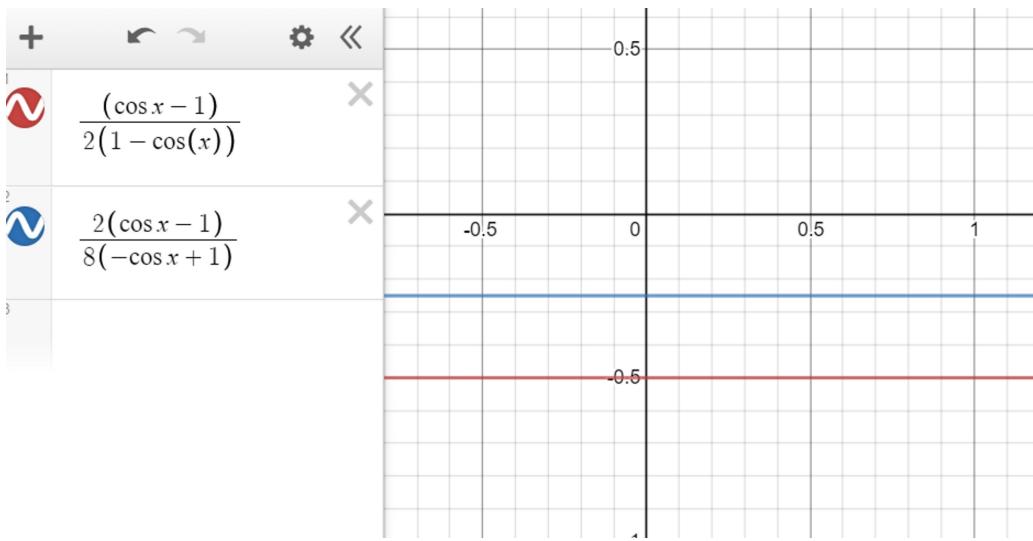
$$\text{when } \theta \in (0, \pi) \Rightarrow \sin\theta > 0$$

these lines intersect @  $f(z) = \infty$

, region bw 2 lns  
becomes region bw  
both crdes.

$$\text{call } f_1(z) = \frac{1}{z-2} = w_1$$

Plot of  $|z-1| = 1$ , real part:



Plot of real part of  $|z|=2$

Part 2 now need to translate strip to an appropriate place:

$$\begin{aligned} w_2 = f_2(z) &= z + i \frac{1}{2}, \quad z = x + iy \\ &\approx \operatorname{Re}(z) + i \operatorname{Im}\left(z + \frac{1}{2}\right) \\ &\text{move both up } \frac{1}{2}. \end{aligned}$$

Part 3 now want to rescale so that

**Exercise 3.** Let  $\text{GL}_2(\mathbb{C})$  be the group of invertible  $2 \times 2$  complex matrices. Show that

the map  $\Phi : \text{GL}_2(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{C}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$  is a group homomorphism, i.e.

$$(1) \quad \Phi(AB) = \Phi(A) \circ \Phi(B).$$

$$(2) \quad \Phi(A^{-1}) = (\Phi(A))^{-1}.$$

Is  $\Phi$  surjective? What is  $\ker \Phi := \Phi^{-1}(\text{id})$ ? You may use the result from Ex 1 in HW 7.

$$\mathcal{C} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{A}{\sim} \text{s.t. } A \text{ is invertible} \rightarrow \frac{az+b}{cz+d}, \quad ab - cd \neq 0$$

$$\textcircled{1} \quad \Phi \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right] = \begin{pmatrix} aa' + bc' & ab' + bd' \\ a'c + dc' & cb' + dd' \end{pmatrix}$$

$\det A \neq 0 \quad \det A' \neq 0$

but from linear  
 algebra  
 we know  
 $\det A \cdot \det A' = \det(AB)$   
 so  $\det(AA') \neq 0$ .

$$\rightarrow \frac{(aa' + bc')z + (ab' + bd')}{(a'c + dc')z + (cb' + dd')}$$

$$= \mathcal{C}(A) \circ \mathcal{C}(A')$$

$$= \left( \frac{az+b}{cz+d} \right) \circ \left( \frac{a'z+b'}{c'z+d'} \right) =$$

$$\underbrace{a \left[ \frac{a'z+b'}{c'z+d'} \right]}_{= f(z)} \stackrel{?}{=} \frac{aa'z+ab'}{c'z+d'} + \frac{bc'z+db'}{c'z+d'} = \frac{\underbrace{aa'z+ab'+bc'z+db'}_{c'z+d'}}{\underbrace{c'z+d'}_{(a'c+dc')z+(cb'+dd')}} = f(a'z+b') + d(f'(z))$$

$$\underbrace{c \begin{bmatrix} c'z + d \\ a'z + b' \\ c'z + d' \end{bmatrix}}_{c'z + d} + d = \frac{c(c'z + d') + d(c'z + d')}{c'z + d'} = \frac{(a'c + dc')z + (cb' + dd')}{c'z + d}$$

=

$$\frac{(aa' + bc')z + (ab' + bd')}{(a'c + dc')z + (cb' + dd')}$$

Thus part 1 is true in both directions.

Part 2 WTS:  $\varphi(A) = (\varphi(A))^{-1}$

$$\varphi: \begin{pmatrix} ab \\ cd \end{pmatrix} \mapsto \frac{az+b}{cz+d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\varphi \left( \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right) = \frac{\frac{dz-b}{ab-cd}}{\frac{-cz+a}{ab-cd}} = \frac{dz-b}{-cz+a}$$

and inverse of  $\frac{az+b}{cz+d}$  is  $\frac{dz-b}{-cz+a}$  (result from HWT.)

From (1), so second part true.

$$= \varphi(A)^{-1} = \left( \frac{az+b}{cz+d} \right)^{-1}$$

Thus,  $\varphi$  is a group homomorphism from  $GL_2(\mathbb{R})$  to  $\text{Aut}(\mathbb{C})$

General  
linear  
gp of



$2 \times 2$  matrix  
with real coeff.  
Complex coefficients

Is  $\varphi$  surjective? If  $\varphi: X \rightarrow U$  then  $\forall Y, \exists x \in X$  s.t.  $\varphi(x) = Y$

Is  $\varphi$  surjective? (i.e.: if  $\varphi: X \rightarrow Y$ , then  $\forall y \in Y, \exists x \in X$  s.t.  $\varphi(x) = y$ )

~~Not surjective.~~ There indeed exist fractional linear maps that are ~~not~~ represented by a  $2 \times 2$  invertible matrix with real coefficients.

ex:

$$\begin{matrix} & a=0 & b=1 & c= \\ a & & & \\ b & & & \\ c & d & 1 & 0 \end{matrix}$$

with  $ab - cd \neq 0$

It is surjective because for all fractional linear maps, there exists an invertible (non zero determinant)  $2 \times 2$  matrix with complex coefficients. Since bilinear transformations require  $ab - cd \neq 0$ , this is equivalent to the condition  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ , which is a condition for membership in  $GL_2(\mathbb{R})$ .

$\ker$

**Exercise 4.** Find all conformal automorphisms of the first octant

$$\mathcal{O} := \left\{ z \in \mathbb{C} \setminus \{0\} \mid 0 < \arg z < \frac{\pi}{4} \right\}.$$

(4)

**Exercise 5.** Let  $\mathbb{D}$  be the unit disk  $\{z \in \mathbb{C} \mid |z| < 1\}$ .

- (1) Show that if a conformal automorphism  $f : \mathbb{D} \rightarrow \mathbb{D}$  has two distinct fixed points in  $\mathbb{D}$ , then it must be the identity map  $f(z) = z$ .
- (2) Is it possible that a conformal automorphism  $f : \mathbb{D} \rightarrow \mathbb{D}$  has no fixed point in  $\mathbb{D}$ ?

**Exercise 6.** Use Schwarz's lemma to show that if  $f : D(0; R) \rightarrow \mathbb{C}$  is analytic with  $|f(z)| \leq M$  for some  $M, R > 0$ , then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)} \right| \leq \frac{|z|}{MR}.$$

Find all possible forms of  $f(z)$  if the equality holds at any point  $z \in D(0; R)$ .