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Summary: The goal for these notes was to show that for any triples of distinct points there exists a unique map such that these points get mapped to each other. So:

WTS: for any $(z_1, z_2, z_3)(w_1, w_2, w_3), \exists! f \in Aut(\overline{C})$ s.t. $f(z_k) = w_k$, for $k = 1, 2, 3$.

More On Fixed Points

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, where $\overline{\mathbb{C}}$ is the extended complex plane. Let $f(z_0) = z_0$ so that z_0 is a fixed point under the map f.

Definition: Index of a Fixed Point z_0:

index of the fixed point z_0 of f is defined by:

if $z_0 \neq \infty$ then

$$ind_{z_0} f := ord_{z_0}[f(z) - z]$$

if $z_0 = \infty$ then set $g(z) = \frac{1}{f(\frac{1}{z})} \to g(0)$ because $f(\infty) = \infty$ so that

$$ind_\infty f = ind_0 g$$

Fact from Geometry, Sum of Indices:

recall that meromorphic functions (analytic when a finite number of singularities are removed) can be written as a ratio of two polynomials:

$$f(z) = \frac{P(z)}{Q(z)}$$

where P, Q have no common roots. Then we set

$$d = max\{degP, degQ\}$$

Then

$$\sum_{f(z_0)=z_o, z_0\in\overline{\mathbb{C}}} Ind_{z_0} f = 1 + d$$

But in particular when f is a fractional linear transformation, then $d = max\{degP, degQ\} = 1$, so for FL:

$$\sum_{f(z_0)=z_o, z_0\in\overline{\mathbb{C}}} Ind_{z_0} f = 2 = \chi(\overline{C})$$

where χ is the Euler Characteristic of the extended complex plane considered as the Reimann Sphere, which is then considered as tetrahedron. Very very very cool.

noteL Should verify this fact if time allows.

Lemma: Existence of an Important and Unique Fractional Linear Map

$\exists!$ fractional linear map (so $f \in Aut(\overline{C})$) sending distinct$z_1 \to \infty, z_2 \to 0, z_3 \to 1$ in $\overline{\mathbb{C}}$.

Proof: The proof is just explicitly providing the map. In general the transformation $T(z)$ is:

$$T(z) = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$$

if $z_1 = \infty$ then

$$T(z) = \frac{z - z_2}{z_3 - z_2}$$

if $z_2 = \infty$ then

$$T(z) = \frac{z_3 - z_1}{z - z_1}$$

if $z_3 = \infty$ then

$$T(z) = \frac{z - z_2}{z - z_1}$$

This map T is the map we wanted.

Definition: Cross Ratio:

The `cross ratio` of four distinct points z_1, z_2, z_3, z_4 is written (z_1, z_2, z_3, z_4).

It is defined as $T(z_4)$ where T is the transformation shown in the Lemma's proof.

$$(z_1, z_2, z_3, z_4) := \frac{(z_4 - z_2)(z_3 - z_1)}{(z_4 - z_1)(z_3 - z_2)}$$

Proposition: Fractional Linear Maps Preserve the Cross Ratio:

Fractional Linear maps preserve the cross ratio.

Proof: 1. Let S be a fractional linear map. $S \in Aut(\overline{C})$. WTS that the cross ratio of z_1, z_2, z_3, z_4 is equal to the cross ratio of Sz_1, Sz_2, Sz_3, Sz_4. 2. Let T be the fractional linear map sending :

$$z_1 \to \infty$$

$$z_2 \to 0$$

$$z_3 \to 1$$

3. Then the composition $T \circ S^{-1}$ maps:

$$S(z_1) \to \infty$$

$$S(z_2) \to 0$$

$$S(z_3) \to 1$$

$$S(z_4) \to (Sz_1, Sz_2, Sz_3, Sz_4)$$

4. Then we get that:

$$(T \circ S^{-1})(Sz_4) = T(x_4) = (z_1, z_2, z_3, z_4)$$

5. Thus, (z_1, z_2, z_3, z_4) is mapped to (Sz_1, Sz_2, Sz_3, Sz_4) under an FL map. This is what we wanted show.

Theorem: Triples Map To Each Other Under Unique F.L Map:

This is the crowning jewel of these notes. This is what all the preliminary material was for:

For any two triples of distinct points z_1, z_2, z_3 and w_1, w_2, w_3, $\exists! f \in Aut(\overline{\mathbb{C}})$ s.t. $f(z_k) = w_k$, for $k = 1, 2, 3$

Proof:

Part 1: Existence

We have shown that there exists a unique map T sending:

$$z_1 \to \infty, z_2 \to 0, z_3 \to 1$$

and a map S sending:

$$w_1 \to \infty, w_2 \to 0, w_3 \to 1$$

such that $S^{-1} \circ T(z_k) = w_k$

Part 2: Uniqueness

For the uniqueness part, we use the cross ratio. Set $w = f(z)$ and then $(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$, which implies that

$$\frac{w - w_2}{w - w_1} \cdot \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_2}{z - z_1} \cdot \frac{z_3 - z_1}{z_3 - z_2}$$

Solving for w in terms of z, we get the formula for $f(z)$

Application

We can use the equation outlined in the existence part of the proof of the above theorem to write down an explicit map sending (z_1, z_2, z_3) to (w_1, w_2, w_3)

TODO on these notes:

1. add formal definition of order as I work
2. get explicit example of unique frac liner map