Reference / Summary Sheet for Complex Analysis

I've attempted to list everything I learned in an upper division course on Complex Analysis here for quick reference:

Complex Identities for a $z = x + iy$

$$cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$sin(z) = \frac{e^{i} - e^{-iz}}{2i}$$

and for $x \in R$:

$$sinh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$cosh(x) = \frac{e^{x} = e^{-x}}{2}$$

$$cosh^2(x) - sinh^2(x) = 1$$

so we can derive the complex hyperbolic functions:

$$sinh(z) = sinh(x)cos(y) + icosh(x)sin(y)$$

$$cosh(z) = TODO$$

$$tanh(z) = TODO$$

more identities of hyperbolic trig functions

List of Important Theorems and Definitions

still in progress, unordered, expanded on below.

- Rouche's Theorem
- Fundamental Theorem of Algebra
- Type I/II Integrals

- singularities
- zeroes
- residues
- Residue Theorem
- $ord_w f(z)$
- meromorphic function
- extended complex plane
- linear fractional
- Cauchy's Residue Theorem
- winding numbers
- Laurent Series
- poles
- order of a pole
- non-isolated singularity
- isolated singularity
- simply connected topological space
- path connected topological space
- hessian matrix and critical points
- Homotopy
- analytic branch of complex log
- principal branch of complex log
- ended hw 5 continue after this

Singular Points

- a singularity of f(z) is a point where it fails to be analytic. You can classify them:

Non-Isolated Singularities

1. Branch points: functions are not analytic here, so not analytic in a deleted neighborhood of a branch point
   - ex 1: $f(z) = \sqrt{z-3}$ has a branch point at $z = 3$
   - ex 2: $f(z) = ln(z^2 + z - 2) = 0$ has branch pts where $f(z) = 0$
2.

Isolated Singularities

- a point z_0 is an isolated singularity if you can find a deleted delta neighborhood around it with no other singularity. If you can't find a deleted neighborhood s.t. f is analytic around z_0, it is a non isolated singularity

1. TYPE 1: **poles/non-essential singularities**: a type of non-essential, isolated singularity. classify its order by : $\lim_{z\to z_0}(z-z_0)^n f(z) \neq 0$ but

$\lim_{z\to z_0}(z-z_0)^{n+1}f(z) \neq 0$ then f has a zero of order n at the pole z_0. The limit of $f(z)$ at a pole is ∞.

- pole classification techniques:
- if $f(z)$ rational function: factor the denominator (here poles are where f is dividing by zero), assured all zeroes of the denominator exist in the complex plane by FTA, if you find simple roots (distinct and constant), poles (the zeroes) have order 1.
  - ex: $\frac{1}{z^4+16}$
- factor the denominator. if you ses a double factor, probably a pole of order 2 (double pole)
  - ex: $\frac{1}{x^2+4x+4}$
- verify that your "guess" for the order of a pole is correct with the limit definition.
- use quadratic equation when you can't easily factor (valid for complex numbers): $z = \frac{-b\pm\sqrt{b^2-4ac}}{2}$
  - ex: $\frac{1}{z^2-z+1}$

2. TYPE 2: **Removable Singularities**: isolated / non essential singularity z_0 is removable if $\lim_{z\to z_0}$ exists and is not ∞. By this fact, f remains continuous and analytic at z_0

3. TYPE 3: **Essential Singularities** Any singular point that is not a pole or a removable singularity is an essential singularity. The limits of the function at essential singularities truly do not exist. (they're not ∞). Also, we cannot find an n s.t. $\lim_{z\to z_0}(z-z_0)^n f(z) \neq 0$

   - ex: $f(z) = e^{\frac{1}{z-2}}$ has an essential singularity at $z = 2$.

Strategy for problems of classification:

Step 1: Visualize the singularity, if provided. If the behavior tends to positive infinity or negative infinity (or both), it's probably a pole. If the behavior at the singularity of the function seems continuous, it might be a removable singularity. An essential singularity looks like neither a pole (strictly tending to positive/negative infinity) or a removable singularity.

Note that a non isolated singularity will have other singularities "really close" to the proposed singularity. If the function behaves erratically (ex: $tan(\frac{1}{z})$ at $z = 0$), it might be a non-isolated singularity. Always formally prove that it is non isolated by finding formulas for the other singularities. Once you have such a formula for "dividing by zero", take the limit

Step 2: After you have a guess, check the limit theorems of each type. A removable singularity should have a limit that exists at that point. A pole will yield an infinite limit. An essential singularity will yield a truly nonexistent limit.

Step 3: If you have a pole, verify the order with the limit equation.

Laurent Series Expansion

Laurent Series are made up of a an analytic sum and a principal sum. When expanding, make sure to account for both parts! Furthermore, while expanding, account for all possible domains of the series.

Essential Theory of Laurent Series:

- if $f(z)$ analytic on an annulus $R_1 < |z - z_0| < R_2$, then the function has a Laurent Series (which in special cases becomes a Taylor Series) $\forall z \in R$, where R is the region defined by the annulus (most generally, a punctured disk).
- Then Laurent Series (LS) are (Analytic part) + (Principal part) $\forall z \in R_1 < |z - z_0| < R_2$, formally:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} \forall z \in R_1 < |z - z_0| < R_2$$