Homework 9

Friday, November 27, 2020 3:44 PM

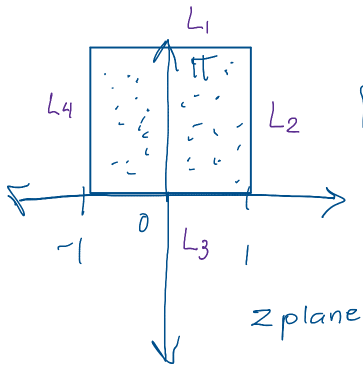
Exercise 1. Find the images of

(1) the rectangle $[-1,1]\times[0,\pi]$ under the map $f(z)=e^z$.

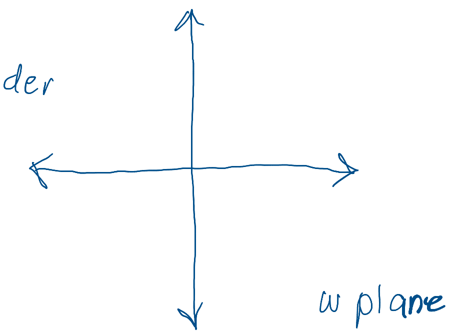
(2) the circle $|z|=R$ under $f(z)=\frac{1}{z-R}$.

(3) the half disk $\{z\in\mathbb{C}\mid |z|<1, \text{Im } z>0\}$ under $f(z)=-\frac{1}{2}(z+\frac{1}{z})$. (Hint: Set $f(z)=w$ and solve for z. For which values of w does this equation have a solution in the half disk?)

$\boxed{1}$



$[-1,1]\times[0,\pi]$ under $\xrightarrow{w=f(z)=e^z}$



Step 1: $w=f(z)=e^z=e^{x+iy}=e^xe^{iy}=e^x(\cos y+i\sin y)=e^x\cos y+ie^x\sin y$

$\Rightarrow w=u+iv$, $\left\{\begin{array}{l} u(x,y)=e^x\cos y \\ v(x,y)=e^x\sin y \end{array}\right.$

$u,v\in\{\mathbb{R}$-val functions$\}$

Step 2: see where w maps the boundary. $\{L_1\ L_2\ L_3\ L_4\}$ case by case.

L_1: line going from $(-1,\pi)$ to $(1,\pi)$, $y\equiv\pi$, $x\in[-1,1]$ varies within the interval.

then

$u:=e^x\cos y=e^x\cos\pi=e^x(-1)=-e^x$, x travels in $[-1,1]$.

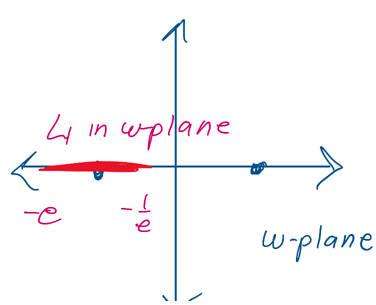
$v:=e^x\sin y=e^x\sin(\pi)=e^x\cdot 0=0$

L_1 mapped to a line where $v\equiv 0$, $-e^{-1}\leq u\leq -e$

L_2: line going from $(1,0)\longrightarrow(1,\pi)$

$u:=e^x\cos y=e\cos(y)$, $y\in[0,\pi]$

$v:=e^x\sin y=e\sin(y)$, $y\in[0,\pi]$



$-e \quad -\frac{1}{e}$ $\downarrow$ w-plane

from $w = u + iv$, $w = e\cos y + ei\sin y$

$= e(\cos y + i\sin y)$

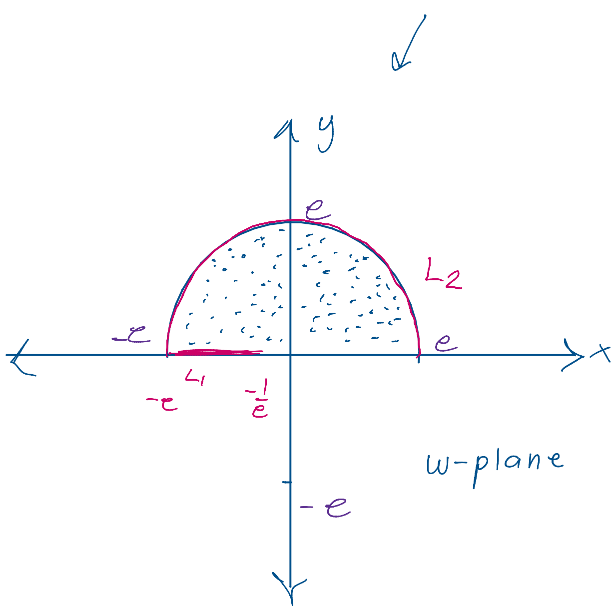
where $|w| = \sqrt{u^2+v^2} = \sqrt{e^2\cos^2 y + e^2\sin^2 y}$

$= \sqrt{e^2(1)}$

$= \pm e$, $\Rightarrow$ w is circle with $u^2+v^2 = e^2$, so

radius $= e$.

- but it's only a half circle because y stays in $[0, \pi]$.



L_3: line going from $(-1,0) \to (1,0)$

$u := e^x\cos y = e^x\cos(0) = e^x(1) = e^x$, $\quad -1 \le x \le 1.$

$v := e^x\sin y = e^x \sin 0 = 0 \quad \forall x \in [-1, 1]$

so L_3 gets mapped to a line $e^{-1} \le u \le e^1$, $v \equiv 0$.

L_4: line going from $(-1, 0)$ to $(-1, \pi)$

$u := e^x\cos y = e^{-1}\cos y \qquad 0 \le y \le \pi$

$u := e^x \cos y = e^{-1} \cos y \qquad 0 \le y \le \pi$

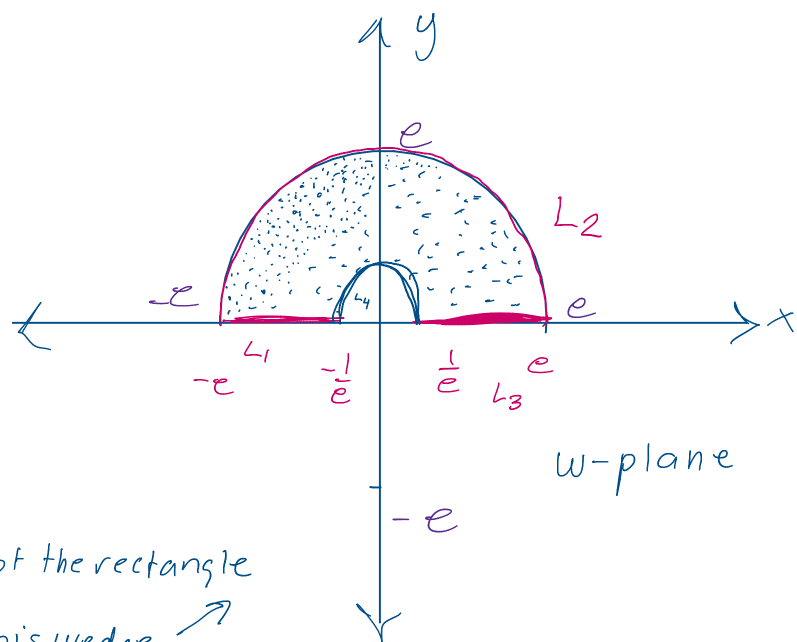
$v := e^x \sin y = e^{-1} \sin y \qquad 0 \le y \le \pi$

then $w = u + iv$

$= e^{-1}\cos y + i e^{-1} \sin y$

where $|w| = \sqrt{u^2+v^2} = \sqrt{e^{-2}\cos^2 y + e^{-2}\sin^2 y} = \sqrt{e^{-2}(1)} = \frac{\sqrt{1}}{\sqrt{e^2}} = \frac{1}{\sqrt{e^2}} = \frac{1}{e}$

so w is a circle centered at 0 with radius $= e^{-1}$



So the image of the rectangle under w is this wedge.

(region inside rectangle became region inside wedge).

$\boxed{2}$ $|z| = R \qquad f(z) = 1/z-R$

$z = \{re^{i\theta} \mid r = R\}$ and also $z = |z|e^{i\theta} = Re^{i\theta}$

so $f(z) = w = 1/Re^{i\theta} - R = (Re^{i\theta} - R)^{-1} = \frac{1}{R\cos\theta + i\sin\theta - R}$

so z becomes the set { TODO }, a circle with radius ___ and center____.

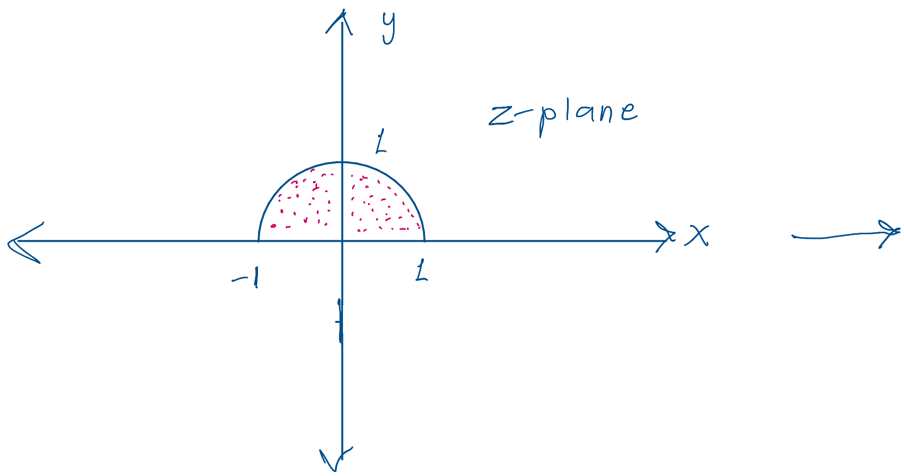
Since $f(z) = \dfrac{1}{z - R}$ is a fractional linear map, with $a=0, b=1, c=1, d=-R$

and we showed F.L maps (in Lecture) map

circles/lines to circles/lines, this also makes sense.

$\boxed{3}$ half disk $\{z \in \mathbb{C} \mid |z| = 1, \ Im(z) > 0\}$

under map $w = f(z) = -\frac{1}{2}\left(z - \frac{1}{z}\right)$



Strategy 1: take $\dfrac{dw}{dz} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$, $\dfrac{dw}{dz} = 0$ when $z = 1, -1$

these are critical points.

Taking the Taylor expansion

Strategy 2: can parametrize the boundary?

$$w = f(z) = -\frac{1}{2}(z) + \frac{1}{2z} = -\frac{z^2}{2z} + \frac{1}{2z} = \frac{1-z^2}{(2z)} = \frac{(z-1)(z+1)(-1)}{2z} = w$$

$$z = \pm\sqrt{w^2+1} - w, \quad \underline{w := u + iv}$$

$$= \pm\sqrt{(u+iv)^2 + 1} - u - iv$$

$= \pm \sqrt{u^2+2ivu-v^2+1} - u - iv$

$|z| = 1 \Rightarrow$???

4.* Find a conformal mapping of the region "between" the circles: $|z| = 2$ and $|z-1| = 1$ onto the unit disc.

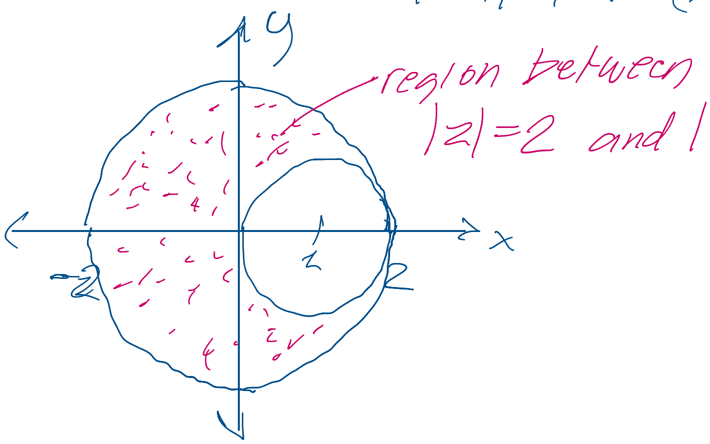
5.* Find a conformal mapping of the semi-infinte strip: $x > 0$, $0 < y < 1$ onto the unit disc.

6.* Find a conformal mapping of the semi-disc $S = \{z : |z| < 1, \operatorname{Im} z > 0\}$ onto the unit disc.

$\boxed{13.4}$ first we sketch: $|z| = 2 \to \sqrt{x^2+y^2} = 2$

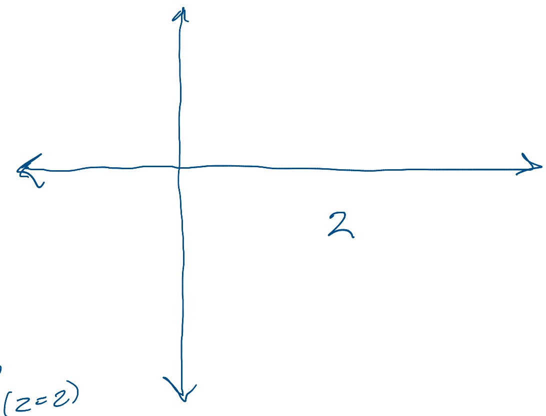
$x^2+y^2 = 4$

$|z-1| = 1 \to \sqrt{(x-1)^2+y^2} = 1$



S1 since the two circles are tangent at $z=2$,
try transformation $f(z) = \dfrac{1}{z-2} = w$

this maps 2 circles to:



$w = \frac{1}{z-2} = \frac{1}{|z|e^{i\theta}-2} = \frac{1}{2e^{i\theta}-2}$

$|z|=2$

$f(2)=\infty$

$f(2+0i)=\infty$

$\arg(z=2)=\pi$

$$\frac{(2\cos\theta - i2\sin\theta - 2)}{(2\cos\theta + i2\sin\theta - 2)(2\cos\theta - 2i\sin\theta - 2)} =$$

$4\cos^2\theta$ =

$-\cancel{4i\sin\theta\cos\theta}$

$-4\cos\theta$

$+\cancel{4i\sin\theta\cos\theta}$

$-(2i)^2\sin^2\theta \quad = 4\sin^2\theta + 4\cos^2\theta - 8\cos\theta + 4$

$-4i\sin\theta \quad = 4(1) - 8\cos\theta + 4$

$-2(2\cos\theta) \quad = -8\cos\theta + 8$

$+2(2i)\sin\theta$

$+2(2)$

separate real/imag parts:

$$\Rightarrow \frac{2\cos\theta - 2i\sin\theta - 2}{8-8\cos\theta} = \frac{2\cos\theta-2}{8+8\cos\theta} - \frac{2i\sin\theta}{8+8\cos\theta}$$

$$= \frac{1}{4}\frac{\cancel{2}(\cos\theta-1)}{\cancel{8}(\cos\theta+1)} - \frac{2i\sin\theta}{8+8\cos\theta}$$

Real $= -1/4$

$|z-1|=1 \rightarrow \sqrt{(x-1)^2+y^2}=1 \qquad r=1, \alpha=1$

$(x-1)^2+y^2=1$

$(\quad i\theta)$

$$(x-1)^2 + y^2 = 1$$

$\longleftrightarrow$ $(1+e^{i\theta})$ is form of z

$$f(w) = \frac{1}{z-2} = \frac{1}{(1+e^{i\theta})-2} = \frac{1}{e^{i\theta}-1} = \frac{1}{\cos\theta + i\sin\theta - 1}\frac{(\cos\theta - i\sin\theta - 1)}{(\cos\theta - i\sin\theta - 1)}$$

$f(1) = \infty$

$$= \frac{\cos\theta - 1 - i\sin\theta}{2 - 2\cos\theta}$$

$$= \frac{\cos\theta - 1}{2(1-\cos\theta)} + \frac{-i\sin\theta}{2(1-\cos\theta)}$$

$= -\frac{1}{2}$ real pt.

now what's the image of these circles?

$f(z)$ for $|z| = 2$ $= \{ z \mid \operatorname{Re}(z) = -\frac{1}{2} \}$

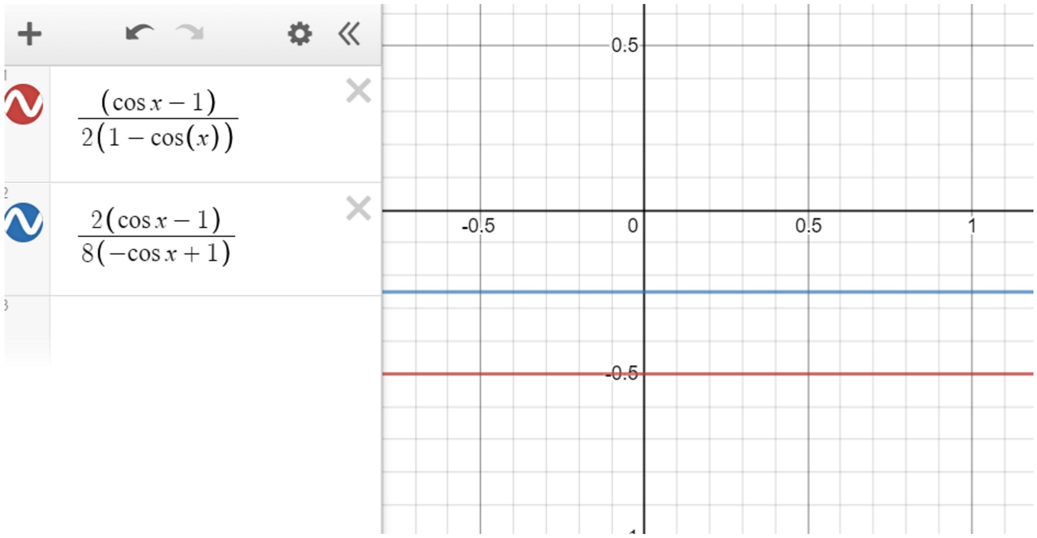
$f(z)$ for $|z| = 1$ $= \{ z \mid \operatorname{Re}(z) = -1/4 \}$

when $\theta \in (0, \pi) \Longrightarrow \sin\theta > 0,$

these lines intersect @ $f(z) = \infty$, region bw 2 lines becomes region bw both circles.

call $f_1(z) = \frac{1}{z-2} = w_1$

Plot of $|z-1| = 1$, real part:



Plot of real part of |z|=2

$\boxed{\text{Part 2}}$ now need to translate strip to an appropiate place:

$$w_2 = f_2(z) = z + i\frac{1}{2}, \quad z = x + iy$$

$$= \mathrm{Re}(z) + \mathrm{Im}\left(z + \tfrac{1}{2}\right)$$

moves both up $\frac{1}{2}$.

$\boxed{\text{Part 3}}$ now want to rescale so that

Exercise 3. Let $\mathrm{GL}_2(\mathbb{C})$ be the group of invertible 2×2 complex matrices. Show that the map $\Phi : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{Aut}\left(\overline{\mathbb{C}}\right), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{az+b}{cz+d}$ is a group homomorphism, i.e.

(1) $\Phi(AB) = \Phi(A) \circ \Phi(B)$.
(2) $\Phi(A^{-1}) = (\Phi(A))^{-1}$.

Is Φ surjective? What is $\ker\Phi := \Phi^{-1}(\mathrm{id})$? You may use the result from Ex 1 in HW 7.

$$\varphi : \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{=A} \text{ s.t. } A \text{ is invertible } \longmapsto \frac{az+b}{cz+d}, \quad ab-cd \neq 0$$

(1) $$\varphi\left[\underset{\det A \neq 0}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \cdot \underset{\det A' \neq 0}{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}\right] = \underbrace{\begin{pmatrix} aa' + bc' & ab' + bd' \\ a'c + dc' & cb' + dd' \end{pmatrix}}$$

but from linear algebra we know $\det A \cdot \det B = \det(AB)$ so $\det(AA') \neq 0$.

$$\longmapsto \frac{(aa' + bc')z + (ab' + bd')}{(a'c + dc')z + (cb' + dd')}$$

$$= \varphi(A) \circ \varphi(A')$$

$$= \left(\frac{az+b}{cz+d}\right) \circ \left(\frac{a'z+b'}{c'z+d'}\right) =$$

$$a\left[\frac{a'z+b'}{c'z+d'}\right] + b = \frac{aa'z + ab'}{c'z+d'} + \frac{bc'z + db'}{c'z+d'} = \frac{aa'z + ab' + bc'z + db'}{c'z+d'}$$

$$= [c(a'z+b') + d(c'z+d')] \quad (a'c + dc')z + (cb' + dd')$$

$$\frac{\left[c'z+d\right]}{c\left[\frac{a'z+b'}{c'z+d'}\right]+d} \quad \frac{c'z+d}{=\frac{c(a'z+b')+d(c'z+d')}{(c'z+d')\,(c'z+d')}} \quad \frac{c'z+d'}{\frac{(a'c+dc')z+(cb'+dd')}{c'z+d}}$$

$$=\frac{(aa'+bc')z+(ab'+bd')}{(a'c+dc')z+(cb'+dd')}$$

Thus part $\underline{1}$ is true in both directions.

$\underline{\text{Part 2}}$ $\underline{\text{WTS}}$: $\varphi(A^{-1})=(\varphi(A))^{-1}$

$\varphi:\begin{pmatrix}a&b\\c&d\end{pmatrix}\longmapsto\frac{az+b}{cz+d}$, $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $A^{-1}=\frac{1}{ab-cd}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$

$$\varphi\begin{pmatrix}\frac{d}{ab-cd}&\frac{-b}{ab-cd}\\\frac{-c}{ab-cd}&\frac{a}{ab-cd}\end{pmatrix}=\frac{\frac{dz-b}{ab-cd}}{\frac{-cz+a}{ab-cd}}=\frac{dz-b}{-cz+a}$$

and inverse of $\frac{az+b}{cz+d}$ is $\frac{dz-b}{-cz+a}$ (result from HW7.)

from sdns. so second part true.

$$=\varphi(A)^{-1}=\left(\frac{az+b}{cz+d}\right)^{-1}.$$

Thus, φ is a group homomorphism from $GL_2(\mathbb{R})$ to $Aut(\hat{\mathbb{C}})$

general liner grp of 2×R matrix w real coeff. complex coefficients

$\blacksquare$

Is φ surjective? (i.e.: if $\varphi: X\to Y$ then $\forall y, \exists x\in X$ s.t. $f(x)=y$)

Is φ surjective? (i.e: if $\varphi: X \to Y$, $\forall y \in Y$, $\exists x \in X$ s.t. $f(x) = y$)

~~Not surjective. There indeed exist fractional linear maps that are not represented by a 2x2 invertible matrix with real coefficients.~~

~~ex: $\longleftrightarrow$ $a = 0$ $b = 1$ $c =$~~

~~$\begin{matrix} a & b \\ c & d \end{matrix}$ $\quad$ $\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$~~

It is surjective because for all fractional liner maps, with $ab - cd \neq 0$ there exists an invertible (non zero determinant) 2x2 matrix with complex coefficients. Since bilinear transformations require $ab - cd \neq 0$, this is equivalent to the condition $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$, which is a condition for membership in $GL_2(\mathbb{R})$.

$\ker \varphi := \varphi^{-1}(ID)$? what is the map that sends all elements of a group homomorphism (φ) to the identity element, of the set of automorphisms on $\mathbb{C}$. (the ID element here is $\mathbb{1}$).

isn't this just the inverse?

$f := \dfrac{az + b}{cz + d}$

$f^{-1} := ?$

$f \circ f^{-1} = \underline{\mathbb{1}}$

Exercise 4. Find all conformal automorphisms of the first octant

$$\mathcal{O} := \left\{ z \in \mathbb{C} \backslash \{0\} \,\middle|\, 0 < \arg z < \frac{\pi}{4} \right\}.$$

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Exercise 5. Let $\mathbb{D}$ be the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

(1) Show that if a conformal automorphism $f : \mathbb{D} \to \mathbb{D}$ has two distinct fixed points in $\mathbb{D}$, then it must be the identity map $f(z) = z$.

(2) Is it possible that a conformal automorphism $f : \mathbb{D} \to \mathbb{D}$ has no fixed point in $\mathbb{D}$?

$\boxed{\textbf{Exercise 6.}}$ Use Schwarz's lemma to show that if $f : D(0;R) \to \mathbb{C}$ is analytic with $|f(z)| \leq M$ for some $M, R > 0$, then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)} \right| \leq \frac{|z|}{MR}.$$

Find all possible forms of $f(z)$ if the equality holds at any point $z \in D(0;R)$.