Lecture 1

Tuesday, August 18, 2020 10:50 PM

1. Historical Background

Why complex numbers?

Quadratic formula.

$ax^2+bx+c=0$, $a\neq 0$ a,b,c are real numbers

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This eqn has $\begin{cases} 2 \text{ rts} & \text{if } \Delta=b^2-4ac>0 \\ 1 \text{ rt} & \Delta=0 \\ 0 \text{ rt} & \Delta<0 \end{cases}$

When solving cubics (§1.3), we need to take sqrt of neg numbers, even when the eqn has 3 real rts.

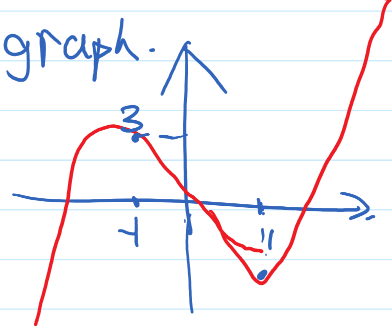
Ex: $x^3-3x+1=0$ set $f(x)=x^3-3x+1$

Claim: Has 3 real rts.

Pf: Find loc max/min sketch graph.

$f'(x)=3x^2-3=0 \Rightarrow x=\pm 1$

$f(1)=-1$ $f(-1)=3$



$\Rightarrow$ We have 3 real rts.

Cardano's formula.

$x=u+v \rightsquigarrow (u+v)^3-3(u+v)+1=0$

$$u^3+3u^2v+3uv^2+v^3-3u-3v+1=0$$

$$u^3+v^3+3(u+v)(uv-1)=-1.$$

$$\Leftarrow \begin{cases} u^3+v^3=-1 & ① \\ uv-1=0 \Rightarrow u^3v^3=1 & ② \end{cases}$$

$\Rightarrow v^3=\frac{1}{u^3}$.

Now solve for u^3 and v^3

plug $v^3=\frac{1}{u^3}$ into ① $u^3+\frac{1}{u^3}=-1$

$$u^6 + u^3 + 1 = 0$$

$$\Rightarrow u^3 = \frac{-1 \pm \sqrt{-3}}{2} \qquad v^3 = \frac{1}{u^3}.$$

$$u = \sqrt[3]{\frac{-1 + \sqrt{-3}}{2}} \Rightarrow v = \sqrt[3]{\frac{-1 - \sqrt{-3}}{2}}$$

$$\Rightarrow x = \sqrt[3]{\frac{-1+\sqrt{-3}}{2}} + \sqrt[3]{\frac{-1-\sqrt{-3}}{2}}$$ is a real solution to $x^3 - 3x + 1 = 0$

Consider $y'' - \lambda y = 0$

when $\lambda > 0$ $\quad y = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}$

$\lambda < 0$ $\quad y = C_1 \cos(\sqrt{-\lambda}\, x) + C_2 \sin(\sqrt{-\lambda}\, x)$

$\rightsquigarrow$ when $\lambda < 0$, indicates some relation between "$e^{\sqrt{\lambda} x}$" and trig fns.

Euler's identity $\quad e^{\sqrt{-1}\theta} = \cos\theta + \sqrt{-1}\sin\theta$

Gauss: answered question: when is an integer the sum of two perfect squares using $\sqrt{-1}$.

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2. Complex numbers.

$$\mathbb{C} := \{ (a,b) \mid a, b \in \mathbb{R} \}$$

\mathbb{C} $\qquad$ \mathbb{R}.

As a set, $\mathbb{C}$ is the same as $\mathbb{R}^2$.

arithmetic operations:

Addition/subtraction.

$$(a_1, b_1) \pm (a_2, b_2) = (a_1 \pm a_2,\ b_1 \pm b_2).$$

multiplication

$$(a_1, b_1)(a_2, b_2) := (a_1 a_2 - b_1 b_2,\ a_1 b_2 + a_2 b_1)$$

Usually we write $(a,b) = a + bi$ $\qquad \underline{i = \sqrt{-1}}$ "imaginary"

$$(a_1+b_1 i)\cdot(a_2+b_2 i) = a_1\cdot(a_2+b_2 i) + b_1 i\cdot(a_2+b_2 i)$$
$$= a_1a_2 + a_1b_2 i + b_1a_2 i + b_1b_2 i\cdot i$$
$$= a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1) i$$

Claim: when $(a,b)\neq(0,0)$, there is a unique (c,d) s.t.
$$(a,b)(c,d) = (c,d)(a,b) = (1,0).$$

$$(a+bi)(c+di) = 1$$
$$c+di = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$
$$\Rightarrow \quad c = \frac{a}{a^2+b^2}$$
$$d = \frac{-b}{a^2+b^2}.$$

Complex numbers have division!
$$\frac{a+bi}{c+di} = \cdots\cdots$$

Check: multiplication is associative.

Now let's solve quadratic eqns. w/ complex coeffs.

$$ax^2+bx+c=0 \qquad a\neq 0$$
$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Now need write $\sqrt{b^2-4ac}$ as $x+yi$

Set: $b^2-4ac = u+vi$ $\qquad u+vi = (x+yi)^2$
$$= x^2-y^2+2xyi$$
$$\Rightarrow \begin{cases} u = x^2-y^2 \\ v = 2xy \end{cases} \qquad \text{solve for } x \text{ and } y$$

Example $\sqrt{i} = \pm\left(\frac{1+i}{\sqrt{2}}\right)$

Cubic eqns: Use Cardano's formula.

Q: How about eqns of higher degrees?
This question has 2 parts.

(1). Given a $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0 \quad a_i \in \mathbb{C}$
Is there a complex number w that solves this eqn.

(2) Is there a formula like in the quadratic/cubic) for w?

(1) was answered by Gauss:

Fundamental thm of alg: (1) is true (Chapter 5).

(2) was answered by Galois:

When $n \geq 5$, there's no formula using $+ - \times \div \sqrt[m]{\ }$ for w.

(Abstract algebra).

3. Complex planes.

$\mathbb{C} = \mathbb{R}^2$ as vector spaces.

$z = x + yi \longleftrightarrow (x, y) \in \mathbb{R}^2$.

$\mathrm{Re}\, z := x$ - coordinate.

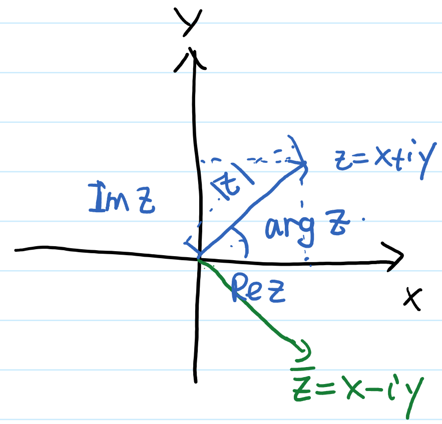
$\mathrm{Im}\, z := y$ - coordinate.

modulus $|z| :=$ length of $(x, y) = \sqrt{x^2+y^2}$.

$= \sqrt{z \cdot \bar{z}}$

Argument $\arg z = \theta \mod 2\pi$

Conjugate $\bar{z} := x - iy$



Sketch some subsets of $\mathbb{C}$

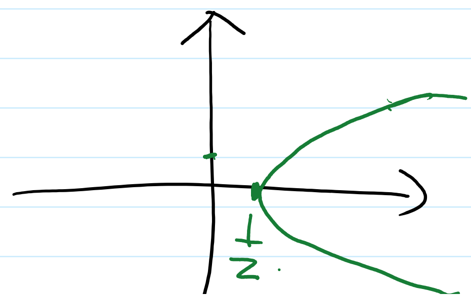
$$\{ z \mid \mathrm{Re}\, z = |z-1| \}$$

$$x = |x - 1 + iy|$$

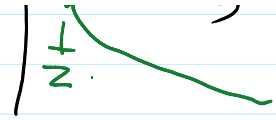
$$= \sqrt{(x-1)^2 + y^2}$$

$$x^2 = (x-1)^2 + y^2$$

$$= x^2 - 2x + 1 + y^2$$

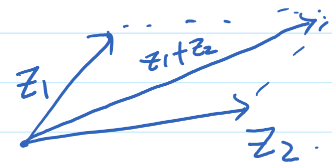


$$x^2 = (x-1)^2 + y^2$$
$$= x^2 - 2x + 1 + y^2$$
$$\Rightarrow \quad y^2 = 2x - 1$$



Geometric meanings of arithmetic ops.

$+$: vector addition



multiplication: $z_1 z_2 = \begin{cases} \text{length } |z_1 z_2| \\ \text{direction } \arg z_1 z_2 \end{cases}$

$$z_j = x_j + i y_j \qquad j = 1, 2.$$

- $|z_j| = \sqrt{x_j^2 + y_j^2}$ $\qquad$ Check $|z_1 z_2| = |z_1|\,|z_2|$
- For $\arg z_1 z_2$ $\qquad z_1 = |z_1| \cdot \frac{z_1}{|z_1|}$

$\frac{z_1}{|z_1|}$ is a unit vector / complex number.

$$\frac{z_1}{|z_1|} = \cos\theta_1 + i\sin\theta_1 \qquad \theta_1 = \arg z_1.$$
$$=: \operatorname{cis}(\theta_1)$$
$$\frac{z_2}{|z_2|} = \cos\theta_2 + i\sin\theta_2.$$

$$\frac{z_1}{|z_1|} \cdot \frac{z_2}{|z_2|} = (\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2).$$

$$= \underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)} + \underbrace{(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)}_{\sin(\theta_1+\theta_2)} i$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2).$$

$$\arg(z_1 \cdot z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

$$\Rightarrow \quad z_1 \cdot z_2 = \begin{cases} \text{length } |z_1 z_2| = |z_1| \cdot |z_2| \\ \text{argument } \arg(z_1 z_2) = \arg z_1 + \arg z_2. \end{cases}$$

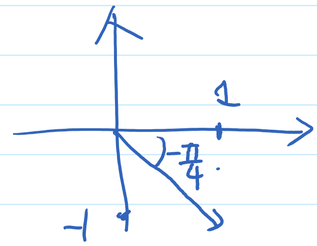
De Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \begin{cases} \text{length } 1 \\ \arg\ n\cdot\theta \end{cases}$$

$$= \cos n\theta + i \sin n\theta.$$

Example: $(1-i)^{10}$ $z = 1-i$ $|z| = \sqrt{2}$

$\arg z = -\frac{\pi}{4} = \frac{7\pi}{4}$



$$z^{10} = \begin{cases} |z^{10}| = |z|^{10} = \sqrt{2}^{10} = 32 \\ \arg z^{10} = 10\cdot\arg z = -\frac{5\pi}{2} = -\frac{\pi}{2} \end{cases}$$

$$(1-i)^{10} = -32i$$

Example: Find all cubic rts of unity $z^3 = 1$

$|1| = 1$ $\arg(1) = 0$

$\Rightarrow$ $|z|^3 = 1$ $\arg z^3 = 0$

$|z| = 1$ $3\arg z = 0$ mod 2π

$\Rightarrow \arg z = 2n\pi,\ 2n\pi + \frac{2\pi}{3},\ 2n\pi + \frac{4\pi}{3}$

$\Rightarrow$ $z = 1,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$= 1,\ \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Alternatively $z^3 = 1 \Rightarrow z^3 - 1 = 0$

$(z-1)(z^2+z+1) = 0$

match!

Now solve $z^2+z+1=0 \Rightarrow z = \frac{-1 \pm \sqrt{-3}}{2}$

More generally if $z^3 = r$, $z = \sqrt[3]{r},\ \omega\sqrt[3]{r},\ \omega^2\sqrt[3]{r}$

For general deg n. if $z^n = 1$

then $\begin{cases} |z| = 1 \\ n\arg z = 0 \Rightarrow \arg z = \frac{2k\pi}{n} \quad 0 \le k \le n-1 \end{cases}$

$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \quad 0 \le k \le n-1.$