Lectures 12-13

Traces * It is useful to have numerical invariants measuring the complexity of linear maps * we already have some discrete (= integer invariants) - for every linear map

$$T : V \mapsto W$$

- we have two integers cpaturing information about T (transformation) + **nullity of T:** = dim Kernel(T) = dim Nullspace(T) = dim of the solution set to

$$Ax = 0$$

+ **Nullspace (T)**: set of all n-dimensional column vectors such that

$$Ax = 0$$

, the solution set of the homogenous linear system. * ***Theorem***: The nullspace N(A) is a subspace of the vector space

$$\mathbb{R}^{\ltimes}$$

* proof: WTS: N(A) is nonempty, closed under addition, closed under scalar multiplication: * S1: the trivial solution is always in N(A)- so it's nonempty.

$$\vec{x} = \vec{0}$$

* S2: WTS:

$$x, y \in N(A) => x + y \in N(A)$$

* Well,

$$Ax = 0, Ay = 0, A(x + y) = A(x) + A(y) = 0 + 0 = 0$$

S3:

$$c \in \mathbb{R}, x \in N(A) => cx \in N(A)$$

Well,

$$A(cx) = c * A(x) = c * 0 = 0$$

* QED + **rank of T**: dim image(T) = ... QUESTION: any other defs? - turns out that for linear operators

$$T : V \mapsto V$$

we also have refined invariants which are scalars of the field

$$\mathbb{F}$$

+ ex: **Trace**:

$$tr : L(V, V) \mapsto \mathbb{F}$$

* the sum of elements on the main diagonal of a square matrix A * the sum of its complex eigenvalues * invariant with respect to change of basis * trace with this def applies to linear operators in general * is a linear mapping:

$$tr(T + S) = tr(T) + tr(S)$$

and

$$tr(cT) = c * tr(T)$$

- notice inside L(V,V) (linear maps from V to V) we have a natural collection of linear operators, from each one we can get a scalar back. * how can we get this scalar? * given any pair (f,v) where *

$$v \in V$$

is a vector *

$$f \in V^v$$

is a linear functional in the dual space = the space of all linear functionals from V to the scalar field * we can construct a linear operator: -

$$s_{f,v} : V \mapsto V, x \mapsto f(x)v$$

QUESTION: doesnt this give me a vector back? * but given (f,v) we can also get a natural scalar: -

$$f(v) \in \mathbb{F}$$

* with this in mind we can form and prove the existence statement: * ***Lemma***: * Suppose V is finite dim vector space over

$$\mathbb{F}$$

* Then there exists a unique linear function: -

$$tr : L(V, V) \mapsto \mathbb{F}$$

- such that for all

$$v \in V$$

and

$$f \in V^v$$

-

$$tr(s_{f,v}) = f(v)$$

* proof of lemma: - fundamental fact: every linear function (any linear transformation) is uniquely determined by what it does to a basis (by its values on a basis) - from this fact, it suffices to constrct a basis of all linear functions from V to V,

$$L(V, V)$$

that consists of operators of the form

$$s_{f,v}$$

for the chosen f's and v's - Let

$$\mathbb{B} = \{b_1.......b_n\} \subset V$$

be any basis of V - Let

$$\mathbb{B}^v = \{b_1^v.......b_n^v\} \subset V$$

be its dual basis - Then we can say that the collection of operators -

$$\mathbb{S} = \{s_{b_1{}^v, b_1}.......s_{b_n{}^v, b_n}\}$$

is a basis of

$$L(V, V)$$

the set of all linear functions from V to V * basis = spanning + linearly independent.

Lecture 14: Row Reduction

Outline 1. Simplifying Linear Systems 2. Row Reduction and Echelon Forms 3. Solving Systems with Row Reduction 4. Corollaries

** Solving a Linear System ** * using row and column operations we can convert every linear system into a system in which all variables separate - *row operation*: - *column operation*:

Extra notes/defs to categorize later

Dual Spaces and Dual Basis * The dual space of V is the set of all linear functionals from V to

$$\mathbb{F}$$

, so :

$$V^v = T : V \mapsto \mathbb{F}$$

- all such elements of dual space are linear functionals - if dim(V) <

$$\infty$$

=>

$$V$$

and

$$V^v$$

are isomorphic + to show this is true, show that they have the same dimension + another way to show the isomorphism is to use the dual basis * linear extension theorem: says if you know what T does on basis vectos, you know what T does on every vector * Let

$$\mathbb{B} =$$

* enough to know what

$$f(v_1), ..., f(v_n)$$

is

Isomorphism * mapppings that are injective and surjective (1:1 and onto)