# Problem of Joint Unit Root Test and Inconsistent Estimators in Over-Differenced Model

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## 1.0 Introduction to the Problem

Stationarity is one of the core assumptions that is required in most time series research methodologies. To this end, researchers often have to ensure the presence of stationarity in the time series variables that they're dealing with and the go-to options to conduct such examination would be using the Augmented Dickey-Fuller (*ADF*) Test introduced in the paper prepared by Said and Dickey (1984) and the Kwiatkowski-Phillips-Schmidt-Shin (*KPSS*) Test, by Kwiatkowski, Phillips, Schmidt and Shin (1992).

However, past studies have shown that both tests have problematic finite sample property and this renders to the practice of using both tests simultaneously in determining if a particular time series is indeed stationary. In order to demonstrate, consider a stochastic process, $\{y_t\}_{t=1}^{T}$, time series researchers tend to subject this series to both *ADF* and *KPSS* Test and would make conclusion based on the decision matrix as below:

```
##                     KPSS: Stationary KPSS: Non-Stationary
## ADF: Stationary     "Stationary"     "Non-Stationary"
## ADF: Non-Stationary "Non-Stationary" "Non-Stationary"
```

Thus, using such *Join-Test* approach, researchers would declare a series as a stationary process *if and only if* both tests indicates so. Otherwise, the series would be treated as a unit root process and to be taken the first difference before moving to the next stage of the analysis. Example of published papers that employed such *Join-Test* approach would be Ha and Shin (2021), Bernardi and Ricciuti (2021), among others.

Intuitively, such practice narrows the likelihood of one concluding stationarity. Thus, this implies that researchers would have to be more tolerant toward the practice of over-differencing. Nevertheless, it is not theoretically justified and in fact only worsens the already poor finite sample performance of these tests, and this project is to demonstrate the downside of using such procedure and also to point out the undesired consequence of over-differencing by conducting Monte Carlo (*MC*) Simulations.

## 2.0 The Downside of *Join-Test* Approach and Monte Carlo (*MC*) Setup

Given a simple first-order autoregressive model as followed:

$$(1-\phi L)y_t = \mu_y + \epsilon_t, \ \ t = 1, 2, \cdots, T \tag{1}$$

Let $\mu_y = E[y_t]$, $L$ be the lag operator and $\{\epsilon_t\}_{t=1}^{T} \sim_{WN} (0, \sigma_\epsilon^2)$ with $WN$ be the short for "White Noise". Then, the particular time series is stationary *if and only if* $|\phi| < 1$ and is a unit root process for when $|\phi| = 1$. To this end, we'll conduct a *MC* Simulation to examine the performance of *ADF*, *KPSS* and the *Join-Test* Approach in distinguishing a stationary from unit root processes.

The *MC* setup is as followed:

- Generate a generic *AR(1)* Model:

$$y_t = \mu_y + \phi y_{t-1} + \epsilon_t, \quad t = 1, 2, \cdots, T \tag{2}$$

- Subject the generic series to three different testing procedures, and create a bernoulli variable, $r$ that:

$$r = \begin{cases} 1, if \;\; correct \;\; classification \\ 0, if \;\; otherwise \end{cases} \tag{3}$$

Then, this process will be repeated for $m$ = 1,000 *MC* trials. Here, this process will be conducted for parameters, T $\in$ {100, 200, 300, 400, 500} and $\phi \in \{0.850, 0.900, 0.925, 0.950, 0.975, 1.000\}$. Specifically, the smaller $T$ is to mimic the scenario in which we've scarce observations and $\phi \rightarrow 1$ to reflect the behavior of these tests for when the autoregressive parameter appraoches the unit root boundary. Also note that, we've $r \in \mathcal{R}^{mX3}$, where each column in matrix $r$ records the prediction of each of the three tests, namely, *ADF*, *KPSS* and *Join-Test* Approach in each *MC* trial. Thus, let $r_{adf}$ denotes the column in matrix $r$ that records the performance of *ADF* test, then, the probability of correct classification is then estimated as:

$$\mathcal{P}(\widehat{Correct}_{adf}) = \frac{\sum_{i=1}^{m} r_{adf,i}}{m} \tag{4}$$

Similar formula applies to compute the estimated probability of correct classification for *KPSS* and the *Join-Test* Approach. In order to replicate this study, R Packages, `tseries` and `tidyverse` are required.