approximations, called Fraunhofer and Fresnel diffraction, will be discussed in the next two chapters.

The appendices of this chapter contain a formal derivation of the Huygens–Fresnel integral. The details contained in these derivations are not required to understand diffraction, but will provide a more formal derivation of the diffraction integral for the reader not satisfied with the intuitive approach taken above.

GAUSSIAN BEAMS

We have just introduced a view of diffraction that treats the propagation of light by an integral method, as modifications made on an input beam. A second approach to describing the propagation of light is to develop the differential equation of the optical system. The output of the system is then obtained, given the input wave from the solutions of the differential equation. For free space, the differential equation that must be satisfied to determine the spatial behavior of a wave is the Helmholtz equation.

We will find that if we make the paraxial approximation, solutions of the Helmholtz equation, with Gaussian amplitude distributions, are obtained. We will show that the waves described by these particular solutions can be characterized by two simple parameters. These parameters are the beam waist and the radius of curvature of the phasefront of the wave. The *beam waist* is defined as the half-width at an amplitude equal to $1/e$ of the maximum amplitude of the wave, transverse to the propagation direction. The *radius of curvature* describes the radius of curvature of the phasefront of the wave, as measured from the position of minimum beam waist.

The mathematical manipulations required to obtain the characteristic parameters of a Gaussian wave are long, but not complicated. The final results are worth the effort in that they allow the inclusion of diffraction in the analysis of complicated optical systems.[19] The transverse amplitude distribution of optical beams from lasers has a Gaussian amplitude distribution, as do the propagation modes of some optical fibers and the cavity modes of Fabry–Perot resonators with spherical mirrors.

To derive the characteristics of a Gaussian wave, we use the paraxial approximation. A plane wave is assumed to be propagating nearly parallel to the z direction, allowing it to be described by a scalar wave of the form

$$\mathcal{E}(\mathbf{r}) = \Psi(x, y, z)\, e^{-ikz} \tag{9-21}$$

i.e., the wave does not propagate in the x or y direction.

We will substitute this general wave into the Helmholtz equation to produce the scalar, paraxial, wave equation that must be solved

$$\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) e^{-ikz} + k^2\Psi e^{-ikz} - 2ik\frac{\partial\Psi}{\partial z}e^{-ikz} - k^2\Psi e^{-ikz} = 0 \tag{9-22}$$

The assumption that Ψ changes very slowly with z (linearly with z will do it) allows $\partial^2\Psi/\partial z^2$ to be neglected. The resulting scalar wave equation is called the *paraxial wave equation*

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - 2ik\frac{\partial\Psi}{\partial z} = 0 \tag{9-23}$$

The paraxial wave equation and its solutions lead to a description that can be shown to be equivalent to the Fresnel description of diffraction.[43]

We assume that the solution of **(9-23)** has the form

$$\Psi = e^{-iQ(z)(x^2+y^2)}\, e^{-iP(z)} \tag{9-24}$$

In this form, we will find it easy to discover the requirements on P and Q when **(9-24)** is a solution of **(9-23)**; however, in this form, **(9-24)** does not appear to be a plane wave with a Gaussian amplitude distribution, but by proper specification of $P(z)$ and $Q(z)$, **(9-24)** will assume a Gaussian form. We will discover that $Q(z)$ must be a complex variable associated with the reciprocal of the Gaussian width and that $P(z)$ must contain information on the phase of the wave.

By the selection of **(9-24)** as the solution of **(9-23)**, we have implicitly assumed that the transverse dependence of the wave is only a function of $(x^2 + y^2)$. The assumption results in the simplest Gaussian wave solutions, i.e., waves that have circular symmetry. We will not discuss the higher-order Gaussian beam modes that arise by eliminating this assumption.[43]

We first determine the derivatives of **(9-24)**, to be substituted into **(9-23)**.

$$\frac{\partial^2\Psi}{\partial x^2} = -4x^2Q^2\Psi - 2iQ\Psi, \qquad \frac{\partial^2\Psi}{\partial y^2} = -4y^2Q^2\Psi - 2iQ\Psi$$

$$\frac{\partial\Psi}{\partial z} = -i\frac{\partial P}{\partial z}\Psi - i(x^2 + y^2)\frac{\partial Q}{\partial z}\Psi \tag{9-25}$$

Substituting **(9-25)** into **(9-23)** results in an expression that will yield the required form of $P(z)$ and $Q(z)$

$$-4(x^2 + y^2)Q^2 - 4iQ - 2k\frac{\partial P}{\partial z} - 2k(x^2 + y^2)\frac{\partial Q}{\partial z} = 0 \tag{9-26}$$

Since **(9-26)** must hold for all values of x and y, we may equate the coefficients of the different powers of x and y to zero

$$k\frac{\partial P}{\partial z} + 2iQ = 0 \tag{9-27}$$

$$2Q^2 + k\frac{\partial Q}{\partial z} = 0 \tag{9-28}$$

[Equation **(9-28)** is called Riccati's equation.]

We make a change of variables

$$q = \frac{k}{2Q} \tag{9-29}$$

In a moment, q will be identified as the desired Gaussian width of the amplitude distribution of the wave. Using the new variable, we may write

$$\frac{\partial q}{\partial z} = -\frac{k}{2Q^2}\frac{\partial Q}{\partial z}$$

We can use this new variable to rewrite **(9-27)** and **(9-28)** as

$$\frac{\partial q}{\partial z} = 1 \tag{9-30}$$

$$\frac{\partial P}{\partial z} = \frac{-i}{q} \qquad \text{(9-3}$$

Equation **(9-30)** integrates to give

$$q = q_0 + z \qquad \text{(9-3}$$

where we have chosen the constant of integration q_0 to be purely imagina By moving q_0 off the real axis, we remove the singularity that would occ in $\partial Q/\partial z$ if $q_0 = -z_0$. Requiring q to be complex also makes it possible interpret **(9-24)** as a wave with a Gaussian amplitude distribution.

If q is known in one plane, we can calculate q in a plane, a distance away, by using **(9-32)**. The solution of the paraxial wave equation provides capability similar to that provided by Huygens' principle. The determinati of new properties of a wave can be made using the old properties of t wave.

The derived form of q **(9-32)** can be substituted into **(9-31)**

$$\frac{\partial P}{\partial z} = \frac{-i}{q_0 + z}$$

to obtain the function $P(z)$.

$$P(z) = -i \ln\left(1 + \frac{z}{q_0}\right) \qquad \text{(9-3}$$

The results obtained for $P(z)$ and $1/Q(z)$ can now be substituted into **(9-2** to produce the wave solution of the paraxial Helmholtz equation

$$\Psi = \exp\left\{-i\left[-i \ln\left(1 + \frac{z}{q_0}\right) + \frac{k}{2(q_0 + z)}(x^2 + y^2)\right]\right\} \qquad \text{(9-3}$$

Since q_0 is imaginary, it is possible to make the following substitution:

$$\ln\left(1 + \frac{z}{q_0}\right) = \ln\left(1 - \frac{iz}{q_0}\right)$$

where q_0 is now a real quantity. The identity

$$\ln(x \pm iy) = \ln\sqrt{x^2 + y^2} \pm i \tan^{-1}\left(\frac{y}{x}\right)$$

can now be used to obtain a new formulation of **(9-34)**

$$\Psi = \frac{1}{\sqrt{1 + (z/q_0)^2}} \exp\left[-\frac{kq_0(x^2 + y^2)}{2(z^2 + q_0^2)}\right] \exp\left[i \tan^{-1}\left(\frac{z}{q_0}\right) - \frac{ikz(x^2 + y}{2(z^2 + q_0^2}\right. \qquad \text{(9-}$$

Evaluating **(9-35)** at $z = 0$ allows the identification of the physi significance of the amplitude of Ψ. At $z = 0$, **(9-35)** reduces to

$$\Psi_0 = \exp\left[-\frac{k(x^2 + y^2)}{2q_0}\right] \qquad \text{(9-}$$

Equation **(9-36)** is a Gaussian function, as we can see by comparing it w a Gaussian spatial amplitude distribution, given by

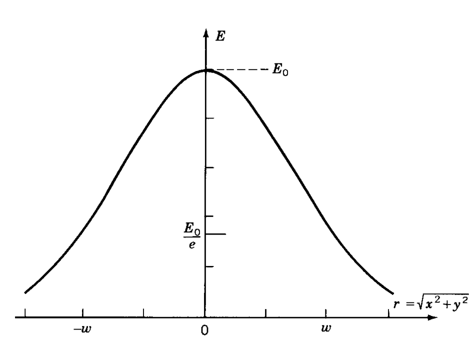

FIGURE 9-10. Gaussian wave of width w and height E_0.

$$E = E_0 \exp[-(x^2 + y^2/w^2)]$$

and plotted in Figure 9-10. The comparison demonstrates that Ψ can be interpreted as a wave, whose transverse amplitude distribution is a Gaussian.

The parameter w of the Gaussian distribution equals the half-width of the Gaussian function at the point where the amplitude is $1/e$ of its maximum value (see Figure 9-10). We define $w = w_0$ as the minimum half-width of the Gaussian function. Comparing the Gaussian function with the equation for Ψ_0 allows the definition of a minimum width for the Gaussian wave, called the *minimum beam waist*, in terms of q_0

$$w_0^2 \equiv \frac{2q_0}{k} \qquad \text{(9-37)}$$

The connection between q_0 and the minimum beam waist has been established when $z = 0$; thus, the coordinate system must have its origin at the minimum beam diameter. The real valued constant

$$q_0 \equiv \frac{kw_0^2}{2} = \frac{\pi w_0^2}{\lambda} \qquad \text{(9-38)}$$

is often called the *confocal parameter* (its significance will be identified in a few moments). The complex constant of integration

$$q_0 = iq_0$$

introduced in **(9-32)** can now be interpreted as the minimum value of the *complex size parameter* of a Gaussian wave

$$q_0 = \frac{i\pi w_0^2}{\lambda} \qquad \text{(9-39)}$$

This analysis allows the association of the complex function q with the physical variables of the beam. With this association, the physical interpretation of **(9-32)** is that as z increases, the beam variables evolve. The minimum value of the parameter is the minimum beam waist w_0 and its position is established as the origin of the coordinate system. The value of q at a distance z from the beam waist is obtained from **(9-32)**

$$q = q_0 + z = z + \frac{i\pi w_0^2}{\lambda}$$

Using the definition of q_0 given in **(9-38)**, we can write **(9-35)** as

$$\Psi = \frac{1}{\sqrt{1 + (\lambda z/\pi w_0^2)^2}}$$

$$\exp\left\{-\frac{(x^2+y^2)}{w_0^2\left[1+(\lambda z/\pi w_0^2)^2\right]}\right\}\exp\left\{i\tan^{-1}\left(\frac{\lambda z}{\pi w_0^2}\right) - \frac{i\pi(x^2+y^2)}{\lambda z\left[1+(\pi w_0^2/\lambda z)^2\right]}\right\} \tag{9-40}$$

We began our analysis by assuming a paraxial wave, i.e., the wave must propagate in a direction that is nearly parallel with the z axis. Examination of that assumption will allow us to assign physical significance to the quantities contained in **(9-40)**. We allow the wave of interest to be a spherical wave, propagating away from the origin, with a wavefront of radius of curvature $R(z)$ a distance z from the origin, as is shown in Figure 9-11.

The assumption of a paraxial wave implies that this spherical wave can be approximated by a *paraxial, spherical wave* of the form

$$\frac{1}{R}e^{-ikR} = \frac{1}{R}\exp\left[-ik\sqrt{z^2+x^2+y^2}\right]$$

$$\approx \frac{1}{R}e^{-ikz}\exp\left[-ik(x^2+y^2/2z)\right]$$

where were have assumed that $z^2 >> x^2 + y^2$. The paraxial assumption is that $z \approx R$, so that

$$\frac{1}{R}e^{-ikR} \approx \frac{1}{z}e^{-ikz}\exp\left[-ik(x^2+y^2/2R)\right]$$

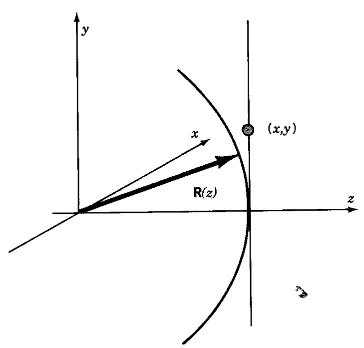

FIGURE 9-11. Propagation of a spherical wave. If point (x, y) is not too far from the z axis, we may approximate the spherical wave by a nearly plane wave called the paraxial, spherical wave.

All of the terms in the equation for a Gaussian wave **(9-40)** can now be given a physical interpretation. Upon examining **(9-40)**, we find that it can be written in the same form as the paraxial, spherical wave if we make the association

$$\frac{k}{2R} = \frac{\pi}{\lambda z\left[1+(\pi w_0^2/\lambda z)^2\right]}$$

The radius of curvature of the phasefront of a paraxial spherical wave, described by **(9-40)**, is therefore determined by the function

$$R(z) = z\left[1+\left(\frac{\pi w_0^2}{\lambda z}\right)^2\right] \tag{9-41}$$

At large values of z, $R \approx z$. The sign convention is such that the beam shown in Figure 9-11 has a positive radius of curvature.

This convention is consistent with the sign convention for the radius of curvature of optical surfaces, defined in Chapter 5. The radius of curvature of an optical surface is measured from the surface to the center of curvature. Here, the radius is measured, in the opposite sense, from the center of curvature to the phase front.

There is an extra phase term in **(9-40)**

$$\phi(z) = \tan^{-1}\left(\frac{\lambda z}{\pi w_0^2}\right) \tag{9-42}$$

$\phi(z)$, defined by **(9-42)**, is the phase difference between an ideal plane wave, shown by the dotted line through the point (x, y) in Figure 9-11, and the "nearly plane," spherical wave of this theory.

The first exponent in **(9-40)** describes the amplitude distribution across the wave. It has the same functional form as a Gaussian distribution with a beam width, sometimes called the beam's *spot size*, given by

$$w(z)^2 = w_0^2\left[1+\left(\frac{\lambda z}{\pi w_0^2}\right)^2\right] \tag{9-43}$$

The curve created by connecting the $1/e$ points of the transverse amplitude of the Gaussian beam, along its propagation path, is described by **(9-43)**. The curve is a hyperbola along the wave's propagation path. At large z, the asymptotic representation of **(9-43)** is a straight line, the geometric ray

$$w(z) = \left(\frac{\lambda}{\pi w_0}\right)z$$

originating at the origin and propagating in the positive z direction. The ray is inclined, with respect to the z axis, at the *diffraction angle*

$$\theta = \frac{\lambda}{\pi w_0} \tag{9-44}$$

We will discover in the next chapter that the Fresnel formalism also yields **(9-44)**.

The diffraction angle can be used to calculate the beam diameter at a distance z from the beam waist

$$w(z)^2 = w_0^2 + \theta^2 z^2 \tag{9-45}$$

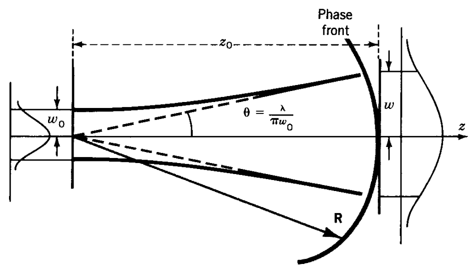

FIGURE 9-12. The propagation of a Gaussian wave from its beam waist to the far field.

The physical interpretation given to **(9-44)** is that diffraction causes a wave of diameter $2w_0$ to spread by an amount

$$\frac{2Z\lambda}{\pi w_0}$$

after propagating a distance Z. The larger the value of the beam waist w_0, the smaller the beam will spread due to diffraction. This is why a megaphone works for a cheerleader. The megaphone changes the effective aperture, producing sound from the small diameter of a human mouth, about 50 mm, to a much larger diameter, something over 30 cm. The propagation properties of laser beams are closely modeled by **(9-44)**. For this reason, lasers are often characterized by their *divergence angle*, which is equal to the diffraction angle.

Figure 9-12 schematically represents the propagation of a Gaussian beam. The dark lines are the hyperbola, given by **(9-43)**, and the dashed straight lines are asymptotes of the hyperbola, i.e., the geometric rays inclined at an angle θ calculated from **(9-44)**.

The significance of the confocal parameter can be identified by examining **(9-43)**. When the definition of the confocal parameter **(9-38)** is substituted into **(9-43)**, the resulting equation

$$w(z)^2 = w_0^2\left[1 + \left(\frac{z}{q_0}\right)^2\right]$$

demonstrates that the confocal parameter, also called the *Rayleigh range*, is the propagation distance z over which the wave's width increases from $w_0 \rightarrow \sqrt{2}w_0$. The confocal parameter, therefore, characterizes a Gaussian beam's convergent or divergent properties. To emphasize the connection with the divergence of the beam, we rewrite the Rayleigh range using **(9-38)** and **(9-44)** as

$$q_0 = \frac{w_0}{\theta}$$

Using the newly defined parameters, we can write **(9-40)** as

$$\Psi = \underbrace{\left[\frac{w_0}{w(z)}\right] \exp\left[-\frac{x^2+y^2}{w(z)^2}\right]}_{\text{amplitude}} \underbrace{\exp\left[-ik\frac{x^2+y^2}{2R(z)}\right]}_{\text{paraxial wave}} \underbrace{\exp[-i\phi(z)]}_{\text{phase}} \qquad (9\text{-}46)$$

The Gaussian wave described by **(9-46)** has as its property that its cross section is everywhere given by the same function, a Gaussian. There are other solutions to **(9-23)** with this property. These additional solutions, arising when the transverse amplitude distribution is not constrained to have circular symmetry, combine, with the function we have been discussing, to form a complete orthogonal set called the *modes of propagation.*

If we divide $R(z)$ by $w(z)$, we can use the result to obtain expressions for w_0 and z in terms of R and w

$$w_0^2 = \frac{w^2}{[1 + (\pi w^2/\lambda R)^2]} \qquad (9\text{-}47)$$

$$z = \frac{R}{[1 + (\lambda R/\pi w^2)^2]} \qquad (9\text{-}48)$$

From **(9-47)**, we see that at the minimum beam waist, the phasefront of the Gaussian wave is a plane, i.e., $R = \infty$. These equations can be used to locate the beam waist of any Gaussian wave. To obtain the parameters of a wave at any point along its propagation path, the *ABCD* matrix can be used, as we will learn in the next section.

THE *ABCD* LAW

We will establish a relationship between the Gaussian beam parameters we have just introduced and geometrical optics that will allow the calculation of Gaussian beam parameters after the wave has passed through an optical system. We first discover how the radius of curvature of a Gaussian wavefront is changed by a thin lens. We will learn that the radius of curvature and the complex beam parameter are governed by the same propagation equations. This finding will allow the construction of an *ABCD* law for a Gaussian wave.

In Chapter 5, we found the *ABCD* matrix, which relates the input and output parameters of an optical system, using the paraxial approximation

$$\begin{pmatrix} x_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x_1 \\ \gamma_1 \end{pmatrix}$$

$$x_2 = Ax_1 + B\gamma_1, \qquad \gamma_2 = Cx_1 + D\gamma_1$$

The variable x_1 is the coordinate position above the optical (z) axis of the ray entering the optical system, x_2 is the coordinate position of the ray leaving the system, and the γ's are the ray slopes. The ray slope for a Gaussian wave of radius R shown in Figure 9-13 is

$$\gamma = \frac{dx}{dz} = \tan\gamma \approx \frac{x}{R} \qquad (9\text{-}49)$$

so that

$$R = \frac{x}{\gamma} \qquad (9\text{-}50)$$

The radius of curvature of the Gaussian wave leaving the optical system described by the *ABCD* matrix is given by

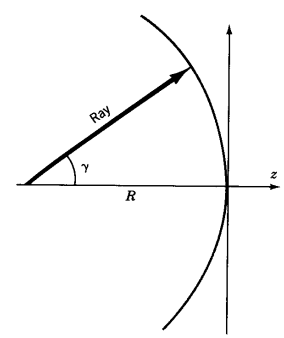

FIGURE 9-13. Geometry for a Gaussian wave in geometrical optics.

$$R_2 = \frac{x_2}{\gamma_2} = \frac{\gamma_1\left(A\frac{x_1}{\gamma_1} + B\right)}{\gamma_1\left(C\frac{x_1}{\gamma_1} + D\right)} = \frac{AR_1 + B}{CR_1 + D} \tag{9-51}$$

where R_1 is the radius of curvature of the Gaussian wave entering the optical system.

To determine the radius of curvature of the phasefront of a Gaussian wave after it has passed through a simple lens, we substitute into **(9-51)** the *ABCD* matrix from Figure 5A-4

$$R_2 = \frac{R_1}{(-R_1/f) + 1}$$

$$\frac{1}{R_2} = \frac{1 - (R_1/f)}{R_1} = \frac{1}{R_1} - \frac{1}{f} \tag{9-52}$$

When the Gaussian wave is propagating through free space, we use the *ABCD* matrix **(5-11)** to obtain the radius of curvature of the phasefront R_2 after the wave has propagated a distance d

$$R_2 = R_1 + d \tag{9-53}$$

The Gaussian wave's complex size parameter q provides a concise description of the propagation of a Gaussian beam through an optical system and it extends easily to higher-order modes. In fact, the higher order modes have the same w, R, and q as the fundamental mode we are discussing; only the phase ϕ is different. For a Gaussian wave propagating through free space, from the beam waist to a position z, the complex size parameter is

$$q_1 = q_0 + z$$

If we propagate from z to $z + d$, the q parameter becomes

$$q_2 = q_1 + d \tag{9-54}$$

The complex size parameter obeys the same rule as the radius of curvature for a wave propagating in free space.

To analyze the effects of a simple lens on a Gaussian wave, recall that the complex size parameter q can be written as

$$\frac{1}{q} = \frac{1}{(z + i\pi w_0^2/\lambda)} = \frac{z - (i\pi w_0^2/\lambda)}{z^2 + (\pi w_0^2/\lambda)^2}$$

Using **(9-41)** and **(9-43)**, we can rewrite this as

$$\frac{1}{q} = \frac{1}{R} - \frac{i\lambda}{\pi w^2} \tag{9-55}$$

For a thin lens, the spot size w is the same at the front and back surfaces of the lens (remember from Appendix 5-A that the front and back vertices of a thin lens define planes of unit magnification); thus, $w_2 = w_1$. The beam radius of curvature should change according to **(9-52)**, allowing us to write **(9-55)** as

$$\frac{1}{q_2} = \frac{1}{R_2} - \frac{i\lambda}{\pi w_2^2} = \left(\frac{1}{R_1} - \frac{1}{f}\right) - \frac{i\lambda}{\pi w_1^2} \tag{9-56}$$

Rearranging the terms yields

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} \tag{9-57}$$

Comparing **(9-52)** and **(9-57)** leads us to the conclusion that the complex beam parameter q plays a role corresponding to the one played by the radius of curvature R of a spherical wave. This should not surprise us because **(9-55)** defines the real part of $1/q$ as being equal to $1/R$. We can rename q as the *complex curvature* of a Gaussian wave.

Because of the formal equivalence between q and R, **(9-51)** can be used to write

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \tag{9-58}$$

Equation **(9-58)** allows a Gaussian beam to be traced through any optical system. Several examples of the application of **(9-58)** will show the usefulness of this result.

Thin Lens

As the first example of the use of **(9-58)**, a Gaussian beam will be followed through a thin lens. Assume that a plane wave uniformly illuminates a lens of diameter D. Because it is a plane wave $R_1 = \infty$ and the aperture of the lens is uniformly illuminated, $w = D/2$. The q parameter at the left surface of the lens is, therefore, given by

$$\frac{1}{q_1} = 0 - \frac{4i\lambda}{\pi D^2}$$

The *ABCD* matrix for a thin lens can be obtained from Figure 5A-4 and can be used to calculate the q parameter, after passing through the lens

$$q_2 = \frac{q_1}{1 - (q_1/f)}$$

$$\frac{1}{q_2} = -\frac{1}{f} - \frac{i\lambda}{\pi (D/2)^2}$$

After passing through the thin lens, the beam waist remains the same size, $w = D/2$, but the radius of curvature of the phase front becomes

$$R_2 = -f$$

i.e., the beam is now converging with the center of curvature of the wavefront on the right of the wavefront.

To find the location of the minimum beam waist, we use **(9-48)**

$$z = \frac{-f}{1 + (4\lambda f/\pi D^2)^2} \approx -f$$

The minus sign signifies that the current position is to the left of the minimum beam waist, i.e., the minimum beam waist lies to the right of the lens, a distance f from the lens. The size of the minimum beam waist is given by **(9-47)**

$$w_0^2 = \frac{D^2/4}{1 + (\pi D^2/4\lambda f)^2} \approx \frac{4\lambda^2 f^2}{D^2}$$

$$w_0 \approx \frac{2\lambda f}{D}$$

The conclusion of this analysis is that parallel light filling the aperture of a thin lens is brought to a focus at the back focal plane of the lens. Diffraction by the aperture of the lens prevents the beam from being focused to a spot smaller than the minimum beam waist. The focal spot size is inversely proportional to the lens aperture and linearly proportional to the focal length of the lens.

Fabry–Perot Resonator

The stability condition of a Fabry–Perot resonator, derived in Chapter 5, can be obtained by using **(9-58)**. For a mode to be stable, we require that the q parameter, at any arbitrary reference plane, must reproduce itself after a round trip in the cavity, i.e.,

$$q = \frac{Aq + B}{Cq + D}$$

where A, B, C, and D are elements of the $ABCD$ matrix for the Fabry–Perot resonator, the matrix elements for a reference plane located at mirror 1 of Figure 5-12 are given in **(5-14)**. The equation may be solved for $1/q$ to obtain

$$\frac{1}{q} = \frac{(D - A) \pm \sqrt{(D - A)^2 + 4BC}}{2B}$$

The $ABCD$ determinant for the resonator must equal 1 because the index of refraction is a constant in the resonator, i.e., $AD - BC = 1$. This fact allows the equation for $1/q$ to be simplified.

$$\frac{1}{q} = \frac{(D - A)}{2B} \pm \frac{i\sqrt{1 - (D + A)^2/4}}{B}$$

This equation is in the standard form of the q parameter

$$\frac{1}{q} = \frac{1}{R} - \frac{i\lambda}{\pi w^2}$$

The real part of the equation for the q parameter can be extracted to discover the radius of curvature of the Gaussian wave in the Fabry–Perot resonator

$$R = \frac{2B}{D - A}$$

In Chapter 5 **(5-14)**, we calculated the $ABCD$ matrix for a ray whose round trip left it at the surface of mirror 1 in Figure 5-12. Using this $ABCD$ matrix, we will find that the radius of curvature of the stable Gaussian wave is equal to the radius of curvature of mirror 1

$$R = \frac{4d\left(1 - \dfrac{d}{R_2}\right)}{\left(1 - \dfrac{2d}{R_2}\right) - 1 + \dfrac{4d}{R_1} + \dfrac{2d}{R_2}\left(1 - \dfrac{2d}{R_1}\right)} = R_1$$

If the $ABCD$ matrix for a wave ending at mirror 2 were used to calculate R, then we would discover that at that mirror, the radius of curvature of a reproducing Gaussian wave is $R = R_2$.

We are led to the conclusion that if a stable, reproducing Gaussian wave exists in a Fabry–Perot resonator, then the radii of curvature of the mirrors making up the resonator match the wave's wavefront curvature. Figure 9-14 shows a typical Gaussian wave. This wave would be a mode of a Fabry–Perot resonator if the mirrors of the resonator were positioned so that their curvature matched the wavefront curvature, shown by the gray lines in Figure 9-14. The mirror's diameter would be selected so that it intercepted, say, 99% of the beam at that location on the optical axis.

In order to insure that the q parameter be complex, we require that

$$\left|\frac{D + A}{2}\right| < 1$$

This is the same stability condition derived using geometrical optics **(5-21)**.

Laser Cavity

As a final example, we will analyze a commercial HeNe laser designed to operate at $\lambda = 632.8$ nm. The optical layout of the laser cavity is shown in Figure 9-15. We will derive the $ABCD$ matrix for this design but will leave the details of the calculation to the reader (see Problem 9-4).

Inside the laser cavity, the phase front curvature of the Gaussian wave

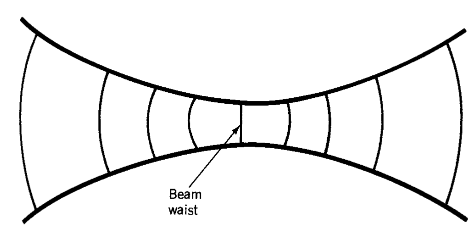

FIGURE 9-14. The black curves are the locus of the beam waists of a Gaussian wave in a Fabry–Perot cavity. The radius of curvature of the wave is shown by the gray curves. The minimum beam waist occurs at the point where the radius of curvature of the phase is infinite, i.e., a plane phasefront.

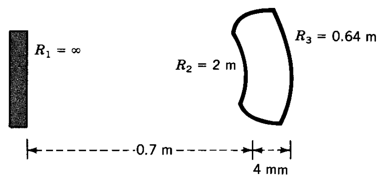

FIGURE 9-15. HeNe laser cavity design.

must match the curvature of the mirrors in the cavity. This means that at the plane mirror, on the left of Figure 9-15, the radius of curvature is infinite. From **(9-47)**, we see that the beam waist always occurs at the point where the radius of curvature is infinite (see Figure 9-14). We will make all of our measurements from the beam waist that we now know to be at the plane mirror. The second mirror is a lens whose concave surface has a reflective dielectric coating. At mirror 2,

$$z = 0.7 \text{ m}, \qquad R = 2 \text{ m}$$

We use **(9-41)** to find the size of the beam waist w_0. From w_0, we can calculate the complex beam parameter q_1.

Light leaves the laser cavity through the lens, whose concave surface serves as one of the Fabry–Perot mirrors. To locate the beam waist and find its size outside the cavity, we must calculate the *ABCD* matrix. From left to right in Figure 9-15, we have the following matrices (see Figure 5A-4 for the proper matrix formulas):

1. The propagation in the laser cavity from mirror 1 to mirror 2. We assume the index of refraction in the cavity is $n_1 = 1.0$

$$\begin{pmatrix} 1 & 0.7 \\ 0 & 1 \end{pmatrix}$$

2. Refraction at the surface of mirror 2. We assume the index of refraction of the lens, which also serves as mirror surface 2, is $n_2 = 1.5$

$$\begin{pmatrix} 1 & 0 \\ \dfrac{n_2 - n_1}{n_2 R_2} & \dfrac{n_1}{n_2} \end{pmatrix}$$

3. Propagation of light through the glass between the surfaces of mirror 2. The mirror thickness is $t = 4$ mm

$$\begin{pmatrix} 1 & \dfrac{t}{n_2} \\ 0 & 1 \end{pmatrix}$$

4. Refraction at the back surface of mirror 2. We assume the index of refraction outside the laser is $n_1 = 1$

$$\begin{pmatrix} 1 & 0 \\ \dfrac{n_1 - n_2}{n_1 R_3} & \dfrac{n_2}{n_1} \end{pmatrix}$$

5. Propagation to the beam waist outside of the laser cavity. This matrix contains the quantity of interest d

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

The *ABCD* matrix for the system is obtained by multiplying all of the above matrices. The resultant matrix is then used in **(9-58)** to find q_2. The beam waist found outside the laser cavity is a minimum beam waist so that, at the minimum waist, $R(z) = \infty$. This means that q_2 must be completely imaginary

$$\frac{1}{\tilde{q}_2} = -\frac{i\lambda}{\pi w_0^2}$$

SUMMARY

Huygens' principle states that the wavefront of a propagating wave can be obtained by treating all of the points on a wavefront, at an earlier time, as sources of spherical waves called Huygens' wavelets. Fresnel was able to describe diffraction by assuming that the new wavefront should be determined by allowing Huygens' wavelets to interfere. The result of Fresnel's theory is the Huygens–Fresnel integral

$$\mathcal{E}(\mathbf{r}_0) = \frac{i}{\lambda}\iint_\Sigma \frac{\mathcal{E}_i(\mathbf{r})}{R} e^{-i\mathbf{k}\cdot\mathbf{R}}\, ds$$

where $\mathcal{E}_i(\mathbf{r})$ is the wave incident on the aperture, denoted by Σ. R is the distance from a point in the aperture to the observation point located at position $\mathbf{r}_0$. The integral is an area integral taken over the aperture Σ.

In this approach to diffraction, free space is assumed to be a linear system with an impulse response given by

$$\frac{i}{\lambda}\frac{e^{-i\mathbf{k}\cdot\mathbf{R}}}{R}$$

The Huygens–Fresnel integral is interpreted as a convolution of the impulse response with the input wave.

A second approach to diffraction is often used when the light wave's transverse amplitude distribution is Gaussian. The Gaussian wave is characterized by its width

$$w(z)^2 = w_0^2\left[1 + \left(\frac{z}{q_0}\right)^2\right]$$

and the radius of curvature of its wavefront

$$R(z) = z\left[1 + \left(\frac{q_0}{z}\right)^2\right]$$

where

$$q_0 = \frac{\pi w_0^2}{\lambda}$$

is the Rayleigh range, the parameter w_0 is the minimum beam width (or waist), and z is the distance from the minimum beam waist to the observation point.

Given the beam waist w and the radius of curvature R at any point, the minimum beam waist and its location can be found by using

$$w_0^2 = \frac{w^2}{\left[1 + \left(\pi w^2/\lambda R\right)^2\right]}$$

$$z = \frac{R}{\left[1 + \left(\lambda R/\pi w^2\right)^2\right]}$$

The *ABCD* matrix, introduced in Chapter 5, can be used to trace the behavior of a Gaussian wave as it propagates through an optical system. The Gaussian wave at any point z is characterized by a q parameter

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{i\lambda}{\pi w(z)^2}$$

The q parameter after propagating through an optical system is given by

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

where A, B, C, and D are the elements of the *ABCD* matrix.

PROBLEMS

9-1. Calculate the *ABCD* matrix for a ray that has made a round trip in the Fabry–Perot resonator of Figure 5-12, ending at mirror 2. Find the radius of curvature of the Gaussian wave at this reference plane.

9-2. Find the location of the beam waist, relative to mirror 1, in the Fabry–Perot resonator of Figure 5-12.

9-3. A Fabry–Perot resonator has a spherical front mirror of radius 2ℓ and a plane back mirror. It is illuminated by a Gaussian beam with a waist w_0 at $z = 0$. The Fabry–Perot's front mirror is located at $z = 3\ell$, its back mirror at $z = 4\ell$. A lens of focal length f at $z = 2\ell$ couples the Gaussian beam into the resonator. What values should f and w_0 have so that the beam will match the resonator's fundamental mode?

9-4. Complete the details of the calculation for the laser cavity design started in the chapter. You will find a second beam waist, the image of the first, outside the cavity. Can you think of any reasons the optical designer placed this waist outside the cavity?

9-5. Find the value of z where the radius of curvature of the phase of a Gaussian wave is a minimum. What Gaussian parameter can be used to specify this position?

9-6. What is the beam waist of a HeNe laser ($\lambda = 632.8$ nm) with a beam divergence of 0.7 mrad? What is the Rayleigh range q_0?

9-7. Use the object image matrix to show that

$$q_2 = \frac{\left(1 - \frac{z_2}{f}\right)q_1 + \left(z_1 + z_2 - \frac{z_1 z_2}{f}\right)}{\left(1 - \frac{z_1}{f}\right) - \frac{q_1}{f}} \tag{9-59}$$

where q_1 is the beam parameter in the object plane of a positive thin lens and q_2 the beam parameter in the conjugate (image) plane. [*Hint:* Use **(9-32)** to follow the beam from the first waist to the lens and from the lens to the second waist. Use **(9-58)** to obtain the effect of the lens. Use **(9-55)** at each waist