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Spectral theory of higher-order discrete vector Sturm–Liouville problems ☆

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Abstract

This paper is concerned with spectral problems of higher-order vector difference equations with self-adjoint boundary conditions, where the coefficient of the leading term may be singular. A suitable admissible function space is constructed so that the corresponding difference operator is self-adjoint in it, and the fundamental spectral results are obtained. Rayleigh's principles and minimax theorems in two special linear spaces are given. As an application, comparison theorems for eigenvalues of two Sturm–Liouville problems are presented. Especially, the dual orthogonality and multiplicity of eigenvalues are discussed. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let the interval $[M, N]$ denote the set of integers $\{M, M+1, \ldots, N\}$. Let $y(t)$ be a d-dimensional column vector-valued function on $[0, N+2n]$, where $n \geqslant 1$ and $N \geqslant 2n-1$. Denote the forward difference operator by Δ, $\Delta y(t) = y(t+1) - y(t)$.

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Consider the following $2n$th-order vector difference equation

$$\sum_{i=0}^{n} \Delta^i \big[r_i(t) \Delta^i y(t-i) \big] = \lambda \omega(t) y(t), \quad t \in [n, N+n], \tag{1.1}$$

with the boundary conditions

$$R \begin{pmatrix} -u(0, y) \\ u(N+1, y) \end{pmatrix} + S \begin{pmatrix} v(0, y) \\ v(N+1, y) \end{pmatrix} = 0, \tag{1.2}$$

where $\omega(t)$ and $r_i(t)$ are $d \times d$-Hermitian matrix-valued functions in t on $[n, N+n]$ and $[n, N+n+i]$, respectively, $0 \leqslant i \leqslant n$, R and S are $2nd \times 2nd$-matrices, $u^{\mathrm{T}}(t, y) = (u_1^{\mathrm{T}}(t, y), \dots, u_n^{\mathrm{T}}(t, y))$, $v^{\mathrm{T}}(t, y) = (v_1^{\mathrm{T}}(t, y), \dots, v_n^{\mathrm{T}}(t, y))$ are nd-vectors, and

$$r_n(t) \text{ is nonsingular on } [n, 2n-1] \cup [N+n+1, N+2n], \tag{1.3}$$

$$\begin{aligned} u_i(t, y) &= \Delta^{i-1} y(t+n-i), \\ v_i(t, y) &= (-1)^{i-1} \textstyle\sum_{k=i}^{n} \Delta^{k-i} \big[r_k(t+n) \Delta^k y(t+n-k) \big]. \end{aligned} \tag{1.4}$$

The nd-vector functions u and v are introduced for the formulation of self-adjointness of the problem.

Our standing hypothesis in this paper is

$$\omega(t) > 0 \quad \text{for } t \in [n, N+n] \quad \text{and} \quad \operatorname{rank}(R, S) = 2nd. \tag{1.5}$$

It is well known that the classical spectral results were obtained for second-order continuous scalar Sturm–Liouville problems by some oscillatory properties of solutions in the parameter λ (cf., e.g., [6,12,20]). The more complicated higher-order or higher-dimensional cases involve the theory of integral equations via Green's functions, the calculus of variation (cf. [4,17]), and the generalized Picone's identities (cf. [11,12]).

Discrete spectral problems have been of growing interest in recent years. Atkinson [1, Chapter 4] and Jirari [9] studied spectral problems of second-order discrete scalar Sturm–Liouville problems (see Eqs. (4.4) and (4.5) in this paper) with spatially separate boundary conditions by investigating some oscillatory properties of solutions in the parameter λ as done in the continuous case. In addition, Atkinson [1, Chapter 6] also considered a vector discrete problem by converting the problem into an equivalent spectral problem of a certain Hermitian matrix. For more references in the second-order case, see [3,8,16].

Higher-order discrete linear problems have also been studied by a few authors (see [5,7,14,15] and their references). In a recent paper [2], Bohner employed discrete quadratic functionals and the Reid roundabout theorems to investigate isolatedness and lower boundedness of eigenvalues for discrete linear Hamiltonian eigenvalue problems, where the parameter λ is implicit in the coefficients.

For degenerate cases, Möller [13] studied the eigenvalue problem for a second-order differential equation in which the coefficient of the leading term changes its sign. Kong et al. [10] studied the dependence of the *n*th Sturm–Liouville eigenvalue on the problem in which the leading coefficient is almost positive on the interval.

We would like to mention here that the most previous results (except [18]) assume that the coefficients in the leading terms of the difference equations are nonsingular so that a solution can be continued both to the right-hand side and to the left-hand side. However, (1.1) is singular since from (1.3) we see that the leading coefficient $r_n(t)$ may be singular in a part of its interval of definition. Besides, there is no restriction on the definiteness of $r_n(t)$. Condition (1.3) can even be weakened if boundary conditions (1.2) are "better" (see Section 4). Therefore, the corresponding difference operator in (1.1) is in general degenerate and all the methods based on continuation of solutions are not applicable. Although the difference operator in (1.1) is formally self-adjoint, it still may not be true self-adjoint even if boundary conditions (1.2) are self-adjoint. Moreover, the eigenvalues of a higher-order problem are not as simple as those of a second-order problem and boundary conditions (1.2) are so general that it is quite difficult to convert the problem into a spectral problem of an equivalent Hermitian matrix. These are the major difficulties we will encounter in this paper.

The paper is organized as follows. In Section 2, a definition of self-adjointness of the boundary conditions will be given and a suitable admissible function space will be constructed so that the corresponding difference operator is self-adjoint. Section 3 will be devoted to the fundamental spectral results, Rayleigh's principles and minimax theorems in two special linear spaces, and comparison results. In Section 4, we will illustrate that some assumptions can be weakened if (1.2) are proper (definition of proper boundary conditions will be given in Section 2), and discuss the dual orthogonality and multiplicity of eigenvalues. For conciseness, the proof of a lemma used in Section 2 will be left to Section 5.

2. The admissible function space and self-adjointness of the difference operator

2.1. Self-adjointness of boundary conditions (1.2)

In the sequel, let

$$L[0, N+2n] = \left\{ y = \{y(t)\}_{t=0}^{N+2n} \subset \mathbb{C}^d \right\}$$

and let $\mathscr{L}$ be the following higher-order difference operator:

$$(\mathscr{L}y)(t) = \omega^{-1}(t) \sum_{i=0}^{n} \Delta^i \left[r_i(t) \Delta^i y(t-i) \right], \quad t \in [n, N+n].$$

Then $\dim L[0, N+2n] = (N+2n+1)d$. Define

$$\langle x, y \rangle = \sum_{t=n}^{N+n} y^*(t)\omega(t)x(t), \quad x, y \in L[0, N+2n], \tag{2.1}$$

where $\omega(t)$ is the weighting function in (1.1) and $y^*(t)$ denotes the complex conjugate transpose of $y(t)$. As usual, $x \perp y$ denotes $\langle x, y\rangle = 0$, y^{T} the transpose of y, and I_d and 0_d the $d \times d$-identity and zero matrices, respectively. For convenience, we write $y \in \mathscr{R}$ if $y \in L[0, N+2n]$ and satisfies boundary conditions (1.2).

Theorem 2.1. *For all* $x, y \in L[0, N+2n]$,

$$\langle \mathscr{L}x, y\rangle - \langle x, \mathscr{L}y\rangle = \big[u^*(t, y)v(t, x) - v^*(t, y)u(t, x)\big]\big|_{t=0}^{N+1}. \tag{2.2}$$

Proof. Let $x, y \in L[0, N+2n]$. Then

$$\langle \mathscr{L}x, y\rangle = \sum_{i=0}^{n} \delta(i, r_i, x, y), \qquad \langle x, \mathscr{L}y\rangle = \sum_{i=0}^{n} \delta^*(i, r_i, y, x) \tag{2.3}$$

with

$$\delta(i, r_i, x, y) = \sum_{t=n}^{N+n} y^*(t)\Delta^i\big[r_i(t)\Delta^i x(t-i)\big]. \tag{2.4}$$

Obviously, we get

$$\delta(0, r_0, x, y) = \sum_{t=n}^{N+n} y^*(t)r_0(t)x(t). \tag{2.5$_0$}$$

Using successively i times the following well-known Abel's equalities

$$a^*(t)\Delta b(s) = \Delta(a^*(t-1)b(s)) - (\Delta a^*(t-1))b(s), \tag{2.6}$$

one can conclude that for $i \geqslant 1$,

$$\begin{aligned}\delta(i, r_i, x, y) = &\sum_{j=0}^{i-1}(-1)^j\big\{\Delta^j y^*(t-j-1)\Delta^{i-j-1}\big[r_i(t)\Delta^i x(t-i)\big]\big\}\big|_{t=n}^{N+n+1}\\ &+ (-1)^i\sum_{t=n}^{N+n}\big\{\Delta^i y^*(t-i)r_i(t)\Delta^i x(t-i)\big\}.\end{aligned} \tag{2.5$_i$}$$

From (2.3) and (2.5), we have

$$\begin{aligned}\langle \mathscr{L}x, y\rangle = &\sum_{t=n}^{N+n}\sum_{i=0}^{n}\big\{(-1)^i\Delta^i y^*(t-i)r_i(t)\Delta^i x(t-i)\big\}\\ &+ u^*(t, y)v(t, x)\big|_{t=0}^{N+1}.\end{aligned} \tag{2.7}$$

Therefore, by the hermiticity of $r_i(t)$ on $[n, N+n+i]$, (2.3) and (2.7) imply that (2.2) holds. The proof is complete. □

Referring to equality (2.2), we naturally give the following definition.

Definition 2.1. Boundary conditions (1.2) are called self-adjoint, if

$$[u^*(t,y)v(t,x) - v^*(t,y)u(t,x)]\,|_{t=0}^{N+1} = 0, \quad \text{whenever } x, y \in \mathscr{R}.$$

The following result follows immediately from Theorem 2.1.

Theorem 2.2. *If boundary conditions* (1.2) *are self-adjoint, then*

$$\langle \mathscr{L}x, y\rangle = \langle x, \mathscr{L}y\rangle \quad \forall x, y \in \mathscr{R}.$$

The two following lemmas are discrete analogs of Propositions 2.1.1 and 2.1.2 in [11], where the coefficient matrices are real.

Lemma 2.1. *Boundary conditions* (1.2) *are self-adjoint if and only if*

$$RS^* = SR^*.$$

Here we point out that the condition $RS^* = SR^*$ is quite strong since it is sufficient for (1.2) to be self-adjoint in some cases where (1.3) is not satisfied.

Lemma 2.2. *Assume that* (1.3) *holds and boundary conditions* (1.2) *are self-adjoint. Then $y \in \mathscr{R}$ if and only if there exists a unique vector $\xi \in \mathbb{C}^{2nd}$ such that*

$$\begin{aligned} &\big(-u^{\mathrm{T}}(0,y), u^{\mathrm{T}}(N+1,y)\big)^{\mathrm{T}} = -S^*\xi, \\ &\big(v^{\mathrm{T}}(0,y), v^{\mathrm{T}}(N+1,y)\big)^{\mathrm{T}} = R^*\xi. \end{aligned} \tag{2.8}$$

2.2. The admissible function space

First, we introduce some notations. Let

$$\begin{aligned} &Y^{\mathrm{T}}(t,k) = \big(y^{\mathrm{T}}(t+k-1), y^{\mathrm{T}}(t+k-2), \ldots, y^{\mathrm{T}}(t)\big), \\ &\quad t \in [0, N+n+1],\ k \in [1,n] \end{aligned} \tag{2.9}$$

and $Y(t) := Y(t,n)$. To express u and v in terms of Y, let L, A, and B be $nd \times nd$-matrices such that

$$\begin{aligned} &u(0,y) = LY(0), \\ &v(0,y) = AY(n) + BY(0), \\ &u(N+1,y) = LY(N+1), \\ &v(N+1,y) = \hat{A}Y(N+n+1) + \hat{B}Y(N+1), \end{aligned} \tag{2.10}$$

where $\hat{A}$ and $\hat{B}$ are the shifts of A and B to the right with $N+1$ units, respectively. More precisely, for $1 \leqslant i, j \leqslant n$,

$$\begin{aligned}
&L = (l_{ij}) \text{ with } l_{ij} = 0_d \text{ if } i < j \text{ and } l_{ij} = (-1)^{j-1} C_{i-1}^{j-1} I_d \text{ if } i \geqslant j, \\
&A = (a_{ij}) \text{ with } a_{ij} = 0_d \text{ if } i > j \text{ and } a_{ii} = (-1)^{i-1} r_n(2n-i), \\
&\hat{A} = (\hat{a}_{ij}) \text{ with } \hat{a}_{ij} = 0_d \text{ if } i > j \text{ and } \hat{a}_{ii} = (-1)^{i-1} r_n(N+2n+1-i).
\end{aligned} \tag{2.11}$$

Clearly, L is nonsingular. If $r_n(t)$ is nonsingular on $[n, 2n-1]$ and $[N+n+1, N+2n]$, then A and $\hat{A}$ are nonsingular, respectively. For convenience, we introduce a “bracket function”

$$[a, b]_A(t, s) = b^*(t)Aa(s) - b^*(s)A^*a(t), \tag{2.12}$$

where $a(t)$ and $b(t)$ are d_1- vector functions, respectively, and A is a given $d_1 \times d_1$-matrix.

Next, we prepare a proposition for construction of the admissible space.

Proposition 2.1. *The matrices L^*B and $L^*\hat{B}$ are Hermitian and*

$$L^*A = D, \qquad L^*\hat{A} = \hat{D}, \tag{2.13}$$

where $\hat{D}$ is the shift of D to the right with $N+1$ units,

$$D = \sum_{k=1}^{n} \begin{pmatrix} 0 & D(k, \hat{r}_k) \\ 0_{(n-k)d} & 0 \end{pmatrix}, \quad \hat{r}_k(t) = r_k(n+t-k), \tag{2.14}$$

$D(k, r) = (d_{ij}(k, r))$ is a $kd \times kd$-matrix, and $d_{ij}(k, r)$ is a $d \times d$-matrix for $1 \leqslant i, j \leqslant k$ with

$$d_{ij}(k, r) = \begin{cases} 0_d & \text{if } i > j, \\ r(2k-i) & \text{if } i = j, \\ (-1)^{j-i} \sum_{l=0}^{j-i} C_k^{j-i-l} C_k^l r(2k-i-l) & \text{if } i < j. \end{cases} \tag{2.15}$$

Furthermore, $d_{ij}(k, r)$ is Hermitian if $r(t)$ is Hermitian on $[k, 2k-1]$.

To prove Proposition 2.1, we need a lemma whose technical proof is left to Section 5.

Lemma 2.3. *The following equalities hold*:

$$u^*(0, y)v(0, x) - v^*(0, y)u(0, x) = [X, Y]_D(0, n), \tag{2.16}$$

$$\begin{aligned}
&u^*(N+1, y)v(N+1, x) - v^*(N+1, y)u(N+1, x) \\
&\quad = [X, Y]_{\hat{D}}(N+1, N+n+1),
\end{aligned} \tag{2.17}$$

where D and $\hat{D}$ are as in Proposition 2.1 *and* $X(t)$ *is defined as* $Y(t)$ *by* (2.9) *with* $y(t)$ *replaced with* $x(t)$.

Proof of Proposition 2.1. We first show that L^*B is Hermitian and the first relation of (2.13) holds. From (2.10), for any $x, y \in L[0, N+2n]$, we have

$$\begin{aligned} &u^*(0, y)v(0, x) - v^*(0, y)u(0, x) \\ &\quad = Y^*(0)(L^*B - B^*L)X(0) + Y^*(0)L^*AX(n) - Y^*(n)A^*LX(0). \end{aligned}$$

Using (2.16) in Lemma 2.3, from the arbitrariness of *x* and *y*, we find

$$L^*B = B^*L, \qquad L^*A = D.$$

Using (2.17), one similarly concludes that $L^*\hat{B}$ is Hermitian and the second relation of (2.13) holds. This completes the proof. □

We are now in a position to construct the admissible function space. In the rest of the section, we always suppose that (1.3) holds and boundary conditions (1.2) are self-adjoint. Let

$$R = (R_1, R_2), \qquad S = (S_1, S_2),$$

where R_j and S_j $(j = 1, 2)$ are $2nd \times nd$-matrices. From (2.10), boundary conditions (1.2) can be rewritten as

$$\Omega \operatorname{diag}\{L, -\hat{A}\} \begin{pmatrix} Y(0) \\ Y(N+n+1) \end{pmatrix} = (S_1A, R_2L + S_2\hat{B}) \begin{pmatrix} Y(n) \\ Y(N+1) \end{pmatrix} \tag{2.18}$$

with

$$\Omega = \left(R_1 - S_1BL^{-1}, S_2\right).$$

Set

$$m = \operatorname{rank} \Omega.$$

Then, $0 \leqslant m \leqslant 2nd$. If Ω is nonsingular, i.e., $m = 2nd$, then (2.18) is solvable for $Y(0)$ and $Y(N+n+1)$ and the linear space

$$\hat{L}[0, N+2n] = \{y \in L[0, N+2n]\colon\ y \in \mathscr{R}\}$$

is admissible since $y(t)$ for $t \in [0, n-1] \cup [N+n+1, N+2n]$ are not weighted by the weighting function ω. In this case, we will call boundary conditions (1.2) *proper*; otherwise *improper*. In the improper case, i.e., $m < 2nd$, we see from (2.18) that $Y(n)$ and $Y(N+1)$ themselves are linked by $2nd - m$ relations. Recall that $y(t)$ is weighted by $\omega(t)$ for $t \in [n, 2n-1] \cup [N+1, N+n]$ in (1.1). So in these $2n$ vector equations, upon some transformation, only m scalar equations are really weighted and the rest $2nd - m$ ones do not involve the parameter λ, and hence, can be viewed as extra conditions for the admissible functions.

By the standard matrix theory (see, e.g., [19]), there exist $2nd \times 2nd$-unitary matrices P and Q such that

$$P^*\Omega Q = \operatorname{diag}\{0, \Omega_0\}, \tag{2.19}$$

where Ω_0 is an $m \times m$-nonsingular matrix. Let

$$P = (P_1, P_2), \qquad Q = (Q_1, Q_2),$$

where P_1 and Q_1 are $2nd \times (2nd - m)$-matrices, P_2 and Q_2 are $2nd \times m$-matrices. From (2.19) and by the unitarity of P and Q, we can find that

$$\begin{aligned} &Q_1^*Q_1 = I_{2nd-m}, \quad Q_1^*Q_2 = 0_{(2nd-m)\times m}, \quad Q_2^*Q_2 = I_m, \\ &P_1^*\Omega = 0, \quad Q_1^*\Omega^* = 0, \quad P_2^*\Omega = \Omega_0 Q_2^*. \end{aligned} \tag{2.20}$$

Suppose that $y \in \hat{L}[0, N+2n]$. From (2.10) and by Lemma 2.2 and Proposition 2.1, we have

$$\begin{pmatrix} AY(n) \\ -LY(N+1) \end{pmatrix} = \Omega^*\xi \tag{2.21}$$

for some $\xi \in \mathbb{C}^{2nd}$. Then we get that

$$Q_1^* \begin{pmatrix} AY(n) \\ -LY(N+1) \end{pmatrix} = 0 \quad \text{whenever } y \in \mathscr{R}. \tag{2.22}$$

For $t \in [n, 2n-1] \cup [N+1, N+n]$, (1.1) can be written as

$$\lambda\big(Y^{\mathrm{T}}(n), Y^{\mathrm{T}}(N+1)\big) = \big(Y^{\mathrm{T}}_{\mathscr{L}y}(n), Y^{\mathrm{T}}_{\mathscr{L}y}(N+1)\big),$$

where $Y_{\mathscr{L}y}(t)$ is defined as $Y(t)$ by (2.9) with $y(t)$ replaced with $(\mathscr{L}y)(t)$. So, from (2.22) and the above relations, we can get the following $2nd - m$ scalar equations

$$Q_1^* \operatorname{diag}\{A, -L\}\big(Y^{\mathrm{T}}_{\mathscr{L}y}(n), Y^{\mathrm{T}}_{\mathscr{L}y}(N+1)\big)^{\mathrm{T}} = 0. \tag{2.23}$$

Since the parameter λ is missing in (2.23), we naturally regard (2.23) as additional conditions to (1.2). For convenience, we write $y \in \mathscr{A}$ if y satisfies (2.23). We can now introduce the admissible function space as follows:

$$L^2_\omega[0, N+2n] = \{y \in L[0, N+2n]\colon\ y \in \mathscr{R} \text{ and } y \in \mathscr{A}\}.$$

2.3. *The elements and the dimension of* $L^2_\omega[0, N+2n]$

We will examine $L^2_\omega[0, N+2n]$ more closely for later discussions. Suppose that $y \in L^2_\omega[0, N+2n]$. From (2.23) and using (2.20), one can conclude that $y \in \mathscr{A}$ if and only if there exists a vector $\zeta \in \mathbb{C}^m$ such that y satisfies

$$\big(Y^{\mathrm{T}}_{\mathscr{L}y}(n), Y^{\mathrm{T}}_{\mathscr{L}y}(N+1)\big)^{\mathrm{T}} = \operatorname{diag}\big\{A^{-1}, -L^{-1}\big\}Q_2\zeta. \tag{2.24}$$

By the definition of $\mathscr{L}$, we find that

$$\begin{aligned} Y_{\mathscr{L}y}(n) ={}& \operatorname{diag}\big\{\omega^{-1}(2n-1), \ldots, \omega^{-1}(n)\big\} \\ &\times \Big\{M_1Y(0) + M_2\big(Y^{\mathrm{T}}(2n), Y^{\mathrm{T}}(n)\big)^{\mathrm{T}}\Big\}, \end{aligned} \tag{2.25}$$

$$
\begin{aligned}
&Y_{\mathscr{L}y}(N+1)\\
&\quad = \operatorname{diag}\left\{\omega^{-1}(N+n), \ldots, \omega^{-1}(N+1)\right\}\\
&\qquad \times \left\{N_1 Y(N+n+1) + N_2\left(Y^{\mathrm{T}}(N+1), Y^{\mathrm{T}}(N+1-n)\right)^{\mathrm{T}}\right\},
\end{aligned}
\tag{2.26}
$$

where M_1 and N_1 are $nd \times nd$-matrices; M_2 and N_2 are $nd \times 2nd$-matrices.

Proposition 2.2. *The following relations hold*:

$$
A^* L = M_1, \qquad L^* \hat{A} = N_1. \tag{2.27}
$$

Furthermore, if (1.3) *holds, then M_1 and N_1 are nonsingular.*

Proof. All that is needed to show is that $M_1 = D^*$ and $N_1 = \hat{D}$. We only show that $M_1 = D^*$ holds. The proof of $N_1 = \hat{D}$ is similar.

Let $M_1^{\mathrm{T}} = (M_{11}^{\mathrm{T}}, M_{12}^{\mathrm{T}}, \ldots, M_{1n}^{\mathrm{T}})$, where M_{1j} $(1 \leqslant j \leqslant n)$ are $d \times nd$-matrices. Since we are interested only in the first n terms of $y(t)$ in (2.25), $0 \leqslant t \leqslant n-1$, we sometimes neglect the other terms to save space. By the expansion of the higher-order difference

$$
\Delta^i x(t) = \sum_{j=0}^{i} (-1)^j C_i^j x(t+i-j), \tag{2.28}
$$

a straightforward calculation gives, for $0 \leqslant l \leqslant n-1$,

$$
\begin{aligned}
&\omega(n+l)(\mathscr{L}y)(n+l)\\
&\quad = \sum_{i=0}^{n} \Delta^i\left[r_i(n+l)\Delta^i y(n+l-i)\right]\\
&\quad = \sum_{i=0}^{n}\sum_{k=0}^{i}\left\{(-1)^k C_i^k r_i(n+l+i-k)\right.\\
&\qquad\qquad \left.\times \sum_{j=0}^{i}\left((-1)^j C_i^j y(n+l+i-j-k)\right)\right\}\\
&\quad = \sum_{i=0}^{n-l-1}\left\{\sum_{k=0}^{n-l-i-1}\sum_{j=0}^{k}(-1)^k C_{p+k}^{p+j} C_{p+k}^{j} r_{p+k}(n+l+k-j)\right\}\\
&\qquad \times y(n-1-i) + \cdots
\end{aligned}
$$

with $p = l + i + 1$. So

$$
M_{1,n-l} = (e_{l0}, e_{l1}, \ldots, e_{l,n-l-1}, 0, \ldots, 0),
$$

$$
e_{li} = \sum_{k=0}^{n-l-i-1}\sum_{j=0}^{k}(-1)^k C_{p+k}^{p+j} C_{p+k}^{j} r_{p+k}(n+l+k-j)
$$

for $0 \leqslant i \leqslant n-l-1$. Obviously, from (2.15), we have

$$e_{li} = \sum_{k=0}^{n-l-i-1} d_{i+1,i+k+1}(l+i+k+1, \hat{r}_{l+i+k+1}).$$

Hence, $M_1 = D^*$ from (2.14) and (2.15). The proof is complete. □

From (2.25) and (2.26) and by Proposition 2.2, we have

$$\begin{aligned} Y(0) = {} & M_1^{-1} \operatorname{diag}\{\omega(2n-1), \ldots, \omega(n)\} Y_{\mathscr{L}y}(n) \\ & - M_1^{-1} M_2 \big(Y^{\mathrm{T}}(2n), Y^{\mathrm{T}}(n)\big)^{\mathrm{T}}, \end{aligned}$$

$$\begin{aligned} Y(N+n+1) = {} & N_1^{-1} \operatorname{diag}\{\omega(N+n), \ldots, \omega(N+1)\} Y_{\mathscr{L}y}(N+1) \\ & - N_1^{-1} N_2 \big(Y^{\mathrm{T}}(N+1), Y^{\mathrm{T}}(N+1-n)\big)^{\mathrm{T}}. \end{aligned}$$

Inserting (2.24) into the above relations, we can find that

$$\begin{aligned} & \begin{pmatrix} Y(0) \\ Y(N+n+1) \end{pmatrix} \\ & \quad = \operatorname{diag}\big\{M_1^{-1}, N_1^{-1}\big\} W_1 \operatorname{diag}\big\{A^{-1}, -L^{-1}\big\} Q_2 \zeta \\ & \qquad - \operatorname{diag}\big\{M_1^{-1} M_2, N_1^{-1} N_2\big\} \\ & \qquad \times \big(Y^{\mathrm{T}}(2n), Y^{\mathrm{T}}(n), Y^{\mathrm{T}}(N+1), Y^{\mathrm{T}}(N+1-n)\big)^{\mathrm{T}} \end{aligned} \tag{2.29}$$

with

$$W_1 = \operatorname{diag}\{\omega(2n-1), \ldots, \omega(n), \omega(N+n), \ldots, \omega(N+1)\}.$$

Next we will show that the vector ζ in (2.29) can be determined by $Y(2n)$, $Y(n)$, $Y(N+1)$, and $Y(N+1-n)$. From (2.29), (2.18) yields

$$\begin{aligned} & \Omega \operatorname{diag}\big\{L M_1^{-1}, -\hat{A} N_1^{-1}\big\} W_1 \operatorname{diag}\big\{A^{-1}, -L^{-1}\big\} Q_2 \zeta \\ & \quad = f(Y(2n), Y(n), Y(N+1), Y(N+1-n)) \end{aligned} \tag{2.30}$$

with

$$\begin{aligned} & f(Y(2n), Y(n), Y(N+1), Y(N+1-n)) \\ & \quad = (S_1 A, R_2 L + S_2 \hat{B}) \big(Y^{\mathrm{T}}(n), Y^{\mathrm{T}}(N+1)\big)^{\mathrm{T}} \\ & \qquad + \Omega \operatorname{diag}\big\{L M_1^{-1} M_2, -\hat{A} N_1^{-1} N_2\big\} \\ & \qquad \times \big(Y^{\mathrm{T}}(2n), Y^{\mathrm{T}}(n), Y^{\mathrm{T}}(N+1), Y^{\mathrm{T}}(N+1-n)\big)^{\mathrm{T}}. \end{aligned} \tag{2.31}$$

Multiplying (2.30) from the left-hand side by P^* and using (2.20), we see that conditions (2.30) can be divided into two parts:

$$P_1^* f(Y(2n), Y(n), Y(N+1), Y(N+1-n)) = 0, \tag{2.32}$$

$$J\zeta = P_2^* f(Y(2n), Y(n), Y(N+1), Y(N+1-n)), \tag{2.33}$$

where

$$J = \Omega_0 Q_2^* \operatorname{diag}\{LM_1^{-1}, -\hat{A}N_1^{-1}\} W_1 \operatorname{diag}\{A^{-1}, -L^{-1}\} Q_2$$

is an $m \times m$-matrix. Since W_1 is positive definite and rank $Q_2 = m$, by Proposition 2.2, J is nonsingular. It follows from (2.33) that

$$\zeta = J^{-1} P_2^* f(Y(2n), Y(n), Y(N+1), Y(N+1-n)). \tag{2.34}$$

Therefore, $y(0), \ldots, y(n-1)$; $y(N+n+1), \ldots, y(N+2n)$ can be determined by $y(n), \ldots, y(3n-1)$; $y(N+1-n), \ldots,$ and $y(N+n)$ from (2.29) and (2.34). However, from (2.20) and (2.31) (particularly $P_1^* S_2 = 0$), conditions (2.32) are equivalent to

$$P_1^*[S_1 A Y(n) + R_2 L Y(N+1)] = 0. \tag{2.35}$$

We now get a useful result for $y \in L^2_\omega[0, N+2n]$.

Proposition 2.3. *Assume that* (1.3) *holds. Then* $y \in L^2_\omega[0, N+2n]$ *if and only if* y *satisfies* (2.29) *and* (2.35) *in which* ζ *is determined by* (2.34).

Proposition 2.4. *If* (1.3) *holds, then* $\mu := \dim L^2_\omega[0, N+2n] = (N+1)d + m - 2nd$.

Proof. By Proposition 2.3 and from (2.35), it suffices to show that

$$\operatorname{rank}[P_1^*(S_1 A, R_2 L)] = 2nd - m. \tag{2.36}$$

Clearly, $\operatorname{rank}(S_1 A, R_2 L, \Omega) = 2nd$ from the nonsingularity of A (by referring to (2.11)) and

$$(S_1 A, R_2 L, \Omega) = (R, S) \begin{pmatrix} 0 & 0 & I_d & 0 \\ 0 & L & 0 & 0 \\ A & 0 & -BL^{-1} & 0 \\ 0 & 0 & 0 & I_d \end{pmatrix}.$$

Again using rank $P_1 = 2nd - m$ and $P_1^* \Omega = 0$ (from (2.20)), one concludes that

$$\operatorname{rank}[P_1^*(S_1 A, R_2 L)] = \operatorname{rank}[P_1^*(S_1 A, R_2 L, \Omega)] = 2nd - m.$$

Therefore, (2.36) holds. This completes the proof. □

Finally, we have the following conclusion by Propositions 2.3 and 2.4.

Theorem 2.3. *If* (1.3) *holds and boundary conditions* (1.2) *are self-adjoint, then* $L^2_\omega[0, N+2n]$ *is a* μ*-dimensional Hilbert space with the inner product defined by* (2.1).

2.4. The self-adjointness of the difference operator

In the end of this section, we will discuss the self-adjointness of the difference operator $\mathscr{L}$ on $L^2_\omega[0, N+2n]$. To do this, we must extend the definition of $(\mathscr{L}y)(t)$ to the intervals $[0, n-1] \cup [N+n+1, N+2n]$. Define

$$\begin{aligned}\begin{pmatrix} Y_{\mathscr{L}y}(0) \\ Y_{\mathscr{L}y}(N+n+1) \end{pmatrix} &= \operatorname{diag}\left\{M_1^{-1}, N_1^{-1}\right\} W_1 \operatorname{diag}\left\{A^{-1}, -L^{-1}\right\} Q_2 \zeta_{\mathscr{L}y} \\ &\quad - \operatorname{diag}\left\{M_1^{-1}M_2, N_1^{-1}N_2\right\} \\ &\quad \times \left(Y^{\mathrm{T}}_{\mathscr{L}y}(2n), Y^{\mathrm{T}}_{\mathscr{L}y}(n), Y^{\mathrm{T}}_{\mathscr{L}y}(N+1), Y^{\mathrm{T}}_{\mathscr{L}y}(N+1-n)\right)^{\mathrm{T}}, \end{aligned} \tag{2.37}$$

where $\zeta_{\mathscr{L}y}$ is defined by (2.34) with $Y(n)$, $Y(2n)$, $Y(N+1-n)$, and $Y(N+1)$ replaced with $Y_{\mathscr{L}y}(n)$, $Y_{\mathscr{L}y}(2n)$, $Y_{\mathscr{L}y}(N+1-n)$, and $Y_{\mathscr{L}y}(N+1)$, respectively.

Proposition 2.5. *The difference operator $\mathscr{L}$ maps $L^2_\omega[0, N+2n]$ into itself.*

Proof. For $y \in L^2_\omega[0, N+2n]$, it is clear that $\mathscr{L}y = \{(\mathscr{L}y)(t)\}_{t=0}^{N+2n}$ satisfies (2.29) from (2.37). By Proposition 2.3, it suffices to show that $Y_{\mathscr{L}y}(n)$ and $Y_{\mathscr{L}y}(N+1)$ satisfy (2.35), i.e.,

$$z = P_1^*[S_1 A Y_{\mathscr{L}y}(n) + R_2 L Y_{\mathscr{L}y}(N+1)] = 0. \tag{2.38}$$

Since y satisfies (2.29), i.e., y satisfies (2.24), we get

$$Q_2\zeta = \begin{pmatrix} AY_{\mathscr{L}y(n)} \\ -LY_{\mathscr{L}y}(N+1), \end{pmatrix},$$

where ζ is determined by (2.34). Then

$$z = P_1^*(S_1, -R_2)Q_2\zeta.$$

In addition, $(S_1, -R_2)\Omega^*$ is Hermitian by Lemma 2.1 and by Proposition 2.1 (particularly L^*B is Hermitian). Hence, from (2.20),

$$z = P_1^*(S_1, -R_2)\Omega^* P_2 \Omega_0^{*-1}\zeta = P_1^*\Omega(S_1, -R_2)^* P_2 \Omega_0^{*-1}\zeta = 0.$$

The proof is complete. □

By Theorems 2.2 and 2.3, and Proposition 2.5, we have the following result directly.

Theorem 2.4. *Assume that all the assumptions in Theorem 2.3 hold. Then the difference operator $\mathscr{L}$ is self-adjoint on $L^2_\omega[0, N+2n]$.*

3. The spectral theory

3.1. The fundamental spectral results

Definition 3.1. A complex number λ is called an eigenvalue of (1.1), (1.2) if there exists $y \in L[0, N+2n]$ with $y \neq 0$ which solves problem (1.1), (1.2) and the non-zero solution y is called an eigenfunction corresponding to λ (denoted by $y(\lambda)$).

According to the discussions in Section 2, every eigenfunction of (1.1), (1.2) is in $L^2_\omega[0, N+2n]$. Therefore, the following fundamental spectral results of (1.1), (1.2) can be obtained by Theorem 2.4 and the spectral theory of self-adjoint linear operators in Hilbert spaces.

Theorem 3.1. *Assume that* (1.3) *holds and boundary conditions* (1.2) *are self-adjoint. Let* $\operatorname{rank}(R_1 L - S_1 B, S_2) = m$ *and* $\mu = (N+1)d + m - 2nd$.

(1) *The eigenvalue problem*, (1.1), (1.2), *has only* μ *real eigenvalues*

$$\lambda_1,\ \lambda_2, \ldots, \lambda_\mu$$

(*multiplicity included*) *and* μ *linearly independent eigenfunctions*

$$(\lambda_1),\ y(\lambda_2), \ldots,\ y(\lambda_\mu), \tag{3.1}$$

which are normalized and orthogonal to each other, i.e., for $1 \leqslant i, j \leqslant \mu$,

$$\langle y(\lambda_i), y(\lambda_j)\rangle = \sum_{t=n}^{N+n} y^*(t, \lambda_j)\omega(t)y(t, \lambda_i) = \delta_{ij}. \tag{3.2}$$

(2) *The eigenfunction basis of* (1.1), (1.2) *consists of* μ *linearly independent eigenfunctions* (3.1) *and is complete for the admissible function space* $L^2_\omega[0, N+2n]$, *i.e., for each* $y \in L^2_\omega[0, N+2n]$, *there exists a unique set of scalars* $\{a_k\}_{k=1}^{\mu} \subset \mathbb{C}$ *such that, for* $t \in [0, N+2n]$,

$$y(t) = \sum_{k=1}^{\mu} a_k y(t, \lambda_k), \tag{3.3}$$

where

$$a_k = \langle y, y(\lambda_k)\rangle = \sum_{t=n}^{N+n} y^*(t, \lambda_k)\omega(t)y(t), \quad 1 \leqslant k \leqslant \mu, \tag{3.4}$$

and the following Parseval's equality holds:

$$\langle y, y\rangle = \sum_{k=1}^{\mu} |a_k|^2. \tag{3.5}$$

(3) *The difference operator* $\mathscr{L}$ *has the following spectral resolution*

$$(\mathscr{L} y)(t) = \sum_{k=1}^{\mu} \lambda_k P_k y(t) = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E_\lambda \, y(t), \quad t \in [0, N+2n],$$

for each $y \in L^2_\omega[0, N+2n]$ *with the projective operators*

$$P_k y(t, \lambda_j) = \delta_{kj} y(t, \lambda_k), \quad t \in [0, N+2n],\ 1 \leqslant k, j \leqslant \mu$$

and the projective operator-valued function

$$E_\lambda = \begin{cases} \sum_{0<\lambda_k \leqslant \lambda} P_k, & \lambda \geqslant 0, \\ -\sum_{\lambda<\lambda_k \leqslant 0} P_k, & \lambda < 0. \end{cases}$$

3.2. *The variational properties of eigenvalues*

We now study the variational properties of eigenvalues for the discrete problem (1.1), (1.2).

Theorem 3.1 provides μ real eigenvalues for problem (1.1), (1.2) arranged in the nondecreasing order, $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_\mu$, and μ orthogonal and normalized eigenfunctions $y(\lambda_1),\ y(\lambda_2), \ldots, y(\lambda_\mu)$. The Rayleigh quotient for the difference equation (1.1) is defined as

$$R(y) = \langle \mathscr{L} y, y\rangle / \langle y, y\rangle$$

for $y \in L[0, N+2n]$ with $y' = \{y(t)\}_{t=n}^{N+n} \neq 0$ (cf. [11] for the continuous case).

Theorem 3.2 (Rayleigh's principle). *If all the assumptions in Theorem* 3.1 *hold, then*

$$\lambda_1 \leqslant R(y) \leqslant \lambda_\mu \quad \forall y \in L^2_\omega[0, N+2n] \ \text{ with } y \neq 0, \tag{3.6}$$

$$\lambda_1 = \min\left\{R(y)\colon\ y \in L^2_\omega[0, N+2n] \ \text{ with } y \neq 0\right\}, \tag{3.7}$$

$$\lambda_\mu = \max\left\{R(y)\colon\ y \in L^2_\omega[0, N+2n] \ \text{ with } y \neq 0\right\}, \tag{3.8}$$

and for $2 \leqslant k \leqslant \mu - 1$,

$$\begin{aligned} \lambda_k = \min\Big\{R(y)\colon\ & y \in L^2_\omega[0, N+2n], \\ & y \perp y(\lambda_j),\ 1 \leqslant j \leqslant k-1,\ y \neq 0\Big\} \qquad (3.9) \\ = \max\Big\{R(y)\colon\ & y \in L^2_\omega[0, N+2n], \\ & y \perp y(\lambda_j),\ \mu - k \leqslant j \leqslant \mu, y \neq 0\Big\}. \qquad (3.10) \end{aligned}$$

Proof. By Theorems 2.3 and 3.1, for every nontrivial $y \in L^2_\omega[0, N+2n]$, there exists a unique set of scalars $\{a_k\}_{k=1}^m \subset \mathbb{C}$ such that (3.3) and (3.4) hold and

$$\sum_{k=1}^{\mu} |a_k|^2 = \langle y, y\rangle > 0.$$

Using (3.2) and (3.3), one can find that

$$\langle \mathscr{L} y, y\rangle = \sum_{k=1}^{\mu} \lambda_k |a_k|^2.$$

Then we see that

$$R(y)=\frac{\sum_{k=1}^{\mu}\lambda_k|a_k|^2}{\sum_{k=1}^{\mu}|a_k|^2}. \tag{3.11}$$

Obviously, $R(y(\lambda_k))=\lambda_k$ $(1\leqslant k\leqslant\mu)$. Hence, (3.6), (3.7) and (3.8) follow directly from (3.11).

Furthermore, if $y\perp y(\lambda_j)$ $(1\leqslant j\leqslant k-1)$, then (3.11) holds with $a_1=\cdots=a_{k-1}=0$. This implies that $R(y)\geqslant\lambda_k$. Thus, (3.9) follows from $R(y(\lambda_k))=\lambda_k$, and (3.10) can be shown similarly. The proof is complete. □

By Lemma 2.2 and by using (2.7), Theorem 3.2 implies the following result.

Corollary 3.1. *Let all the assumptions of Theorem* 3.1 *hold. If*

$$SR^*\leqslant 0,\quad (-1)^i r_i(t)\geqslant 0,\ t\in[n,N+n+i],$$

then all the eigenvalues of (1.1) *and* (1.2) *are nonnegative. Furthermore, if*

$$r_0(t)>0,\quad t\in[n,N+n],$$

then all the eigenvalues of (1.1) *and* (1.2) *are positive.*

Next, we give the variational properties of the eigenvalues of (1.1), (1.2) involving no eigenfunctions.

Theorem 3.3 (Minimax theorem). *Let all the assumptions of Theorem* 3.1 *hold. Then, for* $1\leqslant k\leqslant\mu$,

$$\begin{aligned}\lambda_k&=\min\left\{G\left(z^{(1)},z^{(2)},\ldots,z^{(\mu-k)}\right):\ z^{(j)}\in L^2_\omega[0,N+2n],\ 1\leqslant j\leqslant\mu-k\right\}\\&=\max\left\{g\left(z^{(1)},z^{(2)},\ldots,z^{(k-1)}\right):\ z^{(j)}\in L^2_\omega[0,N+2n],\ 1\leqslant j\leqslant k-1\right\}\end{aligned}$$

with

$$\begin{aligned}&G\left(z^{(1)},z^{(2)},\ldots,z^{(\mu-k)}\right)\\&\quad=\max\left\{R(y):\ y\in L^2_\omega[0,N+2n],\ y\perp z^{(j)},\ 1\leqslant j\leqslant\mu-k,\ y\neq 0\right\}\end{aligned}$$

and

$$\begin{aligned}&g\left(z^{(1)},z^{(2)},\ldots,z^{(k-1)}\right)\\&\quad=\min\left\{R(y):\ y\in L^2_\omega[0,N+2n],\,y\perp z^{(j)},\ 1\leqslant j\leqslant k-1,\ y\neq 0\right\}.\end{aligned}$$

Proof. The proof is similar to that of (3.9) and (3.10) in Theorem 3.2 and therefore omitted. □

We observe that the element of $L^2_\omega[0,N+2n]$ satisfies boundary conditions (1.2) as well as the additional conditions (2.23) in which more data are involved. So

Theorems 3.2 and 3.3 are somewhat more difficult to use in general. Therefore, it is worth giving similar results in the larger space $\hat{L}[0, N+2n]$. To do so, we need two more lemmas.

Lemma 2.2 implies that $\dim \hat{L}[0, N+2n] = (N+1)d$. So the eigenfunction basis $\{y(\lambda_j)\}_{j=1}^{\mu}$ is not complete in $\hat{L}[0, N+2n]$ in the case $m < 2nd$. However, we still have the following partial result.

Lemma 3.1. *Assume all the assumptions in Theorem* 3.1 *hold. Then, for any* $y \in \hat{L}[0, N+2n]$, *there exists a unique set of scalars* $\{a_k\}_{k=1}^{\mu} \subset \mathbb{C}$ *such that* (3.3) *holds for* $t \in [n, N+n]$, *and* (3.4) *and* (3.5) *are valid.*

Proof. Let $y \in \hat{L}[0, N+2n]$. By Lemma 2.2, there exists a unique vector $\xi \in \mathbb{C}^{2nd}$ such that (2.21) holds. Let η be an m-vector consisting of the last m components of $P^*\xi$. From (2.21),

$$\begin{aligned} \begin{pmatrix} Y(n) \\ Y(N+1) \end{pmatrix} &= \mathrm{diag}\left\{A^{-1}, -L^{-1}\right\} Q \; \mathrm{diag}\left\{0, \Omega_0^*\right\} P^*\xi \\ &= \mathrm{diag}\left\{A^{-1}, -L^{-1}\right\} Q_2 \Omega_0^* \eta. \end{aligned} \tag{3.12}$$

It is clear that η, satisfying (3.12) for the given data $Y(n)$ and $Y(N+1)$, is unique. Similarly, for $y(\lambda_j) \in \hat{L}[0, N+2n]$ $(1 \leqslant j \leqslant \mu)$, there exists a unique vector $\eta(\lambda_j) \in \mathbb{C}^m$ such that

$$\begin{pmatrix} Y(n, \lambda_j) \\ Y(N+1, \lambda_j) \end{pmatrix} = \mathrm{diag}\left\{A^{-1}, -L^{-1}\right\} Q_2 \Omega_0^* \eta(\lambda_j), \quad 1 \leqslant j \leqslant \mu, \tag{3.13}$$

where $Y(t, \lambda_j)$ is as $Y(t)$ with $y(t)$ replaced with $y(t, \lambda_j)$. Let

$$\begin{aligned} y'^{\mathrm{T}} &= \left(y^{\mathrm{T}}(N+n), \ldots, y^{\mathrm{T}}(n)\right), \\ y'^{\mathrm{T}}(\lambda_j) &= \left(y^{\mathrm{T}}(N+n, \lambda_j), \ldots, y^{\mathrm{T}}(n, \lambda_j)\right), \\ z^{\mathrm{T}} &= \left(y^{\mathrm{T}}(N), \ldots, y^{\mathrm{T}}(2n), \eta^{\mathrm{T}}\right), \\ z^{\mathrm{T}}(\lambda_j) &= \left(y^{\mathrm{T}}(N, \lambda_j), \ldots, y^{\mathrm{T}}(2n, \lambda_j), \eta^{\mathrm{T}}(\lambda_j)\right). \end{aligned} \tag{3.14}$$

Then

$$y' = Tz, \qquad y'(\lambda_j) = Tz(\lambda_j) \tag{3.15}$$

with

$$T = \begin{pmatrix} 0 & 0 & I_{nd} \\ I_{(N+1-2n)d} & 0 & 0 \\ 0 & I_{nd} & 0 \end{pmatrix} \mathrm{diag}\left\{I_{(N+1-2n)d}, \; \mathrm{diag}\left\{A^{-1}, -L^{-1}\right\} Q_2 \Omega_0^*\right\}.$$

By (3.13) and (3.14), $z(\lambda_1), \ldots, z(\lambda_\mu)$ are linearly independent since $y(\lambda_1), \ldots, y(\lambda_\mu)$ are linearly independent. So there exists a unique set of scalars $\{a_k\}_{k=1}^{\mu} \subset \mathbb{C}$ such that

$$z = \sum_{k=1}^{\mu} a_k z(\lambda_k). \tag{3.16}$$

Then (3.3) holds for $t \in [n, N + n]$ by (3.14), (3.15) and (3.16).

Finally, (3.4) and (3.5) hold by the orthogonality and normalization (3.2) of $y(\lambda_j)$ $(1 \leqslant j \leqslant \mu)$. The proof is complete. $\square$

In the case $m < 2nd$, where boundary conditions (1.2) are improper, $Y(0)$ and $Y(N + n + 1)$ cannot be uniquely determined by $Y(n)$ and $Y(N + 1)$ from (1.2) for $y \in \hat{L}[0, N + 2n]$. Hence, $(\mathscr{L}y)(n)$ and $(\mathscr{L}y)(N + n)$ cannot be determined by y', the inner terms of y. However, in some cases, it suffices to determine $\langle \mathscr{L}y, y\rangle$ with y' and this turns out to be true.

Lemma 3.2. *Assume all the assumptions in Theorem* 3.1 *hold. Then for any* $y \in \hat{L}[0, N + 2n]$, $\langle \mathscr{L}y, y\rangle$ *is determined by* y' *and*

$$\langle \mathscr{L}y, y\rangle = \sum_{k=1}^{\mu} \lambda_k |a_k|^2, \tag{3.17}$$

where $\{a_k\}_{k=1}^{\mu}$ *is the same as in Lemma* 3.1.

Proof. Let $y \in \hat{L}[0, N + 2n]$ and let η and $\eta(\lambda_j)$ $(1 \leqslant j \leqslant \mu)$ be defined as in the proof of Lemma 3.1. Then

$$\begin{aligned}\langle \mathscr{L}y, y\rangle &= \sum_{t=n}^{N+n} \left\{ y^*(t) \sum_{i=0}^{n} \Delta^i \big[r_i(t)\Delta^i y(t-i)\big] \right\} \\ &= (Y^*(n), Y^*(N+1)) W_1 \big(Y^{\mathrm{T}}_{\mathscr{L}y}(n), Y^{\mathrm{T}}_{\mathscr{L}y}(N+1)\big)^{\mathrm{T}} + \beta(y')\end{aligned}$$

with

$$\beta(y') = \sum_{t=2n}^{N} \left\{ y^*(t) \sum_{i=0}^{n} \Delta^i \big[r_i(t)\Delta^i y(t-i)\big] \right\}.$$

From (2.25) and (2.26),

$$\langle \mathscr{L}y, y\rangle = \alpha(y) + \tilde{\beta}(y') \tag{3.18}$$

with

$$\alpha(y) = \big(Y^*(n), Y^*(N+1)\big) \operatorname{diag}\{M_1, N_1\} \big(Y^{\mathrm{T}}(0), Y^{\mathrm{T}}(N+n+1)\big)^{\mathrm{T}}$$

and

$$\begin{aligned}\tilde{\beta}(y') = &\big(Y^*(n), Y^*(N+1)\big) \operatorname{diag}\{M_2, N_2\} \\ &\times \big(Y^{\mathrm{T}}(2n), Y^{\mathrm{T}}(n), Y^{\mathrm{T}}(N+1), Y^{\mathrm{T}}(N+1-n)\big)^{\mathrm{T}}.\end{aligned}$$

From (2.8) and (2.10) and by Propositions 2.1 and 2.2, we find that $\alpha(y) = \xi^* F \xi$ for some $\xi \in \mathbb{C}^{2nd}$, where $F = \Omega(S_1, -(R_2 + S_2 \hat{B} L^{-1}))^*$ is Hermitian. Again from (2.19), we get

$$\alpha(y) = \eta^* E \eta, \tag{3.19}$$

where E is a certain $m \times m$-Hermitian matrix. Similarly,

$$\begin{aligned} \alpha_{jk} = & \left(Y^*(n, \lambda_j), Y^*(N+1, \lambda_j)\right) \operatorname{diag}\{M_1, N_1\} \\ & \times \left(Y^{\mathrm{T}}(0, \lambda_k), Y^{\mathrm{T}}(N+n+1, \lambda_k)\right)^{\mathrm{T}} \\ = & \eta^*(\lambda_j) E \eta(\lambda_k). \end{aligned} \tag{3.20}$$

Hence, $\alpha(y)$, i.e., $\langle \mathscr{L} y, y \rangle$ is determined by y' from (3.12), (3.18), and (3.19).

Moreover, from (3.14) and (3.16), we have

$$\eta = \sum_{k=1}^{\mu} a_k \eta(\lambda_k), \tag{3.21}$$

where $\{a_k\}_{k=1}^{\mu}$ are as in (3.16) or in Lemma 3.1. So, from (3.18)–(3.21) and Lemma 3.1,

$$\langle \mathscr{L} y, y \rangle = \sum_{j,k=1}^{\mu} a_j^* a_k \langle \mathscr{L} y(\lambda_k), y(\lambda_j) \rangle = \sum_{j,k=1}^{\mu} a_j^* a_k \lambda_k \delta_{jk} = \sum_{k=1}^{\mu} \lambda_k |a_k|^2.$$

This completes the proof. □

Now we can establish a Rayleigh's principle and a minimax theorem on $\hat{L}[0, N + 2n]$ by Lemmas 3.1 and 3.2, whose proofs are similar to those of Theorems 3.2 and 3.3, respectively, and so omitted.

Theorem 3.4 (Rayleigh's principle). *If all the assumptions in Theorem* 3.1 *hold, then*

$$\lambda_1 \leqslant R(y) \leqslant \lambda_\mu \quad \forall y \in \hat{L}[0, N+2n] \textit{ with } y' \neq 0,$$

$$\lambda_1 = \min\{R(y)\colon\ y \in \hat{L}[0, N+2n] \textit{ with } y' \neq 0\},$$

$$\lambda_\mu = \max\{R(y)\colon\ y \in \hat{L}[0, N+2n] \textit{ with } y' \neq 0\}$$

and for $2 \leqslant k \leqslant \mu - 1$,

$$\begin{aligned} \lambda_k = & \min\{R(y)\colon\ y \in \hat{L}[0, N+2n] \\ & \qquad \textit{with } y \perp y(\lambda_j)\ (1 \leqslant j \leqslant k-1) \textit{ and } y' \neq 0\} \\ = & \max\{R(y)\colon\ y \in \hat{L}[0, N+2n] \\ & \qquad \textit{with } y \perp y(\lambda_j)\ (\mu - k \leqslant j \leqslant \mu) \textit{ and } y' \neq 0\}. \end{aligned}$$

Theorem 3.5 (Minimax theorem). *Let all the assumptions of Theorem* 3.1 *hold. Then, for* $1 \leqslant k \leqslant \mu$,

$$\begin{aligned}\lambda_k &= \min\left\{G\left(z^{(1)}, z^{(2)}, \ldots, z^{(\mu-k)}\right):\ z^{(j)} \in \hat{L}[0, N+2n],\ 1 \leqslant j \leqslant \mu-k\right\}\\ &= \max\left\{g\left(z^{(1)}, z^{(2)}, \ldots, z^{(k-1)}\right):\ z^{(j)} \in \hat{L}[0, N+2n],\ 1 \leqslant j \leqslant k-1\right\}\end{aligned}$$

with

$$\begin{aligned}&G\left(z^{(1)}, z^{(2)}, \ldots, z^{(\mu-k)}\right)\\ &\quad= \max\left\{R(y):\ y \in \hat{L}[0, N+2n],\ y \perp z^{(j)},\ 1 \leqslant j \leqslant \mu-k,\ y' \neq 0\right\}\end{aligned}$$

and

$$\begin{aligned}&g\left(z^{(1)}, z^{(2)}, \ldots, z^{(k-1)}\right)\\ &\quad= \min\left\{R(y):\ y \in \hat{L}[0, N+2n],\ y \perp z^{(j)},\ 1 \leqslant j \leqslant k-1,\ y' \neq 0\right\}.\end{aligned}$$

3.3. *Comparison of eigenvalues*

In the final part of the section, we compare the eigenvalues of two discrete problems by applying Theorem 3.5.

Consider the following two difference equations

$$\sum_{i=0}^{n} \Delta^i\left[r_i^{(j)}(t)\Delta^i y(t-i)\right] = \lambda\omega^{(j)}(t)y(t), \quad t \in [n, N+n] \tag{3.22_j}$$

with the boundary conditions

$$R\begin{pmatrix}-u^{(j)}(0, y)\\ u^{(j)}(N+1, y)\end{pmatrix} + S\begin{pmatrix}v^{(j)}(0, y)\\ v^{(j)}(N+1, y)\end{pmatrix} = 0, \tag{3.23_j}$$

where $j = 1, 2$, $u^{(j)}(t, y)$ and $v^{(j)}(t, y)$ are as in (1.4) with $r_k(t)$ replaced with $r_k^{(j)}(t)$, $j = 1, 2$. Denote $y \in \mathscr{R}^{(j)}$ if y satisfies (3.23_j), $j = 1, 2$.

Assume that all the assumptions in Theorem 3.1 hold for (3.22_j), (3.23_j) ($j = 1, 2$) and

$$r_i^{(1)}(t) = r_i^{(2)}(t), \quad t \in [n, n+i-1] \cup [N+n+1, N+n+i] \tag{3.24}$$

for $1 \leqslant i \leqslant n$. Then, from (2.10), $B^{(1)} = B^{(2)}$ and

$$\begin{aligned}\operatorname{rank} \Omega^{(1)} &= \operatorname{rank}\left(R_1 - S_1 B^{(1)} L^{-1}, S_2\right)\\ &= \operatorname{rank}\left(R_1 - S_1 B^{(2)} L^{-1}, S_2\right)\\ &= \operatorname{rank} \Omega^{(2)}\\ &=: m,\end{aligned}$$

and then problem (3.22_j), (3.23_j) has $\mu = (N+1)d + m - 2nd$ real eigenvalues $\lambda_k^{(j)}$ $(1 \leqslant k \leqslant \mu,\ j = 1, 2)$ by Theorem 3.1, ordered increasingly.

Theorem 3.6. *Let all the assumptions hold in Theorem* 3.1 *for problems* (3.22_j), (3.23_j) ($j = 1, 2$) *and let* (3.24) *hold. If*

$$(-1)^i r_i^{(2)}(t) \geqslant (-1)^i r_i^{(1)}(t) \geqslant 0, \quad t \in [n, N+n+i],\ 0 \leqslant i \leqslant n, \tag{3.25}$$

$$SR^* \leqslant 0, \quad \omega^{(2)}(t) \leqslant \omega^{(1)}(t),\ t \in [n, N+n], \tag{3.26}$$

then

$$\lambda_k^{(2)} \geqslant \lambda_k^{(1)} \geqslant 0, \quad 1 \leqslant k \leqslant \mu. \tag{3.27}$$

Furthermore, all the first inequalities of (3.27) *are strict if one of the following conditions is satisfied*:

(1) $r_0^{(2)}(t) > r_0^{(1)}(t) \geqslant 0, \quad t \in [n, N+n]$;

(2) $\omega^{(2)}(t) < \omega^{(1)}(t)$ *and* $r_0^{(1)}(t) > 0$ *for* $t \in [n, N+n]$.

Proof. By Corollary 3.1, all the eigenvalues of problems (3.22_j), (3.23_j) ($j = 1, 2$) are nonnegative. Let

$$\hat{L}^{(j)}[0, N+2n] = \left\{y \in L[0, N+2n]:\ y \in \mathscr{R}^{(j)}\right\}.$$

Then, from (3.24),

$$\hat{L}^{(1)}[0, N+2n] = \hat{L}^{(2)}[0, N+2n] =: \hat{L}[0, N+2n].$$

For $y \in \hat{L}[0, N+2n]$ and $y' \neq 0$, let

$$\left(\mathscr{L}^{(j)}y\right)(t) = \omega^{(j)-1}(t) \sum_{i=0}^{n} \Delta^i \left[r_i^{(j)}(t) \Delta^i y(t-i)\right],$$

$$R^{(j)}(y) = \left\langle \mathscr{L}^{(j)}y, y\right\rangle_j \Big/ \langle y, y\rangle_j, \quad j = 1, 2,$$

where the inner product $\langle , \rangle_j$ is defined by (2.1) with $\omega(t)$ replaced with $\omega^{(j)}(t)$.

From (2.7), (3.24), and (3.25), by Lemma 2.2, we get that

$$\langle y, y\rangle_2 - \langle y, y\rangle_1 = \sum_{t=n}^{N+n} \left\{y^*(t)\left(\omega^{(2)} - \omega^{(1)}\right)(t) y(t)\right\} \leqslant 0,$$

$$\begin{aligned}&\left\langle \mathscr{L}^{(2)}y, y\right\rangle_2 - \left\langle \mathscr{L}^{(1)}y, y\right\rangle_1 \\ &\quad = \sum_{t=n}^{N+n} \sum_{i=0}^{n} \left\{(-1)^i \Delta^i y^*(t-i)\left(r_i^{(2)} - r_i^{(1)}\right)(t) \Delta^i y(t-i)\right\} \geqslant 0,\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\left\langle \mathscr{L}^{(1)}y, y\right\rangle_1 \\ &\quad = \sum_{t=n}^{N+n} \sum_{i=0}^{n} \left\{(-1)^i \Delta^i y^*(t-i) r_i^{(1)}(t) \Delta^i y(t-i)\right\} - \xi^* SR^* \xi \geqslant 0\end{aligned}$$

for some $\xi \in \mathbb{C}^{2nd}$. Then

$$\begin{aligned} & R^{(2)}(y) - R^{(1)}(y) \\ & \quad = \left(\left\langle \mathscr{L}^{(2)}y, y\right\rangle_2 - \left\langle \mathscr{L}^{(1)}y, y\right\rangle_1\right)/\langle y, y\rangle_2 \\ & \qquad + \left\langle \mathscr{L}^{(1)}y, y\right\rangle_1 (\langle y, y\rangle_1 - \langle y, y\rangle_2)/(\langle y, y\rangle_2 \langle y, y\rangle_1) \geqslant 0, \\ & \qquad\quad \text{for } y \in \hat{L}[0, N+2n],\ \ y' \neq 0, \end{aligned} \tag{3.29}$$

which implies (3.27) by Theorem 3.5.

Moreover, for $y \in \hat{L}[0, N+2n]$ with $y' \neq 0$, from (3.28), we have

$$\begin{aligned} & \left\langle \mathscr{L}^{(2)}y, y\right\rangle_2 - \left\langle \mathscr{L}^{(1)}y, y\right\rangle_1 \\ & \quad \geqslant \sum_{t=n}^{N+n} y^*(t)\left(r_0^{(2)} - r_0^{(1)}\right)(t)y(t) > 0 \quad \text{if (i) holds;} \end{aligned}$$

$$\left\langle \mathscr{L}^{(1)}y, y\right\rangle_1 > 0, \quad \langle y, y\rangle_2 - \langle y, y\rangle_1 < 0 \quad \text{if (ii) holds.}$$

Hence, $R^{(2)}(y) > R^{(1)}(y)$ from (3.29) and all the first inequalities of (3.27) are strict if one of (i) and (ii) holds. This completes the proof. □

Theorem 3.7. *Let all the assumptions hold in Theorem* 3.1 *for problems* (3.22_j), (3.23_j) $(j = 1, 2)$ *and let* (3.24) *hold. If* $\omega^{(2)}(t) = \omega^{(1)}(t)$ *for* $t \in [n, N+n]$ *and*

$$(-1)^i r_i^{(2)}(t) \geqslant (-1)^i r_i^{(1)}(t), \quad t \in [n+i, N+n],\ 0 \leqslant i \leqslant n, \tag{3.30}$$

then

$$\lambda_k^{(2)} \geqslant \lambda_k^{(1)}, \quad 1 \leqslant k \leqslant \mu. \tag{3.31}$$

Furthermore, if the inequalities in (3.30) *are strict for* $i = 0$, *then all the inequalities of* (3.31) *are strict.*

Proof. The proof is similar to that of Theorem 3.6 and so omitted. □

4. The proper case and the multiplicity of eigenvalues

In this section, we will more closely discuss the conditions which assure the spectral results in the proper case and consider the multiplicity of eigenvalues in general cases.

4.1. The proper case

In the proper case $m = \text{rank } \Omega = 2nd$, condition (1.3) on $r_n(t)$ and the restriction on the integer N can both be relaxed.

Rewrite boundary conditions (1.2) as (2.18). We notice that the condition "$N \geqslant n-1$" is required by (2.18) and the coefficient matrix $H = \Omega \, \text{diag}\{L, -\hat{A}\}$ is nonsingular if Ω is nonsingular and

$$r_n(t) \text{ is nonsingular on } [N+n+1, N+2n]. \tag{4.1}$$

If this is the case, $Y(0)$ and $Y(N+n+1)$ can be determined by $Y(n)$ and $Y(N+1)$ from (2.18) for $y \in \mathscr{R}$. Then $L^2_\omega[0, N+2n] = \hat{L}[0, N+2n]$ and $\mu = (N+1)d$. One can show that the difference operator $\mathscr{L}$ maps $\hat{L}[0, N+2n]$ into itself by extending the definition of $(\mathscr{L}y)(t)$ to $[0, n-1] \cup [N+n+1, N+2n]$ as

$$\begin{pmatrix} Y_{\mathscr{L}y}(0) \\ Y_{\mathscr{L}y}(N+n+1) \end{pmatrix} = \text{diag}\left\{L^{-1}, -\hat{A}^{-1}\right\}\Omega^{-1}(S_1 A, R_2 L + S_2 \hat{B}) \times \begin{pmatrix} Y_{\mathscr{L}y}(n) \\ Y_{\mathscr{L}y}(N+1) \end{pmatrix}.$$

Hence, $\mathscr{L}$ is self-adjoint on $\hat{L}[0, N+2n]$ by Theorem 2.2. This implies the following spectral results in the proper case.

Theorem 4.1. *Assume that* $N \geqslant n-1$ *and* (4.1) *hold. If boundary conditions* (1.2) *are self-adjoint and proper, then all the results in Theorems* 3.1–3.3 *are true and Theorems* 3.2 *and* 3.3 *agree with Theorems* 3.4 *and* 3.5*, respectively. If* (3.24) *and the above assumptions for* (3.22$_j$), (3.23$_j$) $(j = 1, 2)$ *hold, then the results in Theorems* 3.6 *and* 3.7 *hold.*

Moreover, the following is the dual orthogonality of eigenfunctions in the proper case.

Theorem 4.2 (Dual orthogonality). *If all the assumptions in Theorem* 4.1 *hold, then the* $\mu = (N+1)d$ *eigenfunctions satisfy the dual orthogonality*

$$\sum_{k=1}^{\mu} y(t, \lambda_k) y^*(s, \lambda_k) = \delta_{ts}\omega^{-1}(s), \quad t, s \in [n, N+n]. \tag{4.2}$$

Proof. Let

$$K = (y(i, \lambda_j))_{\mu\times\mu} \quad \text{for } i \in [n, N+n] \quad \text{and} \quad 1 \leqslant j \leqslant \mu.$$

Then (3.2) can be written as

$$K^* \text{diag}\left\{\omega(n), \omega(n+1), \ldots, \omega(N+n)\right\} K = I_\mu.$$

So K is nonsingular and

$$KK^* = \operatorname{diag}\{\omega^{-1}(n), \omega^{-1}(n+1), \ldots, \omega^{-1}(N+n)\},$$

which implies (4.2). The proof is complete. □

Remark 4.1. By Theorem 4.1, we see that the dimension μ of $L^2_\omega[0, N+2n]$, i.e., the number of eigenvalues, i.e., the number of the linearly independent eigenfunctions, is independent of the order of (1.1) in the proper case.

Remark 4.2. The Dirichlet boundary conditions

$$y(0) = \cdots = y(n-1) = 0, \quad y(N+n+1) = \cdots = y(N+2n) = 0 \tag{4.3}$$

are special cases of (1.2) with

$$R = \begin{pmatrix} I_{nd} & 0 \\ 0 & -BL^{-1} \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 0 \\ 0 & I_{nd} \end{pmatrix}.$$

Clearly, by Proposition 2.1, we have

$$\operatorname{rank}(R, S) = 2nd, \quad RS^* = SR^*, \ \Omega = I_{2nd}.$$

Hence, conditions (4.3) are self-adjoint and proper, and Theorems 4.1 and 4.2 hold for the Dirichlet boundary value problem (1.1) and (4.3).

4.2. The multiplicity of eigenvalues

Jirari in [9] studied the second-order scalar Sturm–Liouville problem

$$\Delta[r_1(t)\Delta y(t-1)] + r_0(t)y(t) = \lambda\omega(t)y(t), \quad t \in [1, N+1], \tag{4.4}$$

$$y(0) + hy(1) = 0, \quad y(N+2) + ky(N+1) = 0, \tag{4.5}$$

where

$$\begin{aligned} &\omega(t) > 0, \ r_1(t) \text{ and } r_0(t) \text{ are real-valued functions on } [1, N+2] \\ &\text{and } [1, N+1], \ \text{respectively, and the constants } h \text{ and } k \text{ are real.} \end{aligned} \tag{4.6}$$

By the oscillation method, he obtained that all the eigenvalues are simple and each eigenvalue corresponds to a unique eigenfunction (up to nonzero multiples) provided that $r_1(t) > 0$ for $t \in [1, N+2]$. However, for second-order scalar Sturm–Liouville problems with non-separate boundary conditions, for second-order vector Sturm–Liouville problems, and for higher-order problems, (1.1), (1.2), whether in the scalar or vector case, eigenvalues may not be simple and hence a given eigenvalue may correspond to more than one linearly independent eigenfunctions.

We now proceed to discuss the multiplicity of eigenvalues and the number of the corresponding linearly independent eigenfunctions for problem (1.1), (1.2) in general cases.

Rewrite (1.1) as

$$\begin{aligned} &r_n(t+n)y(t+n)+r_n(t)y(t-n)\\ &\quad =\lambda\omega(t)y(t)+l(t,y(t+1-n),\dots,y(t+n-1)) \end{aligned} \tag{4.7}$$

for $t\in[n,N+n]$, where $l(t,x_1,\dots,x_{2n-1})$ is linear in $x_1,\dots,x_{2n-1}$. Suppose that

$$r_n(t) \text{ is nonsingular on } [2n,N+2n]. \tag{4.8}$$

Then, for any given initial values $Y(0)$ and $Y(n)$, the initial value problem has a unique solution $y(t,\lambda)$, which can be represented by

$$Y(t,\lambda)=\Phi(t,\lambda)\big(Y^{\mathrm T}(0),Y^{\mathrm T}(n)\big)^{\mathrm T},\quad 2n\leqslant t\leqslant N+n+1, \tag{4.9}$$

where $\Phi(t,\lambda)$ is an $nd\times 2nd$-matrix, and its elements are $d\times d$-matrix coefficient polynomials of λ.

Inserting $Y(t,\lambda)$ into boundary conditions (1.2) or (2.18), we get

$$\mathscr{M}(\lambda)\big(Y^{\mathrm T}(0),Y^{\mathrm T}(n)\big)=0, \tag{4.10}$$

where

$$\begin{aligned}\mathscr{M}(\lambda)=&(-R_1L+S_1B,S_1A)+(R_2L+S_2\hat B,S_2\hat A)\\ &\times\big(\Phi^{\mathrm T}(N+1,\lambda),\Phi^{\mathrm T}(N+n+1,\lambda)\big)^{\mathrm T}\end{aligned}$$

is an $2nd\times 2nd$-matrix and its elements are polynomials of λ. Hence, λ_0 is an eigenvalue for problem (1.1), (1.2) if and only if $\det\mathscr{M}(\lambda_0)=0$.

Definition 4.1. λ_0 is called an eigenvalue of algebraic multiplicity $k(\lambda_0)$ for problem (1.1), (1.2) if λ_0 is a zero of multiplicity $k(\lambda_0)$ of $\det\mathscr{M}(\lambda)$.

According to the discussions in Section 2, we see that the additional conditions (2.23) are intrinsic for problem (1.1), (1.2) and then the eigenvalue problem, (1.1), (1.2), is equivalent to the spectral problem of the difference operator $\mathscr{L}$ on $L^2_\omega[0,N+2n]$. Hence, we have the following results by Theorems 3.1 and 4.1.

Theorem 4.3. *Suppose that* $N\geqslant 2n-1$ *and*

$$r_n(t) \textit{ is nonsingular on } [n,N+2n] \tag{4.11}$$

in the general case $m\leqslant 2nd$ *and that* $N\geqslant n-1$ *and* (4.8) *holds in the proper case. Then the degree of* $\det\mathscr{M}(\lambda)$ *in* λ *is equal to* μ *and the geometric multiplicity of each eigenvalue* λ *of problem (1.1), (1.2) is equal to its algebraic multiplicity* $k(\lambda)$*.*

Theorem 4.4. *If* (4.11) *holds, then, for any eigenvalue* λ *of problem,* (1.1), (1.2),

$$k(\lambda)\leqslant\min\{\operatorname{rank}(R_1,S_1),\ \operatorname{rank}(R_2,S_2)\}. \tag{4.12}$$

Proof. Let λ be any given eigenvalue of problem, (1.1), (1.2), and $y(t, \lambda)$ be a eigenfunction corresponding to λ.

Since $y(t, \lambda) \in \mathscr{R}$ and A is nonsingular, by Lemma 2.2 and (2.10), there exists $\xi \in \mathbb{C}^{2nd}$ such that

$$\begin{pmatrix} Y(0, \lambda) \\ Y(n, \lambda) \end{pmatrix} = \operatorname{diag}\left\{L^{-1}, A^{-1}\right\}\left(S_1, R_1 - S_1 B L^{-1}\right)^* \xi. \tag{4.13}$$

Obviously, $\operatorname{rank}(S_1, R_1 - S_1 B L^{-1}) = \operatorname{rank}(R_1, S_1) =: k_1$. Thus, there exist at most k_1 linearly independent vectors $(Y^{\mathrm{T}}(0, \lambda), Y^{\mathrm{T}}(n, \lambda))^{\mathrm{T}}$ satisfying (4.13). Again from (4.9), there exist at most k_1 linearly independent eigenfunctions corresponding to λ. By Theorem 4.3, we get

$$k(\lambda) \leqslant k_1. \tag{4.14}$$

On the other hand, by referring to the nonsingularity of $r_n(t)$ on $[n, N + 2n]$, (1.1) or (4.7) has a unique solution $y(t, \lambda)$ for any given initial values $Y(N + 1)$ and $Y(N + n + 1)$, and $y(t, \lambda)$ can be represented as

$$Y(t, \lambda) = \Psi(t, \lambda)\left(Y^{\mathrm{T}}(N + 1), Y^{\mathrm{T}}(N + n + 1)\right)^{\mathrm{T}}, \quad 0 \leqslant t \leqslant N.$$

For the given eigenvalue λ and its eigenfunction $y(t, \lambda)$, by Lemma 2.2, (2.10), and the nonsingularity of $\hat{A}$, there exists $\xi \in \mathbb{C}^{2nd}$ such that

$$\begin{pmatrix} Y(N + 1, \lambda) \\ Y(N + n + 1, \lambda) \end{pmatrix} = \operatorname{diag}\left\{-L^{-1}, \hat{A}^{-1}\right\}\left(S_2, R_2 + S_2 \hat{B} L^{-1}\right)^* \xi.$$

Hence, similarly to the above discussions, one concludes that

$$k(\lambda) \leqslant \operatorname{rank}(R_2, S_2),$$

which with (4.14) implies (4.12). The proof is complete. □

We now consider the special case, problem (1.1) with the following spatially separate boundary conditions:

$$\begin{aligned} &R^{(1)} u(0, y) - S^{(1)} v(0, y) = 0, \\ &R^{(2)} u(N + 1, y) + S^{(2)} v(N + 1, y) = 0, \end{aligned} \tag{4.15}$$

where $R^{(j)}$ and $S^{(j)} (j = 1, 2)$ are $nd \times nd$-matrices and satisfy

$$R^{(j)} S^{(j)*} = S^{(j)} R^{(j)*}, \quad \operatorname{rank}(R^{(j)}, S^{(j)}) = nd, \ \ j = 1, 2. \tag{4.16}$$

It is evident that boundary conditions (4.15) can be written as (1.2) with

$$R = \begin{pmatrix} R^{(1)} & 0 \\ 0 & R^{(2)} \end{pmatrix}, \qquad S = \begin{pmatrix} S^{(1)} & 0 \\ 0 & S^{(2)} \end{pmatrix},$$

and hence $\operatorname{rank}(R, S) = 2nd$ and $RS^* = SR^*$ from (4.16). Then boundary conditions (4.15) are self-adjoint by Lemma 2.1.

For boundary conditions (4.15), we have

$$\Omega = \begin{pmatrix} R^{(1)} - S^{(1)} B L^{-1} & 0 \\ 0 & S^{(2)} \end{pmatrix}.$$

So the following consequence directly follows from Theorems 3.1–3.7 and 4.1–4.4.

Corollary 4.1. *Assume that* (4.16) *holds. Let* $\operatorname{rank}[R^{(1)}L - S^{(1)}B] = m_1$, $\operatorname{rank} S^{(2)} = m_2$, *and* $\mu = (N+1)d + m_1 + m_2 - 2nd$. *Then, for problem* (1.1) *and* (4.15),

(1) *Theorems* 3.1–3.7 *are true if* $N \geqslant 2n-1$ *and* (1.3) *holds*;
(2) *Theorems* 4.1 *and* 4.2 *are true if* $m_1 = m_2 = nd,\ N \geqslant n-1$, *and* (4.1) *holds*;
(3) *Theorem* 4.3 *is true if* $N \geqslant 2n-1$ *and* (4.11) *holds or* $m_1 = m_2 = nd,\ N \geqslant n-1$, *and* (4.8) *holds*;
(4) $k(\lambda) \leqslant nd$ *for every eigenvalue* λ *of problem* (1.1) *with* (4.15) *if* (4.11) *holds.*

Remark 4.3. Clearly, if

$$r_1(t) \neq 0 \quad \text{for } t \in [1, N+2], \tag{4.17}$$

and $n = 1$ and $d = 1$, then problem (1.1) with (4.15) includes the second-order scalar problem, (4.4), (4.5), and the assumptions in Corollary 4.1 hold. Therefore, Theorems 2.3.1–2.3.3 of [9] are improved and $k(\lambda) = 1$ for any eigenvalue λ, i.e., all eigenvalues of problem (4.4), (4.5) are simple if (4.6) and (4.17) hold. Then Theorem 2.2.6 of [9] is improved.

5. Proof of Lemma 2.3

Lemma 5.1. *If* $r(t)$ *is a* $d \times d$*-Hermitian matrix on* $[k, 2k-1] (k \geqslant 1)$ *and* $D(k, r)$ *is defined as in Proposition* 2.1, *then*

$$\begin{aligned} \tau(k, r, x, y) :=& \sum_{j=0}^{k-1} (-1)^j \Big\{ \Delta^j y^*(k-1-j) \Delta^{k-1-j} \big[r(k) \Delta^k x(0) \big] \\ & \qquad - \Delta^{k-1-j} \big[r(k) \Delta^k y(0) \big]^* \Delta^j x(k-1-j) \Big\} \\ =& [X(\cdot, k), Y(\cdot, k)]_{D(k,r)}(0, k), \end{aligned} \tag{5.1$_k$}$$

where the bracket function is defined by (2.12) *and* $Y(t, k)$ *by* (2.9).

Proof. We will show that (5.1) hold by induction.

Let $k = 1$. Clearly,

$$\begin{aligned} \tau(1, r, x, y) &= y^*(0) r(1) \Delta x(0) - [r(1) \Delta y(0)]^* x(0) \\ &= y^*(0) r(1) x(1) - [x^*(0) r(1) y(1)]^*, \end{aligned}$$

that is, (5.1_1) holds.

Assume that (5.1_k) holds for a positive integer k, and we will show that (5.1_{k+1}) holds. To do so, we rewrite $\tau(k+1, r, x, y)$ as

$$\tau(k+1, r, x, y) = \alpha(k+1, r, x, y) - \beta(k+1, r, x, y) \tag{5.2}$$

with

$$\begin{aligned}
&\alpha(k+1, r, x, y) \\
&\quad = y^*(k)\Delta^k\big[r(k+1)\Delta^{k+1}x(0)\big] - \Delta^k\big[r(k+1)\Delta^{k+1}y(0)\big]^* x(k)
\end{aligned}$$

$$\begin{aligned}
&\beta(k+1, r, x, y) \\
&\quad = \sum_{j=1}^{k}(-1)^{j-1}\big\{\Delta^j y^*(k-j)\Delta^{k-j}\big[r(k+1)\Delta^{k+1}x(0)\big] \\
&\qquad\qquad - \Delta^{k-j}\big[r(k+1)\Delta^{k+1}y(0)\big]^*\Delta^j x(k-j)\big\}.
\end{aligned}$$

By a straightforward calculation and (2.28), we find that

$$\begin{aligned}
\alpha(k+1, r, x, y) = {}& y^*(k)a(r)X(k+1, k+1) \\
& - Y^*(k+1, k+1)a^*(r)x(k) \\
& + y^*(k)b(r)X(0, k+1) \\
& - Y^*(0, k+1)b^*(r)x(k)
\end{aligned} \tag{5.3}$$

with $a(r) = (a_1, a_2, \dots, a_{k+1})(r)$, $b(r) = (b_1, b_2, \dots, b_{k+1})(r)$ and

$$a_i(r) = (-1)^{i-1}\sum_{j=0}^{i-1} C_k^j C_{k+1}^{i-j-1} r(2k+1-j), \tag{5.4}$$

$$b_i(r) = (-1)^{k+i}\sum_{j=0}^{k+1-i} C_k^{i+j-1} C_{k+1}^{k+1-j} r(2k+2-i-j). \tag{5.5}$$

However, $\beta(k+1, r, x, y)$ can be rewritten as

$$\beta(k+1, r, x, y) = \tau(k, r', x', y')$$

by letting

$$r'(t) = r(t+1), \quad x'(t) = \Delta x(t), \quad y'(t) = \Delta y(t). \tag{5.6}$$

Thus, we get that

$$\beta(k+1, r, x, y) = [X'(\cdot, k), Y'(\cdot, k)]_{D(k,r')}(0, k),$$

where $X'(t, k)$ and $Y'(t, k)$ are as $X(t, k)$ and $Y(t, k)$ with $x(t)$ and $y(t)$ replaced with $x'(t)$ and $y'(t)$, respectively. From (5.6),

$$\begin{aligned}
&Y'(0, k) = L_1 Y(0, k+1), \\
&Y'(k, k) = L_2 Y(k+1, k+1) - \big(0, \dots, 0, y^{\mathrm{T}}(k)\big)^{\mathrm{T}}
\end{aligned} \tag{5.7}$$

with

$$L_1 = (I_{kd}, 0_{kd\times d}) - (0_{kd\times d}, I_{kd}),$$

$$L_2 = (0_{kd\times d}, I_{kd}) - \begin{pmatrix} 0 & I_{(k-1)d} \\ 0_{d\times 2d} & 0 \end{pmatrix}.$$

Moreover, relations (5.7) hold for X'. Therefore,

$$\begin{aligned} &\beta(k+1, r, x, y) \\ &\quad = [X(\cdot, k+1), Y(\cdot, k+1)]_{L_1^* D(k,r')L_2}(0, k+1) \\ &\qquad - Y^*(0, k+1)E(k, r')x(k) + y^*(k)E^*(k, r')X(0, k+1) \end{aligned} \tag{5.8}$$

with

$$E^*(k, r') = \left(d_{1k}^*, -d_{1,k}^* + d_{2,k}^*, \ldots, -d_{k-1,k}^* + d_{k,k}^*, -d_{k,k}^*\right)(k, r').$$

From (2.15), (5.4), and (5.5) and by the Hermiticity of $r(t)$ on $[k, 2k-1]$, one concludes that for $2 \leqslant j \leqslant k$,

$$\begin{aligned} &-d_{j-1,k}(k, r') + d_{j,k}(k, r') = b_j^*(r), \\ &d_{1,k}(k, r') = d_{1,k}^*(k, r'), \\ &b_1(r) = b_1^*(r), \\ &-d_{k,k}(k, r') = b_{k+1}^*(r), \\ &\left(a^{\mathrm{T}}(r), 0_{kd\times(k+1)d}^{\mathrm{T}}\right)^{\mathrm{T}} - L_1^* D(k, r')L_2 = D(k+1, r). \end{aligned} \tag{5.9}$$

Therefore, (5.1_{k+1}) follows from (5.2), (5.3), (5.8) and (5.9). By induction, the proof is complete. □

Proof of Lemma 2.3. Since (2.17) can be obtained by shifting all the functions in (2.16) to the right-hand side with $N+1$ units, it suffices to show that (2.16) holds.

Changing the order of the summation, from (1.4), we have

$$\begin{aligned} \gamma &= u^*(0, y)v(0, x) - v^*(0, y)u(0, x) \\ &= \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} (-1)^j \\ &\quad \times \left\{\Delta^j y^*(n-1-j)\Delta^{k-1-j}\left[r_k(n)\Delta^k x(n-k)\right]\right. \\ &\qquad \left. - \Delta^{k-1-j}\left[r_k(n)\Delta^k y(n-k)\right]^* \Delta^j x(n-1-j)\right\} \\ &= \sum_{k=1}^{n} \theta(k, x, y) \end{aligned} \tag{5.10}$$

with

$$\theta(k,x,y)=\sum_{j=0}^{k-1}(-1)^j\Big\{\Delta^j y^*(n-1-j)\Delta^{k-1-j}\big[r_k(n)\Delta^k x(n-k)\big] - \Delta^{k-1-j}\big[r_k(n)\Delta^k y(n-k)\big]^*\Delta^j x(n-1-j)\Big\}.$$

Let $k \in [1,n]$ and let

$$\tilde{r}(t)=r_k(n+t-k), \quad \tilde{x}(t)=x(n+t-k), \quad \tilde{y}(t)=y(n+t-k).$$

By Lemma 5.1, we then have that

$$\begin{aligned}\theta(k,x,y) &= \tau(k,\tilde{r},\tilde{x},\tilde{y})\\ &= [\tilde{X}(\cdot,k),\tilde{Y}(\cdot,k)]_{D(k,\tilde{r})}(0,k)\\ &= [X(\cdot,k),Y(\cdot,k)]_{D(k,\hat{r}_k)}(n-k,n)\\ &= Y^*(0)\begin{pmatrix}0 & D(k,\hat{r}_k)\\ 0_{(n-k)d} & 0\end{pmatrix}X(n)\\ &\quad - Y^*(n)\begin{pmatrix}0 & D(k,\hat{r}_k)\\ 0_{(n-k)d} & 0\end{pmatrix}^* X(0),\end{aligned} \tag{5.11}$$

where $\tilde{X}(t,k)$ and $\tilde{Y}(t,k)$ are defined as $X(t,k)$ and $Y(t,k)$ by (2.9) with $x(t)$ and $y(t)$ replaced with $\tilde{x}(t)$ and $\tilde{y}(t)$, respectively; $D(k,\hat{r}_k)$ and $\hat{r}_k$ are as in Proposition 2.1. Hence, (5.10) and (5.11) imply (2.16). This completes the proof. □

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