Normal Maps and Zero Discord States

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Abstract

In this paper we develop some tools to detect quantum discord using normal maps, whose role in the theory is analogous to the role that positive maps play in detecting entanglement. We provide a necessary condition that zero-quantum discord states must satisfy with respect to normality-preserving maps, that mirrors the relation between positive maps and separable states, supplying examples to show that this condition is not sufficient. Finally, we use said necessary condition to develop a computationally simple test for zero quantum discord, that has all the qualities of the positive partial trace

1 Introduction

The presence of non-classical correlations in quantum mechanics has been known for a very long time; these correlations play a fundamental role in the theory of quantum information and make it strikingly different from the classical theory.

Quantum discord is a type of quantum correlation that presents itself when one tries to compute the relative information between two subsystems A and B of a larger system.
In classical probability theory, there are two definitions of this quantity:

$$I(A;B) = H(A) + H(B) - H(A,B)$$

$$J(A;B) = H(A) - H(A|B)$$

Where $H(A)$ is the entropy of subsystem A, $H(A,B)$ is the joint entropy, and $H(A|B)$ is the conditional entropy. It is a standard exercise in classical probability theory to show that these two quantities are the same.

In quantum probability theory, however, the classical proof down breaks due to noncommutativity they can take different values. The difference between these two quantities is known as quantum discord.

Quantum discord turns out to be a more sensitive quantity than entanglement. This can be seen from the fact that, while every unentangled state X over a bipartite system $\mathcal{H}_a \otimes \mathcal{H}_b$ can be written as $\sum_i p_i \rho_i^A \otimes \rho_i^B$, where p_i add up to one and ρ_i^A, ρ_i^B are states over their respective spaces, a zero quantum discord (ZQD) state ρ has a more constrained representation:

$$\rho = \sum_i p_i \Pi_i \otimes \rho_i^B \tag{1}$$

where $\sum_i p_i = 1$, ρ_i^B is a state in B, and $\{\Pi_i\}$ is a complete set of rank-one orthogonal projectors on $\mathcal{H}_a$ so that $\Pi_i \Pi_j = \delta_{ij} \Pi_i$ and $\Pi_i^\dagger = \Pi_i$ [KMV12].

1.1 Normal operators

The second important notion in this paper is that of normal operators, operators that commute with their hermitian conjugates.

$$[N, N^\dagger] = 0 \implies NN^\dagger = N^\dagger N$$

While quantum mechanics typically deals with hermitian operators due to their correspondence with physical observables, here we will focus on normal ones. It turns out that, just like hermitian operators, normal operators can be diagonalized (in fact, every diagonalizable operator is normal); their eigenvalues, however, don't need to be real.

Despite their lesser fame, normal operators are still useful in quantum information theory: [YG13] showed a link between maps that send normal operators to normal operators: if $\Lambda \in \mathcal{B}(\mathcal{B}(H_a))$ is a map of this kind and ρ is a ZQD state over $\mathcal{H}_a \otimes \mathcal{H}_a$, then $(I \otimes \Lambda)\rho$ is also a ZQD over the same bipartition.

1.2 Positive operators and entanglement

Positive maps play an important role in quantum information theory, related to their deep connection with entanglement [DC14]. Here we briefly review the aspects of this connection that bear resemblance to our results.

Recall that an unentangled (*separable*) state ρ in $\mathcal{H}_{ab}$ is defined as a state that can be represented as $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$.

The importance of positive maps in the theory of entanglement is given by the following theorem:

Theorem 1. *A state $\rho \in \mathcal{H}_{ab}$ is separable if and only if*

$$(I \otimes \Phi)\rho \geq 0 \tag{2}$$

For all separable states ρ.

If one wishes to show that a given state is entangled, it suffices to exibit a positive map Φ for which 2 fails.

In this paper, we will develop a parallel theory, but we will replace positive maps with normal ones, and we will use them to detect quantum discord.

2 Normal maps

We present a brief review of normal matrices and normality preserving maps. An operator $X : \mathcal{H}_a \to \mathcal{H}_b$ is called normal if it satisfies $[X, X^\dagger] = 0$.
A map (superoperator) $\Lambda : \mathcal{B}(\mathcal{H}_{ab}) \to \mathcal{B}(\mathcal{H}_{ab})$ is said to be normality-preserving if it maps normal operators to normal operators. In symbols:

$$[\Lambda(X), \Lambda(X)^\dagger] = 0 \tag{3}$$

For every normal operator X.
A related, but surprisingly distinct, property is hermiticity preservation. A map is hermiticity-preserving if it maps hermitian operators to hermitian operators.
It is well known that one can decompose operators via the Toeplitz decomposition $A = H + iK$ into a hermitian and anti-hermitian part. Barker, Hill, and Haertel [GPB84] showed that there is an analogous decomposition for maps:

$$\Lambda = \mathcal{R}\Lambda + i\mathcal{I}\Lambda$$

where $\mathcal{R}\Lambda$ and $\mathcal{I}\Lambda$ are hermiticity preserving. Kunicki and Hill [CMK91] use this decomposition to prove many equivalent conditions for Λ to be normality preserving, and furthermore derive a structural theorem on normality preserving maps which we state here:

Theorem 2. *Let $n \geq 3$. Then L is normality preserving if and only if the range of Λ is normal or there exist a unitary matrix U, a scalar c, and a linear function f such that Λ has one of the following forms:*
(i)$\Lambda(X) = cU^{\dagger}XU + f(X)I$ for all $X \in \mathcal{M}_n$
(ii)$\Lambda(X) = cU^{\dagger}X^{tr}U + f(X)I$ for all $X \in \mathcal{M}_n$
where $\mathcal{M}_n$ is the set of all complex n by n matrices.

For simplicity, we will restrict our attention to form (i) above. After also requiring that these maps be trace preserving, we obtain the following representation:

$$\Lambda(\rho) = cU^{\dagger}\rho U + (1-c)Tr[\rho]\frac{I}{d}$$

where c is a complex scalar and U is a unitary matrix. Note that in the case when c is a real number, this becomes identical with the notion of an isotropic channel [YG13], the difference being that we are not restricting Λ to be a CP-map. In fact, the tests we develop in the following section rely on c not being a real number. Notice that these results also depend on the dimension of the Hilbert space on which Λ acts being greater than 2.

3 Normal maps as tests for zero discord states

As mentioned earlier, given a positive map Φ, it is not always the case that $I \otimes \Phi$ is also a positive map. Furthermore, we can use such a map as a test for entanglement due to the fact that $(I \otimes \Lambda)\rho$ is always positive when ρ is separable. As we will show in 5, a similar test for quantum discord can constructed from normality preserving maps Λ. This is because $I \otimes \Lambda$ is not normality preserving in general, yet it does preserve the normality of all states with zero quantum discord. These results are summarized in the following table.

Positive maps	Normal maps
$\Phi \geq 0$	Λ is normality preserving
$I \otimes \Phi \ngeq 0$	$I \otimes \Lambda$ is not normality preserving
$(I \otimes \Phi)\rho \geq 0\ \forall \rho$ separable	$[(I \otimes \Lambda)\rho, ((I \otimes \Lambda)\rho)^{\dagger}] = 0\ \forall \rho$ ZQD

In order to show these results, we first prove a few technical lemmas.

Lemma 3. the action of $(I \otimes \Lambda)$ on ρ can be represented as

$$(I \otimes \Lambda)\rho = c(I \otimes U)\rho(I \otimes U^{\dagger}) + (1-c)Tr_B[\rho] \otimes \frac{I}{d} \tag{4}$$

Proof. To prove this we first write $\rho = \sum_i A_i \otimes B_i$ where A_i and B_i are operators on $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively. Now we have:

$$\begin{aligned}(I \otimes \Lambda)\rho &= \sum_i A_i \otimes (cUB_iU^{\dagger} + (1-c)Tr[B_i]\frac{I}{d}) \\ &= c\sum_i A_i \otimes UB_iU^{\dagger} + \frac{(1-c)}{d}(\sum_i A_iTr[B_i]) \otimes I \\ &= c(I \otimes U)\rho(I \otimes U^{\dagger}) + (1-c)Tr_B[\rho] \otimes \frac{I}{d}\end{aligned}$$

Lemma 4. Consider a state ρ and a normality-preserving map $\Lambda = cU[\cdot]U^\dagger + (1-c)\mathrm{Tr}[\cdot]\frac{I}{d}$ with non-real c.
Then $(I \otimes \Lambda)\rho$ is normal if and only if $[(I \otimes U)\rho(I \otimes U^\dagger), \mathrm{Tr}_B[\rho] \otimes I] = 0$

Proof:
$0 = [(I \otimes \Lambda)\rho, ((I \otimes \Lambda)\rho)^\dagger] =$

$$
\begin{aligned}
&= [c(I \otimes U)\rho(I \otimes U^\dagger) + (1-c)\mathrm{Tr}_B[\rho] \otimes \frac{I}{d}, c^*(I \otimes U)\rho(I \otimes U^\dagger) + (1-c^*)\mathrm{Tr}_B[\rho] \otimes \frac{I}{d}] \\
&= c(1-c^*)[(I \otimes U)\rho(I \otimes U^\dagger), \mathrm{Tr}_B[\rho] \otimes I] - c^*(1-c)[(I \otimes U)\rho(I \otimes U^\dagger), \mathrm{Tr}_B[\rho] \otimes I] \\
&= (c(1-c^*) - c^*(1-c))[(I \otimes U)\rho(I \otimes U^\dagger), \mathrm{Tr}_B[\rho] \otimes I] \\
&\propto [(I \otimes U)\rho(I \otimes U^\dagger), \mathrm{Tr}_B[\rho] \otimes I]
\end{aligned}
$$

Note that c being nonreal is a necessary condition for the lemma to hold.

We can now state our main theorem and provide some examples.

Theorem 5. *If ρ is a zero quantum discord state, $(I \otimes \Lambda)\rho$ is normal for all normal maps Λ.*

Proof. If ρ is a ZQD state, then it can be written as $\rho = \sum_i p_i \Pi_i \otimes \rho_i$ thus $(I \otimes \Lambda)\rho = \sum_i p_i \Pi_i \otimes \Lambda(\rho_i)$. Now we may simply calculate the commutator:

$$
\begin{aligned}
[(I \otimes \Lambda)\rho, ((I \otimes \Lambda)\rho)^\dagger] &= \sum_{ij} p_i p_j (\Pi_i \Pi_j^\dagger \otimes \Lambda(\rho_i)\Lambda(\rho_j)^\dagger - \Pi_j^\dagger \Pi_i \otimes \Lambda(\rho_j)^\dagger \Lambda(\rho_i)) \\
&= \sum_i p_i^2 \Pi_i \otimes (\Lambda(\rho_i)\Lambda(\rho_i)^\dagger - \Lambda(\rho_i)^\dagger \Lambda(\rho_i)) \\
&= \sum_i p_i^2 \Pi_i \otimes [\Lambda(\rho_i)), \Lambda(\rho_i)^\dagger] = 0
\end{aligned}
$$

where the last equality comes from Λ being normality preserving and the ρ_i being hermitian and therefore normal.

Example 6. Hermitian operator mapped into a not-normal one.
An example is provided by the following setup.
$|\psi\rangle = \frac{1}{\sqrt{5}}(|00\rangle + 2i|11\rangle)$, $U = I$.
We defer this computation to example 9.

Unfortunately, unlike theorem 1 positive maps, the converse of theorem 5 isn't true, as shown by the folloing example.

Example 7. $(I \otimes \Lambda)|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a maximally entangled state.
If $|\psi\rangle$ is maximally entangled, then $\mathrm{Tr}_B[\rho] = I/d$ and, by lemma 4, we have that $(I \otimes \Lambda)\rho$ is normal.

In this case, the converse of the theorems fails to hold in a spectacular way: $|\psi\rangle$ is maximally entangled (and thus maximally discordant), yet normality is preserved.

While the normality-preserving criterion is necessary but not sufficient for ZQD, it turns out that it can be used to derive quite a useful (and easy to compute) test for ZQD.

Recall that any matrix can be expanded as $\rho = \sum_{ij} E_{ij} \otimes B_{ij}$ and that, by lemma 4, we have that $(I \otimes \Lambda)\rho$ is normal if and only if $[(I \otimes U)\rho(I \otimes U^\dagger), \mathrm{Tr}_B[\rho] \otimes I] = 0$.
Then,
$[(I \otimes U)\rho(I \otimes U^\dagger), \mathrm{Tr}_B[\rho] \otimes I] =$

$$
= \sum_{ijkl} [E_{ij} \otimes UB_{ij}U^\dagger, \mathrm{Tr}[B_{kl}] \cdot E_{kl} \otimes \frac{I}{d}]
$$

$$= \frac{1}{d} \sum_{ijkl} [E_{ij}, \mathrm{Tr}[B_{kl}] \cdot E_{kl}] \otimes U B_{kl} U^\dagger$$

The form of the expression above implies that the choice of unitary U does not affect the normality of $(I \otimes \Lambda)\rho$. We thus can choose any unitary, and still get a valid criterion. The simplest possible choice is of course the identity. This yields our zero-discord test:

Theorem 8. *Test for zero discord. Let ρ be a zero-discord quantum state, then*

$$[\rho, Tr_B[\rho] \otimes I]] = 0 \tag{5}$$

While the validity of the test was technically proved in the previous paragraph, we state a clearer, self contained, proof below.

Let $\rho \in \mathcal{B}(\mathcal{H}_{ab})$ be a zero-discord quantum state. Then there exists a basis $\{|i\rangle\}$ for $\mathcal{H}_a$ such that $\rho = \sum_i p_i |i\rangle\langle i| \otimes \rho_i$, where ρ_i are density operators on H_b.
So we can compute $\mathrm{Tr}_B[\rho] = \sum_i \mathrm{Tr}[\rho_i] |i\rangle\langle i| = \sum_i p_i |i\rangle\langle i|$.
Hence:

$$[\rho, \mathrm{Tr}_B[\rho] \otimes I] = \sum_{ij} p_i p_j [|i\rangle\langle i| \otimes \rho_i, |j\rangle\langle j| \otimes I] = \sum_{ij} p_i p_j [|i\rangle\langle i|, |j\rangle\langle j|] = 0$$

For all zero discord states.

This test provides a quantum discord analogous to the Positive Partial Transpose (PPT) test for entanglement. While this test is not guaranteed to always detect discord, its strength relies on the fact that computing it is extremely simple. Furthermore, it does not depend on the structure theorem for normality preserving maps, and hence is applicable for all dimensions including the qubit case.

Clearly, this test still fails to detect QD for maximally entangled state.
For an example of a successful detection, we return to example 6:

Example 9. $|\psi\rangle = \frac{1}{\sqrt{5}}(|00\rangle + 2i\,|11\rangle)$

We seek to determine if the maximally entangled state given in Example 6 satisfies our test to determine whether the given state is a zero discord state or not. Since it is a maximally entangled state, it cannot have zero discord and we expect to find that the commutator of ρ and $\mathrm{Tr}_B[\rho] \otimes I$ does not vanish.

The density matrix ρ for our state is given as follows:

$$|\psi\rangle\langle\psi| = \rho = \frac{1}{\sqrt{5}}(|00\rangle\langle 00| - 2i\,|00\rangle\langle 11| + 2i\,|00\rangle\langle 11| + 4\,|11\rangle\langle 11|)$$

Next we need to calculate $\mathrm{Tr}_B[\rho] \otimes I$ where I = $|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|$

$$\mathrm{Tr}_B[\rho] = \frac{1}{5}(|0\rangle\langle 0| + 4\,|1\rangle\langle 1|)$$

$$\mathrm{Tr}_B[\rho] \otimes I = \frac{1}{\sqrt{5}}(|00\rangle\langle 00| + |01\rangle\langle 01| + |02\rangle\langle 02| + 4\,|10\rangle\langle 10| + 4\,|11\rangle\langle 11| + 4\,|12\rangle\langle 12|)$$

The commutator of ρ and $\mathrm{Tr}_B[\rho] \otimes I$ is given by:

$$[\rho, \mathrm{Tr}_B[\rho] \otimes I] = \frac{2i}{5}\,|11\rangle\langle 00| - \frac{6i}{25}\,|00\rangle\langle 11| \neq 0$$

As expected, the commutator turns out to be not 0 and our test has successfully detected that the state is not a zero discord state.

4 Conclusions

We drew an analogy between the role that positivity-preserving maps play in the theory of entanglement and the role of normality-preserving maps in the theory of quantum discords by means of theorem 5. We then elaborated on this result to develop a test for zero quantum discord, in the spirit of the PPE test for entanglement. Just like the PPE criterion doesn't always detect entanglement has the advantage of being easy to calculate, our test doesn't always detect quantum discord, but it can be easily implemented, both numerically and analytically.

There remain to be explored topics such as a classification of states for which the test gives wrong results. Furthermore, this paper only explored the $d \geq 3$ case, but it didn't address the very important $d = 2$ (qubit) case. While extending these results to the qubit case shouldn't present great difficulties, it needs to be done, given the importance of such a case.

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