EE 515 Report

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1 Introduction

We summarize the results of [1], which uses techniques from quantum error correction (QEC) to recover the Heisenberg limit in quantum sensing even in the presence of noise.

2 Background

As a warm-up example, consider the case of estimating a qubit's precession frequency via Ramsey inteferometry, as explained in [2]. One applies a π-pulse to a qubit which begins in $|0\rangle$ to produce the state $|+\rangle$, waits for some free evolution time t, and then measures in the X-basis. Since these qubit pulses are calibrated to work in a frame which rotates with the qubit, if the pulses are exactly resonant with the qubit's frequency ω_0, then the $\Pr(+) = \langle +|\rho(T)|+\rangle = 1$. If we intentionally apply some detuning to the original pulse, then the probability is instead

$$\Pr(+) \equiv P = (1 + \cos(\delta t))/2 \tag{1}$$

where $\delta = \omega - \omega_0$. Since we can choose the frequency of the pulse ω, we can estimate δ from the probability of obtaining a $|+\rangle$, which itself is obtained simply by repeating the procedure. If the total time allowed is T, then we can use the procedure T/t times and apply it to n qubits in parallel, leading to a total number of repetitions $N = nT/t$. Since the measurement outcome x is a coin toss with bias P (i.e. Bernoulli R.V.), the standard deviation–or *error*–in our estimate of the mean of x is

$$\Delta P = \sqrt{P(1-P)/N} \tag{2}$$

Thus, the corresponding standard deviation of our estimate of δ is found via error propagation on the function $\delta(P) = \cos^{-1}(2P-1)/t$:

$$\Delta\delta = |\delta'(P)|\Delta P \tag{3}$$

$$= \sqrt{\frac{P(1-P)}{1-(2P-1)^2}}\frac{2}{\sqrt{t^2N}} = \frac{1}{\sqrt{nTt}} \tag{4}$$

whose dependence on P (and therefore δ itself) actually cancels out. If the input state over the n qubits is entangled (i.e. GHZ), then

$$\Pr(+) = (1 + \cos(n\delta t))/2 \tag{5}$$

which implies $t \to tn$ and thus $\Delta\delta \to 1/n\sqrt{Tt}$ which is the Heisenberg limit. However, if the evolution is noisy and follows a simple dephasing model (for 1 qubit)

$$\dot{\rho}(t) = -i\delta[H, \rho] + \frac{\gamma}{2}(Z\rho Z - \rho) \tag{6}$$

where $H = |1\rangle\langle 1|$, then we find

$$\Pr(+) = (1 + \cos(\delta t)e^{-\gamma t})/2 \tag{7}$$

$$\Delta\delta = \sqrt{\frac{1-\cos^2(\delta t)e^{-2\gamma t}}{nTte^{-2\gamma t}\sin^2(\delta t)}} \tag{8}$$

Now the dependence on δ is explicit, so only certain measurement times will minimize $\Delta\delta$. The minimum value ends up being

$$\Delta\delta = \sqrt{2\gamma e/nT} \tag{9}$$

which actually *doesn't* improve when using an n-qubit GHZ state.

3 QEC for Quantum Sensing

In the above example, one might propose to use QEC to protect the probe from quantum noise. Intuitively, however, it should not be possible to remedy the example of Ramsey interferometry since the noise operator commutes with the Hamiltonian, so anything we do to suppess the effect of noise will also reduce the probe's sensitivity to the Hamiltonian. Specifically, [3] derived the following bound on the quantum Fisher information for the Ramsey problem, which holds for any input state and possible feedback scheme

$$F_Q^{\max} \leq \frac{n^2t^2}{1 + n(e^{2\gamma t} - 1)} \tag{10}$$

where the maximum is taken over all Kraus channel decompositions of the noisy evolution, since it is not unique. Thus, the estimation error obeys

$$\Delta\delta \geq \frac{\sqrt{1+n(e^{2\gamma t}-1)}}{nt} \tag{11}$$

which always gives the SQL (w.r.t. n) when $\gamma > 0$, and otherwise produces the HL. Generally, it is known that for most parameterized quantum channels (i.e. those which are full-rank, having at least d^2 Kraus operators), it is impossible to achieve the Heisenberg limit in the presence of noise [3], even when using entangled states and/or feedback, i.e. QEC. However, what if the Kraus rank is not maximal *and* the noise operator does not commute with the Hamiltonian? This case was considered by [4], which showed that with $H = X$ in the presence of dephasing, it is possible to use QEC to recover the HL. It is sufficient for the noise operators and code space to obey the Knill-Laflamme error-correction conditions, and for the Hamiltonian to generate a nontrivial logical evolution in the code space so that some signal remains after applying corrections. As long as the QEC is applied frequently compared to the dephasing rate γ, then the noise channel never has enough time to become full-rank, skirting the limitations mentioned above. Finally, the authors of [1] took this idea even further, showing how to achieve the HL for a qudit evolving under some arbitrary sensing Hamiltonian and set of Lindblad collapse operators, which we now discuss in more detail.

3.1 Setup

Assume an open quantum evolution of the form

$$\frac{d\rho_p}{dt} = -i[\omega G, \rho_p] + \sum_{k=1}^{r} L_k \rho_p L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_p\} \tag{12}$$

where ρ_P is the probe state, $H = \omega G$ is a Hamiltonian parameterized by ω which we wish to sense, and r is the rank of the noise channel affecting the probe. Generalizing the "non-commutation" requirement, the authors prove that the Hamiltonian-not-in-Lindblad span (HNLS) condition is both necessary and sufficient to achieve the HL for any such open evolution.

3.2 Sufficient condition

First, denote the Lindblad span by

$$S = \mathrm{span}(\{I, L_k, L_k^\dagger, L_k^\dagger L_j\} \ \forall\, j,k) \tag{13}$$

and then decompose the generator G into its components w.r.t S, i.e. $G = G_\parallel + G_\perp$. If HNLS holds, then $G_\perp \neq 0$, so we can say

$$G_\perp = UDU^\dagger = U(D_+ + D_-)U^\dagger \tag{14}$$

$$= U(D_+ - |D_-|)U^\dagger \tag{15}$$

$$\mathrm{Tr}(G_\perp) = 0 \tag{16}$$

$$\implies \mathrm{Tr}(D_+) = \mathrm{Tr}(|D_-|) = \mathrm{Tr}(|G_\perp|)/2 \tag{17}$$

$$\therefore G_\perp = \mathrm{Tr}(G_+)U\Big(\frac{D_+}{\mathrm{Tr}(G_+)} - \frac{|D_-|}{\mathrm{Tr}(G_+)}\Big)U^\dagger \tag{18}$$

$$\equiv \mathrm{Tr}(|G_\perp|)(\rho_0 - \rho_1)/2 \tag{19}$$

where we used the fact that $I \in S$, therefore $\langle\langle I|G_\perp\rangle\rangle = \mathrm{Tr}(G_\perp) = 0$. $D_\pm$ denotes the restriction of the diagonal matrix D to the positive (negative) eigenvalues, which must correspond to orthogonal eigenspaces. Thus, any traceless Hermitian matrix can be written as the (scaled) difference of two density matrices which obey $\mathrm{Tr}(\rho_0\rho_1) = 0$. Define the purifications of these "states" over an ancilla register by $\rho_i = \mathrm{Tr}_A |\bar{i}\rangle\langle\bar{i}|$ with $i \in \{0,1\}$:

$$\rho_i = \sum_k p_{ik}|\phi_{ik}\rangle\langle\phi_{ik}| \tag{20}$$

$$|\bar{i}\rangle = \sum_k \sqrt{p_{ik}}|\phi_{ik}\rangle \otimes U_i|k\rangle \tag{21}$$

for any U_i on the ancilla (the purification is not unique). For an observable $O_S I_A = O \otimes I$, we can write

$$\langle\bar{i}|O\otimes I|\bar{j}\rangle = \sum_{kl} \sqrt{p_{ik}}\sqrt{p_{jl}}\langle\phi_{ik}|O|\phi_{jl}\rangle\langle k|U_i^\dagger U_j|l\rangle \tag{22}$$

The RHS is 0 if $i \neq j$ because we can purify into the same ancilla basis, where the support of the code states must be disjoint. Otherwise, we obtain

$$\langle\bar{i}|O\otimes I|\bar{i}\rangle = \mathrm{Tr}(\rho_i O) \tag{23}$$

$$\therefore \mathrm{Tr}((\rho_0-\rho_1)O) = \frac{2\,\mathrm{Tr}(G_\perp O)}{\mathrm{Tr}(|G_\perp|)} \tag{24}$$

which is 0 if $O \in S$. Thus, the effective logical evolution in the code space

$$G_{\text{eff}} = \Pi_C G \Pi_C = G_{\bar{0}\bar{0}}|\bar{0}\rangle\langle\bar{0}| + G_{\bar{1}\bar{1}}|\bar{1}\rangle\langle\bar{1}| \propto \Pi_C \tag{25}$$

iff $G = G_\parallel$, which violates the HNLS condition and results in an effective identity evolution in the logical subspace, which is not useful. However, if $G = G_\parallel + G_\perp$, then

$$G_{\text{eff}} = G_{\perp\bar{0}\bar{0}}|\bar{0}\rangle\langle\bar{0}| + G_{\perp\bar{1}\bar{1}}|\bar{1}\rangle\langle\bar{1}| \tag{26}$$

up to an overall additive constant. We know that the following state

$$|\psi\rangle = (|\lambda_{\max}\rangle + |\lambda_{\min}\rangle)/\sqrt{2} \tag{27}$$

maximizes the QFI for a Hamiltonian with some maximal and minimal eigenvalues in the case of unitary evolution on a pure state (which holds here if we neglect terms of order dt^2 in the error-corrected evolution). Thus,

$$\begin{aligned} \mathrm{QFI}[\rho(t)] &= t^2(\lambda_{\max} - \lambda_{\min})^2 && (28)\\ &= t^2(\mathrm{Tr}((\rho_0 - \rho_1)G_\perp))^2 && (29)\\ &= 4t^2(\mathrm{Tr}\big(G_\perp^2\big)/\,\mathrm{Tr}(|G_\perp|))^2 && (30)\end{aligned}$$

so the more of G which lies outside the span of S, i.e. the more the HNLS condition holds, the greater the QFI. Note that $G_\perp$ defined the code states and also the effective evolution. But any operator $\tilde{G} \notin S$ would have sufficed to define the code states via some new $\tilde{\rho}_i$, leading to a different QFI via

$$\lambda_{\max} - \lambda_{\min} \propto \mathrm{Tr}\Big(\tilde{G}G_\perp\Big) \tag{31}$$

The authors derive a semidefinite program to find an optimal $\tilde{G}$ which maximizes the QFI. If ρ_i is rank-1, then the optimal choice is $\tilde{G} = G_\perp$. If the noise is full-rank, it is now clear that the size of $G_\perp$ must be 0, and so the HNLS condition always fails and the QFI is 0.

3.3 Necessary condition

In the above construction, it is clear how the failure of the HNLS condition leads to a vanishing QFI. But to prove necessity, we cannot refer to any specific scheme, and must rely on the properties of the HNLS itself. It turns out that one can upper bound the QFI for a noisy Kraus evolution by

$$\mathrm{QFI}[\rho(t)] \leq 4\frac{t}{dt}\|\alpha_{dt}\| + 4\big(\frac{t}{dt}\big)^2\|\beta_{dt}\|(\|\beta_{dt}\| + 2\sqrt{\|\alpha_{dt}\|}) \tag{32}$$

where α and β depend on the derivative of the Kraus operators w.r.t. the sensing parameter ω. The authors expand α and β in a power series in terms of $\sqrt{dt}$. The violation of the HNLS conditions, combined with the non-uniqueness of the Kraus form, allows one to cancel all terms in the power series of β up to $O(dt^2)$, which is 0 in their approximation limit, but the same does not hold for the series of α, which to leading order scales like $O(dt)$. Thus, we have

$$\mathrm{QFI}[\rho(t)] \leq 4\|\alpha^{(2)}\|t \tag{33}$$

which (apparently) is the SQL, in terms of t.

3.4 Qubit Example

The authors illustrate their idea with several examples, the most intuitive (but perhaps too simple) case being that of a single qubit probe and a single qubit ancilla. Note the following conditions on the Lindblad jump operators

- If there is even one non-Hermitian L_k, then L_k, $L_k^\dagger$, and $L_k^\dagger L_k$ will all be linearly independent and along with I, we have that $\dim(S) = 4$ and no part of H may lie outside of that span, so each L_k must be Hermitian
- If there are even 2 linearly independent Hermitian jump operators L_k and L_j, then $L_k^\dagger L_j$ is a new operator and so $\dim(S) = 4$ again, so we cannot even have 2 such operators

Thus, HL scaling is only achievable in the single qubit case when there is just one Hermitian jump operator, which necessarily will not commute with L. Since G and L are both traceless and Hermitian, they each can be decomposed into Bloch vectors $\vec{v}\cdot\vec{\sigma}$, then the HNLS condition just means that the two Bloch vectors must not be parallel. One can then choose the code states

$$|\bar{0}\rangle = |+0\rangle \quad |\bar{1}\rangle = |-1\rangle \tag{34}$$

where $|\pm\rangle$ is the positive (negative) eigenstate of $G_\perp$, whose Bloch vector is chosen to be orthogonal to L's Bloch vector. In the case that $G = G_\perp$, then the signal eigenstates are the code states, which just accrue phases at different rates–this precession is the desired signal. Moreover, random jumps caused by L will act like bit flips on the probe, taking it out of the code space:

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{\sqrt{2}}(e^{-i\omega t/2}|+0\rangle + e^{i\omega t/2}|-1\rangle && (35)\\ &\to \frac{1}{\sqrt{2}}(e^{-i\omega t/2}|-0\rangle + e^{i\omega t/2}|+1\rangle && (36)\end{aligned}$$

where the arrow indicates an instantaeous jump. The POVM $\{|+0\rangle\langle+0|+|-1\rangle\langle-1|, |-0\rangle\langle-0|+|+1\rangle\langle+1|\}$ can detect this error, and the recovery operation is just to (rapidly) apply another bit flip on the probe. When $G \neq G_\perp$, then actually the evolution under G *without* error still causes the code states to evolve away from the code space, but since the timescale of the free evolution between QEC rounds is dt, this deviation ends up being $O(dt^2)$, which is negligible.

References

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