

# The Casimir Effect

Notes by Asaf Szulc

## 1 Introduction

The Casimir effect describes the attraction between two perfectly conducting and uncharged parallel plates confined in vacuum. It was the Dutch physicist Hendrik B. G. Casimir who first predicted this phenomena in his paper from 1948 which described it as the pressure difference on both sides of a plate caused by the difference in the vacuum fluctuations outside and amid the plates.

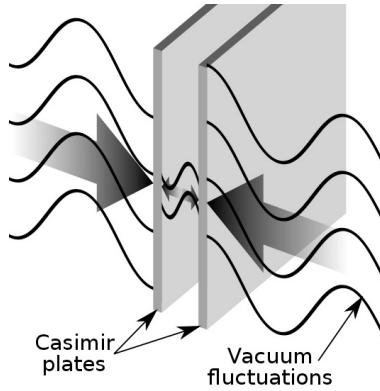


Figure 1: Casimir force on parallel plates. The pressure from outside the plates is greater than the pressure amid them, resulting an attracting force acting on both plates. (picture from wikipedia).

Let us start from the end by presenting the astonishing result, as presented by Casimir [1]. The force per  $cm^2$ , generated by the vacuum vibrations alone, acting on two parallel plates at distance  $R$  microns apart is given by:

$$F = -\frac{\hbar c}{240} \frac{\pi^2}{R^4} \frac{1}{R^4} = -\frac{0.013}{R^4} \text{ dyne/cm}^2 \quad (1)$$

For better understanding the number that Casimir found, let us make some comparisons. The force acting on a  $1 \times 1 cm$  plates separated by  $1\mu m$  apart is  $F = -0.013 \text{ dyne}$ . This force is comparable to the Coulomb force on the electron in the Hydrogen atom, the gravitational force between two  $0.5kg$  weights separated by  $1cm$ , or about  $1/1000$  the weight of a housefly.

While the Casimir force is too small to be observed when dealing with long distances of several meters, when dealing with small distances of several microns the Casimir force becomes dominant.

## 2 Classical treatment of the Casimir force

The idea of a force acting on neutral atoms or molecules is not new to the physics world. The so called *Van der Waals (VdW) interaction* is a force acting between two neutral atoms [2].

One can make a Casimir force approximate derivation based on the VdW interactions between atoms in a two half-spaces. By summing the interactions between all the atoms and using *Clausius–Mossotti relation* one can get the total force:

$$F(R) = -\frac{23\hbar c N^2 \alpha^2}{40R^4} = -\frac{23}{40} \frac{\hbar c}{R^4} \left(\frac{3}{4\pi}\right)^2 \left(\frac{\epsilon-1}{\epsilon+2}\right)^2 \quad (2)$$

And by taking the limit for a perfect conductor ( $\epsilon \rightarrow \infty$ ) one will get:

$$F(R) = -\frac{23}{40} \frac{\hbar c}{R^4} \left(\frac{3}{4\pi}\right)^2 \quad (3)$$

This result is about 20% within the exact Casimir force given in (1).

So why didn't we get exactly the Casimir force?

The answer is that the VdW forces are not pairwise additive due to multiple scattering. Theoretically, it is possible to make the exact calculation using the VdW approach, however, the calculation is very complicated and one should try and find an alternative way. Fortunately, the solution relies on the quantum mechanics phenomena known as *zero – point energy*.

## 3 Zero-Point Energy (Vacuum Energy)

From the quantization of the Electromagnetic (EM) field (See Appendix A) one can write the Hamiltonian of the field as a sum of independent harmonic oscillators:

$$H = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} (a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}) \quad (4)$$

The ground state of the Hamiltonian is defined as:

$$a_{\mathbf{k}} |0\rangle = 0 \quad (5)$$

Meaning that it is the state from which no more photons can be annihilated. This is an empty EM field, every mode has no photons present, and is in its lowest possible energy state.

Knowing that  $[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'} \mathbf{1}$  one can decompose  $|0\rangle$  into the product of each independent oscillator for a given  $\mathbf{k}$ :

$$|0\rangle = |0_{\mathbf{k}_1}\rangle \otimes |0_{\mathbf{k}_2}\rangle \otimes |0_{\mathbf{k}_3}\rangle \otimes \dots \quad (6)$$

So one can take the expectation value of the Hamiltonian acting on  $|0\rangle$ :

$$\langle E \rangle = \langle 0 | H | 0 \rangle = \langle 0 | \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} (a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}) | 0 \rangle = \frac{1}{2} \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \quad (7)$$

Hence the energy of the EM field in the ground state is the sum of many uncoupled harmonic oscillators, each with energy  $\hbar\omega/2$ .

Surprisingly, the zero-point energy is infinite, being a sum over all possible frequencies. Under ordinary conditions this infinity is not physically measurable, as it is present at each point in space. As we will see, however, imposing boundary conditions on the EM field can generate differences in the zero-point energy, which will lead to surprising results.

## 4 The Casimir Force

Let us consider a large cavity of the dimensions  $L \times L \times L$  bounded by conducting walls. A conducting plate is then inserted at distance  $R$  parallel to one of the  $xy$  walls ( $R \ll L$ ). We shall now calculate the energy shift due to the conducting plate at  $z = R$  (Figure 2):

$$\Delta E = E_R + E_{L-R} - E_L \quad (8)$$

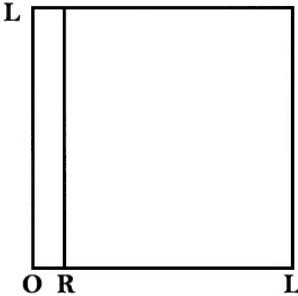


Figure 2: A large cavity with a plate inserted near one end at distance  $R$ .

The conducting walls of the box impose Dirichlet boundary conditions on the possible EM modes of the cavity. The tangential component of the electromagnetic field must vanish there.

Solving the wave equation with these conditions reveals that the boundaries quantize the x, y, and z components of the wavevectors  $\mathbf{k}$ , such that they take on the discrete values:

$$k_x = \frac{n_x \pi}{L}, \quad k_y = \frac{n_y \pi}{L}, \quad k_z = \frac{n_z \pi}{R} \quad (9)$$

In a vacuum, angular frequency  $\omega$  is related to the wave number  $k$  by the speed of light  $c$ :

$$\omega = ck \quad (10)$$

Allowing one to write the zero-point energy of the field as:

$$\langle E \rangle = \frac{1}{2} \hbar c \sum \sqrt{k_x^2 + k_y^2 + k_z^2} \quad (11)$$

By assuming that the cavity is large  $R \ll L$  one can replace the summation with an integral and get the energies:

$$E_L = \frac{2L^3}{\pi^3} \int \int \int_0^\infty \frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2} dk_x dk_y dk_z \quad (12)$$

$$E_{L-R} = \frac{2L^2(L-R)}{\pi^3} \int \int \int_0^\infty \frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2} dk_x dk_y dk_z \quad (13)$$

$$E_R = \sum_{n=0}^{\infty} 2\theta_n \frac{L^2}{\pi^2} \int \int_0^\infty \frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{R}\right)^2} dk_y dk_z \quad (14)$$

Where  $n \equiv n_z$ , in order not to carry the index through all the calculations, and  $\theta_n = 1$  for  $n > 0$ ,  $\theta_n = 1/2$  for  $n = 0$  due to the different number of polarization states of  $\mathbf{k}$ . The notation of  $\theta_n$  can be discarded by taking the limits of the summation from minus infinity instead of zero, however, this notation make the use of Euler-Maclaurin formula later in (23) much clearer.

The sum and the integrals are divergent hence it is necessary to introduce a smooth cutoff function,  $f(k/k_c)$ , having the properties:

$$f(k/k_c) \rightarrow 1, \quad k \ll k_c \quad (15)$$

$$f(k/k_c) \rightarrow 0, \quad k \gg k_c \quad (16)$$

One can justify forcing the cutoff function into the equation by noting that each conductor has a plasma frequency

$$\omega_p^2 = \frac{Nq^2}{m_e \epsilon_0} \quad (17)$$

which is the minimum oscillation frequency the electrons in the conductor can support. Below the plasma frequency, the conductor acts as a reactive medium and reflects electromagnetic waves giving rise to the boundary conditions discussed above. Above the plasma frequency, however, the electrons are capable of oscillating in resonance with the waves. This means that the conductor is effectively transparent to photons above a certain frequency, and the boundary conditions no longer hold.

Now the equation for the energy difference (8) will take the form:

$$\Delta E = \hbar c \frac{L^2}{\pi^2} \left[ \sum_{n=0}^{\infty} \theta_n g \left( \frac{n\pi}{R} \right) - \frac{R}{\pi} \int_0^{\infty} g(k_z) dk_z \right] \quad (18)$$

Where

$$g(k_z) = \int \int_0^{\infty} f \left( \frac{k}{k_c} \right) \sqrt{k_x^2 + k_y^2 + k_z^2} dk_x dk_y \quad (19)$$

The expression for  $g(k_z)$  can be simplified by some simple transformations. One should first introduce  $\kappa = \sqrt{k_x^2 + k_y^2}$  and use polar coordinates in the  $xy$  plane to obtain:

$$g(k_z) = \frac{4\pi}{8} \int_0^{\infty} \sqrt{k_z^2 + \kappa^2} f \left( \frac{\sqrt{k_z^2 + \kappa^2}}{k_c} \right) k dk$$

Next to introduce a dimensionless variables  $n, \alpha$  so  $k_z = n\pi/R$  and  $\kappa = \alpha\pi/R$  to get:

$$g(k_z) = \frac{\pi}{2} \frac{\pi^3}{R^3} \int_0^{\infty} \sqrt{n^2 + \alpha^2} f \left( \frac{\pi\sqrt{n^2 + \alpha^2}}{Rk_c} \right) \alpha d\alpha$$

And finally,  $\omega = n^2 + \alpha^2$ ,  $d\omega = 2\alpha d\alpha$  to get:

$$g(k_z) = \frac{\pi^4 F(n)}{4R^3} \quad (20)$$

$$F(n) = \int_{n^2}^{\infty} \sqrt{\omega} f \left( \frac{\pi\sqrt{\omega}}{Rk_c} \right) d\omega \quad (21)$$

Thus the final expression for the energy difference is:

$$\Delta E = \frac{\hbar c L^2 \pi^2}{4R^3} \left[ \sum_{n=0}^{\infty} \theta_n F(n) - \int_0^{\infty} F(n) dn \right] \quad (22)$$

The difference between the summation and the integral can be approximated by using *Euler-Maclaurin formula*:

$$\sum_{n=0}^{\infty} \theta_n F(n) - \int_0^{\infty} F(n) dn = -\frac{1}{6 \times 2!} F'(0) + \frac{1}{30 \times 4!} F'''(0) - \dots \quad (23)$$

Using (21) one can obtain:

$$F'(n) = -2n^2 f\left(\frac{n^2 \pi}{Rk_c}\right)$$

$$F'(0) = 0$$

$$F'''(0) = -4$$

The higher derivatives contain powers of  $(\pi/Rk_c)$ , thus one gets:

$$\Delta E = -\hbar c \frac{\pi^2}{720} \frac{L^2}{R^3} \quad (24)$$

And the force per  $cm^2$  is easily calculated to be:

$$F = -\frac{1}{L^2} \frac{\partial \Delta E}{\partial R} = -\frac{\hbar c}{240} \frac{\pi^2}{R^4} \quad (25)$$

And it is the desired result as stated at the beginning (1).

## 5 A modern derivation of the Casimir force

While the "old school" derivation made by Casimir in his paper from 1948 was rooted in sound physical intuition, many modern derivations of the Casimir effect make use of a more advanced mathematical transformation known as *zeta function regularization* [3].

One can write the mean energy of the modes between the two parallel Casimir plates:

$$\langle E \rangle = \frac{\hbar}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{L^2}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \omega dk_x dk_y \quad (26)$$

Here the limits of the sum take into consideration the two polarization states for  $k \neq 0$ . In order to prevent the sum of not being convergent one must divide by powers  $s$  of  $\omega$  and at the end taking  $s = 0$ . The regularized expression is:

$$\langle E(s) \rangle = \frac{\hbar}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{L^2}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \omega \cdot \omega^{-2s} dk_x dk_y \quad (27)$$

Switching to polar coordinates and substituting  $y \equiv Rx/n\pi$  to get:

$$\langle E(s, R) \rangle = \frac{\hbar c L^2}{2\pi} \int_0^{\infty} y (y^2 + 1)^{1/2-s} \sum_{n=1}^{\infty} \left(\frac{n\pi}{R}\right)^{3-2s} dy \quad (28)$$

This expression becomes divergent as  $s \rightarrow 0$ , but it does converge for  $s = 3/2$  and larger so one should evaluate each expression in its domain of convergence, and then analytically continue to  $s = 0$ . The integral is easily calculated:

$$\begin{aligned} \int_0^\infty y(y^2 + 1)^{1/2-s} dy &= \frac{1}{2} \int_0^\infty (u + 1)^{1/2-s} du = \frac{1}{2} \int_1^\infty w^{1/2-s} dw = \\ \frac{1}{2} \left( \frac{1}{3/2-s} \right) w^{3/2-s} \Big|_1^\infty &= -\frac{1}{2} \left( \frac{1}{3/2-s} \right) \Big|_{s=0} = -\frac{1}{3} \end{aligned} \quad (29)$$

One can now write:

$$\langle E(s, R) \rangle = -\frac{\hbar c \pi^2 L^2}{6} \frac{1}{R^3} \sum_{n=1}^{\infty} \frac{1}{n^{2s-3}} = -\frac{\hbar c \pi^2 L^2}{6} \frac{1}{R^3} \zeta(2s-3) \Big|_{s=0} = -\frac{\hbar c \pi^2 L^2}{6} \frac{1}{R^3} \zeta(-3) \quad (30)$$

It is known that  $\zeta(-3) = 1/120$  hence the final energy is:

$$\langle E(R) \rangle = -\frac{\hbar c \pi^2 L^2}{720} \frac{1}{R^3} \quad (31)$$

Which is exactly the same as (24) that was derived by Casimir.

Note, however, that we have not subtracted the plate energies in the near and far configurations, as we did in Sec. 4. This suggests an important physical intuition for zeta renormalization: using the analytic continuation from  $s = 3/2$  to  $s = 0$  in some sense corresponds to subtracting the EM field's inherent contribution to the ground state energy  $\langle E \rangle$ .

## 6 Repulsive Casimir force

As Casimir's theory of a force from nothing arises, one must evaluate the possibility of achieving a repulsive Casimir force. In fact, it has already been done. This so called *Casimir-Lifshitz force* was developed theoretically by Lifshitz, who used different dielectric materials instead of the vacuum, later to be tested experimentally.

For one to understand the phenomena let us consider a single flat and thin metallic plate of thickness  $d$ , In this case, the phenomena that must be considered is the surface plasmons [4]. To calculate the pressure made by the surface plasmons one should consider a metallic slab with a dialectic constant:

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \quad (32)$$

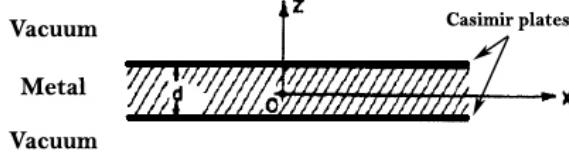


Figure 3: The Casimir effect with a dielectric medium between the plates as a model of a metallic bar in vacuum.

The absence of external charges leads to the continuity equation satisfied by the electric displacement vector [5]:

$$\nabla \cdot \mathbf{D} = 0 \quad (33)$$

Where  $\mathbf{D} = \epsilon \mathbf{E}$ . Following (33) one knows that the normal component of  $\mathbf{D}$  is continuous across the boundaries and that  $\nabla \cdot \mathbf{E} = 0$ , except at the boundaries where  $\epsilon$  is discontinuous. Introducing the electric scalar potential,  $\mathbf{E} = -\nabla\phi$ , one gets Laplace's equation

$$\nabla\phi = 0 \quad (34)$$

Due to the symmetry relative to the  $z = 0$  plane there are even and odd solutions allowing us to consider for now only positive values of  $z$  and get:

$$\phi = \begin{cases} \phi_0 \cos(\mathbf{k} \cdot \mathbf{r} + \delta)(e^{kz} \pm e^{-kz}), & 0 < z < d/2 \\ \phi_0 \cos(\mathbf{k} \cdot \mathbf{r} + \delta)e^{-kz}(e^{kd} \pm 1), & z > d/2 \end{cases} \quad (35)$$

Where  $\mathbf{k}$  and  $\mathbf{r}$  are vectors in the  $x - y$  plane and  $\delta$  is an arbitrary phase. Now one can derive the normal component of  $\mathbf{E}$  at the boundary and get:

$$\mathbf{E}_z = \begin{cases} -k\phi_0 \cos(\mathbf{k} \cdot \mathbf{r} + \delta)(e^{kd/2} \mp e^{-kd/2}), & 0 < z < d/2 \\ k\phi_0 \cos(\mathbf{k} \cdot \mathbf{r} + \delta)(e^{kd/2} \pm e^{-kd/2}), & z > d/2 \end{cases} \quad (36)$$

Knowing that the normal component of  $\mathbf{D}$  is continuous across the boundaries, the relation for  $\epsilon$  is calculated by matching boundary conditions:

$$\epsilon(\omega) = - \left( \frac{e^{kd/2} \pm e^{-kd/2}}{e^{kd/2} \mp e^{-kd/2}} \right) \quad (37)$$

One can now calculate the dispersion relation and get:

$$\omega_{\pm}(k) = \frac{\omega_p}{\sqrt{2}} \sqrt{1 \mp e^{-kd}} \quad (38)$$

The zero-point energy of the plasmons is, therefore:

$$E_0(d) = \hbar \left( \frac{L}{2\pi} \right)^2 \int \left[ \left( \omega_+(k) - \frac{\omega_p}{\sqrt{2}} \right) + \left( \omega_-(k) - \frac{\omega_p}{\sqrt{2}} \right) \right] d^2 k \quad (39)$$

Note the subtraction of an infinite constant from each branch, as we did in the early calculations, here we took the limit  $k \rightarrow \infty$ . Finally, the force per area can then be calculated:

$$F(d) = -\frac{1}{L^2} \frac{\partial E_0(d)}{\partial d} = 0.0078 \frac{\hbar \omega_p}{d^3} \quad (40)$$

The most significant property of the result is not the number itself but its sign, a positive number means a repulsive force and that was the desired result.

## 7 Summary

In this work I reviewed a phenomena that gets more and more relevant as the technology advances called the Casimir effect. Even though one can use the "classical" method and try to calculate the force by summing atom-atom interactions (VdW interactions) I showed that quantum mechanics gives us an easier way to make the calculation both in the old straightforward method and the new zeta function method. Finally I introduced a method one can use to approximate repulsive Casimir force on a thin dialectic material, showing that a repulsive Casimir force is achievable.

## References

- [1] H. G. B. Casimir, *On the attraction between two perfectly conducting plates*, Proc. K. Ned. Akad. Wet. 51,793, (1948).
- [2] Peter W. Milonni, *The Quantum Vacuum - An Introduction to Quantum Electrodynamics*, ACADEMIC PRESS, INC, (1994).
- [3] Kyle Kingsbury, *The Casimir Effect - A Comprehensive Exercise*, Version 3, (2009).
- [4] Yoseph Imry, *The Casimir zero-point radiation pressure*, Phys. Rev. Lett. 95, 080404, (2005).
- [5] Richard A. Ferrell, *Predicted Radiation of Plasma Oscillations in Metal Films*, Phys. Rev. 111, 1214–1222, (1958).
- [6] Leslie E. Ballentine, *Quantum Mechanics - A Modern Development*, SFU, (2000).

## A Quantization of the Electromagnetic Field

I started my work assuming that the EM field can be presented by a sum of harmonic oscillators, here I will present the full derivation [6].

First, One can derive the wave equation of the Electric field in vacuum ( $\mathbf{E}$ ) from Maxwell's Equations:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad (41)$$

The solution for the wave equation is a sum of *mode functions*,  $\mathbf{u}_m(\mathbf{x})$ , which satisfy the conditions:

$$\nabla^2 \mathbf{u}_m(\mathbf{x}) = -k_m^2 \mathbf{u}_m(\mathbf{x}) \quad (42)$$

$$\nabla \cdot \mathbf{u}_m(\mathbf{x}) = 0 \quad (43)$$

$$\hat{n} \times \mathbf{u}_m(\mathbf{x}) = 0 \quad (44)$$

Where  $\hat{n}$  is the unit vector normal to the surface. The last condition (44) must be satisfied because the tangential component of  $\mathbf{E}$  must vanish on a conducting surface. By choosing a set of orthonormal mode functions one can write the electric field as a sum of the mode functions:

$$\mathbf{E}(\mathbf{x}, t) = \sum_m f_m(t) \mathbf{u}_m(\mathbf{x}) \quad (45)$$

Substituting it into the wave equation (41) to get:

$$\frac{d^2}{dt^2} f_m(t) + c^2 k_m^2 f_m(t) = 0 \quad (46)$$

This is the equation of motion for a harmonic oscillator with angular frequency of  $\omega_m = ck_m$ . The magnetic field could be derived from the electric field and after some algebra, the result is the same.

$$\mathbf{B}(\mathbf{x}, t) = \sum_m h_m(t) \nabla \times \mathbf{u}_m(\mathbf{x}) \quad (47)$$

With:

$$\frac{dh_m(t)}{dt} - cf_m(t) = 0 \quad (48)$$

Leading to the condition:

$$\frac{d^2}{dt^2} h_m(t) + c^2 k_m^2 h_m(t) = 0 \quad (49)$$

One can conclude and say that the analysis into normal modes had shown that the EM field is equivalent to an infinite number of independent harmonic oscillators.

The Hamiltonian can be written by calculating the total energy of the EM field:

$$\begin{aligned} H_{EM} &= \frac{1}{8\pi} \int (E^2 + B^2) d\mathbf{x} \\ &= \frac{1}{8\pi} \sum_{m'm} f_{m'} f_m \int \mathbf{u}_{m'} \cdot \mathbf{u}_m d\mathbf{x} + \frac{1}{8\pi} \sum_{m'm} h_{m'} h_m \int (\nabla \times \mathbf{u}_{m'}) \cdot (\nabla \times \mathbf{u}_m) d\mathbf{x} \\ &= \frac{1}{8\pi} \sum_m (f_m^2 + k_m^2 h_m^2) \end{aligned} \quad (50)$$

One can clearly see the resemblance to the Hamiltonian for a set of harmonic oscillators with unit mass:

$$H_{osc} = \sum_m \frac{1}{2} (P_m^2 + \omega_m^2 X_m^2) = \sum_m \hbar \omega_m (a_m^\dagger a_m + \frac{1}{2}) \quad (51)$$

Giving that:

$$X_m = \left( \frac{\hbar}{2\omega_m} \right)^{\frac{1}{2}} (a_m^\dagger + a_m) \quad (52)$$

$$P_m = \left( \frac{\hbar \omega_m}{2} \right)^{\frac{1}{2}} (a_m^\dagger - a_m) \quad (53)$$

Assuming that this form holds for the Hamiltonian of the modes of the EM field By using (48) one can get the relation:

$$X_m \leftrightarrow \frac{h_m}{2c\sqrt{\pi}} \quad (54)$$

Inserting  $h_m = 2c\sqrt{\pi}X_m$  to (48) to get the relation of  $f_m(t)$ :

$$f_m(t) = -\frac{1}{c} \frac{dh_m(t)}{dt} = -(2\sqrt{\pi}) \frac{dX_m}{dt} = -(2\sqrt{\pi}) P_m = -i(2\pi\hbar\omega_m)^{1/2} (a_m^\dagger - a_m)$$

And by inserting this identity into (45) one gets the equation for the electric field:

$$E(\mathbf{x}, t) = \sum_m -i(2\pi\hbar\omega_m)^{1/2} (a_m^\dagger - a_m) \mathbf{u}_m(\mathbf{x}) \quad (55)$$

and equivalently, the magnetic field will be:

$$B(\mathbf{x}, t) = \sum_m \left( \frac{2\pi\hbar c^2}{\omega_m} \right)^{1/2} (a_m^\dagger + a_m) \mathbf{u}_m(\mathbf{x}) \quad (56)$$