Entropy Change of Thermal Equlibrium

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1 Abstract

This answer is written to answer user Wow#5102's question on the discord server Physics Olympiads.

2 Problem

A generalized statement is as follows: two bricks A and B are touching each other. Their initial temperatures are T_{A0} and T_{B0}, where $T_{A0} > T_{B0}$, and their specific heat capacities are C_A and C_B. Suppose that the heat exchange between them is proportional to their temperature difference. Show that the second law of thermodynamics holds.

3 Solution

First we write out the heat transfer equations as follows.

$$\frac{\mathrm{d}}{\mathrm{dt}} T_A = C_A(T_B - T_A)$$

$$\frac{\mathrm{d}}{\mathrm{dt}} T_B = C_B(T_A - T_B)$$

Subtracting them gives

$$\frac{\mathrm{d}}{\mathrm{dt}} U = -(C_A + C_B)U$$

where $U = T_A - T_B$. Solving the differential equation yields

$$T_A - T_B = U = U_0 \exp\left(C_A - C_B\right)$$

we can find U_0 by substituting the initial conditions at $t = 0$ to find $U_0 = T_{A0} - T_{B0}$.

Our final goal is to calculate

$$S = \int \frac{dQ}{T}$$

for both bricks. For brick A we can write $dQ = C_A dT$, so

$$S_A = \int_{T_{A0}}^{T_{eq}} \frac{C_A dT_A}{T_A} = C_A \ln \frac{T_{eq}}{T_{A0}}$$

similarly, for brick B,

$$S_B = \int_{T_{B0}}^{T_{eq}} \frac{C_B dT_B}{T_B} = C_B \ln \frac{T_{eq}}{T_{B0}}$$

where $T_{eq} = \frac{C_A T_{A0} + C_B T_{B0}}{C_A + C_B}$. Hence,

$$S_{sys} = C_A \ln \frac{T_{eq}}{T_{A0}} + C_B \ln \frac{T_{eq}}{T_{B0}}$$

Our job is prove $S_{sys} > 0$. In other words,

$$\ln \left(\left(\frac{T_{eq}}{T_{A0}} \right)^{C_A} \left(\frac{T_{eq}}{T_{B0}} \right)^{C_B} \right) > 0$$

$$\left(\frac{T_{eq}}{T_{A0}} \right)^{C_A} \left(\frac{T_{eq}}{T_{B0}} \right)^{C_B} > 1$$

This could be very hard to prove. However, we can veryfy this by using the special case $C_A = C_B = C$, then

$$T_{eq} = \frac{T_{A0} + T_{B0}}{2}$$

the LHS of the above inequality is now

$$\frac{T_{eq}^2}{T_{A0} T_{B0}}$$

For simplicity I'll write $a = T_{A0}$ and $b = T_{B0}$, we can find

$$\frac{T_{eq}^2}{T_{A0} T_{B0}} = \frac{\left(\frac{a+b}{2}\right)^2}{ab} = \frac{a^2 + 2ab + b^2}{4ab} > 1$$

in the last step I used $(a - b)^2 > 0$ which leads to $a^2 + 2ab + b^2 > 4ab$. Therefore, we have verified that the change in entropy S_{sys} is always positive, in agreement with the second law for non-reversible processes.