Intuitive Understanding of the Behavior of the Mechanical Pendulum Beyond the Small-Angle Approximation

CHENG-YOU HO, Chiayi Senior High School, Taiwan

The discussion of the mechanical pendulum in freshman years and below are often limited to a small initial angle satisfying the approximation $\sin\theta \approx \theta$. Mathematical techniques beyond the knowledge of undergrads are often necessary to tackle this problem for larger initial angles. A intuitive picture of the behavior for such generalization of the system is provided in this paper. With the aid of Python simulations and graphs, it is concluded that oscillations with larger initial angles still follow a sinusoidal trend, while their periods are longer compared to those obtained with the small-angle approximation. Practically, it is also justified that averaging over more oscillations is way more important than releasing the pendulum at a small 5° angle or smaller when measuring the period of a pendulum in an experiment manually.

Additional Key Words and Phrases: Pendulum, Numerical, Python, Simulation, Mechanical, Physics

1 INTRODUCTION

The understanding of the mechanical pendulum, consisting of a point mass suspended under a massless string of length l, dates back several centuries. As taught in textbooks in the freshman year [Halliday 1997], given that the gravitational acceleration is g, the equation of motion is written as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \sqrt{\frac{g}{l}}\sin\theta = 0 \tag{1}$$

By using the *small-angle approximation*:

$$\theta < 5^o \rightarrow \theta \approx \sin\theta. \tag{2}$$

the above equation turns into a nice ODE, and the oscillation period of such system is then found to be

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{3}$$

Texts regarding the motion of the pendulum for larger initial angles often require burdensome mathematical techniques. Such tedious approach to the generalization of this matter drives away the interest of most undergraduate students. The paper will provide a intuitive understanding of the motion of the mechanical pendulum with the aid of Python simulations. By displaying graphs instead of complex math equations, it is certain that this approach to the problem will favor more students with little math background.

Author's address: Cheng-You Ho, chengyou1368@gmail.com, Chiayi Senior High School, Chiayi CIty, Taiwan, 600.

XXXX-XXXX/2023/11-ART $15.00
https://doi.org/10.1145/nnnnnnn.nnnnnnn

2 THEORETICAL ANALYSIS

Before diving into the numerical solution, a brief theoretical analysis is presented here. Present researches related to the exact solution of pendulum motion mostly approach it from elliptic integrals [AA. Bel´endez and Neipp 2007]. Solving the motion analytically is beyond the scope of this paper, and a simple explanation of why this problem requires such a technique is discussed here. Those who seek such result could refer to the references.

Defining the height at which the pendulum is at rest ($\theta = 0$) having zero gravitational potential energy, the equation of motion can be written as

$$\frac{1}{2}ml^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = mgl(\cos\theta - \cos\theta_0) \tag{4}$$

where θ_0 denotes the initial release angle. A differential equation can be obtained:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\cos\theta - \cos\theta_0} = \frac{2g}{l}\mathrm{d}t \tag{5}$$

Here it is clear that the difficulty lies in evaluating the integral, Unless $\theta_0 = 90°$, which the numerical value for the integral[物理奧林匹亞國家代表隊選訓工作小組 2011]

$$\int_0^{\pi/2}\frac{1}{\sqrt{\cos x}}x\mathrm{d}x \approx 2.62$$

may be present in some books, an alternative approach could be needed to find the exact solution.

3 THE NUMERICAL SOLUTION

A python program is used to determine the solution of the differential equation numerically. The script, completed with the help of ChatGPT 3.5, is attached in the Appendix. The value of omega is chosen to be 2 plainly for the sake of the layout of the graph. Increasing omega while shortening the time interval of analysis would yield the same result. Factors that may affect the behavior of the system include the initial angle and the initial speed at which the bob is released. However, from the perspective of conservation of energy, releasing the bob at a faster speed simply corresponds to releasing it still at a larger amplitude (then waiting for a split moment). Hence, only the effect of initial angle is tackled in this paper.

4 INITIAL ANGLE

One should expect that the initial angle is the primary factor that makes the motion of the system differ from that of a system applicable of the small-angle approximation. Graphs of the solutions of the pendulum at 1°, 5°, 45°, 90°, 135° and 170° are shown in Figure 2. It can be clearly observed that the period of oscillation increases as the initial release angle is enlarged. The resulting motion beyond

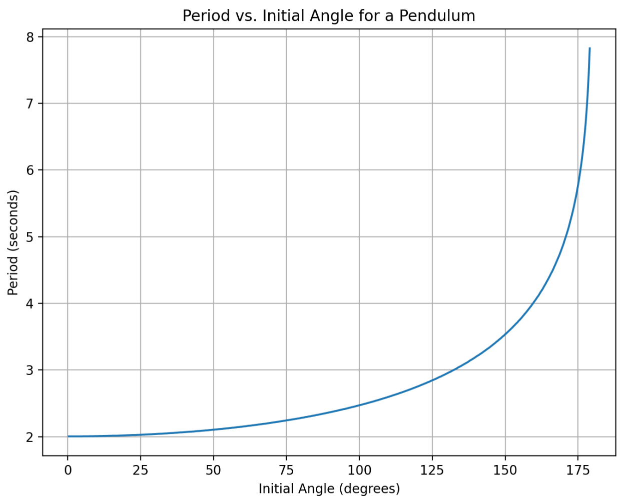

Fig. 1. Period vs. θ
This graph presents the trend that the oscillation period increases as θ increases.

the small-angle condition is curiously similar to a sinusoid. This behavior, however, is less observed at larger initial angles. Put literally, the valleys and apexes of the curve are flattened out.

A plot of period versus initial angle can be made to clearly display the trend of the period of the pendulum increasing with the initial angle, which is shown in Figure 1. The theoretical period calculated from the small-angle assumption is

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{1.000}{9.810}} \approx 2.006 \text{ seconds.}$$

5 A PRACTICAL PERSPECTIVE

In Figure 2, for $\theta < 5^\circ$, the difference between the two solutions are indistinguishable, as the dotted orange line and the solid blue line overlap perfectly. However, the upper bound for which these two lines "overlap" can be quite difficult to deal with just from Figure 2 [Prallax 2021]. To qualitatively analyze this issue, Figure 1 can be enlarged to better display the period-θ relationship at smaller angles, as shown in Figure 3. The staircase behavior of the figure is consequent of the limit of the ODE numerical solution. One can observe that the period from $\theta = 0$ to $\theta \approx 5^\circ$ is a flat line, indicating that the difference is less than 10^{-3} seconds. At around $\theta = 15^\circ$, the error of the period as calculated under the small-angle approximation becomes 0.01 seconds. At $\theta \approx 25^\circ$, this error rises to 0.02 seconds. The rate at which this error changes with initial angle increases with θ.

The essential key to observe from this graph should not be the exact periods of a given initial angle, but rather the fact that the error is surprisingly insignificant from an experimental perspective. To a student measuring the period of the pendulum, considering that the average human reaction time for a visual stimulus is 0.2 to 0.4 seconds[Aditya Jain and Singh 2015], the measurement's accuracy wouldn't differ much even if the student decided to release the pendulum at $\theta = 15^\circ$ and averaging 20 oscillations, nor choosing a seemingly-ridiculous $\theta = 25^\circ$ and averaging over 10 periods, as the period difference between the experiment and that using the small-angle approximation is smaller than what could have been caused by inevitable reaction-time-related errors.

Hence, in a freshman physics lab, it is advised to encourage students to average more than 20, say, periods, rather than to repeatedly emphasize that one should release the system at a small angle in order to satisfy the small-angle approximation. If students doesn't take the average of more than 20 oscillations, the point of releasing at a small angle $\theta < 5^\circ$ would be meaningless, and doing it at $\theta = 15^\circ$ would be just fine. What's more, since the speed of the bob would be larger if it were to be released at a larger initial angle, the period could be measured more accurately with a stopwatch from a experimental perspective, if the student starts and ends the stopwatch at the moment when the bob reaches the bottommost position.

6 STRING LENGTH

It must be noted that the above discussion is concerned with a pendulum of length $l = 1$ meters. A analog to Figure 3 is shown in 4, where the only difference between these two cases is that the latter is plotted with $l = 0.25$m while the former uses $l = 1$m. In a lab setting, 4 may better represent the usual situation that students deal with, as a pendulum with a one-meter string is more difficult to set up. It can be observed from this figure that the period difference is even less significant when choosing initial angles beyond five degrees. For example, the period error at $\theta = 15^\circ$ drops from the previous 0.01 seconds to 0.005 seconds. In continuation of the previous discussion, if the human reaction time for a visual stimulus is taken to be 0.2 seconds, one should average for 40 periods or more to make it meaningful to perform the same experiment under the small-angle condition. Considering that the experiment could be easier performed with a larger initial angle, this further highlights the fact that unless a certain number of periods is average to obtain the final result, the cons of measuring the period of the system at $\theta < 5^\circ$ may outweigh the pros.

7 CONCLUSION

The behavior of the mechanical pendulum beyond the small-angle condition is reviewed from a intuitive context. Based on numerical solutions, it has been confirmed that its motion is oscillatory with a larger period compared to that of a system with a small initial angle. Practically, the error caused by not releasing the pendulum at $\theta < 5^\circ$ is also analyzed, and it can be concluded that such difference is small enough so that one can claim that averaging over more periods is more important over releasing it at a small angle.

ACKNOWLEDGMENTS

I thank my friend Cody (Cheng En, Chiayi Senior High School) for reviewing this paper.

REFERENCES

D.I. M´endez T. Bel´endez AA. Bel´endez, C. Pascual and C. Neipp. 2007. Exact solution for the nonlinear pendulum. *Revista Brasileira de Ensino de F´ısica* 29, 4 (Aug. 2007),

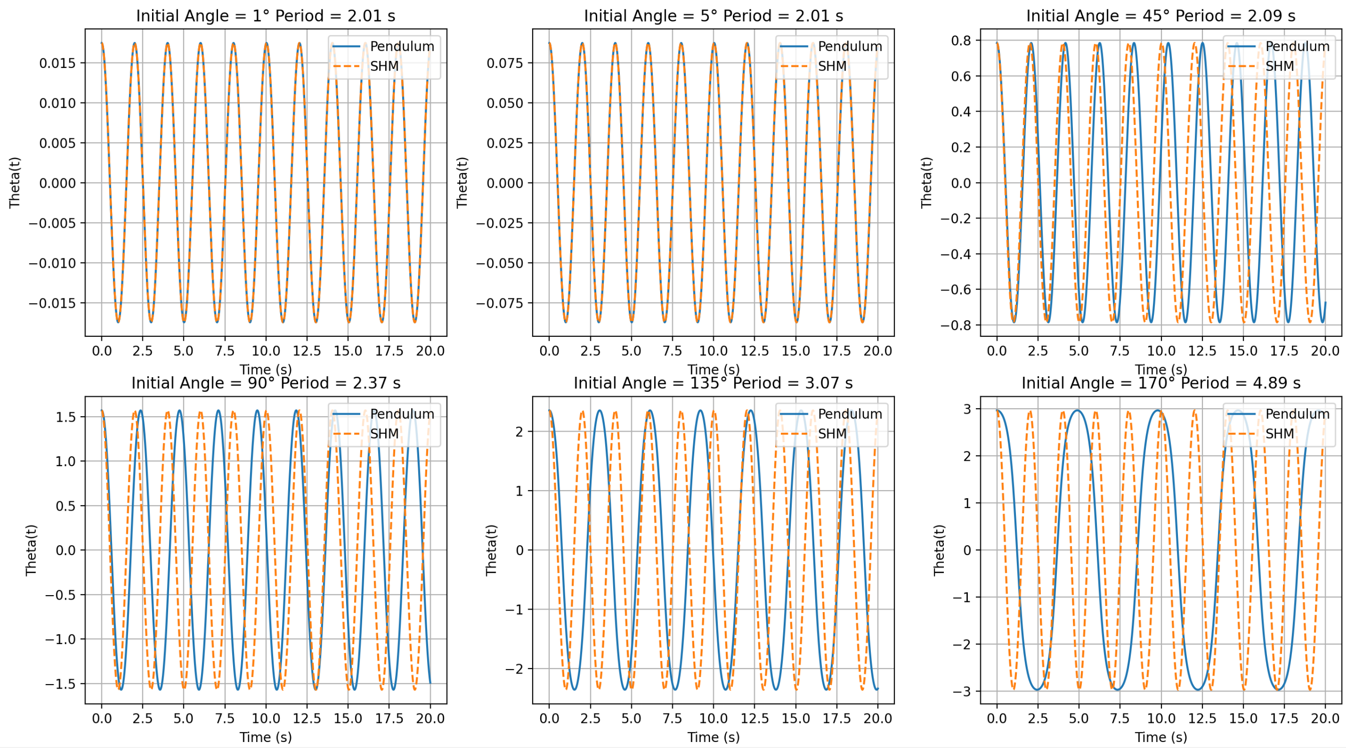

Fig. 2. Numerical Solutions
Graphs of θ as a function of time. The solid blue lines are the numerical solutions for their respective initial angles, while the dashed orange line depicts the solution obtained with the small angle approximation.

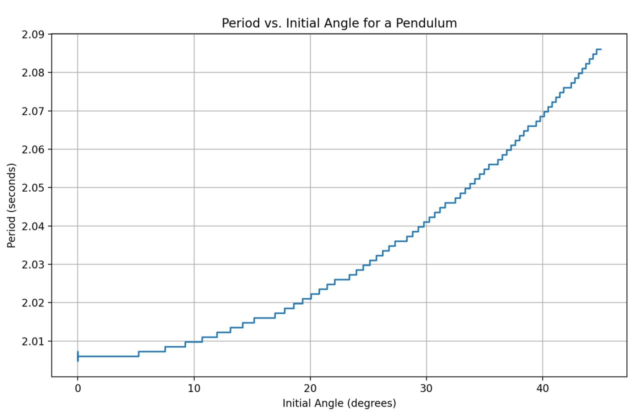

Fig. 3. Period vs. θ, Enlarged
This graph presents an enlarged version of Figure 1.

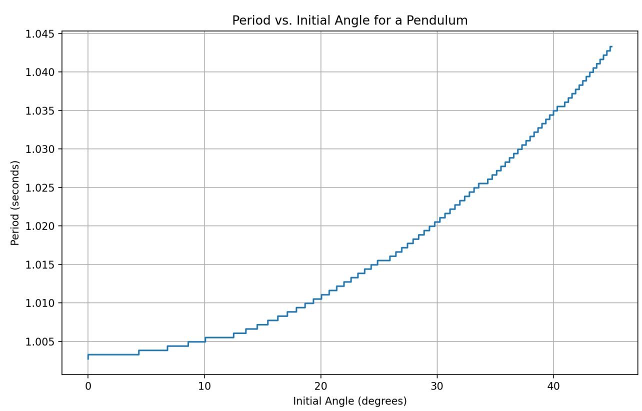

Fig. 4. Period vs. θ for $l = 0.25$m
This graph presents Figure 3 with a smaller string length of 0.25 meters.

4 pages. http://www.sbfisica.org.br/rbef/pdf/070707.pdf

Avnish Kumar Aditya Jain, Ramta Bansal and KD Singh. 2015. A comparative study of visual and auditory reaction times on the basis of gender and physical activity levels of medical first year students. *International Journal of Applied and Basic Medical Research* 5, 2 (May 2015), 4 pages. https://doi.org/10.4103/2229-516X.157168

David; Robert Resnick; Jearl Walker Halliday. 1997. *Fundamentals of Physics* (5th ed.). Vol. 1. John Wiley Sons. (book).

Prallax. 2021. Solution to pendulum differential equation. *Physics Stack Exchange* (July 2021). https://physics.stackexchange.com/a/653874/292866

物理奧林匹亞國家代表隊選訓工作小組. 2011. *2011* 賽前集訓教材.

A PYTHON SCRIPT FOR THE NUMERICAL SOLUTION

Backslashes are used for the purpose of formatting and may be removed.

A.1 $\theta - t$ Graph

```
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt

# Define the ODE for the pendulum
def pendulum_ode(y, t, omega):
    theta, theta_prime = y
    theta_double_prime = -omega**2 * np.sin(theta)
    return [theta_prime, theta_double_prime]

# Constants
# Acceleration due to gravity (m/s^2)
g = 9.81
# Length of the pendulum (m)
L = 1.0
omega = np.sqrt(g / L)

# Initial conditions: theta(0) and theta'(0)
# The value of theta(0) can be modified
initial_conditions = [np.pi / 4, 0.0]

# Time points
t = np.linspace(0, 20, 2000)

# Solve the ODE using odeint
solution = odeint(pendulum_ode, \
initial_conditions, t, args=(omega,))

# Extract the solutions for theta and theta'
theta_solution, theta_prime_solution = \
solution[:, 0], solution[:, 1]

# Calculate the corresponding
# sine curve (small angle approximation)
sine_solution = initial_conditions[0] * \
np.cos(omega * t)

# Find the peaks (or troughs) of theta_solution
from scipy.signal import find_peaks
peaks, _ = find_peaks(theta_solution)

# Calculate the period of oscillation
period = np.mean(np.diff(t[peaks]))

# Plot the angle vs. time
plt.figure(figsize=(8, 6))
plt.plot(t, theta_solution, label='Pendulum Motion')
plt.plot(t, sine_solution,
label='Simple Harmonic Approximation', linestyle='dashed')
plt.xlabel('Time (s)')
plt.ylabel('Theta(t)')
plt.xlim(0, 10)
plt.title('Pendulum Motion vs. Simple Harmonic Approximation')
plt.legend(loc='lower right')
plt.grid()
plt.show()

print(f"Period of oscillation: {period} seconds")
```

A.2 Period $- \theta$ Graph

```
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt

# Define the ODE for the pendulum
def pendulum_ode(y, t, omega):
    theta, theta_prime = y
    theta_double_prime = -omega**2 * np.sin(theta)
    return [theta_prime, theta_double_prime]

# Constants
g = 9.81
L = 0.25

# Range of initial angles in degrees
# Vary initial angles from 0 to 45 degrees
initial_angles_degrees = np.linspace(0, 45, 50000)
initial_angles = np.deg2rad(initial_angles_degrees)
periods = []

# Time points
t = np.linspace(0, 20, 2000)

for initial_angle in initial_angles:
    # Calculate omega for each initial angle
    omega = np.sqrt(g / L)

    # Initial conditions: theta(0) and theta'(0)
    initial_conditions = [initial_angle, 0.0]

    # Solve the ODE using odeint
    solution = odeint(pendulum_ode, \
    initial_conditions, t, args=(omega,))

    # Extract the solutions for theta and theta'
    theta_solution, _ = solution[:, 0], solution[:, 1]

    # Find the peaks (or valleys) of theta_solution
    from scipy.signal import find_peaks
    peaks, _ = find_peaks(theta_solution)

    # Calculate the period of oscillation
    period = np.mean(np.diff(t[peaks]))

    # Append the period to the list
```

```
    periods.append(period)

# Plot initial angle (in degrees) vs. period
plt.figure(figsize=(10, 6))
plt.plot(initial_angles_degrees, \
periods, label='Pendulum Motion')
plt.xlabel('Initial Angle (degrees)')
plt.ylabel('Period (seconds)')
plt.title('Period vs. Initial Angle for a Pendulum')
plt.grid()
plt.show()
```