Average Value of $\sin^2\theta$

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1 Introduction

The aim of this article is to find the average value of $\sin^2 x$. This value is often used in physics, closely related to the definition of the root-mean-square.

2 The Proof

We seek to find

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} \sin^2\theta \, d\theta \tag{1}$$

So we must first evaluate the integral part. However, $\int \sin^2 x \, dx$ is difficult to find. Therefore, we should seek to using the trig identity

$$\cos 2x = 1 - 2 \cdot \sin^2 x \tag{2}$$

After rearranging, we obtain

$$\sin^2 x = \frac{1 - \cos 2x}{2} \tag{3}$$

Substitute (3) into (2),

$$\int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\, d\theta \; + \; \int_0^{2\pi} \frac{\cos 2\theta}{2}\, d\theta$$

$$= \frac{1 \cdot 2\pi}{2} \; + \; \frac{1}{2} \cdot \left(\frac{1}{2} \cdot \sin 2\theta \right|_0^{2\pi}$$

$$= \pi \; + \; \frac{1}{4} \cdot (0 - 0)$$

$$= \pi \tag{4}$$

We then substitute (4) into (2) and find

$$\boxed{\frac{1}{2\pi} \cdot \pi \; = \; \frac{1}{2}} \tag{5}$$

Which is our final answer. Note that the average value of $\sin^2 x$ has no relationship with x. Therefore, the root-mean-square value of a sinusoidal-alternating variable A, in physics, is defined to be

$$A_{rms} = \frac{A}{\sqrt{2}}$$