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2023 Physics Cup Problem 2

General Idea

1. A real image is formed, so the thin lens must be a convex lens located to the left of the ellipse. The lens and the axis are not necessarily perpendicular to the x axis as intuition may suggest.
2. The axis passes through focus F. Additionally, drawing the two tangent lines from F to the ellipse gives us two points. As will be shown later, they are in fact the topmost and the bottommost points on the circle, enabling us to determine the axis.
3. Suppose we have found the axis. By $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$, the furthest point to the lens C on the circie would end up at the closest point to the lens C' on the ellipse after passing through the thin lens. Simularly, the image of the closest point to the lens D on the circle would be at D', the furthest point to the lens on the ellipse. The topmost point A, bottommost point B and center P of the circle would end up on the same vertical line. Note that *topmost, bottommost* and *vertical* only make sense after finding the axis.
4. The coordinates of the center of the lens can be determined by drawing light rays emerging from points A', B', C' and D' and noting the relationships between their corresponding points on the object circle.

Determining the Axis

Draw two tangent lines from F to the ellipse gives two points $G = B'$ (corresponding to bottommost point on the circle) and $H = A'$ (corresponding to topmost point on the circle).

Claim. GH is perpendicular to the axis.

Proof. Since G and H are the tangent points, no ray emerging from the lens shoots higher than FG or lower than FH. Since rays through the focus becomes parallel after passing the lens, ray FG must end up at the topmost point A on the circle, whereas FH ends up at the bottommost point B. Since A, B are at the same position on the axis, G and H must be also. Hence, GH is perpendicular to the axis.

The axis, colored in green in the figure below, is then found to be the line passing through F and perpendicular to line GH.

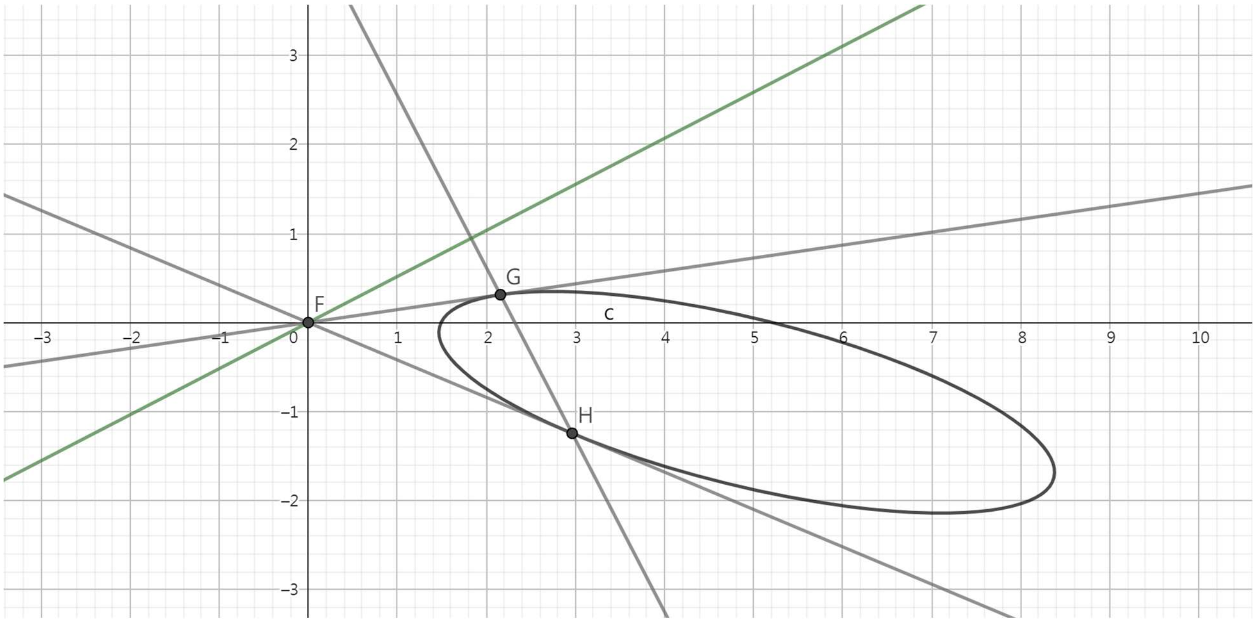

Figure 1

Closest and Furthest Points

By considering the location on the axis only (p as in the lens formula), C' and D' can be found by drawing tangent lines to the ellipse that are also perpendicular to the axis. Connecting them, the intersection with GH corresponds to the point that is both

1. On the same height with the closest and furthest points to the lens on the circle.
2. On the same position on the axis with the topmost and bottommost points on the circle.

Hence, this point must be the image of the center of the circle P'.

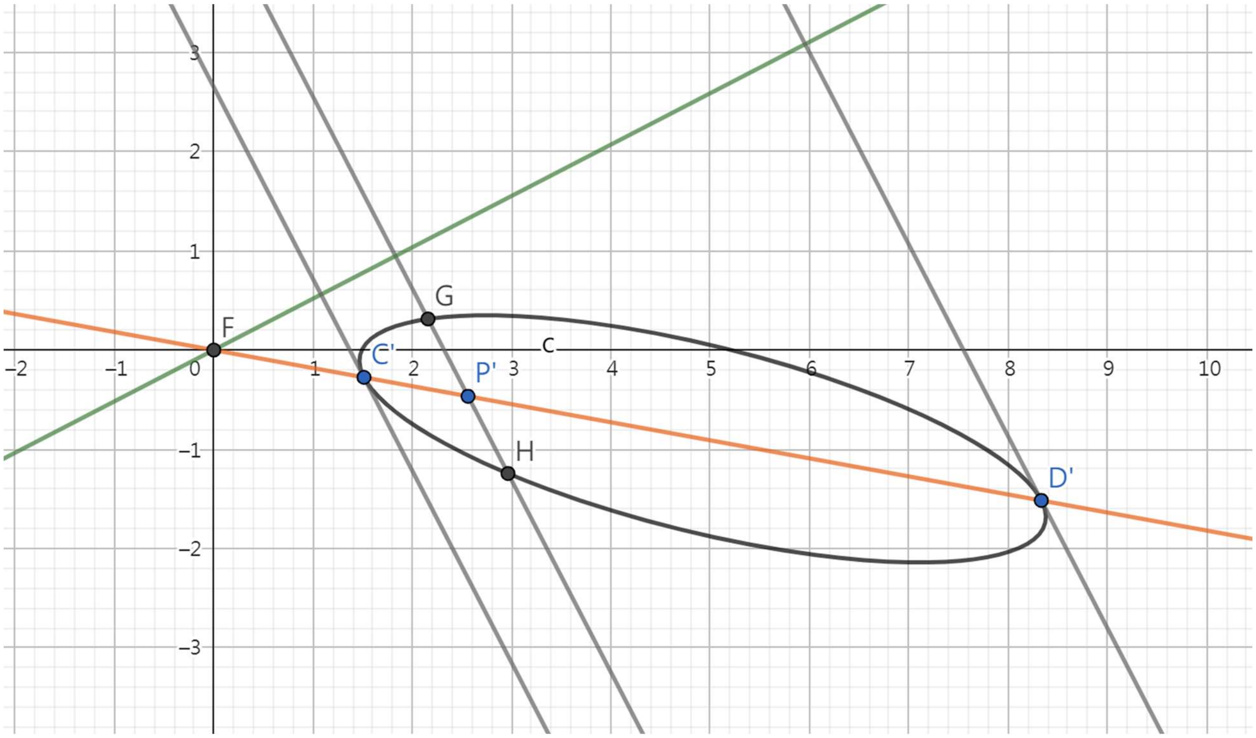

Figure 2

Final Steps

Figure 3 below depicts, for a given lens center O lying on the axis, the points A, B, C, D and their corresponding images A', B', C', D'. This diagram is made numerically and is only for the purpose of demonstration. The axis is colored in green, important orthogonal lines in blue, and light rays in orange. The rest of the solution is inspired by Richard Luhtaru's solution[1] to the 2017 Physics Cup problem 4.

Let MO be a line parallel to AD and, hence, BC. Parallel lines must intersect at the same point after refraction, so $A'D'$, $B'C'$ and MO are concurrent. As MO passes through the center of the lens, it doesn't refract. Let their intersection be N. Since AD makes a 45^o angle with the green axis, MO can be constructed by drawing the perpendicular line to the axis at F. Since $\angle FNO = 45^o$, the center can finally be found by drawing the angle bisector of $\angle FNV$ where VN is parallel to the axis. Figure 4 shows the final sketch. The coordinate of O reads $\boldsymbol{O(-1.12358, -0.58132)}$.

[1] https://www.ioc.ee/~kalda/ipho/PhysicsCup2017/4luhtaru.pdf

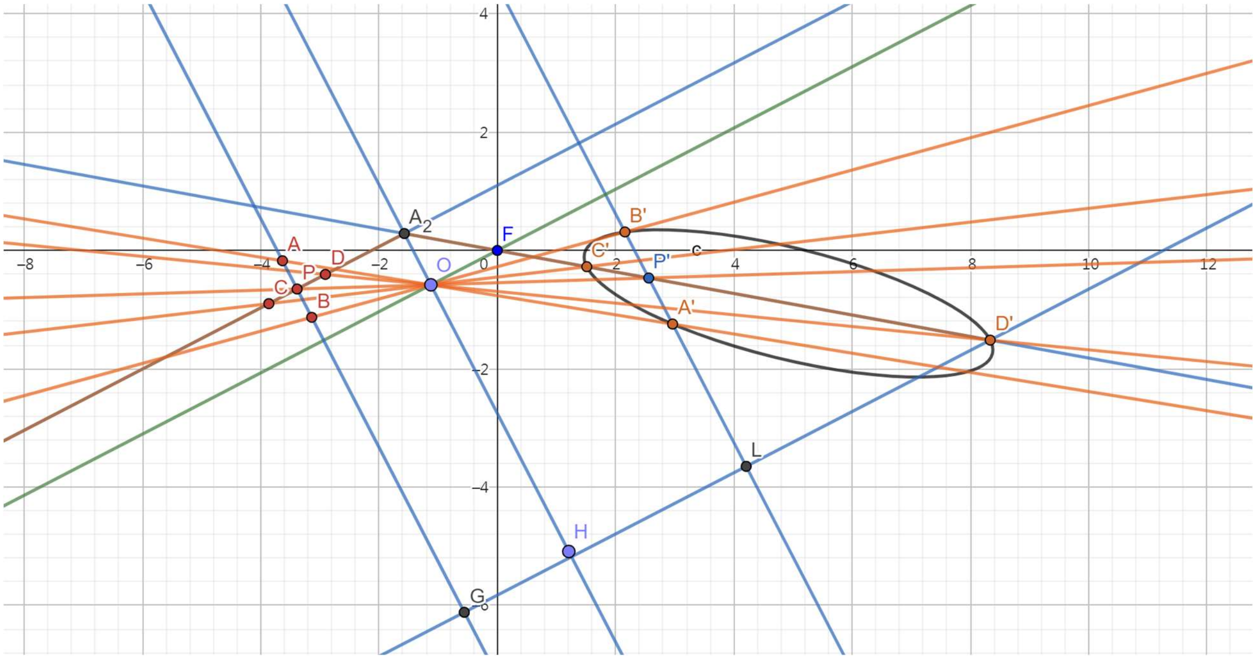

Figure 3

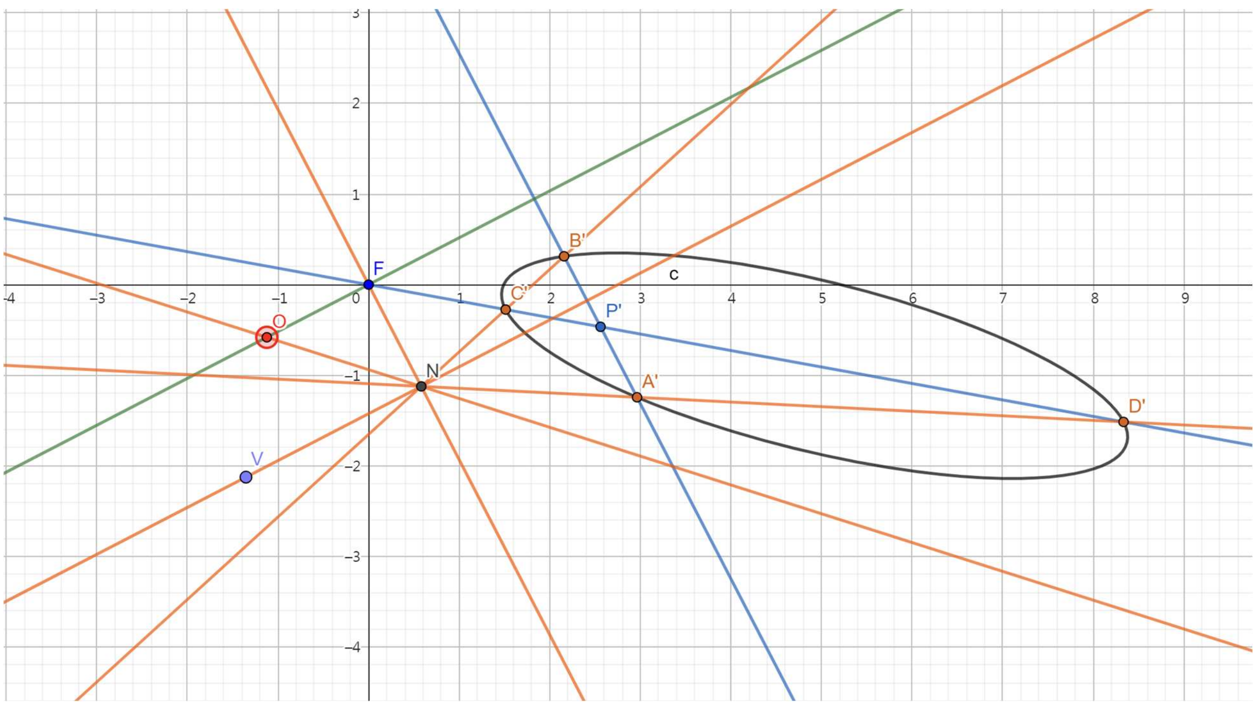

Figure 4

Remarks

This solution was submitted after the release of hint 4. I proposed a solution that requires fine-tuning a certain value in order to find O, which I got -1 points for and was rejected:

> *"…the score -1 means that you have tried to solve the problem by trial and error (looking for such a position of some point by which three points are collinear or two points coincide etc). This is not what "construct" means. "Construct" means by just drawing tangents, line through two points, etc, you will be able to find the position of the point. -1 means also that you will not get a penalty as you misunderstood the problem." --- Mr. Kalda's Message on Week 1*

The "numerical" attempt is shown below, replacing everything starting from "Figure 2" of this document:

Finding the Center

Figure 3 below depicts, for a given lens center O lying on the axis, the points A, B, C, D and their corresponding images A', B', C', D'. The axis is colored in green, important orthogonal lines in blue, and light rays in orange. $CP = PD$ and $AP = PB$ are trivial relationships, regardless of where O is. However, there is only one location of the lens center that $CP = PD = AP = PB$. This can be observed by moving O along the green line. Note that the condition $CP = PD = AP = PB$ must be satisfied for the original shape to be a circle (it would be an ellipse otherwise). Therefore, the lens center can be found by dragging point O along the axis until reaching the point where, without loss of generality, $CP = AP$ is satisfied. The lengths of CP and AP could be measured in GeoGeBra. To find the answer, point O is fine tuned until the difference between CP and AP is less than 10^{-5}, as this would imply that the coordinates of the lens center is accurate to at least 5 decimal places, which the problem requires.

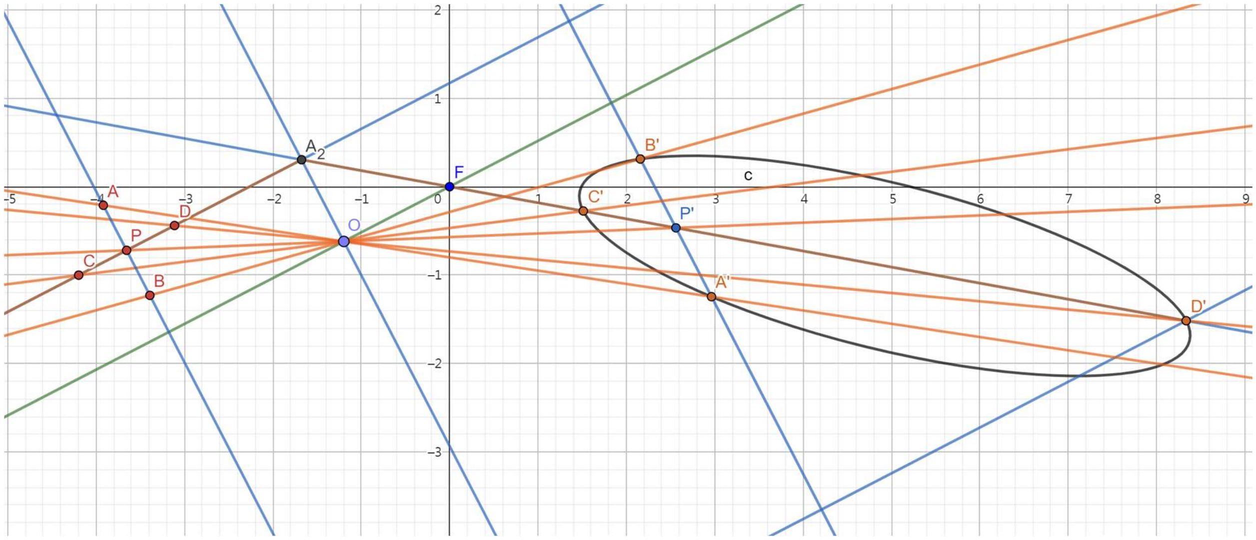

Figure 4

The final diagram, whose .ggb file is attached with this solution, is is shown in Figure 4. When point O is positioned at $(-1.1235808356, -0.5813214324)$, one finds

$$AP = 0.538098037$$

$$CP = 0.5380984372$$

Since $|AP - CP| \approx 4 \times 10^{-7} < 10^{-5}$, we can assume that the coordinates of O is accurate to at least five significant digits. Therefore, the center of the lens lies at

$$(\mathbf{-1.12358, -0.58132})$$

Remarks

I acknowledge that this method may not be what the problem creater originally seeks for. There may be better solutions that only require geometrical constructions to determine the center of the lens. However, the problem statement only requires that one

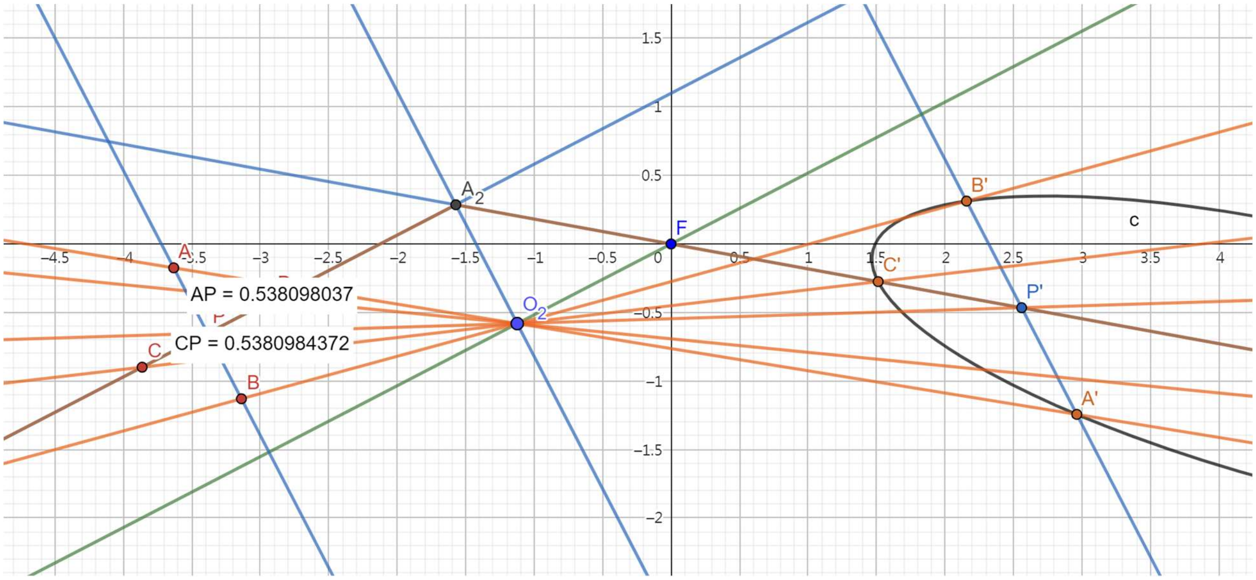

Figure 5

… may use the functionality available through the menu buttons (except the Locus, Polar, Complex numbers, Best Fit Line, functions tools, and tools with numeric input). You are not allowed to make numerical calculations in Geogebra and enter calculated point coordinates. The available tools are shown below.

This solution does not use the forbidden functions mentioned above, as measuring lengths and comparing them should not be "numberical calculations in GeoGeBra", nor "entering calculated point coordinates". It is also not a tool "with numeric input", since it should be one with a numerical output. In short, I suggest that this solution complies with all the requirements.

Furthermore, this approach may generally be more intuitive, as it requires nothing more than drawing critical light rays and making use of the qualitative concept that objects with a larger object distance has an image at a smaller image distance. In conclusion, although this solution may not be the most mathematically rigorous, I sincerely hope that it would be considered a viable approach to this problem.