Physics Cup Problem 1

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Solution

The density of the cylinder is

$$\boxed{3320 \text{ kg/m}^3}$$

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Formula

$$\rho_{cylinder} = \rho_{water} \cdot \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right)$$

where

$$a = \frac{2h^2}{1.15}$$

$$b = -\frac{U_0^2 h}{g} - \frac{2h^2}{1.15}$$

$$c = y_c^2$$

Symbols

y_c is the maximum depth of the cavity formed measured to the bottom of the cylinder. h is the height of the cylinder. U_0 is the initial impact speed, measured at or close to frame 23. g is the acceleration due to gravity.

Measurements

All lengths will be expressed in pixels and time in frames. h will be measured directly from frame 20. U_0 will be found by measuring the y pixel difference between frame 20 and frame 23, then dividing it by 3 frames (time unit). y_c will be found by first finding the two cavity wall lines' equation, calculating their intersection, then finding the distance of that point to the water surface. g will be found near 00:31 by calculating the second order time derivative of the y position of a particular water droplet.

1 Assumptions and Observations

- From [1], one can observe that the scenario in the video is most similar to the *quasi-static* case discussed in the paper, where the cavity collapses near the top of the immersed object. Thus, it is assumed that there is no energy loss for the water-cylinder system.
- The cylinder has mass m, radius r and height h, the initial impact speed is U_0, the density of the cylinder is ρ_s and the density of water is ρ. Note that $m = \pi r^2 h \rho_s$.
- No absolute length, nor time, reference can be found in the video. Hence, all quantities should eventually be non-dimensional and only the ratio between them should determine the density of the cylinder.
- The height of the container is much larger than that of the cylinder.
- Neglect viscosity.

2 The General Idea

The cylinder sinks into the water, creating a cavity. Intuitively, after the cavity is formed, the object wouldn't be decelerating, or one would expect a larger cavity to be formed. Therefore, the formation of the cavity ends when the cylinder reaches its terminal velocity determined by the buoyant force and the drag force.

1. As the hydrostatic pressure is used to slow the cylinder's descent, the depth at which the terminal velocity is achieved can be calculated by conservation of energy.
2. By measuring the depth from the video, we can find the density of the cylinder.

3 Deceleration Phase

When the cylinder has not yet reached its terminal velocity, it would only be partially immersed in the water. As no hydrostatic pressure is exerted on the top surface, when the bottom of the cylinder is at depth y, a hydrostatic force

$$\vec{F_H} = -\rho g y \hat{y} \tag{1}$$

slows the cylinder down, where we take the initial location of the water surface to be at $y = 0$ and the $+y$ axis pointing downwards.

To calculate the terminal velocity, we consider the constant buoyant force

$$\vec{F_B} = -\rho \pi r^2 h \hat{y}$$

and the drag force

$$\vec{F_D} = -\frac{1}{2}\rho v^2 C_D \pi r^2 \hat{y}$$

The drag force is taken into account due to the cylinder's flat shape. The value of the drag coefficient C_D for short cylinders is 1.15 [2]. From the equilibrium equation

$$F_D + \pi r^2 h \rho g = \pi r^2 h \rho_s g \tag{2}$$

the terminal velocity can be found as

$$v_t = \sqrt{\frac{2gh(\rho_s - \rho)}{\rho C_D}} \tag{3}$$

Since the cylinder sinks, $\rho_s > \rho$, and (3) is reasonable.

The work done by the hydrostatic force on the cylinder

$$W_H(y) = -\int_0^y (-\rho g y \hat{y}) \cdot (\hat{y}\, \mathrm{d}y) = \frac{1}{2}\rho g \pi r^2 y^2$$

is equal to the loss of kinetic energy, so that

$$\frac{1}{2}\rho_s \pi r^2 h(U_0^2 - v_t^2) = \frac{1}{2}\rho g \pi r^2 y_c^2$$

thus

$$y_c = \sqrt{\frac{\rho_s h(U_0^2 - v_t^2)}{\rho g}} = \sqrt{\frac{\rho_s h\left(U_0^2 - \frac{2gh(\rho_s - \rho)}{\rho C_D}\right)}{\rho g}} \tag{4}$$

Rearranging yields a quadratic equation

$$a\left(\frac{\rho_s}{\rho}\right)^2 + b\left(\frac{\rho_s}{\rho}\right) + c = 0$$

where

$$a = \frac{2h^2}{1.15}$$

$$b = -\frac{U_0^2 h}{g} - \frac{2h^2}{1.15}$$

$$c = y_c^2$$

so the formula is

$$\rho_{cylinder} = \rho_{water} \cdot \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \tag{5}$$

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As a practical side note, the achievement of y_c corresponds to the moment when the shape of the cavity (which is easily observed from the video) changes from expanding to collapsing. We assume that at that moment, the velocity of the water near the cavity is 0.

4 Before we start measuring stuff...

We can find y_c just from the geometry of the water cavity. At frame 23 the cylinder impacts the water surface. For a few frames near frame 64, the cavity borders seems unchanged. This signifies that the cylinder has reaches y_c. Analyzing this picture with ImageJ and approximating the two cavity walls (the left one labeled as L and the right one R) as straight lines, we can find y_c!

5 Measurement

Measurements are taken by selecting individual pixels from screenshots on my laptop. Performing these measurements on devices of different screen resolution may yield different numerical values, but the final answer should be the same.

All lengths are in units of pixels and time in units of frames.

5.1 h

At frame 20, the top and bottom of the cylinder are at y coordinates of 727 and 341 repectively, so

$$h = 727 - 341 = 386 \text{ px} \tag{6}$$

5.2 U_0

From frame 17 to 20, the top of the cylinder went from $y = 477$ to $y = 327$. Hence

$$U_0 = \frac{477 - 327}{20 - 17} = 50 \text{ px/frame} \tag{7}$$

5.3 g

The acceleration due to gravity could be difficult to find. However, from the impact lots of water droplets are created. By choosing one that doesn't go out of the screen, we can find g by taking the second time derivative. Therefore, we turn our attention to a water droplet launched up near 00:31. After it has separated from the rest of the water (so that no additional mass would be added to the droplet), its location is tracked every 5 frames.

Figure 1: A snapshot near 00:31. The droplet being analyzed is circled in red.

Relevant data are displayed in the Appendix. The value of g, averaging over 6 values, is

$$g = 0.367 \text{ px/frame}^2 \tag{8}$$

5.4 y_c

Let the water surface be denoted in S, the equations for the lines are

$$S : y = 859 \tag{9}$$

$$L : y = 2.2578x - 636.94 \tag{10}$$

$$R : y = -2.532x + 7223 \tag{11}$$

The data points by which the equations are calculated are listed in the Appendix. Using GeoGeBra to find the distance between the intersection point of L and R and line S, we get 2209.052. However, this is the depth measured to the top of the cylinder. To find y_c, we need the depth measured to the bottom of the cylinder, hence

$$y_c = 2209.052 + h = 2595.052 \tag{12}$$

6 Calculation

$$a = \frac{2h^2}{1.15} = 259123.478$$
$$b = -\frac{U_0^2 h}{g} - \frac{2h^2}{1.15} = -2888551.271$$
$$c = y_c^2 = 6734294.883$$

$$\rho_s = \rho \cdot \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right) = \rho_+, \rho_- \tag{13}$$

where

$$\rho_+ = 7.827\rho \tag{14}$$

$$\rho_- = 3.320\rho \tag{15}$$

These two solutions both seem valid at first because they say the density of the cylinder is larger than that of water. However, since it is obvious from the video that the cylinder is made of none-metalic material, it is way more likely that $\rho_s = \rho_-$ as a specific weight higher than 7 is quite typical of metals. Therefore,

$$\boxed{\rho_{cylinder} = 3320 \text{ kg/m}^3} \tag{16}$$

References

[1] Jeffrey M. Aristoff and John W. M. Bush. Water entry of small hydrophobic spheres. 2008.

[2] Wikipedia. List of Drag Coefficients.

Appendix

6.1 Points For Lines L and R

Table 1: Coordinates for points used to construct lines L and R

(a) Line L

X	Y
690	901
786	1155
901	1399
880	1360
836	1265
965	1526
926	1448

(b) Line R

X	Y
2370	1220
2337	1316
2288	1419
2257	1502
2230	1583
2241	1552
2314	1364

6.2 Table For Calculation of g

Table 2: Relevant data to calculate acceleration due to gravity. The 6 values of g are averaged to obtain the value used in the analysis above. Values of -1 indicate null values.

Y	ΔY	v	Δv	g
232	-1	-1	-1	-1
216	-16	-3.2	-1	-1
193	-23	-4.6	-1.4	0.28
182	-11	-2.2	2.4	0.48
180	-2	-0.4	1.8	0.36
181	1	0.2	0.6	0.12
193	12	2.4	2.2	0.44
218	25	5	2.6	0.52
250	32	6.4	1.4	0.28