Drifting Gas System

Consider a one-dimensional gas system composed of ions and molecules. The ions have mass m and charge $q > 0$, while the molecules are neutral with mass $M \gg m$. The system is subject to an external electric field $\mathbf{E}$ parallel to the 1D system, causing the ions to drift in addition to thermal motion. Assuming that the probability of an ion having a mean free path of L is

$$P(L) = e^{-L/\lambda}$$

where λ is a constant. Find the average drift speed v of the ions.
Hints:

1. As the ions mutually repel, interactions between them are neglected. You also don't have to consider gravitational attraction between masses.
2. You may need the integral
$$I_n = \int_0^\infty x^n e^{-x^2}\,dx$$
some known values are $I_0 = \frac{\sqrt{\pi}}{2}$, $I_1 = \frac{1}{2}$, and $I_2 = \frac{\sqrt{\pi}}{4}$
3. Given the previous assumptions, the mean free path of an ion is the distance it has travelled before colliding with a molecule.
4. Take all collisions as perfectly inelastic.

Source: 2023 Taiwan Physics Olympiad Preliminary Selection.

Solution

Consider an ion at rest relative to a molecule that it will eventually collide with. Let the distance between them be initially L. As the ion accelerates with

$$a = \frac{qE}{m}$$

it picks up speed. Until its encounter with the molecule, the average speed is

$$v_{avg}(L) = \frac{1}{2}\sqrt{2aL} = \sqrt{\frac{qEL}{2m}}$$

The probability of this particular average speed be obtained is

$$P(L)v_{avg}(L)\,dL$$

to find the drift velocity, we integrate through all the possible mean free paths, that is,

$$v_{drift} = \int_0^\infty P(L)v_{avg}(L)\,dL = \int_0^\infty e^{-L/\lambda}\sqrt{\frac{qEL}{2m}}\,dL = \sqrt{\frac{qE}{2m}}\int_0^\infty e^{-x/\lambda}\sqrt{x}\,dx$$

to evaluate the integral we substitute

$$u = \sqrt{\frac{x}{\lambda}}$$

$$dx = 2\lambda u du$$

so that

$$\int_0^\infty e^{-x/\lambda}\sqrt{x}\,dx = \int_0^\infty 2\lambda\sqrt{\lambda}u^2 e^{-u^2}\,du = 2\lambda\sqrt{\lambda}\frac{\sqrt{\pi}}{4}$$

hence the drift speed is

$$v_{drift} = \sqrt{\frac{qE}{2m}}\int_0^\infty e^{-x/\lambda}\sqrt{x}\,dx = \sqrt{\frac{\pi\lambda^3 qE}{8m}}$$

Note: $P(L)$ is meant to be a probability density function so that $P(L)\,dL$ is a dimensionless number. To fix this ambiguity, we can redefine $P(L)$ as

$$P(L) = \frac{1}{\lambda}e^{-L/\lambda}$$

so the final answer would be $\sqrt{\frac{\pi\lambda qE}{8m}}$.

For this problem, all answers with the correct variables would be correct. That is, answers with the form

$$\boxed{k\sqrt{\frac{\lambda qE}{m}}}$$

are considered right, where k is a numerical constant. However, you need to explain how you got the answer with your physics model besides simply using dimensional analysis.