Matrix Algebra for Educational Scientists

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# 1 Foreword

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The contents of this book constitute information from several set of notes from a variety of QME courses, including from an old course called EPsy 8269. We are making this book available as a resource for anyone who wants to use it. We will be adding and revising the content for awhile. Feel free to offer criticism, suggestion, and feedback. You can either [open an issue](https://github.com/zief0002/matrix-algebra/issues) on the book’s github page or [send us an email](mailto:%20zief0002@umn.edu) directly.

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## 1.1 Colophon

Artwork by [@allison\_horst](https://twitter.com/allison_horst)

Icon and note ideas and prototypes by [Desirée De Leon](http://desiree.rbind.io/).

The book is typeset using [EB Garamond](https://fonts.google.com/specimen/EB+Gara) for the body font, [Raleway](https://fonts.google.com/specimen/Raleway) for the headings and [Sue Ellen Francisco](https://fonts.google.com/specimen/Sue+Ellen+Francisco) for the title. The color palette was generated using [coolors.co](https://coolors.co/).

## 1.2 Acknowledgments

Many thanks to James Terwilliger, whose initial notes gave rise to some of this material. Also, thank you to all the students in our courses who have been through previous iterations of this material. Your feedback has been invaluable, and you are the world’s greatest copyeditors.

# 2 Introduction

Having a basic understanding of the vocabulary, notation, and ideas of matrix algebra, is important for all educational scientists who use quantitative methods in their work. The statistical and psychometric models underlying many quantitative methodologies employed in educational research rely on matrix algebra. Subsequently, educational scientists use the language and notation of matrix algebra to communicate in the scientific literature. Moreover, matrix algebra forms the bedrock of statistical computation. Having fundamental knowledge of matrix algebra can often help an educational scientist troubleshoot problems that arise in their own work, and devise solutions for those issues.

For quantitative methodologists, it is important to have a much deeper understanding of matrix algebra, as it is foundational to the computational estimation and optimization used in methodological work. Statistical programming, formulating the mathematics of quantitative methods, and even back-of-the-napkin calculations are all made easier (and more efficient) through matrix algebra.

## 2.1 Prerequisites

ADD PREREQUISITES

# 3 Data Structures

In this chapter you will be introduced to four common data structures that form the building blocks of matrix algebra: scalars, vectors, matrices, and tensors. You will also be introduced to some of the vocabulary that we use to describe these structures. In future chapters, we will examine these structures in more detail and learn how to mathematically manipulate and operate on these structures.

## 3.1 Scalars

A *scalar* is a single real number. You have likely had a lot of previous experience with scalars, as they are emphasized in much of the mathematics taught in high schools in the United States. Here are three examples of scalars:

*Scalar arithmetic* is the arithmetic operations (addition, subtraction, multiplication, and division) we perform using real numbers. For example,

Notice that scalar arithmetic also produces a scalar. For example the scalar addition in the example, produces the scalar .

## 3.2 Vectors

A *vector* is a specifically ordered one-dimensional array of values. Here are three examples of vectors:

A vector may be written as a row or a column, and are respectively referred to as *row vectors* or *column vectors*. Each value in a vector is called an *element* or *component*. Vectors are also typically described in terms of the number of elements they have. For example, the following is a two-element row vector:

Here, **a** is a four-element column vector:

## 3.3 Matrices

A *matrix* is specifically ordered two-dimensional array of values. Here are three examples of matrices:

Matrices have both rows and columns, and are typically described by the number of rows and columns they have. For example, the matrix **B** (below) has 3 rows and 2 columns:

We say that **B** is a “3 by 2” matrix. The number of rows and columns are referred to as the *dimensions* or *order* of the matrix, and, for matrix **B**, is denoted as . The dimension is often appended to the bottom of the matrix (e.g., ).

The elements within the matrix are indexed by their row number and column number, respectively. For example, since the element in the first row and second column is 1. The subscripts on each element indicate the row and column positions of the element.[[1]](#footnote-36)

More generally, we define matrix **A**, which has *n* rows and *k* columns as:

where element is in the row and column of **A**.

## 3.4 Tensors

*Tensors*, generally speaking, are the generalization of the matrix to three or more dimensions. Here is an example of a tensor:

Technically, this definition is not quite true, but for our purposes it will be adequate to think of a tensor as a structure for data in *N* dimensions.

This book will primarily deal with scalars, vectors, and matrices, but tensors do come up in statistical work. For example, image data are often represented as tensors; with each pixel in a two-dimensional image having multiple values associated with it to represent the color information for the pixel. In longitudinal data analysis, the variance–covariance matrix of the responses for multiple subjects is also represented as a tensor. [Bi et al.](#ref-bi_tensors_2021) ([2021](#ref-bi_tensors_2021)) and [McCullagh](#ref-mccullagh_tensor_2018) ([2018](#ref-mccullagh_tensor_2018)) are good resources to learn more about working with and operating on tensors.

## 3.5 A Word about Notation

Authors and textbooks use a wide variety of notation to represent scalars, vectors, and matrices. For example, vectors might be denoted using a lower-case underlined letter (), a lower-case bolded letter (**a**), or a lower-case letter with an overset arrow (). In this book, we will try to use a consistent notation to denote each of these structures.

* Scalars will be denoted with an italicized lower-case letter (e.g., ) or a non-bolded lower-case Greek letter (e.g., ).
* Vectors will be denoted using a bold-faced lower-case letter (e.g., **a**.
* Matrices will be denoted using a bold-faced upper-case letter (e.g., **A**) or a bold-faced upper-case Greek letter (e.g., ).

### 3.5.1 Exercises

**Identify each of the following as a scalar, row vector, column vector, matrix, or tensor.**

Show/Hide Solution

Scalar

Show/Hide Solution

Column vector

1. 13

Show/Hide Solution

Scalar

Show/Hide Solution

Row vector

Show/Hide Solution

Matrix

Show/Hide Solution

Row vector

**How many elements are in each of the following vectors?**

Show/Hide Solution

4 elements

Show/Hide Solution

3 elements

Show/Hide Solution

*k* elements

**What is the order (dimensions) of each of the following matrices?**

Show/Hide Solution

Show/Hide Solution

# References

Bi, X., Tang, X., Yuan, Y., Zhang, Y., & Qu, A. (2021). Tensors in Statistics. *Annual Review of Statistics and Its Application*, *8*(1), 345–368. <https://doi.org/10.1146/annurev-statistics-042720-020816>

McCullagh, P. (2018). *Tensor methods in statistics*. <http://public.ebookcentral.proquest.com/choice/publicfullrecord.aspx?p=5228165>

1. Many authors do not include the comma between the row and column parts of the subscript (e.g., ). [↑](#footnote-ref-36)