Julius-Maximilians-
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WÜRZBURG

Institute of Computer Science
Chair of Communication Networks
Prof. Dr. Tobias Hoßfeld

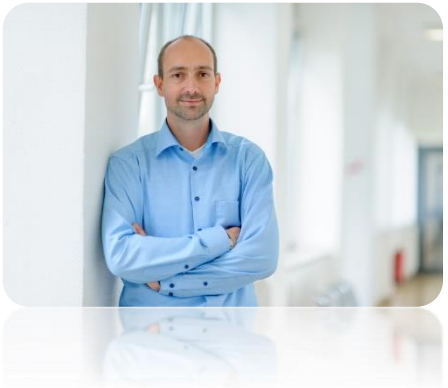
Chapter 2.2

Probabilities and Random Variables

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022



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The book must be cited and the disclaimer attached when using lectures slides or scripts.

Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/

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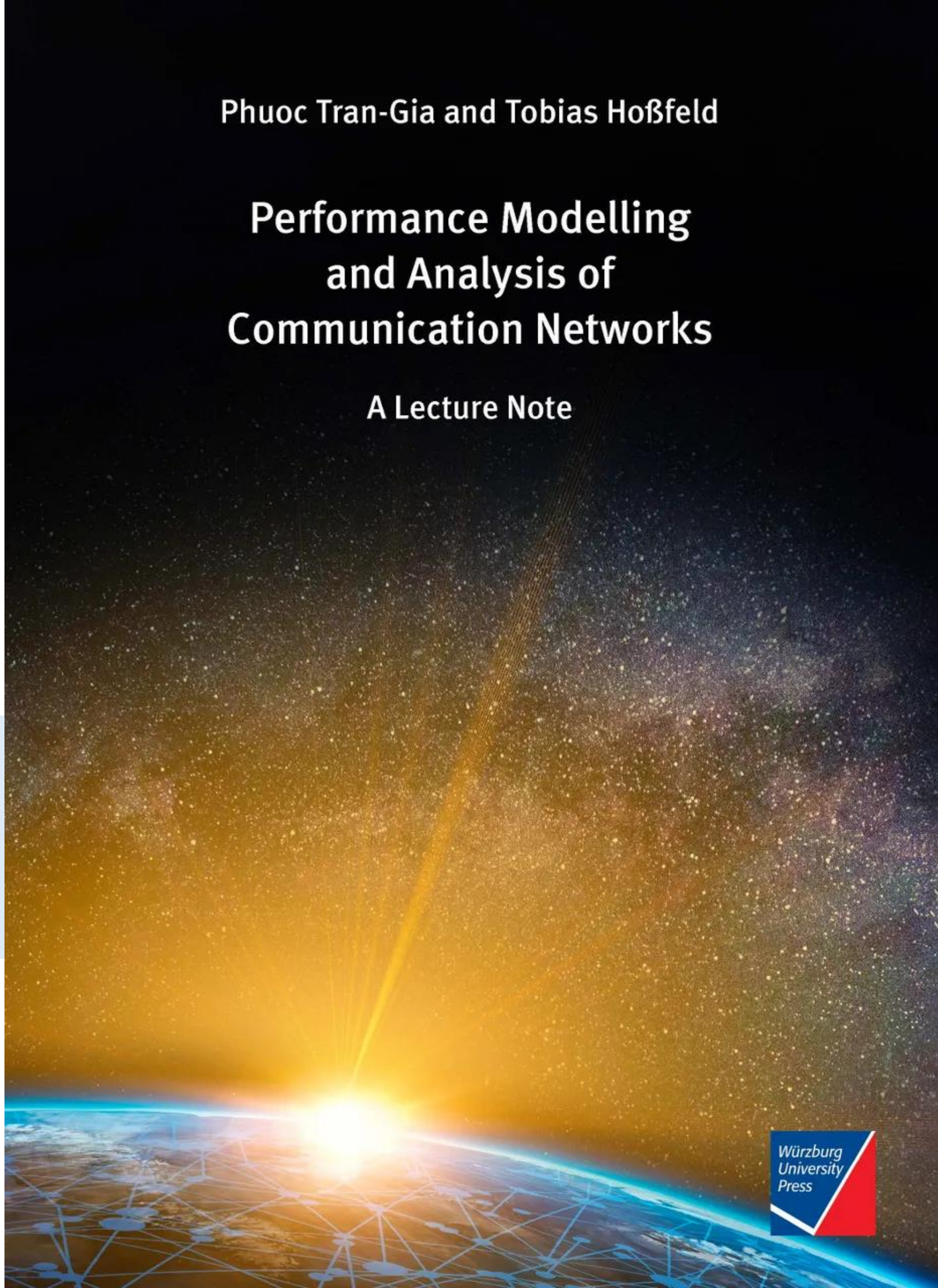


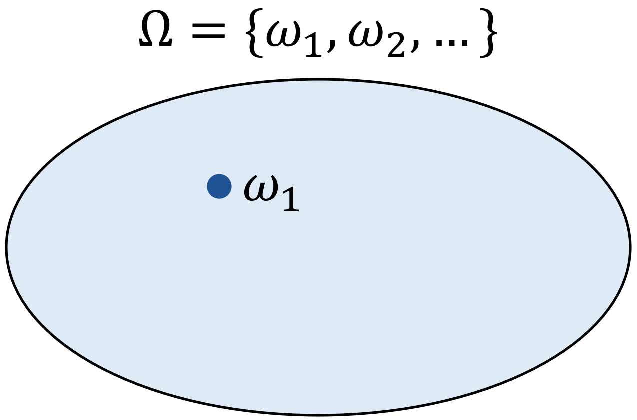
Chapter 2

2 Fundamentals and Prerequisites

Random Experiments and Probabilities

Random Events

- ωi sample or (elementary) experiment results
- Ω sample space
  set containing all possible experiment results,
  $\Omega = \{\omega_1, \omega_2, \dots\}$, can be infinite or finite



- Example: dice $\Omega = \{1,2,3,4,5,6\}$

- Example: transmitting a data packet over the Internet
  - Two possible outcomes: correct (successful packet transmission) or erroneous (packet error) transmission
  - Any number can be assigned for the experiment outcome: e.g. zero {0} erroneous, one {1} for successful transmission; then $\Omega = \{\text{erroneous}, \text{successful}\}$
  - A random variable (r.v.) assigns a real number to each elementary event ω_i

Probability as Limit of Relative Frequency

- n number of experiments
- A_i event (or feature), consisting of a subset of (elementary) experiment results
- n_i number of performed experiments, which delivers experiment results belonging to event A_i

- Relative frequency for the event A_i
$$h(A_i) = \frac{n_i}{n}$$

- Probability of the event A_i
$$P(A_i) = \lim_{n \to \infty} \frac{n_i}{n}$$

Properties of the Probability of an Event

- Bounded in the interval [0;1]: $0 \leq h(A_i) \leq 1 \;\Rightarrow\; 0 \leq P(A_i) \leq 1$

- If A_i and A_j are mutually exclusive events ($A_i \cap A_j = \emptyset$), then either event A_i or A_j can occur but never both simultaneously: $h(A_i \cup A_j) = h(A_i) + h(A_j) \;\Rightarrow P(A_i \cup A_j) = P(A_i) + P(A_j)$

- For a set $\{A_i\}$ of mutually exclusive events with $\bigcup_i A_i = \Omega$, the sum of all probabilities is 1:

$$\sum_i h(A_i) = 1 \;\Rightarrow\; \sum_i P(A_i) = 1$$

A-priori Probability

- Define probability by deductive reasoning
  - if there are a number of equivalent mutually exclusive alternatives and if they are equally likely
  - e.g. due to symmetric property like coins, dices, etc.
- Experiment with m aquivalent alternatives
  - m number of all alternatives
  - m_i number of alternatives belonging to an event A_i
- Definition of **a-priori probability** – or **probability with Laplace assumption**

$$P(A_i) = \frac{m_i}{m}$$

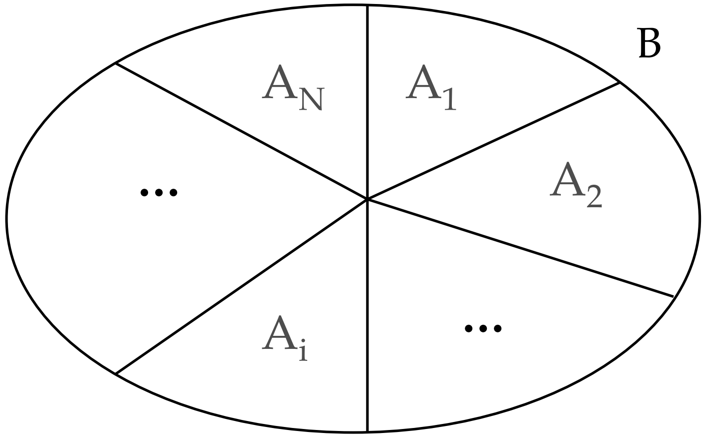
- Example: tossing a six-sided dice (fully symmetric)
  - $m = 6$ alternatives
  - event A_1 that the outcome is an odd number: $A_1 = \{1,3,5\}$
  - $m_1 = 3 \Rightarrow P(A_1) = \frac{3}{6} = 0.5$

Mutually Exclusive Event System

- Given a set of events $\{A_i, i = 1,2,\dots,N\}$ which belong to a sample space Ω.

- Events are mutually exclusive: $A_i \cap A_j = \emptyset$ for any $i \neq j$
- The probability of the union set $B = A_1 \cup A_2 \cup \cdots \cup A_N$ is

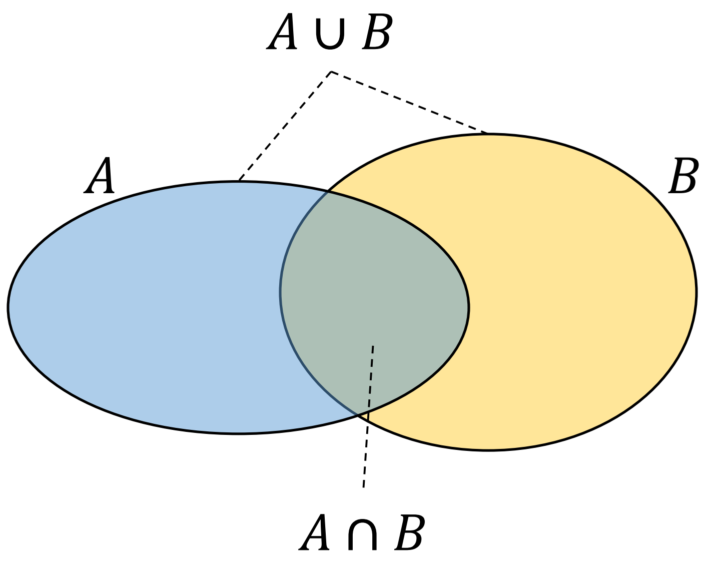
$$P(B) = \sum_{i=1}^{N} P(A_i)$$



- If $B = \Omega$, i.e. B contains all elementary events in Ω,
  - then $P(B) = 1$
  - $\{A_i, i = 1,2,\dots,N\}$ form a **mutually exclusive event system**

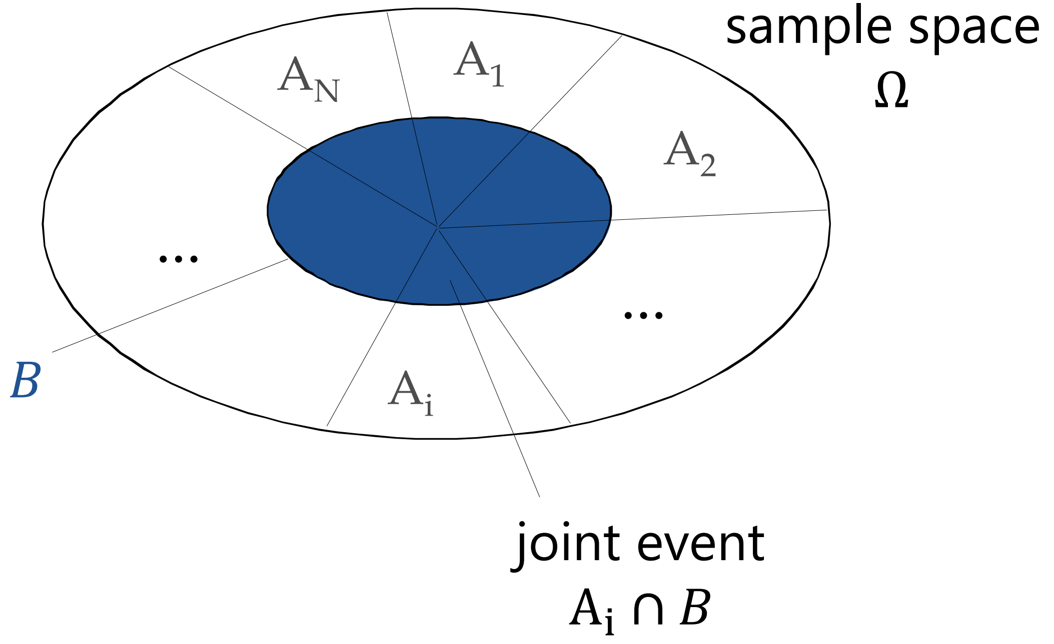
Joint Probability

- Consider two not necessarily disjoint events A and B.
- Joint event: $A \cap B = (A, B)$
- Joint Probability: $P(A \cap B) = P(A, B) = P(B, A)$
- Furthermore: $P(A \cup B) = P(A) + P(B) - P(A, B)$



Joint Probability (f.)

mututally exclusive
event system
$\{A_i, i = 1, \dots, N\}$



- If $\{A_i, i = 1, \dots, N\}$ represents a mutually exclusive event system, the probability of an arbitrary event B can be determined from the probabilities of the joint events (B, A_i).

$$P(B) = \sum_{i=1}^{N} P(A_i, B)$$

Conditional Probability and Statistical Independence

- Conditional event $(A|B)$: an event A occurs under the condition that another event B has occurred ($P(B) > 0$).

- **Conditional probability** $P(A|B)$:

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

- **Statistical independence**
  - Two events A and B are said to be statistically independent, if:

$$P(A|B) = P(A)$$

or

$$P(A,B) = P(A) \cdot P(B)$$

Statistical Independence and Disjointness

Lecture

Example: Joint and Conditional Probability

Law of Total Probability

- $\{A_i, i = 1, \dots, N\}$ represents a mutually exclusive event system
- Arbitrary event $B \subset \Omega$
- **Law of total probability**

$$P(B) = \sum_{i=1}^{N} P(A_i, B) = \sum_{i=1}^{N} P(B|A_i) \cdot P(A_i)$$

- **Bayes‘ theorem** using $P(A_i, B) = P(B|A_i) \cdot P(A_i) = P(A_i|B) \cdot P(B)$

$$P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{P(B)} = \frac{P(B|A_i) \cdot P(A_i)}{\sum_{i=1}^{N} P(B|A_i) \cdot P(A_i)}$$

Example: Bayes' Theorem

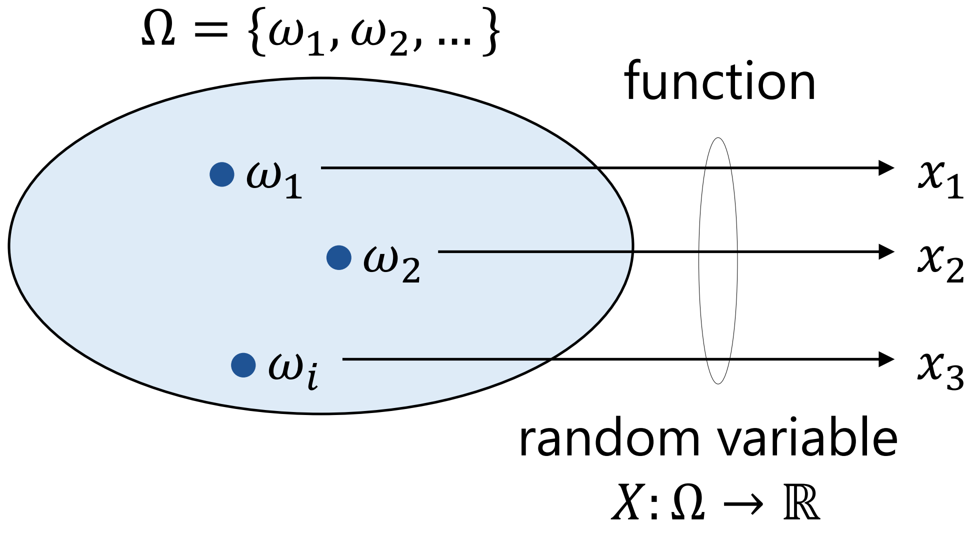
Lecture

Random Variables (R.V.)

Random Variable, Distribution and Distribution Function

Random Variables (R.V.s)

- **Random variable** is a function
  - assigns a real number to each complementary event in the sample space
  - range of values (codomain of the r.v.) determines discrete or continous r.v.



- **Discrete random variable**
  - a discrete r.v. only takes on discrete values, e.g. integer values
  - unless otherwise noted, we consider non-negative integer random variables, including zero
  - example: the discrete r.v. X reflects the sum of numbers when tossing two dice

- **Continuous random variable**
  - a continuous r.v. has a real value range
  - example: round-trip time of an IP packet is described with the r.v. T

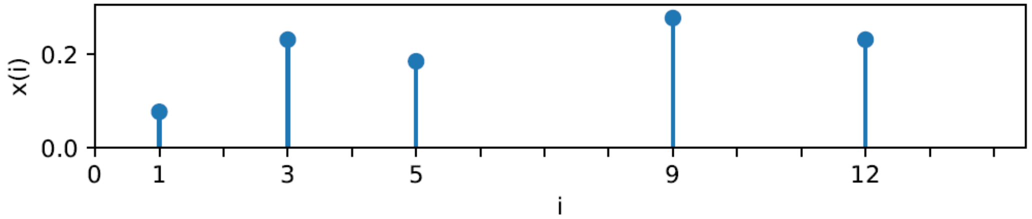
Distribution or Probability Mass Function (PMF)

- X : non-negative discrete random variable
- Realization of r.v. X occurs with probability

$$x(i) = P(X = i), \quad i = 0,1,\dots,X_{max}$$

(distribution or PMF)

- X_{max} does not have to be finite



(a) Probability mass function (PMF).

- **Normalization condition**

$$\sum_{i=0}^{X_{max}} x(i) = 1$$

or $\boldsymbol{X} \cdot \mathbf{1} = 1$ with all-ones vector $\mathbf{1} = (1,1,\dots,1)$
and probability vector $\boldsymbol{X} = \left(x(0), x(1), \dots, x(X_{max})\right)$

Cumulative Distribution Function (CDF)

- A : (discrete or continuous) random variable
- **Cumulative distribution function (CDF)**

$$A(t) = P(A \leq t)$$

- **Complementary cumulative distribution function (CCDF)**

$$A^c(t) = 1 - A(t) = P(A > t)$$

- **Fundamental properties** of the CDF
  - $t_1 < t_2 \;\Rightarrow\; A(t_1) \leq A(t_2)$ **monotony**
  - $t_1 < t_2 \;\Rightarrow\; P(t_1 < A \leq t_2) = A(t_2) - A(t_1)$
  - $A(-\infty) = 0,\; A(\infty) = 1$

Probability Density Function (PDF)

- First derivative of the CDF $A(\mathrm{t})$

$$a(t) = \frac{d}{dt} A(t)$$

- Normalization condition holds

$$\int_{-\infty}^{+\infty} a(t)\, dt = 1$$

- Remark:
  - $a(t)$ may be larger than 1
  - $a(t)$ is not a probability

- If $A(t)$ is discontinuous at t_0 with a step of height A_0, PDF $a(t)$ can be specified with the Dirac delta function $\delta(t)$:
  $a(t_0) = A_0 \cdot \delta(t - t_0)$

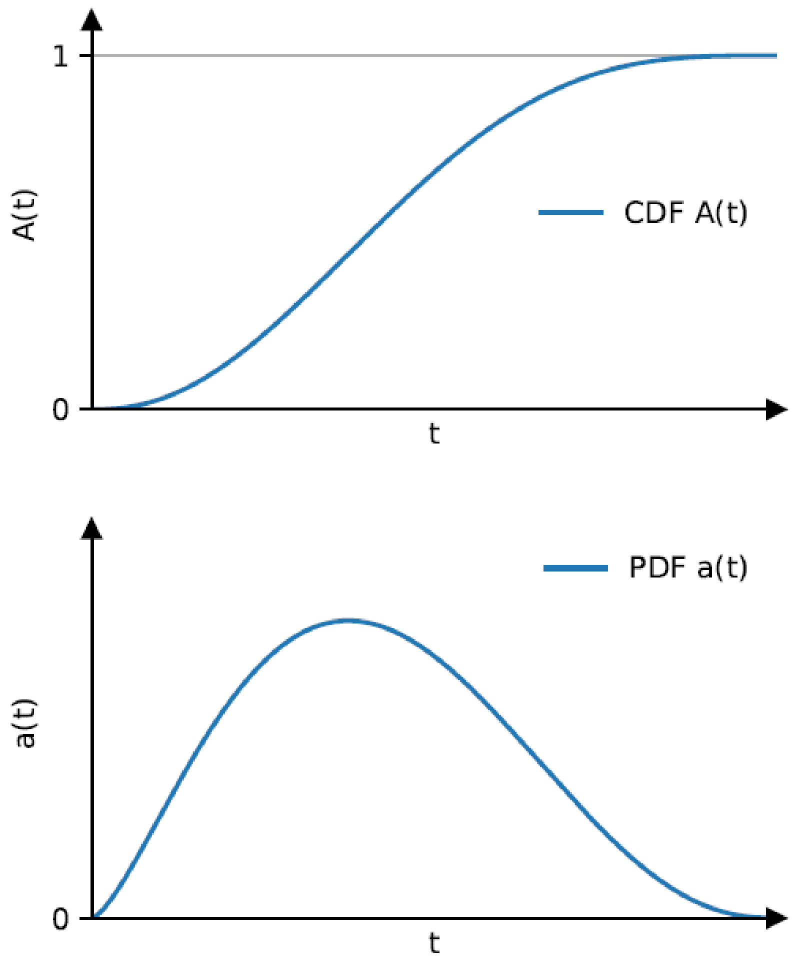
- Dirac delta function

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

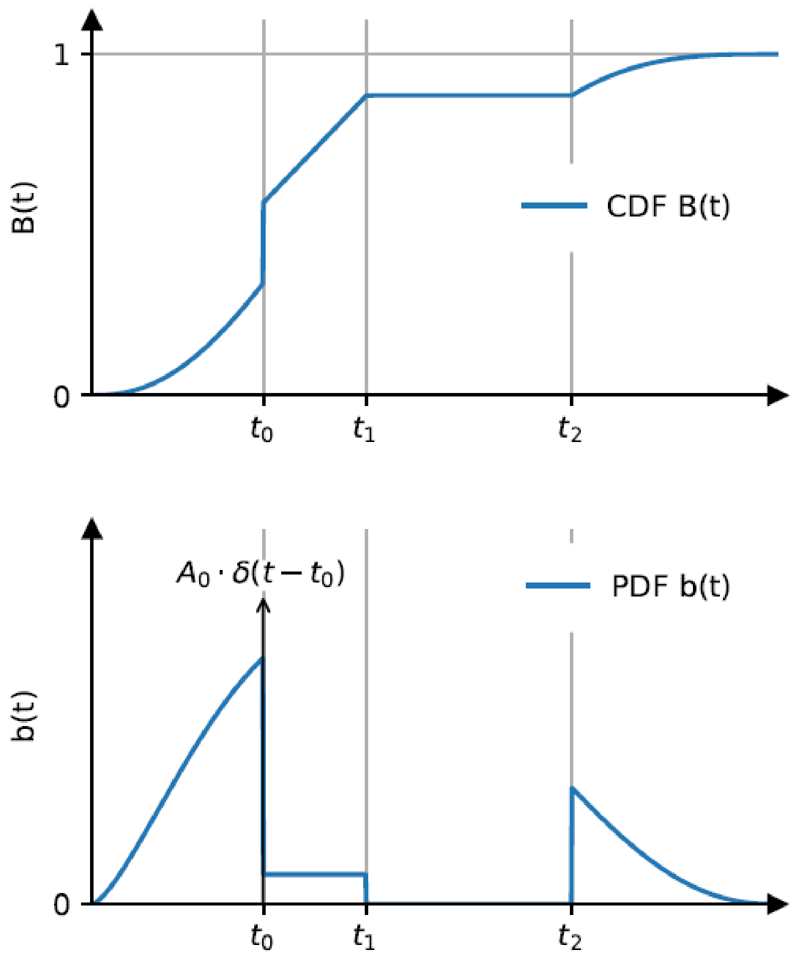
while satisfying

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

Illustration of CDF and PDF



(a) Continuous differentiable function.



(b) Piece-wise continuous function.

Probability Density

Lecture

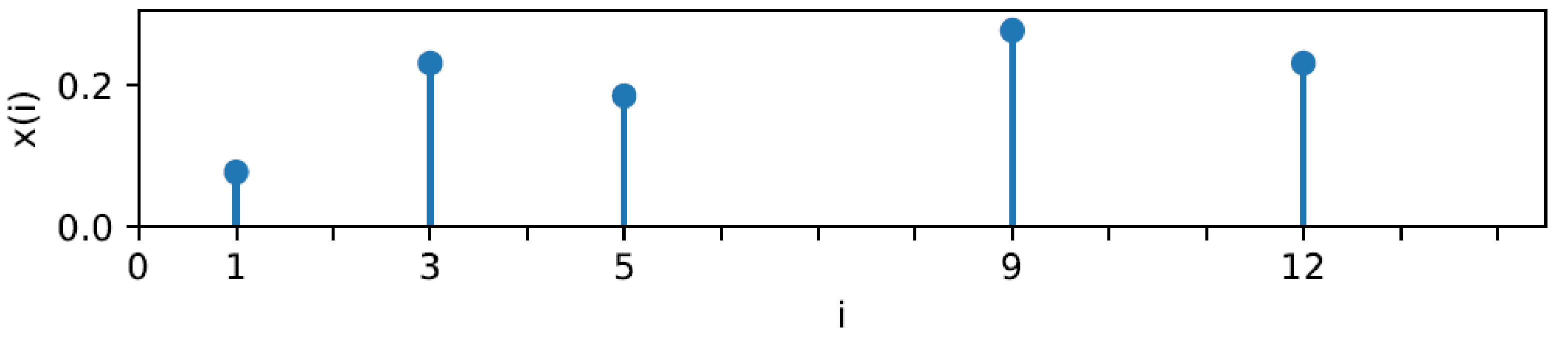
Example: Exponential Distribution

Lecture

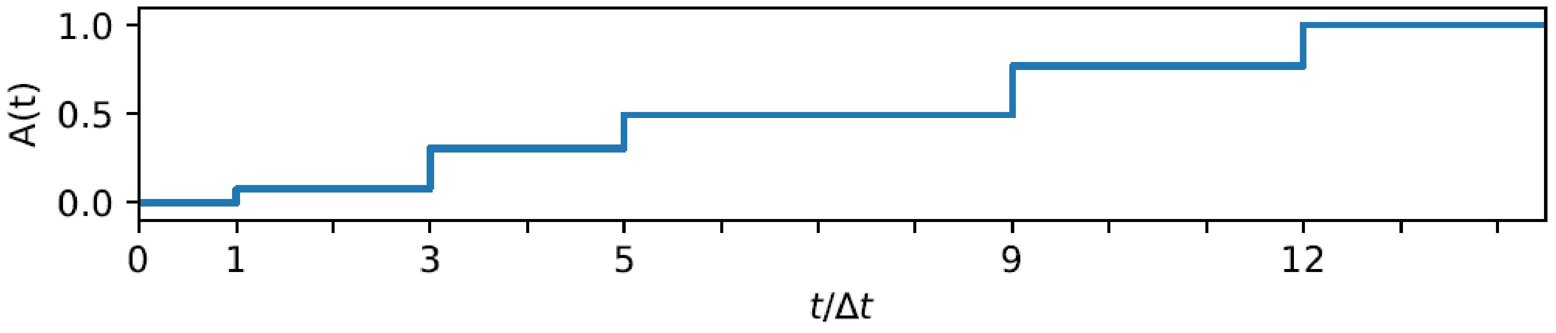
Discrete R.V.

- PMF
  $x(i) = P(X = i)$
  $= \mathrm{P}(\mathrm{A} = \mathrm{i} \cdot \Delta t)$
  with discretization constant Δt

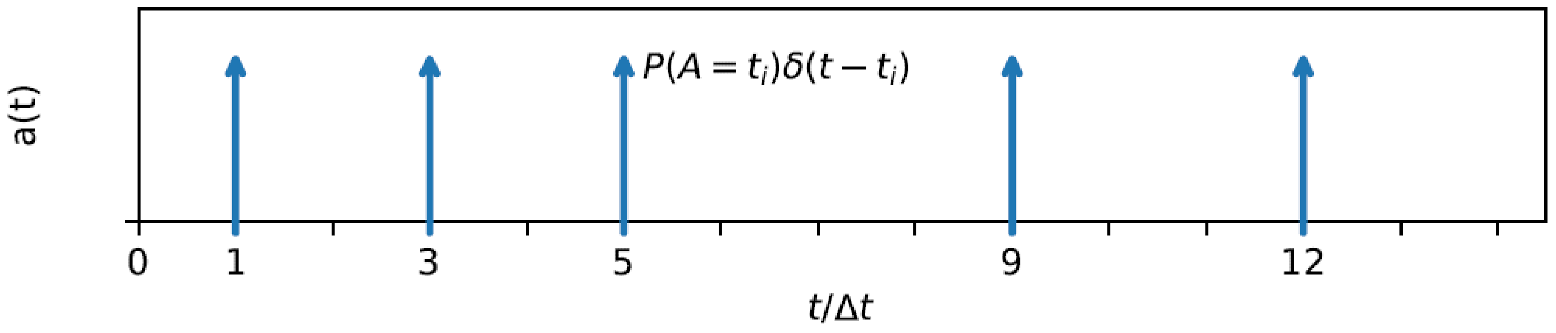
- In practice, measurement accuracy may lead to discrete r.v.s, e.g. $\Delta t = 1\mu s$



(a) Probability mass function (PMF).



(b) Cumulative distribution function (CDF).



(c) Probability density function (PDF).

Expected Value and Moments

Expected value, variance, standard deviation, coefficient of variation

Expected Value and Mean of a Random Variable

- A is a random variable with PDF $a(t)$ and CDF $A(t)$
- $g(A)$ is a function of the r.v. A; then $g(A)$ is another r.v.
- **Expected value** of $g(A)$ is defined as

$$E[g(A)] = \int_{-\infty}^{+\infty} g(t) \cdot a(t)\, dt$$

- **Mean** of a r.v. A for $g(A) = A$:

$$E[A] = \int_{-\infty}^{+\infty} t \cdot a(t)\, dt$$

Another notation

$$a(t)\, dt = dA(t)$$

- For a non-negative random variable A, the mean can also be derived using the complementary cumulative distribution function (CCDF) $P(A > t) = 1 - A(t)$

$$E[A] = \int_{0}^{+\infty} P(A > t)\, dt$$

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Expected Value and Mean of a Discrete Random Variable

- X is a discrete random variable with PMF $x(i)$ and CDF $X(i)$
- $g(X)$ is a function of the r.v. X; then $g(X)$ is another r.v.
- **Expected value** of $g(X)$ is defined as

$$E[g(X)] = \sum_{i=-\infty}^{+\infty} g(i) \cdot x(i)$$

- **Mean** of a r.v. X for $g(X) = X$:

$$E[X] = \sum_{i=-\infty}^{+\infty} i \cdot x(i)$$

- For a non-negative random variable X, the mean can also be derived using the complementary cumulative distribution function (CCDF) $P(X > i) = 1 - X(i)$

$$E[A] = \sum_{i=0}^{+\infty} P(X > i)$$

CCDF and Mean Value

(Ordinary) Moments and Central Moments

- ***k*-th (ordinary) moment**: expected value of $g(A) = A^k$ for $k = 0,1,2,\ldots$

$$m_k = E[A^k] = \int_{-\infty}^{+\infty} t^k \cdot a(t)\, dt$$

- ***k*-th central moment**: expected value of $g(A) = (A - m_1)^k$ for $k = 0,1,2,\ldots$

$$\mu_k = E[(A - m_1)^k] = \int_{-\infty}^{+\infty} (t - m_1)^k \cdot a(t)\, dt$$

Variance, Standard Deviation, Coefficient of Variation

- **Variance** (second central moment)

$$VAR[A] = E[A^2] - E[A]^2$$

$$VAR[A] = \mu_2 = E[(A - m_1)^2] = E[A^2 - 2m_1 A + m_1^2]$$
$$= E[A^2] - 2m_1 E[A] + m_1^2 = E[A^2] - E[A]^2$$

with $m_1 = E[A]$

- **Standard deviation**

$$STD[A] = \sigma_A = \sqrt{\mathrm{VAR}[A]}$$

- **Coefficient of variation**

$$c_A = \frac{\sigma_A}{\mu_A} = \frac{STD[A]}{E[A]}$$

Functions of Random Variables and Inequalities

Functions of a single R.V., Jensen‘s inequality, Markov‘s inequality

Function of a Random Variable

- Random variable X is mapped to a r.v. $Y = f(X)$ with function X with inverse function f^{-1}
  - r.v. $Y = f(X)$
  - CDF $Y(t) = P(Y \leq t) = P(f(X) \leq t) = P\left(X \leq f^{-1}(t)\right) = X\left(f^{-1}(t)\right)$
  - PDF $y(t) = \frac{d}{dt} Y(t)$

- Example 1: energy consumption Y is a linear function of the sojourn time W of IoT packets when transmitting over the air interface, $Y = f(W) = aW + b$ with constants a, b
- Example 2: number of video stalls X is mapped to QoE Y with nonlinear function, $Y = f(X)$

- **In general**: $E[Y] = E[f(X)] \neq f(E[X])$

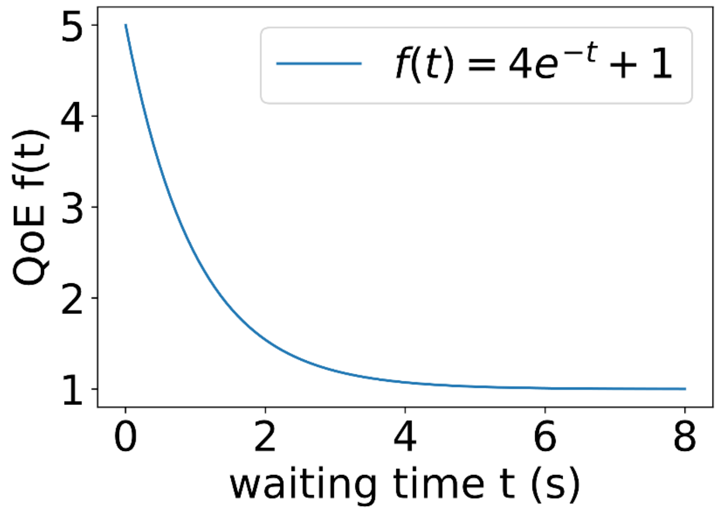
Example

Lecture

Jensen‘s Inequality for Mean Values

- Jensen‘s inequality is valid for discrete and continuous r.v.s

- $f(E[X]) \leq E[f(X)]$ for convex function f
  - $f'(x) = \frac{d}{dx} f(x)$ is monotonically decreasing or
  - $f''(x) \leq 0$ for a convex function



- $f(E[X]) \geq E[f(X)]$ for concave function f
  - $f'(x) = \frac{d}{dx} f(x)$ is monotonically increasing or
  - $f''(x) \geq 0$ for a concave function

- Linear function $f(x) = ax + b$: Then $E[f(X)] = E[aX + b] = aE[X] + b = f(E[X])$

Example: Network Throughput

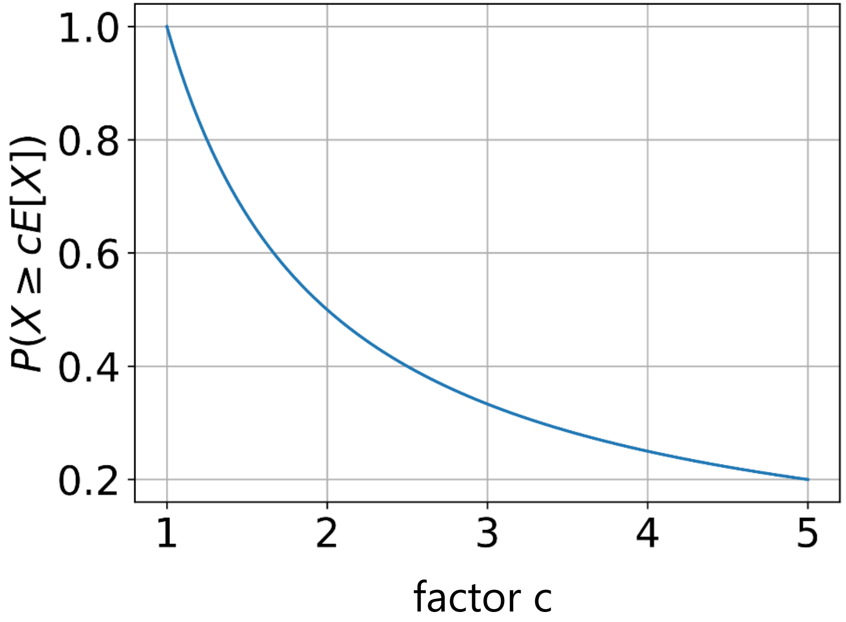
Example: QoE of Video Streaming

Lecture

Markov‘s Inequality

- Relation between the probability that a non-negative random variable is larger than a certain value $a > 0$ and its mean value.

- **Markov‘s Inequality**
  - $P(X \geq a) \leq \frac{E[X]}{a}$ for $a > 0$ and $E[X] < \infty$
  - Probability that the random variable is larger than the c-fold of the mean value $P(X \geq cE[X]) \leq \frac{1}{c}$



- Note: No information on the distribution is requierd

Functions of Two Random Variables

Moments, correlation coefficient, sum, difference, maximum, minimum of r.v.s

Two-dimensional Random Variables

- General case: multi-dimensional r.v.s with $A_1, A_2, \dots, A_i$ arbitray, non-negative r.v.s
  - joint event: $\{A_1 \le t_1, A_2 \le t_2, \dots, A_i \le t_i\}$
  - joint CDF: $A(t_1, t_2, \dots, t_i) = P(A_1 \le t_1, A_2 \le t_2, \dots, A_i \le t_i)$

- Two-dimensional r.v. $A = (A_1, A_2)$
  - with **joint cumulative distribution function** and **joint density function**

$$A(t_1, t_2) = P(A_1 \le t_1, A_2 \le t_2)$$

$$a(t_1, t_2) = \frac{\partial^2 A(t_1, t_2)}{\partial t_1 \, \partial t_2}$$

  - **marginal cumulative distribution function** for the limits $t_1 \to \infty$ or $t_2 \to \infty$

$$A_1(t_1) = \lim_{t_2 \to \infty} A(t_1, t_2)$$

$$A_2(t_2) = \lim_{t_1 \to \infty} A(t_1, t_2)$$

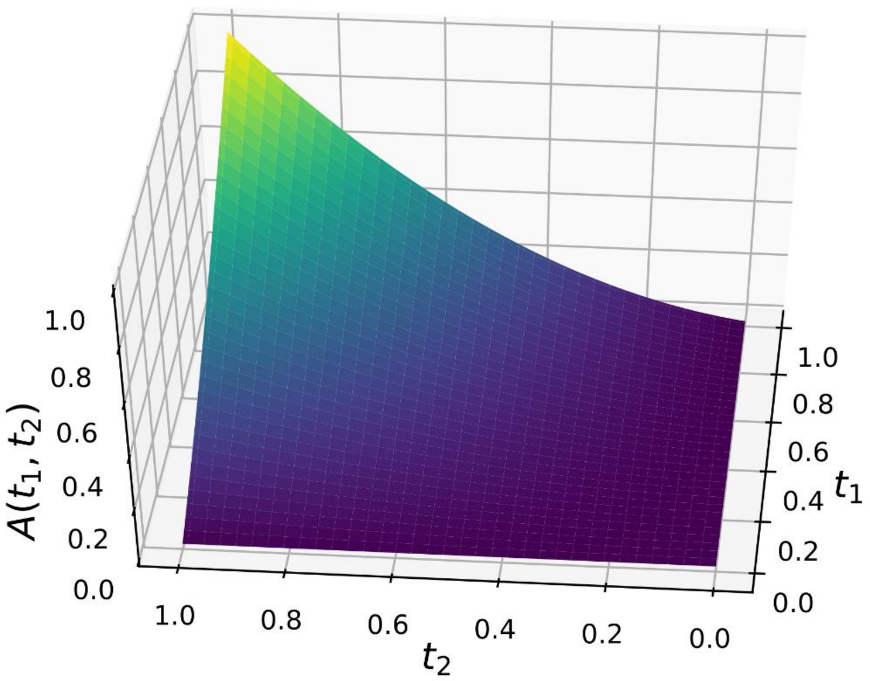
  - **marginal probability density function** for the limits $t_1 \to \infty$ or $t_2 \to \infty$

$$a_1(t_1) = \frac{d}{dt_1} A_1(t_1)$$

$$a_2(t_2) = \frac{d}{dt_2} A_2(t_2)$$

Example

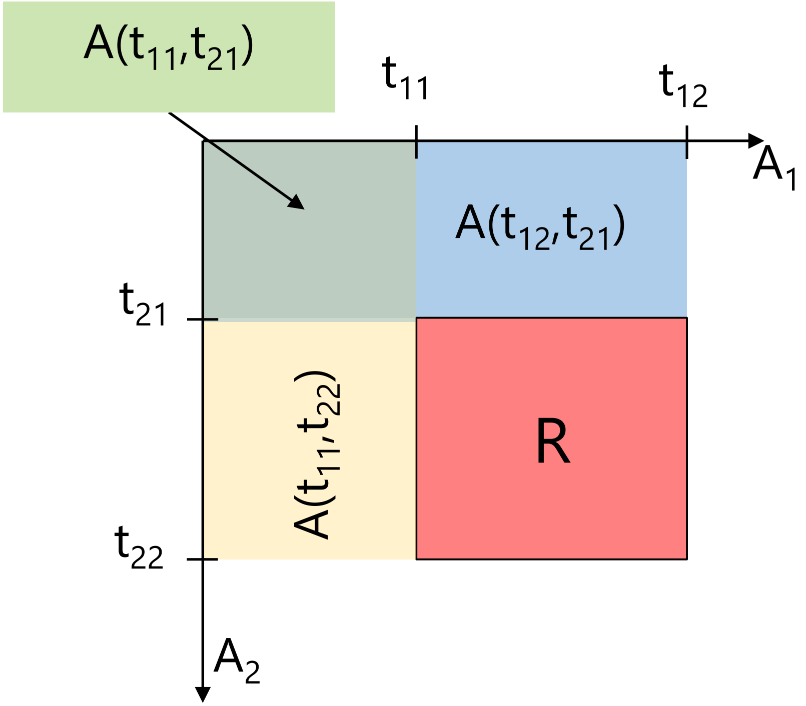
Lecture



Two-dimensional Random Variables (f.)

- Definition of an event $R = \{t_{11} < A_1 \leq t_{12}, t_{21} < A_2 \leq t_{22}\}$
- Probability of event R:
  $P(R) = A(t_{12}, t_{22}) - A(t_{12}, t_{21}) - A(t_{11}, t_{22}) + A(t_{11}, t_{21})$
- Computation with joint density $a(t_1, t_2) = \dfrac{\partial^2 A(t_1, t_2)}{\partial t_1 \, \partial t_2}$

$$P(R) = P(t_{11} < A_1 \leq t_{12}, t_{21} < A_2 \leq t_{22})$$
$$= \int_{t_2=t_{21}}^{t_{22}} \left(\int_{t_1=t_{11}}^{t_{12}} a(t_1, t_2) \, dt_1 \right) dt_2$$



Visualization

- Notebook
  in WueCampus or at
  https://modeling.systems

Chapter 2.2

Two-dimensional Random Variables

> (c) Tobias Hossfeld (Aug 2021)
>
> This script and the figures are part of the following book. The book is to be cited whenever the script is used (copyright CC BY-SA 4.0):
> *Tran-Gia, P. & Hossfeld, T. (2021). Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.*
> *https://doi.org/10.25972/WUP-978-3-95826-153-2*

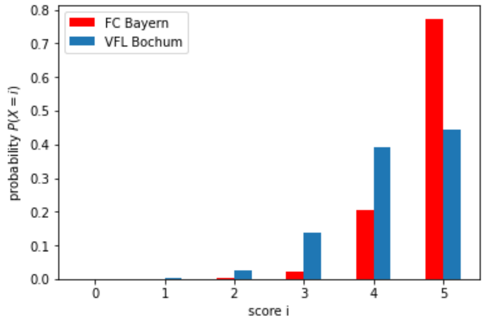
Penalty shootout We are looking at a penalty shootout between two teams: FC Bayern München and VFL Bochum. Every team has to shoot five penalties. FCB scores with a probability of 95%, while VFL scores with a probability of 85%. What is the joint probability that FCB scores i-times and VFL scores j-times. What is the probability that VFL scores more often than FCB?

In [5]:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import binom

n = 5 # number of penalties
x = np.arange(n+1)
p_fcb = binom.pmf(x, n, 0.95) # number of goals follows a binomial distribution
p_vfl = binom.pmf(x, n, 0.85)

plt.bar(x, p_fcb, width=-0.25, align='edge', color='r', label='FC Bayern')
plt.bar(x, p_vfl, width=0.25, align='edge', label='VFL Bochum')
plt.xlabel('score i')
plt.ylabel('probability $P(X=i)$')
plt.legend();
```



Properties of Two-dimensional R.V.s

- Properties

$$\int_0^{\infty}\int_0^{\infty} a(\xi_1,\xi_2)\,d\xi_1 d\xi_2 = 1$$

$$\int_0^{t_2}\left(\int_0^{t_1} a(\xi_1,\xi_2)\,d\xi_1\right)d\xi_2 = A(t_1,t_2)$$

$$\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} a(\xi_1,\xi_2)\,d\xi_1\right)d\xi_2 = P(a_1 < A_1 \le b_1, a_2 < A_2 \le b_2)$$

- Moments

$$E\left[A_1^{k_1}A_2^{k_2}\right] = \int_0^{+\infty}\int_0^{+\infty} t_1^{k_1}t_2^{k_2}\cdot a(t_1,t_2)\,dt_1\,dt_2$$ **moment of (k_1, k_2)-th order**

$$\mu_{k_1k_2} = E\left[(A_1-m_1)^{k_1}(A_2-m_2)^{k_2}\right]$$ **central moment of (k_1, k_2)-th order**

Covariance and Correlation Coefficient

- Covariance

$$\begin{aligned} \mathrm{COV}[A_1, A_2] &= \mu_{11} = E\left[(A_1 - m_1)\cdot(A_2 - m_2)\right] \\ &= E[A_1 \cdot A_2] - E[A_1]\cdot E[A_2]. \end{aligned}$$

- Correlation Coefficient

$$\begin{aligned} r &= \mathrm{COR}[A_1, A_2] = \frac{\mu_{11}}{\sigma_{A_1}\sigma_{A_2}} = \frac{E\left[(A_1 - m_1)(A_2 - m_2)\right]}{\sqrt{E\left[(A_1 - m_1)^2\right]}\sqrt{E\left[(A_2 - m_2)^2\right]}} \\ &= \frac{E[A_1 \cdot A_2] - E[A_1]E[A_2]}{\sqrt{E\left[(A_1 - m_1)^2\right]}\sqrt{E\left[(A_2 - m_2)^2\right]}}. \end{aligned}$$

Calculation Rules for R.V.s

Lecture

Example: Linear Transformation

Statistical Dependence and Correlation

- Covariance

$$\mathrm{COV}[A_1, A_2] = \mu_{11} = E\left[(A_1 - m_1)\cdot(A_2 - m_2)\right] \\ = E[A_1 \cdot A_2] - E[A_1]\cdot E[A_2].$$

- For **two statistically independent** r.v.s

$$E[A_1 \cdot A_2] = E[A_1]\cdot E[A_2]$$

  - covariance and correlation coefficient vanish: $\mathrm{COV}[A_1, A_2] = 0$ and $r = 0$

- Note: statistical independence implies uncorrelation
- Conversely, the uncorrelated nature of two stochastic processes does not always result in statistical independence

Sum of Random Variables

$A = A_1 + A_2$

Sum of Two Continuous R.V.s

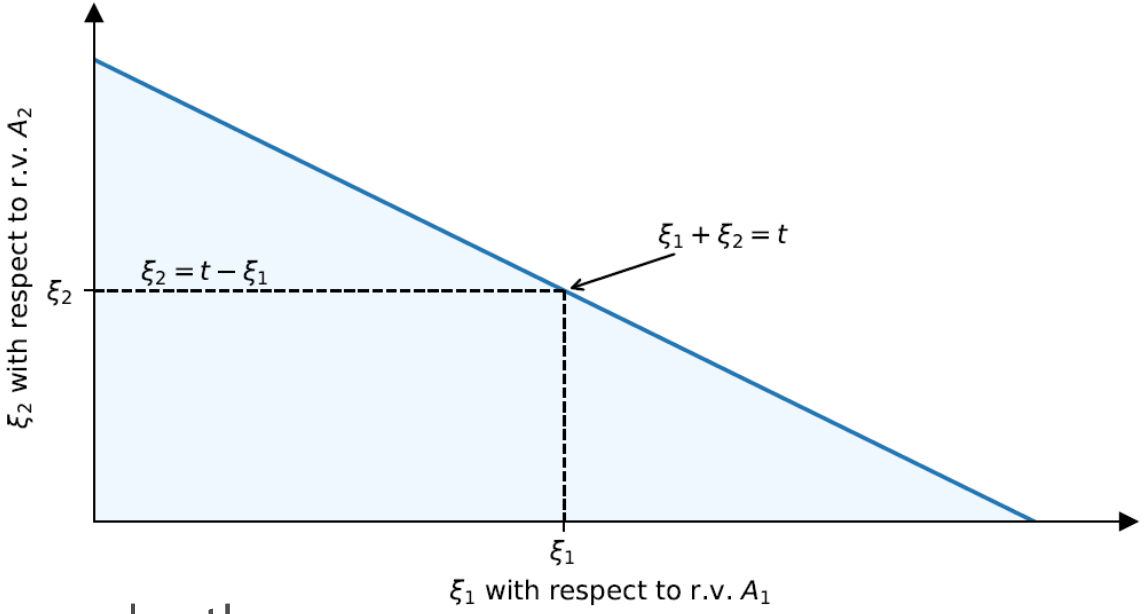
- Given A is the sum of two non-negative random variables A_1 and A_2 with $A_1, A_2 \geq 0$

$$A = A_1 + A_2$$

- Joint density function $a(t_1, t_2)$ and marginal density functions $a_1(t)$ and $a_2(t)$

- CDF follows by integrating over the triangle

$$\begin{aligned} A(t) &= \int_{\xi_1+\xi_2\le t} a(\xi_1,\xi_2)\,d\xi_1\,d\xi_2 \\ &= \int_{\xi_1=0}^{t} \left(\int_{\xi_2=0}^{t-\xi_1} a(\xi_1,\xi_2)\,d\xi_2 \right) d\xi_1 \\ &= \int_{u=0}^{t} \int_{v=u}^{t} a(u, v-u)\,dv\,du\,. \end{aligned}$$



- Note: A_1 and A_2 can be statistically dependent on each other.

Sum of Two Continuous R.V.s: Mean and Variance

- Mean

$$
\begin{aligned}
E[A] &= E[A_1 + A_2] = \int_0^\infty \int_0^\infty (t_1 + t_2)\, a(t_1, t_2)\, dt_1\, dt_2 \\
&= \int_0^\infty t_1 \left[\int_0^\infty a(t_1, t_2)\, dt_2 \right] dt_1 + \int_0^\infty t_2 \left[\int_0^\infty a(t_1, t_2)\, dt_1 \right] dt_2 \\
&= \underbrace{\int_0^\infty t_1\, a_1(t_1)\, dt_1}_{E[A_1]} + \underbrace{\int_0^\infty t_2\, a_2(t_2)\, dt_2}_{E[A_2]} = E[A_1] + E[A_2],
\end{aligned}
$$

$$E[A] = E[A_1] + E[A_2]$$

- Second Moment

$$
\begin{aligned}
E\left[A^2\right] &= E\left[(A_1 + A_2)^2\right] = E\left[A_1^2 + 2A_1A_2 + A_2^2\right] \\
&= E\left[A_1^2\right] + 2E[A_1 \cdot A_2] + E\left[A_2^2\right],
\end{aligned}
$$

- Variance

$$
\begin{aligned}
VAR[A] &= E[A^2] - E[A]^2 \\
&= VAR[A_1] + VAR[A_2] + 2\underbrace{\left(E[A_1 \cdot A_2] - E[A_1] \cdot E[A_2]\right)}_{COV[A_1, A_2]} \\
&= VAR[A_1] + VAR[A_2] + 2\, COV[A_1, A_2].
\end{aligned}
$$

$$VAR[A] = VAR[A_1] + VAR[A_2] + 2COV[A_1, A_2]$$

Special Case: Statistically Independent R.V.s

- If A_1 and A_2 are statistically independent of each other, their joint density function is the product of the marginal density functions

$$a(t_1, t_2) = a_1(t_1) \cdot a_2(t_2)$$

- For $A = A_1 + A_2$, it is

$$A(t) = \int_{u=0}^{t} \int_{v=u}^{t} a(u, v-u)\,dv\,du = \int_{u=0}^{t} \int_{v=u}^{t} a_1(u) \cdot a_2(v-u)\,dv\,du = \int_{u=0}^{t} a_1(u) \cdot A_2(t-u)\,du$$

- We obtain

$$a(t) = \frac{dA(t)}{dt} = \int_{u=0}^{t} a_1(u) \cdot a_2(t-u)\,du = a_1(t) * a_2(t),$$

(continuous) convolution

- Both notations $a_1(t) * a_2(t)$ and $(a_1 * a_2)(t)$ are common.
- Convolution is an operation between functions (not between numbers)

Special Case: Statistically Independent R.V.s (f.)

- Mean

$$E[A_1 \cdot A_2] = \int_0^\infty \int_0^\infty t_1\, t_2\, a(t_1, t_2)\, dt_1 dt_2 = \int_0^\infty \int_0^\infty t_1\, t_2\, a(t_1) a(t_2) dt_1 dt_2$$

$$= \int_0^\infty t_1 a_1(t_1) \int_0^\infty t_2\, a_2(t_2) dt_2 dt_1 = E[A_1] \cdot E[A_2],$$

- Variance

$$\begin{aligned} VAR[A] &= E[A^2] - E[A]^2 \\ &= VAR[A_1] + VAR[A_2] + 2\underbrace{\left(E[A_1 \cdot A_2] - E[A_1] \cdot E[A_2]\right)}_{COV[A_1, A_2]} \\ &= VAR[A_1] + VAR[A_2] + 2\, COV[A_1, A_2]. \end{aligned}$$

but $COV[A_1, A_2] = 0$

$$VAR[A] = VAR[A_1 + A_2] = VAR[A_1] + VAR[A_2]$$

- In general

$$A = \sum_{i=1}^{k} A_i$$

$$E[A] = \sum_{i=1}^{k} E[A_i]$$

$$VAR[A] = \sum_{i=1}^{k} VAR[A_i]$$

Summary: Sum of Two Statistically Independent R.V.s

- Sum of two statistically independent r.v.s

$$A = A_1 + A_2$$

- PDF of the sum obtained by convolution

$$a(t) = a_1(t) * a_2(t) = (a_1 * a_2)(t)$$

- Mean is also valid for dependent r.v.s

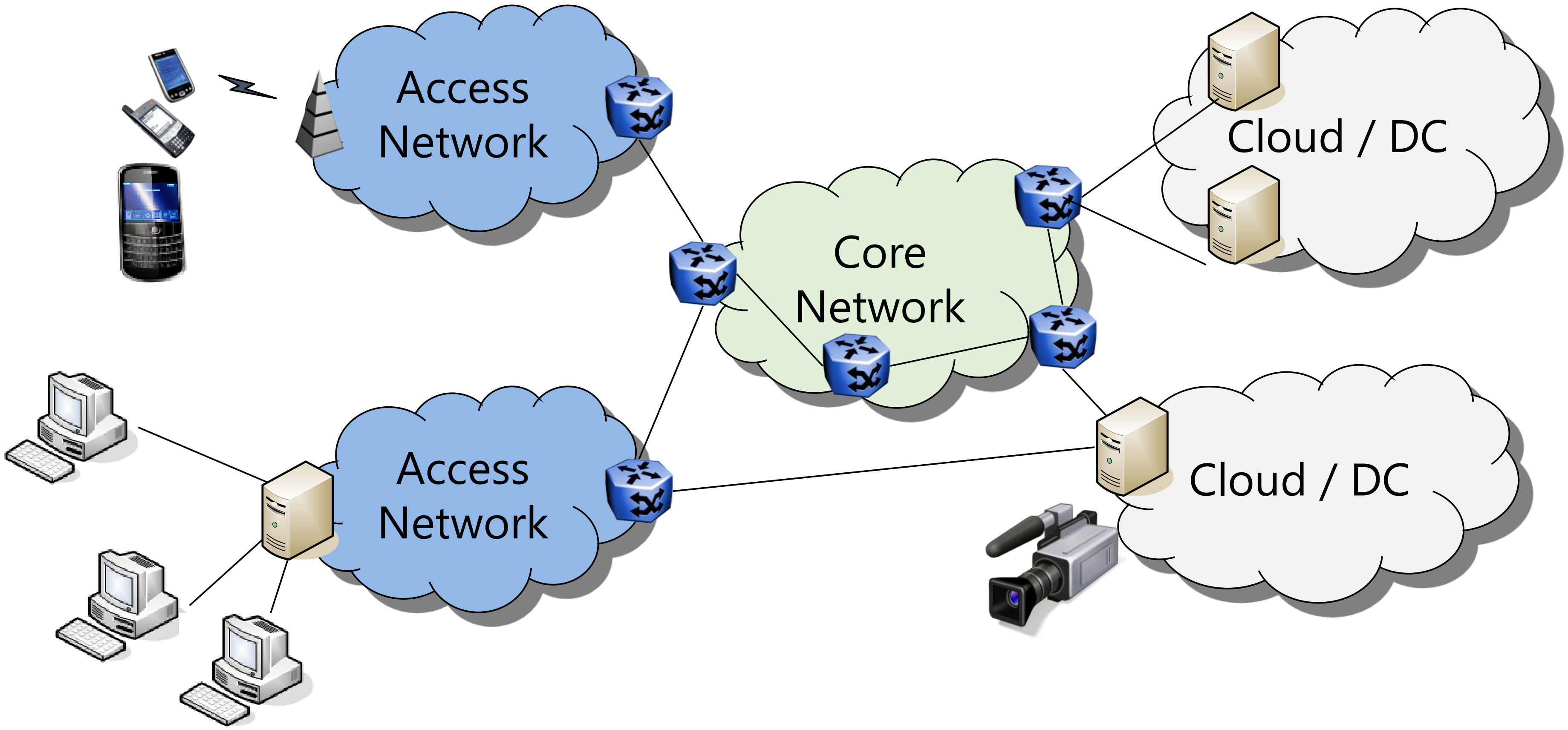
$$E[A] = E[A_1] + E[A_2]$$

- Variance for independent r.v.s

$$VAR[A] = VAR[A_1] + VAR[A_2]$$

Example: Transmission Delays

$$X_1 \quad + \quad X_2 \quad + \quad X_3$$



Sum of Discrete Random Variables

- X is the sum of two independent non-negative discrete random variables X_1 and X_2

$$X = X_1 + X_2$$

- Distribution of X

$$\begin{aligned} x(i) &= P(X = i) = P(X_1 + X_2 = i) \\ &= \sum_{j=0}^{i} P(X_1 = i - j \mid X_2 = j) \cdot P(X_2 = j) \\ &= \sum_{j=0}^{i} x_1(i-j) \cdot x_2(j) \end{aligned}$$

- **Discrete convolution**

$$x(i) = x_1(i) * x_2(i) = (x_1 * x_2)(i)$$

Example: Sum of Bernoulli Distributions

Difference, Minimum, Maximum of R.V.s

$A = A_1 - A_2, \quad A = \min(A_1, A_2)\,, \quad A = \max(A_1, A_2)$

Difference of Discrete Random Variables

- Difference X of two independent non-negative discrete random variables X_1 and X_2

$$X = X_1 - X_2$$

- Distribution of X

$$\begin{aligned} x(i) &= P(X=i) = P(X_1 - X_2 = i) \\ &= \sum_{j=0}^{\infty} P(X_1 = i+j \mid X_2 = j) \cdot P(X_2 = j) \\ &= \sum_{j=0}^{\infty} x_1(i+j)\, x_2(j) \\ &= x_1(i) * x_2(-i), \end{aligned}$$

where $x(i)$ can exist for negative values of i

- Notation with discrete convolution

$$x(i) = x_1(i) * x_2(-i) = (x_1 * -x_2)(i)$$

Maximum of Random Variables

- Let A be the maximum of two statistically independent random variables:

$$A = \max(A_1, A_2)$$

- Maximum can be formulated as follows:

$$\{A \le t\} \text{ for } \{A_1 \le t \text{ and } A_2 \le t\}$$

$$P(A \le t) = P(A_1 \le t) \cdot P(A_2 \le t)$$

- Thus, we obtain

$$A(t) = A_1(t) \cdot A_2(t)$$

$$a(t) = \frac{d}{dt} A(t) = a_1(t) A_2(t) + a_2(t) A_1(t)$$

- For k statistically independent r.v.s:

$$A = \max(A_1, A_2, \dots, A_k)$$

$$A(t) = \prod_{i=1}^{k} A_i(t)$$

Minimum of Random Variables

- Let A be the minimum of two statistically independent random variables:

$$A = \min(A_1, A_2)$$

- Minimum can be formulated as follows:

$$\{A > t\} \text{ for } \{A_1 > t \text{ and } A_2 > t\}$$

$$P(A > t) = P(A_1 > t) \cdot P(A_2 > t)$$

- Thus, we obtain

$$1 - A(t) = (1 - A_1(t)) \cdot (1 - A_2(t))$$

$$A(t) = 1 - (1 - A_1(t)) \cdot (1 - A_2(t))$$

- For k statistically independent r.v.s:

$$A = \min(A_1, A_2, \dots, A_k)$$

$$A(t) = 1 - \prod_{i=1}^{k} \left(1 - A_i(t)\right)$$

Minimum of Exponential Distributions