Institute of Computer Science
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Chapter 3.5

Markov State Process

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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The book must be cited and the disclaimer attached when using lectures slides or scripts.

Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/

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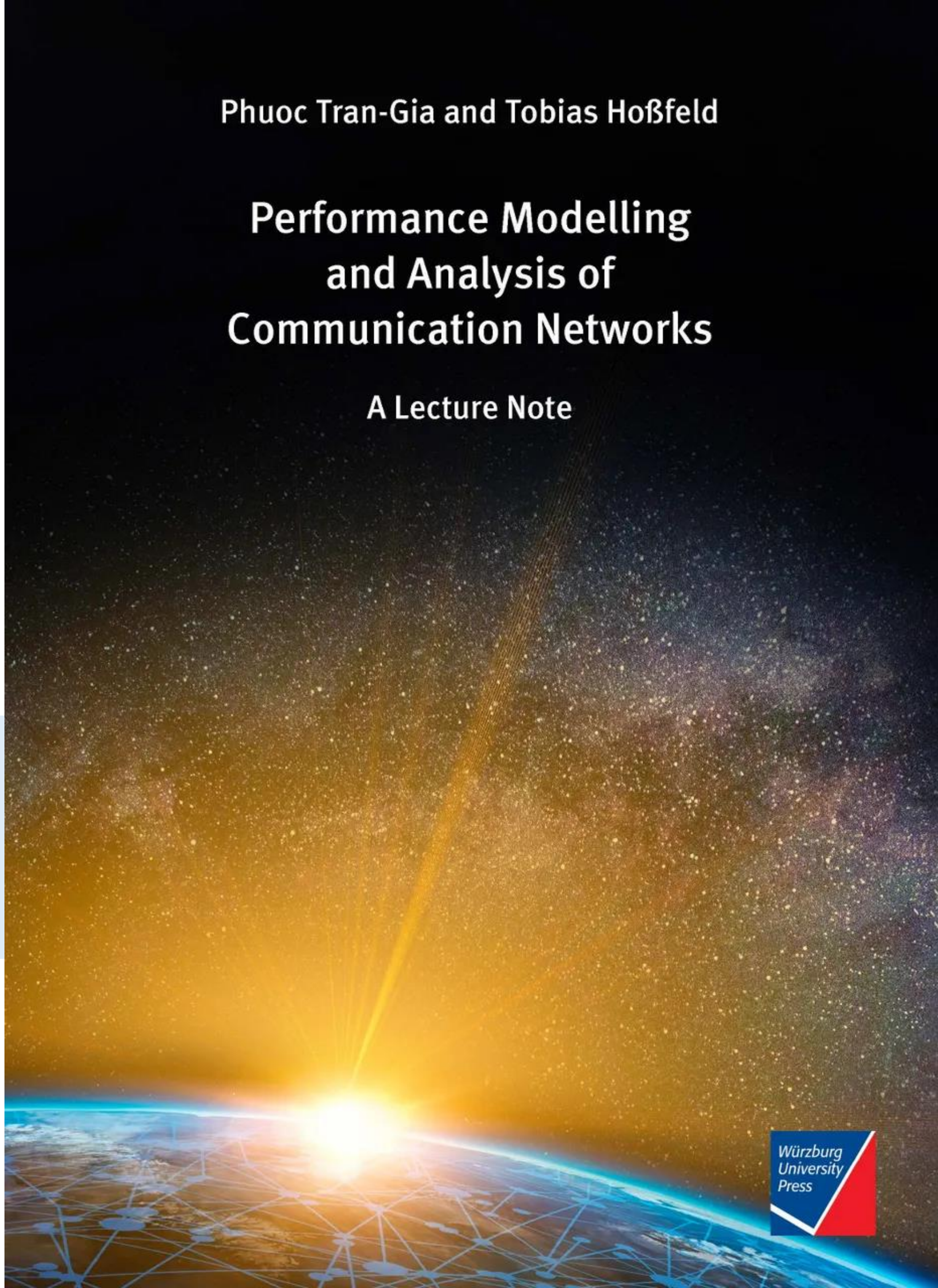


Chapter 3

Definition of Continuous-Time Markov Chain

Continuous-Time Markov Chain (CTMC)

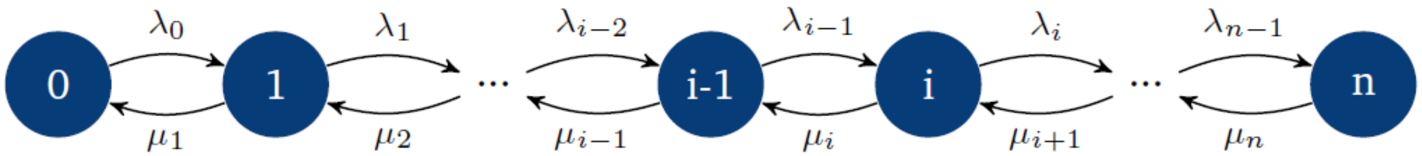
- **Continuous-time Markov process**
  - Stochastic process $\{X(t), t \geq 0\}$ with Markov property
- **Continuous-time Markov chain (CMTC)** is defined by
  - discrete state space S is finite or countable; e.g. number of customers in system
  - transition rates $q_{ij} \geq 0$ for $i \neq j$ and $i, j \in S$
  - initial state $X(0)$, i.e. probability distribution of initial state

- Probability $x(i,t) = P(X(t) = i)$ that the system is in state $[X = i]$ at time t
- State vector $\boldsymbol{X}(t) = (x(0,t), x(1,t), \dots)$

- Definition of **rate matrix** $\boldsymbol{Q}$ with $q_{ii} = -\sum_{i \neq j} q_{ij}$
  - allows compact notation (Kolmogorov equations.)
  - row-wise sums of $\boldsymbol{Q}$ are 0

Illustration of CTMC: Example

- State space $S = \{0,1,2,\dots,n\}$



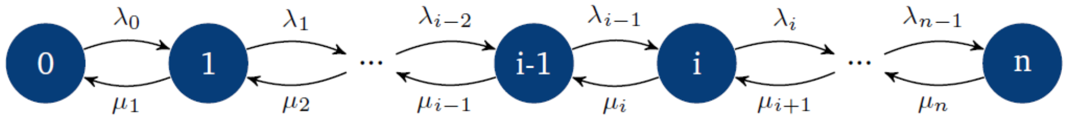
- Transition rates
  - $q_{i,i+1} = \lambda_i$ for $i = 0,1,\dots,n-1$
  - $q_{i,i-1} = \mu_i$ for $i = 1,\dots,n$
  - otherwise: $q_{i,j} = 0$ for $i \neq j$
  - $q_{ii} = -\sum_{i \neq j} q_{i,j}$

- Transition matrix

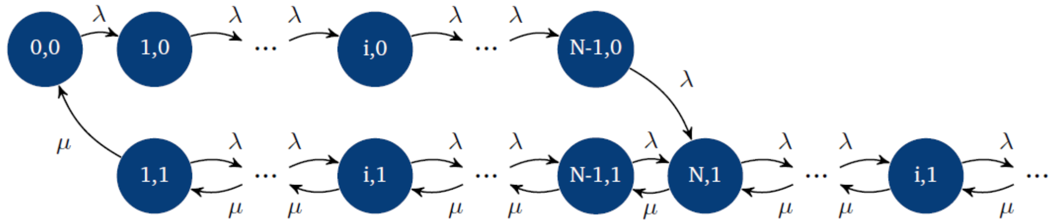
$$\mathcal{Q} = \begin{pmatrix} q_{00} & q_{01} & \cdots & q_{0j} & \cdots \\ q_{10} & q_{11} & \cdots & q_{1j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ q_{j0} & q_{j1} & \cdots & q_{jj} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}$$

Transition Behavior of CTMC

- System remains in state i for time T_i
  - exponentially distributed with rate $q_i = -q_{ii} > 0$
  - time to change from i to j: $T_{ij} \sim \mathrm{EXP}(q_{ij})$
  - $T_i = \min_{i \neq j}\{T_{ij}\} \sim \mathrm{EXP}(q_i)$

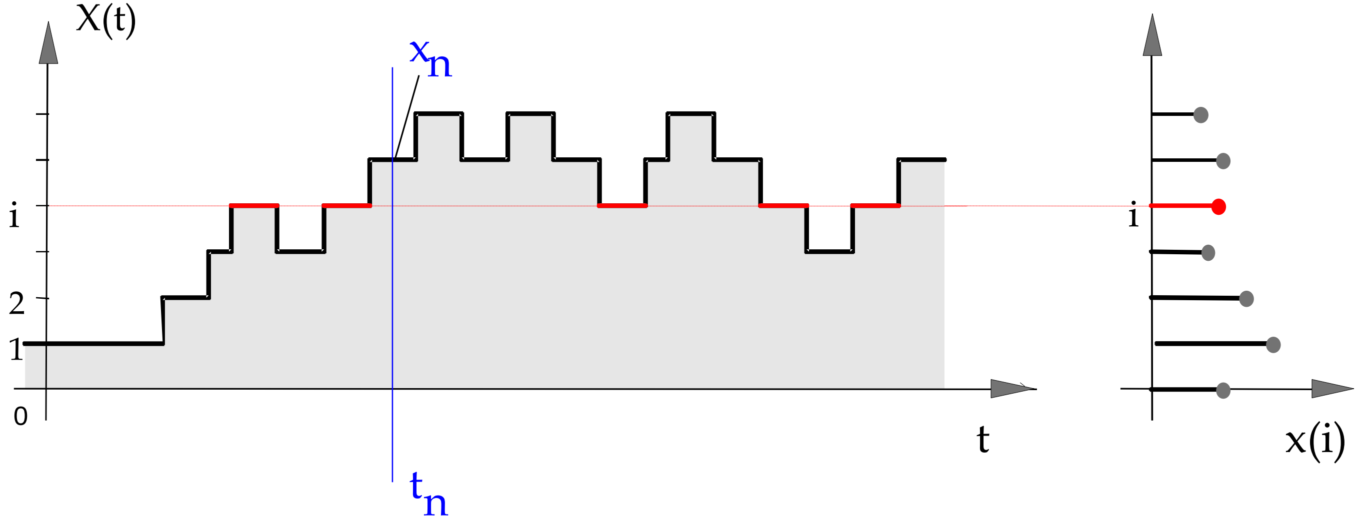


- When process leaves state i,
  the state j is reached with probability
  - $p_{ij} = \frac{q_{ij}}{q_i}$

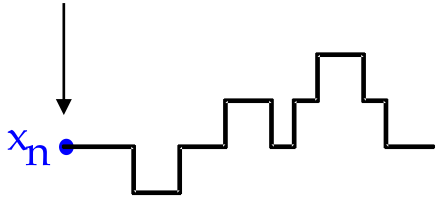


Analysis of Continuous-time Markov Chain (CMTC)

independent outside observer at time
t_n sees system state probability x_n



state $[X(t_n) = x_n]$



future development of process
only depends on state x_n

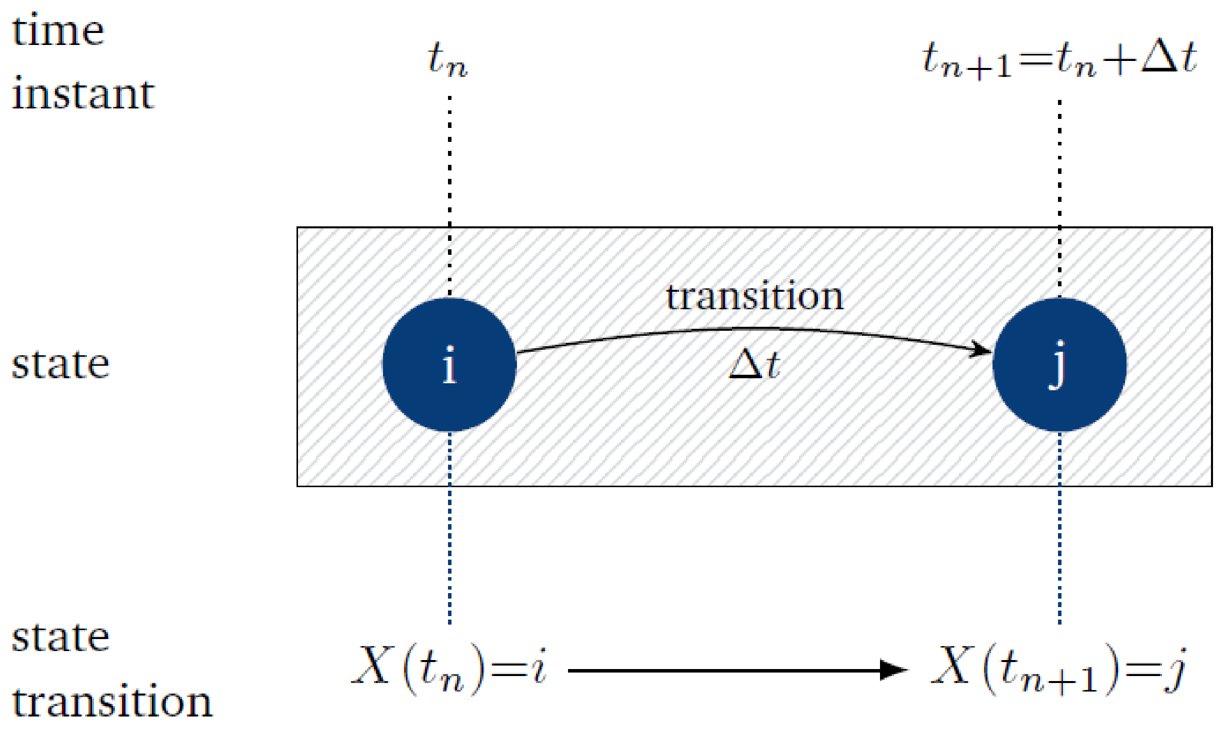
Overview on the Analysis

- Markov state process and memoryless property
- **Homogeneous systems**: transition probabilities independent of time instant
- **Chapman-Kolmogorov equation** for state transition probabilities: $\mathcal{P}(t+\Delta t) = \mathcal{P}(t) \cdot \mathcal{P}(\Delta t)$
  - Kolmogorov forward equation for **probability densities** $\lim\limits_{\Delta t \to 0} \mathcal{P}(t+\Delta t) \Rightarrow \frac{d\mathcal{P}(t)}{dx} = \mathcal{P}(t) \cdot Q$
  - Kolmogorov forward equation for **state probabilities** $\mathcal{X}(t) = \mathcal{X}(0) \cdot \mathcal{P}(t) \Rightarrow \frac{d\mathcal{X}(t)}{dx} = \mathcal{X}(t) \cdot Q$
- **Stationary system** $\lim\limits_{t \to \infty} \Rightarrow \frac{\partial}{\partial t} x(j,t)=0 \qquad \mathcal{X} \cdot Q = 0$
- Example: **birth-and-death process**

Transition Behavior of Markovian State Processes

Chapman-Kolmogorov equation

Transition Behavior of Markovian State Processes



Transition Probability

- State transition $i \to j$ during interval $\Delta t = t_{n+1} - t_n$ occurs with probability

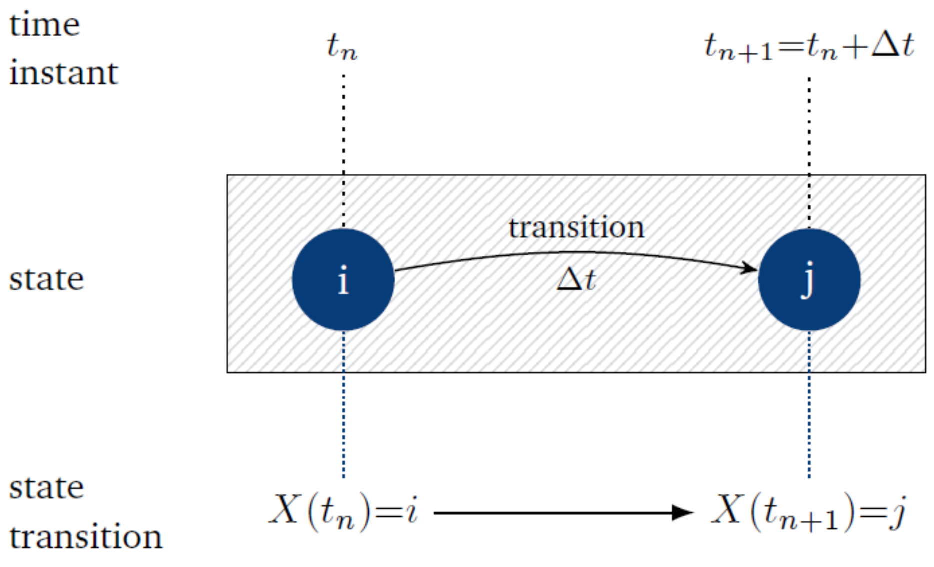
$$p_{ij}(t_n, t_{n+1}) = P\left(X(t_{n+1}) = j \,\middle|\, X(t_n) = i\right)$$

- **Time-homogeneous state process**
  - transition behavior is identical for each process point in time
  - transition probability is independent of the observation instant

$$p_{ij}(t_n, t_{n+1}) = p_{ij}(t_{n+1} - t_n) = p_{ij}(\Delta t)$$

  - standardization condition for all i

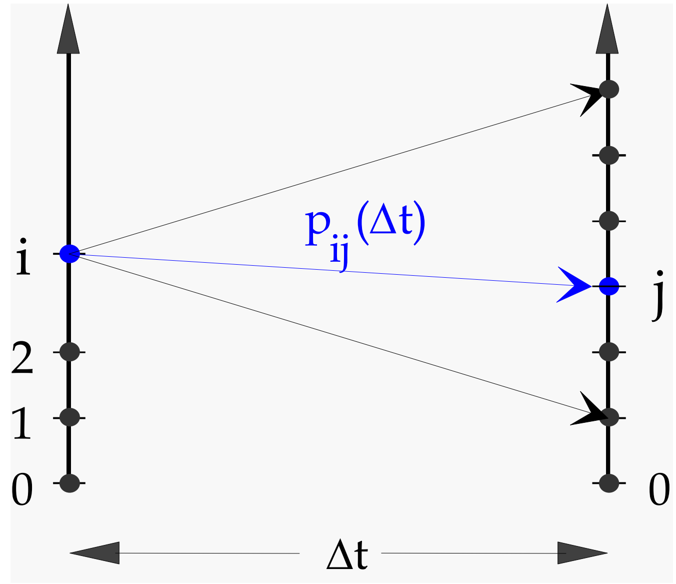
$$\sum_j p_{ij}(\Delta t) = 1, \qquad \Delta t \geq 0$$



Transition Matrix

- Transition probabilities $\{p_{ij}(\Delta t),\ i,j = 0,1,\ldots\}$ form the transition matrix

$$\mathcal{P}(\Delta t) = \begin{pmatrix} p_{00}(\Delta t) & p_{01}(\Delta t) & \ldots & p_{0j}(\Delta t) & \ldots \\ p_{10}(\Delta t) & p_{11}(\Delta t) & \ldots & p_{1j}(\Delta t) & \ldots \\ \vdots & \vdots & & \vdots & \\ p_{i0}(\Delta t) & p_{i1}(\Delta t) & \ldots & p_{ij}(\Delta t) & \ldots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

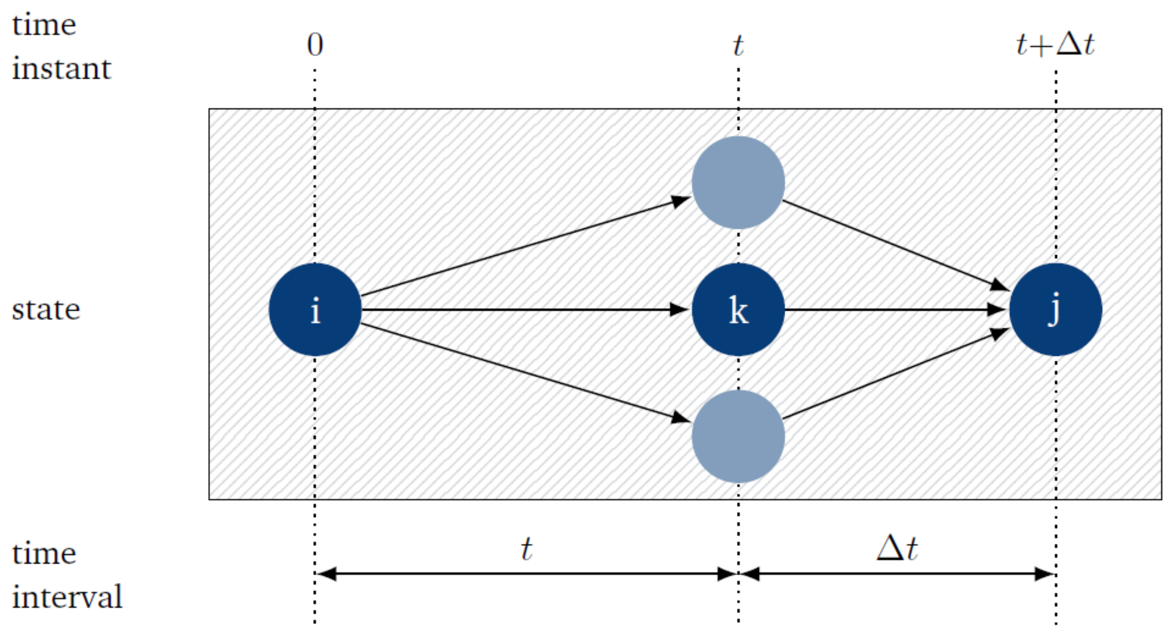


Chapman-Kolmogorov Equation

$$\mathcal{P}(t+\Delta t) = \mathcal{P}(t) \cdot \mathcal{P}(\Delta t)$$

or

$$p_{ij}(t+\Delta t) = \sum_k p_{ik}(t) p_{kj}(\Delta t)$$



Chapman-Kolmogorov Equation (f.)

Kolmogorov Forward Equation for Transition Probabilities

Kolmogorov Forward Equation for Transition Probabilities

- Chapman-Kolmogorov equation

$$p_{ij}(t+\Delta t) = \sum_{k} p_{ik}(t) p_{kj}(\Delta t)$$

- can be formulated as

$$p_{ij}(t+\Delta t) = \sum_{k \neq `j} p_{ik}(t) p_{kj}(\Delta t) + p_{ij}(t) p_{jj}(\Delta t)$$

$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \sum_{k \neq j} p_{ik}(t) \cdot \frac{p_{kj}(\Delta t)}{\Delta t} - p_{ij}(t) \cdot \frac{1 - p_{jj}(\Delta t)}{\Delta t}$$

- Next: limiting process $\Delta t \to 0$

Kolmogorov Forward Equation: Limiting Process

$$\frac{p_{ij}(t+\Delta t)-p_{ij}(t)}{\Delta t} = \sum_{k\neq j} p_{ik}(t)\cdot\frac{p_{kj}(\Delta t)}{\Delta t} - p_{ij}(t)\cdot\frac{1-p_{jj}(\Delta t)}{\Delta t}$$

- Limiting process $\Delta t \to 0$

$$\lim_{\Delta t\to 0}\frac{p_{ij}(t+\Delta t)-p_{ij}(t)}{\Delta t} = \frac{d}{dt}p_{ij}(t)$$

first derivative of transition probabilities $p_{ij}(t)$ at time t

$$\lim_{\Delta t\to 0}\frac{p_{kj}(\Delta t)}{\Delta t} = q_{kj}, \quad k\neq j$$

transition probability density for the transition $k \to j$

$$\lim_{\Delta t\to 0}\frac{1-p_{jj}(\Delta t)}{\Delta t} = q_j = \sum_{k\neq j} q_{jk}$$

transition probability density for leaving the state j

- **Kolmogorov forward equation for transition probabilities**

$$\frac{d}{dt}p_{ij}(t) = \sum_{k\neq j} q_{kj}\, p_{ik}(t) - q_j\, p_{ij}(t)$$

Kolmogorov Forward Equation: Matrix Notation

- Kolmogorov forward equation for transition probabilities

$$\frac{d}{dt} p_{ij}(t) = \sum_{k \neq j} q_{kj}\, p_{ik}(t) - q_j\, p_{ij}(t)$$

- Matrix for transition probability densities is defined

$$Q = \begin{pmatrix} q_{00} & q_{01} & \cdots & q_{0j} & \cdots \\ q_{10} & q_{11} & \cdots & q_{1j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ q_{j0} & q_{j1} & \cdots & q_{jj} & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

with $\sum_k q_{jk} = 0$ row-wise sum is 0

and $q_{jj} = -\sum_{k \neq j} q_{jk} = -q_j$ probability density to remain in state *j*

$$\Rightarrow \quad \frac{d\mathcal{P}(t)}{dt} = \mathcal{P}(t) \cdot Q$$

- Note:
  - „$-q_j$" does not imply that there is a negative rate.
  - solely notation for the rate to stay in state

Solution of Kolmogorov Forward Equation

- Kolmogorov forward equation for transition probabilities in matrix notation

$$\frac{d\mathcal{P}(t)}{dt} = \mathcal{P}(t) \cdot Q$$

- Solution requires the computation of the matrix exponential of the matrix $t \cdot Q$

$$\mathcal{P}(t) = e^{tQ} = \sum_{k=0}^{\infty} \frac{(tQ)^k}{k!}$$

- See notebook script "3.5 Markov processes: nonstationary and stationary analysis [ipynb]"
https://modeling.systems/

State Equations and State Probabilities

Kolmogorov Forward Equation for State Probabilities

Kolmogorov Forward Equation for State Probabilities

- Probability $x(j,t)$ for the system to be in state j at time t.

$x(j,t) = P(X(t) = j)$ process is in state j at time t

$x(i,0)$ initial state at time $t = 0$

$$\Rightarrow \quad x(j,t) = \sum_i P(X(t)=j \mid X(0)=i) \cdot P(X(0)=i) = \sum_i x(i,0) \cdot p_{ij}(t)$$

- Kolmogorov forward equation **for transition probabilities**

$$\frac{d}{dt} p_{ij}(t) = \sum_{k \neq j} q_{kj}\, p_{ik}(t) - q_j\, p_{ij}(t)$$

$$\Rightarrow \quad \sum_i \frac{d}{dt} p_{ij}(t)\, x(i,0) = \sum_{k \neq j} q_{kj} \sum_i \left(p_{ik}(t)\, x(i,0)\right) - \sum_i \left(q_j p_{ij}(t)\, x(i,0)\right)$$

- Kolmogorov forward equation **for state probabilities**

$$\frac{\partial}{\partial t} x(j,t) = \sum_{k \neq j} q_{kj}\, x(k,t) - q_j x(j,t)\,, \qquad j = 0,1,\ldots, \qquad \left(\sum_j x(j,t) = 1\right)$$

Kolmogorov Forward Eq. for State Probabilities: Matrix Notation

- Kolmogorov forward equation **for state probabilities** is system of differential equations

$$\frac{\partial}{\partial t} x(j,t) = \sum_{k \neq j} q_{kj}\, x(k,t) \;-\; q_j\, x(j,t)\,, \quad j = 0, 1, \ldots\ ,$$

- State probabilities as vector $\mathbf{X}(t) = \big(x(0,t), x(1,t), \ldots, x(j,t), \ldots\big)$

- Initial state and transition probability matrix $\mathbf{X}(t) = X(0) \cdot \mathcal{P}(t)$

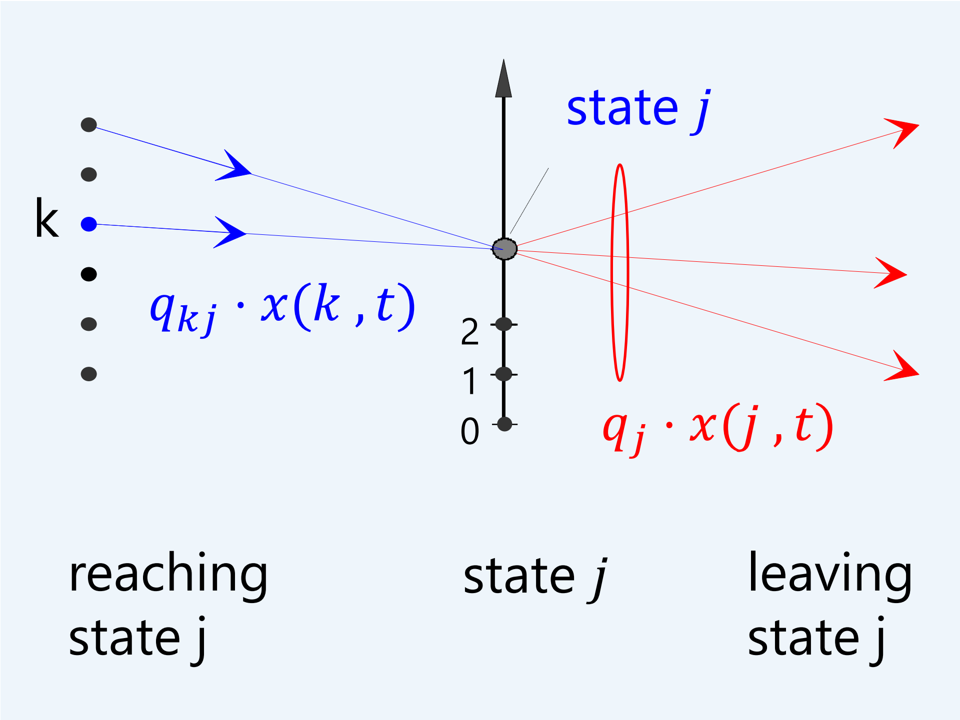
- **Matrix notation** of Kolmogorov forward equation for state probabilities

$$\frac{d}{dt}\mathbf{X}(t) = \mathbf{X}(t) \cdot \mathcal{Q}$$

Stationary State Probabilities

- State process is **stationary** if
  - system state stops changing (statistical equilibrium)
  - state probability no longer depends on time t

$$\frac{d}{dt} P(X(t) = j) = \frac{\partial}{\partial t} x(j, t) = 0$$



- Stationary state probability

$$x(j) = \lim_{t \to \infty} P(X(t) = j)$$

- Results in stationary equation system

$$\frac{\partial}{\partial t} x(j, t) = \sum_{k \neq j} q_{kj} \, x(k, t) - q_j x(j, t) \, , \qquad j = 0, 1, \ldots ,$$

$$\Rightarrow$$

$$q_j \, x(j) = \sum_{k \neq j} q_{kj} \cdot x(k) \, , \quad j = 0, 1, \ldots \, ,$$

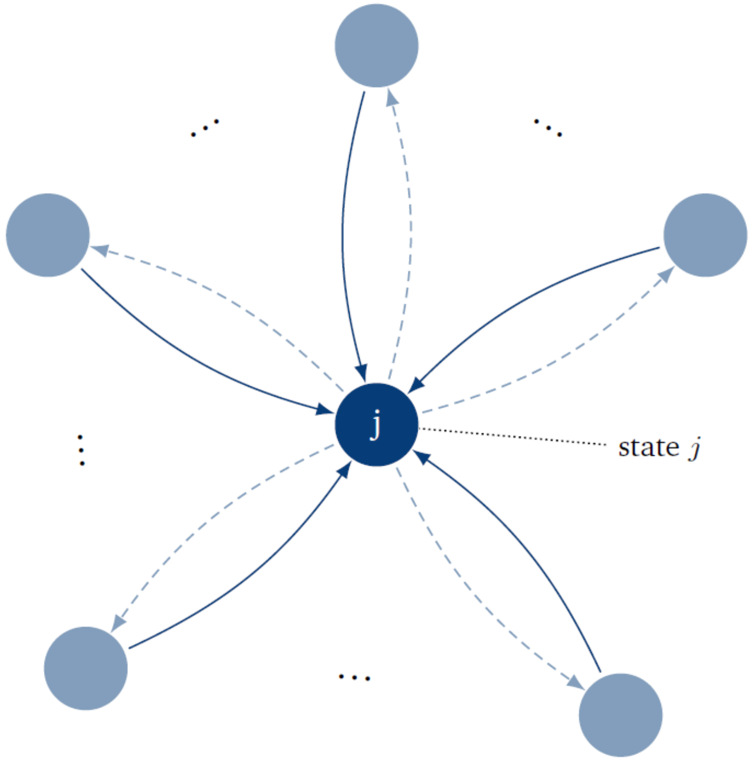
$$\sum_j x(j) = 1 \, .$$

Principle of Maintaining Statistical Equilibrium

$$\sum_j x(j) = 1$$

$$\boxed{q_j\, x(j)} = \boxed{\sum_{k \neq j} q_{kj}\, x(k)}$$

rate for leaving state j = rate for reaching state j



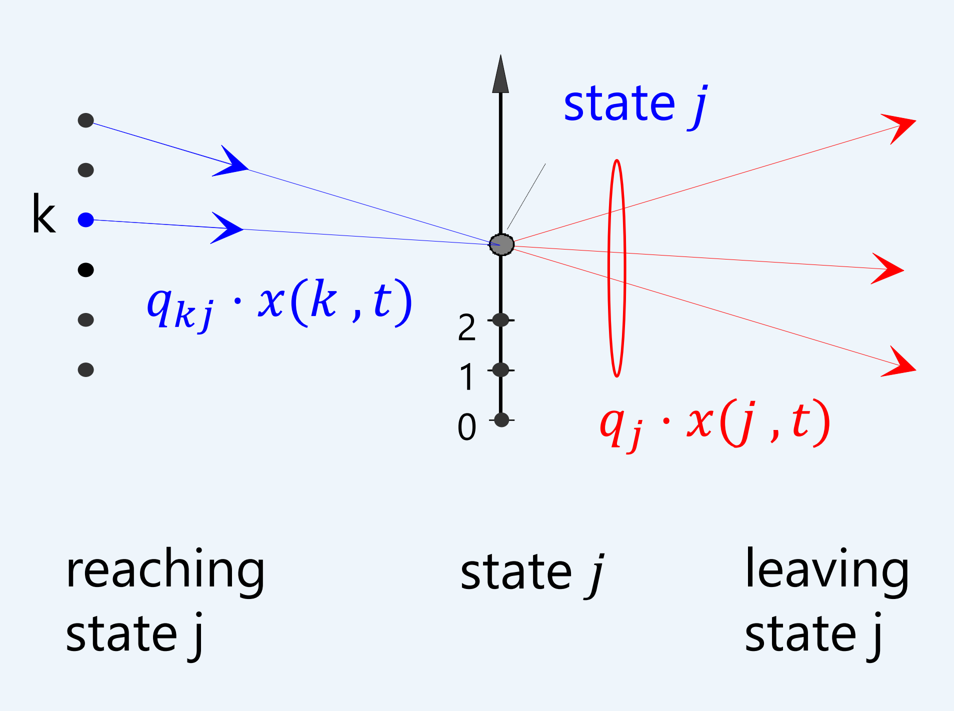
- **Principle of maintaining statistical equilibrium**
  - Stationary system: flows of weighted probability densities for reaching and leaving a state must be in equilibrium, i.e., they are the same
  - state probability no longer changes in time

Stationary Equation System: Matrix Notation

$$\sum_j x(j) = 1$$

$$\underbrace{q_j\, x(j)}_{\text{rate for leaving state } j} = \underbrace{\sum_{k\neq j} q_{kj}\, x(k)}_{\text{rate for reaching state } j}$$

rate for leaving state *j* = rate for reaching state *j*



- Stationary state probability vector $X = \{x(0), x(1), \ldots, x(j), \ldots\}$
- **State equation system** $X \cdot Q = 0 \quad \left(\sum_j x(j) = 1 \text{ or } X\, e = 1 \right)$

Linear Dependency of State Equations

- Consider a finite state space $\{0,1,2,\ldots,N\}$

- Stationary system of equations

$$\sum_{j=1}^{N} x(j) = 1$$

$$q_j \, x(j) = \sum_{k \neq j} q_{kj} \, x(k) \; , \quad j=0,1,\ldots,N$$

  - $(N+2)$ equations for $(N+1)$ unknowns (state probabilities)
  - Any arbitrarily chosen equation can be omitted due to linear dependency to solve the equation system

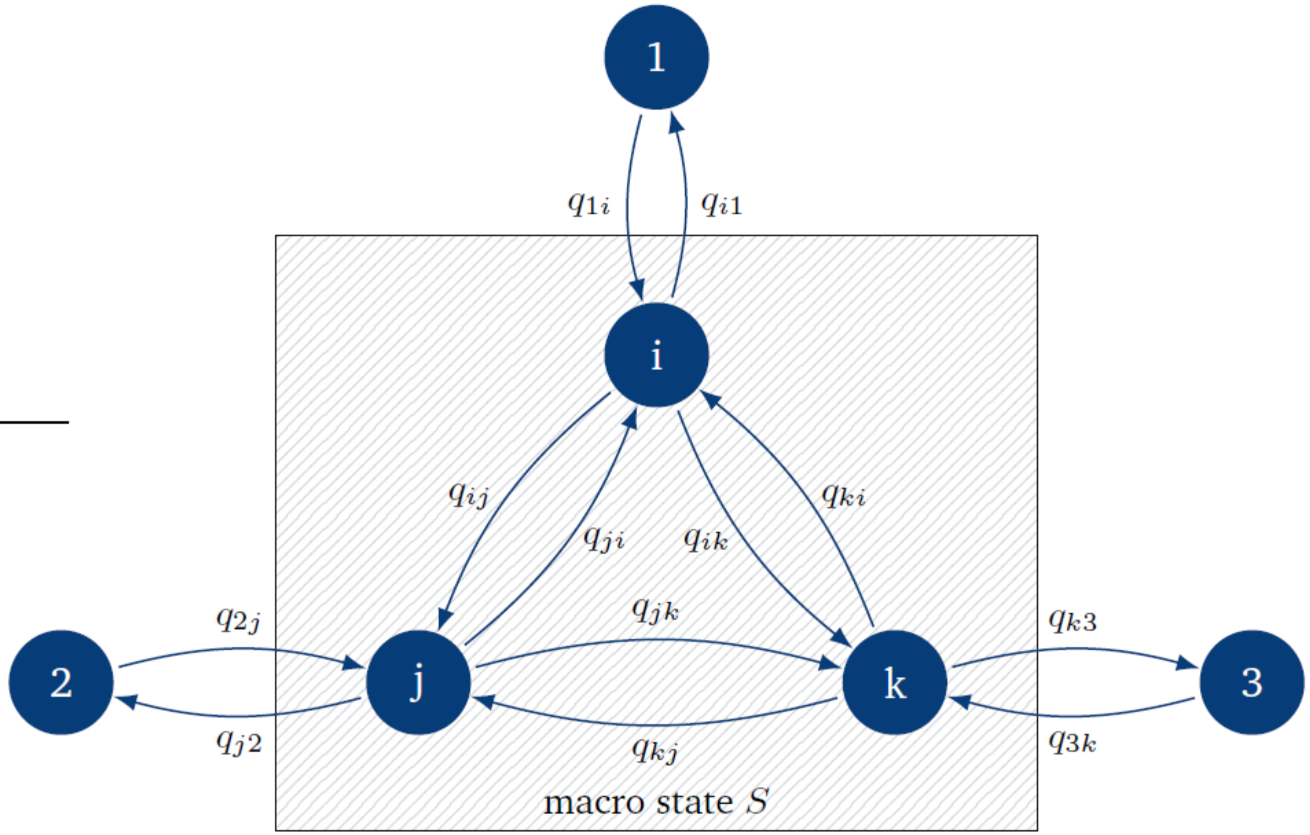
Macro States

- Single state that cannot be further decomposed is also called a **micro state**
- Combination of any set of micro states leads to a **macro state**

- Example

$$+\begin{cases} (q_{i1}+q_{ij}+q_{ik})x(i) = q_{1i}x(1)+q_{ji}x(j)+q_{ki}x(k) \\ (q_{j2}+q_{jk}+q_{ji})x(j) = q_{2j}x(2)+q_{kj}x(k)+q_{ij}x(i) \\ (q_{k3}+q_{ki}+q_{kj})x(k) = q_{3k}x(3)+q_{ik}x(i)+q_{jk}x(j) \end{cases}$$

$$q_{i1}x(i)+q_{j2}x(j)+q_{k3}x(k) = q_{1i}x(1)+q_{2j}x(2)+q_{3k}x(3)$$



- Appropriate choice of macro states often provides simpler system of equations for computing the (micro) state probabilities

Global Equilibrium Equation

- State equation for an arbitrary macro state

$$\underbrace{\sum_{j \in S, u \notin S} q_{ju} x(j)}_{\substack{\text{weighted rates for leaving} \\ \text{the macro state } S}} = \underbrace{\sum_{u \notin S, j \in S} q_{uj} x(u)}_{\substack{\text{weighted rates for reaching} \\ \text{the macro state } S}}$$

- Relates transition probability densities between a macro state and rest in the state space
- Global equilibrium equations are also referred to as full or **global balance equations**

Example: Stationary State Equations

Lecture

Example: Stationary Macro State Equations

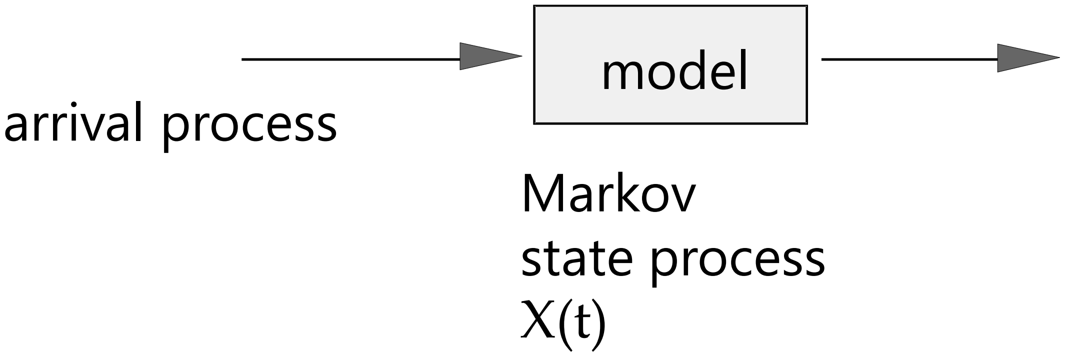
Lecture

Summary: Stationary Equation System

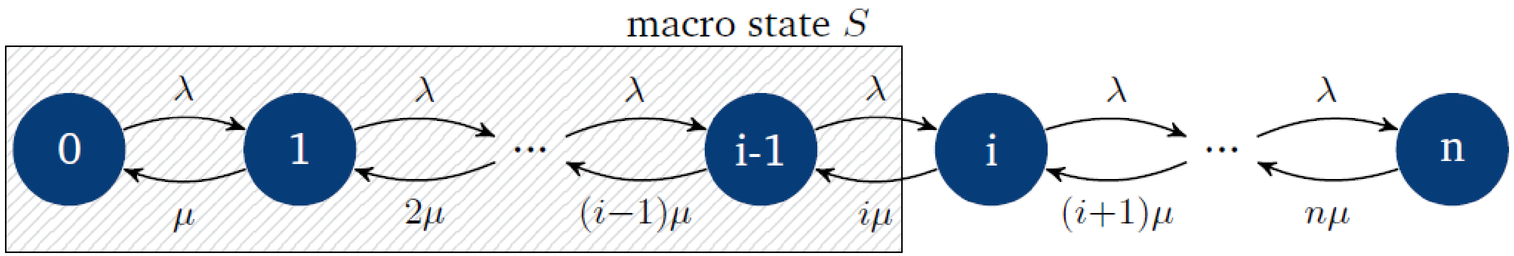
Examples of Transition Probability Densities

Transition Probability Densities

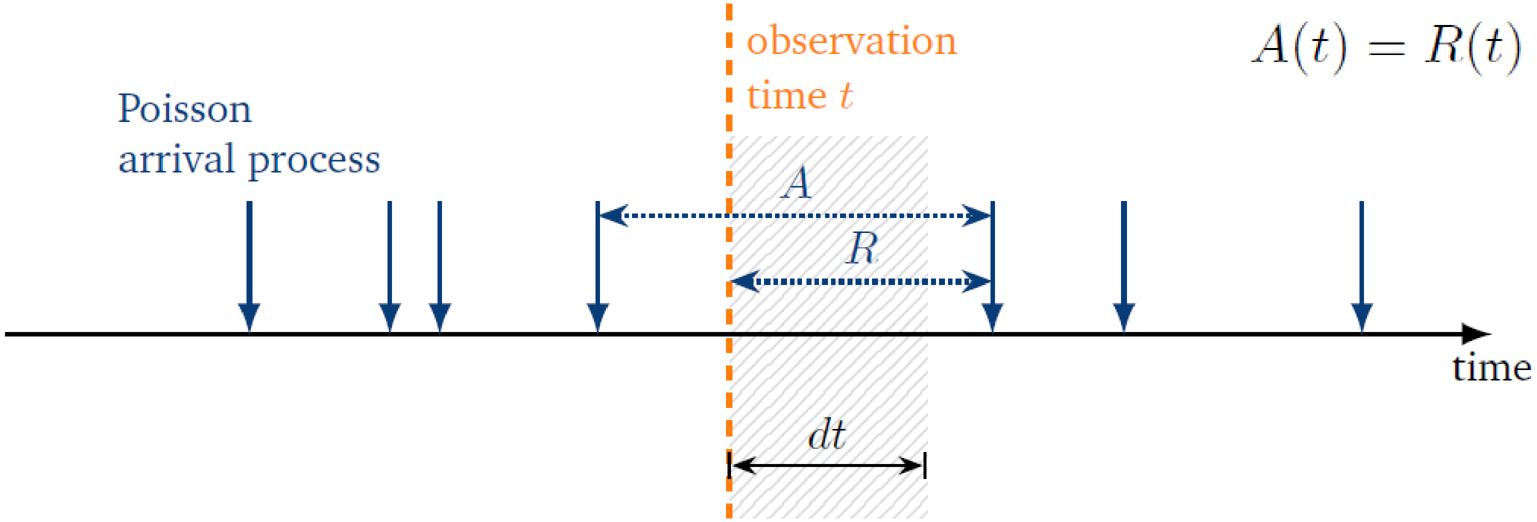
- What is the transition probability density of a Markov state process?



- Example: M/M/n-0
  - transition rates of a Poisson process? $q_{i,i+1} = \lambda\,.$



Transition Probability Density of Poisson Arrival Process



$$A(t) = R(t) = 1 - e^{-\lambda t}$$

- Transition $i \quad i + 1$: arriving customer is accepted

$$q_{i,i+1} = \lim_{dt \to 0} \frac{p_{i,i+1}(dt)}{dt}$$

$$= \lim_{dt \to 0} \frac{P(R \leq dt)}{dt} = \lim_{dt \to 0} \frac{1 - e^{-\lambda dt}}{dt} \underset{\text{L'Hospital}}{=} \lambda$$

$$p_{i,i+1}(dt) = P(X(t + dt) = i + 1 | X(t) = i)$$

Transition Probability Density for Exponential Service Time

- k servers with exponential service times

$$R(t) = B(t) = 1 - e^{-\mu t}$$

- Interval R^* until next service termination

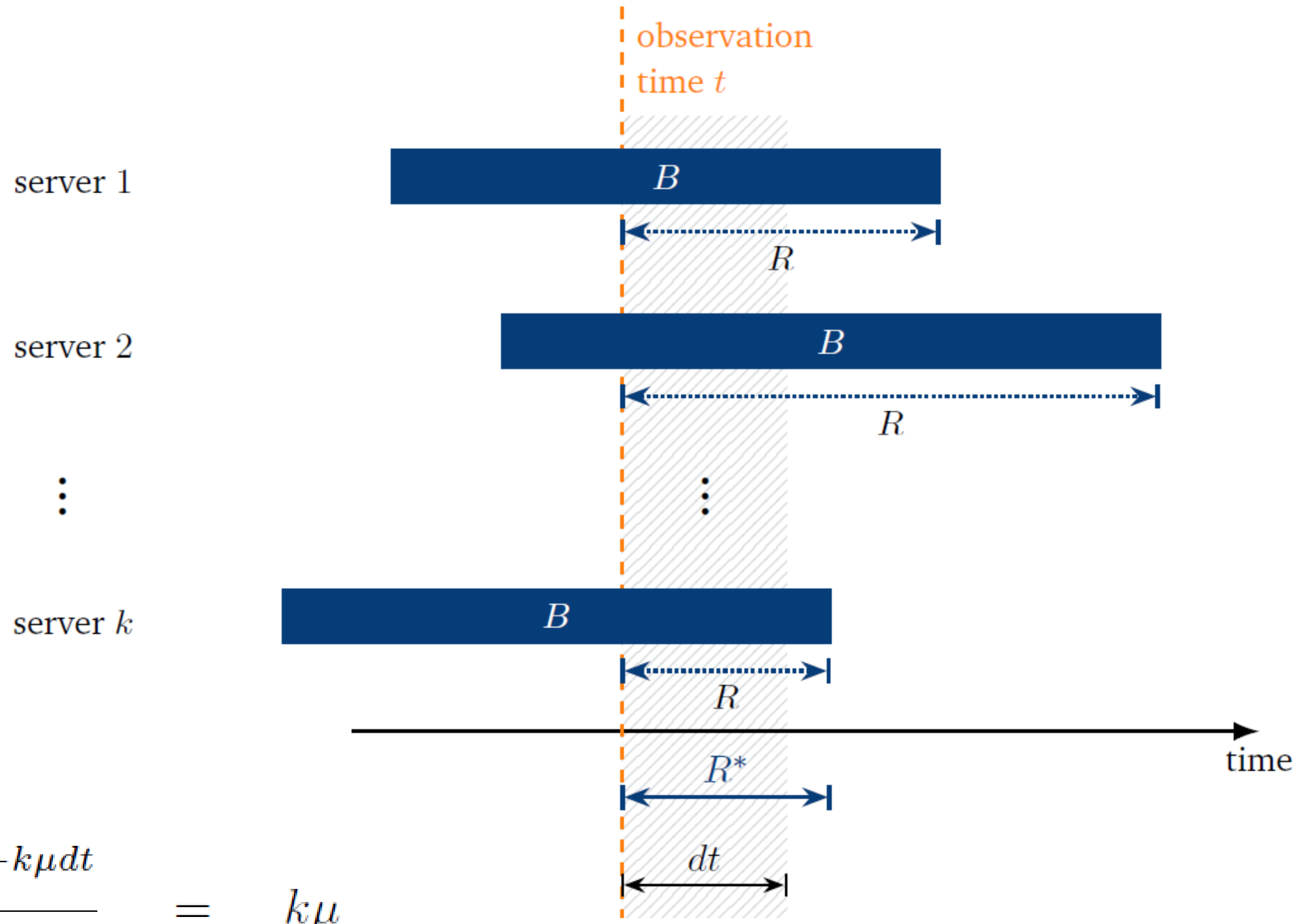
$$R^* = \min\{\underbrace{R, \ldots, R}_{\text{k-times}}\}$$

$$R^*(t) = 1 - \prod_{i=1}^{k}(1 - R(t)) \; = \; 1 - e^{-k\mu t}$$

- Transition $k \to k-1$

$$q_{k,k-1} = \lim_{dt \to 0} \frac{p_{k,k-1}(dt)}{dt}$$

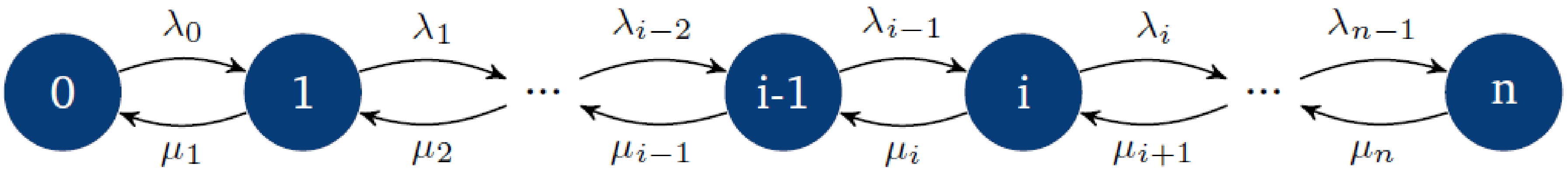
$$= \lim_{dt \to 0} \frac{P(R^* \leq dt)}{dt} = \lim_{dt \to 0} \frac{1 - e^{-k\mu dt}}{dt} \underset{\text{L'Hospital}}{=} k\mu$$



Birth-and-Death Processes

Birth-and-Death Processes

- Markov processes in which only transitions between neighboring states occur
- State transition diagram of a finite-state, one-dimensional BD process



$$q_{ij} = \begin{cases} \lambda_i & i = 0, 1, \ldots, n-1, \quad j = i+1 \quad \text{birth rate} \\ \mu_i & i = 1, 2, \ldots, n, \quad j = i-1 \quad \text{death rate} \\ 0 & \text{otherwise} \end{cases}$$

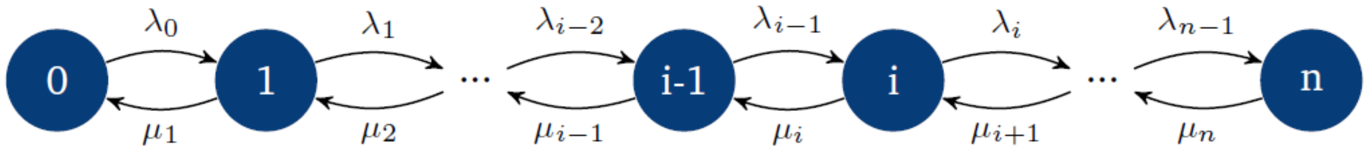
- Pure birth process: all $\mu_i = 0$
  - equilibrium: $x(n) = 1\,, x(i) = 0$ otherwise
- Pure death process: all $\lambda_i = 0$
  - equilibrium: $x(0) = 1\,, x(i) = 0$ otherwise

Example: No BD Process

Non-stationary Birth-and-Death Processes

- Transient (time-dependent) state probabilities of the BD process

$$x(i,t) = P(X(t) = i)$$



$$\Rightarrow \begin{cases} \frac{\partial}{\partial t} x(0,t) &= -\lambda_0 x(0,t) + \mu_1 x(1,t) \\ \frac{\partial}{\partial t} x(i,t) &= -(\lambda_i + \mu_i)\, x(i,t) + \lambda_{i-1} x(i-1,t) + \mu_{i+1} x(i+1,t), \quad i = 1,\ldots,n-1 \\ \frac{\partial}{\partial t} x(n,t) &= -\mu_n\, x(n,t) + \lambda_{n-1}\, x(n-1,t) \end{cases}$$

- Solution of this differential equation system with the initial conditions $\{x(i,0)\ , i = 0, \ldots, n\}$

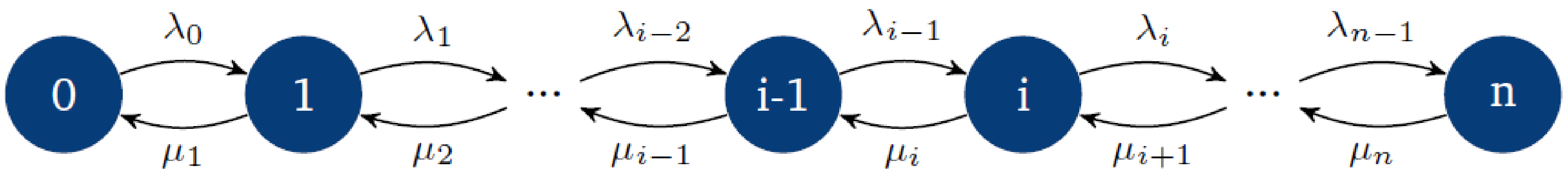
$$\mathbf{X}(t) = \big(x(0,t), x(1,t), \ldots, x(n,t)\big)$$

- Example: Poisson Process as pure birth process

Example: Poisson Process as Pure Birth Process

Stationary Birth-and-Death Processes

- Equation system for the micro states of the state space

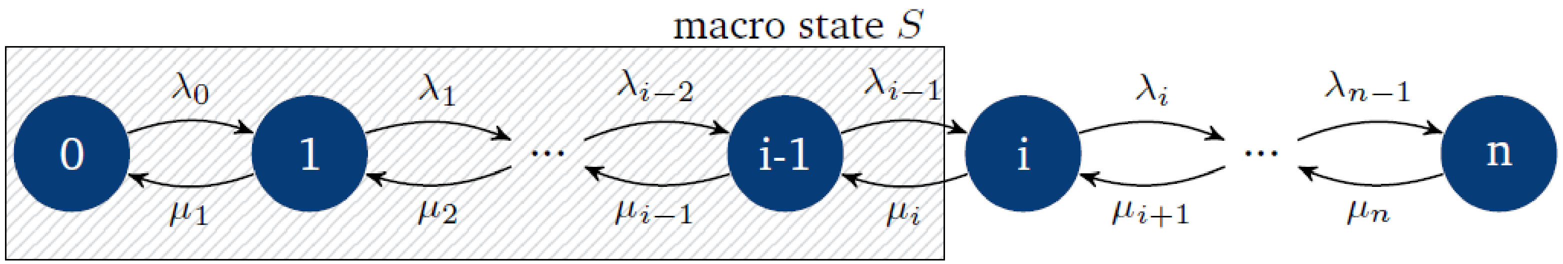


$$\left.\begin{array}{ll} 0: & \lambda_0\, x(0) = \mu_1\, x(1) \\ i \in (1,n\text{-}1): & (\lambda_i + \mu_i)\, x(i) = \lambda_{i-1}\, x(i-1) + \mu_{i+1}\, x(i+1), \\ n: & \lambda_{n-1}\, x(n-1) = \mu_n\, x(n) \end{array}\right\}$$

because of linear dependency, one of these equations can be omitted

$$\sum_{i=0}^{n} x(i) = 1$$

Macro State Equation System



- State equation for macro state S

$$i \in (1,n) \qquad \lambda_{i-1}\, x(i-1) = \mu_i\, x(i),$$

$$\left(\sum_{i=0}^{n} x(i) = 1 \right)$$

$$\Rightarrow \quad x(i) = x(0) \cdot \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=1}^{i} \mu_k}, \qquad i = 1, 2, \ldots, n$$

- Normalization yields $x(0)$

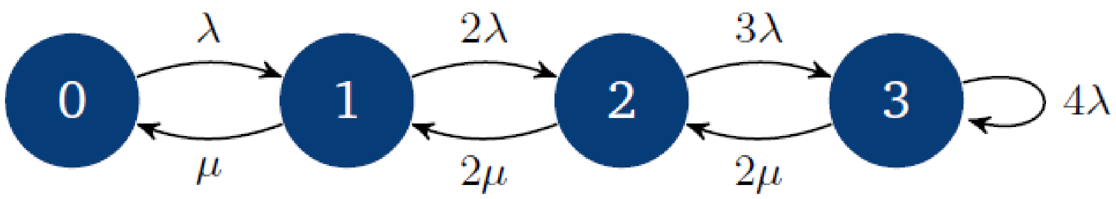
$$1 = \sum_{i=0}^{n} x(i) = x(0) + x(0) \sum_{i=1}^{n} \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=1}^{i} \mu_k}$$

$$\Rightarrow \quad x(0) = \left(1 + \sum_{i=1}^{n} \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=1}^{i} \mu_k} \right)^{-1}$$

Solving the Macro State Equation System

Example: Delay-Loss System with State-dependent Rates

- Delay-loss system M(x)/M/2-1 as an example for birth-and-death process



- Customers arrive with state-dependent arrival rates
  - not a Poisson process
  - PASTA property is not valid: $x_A(i) \neq x^*(i)$

- State equation system yields arbitrary-time state probabilities $x^*(i)$
  - e.g. blocking probability requires $x_A(i)$ for arriving customers
  - strong law of large numbers for Markov chains can be applied

Strong Law of Large Numbers for Markov Chains

- Expected number of arrivals when the system is in state [X = i] in interval of length T is $n_A(i,T)$
- Expected total number $n_A(T)$ of arrivals

- **Strong law of large numbers for Markov chains**

$$x_A(i) = \lim_{T\to\infty} \frac{n_A(i,T)}{n_A(T)} = \frac{\lambda_i \cdot x(i)}{\sum_k \lambda_k \cdot x(k)}$$

- For birth-and-death process, as in the example, $x_A(i)$ and characteristics like blocking probability or waiting probability can be derived accordingly