Chapter 5.3

Delay System M/GI/1

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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The book must be cited and the disclaimer attached when using lectures slides or scripts.

Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/

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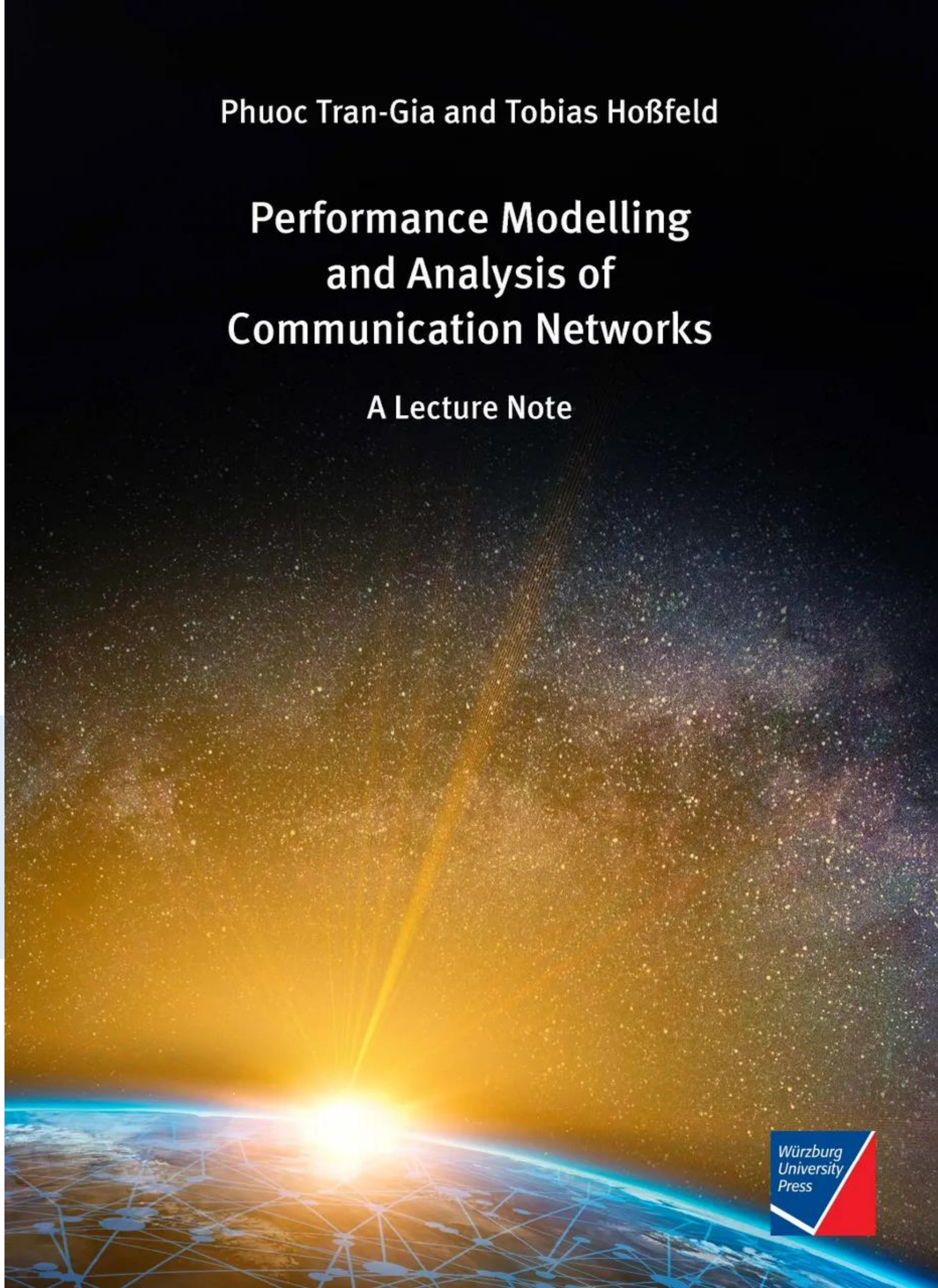
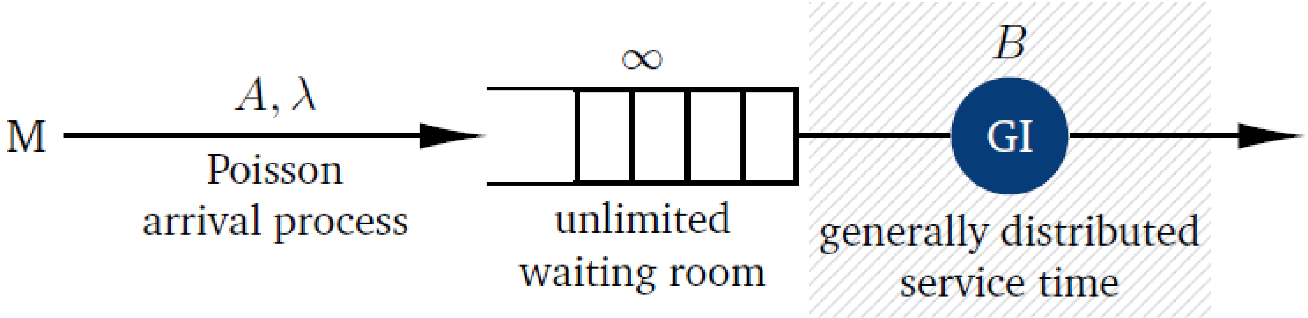


Chapter 5

5 Analysis of Non-Markovian Systems

Model Structure and Parameters

Delay System M/GI/1



- Interarrival time A with arrival rate λ
- Service time B with service rate μ is generally distributed

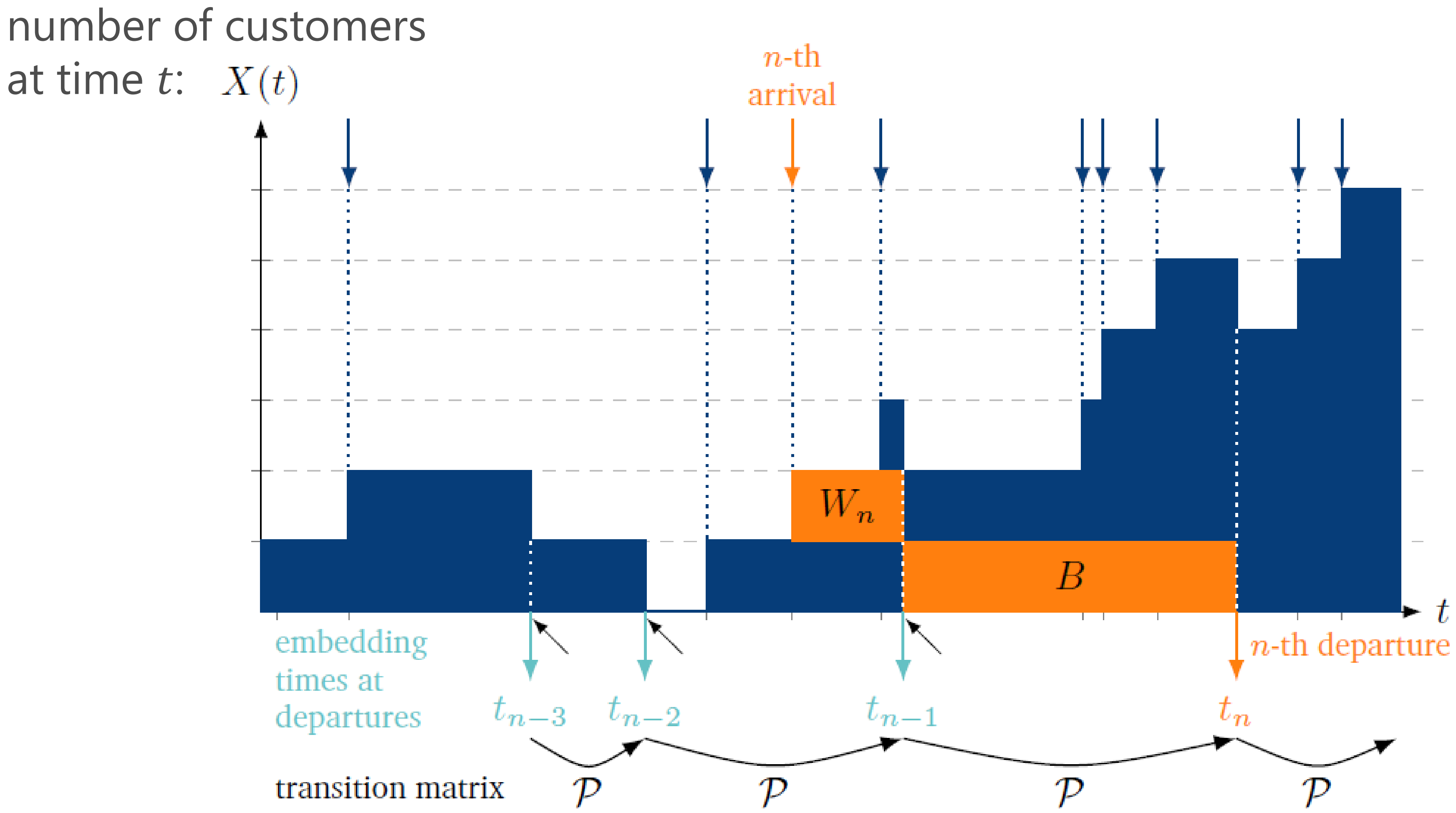
$$A(t) = P(A \leq t) = 1 - e^{-\lambda t}, \quad E[A] = \frac{1}{\lambda}$$

- Offered traffic a identical to server utilization ρ : $\rho = a = \frac{\lambda}{\mu}$ in pseudo-unit Erlang [Erl]

- Pure delay system: number of waiting places is assumed to be unlimited
- FIFO queue: first-in first-out queuing discipline

- **Stability condition** $\rho < 1$

State Process of M/GI/1 (FIFO)



Markov chain: $\{X(t_0), X(t_1), \dots, X(t_n), X(t_{n+1}), \dots\}$

Markov Chain and State Transition

M/GI/1: Embedded Markov Chain

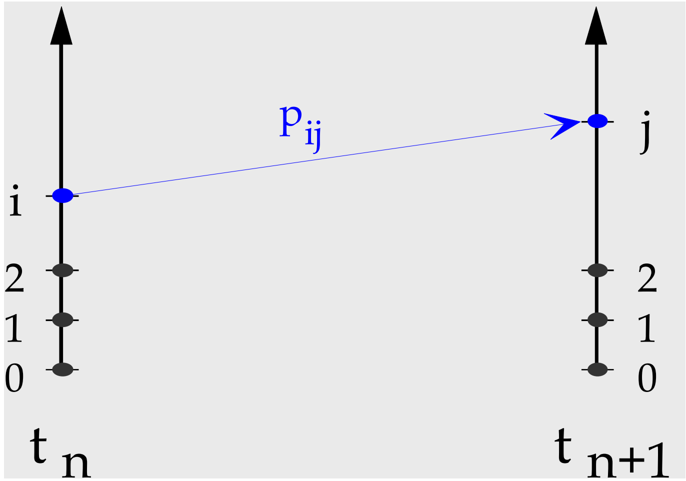
- Service process is the only non-Markovian model component (not memoryless)
- State process becomes memoryless at instances of service ends
  - regeneration points: immediately before or immediately after
  - easier for the analysis of M/GI/1: **immediately after service ends**

- **System state at embedding time** t_n: $X(t_n) = X_D(t_n)$
- State probability at time t_n: $x(j,n) = P(X(t_n) = j)$

- **State transition probability**

$$p_{ij} = P\left(X(t_{n+1}) = j \,\middle|\, X(t_n) = i\right)$$

state transition



Number of Arrivals During Service Time

- Random variable Γ for the number of arrivals during a service duration B

  - distribution $\gamma(i) = P(\Gamma = i)$

  - generating function (GF transform of Γ)

$$\Gamma_{GF}(z) = \sum_{j=0}^{\infty} \gamma(j)\, z^j = \sum_{j=0}^{\infty} \int_0^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t}\, b(t)\, dt\, z^j$$

$$\underset{(3.20)}{=} \Phi_B(s)\Big|_{s\,=\,\lambda(1-z)} = \Phi_B(\lambda(1-z))\,.$$

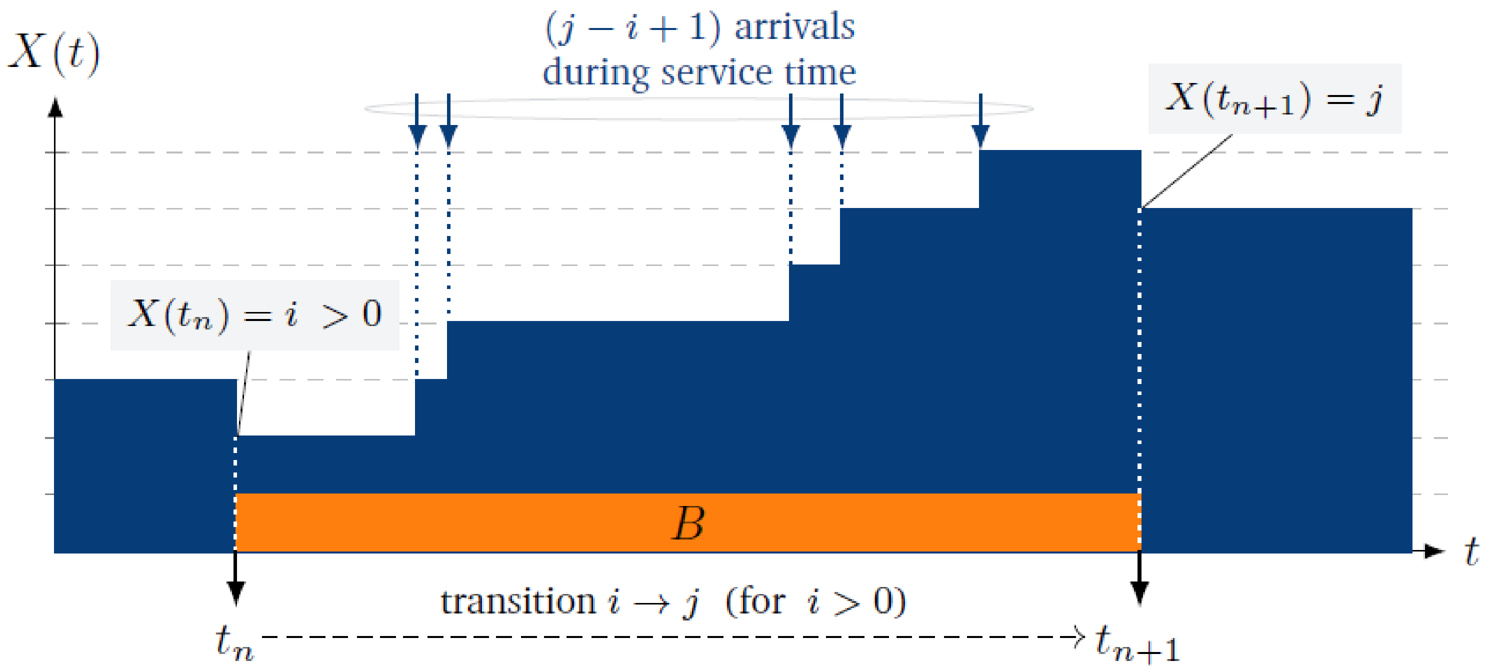
Ch. 3.3 Poisson arrivals during an arbitrarily distributed Interval

  - mean value $E[\Gamma] = \left.\frac{d\Gamma_{GF}(z)}{dz}\right|_{z=1} = \lambda E[B] = \rho.$

- Example: Poisson arrivals during deterministic service time
  - $B \sim D(t_0)$ and arrival rate λ
  - Γ follows a Poisson distribution: $\Gamma \sim POIS(\lambda \cdot t_0)$

State Transition Probability (Case a)

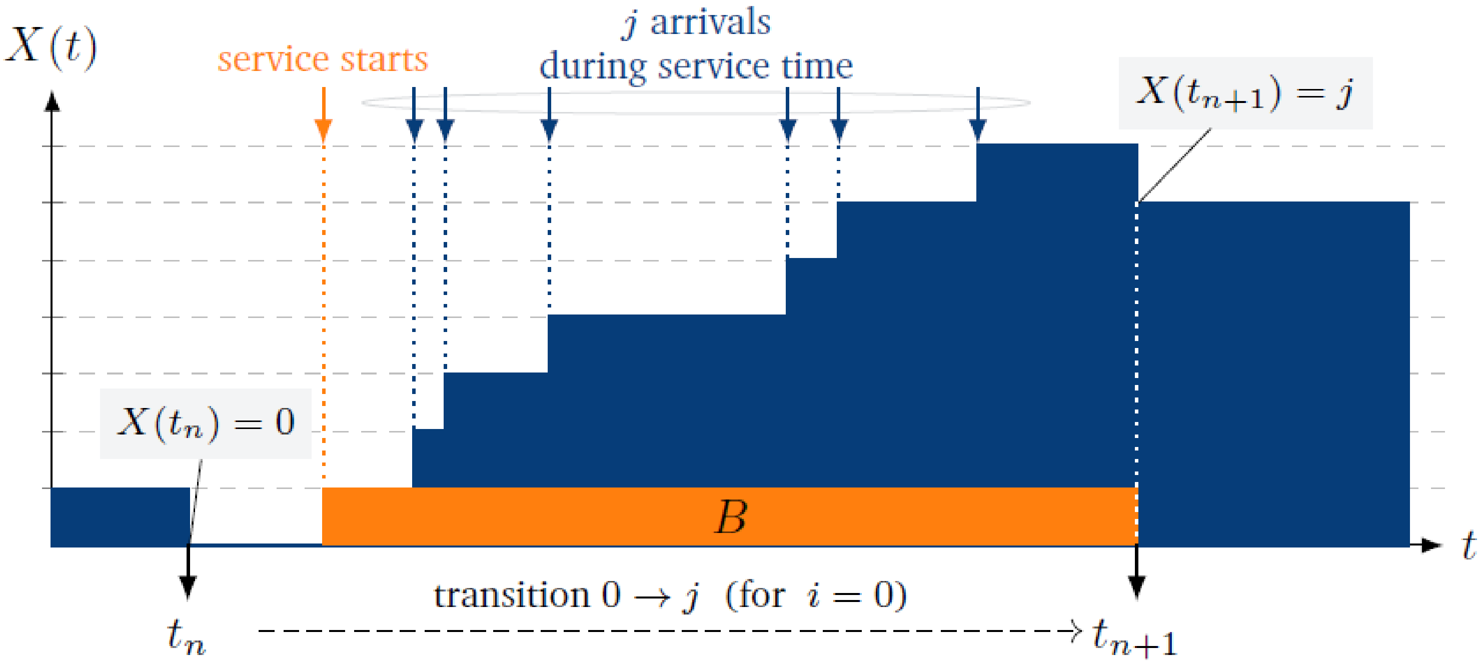
- For $i > 0$: $\quad p_{ij} = \gamma(j-i+1), \quad i = 1, \ldots, \quad j = i-1, i, \ldots$



(a) Transition $[X(t_n) = i] \to [X(t_{n+1}) = j]$ for $(i > 0)$.

State Transition Probability (Case b)

- For $i = 0$: $p_{0j} = \gamma(j), \quad j = 0, \ldots$.



(b) Transition $[X(t_n) = 0] \to [X(t_{n+1}) = j]$ for $(i = 0)$.

State Transition Matrix

- Case a: $i > 0$ $\quad p_{ij} = \gamma(j-i+1), \quad i = 1, \ldots, \quad j = i-1, i, \ldots$
- Case b: $i = 0$ $\quad p_{0j} = \gamma(j), \quad j = 0, \ldots\ .$
- State transition matrix

$$\mathcal{P} = \{p_{ij}\} = \begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \cdots \\ \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \cdots \\ 0 & \gamma(0) & \gamma(1) & \gamma(2) & \cdots \\ 0 & 0 & \gamma(0) & \gamma(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

State Equation and State Probabilities

General State Transition Equation

- State probabilities at the regeneration point t_n

$$\mathcal{X}_n = \{x(0,n), x(1,n), \ldots, x(j,n), \ldots\}$$

$$x(j,n) = P(X(t_n) = j), \qquad j = 0,1,\ldots$$

- **General state transition equation**

$$\begin{cases} \mathcal{X}_n \cdot \mathcal{P} = \mathcal{X}_{n+1} \\ x(j,n+1) = x(0,n)\,\gamma(j) + \sum_{i=1}^{j+1} x(i,n) \cdot \gamma(j-i+1)\,, \quad j=0,1,.. \end{cases}$$

case b: i=0 *case a: i>0*

$$\mathcal{P} = \{p_{ij}\} = \begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \cdots \\ \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \cdots \\ 0 & \gamma(0) & \gamma(1) & \gamma(2) & \cdots \\ 0 & 0 & \gamma(0) & \gamma(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- **Non-stationary analysis**
  - With start vector $\boldsymbol{X}_0$, future time-dependent state probability vectors can be derived

$$\mathcal{X}_1, \ldots, \mathcal{X}_n, \mathcal{X}_{n+1}$$

Stationary Analysis

- In statistical equilibrium

$$X_{\mathrm{n}} = X_{\mathrm{n+1}} = \ldots = X$$

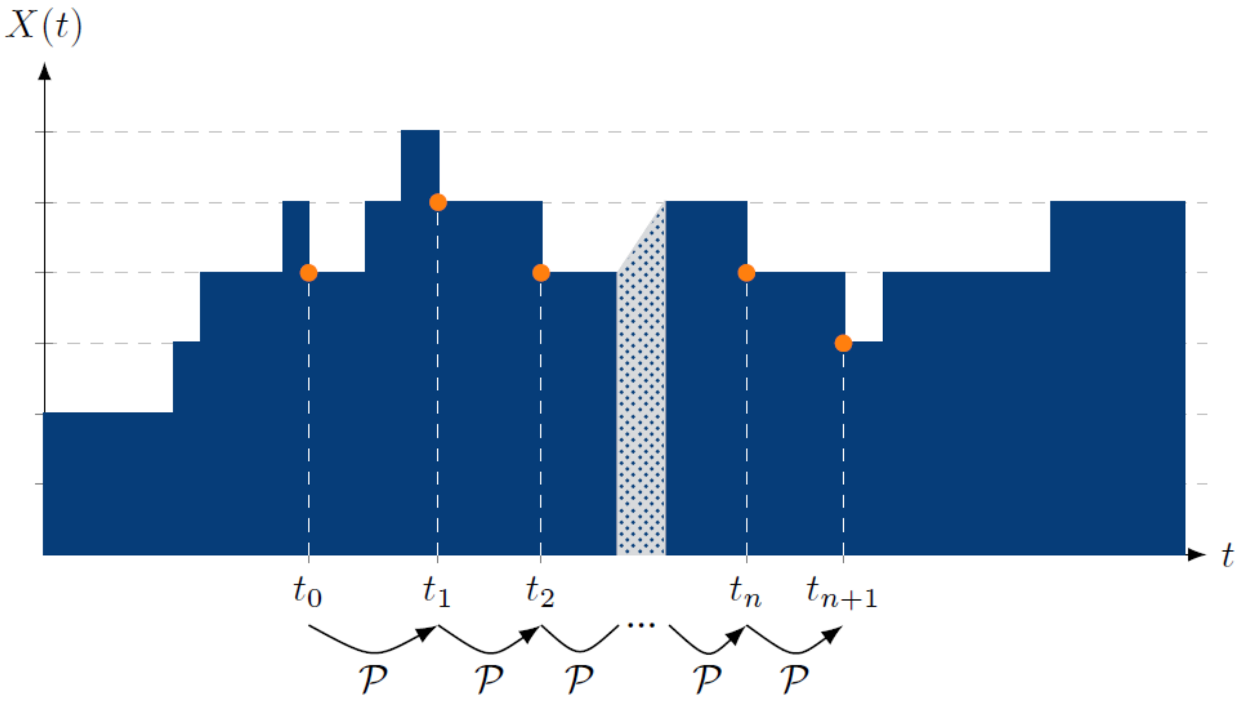
$$X = \{x(0), x(1), \ldots, x(j), \ldots\}$$

- Stationary state transition equation

$$X \cdot \mathcal{P} = X$$

- Components of state probability vector

$$x(j) = \underbrace{x(0)\,\gamma(j)}_{\text{case b: } i=0} + \underbrace{\sum_{i=1}^{j+1} x(i)\cdot\gamma(j-i+1)}_{\text{case a: } i>0}, \quad j=0,1,\ldots$$



M/GI/1: Analysis of States

- Analysis using generating function
$$X_{GF}(z) = \frac{(1-\rho)(1-z)\Gamma_{GF}(z)}{\Gamma_{GF}(z) - z}$$

- Generating function of number of arrivals Γ during random service time
$$\Gamma_{GF}(z) = \sum_{j=0}^{\infty} \gamma(j)\, z^j = \sum_{j=0}^{\infty} \int_0^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t}\, b(t)\, dt\, z^j$$
$$\underset{(3.20)}{=} \Phi_B(s)\Big|_{s\,=\,\lambda(1-z)} = \Phi_B(\lambda(1-z))\,.$$

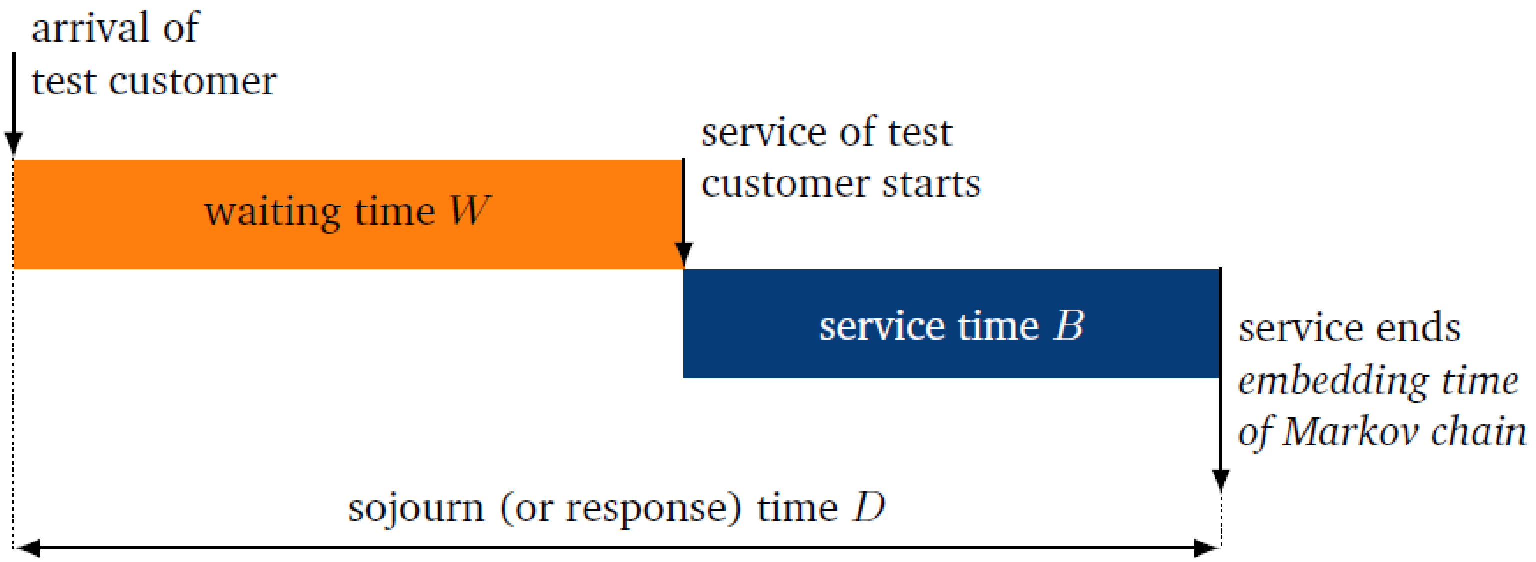
- **Pollaczek-Khintchine formula** for system state

$$X_{GF}(z) = \frac{(1-\rho)\,(1-z)\,\Phi_B(\lambda(1-z))}{\Phi_B(\lambda(1-z))\; - \; z}$$

M/GI/1: Analysis of States (Proof)

Delay Distribution

Sojourn Time of Customer



$$D = W + B$$

$$d(t) = w(t) * b(t) \Rightarrow \Phi_D(s) = \Phi_W(s) \cdot \Phi_B(s)$$

Key Idea to Derive Delay Distribution

- Interpretations of the state probabilities at embedding times (after service ends)

  $x(k)$ = *P(test customer left behind X = k customers in system)*

  = *P(k arrivals during the sojourn time of test customer)*

Delay Distribution: Analysis

- Number of Poisson arrivals X during sojourn time D

$$X_{GF}(z) = \sum_{k=0}^{\infty} x(k)\, z^k = \sum_{k=0}^{\infty} \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t}\, d(t)\, dt\, z^k$$

$$\underset{(3.20)}{=} \Phi_D(s)\Big|_{s=\lambda(1-z)} = \Phi_D(\lambda(1-z))\,.$$

- Already derived: Pollaczek-Khintchine formula for system state

$$X_{GF}(z) = \frac{(1-\rho)\,(1-z)\,\Phi_B(\lambda(1-z))}{\Phi_B(\lambda(1-z)) \; - \; z}$$

- Finally

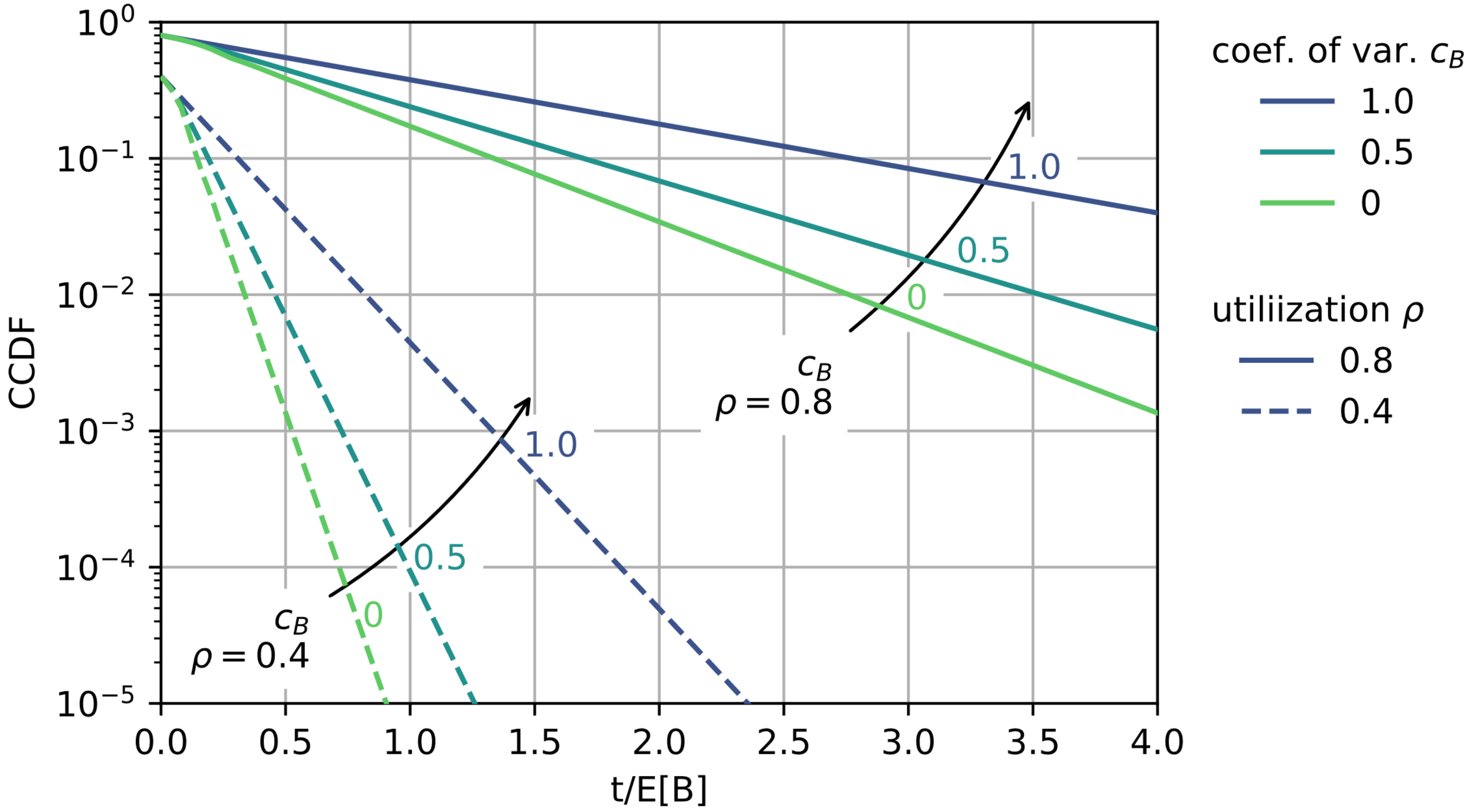
$$\Phi_D(\lambda(1-z)) = \frac{(1-\rho)\,(1-z)\,\Phi_B(\lambda(1-z))}{\Phi_B\big(\lambda(1-z)\big) \; - \; z} \quad \xrightarrow[s=\lambda(1-z)]{} \quad \Phi_D(s) = \frac{s(1-\rho)}{s-\lambda+\lambda\Phi_B(s)}\,\Phi_B(s)$$

- **Pollaczek-Khintchine formula for waiting time**

$$\xrightarrow[\Phi_D(s) = \Phi_W(s)\cdot\Phi_B(s)]{} \quad \Phi_W(s) = \frac{s\,(1-\rho)}{s-\lambda+\lambda\,\Phi_B(s)}\,.$$

Convolution Theorem for Continuous R.V.s

CCDF of the Waiting Time



Other System Characteristics

Waiting probability, mean waiting times, higher moments of waiting time

Waiting Probability

- Initial value theorem of Laplace transform applied

$$w(t) \;\circ\!\!\xrightarrow{\text{LT}}\!\!\bullet\; \Phi_W(s) \quad \Rightarrow \quad W(t) \;\circ\!\!\xrightarrow{\text{LT}}\!\!\bullet\; \frac{\Phi_W(s)}{s}$$

$$P(W=0) = \lim_{t\to 0} W(t) = \lim_{s\to\infty} s \cdot \frac{\Phi_W(s)}{s}$$

$$\Phi_W(s) = \frac{s(1-\rho)}{s-\lambda+\lambda\,\Phi_B(s)}$$

- **Waiting probability**

$$p_W = P(W>0) = 1-P(W=0) = 1-W(t)\big|_{t\to 0} = \rho\,.$$

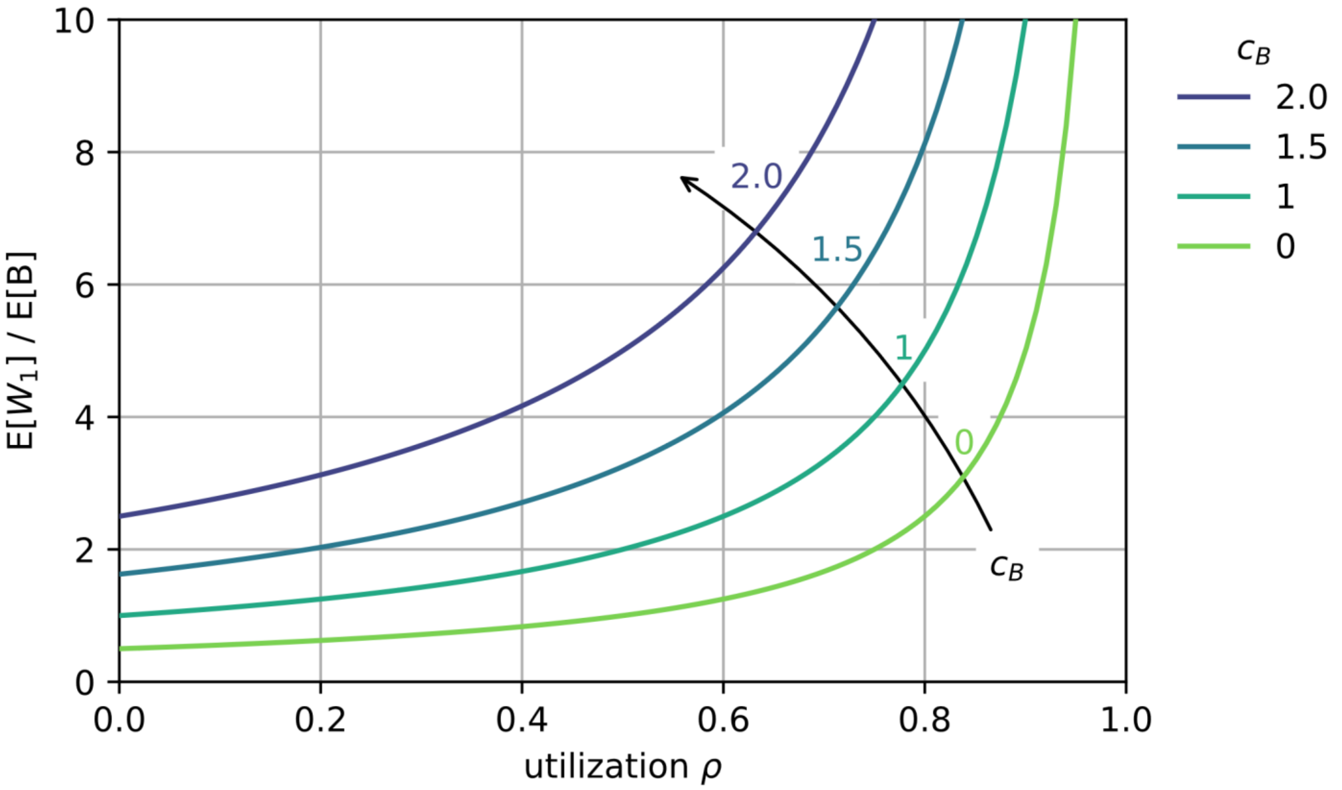
Mean Waiting Times

- Mean waiting time of **all customers $E[W]$**

$$E[W] = E[B] \cdot \frac{\rho\left(1+c_B^2\right)}{2(1-\rho)} = \frac{\lambda\, E\left[B^2\right]}{2(1-\rho)}$$

- Mean waiting time of **waiting customers $E[W_1]$**

$$E[W_1] = \frac{E[W]}{p_W} = E[B] \cdot \frac{1+c_B^2}{2(1-\rho)}$$



- Waiting time W is a mixture distribution
  - $W = 0$ with probability $1 - p_W$
  - $W = W_1$ with probability p_W

Higher Moments of Waiting Time

- **Takács recursion formula**

$$E[W^k] = \frac{\lambda}{1-\rho} \sum_{i=1}^{k} \binom{k}{i} \frac{E[B^{i+1}]}{i+1} E[W^{k-i}],$$

$$E[W^0] = 1 .$$

- Especially for the two first moments

$$E[W] = \frac{\lambda\, E[B^2]}{2(1-\rho)}$$

$$E[W^2] = 2\, E[W]^2 + \frac{\lambda\, E[B^3]}{3\,(1-\rho)}$$

State Probabilities at Arbitrary Time

State Probabilities at Arbitrary Time

- Embedded Markov chain: $x(i) = x_D(i)$
  - state probabilities given in the Pollaczek-Khintchine formula hold at regeneration points of the Markov chain
  - Markov chain: embedded immediately after service ends

- **Kleinrock's (Burke's) result**: $x_D(i) = x_A(i)$
  - M/GI/1 system state can change at most by +1 or -1

- **PASTA property**: $x^*(i) = x_A(i)$
  - arrival process is a Poisson process

- In summary: $x^*(i) = x_A(i) = x_D(i) = x(i)$
  - E.g., $x^*(0) = 1 - \rho$ (utilization law) and $p_W = 1 - x_A(0) = \rho$ (waiting probability)