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Institute of Computer Science
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Chapter 5.4

Delay System GI/M/1

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
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Website to download book, exercises, slides and scripts:
https://modeling.systems/

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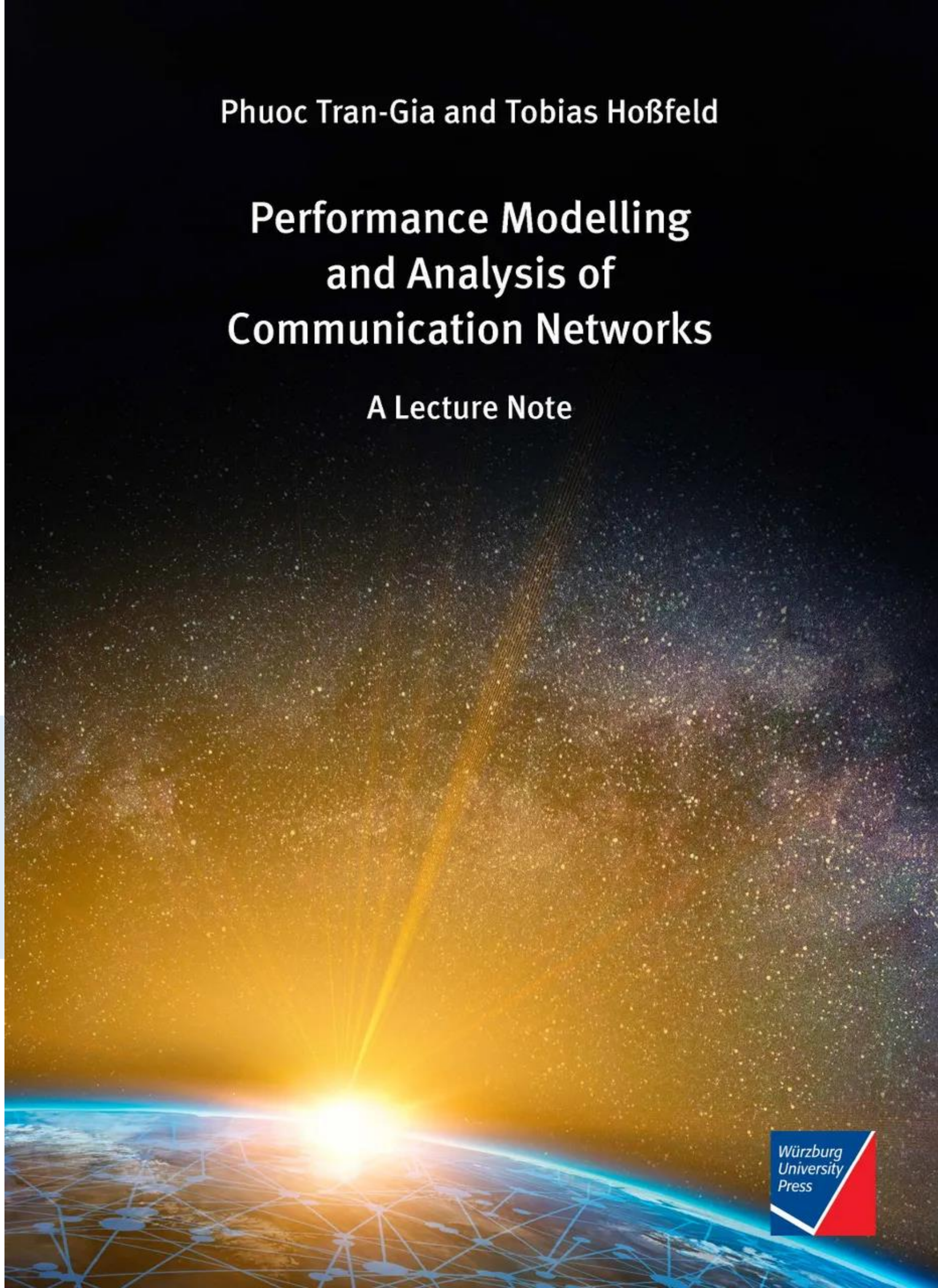
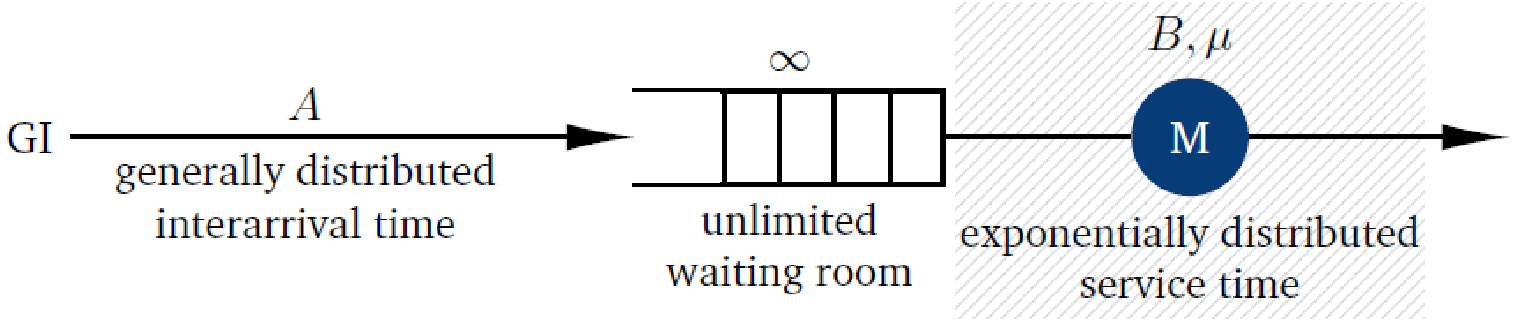


Chapter 5

5 Analysis of Non-Markovian Systems

MODEL STRUCTURE AND PARAMETERS

Delay System GI/M/1

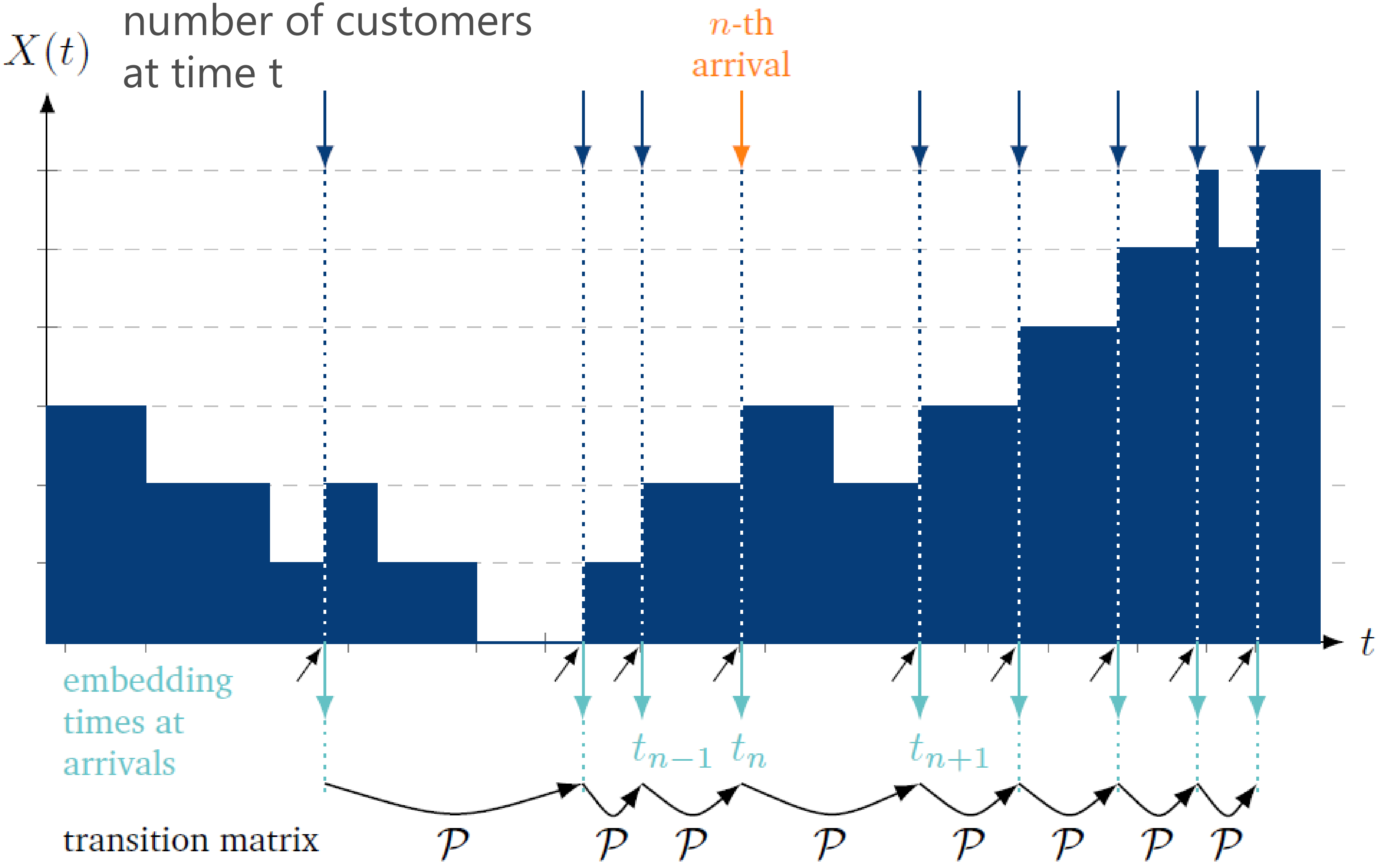


- Generally distributed interarrival time A
- Exponentially distributed service time B with rate μ $\quad B(t) = P(B \le t) = 1 - e^{-\mu t}, \quad E[B] = \frac{1}{\mu}$
- Offered traffic a identical to server utilization ρ : $\rho = a = \frac{E[B]}{E[A]} = \frac{1}{\mu E[A]}$ in pseudo-unit Erlang [Erl]

- Pure delay system: number of waiting places is assumed to be unlimited
- FIFO queue: first-in first-out queuing discipline

- **Stability condition** $\rho < 1$

State Process of GI/M/1 Delay System



Markov chain: $\{X(t_0), X(t_1), \ldots, X(t_n), X(t_{n+1}), \ldots\}$

Markov Chain and State Transition

Embedded Markov Chain

- Γ number of customers with service terminations during an interarrival time A

$$\gamma(i) = P(\Gamma = i) \qquad \Gamma_{GF}(z) = \sum_{i=0}^{\infty} \gamma(i) z^i$$

$$E[\Gamma] = \left. \frac{d\Gamma_{GF}(z)}{dz} \right|_{z=1} = \mu \cdot E[A] = \frac{1}{\rho}$$

- **Embedded Markov chain**
  - arrival process is only non-Markovian model component; IAT A is not memoryless
  - regeneration point **immediately before customer arrivals** → arriving customer observes number of customers in queue corresponding to the waiting time
  - $X(t_n)$ is system state immediately before arrival time t_n of n-th customer: $X(t_n) = X_A(t_n)$
  - $x(j,n) = P(X(t_n) = j)$ is probability that system is in state j at time t_n

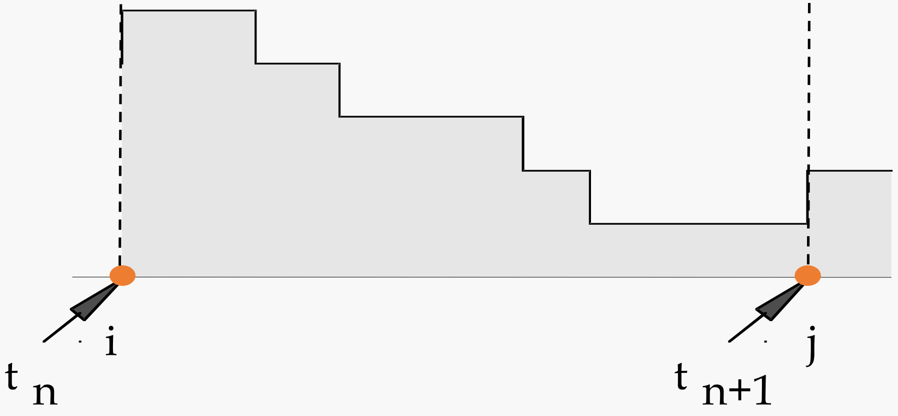
Transition Probability

- Transition probability between two successive regeneration points (embedding immediately before arrivals) $p_{ij} = P(X(t_{n+1}) = j \mid X(t_n) = i)$

Case 1: $\underline{j \neq 0:}$

$$p_{ij} = \gamma(i+1-j), \quad i = 0,1,\ldots, \quad j = 1,\ldots,i+1$$

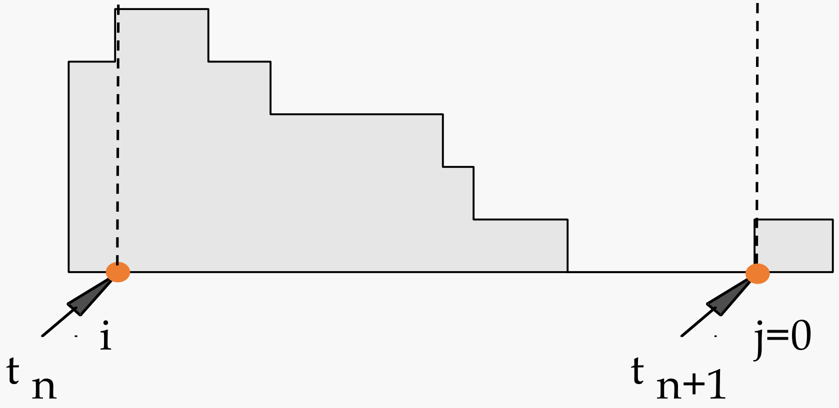
(i+1-j) service ends during an interarrival time



Case 2: $\underline{j = 0:}$

$$p_{i0} = \sum_{k=i+1}^{\infty} \gamma(k) = 1 - \sum_{k=0}^{i} \gamma(k), \quad i = 0,1,\ldots$$

at least (i+1) service ends during an interarrival time



State Transition Probability and State Transition Matrix

- Two types of transitions are distinguished
  - $j \neq 0$ $\quad p_{ij} = \gamma(i+1-j), \quad i = 0,1,\dots, \quad j = 1,\dots,i+1$
  - $j = 0$ $\quad p_{i0} = \sum_{k=i+1}^{\infty} \gamma(k) = 1 - \sum_{k=0}^{i} \gamma(k), \quad i = 0,1,\dots$
- **State transition matrix** of the GI/M/1 delay system

$$\Rightarrow \quad \mathcal{P} = \{p_{ij}\} = \begin{pmatrix} 1-\gamma(0) & \gamma(0) & 0 & 0 & \cdots \\ 1-\sum_{k=0}^{1}\gamma(k) & \gamma(1) & \gamma(0) & 0 & \cdots \\ 1-\sum_{k=0}^{2}\gamma(k) & \gamma(2) & \gamma(1) & \gamma(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

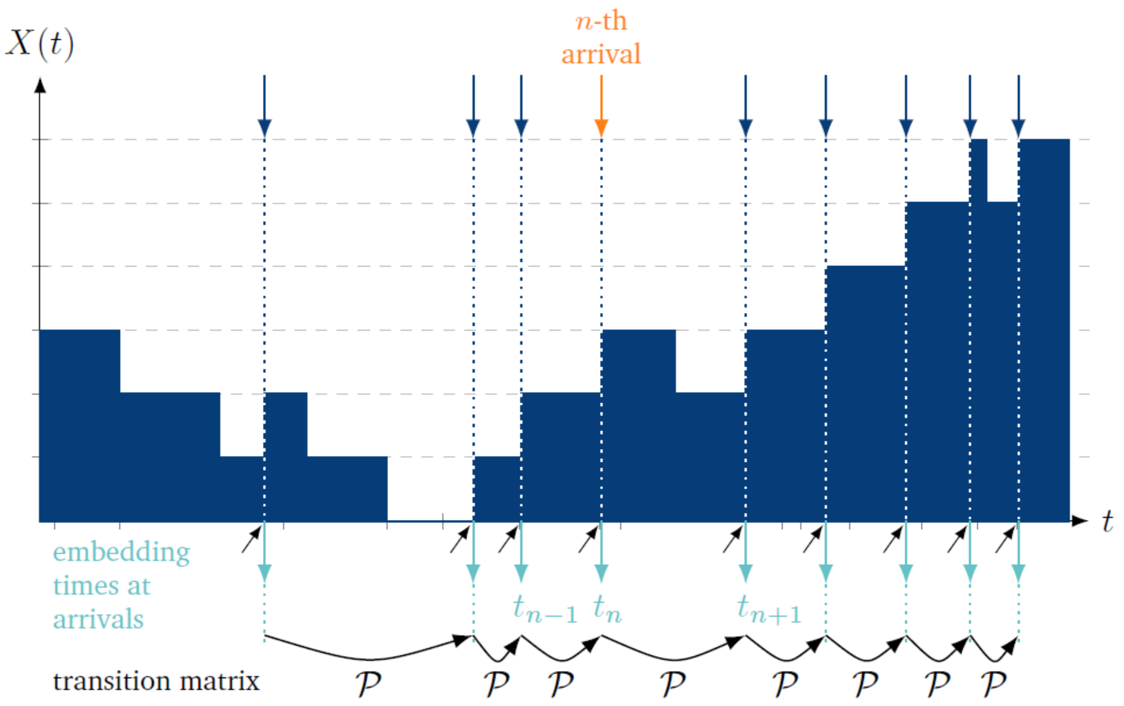
State Probabilities

General State Transition Equation

- State probabilities at the regeneration point t_n

$$\mathcal{X}_n = \{x(0,n), x(1,n), \ldots, x(j,n), \ldots\}$$

$$x(j,n) = P(X(t_n) = j), \qquad j = 0,1,\ldots$$



- **General state transition equation**

$$\mathcal{X}_n \cdot \mathcal{P} = \mathcal{X}_{n+1}$$

- **Non-stationary analysis**
  - With start vector $\boldsymbol{X}_0$, future time-dependent state probability vectors can be derived

$$\mathcal{X}_1, \ldots, \mathcal{X}_n, \mathcal{X}_{n+1}$$

Stationary Analysis GI/M/1

- In statistical equilibrium

$$\mathcal{X}_n = \mathcal{X}_{n+1} = \ldots = \mathcal{X}$$

$$\mathcal{X} = \{x(0), x(1), \ldots, x(j), \ldots\}$$

- **Stationary state transition equation**
  - $\mathcal{X} \cdot \mathcal{P} = \mathcal{X}$
  - left eigenvector of transition probability matrix to eigenvalue 1

$$\mathcal{P} = \{p_{ij}\} = \begin{pmatrix} 1-\gamma(0) & \gamma(0) & 0 & 0 & \cdots \\ 1-\sum_{k=0}^{1}\gamma(k) & \gamma(1) & \gamma(0) & 0 & \cdots \\ 1-\sum_{k=0}^{2}\gamma(k) & \gamma(2) & \gamma(1) & \gamma(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$x(0)$

$x(j)$
here: $j = 2$

- Components of state probability vector

$$x(0) = \sum_{i=0}^{\infty} x(i)\left(1-\sum_{k=0}^{i}\gamma(k)\right) = \sum_{i=0}^{\infty} x(i) \sum_{k=i+1}^{\infty} \gamma(k)$$

$$x(j) = \sum_{i=j-1}^{\infty} x(i)\,\gamma(i+1-j) = \sum_{i=0}^{\infty} x(i+j-1)\,\gamma(i), \quad j = 1, 2, \ldots$$

State Analysis with Geometric Approach

State Analysis with Geometric Approach

- State equation in component notation

$$x(0) = \sum_{i=0}^{\infty} x(i)\left(1-\sum_{k=0}^{i}\gamma(k)\right) = \sum_{i=0}^{\infty} x(i)\sum_{k=i+1}^{\infty}\gamma(k)$$

$$x(j) = \sum_{i=j-1}^{\infty} x(i)\,\gamma(i+1-j) = \sum_{i=0}^{\infty} x(i+j-1)\,\gamma(i), \quad j=1,2,\ldots$$

- **Geometric Approach**: consider whether the following assumption would be valid and lead to a valid solution for σ

$$x(j+1) = \sigma\cdot x(j), \quad j=0,1,\ldots \quad \Rightarrow \quad x(j+1) = \sigma^{j+1}\cdot x(0)$$

- Then:

$$x(j)-\left[x(j-1)\gamma(0)+x(j)\gamma(1)+x(j+1)\,\gamma(2)+\ldots\right] = 0$$

$$\sigma x(j-1)-x(j-1)\gamma(0)-\sigma x(j-1)\gamma(1)-\sigma^2 x(j-1)\gamma(2)-\ldots = 0$$

$$x(j-1)\left[\sigma-\left(\gamma(0)+\sigma\gamma(1)+\sigma^2\gamma(2)+\ldots\right)\right] = 0$$

$$\boxed{x(j-1)\left[\sigma-\sum_{i=0}^{\infty}\gamma(i)\sigma^i\right] = 0\,.}$$

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State Analysis with Geometric Approach: Non-Trivial Root

- A non-trivial solution to the equation

$$x(j-1)\left[\sigma - \sum_{i=0}^{\infty} \gamma(i)\sigma^i\right] = 0$$

- is identical to a non-trivial root of

$$\sigma = \sum_{i=0}^{\infty} \gamma(i)\sigma^i$$

- i.e., a non-trivial root for $z = \sigma$ of the equation

$$z = \Gamma_{GF}(z)$$

- Trivial solution is excluded

$$1 = \Gamma_{GF}(1) \qquad 1 = \sum_{i=0}^{\infty} \gamma(i) 1^i$$

- For real valued $z \geq 0$

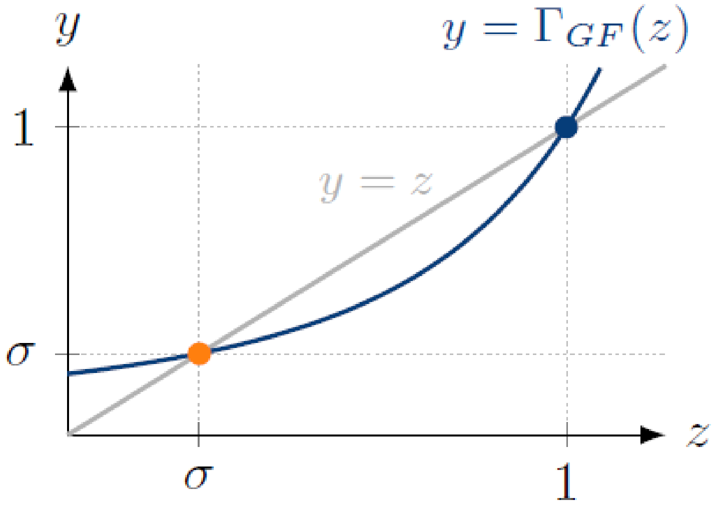
$$\frac{d}{dz}\Gamma_{GF}(z) = \sum_{i=1}^{\infty} i\,\gamma(i)\,z^{i-1} \geq 0$$

$$\frac{d^2}{dz^2}\Gamma_{GF}(z) = \sum_{i=2}^{\infty} i(i-1)\gamma(i)z^{i-2} \geq 0$$

$\Gamma_{GF}(z)$ is convex and monotonically increasing in $]0;1[$

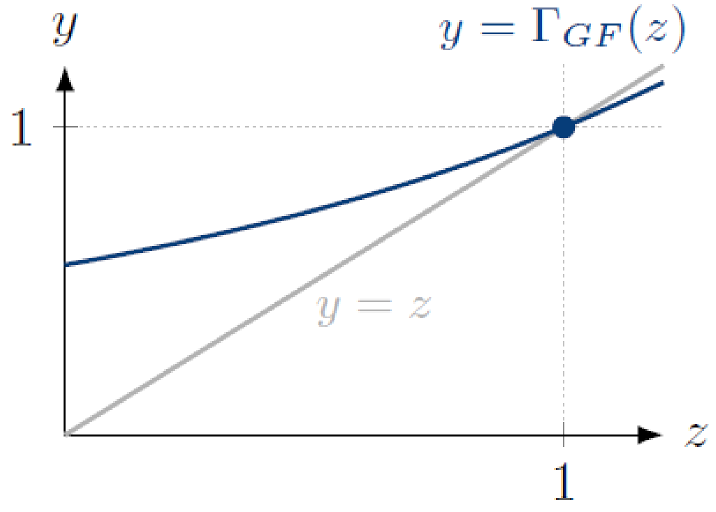
Visualization: Non-Trivial Root

non-trivial solution



(a) $\left.\frac{d\Gamma_{GF}(z)}{dz}\right|_{z=1} > 1.$

only trivial solution $1 = \Gamma_{\mathrm{GF}}(1)$



(b) $\left.\frac{d\Gamma_{GF}(z)}{dz}\right|_{z=1} \leq 1.$

Summary: Analysis with Geometric Approach

- Non-trivial solution

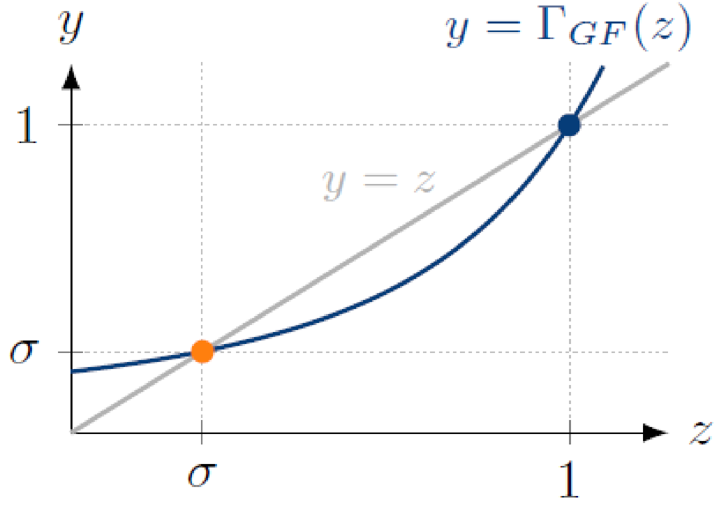
$$\frac{d}{dz}\Gamma_{GF}(z)\Big|_{z\to 1} = E[\Gamma] = \frac{1}{\rho} > 1$$

- or $\rho < 1$ (**stability condition**)

- State probability

$$\left.\begin{array}{l} x(j) = \sigma^j\ x(0) \\ \sum_{j=0}^{\infty} x(j) = 1 \end{array}\right\} \quad \to \quad x(j) = (1-\sigma)\,\sigma^j$$

(j=0, 1, ... ; ρ<1)



(a) $\left.\frac{d\Gamma_{GF}(z)}{dz}\right|_{z=1} > 1.$

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Scripts

An implementation of the models in the book is available as interactive notebooks. The scripts will help students to better understand the impact of parameters on performance characteristics, will avoid common pitfalls in the implementation, and provide means for numerical robust and efficient implementations for researchers in the domain.

Chapter 1: Introduction +

Chapter 2: Fundamentals +

Chapter 3: Stochastic Processes +

Chapter 4: Markovian Systems +

Chapter 5: Non-Markovian Systems −

- 5.2 Power method for DTMC [ipynb]
- 5.3 M/GI/1-S delay-loss system: power method and Eigenvalue problem [ipynb]
- 5.4 GI/M/1 delay system: geometric approach [ipynb]
- 5.5 Model with batch service and threshold control [ipynb]
- 5.6 Kingman's approximation [ipynb]

Chapter 6: Discrete-Time Analysis +

Chapter 7: Applications +

Chapter 5.4

GI/M/1 Delay System with Geometric Approach

> (c) Tobias Hossfeld (Aug 2021)
>
> This script and the figures are part of the following book. The book is to be cited whenever the script is used (copyright CC BY-SA 4.0):
> *Tran-Gia, P. & Hossfeld, T. (2021). Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.*
> https://doi.org/10.25972/WUP-978-3-95826-153-2

The geometric approach is used to analyze the condition. The state probability is

$x(j) = (1-\sigma)\sigma^j, j >= 0, \rho < 1$

The parameter σ can be determined numerically by solving the following equation. The random variable Γ is the number of requests that can be served during an interarrival time A. While A follows a general distribution, the service time B is described by an exponential distribution with rate μ, i.e. $B \sim \text{EXP}(\mu)$.

$z = \Gamma_{GF}(z)$

In the following we consider a uniform distribution in the interval $[0, 2/\lambda]$ for the interarrival time A with the mean value $E[A] = 1/\lambda$. It is $A \sim U(0, 2/\lambda)$. The Laplace transform of the continuous uniform distribution is

$\Phi_A(s) = \frac{e^{-sa}-e^{-sb}}{s(b-a)} = \frac{1-e^{-sb}}{s b}$

with $a = 0$ and $b = 2/\lambda$.

The generating function is obtained with the help of the Laplace transform of the uniform distribution.

$\Gamma_{GF}(z) = \phi_A(\mu(1-s))$

Now we need to solve

$z = \Gamma_{EF}(z)$.

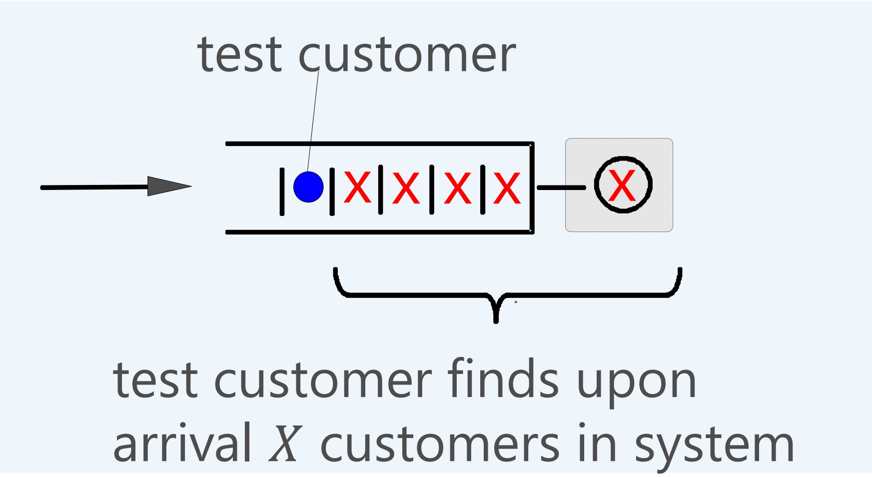
To do this, we calculate the solution of $\Gamma_{EF}(z) - z = 0$. There are numerical methods such as fsolve.

```
import numpy as np
import matplotlib.pyplot as plt

lam = 0.4
mu = 1

def gamEF(z, lam=0.5, mu=1):
    return (1-np.exp(-2/lam*mu*(1-z)))/(2/lam*mu*(1-z))

z = np.linspace(0,0.999,100)
plt.plot(z, gamEF(z, lam=lam), label='y=$\Gamma_{EF}(z)$')
plt.plot(z, z, label='y=z')
plt.grid()
```

Waiting Time Distribution

Waiting Time Distribution

- Derivation in the same way as the waiting time analysis of the M/M/n delay system



$$P(W>t \mid W>0) = \frac{P(W>t, W>0)}{P(W>0)}$$

$$= \frac{P(W>t)}{P(W>0)}$$

$$P(W>0) = \sum_{i=1}^{\infty} x(i)$$

$$= 1 - x(0) = \sigma$$

$$P(W>t) = \sum_{i=1}^{\infty} P(W>t \mid X=i) \cdot \underbrace{P(X=i)}_{(1-\sigma)\,\sigma^i}$$

$$= \sum_{i=1}^{\infty} P(W>t \mid X=i) \cdot (1-\sigma)\,\sigma^i$$

Waiting Time Distribution (f.)

$$P(W>t|W>0) = \frac{P(W>t, W>0)}{P(W>0)} = \frac{P(W>t)}{P(W>0)}$$

$$P(W>t) = \sum_{i=1}^{\infty} P(W>t\,|\,X=i)\cdot \underbrace{P(X=i)}_{(1-\sigma)\,\sigma^i} = \sum_{i=1}^{\infty} P(W>t\,|\,X=i)\cdot(1-\sigma)\,\sigma^i$$

$$P(W>0) = \sum_{i=1}^{\infty} x(i) = 1 - x(0) = \sigma$$

$$P(W>t\,|\,W>0) = \sum_{i=1}^{\infty} \underbrace{P(W>t\,|\,X=i)}_{\substack{\text{Erlang-i} \\ e^{-\mu t}\sum_{k=0}^{i-1}\frac{(\mu t)^k}{k!}}}\ (1-\sigma)\,\sigma^{i-1} = e^{-(1-\sigma)\,\mu t}$$

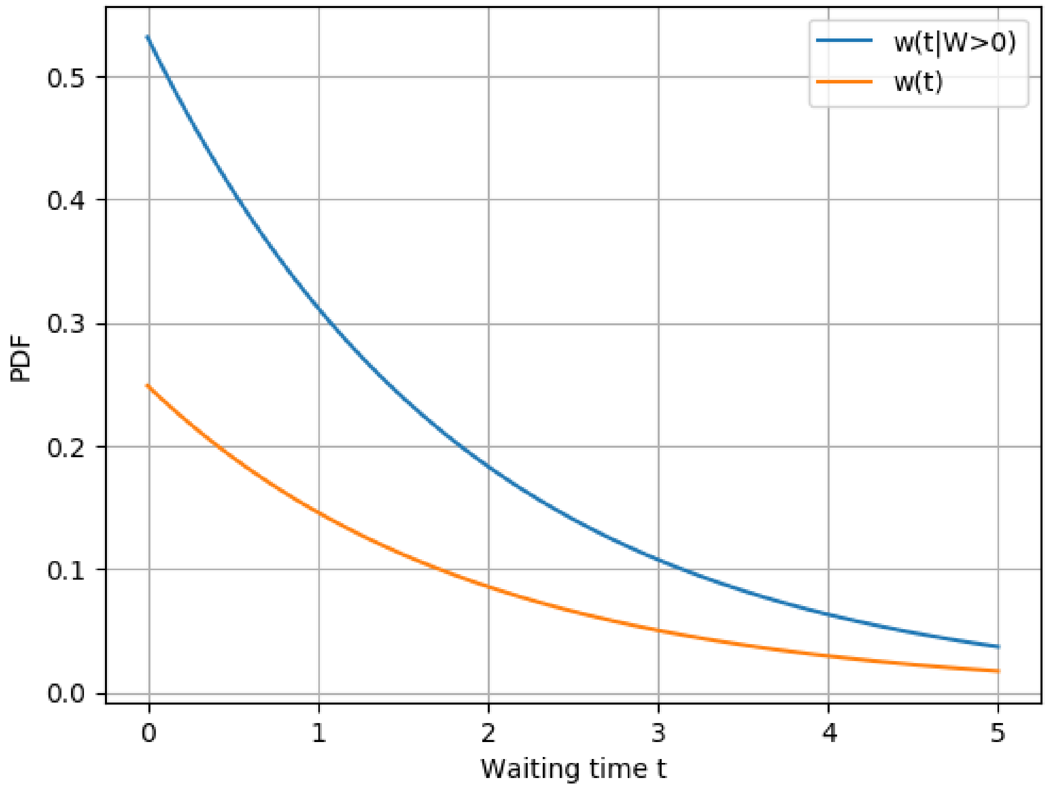
$$P(W>t) = P(W>t\,|\,W>0)\cdot P(W>0) = \sigma\, e^{-(1-\sigma)\,\mu\, t} = 1 - W(t)$$

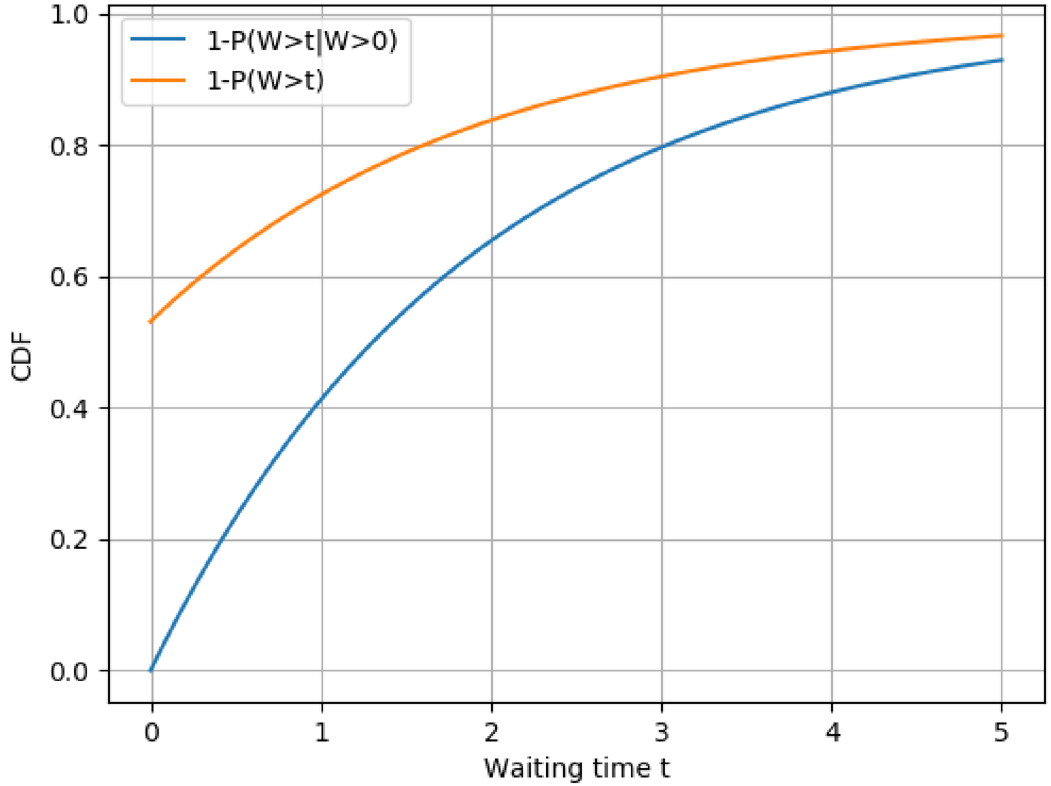
Waiting time of GI/M/1 delay system for all customers

$$W(t) = 1 - \sigma\cdot e^{-(1-\sigma)\mu t}$$

Example 1: Uniform Interarrival Times

- G/M/1 with $\rho = 0.6$
  - $B \sim Exp(\mu)$, $\mu = 1$
  - $A \sim U(0, \frac{2}{\lambda})$, $\lambda = 0.6$
- Numerically derived non-trivial root $\sigma = P(W > 0) = 0.4684$
- Mean values
  - $E[W|W > 0] = \frac{1}{(1-\sigma)\mu} = 1.88$
  - $E[W] = \frac{\sigma}{(1-\sigma)\mu} = 0.88$





Example 2: Parameter Study

- The higher the coefficient of variation, the higher the probability $P(W > t)$

- Note:
  - c_A =2 corresponds to $H_2/M/1$
  - $c_A = 1$ corresponds to M/M/1
  - c_A =0.5 corresponds to $E_2/M/1$
  - c_A =0 corresponds to D/M/1

- Waiting probability
  - GI/M/1: $p_W = \sigma$ depends on ρ and c_A
  - M/GI/1: $p_W = \rho$ is independent of c_B