Phuoc Tran-Gia and Tobias Hoßfeld

Performance Modeling and Analysis of Communication Networks

A Lecture Note

Version date: March 23, 2022
Online access: http://modeling.systems

4 Analysis of Markovian Systems

Question and Answer

Question 4.1. What is the "offered traffic" of an M/M/n-0 loss system? The arrival rate is λ, the service rate is μ. The blocking probability is p_B.

a. $(1 - p_B)\lambda$

b. $(1 - p_B)\lambda/\mu$

c. λ/μ

d. $\lambda/(n\mu)$

Question 4.2. What is the "carried traffic" of an M/M/n-0 loss system? The arrival rate is λ, the service rate is μ. The blocking probability is p_B.

a. Arrival rate of accepted customers

b. Utilization per server

c. Utilization of the entire system

d. Average number of occupied servers

Question 4.3. What is the blocking probability of an M/M/1-0 loss system with arrival rate λ and service rate μ? The r.v. X denotes the number of customers in the system.

a. $x(0)$

b. $x(1)$

c. $x(0) + x(1)$

d. $1 - x(1)$

Question 4.4. What is the blocking probability of an M/M/1-0 loss system with arrival rate λ and service rate μ? The offered traffic is denoted as a.

a. $\dfrac{a}{1+a}$

b. $\dfrac{1+a}{a}$

c. $\dfrac{\mu}{\lambda+\mu}$

d. $\dfrac{\lambda}{\lambda+\mu}$

Question 4.5. System I consists of two identical, but independent subsystems I.a and I.b which are both modeled as M/M/1-0 loss system with arrival rate λ and service rate μ. The blocking probability of the entire system I is $p_{B,I}$. The system operator thinks of implementing another system II instead of system I. The system II consists of two service units with service rate μ each: M/M/2-0. What is the blocking probability $p_{B,II}$ of the system II in relation to $p_{B,I}$?

a. $p_{B,I} > p_{B,II}$

b. $p_{B,I} < p_{B,II}$

c. $p_{B,I} = p_{B,II}$

d. $p_{B,I} = 2 \cdot p_{B,II}$

Question 4.6. Are the steady state probabilities $x(i)$ for an M/M/n-0 loss system also applicable to the following systems because of the robustness / insensitivity property?

a. M/M/n-∞

b. M/GI/n-0

c. M/M/∞-0

d. M/GI/∞-∞

Question 4.7. What is the “offered load” of an M/M/n-∞ delay system? The interarrival time A follows a Poisson process with rate $\lambda = 1/\mathrm{E}[\,A\,]$. The service time B is on average $\mathrm{E}[\,B\,]$ with the service rate $\mu = 1/\mathrm{E}[\,B\,]$.

a. $\lambda \cdot \mu$

b. $\lambda \cdot \mathrm{E}[\,B\,]$

c. $\lambda \cdot \mathrm{E}[\,B\,]/n$

d. λ/n

Question 4.8. What is the “utilization” of an M/M/n-∞ delay system with arrival rate λ and service rate μ?

a. $\lambda \cdot \mu$

b. $\lambda \cdot \mathrm{E}[\,B\,]$

c. $\lambda \cdot \mathrm{E}[\,B\,]/n$

d. λ/n

Question 4.9. Is the offered load a_∞ of an M/M/n-∞ delay system with arrival rate λ and service rate μ identical to the offered load a_0 of an M/M/n-0 loss system with λ and μ?

a. Yes, $a_\infty = a_0$

b. No, $a_\infty \neq a_0$

c. No, $a_\infty > a_0$

d. No, $a_\infty < a_0$

Question 4.10. Is the utilization U_∞ of an M/M/n-∞ delay system with arrival rate λ and service rate μ identical to the utilization U_0 of an M/M/n-0 loss system with λ and μ?

a. Yes, $U_\infty = U_0$

b. No, $U_\infty \neq U_0$

c. No, $U_\infty > U_0$

d. No, $U_\infty < U_0$

Question 4.11. Under which conditions is the M/M/n-∞ delay system not stable?

a. $n/a < 1$

b. $n/a > 1$

c. $a < 1$

d. $a < n$

Question 4.12. What is the relation between the waiting time W of all customers in an M/M/n delay system and the waiting time W_1 of waiting customers? The waiting probability of customers is p_W.

a. $W = W_1$

b. $W = (1 - p_W) \cdot 0 + p_W \cdot W_1$

c. $W \sim W_1 + D(0)$

d. $W \sim \mathrm{MIX}((0, W_1), (1 - p_W, p_W))$

Question 4.13. What is the waiting probability p_W of an M/M/1 delay system with arrival rate λ and service rate μ?

a. $p_W = 0$

b. $p_W = \lambda$

c. $p_W = \mu$

d. $p_W = \lambda/\mu$

Question 4.14. Consider an M/M/1 delay system with utilization $\rho < 1$. Which distribution does the number X of customers in the system follow?

a. $X \sim \mathrm{EXP}(\rho)$

b. $X \sim \mathrm{EXP}(1 - \rho)$

c. $X \sim \mathrm{GEOM}(\rho)$

d. $X \sim \mathrm{GEOM}(1 - \rho)$

Question 4.15. The waiting time W_1 of waiting customers in an M/M/n delay system follows an exponential distribution: $W_1 \sim \mathrm{EXP}\left((1 - \rho)n\mu\right)$. What is the mean waiting time of waiting customers?

a. $(1 - \rho)n\mu$

b. $(\mu - \lambda)n$

c. $\frac{1-\rho}{n\mu}$

d. $\frac{1}{(1-\rho)n\mu}$

Question 4.16. What is the mean waiting time $\mathrm{E}[\,W_1\,]$ of waiting customers in an M/M/10 delay system with arrival rate $\lambda = 20\,\mathrm{s}^{-1}$ and service rate $\mu = 1\,\mathrm{s}^{-1}$?

a. 5

b. 50

c. 500

d. ∞

Question 4.17. The robustness property applies to the Engset model (M/M/n/0/m) for a finite number m of customers with the offered traffic a^* of an idle customer. The steady state probabilities are $x(i)$. The blocking probability is p_B. What does the robustness property mean in that context?

a. System is stable for $a^* > 0$.

b. Erlang-B formula can be used for p_B.

c. p_B is always less than 10^{-3}.

d. $x(i)$ applies to M/GI/n/0/m.

Question 4.18. Consider a system with steady state probabilities $x^*(i)$ as seen by a random observer. The probability of the state seen by an arriving customer is $x_A(i)$. The probability of the state seen by a departing customer is $x_D(i)$. What does the PASTA property mean?

a. $x_A(i) = x_D(i)$

b. $x^*(i) = x_D(i)$

c. $x^*(i) = x_A(i)$

d. $x_A(i) = x_D(i) = x^*(i)$

Question 4.19. Is the PASTA property valid for the Engset model (M/M/n/0/m) with exponentially distributed interarrival times (IATs)? The notation from Question 4.18 is used.

a. Yes, $x_A(i) = x^*(i)$.

b. No, this is not a Poisson process.

c. Yes, IATs are exponentially distributed.

d. No, it is a loss system.

Question 4.20. For systems with a finite number of sources, a random source upon arrival observes the state of the system as if the source itself does not belong to the system. How is this called in literature?

a. Arrival theorem

b. PASTA property

c. Kleinrock's (Burke's) result

d. Palm-Khintchine theorem

Question 4.21. Consider an Engset model (M/M/n/0/m) with $n = 10\,000$, offered traffic $a^* = 10$ of an idle customer, $m = 2000$. What is the blocking probability p_B?

a. Compute p_B with Erlang-B formula.

b. Compute p_B with Engset formula.

c. It is $p_B = 1 - m/n$.

d. It is $p_B = 0$.

Question 4.22. How many states has an M/M/2-1 delay-loss system?

a. 1

b. 2

c. 3

d. 4

Exercises and Problems

Problem 4.1. Macro state of an M/M/n Loss System We consider the state transition diagram of an M/M/n loss system with arrival rate λ and service rate μ. Let us consider the macro state S which consists of the micro states $\{X = 0, 1, \ldots, n-1\}$, as depicted in Figure 4.1.

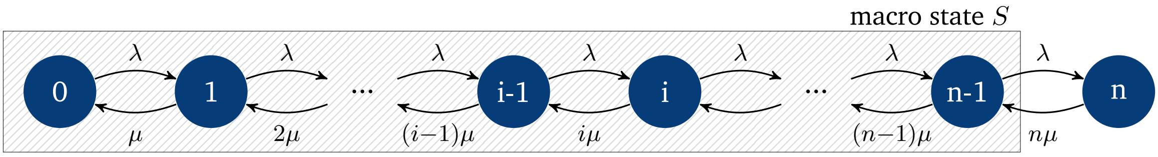

Figure 4.1: State transition diagram of an M/M/n loss system with macro state S.

Let $p(S)$ be the probability that the system is in the macro state S, i.e., $p(S) = \sum_{i=0}^{n-1} x(i)$. The blocking probability is $p_B = x(n)$. Consider the following equation system.

$$\lambda \cdot p(S) = n\mu \cdot x(n) \tag{4.1a}$$

$$p(S) + x(n) = 1 \tag{4.1b}$$

Can you compute the blocking probability by solving the equation system (4.1)? Why?

Problem 4.2. Dimensioning a Game Server We consider a massively multiplayer online game (MMOG) in which players can connect to a game server to join the MMOG. The game is designed for casual gamers. A player logs in on average three times a week for an average of $90\,\mathrm{min}$ each time. There are 3360 players which are registered at the server. These 3360 players are not necessarily all playing the game at the same time, but there are 3360 potential players on this server. It is valid to assume an infinite number of sources. Measurements by the operator show that the player arrival process can be modeled by a Poisson process. If there are n players actively playing on the game server, login requests from other players will be blocked.

4.2.1. What is the offered traffic at the server?

4.2.2. How to dimension the server properly? Specify how many active players can be served simultaneously, under the condition that a player will be rejected with at most 1% chance. ☞ *Hint:* Use the corresponding figure in the book for dimensioning loss systems.

Problem 4.3. Loss System with Finite Sources Consider an M/M/n loss system with a finite number q of sources. It is $q \geq n$. The sources are identical and have a call rate α per source. The service rate of each of the n identical service units is μ.

4.3.1. Define appropriate system states and sketch the state transition diagram.

4.3.2. Define suitable macro states and determine the steady state probabilities.

4.3.3. Show the Erlang formula for loss systems by taking the limit of the solution in the previous Problem 4.3.2.: $\lim\limits_{\substack{q\to\infty \\ \alpha\to 0}} (q\alpha) = \lambda$.

Problem 4.4. Cellular IoT System with Finite Number of Sensors A cellular communication system is used to deploy an IoT system. Each cell is served by an IoT gateway, which is connected to the mobile core network of the mobile provider. The IoT sensors located within a cell are attached to the corresponding gateway. Thereby, the IoT devices are sensing the environment. There are two different sensing modes: L (low resolution) and H (high resolution). In state H, a data packet of $256\,\mathrm{bit}$ is sent every $20\,\mathrm{ms}$ from the IoT sensor to the gateway. In state L, a data packet of $96\,\mathrm{bit}$ is sent every $20\,\mathrm{ms}$. The transition rate from state L to H is λ; the transition rate from H to L is μ. The time per state is assumed to be exponentially distributed. For dimensioning the required radio access resources, we assume that there are n active IoT sensors within a cell.

4.4.1. The scenario can be modeled as a birth-and-death process. How can you properly describe the state of a single cell? Specify the state space and sketch the state transition diagram with its transition rates.

4.4.2. Use appropriate macro states to determine the steady state probabilities.

4.4.3. Compute the expected data rate in a single cell with $\lambda = 0.1\,\mathrm{s}^{-1}$, $\mu = 1.5\lambda$, $n = 10$.

Problem 4.5. Non-stationary M/M/1 Loss System Consider the M/M/1-0 loss system in the non-stationary case. The arrival process has rate λ and the service process has rate μ.

4.5.1. What are the non-stationary state equations $x(j,t)$ at time t for the possible states j?

4.5.2. Solve the state equations assuming that the system is empty at time $t_0 = 0$, i.e., the state $[X(t_0) = 0]$ is observed with probability 1: $x(0, t_0) = 1$ and $x(1, t_0) = 0$.

☞ *Hint:* The solution of the differential quation $f'(t) = a + b \cdot f(t)$ with $f(0) = 0$ is

$$f(t) = \frac{a(e^{bt} - 1)}{b}.$$

4.5.3. Show that the formula for the stationary case follows for $t \to \infty$:

$$x(0) = \frac{\mu}{\lambda + \mu}\,, \qquad x(1) = \frac{\lambda}{\lambda + \mu}\,.$$

Problem 4.6. M/M/n-S Delay-Loss System Consider the delay-loss system $M/M/n - S$ with n servers and S waiting places.

4.6.1. Specify the state space of the system and sketch the state transition diagram with the corresponding transition rates.

4.6.2. Give the micro state equations.

4.6.3. Develop the macro state equations and calculate the corresponding steady state probabilities depending on $a = \frac{\lambda}{\mu}$ and $\rho = \frac{a}{n}$ in for $\rho < 1$.

4.6.4. Calculate the following performance characteristics:

(a) blocking probability p_B, i.e. the probability that an incoming request will be rejected,

(b) waiting probability p_W,

(c) carried traffic Y, i.e., the mean number of occupied servers in the system,

(d) utilization per server ρ^*,

(e) mean queue length Ω,

(f) mean waiting time considering all customers $\mathrm{E}[W]$,

(g) mean waiting time considering waiting customers $\mathrm{E}[W_1]$.

4.6.5. Derive the cumulative distribution function $W_1(t)$ of the waiting time of waiting customers assuming the FIFO discipline.

4.6.6. Consider the M/M/5-10 delay loss system with offered traffic $a = \lambda/\mu = 3$. For all $n = 5$ servers, the service rate $\mu_i = \mu$ is identical. Compare the following two alternatives assuming that there are $S = 10$ waiting spaces in total.

System 1. Central storage for all n servers, i.e., a single queue with S waiting places, as indicated in Figure 4.2:

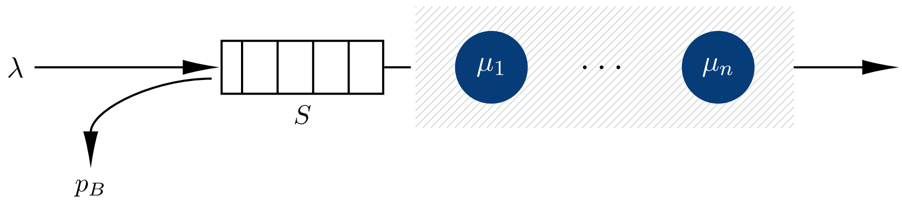

Figure 4.2: Central storage for all n servers.

System 2. Local storage per server, i.e., a waiting queue for each server with $\tilde{S} = \frac{S}{n}$ waiting places. The job requests are randomly arriving at a queue with probability $p = \frac{1}{n}$, see Figure 4.3.

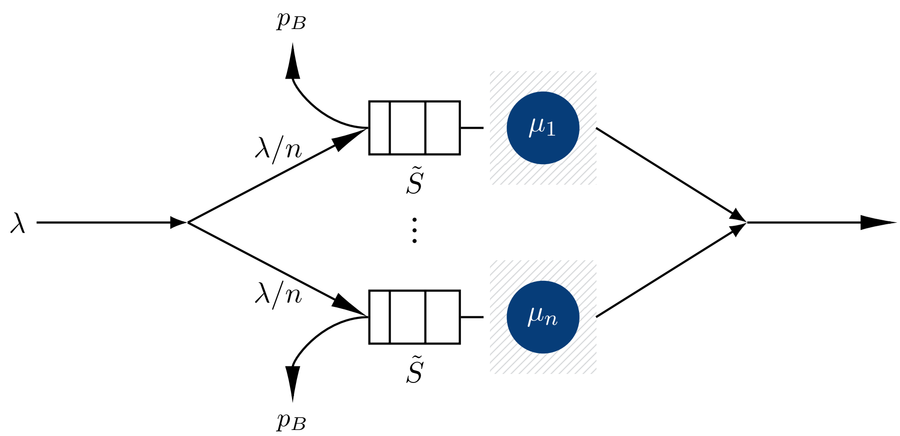

Figure 4.3: Local storage per server.

Compute the blocking probability p_B for both cases. How to dimension the number of waiting places in $\tilde{S}$ for the local storage system 2, such that the blocking probabilities for system 1 and system 2 are similar?

Problem 4.7. Delay-Loss System with Bulk Service Consider an $M/M^{[2,2]}/2-3$ queueing system, which is used for modeling a production system. Arrivals of job requests follow a Poisson process with arrival rate λ and are served in batches. There is a fixed number of two jobs per batch. Each job is then processed by one of the two service units. The service time for simultaneously processing two jobs is exponentially distributed with service rate μ. Note that two jobs are always processed together. If there is only one job in the waiting queue, which consists of three waiting places, the system waits for another job to arrive.

4.7.1. Define a suitable state description for this queueing system.

4.7.2. Sketch the state transition diagram with its transition rates.

4.7.3. Is it a birth-and-death process?

4.7.4. Give the stationary equation system for micro states.

4.7.5. Use appropriate macro states to determine the state probabilities depending on $a = \lambda/\mu$.

4.7.6. Derive the blocking probability p_B depending on a.

4.7.7. What is the utilization ρ^* of the entire system?

4.7.8. What is the mean waiting time $\mathrm{E}[\,W\,]$ of customers depending on $1/\mu$?

4.7.9. Plot the performance measures p_B, ρ^*, $\mathrm{E}[\,W\,]$ for $\rho = \frac{a}{n} \in (0, 1.5]$ with $n = 4$.
Compare p_B, ρ^*, $\mathrm{E}[\,W\,]$ with the performance of the corresponding M/M/n-S system.

☞ *Hint:* Performance characteristics of the M/M/n-S system.

$$p_B = \frac{\frac{a^n}{n!} \cdot \rho^S}{\sum\limits_{i=0}^{n-1} \frac{a^i}{i!} + \frac{a^n}{n!} \cdot \frac{1-\rho^{S+1}}{1-\rho}}$$

$$\Omega = \frac{a^n \cdot \rho}{n!(1-\rho)^2} \cdot \frac{(1+\rho^S(S\cdot(\rho-1)-1))}{\sum\limits_{i=0}^{n-1} \frac{a^i}{i!} + \frac{a^n}{n!}\frac{1-\rho^{S+1}}{1-\rho}}$$

$$\rho^* = \frac{(1-p_B)\cdot a}{n}$$

Problem 4.8. Output Process and Blocked Customers Consider the M/M/1-S queueing system. The arrival process is a Poisson process with rate λ. The service time is exponentially distributed with parameter μ. The output process of the system is denoted by P_1 and the process of rejected requests by P_2. Show that the processes P_1 and P_2 are Markovian arrival processes.

Problem 4.9. Two Servers vs. High Capacity Server The following two delays systems are to be examined. An infinite waiting room is assumed.

Two servers system $M/M/2_\mu$: A system with two servers, each with service rate μ.

High capacity server $M/M/1_{2\mu}$: Single server with service rate 2μ.

4.9.1. Sketch the state transition diagrams for both systems.

4.9.2. For both systems, derive the probability $x(0)$ that the system is empty and the waiting probability p_W depending on the normalized offered load ρ.

4.9.3. For both systems, derive the expected waiting time $\mathrm{E}[\,W\,]$ of all jobs and the expected response time $\mathrm{E}[\,D\,]$ depending on ρ and μ.

4.9.4. For which range $0 < \rho < 1$ is the two servers system $M/M/2_\mu$ preferable to the high capacity server $M/M/1_{2\mu}$ in terms of average waiting time $\mathrm{E}[\,W\,]$? What can you observe regarding the mean response time $\mathrm{E}[\,D\,]$? Describe and explain the result.

Problem 4.10. Loss System with Two Service Classes Consider a pure loss system with n identical service units, each of which requires an exponentially distributed service time B in order to serve a job. The mean service time $\mathrm{E}[B]$ is μ^{-1}. Incoming jobs belong to two independent classes. Jobs in the first class require one service unit. Jobs in the second class require two service units for processing, with processing on the two processors being assumed to be independent of each other. The interarrival times of the jobs of both classes are exponentially distributed with arrival rates λ_1 and λ_2, respectively. If only *one* service unit is free and a second class job arrives, this job cannot occupy this service unit and is rejected.

4.10.1. Specify the state space and sketch the state transition diagram with its transition rates.

4.10.2. Is it a birth-and-death process? Why?

4.10.3. Give the stationary equation system for micro states.

4.10.4. Use appropriate macro states to determine the steady state probabilities.

In the following, $n = 3$ and $\lambda_1 = \lambda_2$. Further, we define $\lambda = \lambda_1 + \lambda_2$, $a = \lambda/\mu$.

4.10.5. Calculate the steady state probabilities.

4.10.6. Derive the blocking probabilities p_{B_1} and p_{B_2} for jobs of class 1 and class 2, respectively. Give p_{B_1} and p_{B_2} depending on a.

4.10.7. What is the mean number $\mathrm{E}[\,X\,]$ of occupied service units?

Problem 4.11. Machines with Repairs An industrial production hall has N failure-prone machines which are looked after by a mechanic. The mechanic's job is to repair failed machines in the order in which they failed. If a machine fails and the mechanic is busy, this machine has to wait before it is put back into operation. The random variable A indicates the time between the start of repair and the subsequent failure. The repair time B corresponds to the mechanic's service time to repair the machine. The repair time is the same for all machines and follows an exponential distribution with mean value $\mathrm{E}[B] = \mu^{-1}$. The interarrival time of failures is the same for all machines. They are independently and identically distributed and follow an exponential distribution with $\mathrm{E}[A] = \lambda^{-1}$.

4.11.1. Give two alternative one-dimensional state descriptions of the system.

4.11.2. Sketch the state transition diagram.

4.11.3. What are the equation systems of the steady state probabilities for both state descriptions? Derive the steady state probabilities depending on λ, μ and N.

4.11.4. How busy is the mechanic?

Problem 4.12. Football Stadium During a football game, fans get thirsty and go to one of the beverage stands to order a refreshing drink. We assume that the fans arrive at a single beverage stand according to a Poisson process with arrival rate λ. The time to prepare and sell the drink is exponentially distributed with service rate μ. Two different cases S1 and S2 are considered.

S1 If the football game is very exciting and thrilling, the fans won't wait at the beverage stand and immediately go back instead to their seat in the stadium.

S2 If the game is "characterized by tactics in midfield", the fans will wait to get some drinks. However, there is only space for three waiting fans and the fan currently being served.

Determine for both systems respectively:

4.12.1. Kendall's notation.

4.12.2. State space and state transition diagram.

4.12.3. Equation system for the micro states.

4.12.4. Probability that the beverage stand is empty for $\lambda = 10\,\mathrm{min}^{-1}$ and $\mu = 2\,\mathrm{min}^{-1}$.

Problem 4.13. Dimensioning a DHCP Server The Dynamic Host Configuration Protocol (DHCP) allows computers on a network to be dynamically assigned an IP address. A computer that needs an IP address requests the associated DHCP server. The DHCP server then automatically assigns an IP address to the requesting client. When the client disconnects from the network, the IP address is released and available for other requesting clients.

We consider the following fictitious scenario. The chair of communication networks is providing a DHCP server. On average, four students arrive per hour and start surfing in the Internet. The arrival process is modeled as a Poisson process. An IP address is then occupied on average for $4\,\mathrm{h}$ and the occupancy time is exponentially distributed.

4.13.1. What is Kendall's notation of the system?

4.13.2. Sketch the state transition diagram including all rates.

4.13.3. What is the offered traffic of the system? What is the probability that a student's device is not assigned an IP address? What is the mean number of occupied IP addresses?

4.13.4. How many IP addresses must be available so that, on average, a student's device does not get an IP address assigned more than once in 100 attempts?

Now, we assume that the institute of computer science runs a central DHCP server, instead of each chair running its own server. There are 15 chairs at the institute. Each chair provides 20 IP addresses which are then available in the pool of IP addresses of the central DHCP server. The arrival rate of students and the mean occupancy time are the same for all chairs as described above.

4.13.5. What is the offered traffic and the carried traffic?

4.13.6. What is the blocking probability?

4.13.7. How many IP addresses does each chair have to provide now to guarantee that every student has a probability of at most 1% of not getting an address?

4.13.8. Compare the results for the central DHCP server with the results in 4.13.4. How do you interpret the differences?

☞ *Hint:* Calculating $\frac{a^k}{k!}$ is numerically inefficient if you first calculate a^k and $k!$ and then do the division. Use the relation $\frac{a^k}{k!} = \frac{a}{1} \cdot \frac{a}{2} \cdot \frac{a}{3} \cdot \ldots \cdot \frac{a}{k}$.