

Chapter 5.6

Results for Continuous-Time GI/GI/1- ∞

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

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*Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication
Networks - A Lecture Note. Würzburg University Press.
<https://doi.org/10.25972/WUP-978-3-95826-153-2>*

Website to download book, exercises, slides and scripts:
<https://modeling.systems/>

Chapter 5

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5.6.1 Characteristics of GI/GI/1 Delay Systems

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Characteristics of GI/GI/1 Delay Systems

► **Stability condition** $\rho = \frac{E[B]}{E[A]} = \lambda E[B] < 1$

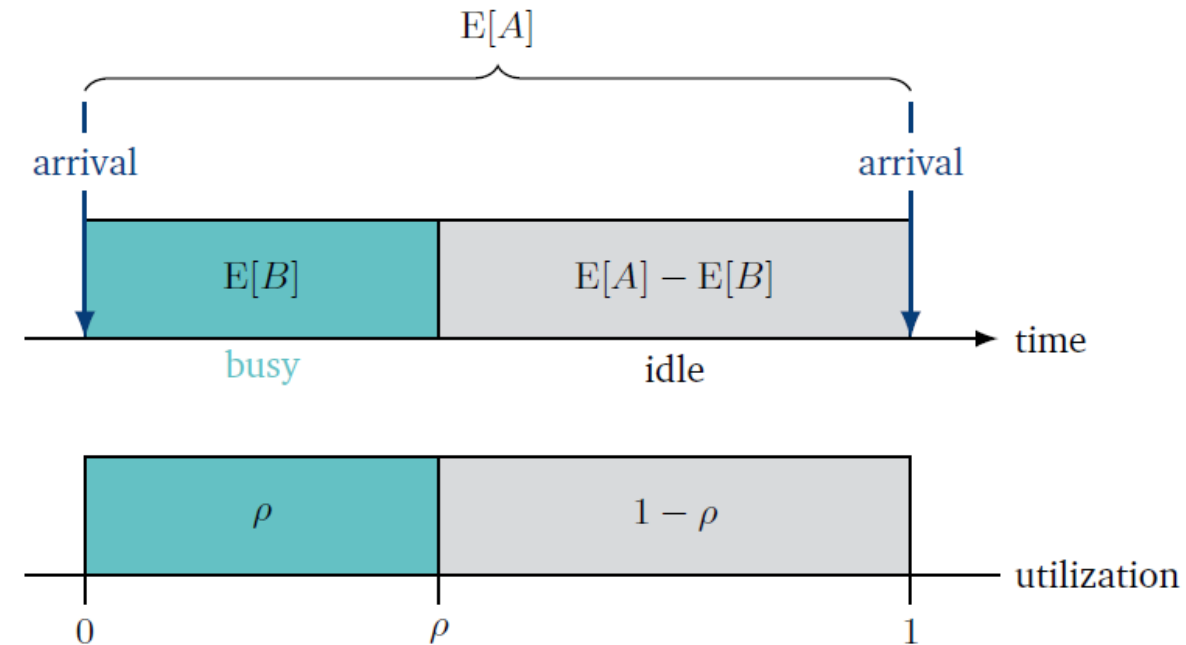
► **Utilization** of the server $\rho = \lambda E[B]$

► Probability that **server is idle** =
server is empty at arbitrary point in time

$$x(0) = P(X = 0) = 1 - \rho = \frac{E[A] - E[B]}{E[A]}$$

► System state from perspective of arrivals

$$x_A(0) = P(X_A = 0) = \frac{(1 - \rho)E[A]}{E[I]} = \frac{E[A] - E[B]}{E[I]}$$



Characteristics of GI/GI/1 Delay Systems (f.)

- ▶ **Waiting probability** $p_w = 1 - x_A(0) = \frac{E[I] - (E[A] - E[B])}{E[I]}$
- ▶ Mean number of customers in the system due to Little's law
 $E[X] = \lambda(E[W] + E[B])$
- ▶ **Mean waiting time** $E[W] = E[X] \cdot E[A] - E[B]$
- ▶ Note: There are no simple equations available for $x_A(0), E[I], E[X]$
- ▶ Solution of Lindley integral equation required to derive those characteristics

$$W_{n+1} = \max(W_n + B_n - A_n, 0)$$

$$W = \max(W + B - A, 0) = \max(W + C, 0)$$

**derivation of waiting
times for discrete-time
GI/GI/1 queue**

Lindley Integral Equation for GI/GI/1 Systems

- ▶ For a GI/GI/1 system under stationary conditions, the following functional relationship for the waiting time distribution function are obtained $W(t)$: **Lindley integral equation**

$$W(t) = \begin{cases} 0 & t < 0 \\ W(t) * c(t) & t \geq 0 \end{cases}$$

where

$$c(t) = b(t) * a(-t)$$

- ▶ **System function** $C = B - A$ (r.v.) contains all parameters of stochastic process of GI/GI/1 queue

- ▶ Probability density function

$$w(t) = \begin{cases} 0 & t < 0 \\ \delta(t) \int_{-\infty}^{0^+} (w(u) * c(u)) du & t = 0 \\ w(t) * c(t) & t > 0 \end{cases}$$

Compact notation $w(t) = \pi_0(w(t) * c(t))$

$$\text{with } \pi_0(f(t)) = \begin{cases} 0 & t < 0 \\ \delta(t) \int_{-\infty}^{0^+} f(u) du & t = 0 \\ f(t) & t > 0 \end{cases}$$

Kingman's Approximation of Mean Waiting Times

- Kingman provides an **approximation** for the mean waiting time: **Kingman's formula**

$$E[W] \approx \left(\frac{\rho}{1-\rho} \right) \left(\frac{c_A^2 + c_B^2}{2} \right) E[B] \stackrel{\text{def}}{=} \widetilde{W}$$

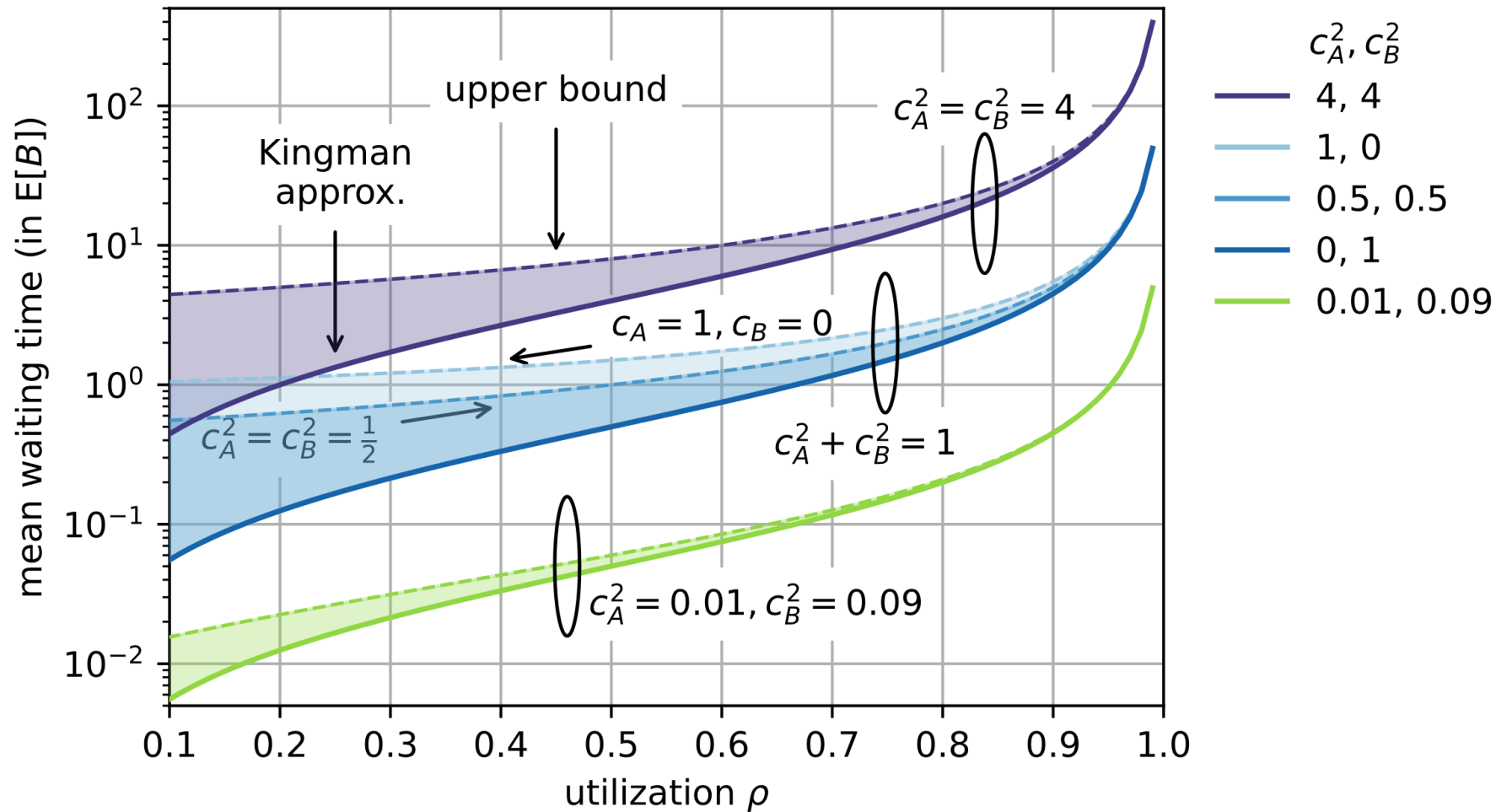
- For a Poisson process, Kingman's formula is exact $E[W] = \left(\frac{\rho}{1-\rho} \right) \left(\frac{1 + c_B^2}{2} \right) E[B] = \widetilde{W}$
- Tighter upper bound** of the mean waiting time is provided by Daley

$$E[W] \leq \frac{(2-\rho)c_A^2 + \rho c_B^2}{2(1-\rho)} \cdot E[B] \stackrel{\text{def}}{=} \widehat{U}$$

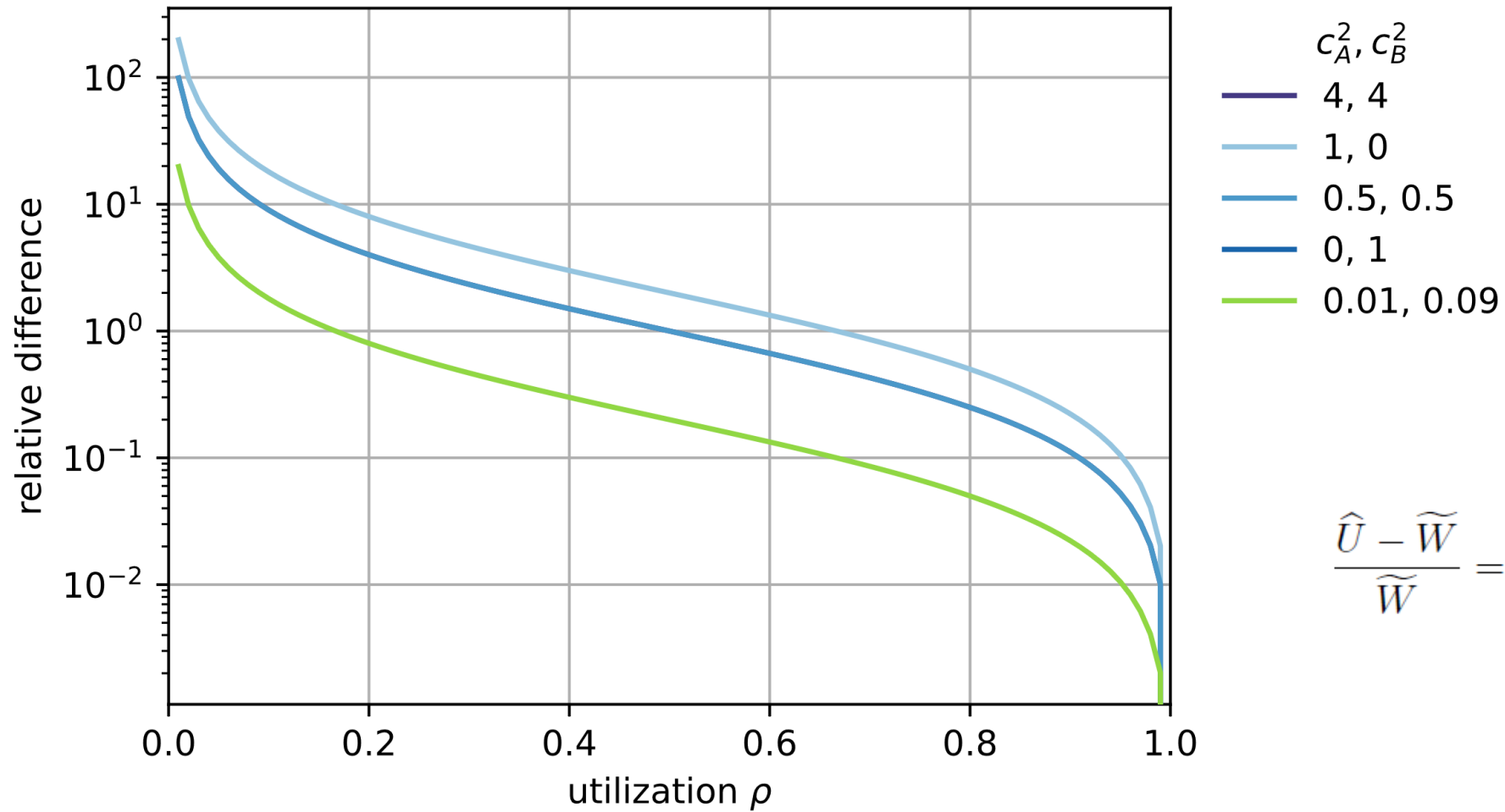
- Difference between upper bound and approximated mean $\widehat{U} - \widetilde{W} = E[B] \cdot c_A^2$

- Relative difference $\frac{\widehat{U} - \widetilde{W}}{\widetilde{W}} = \frac{2c_A^2}{c_A^2 + c_B^2} \cdot \left(\frac{1-\rho}{\rho} \right)$

Kingman's Approximation and Tight Upper Bounds



Relative Difference



$$\frac{\hat{U} - \widetilde{W}}{\widetilde{W}} = \frac{2c_A^2}{c_A^2 + c_B^2} \cdot \left(\frac{1 - \rho}{\rho} \right)$$