Chapter 5.6

Results for Continuous-Time GI/GI/1-∞

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

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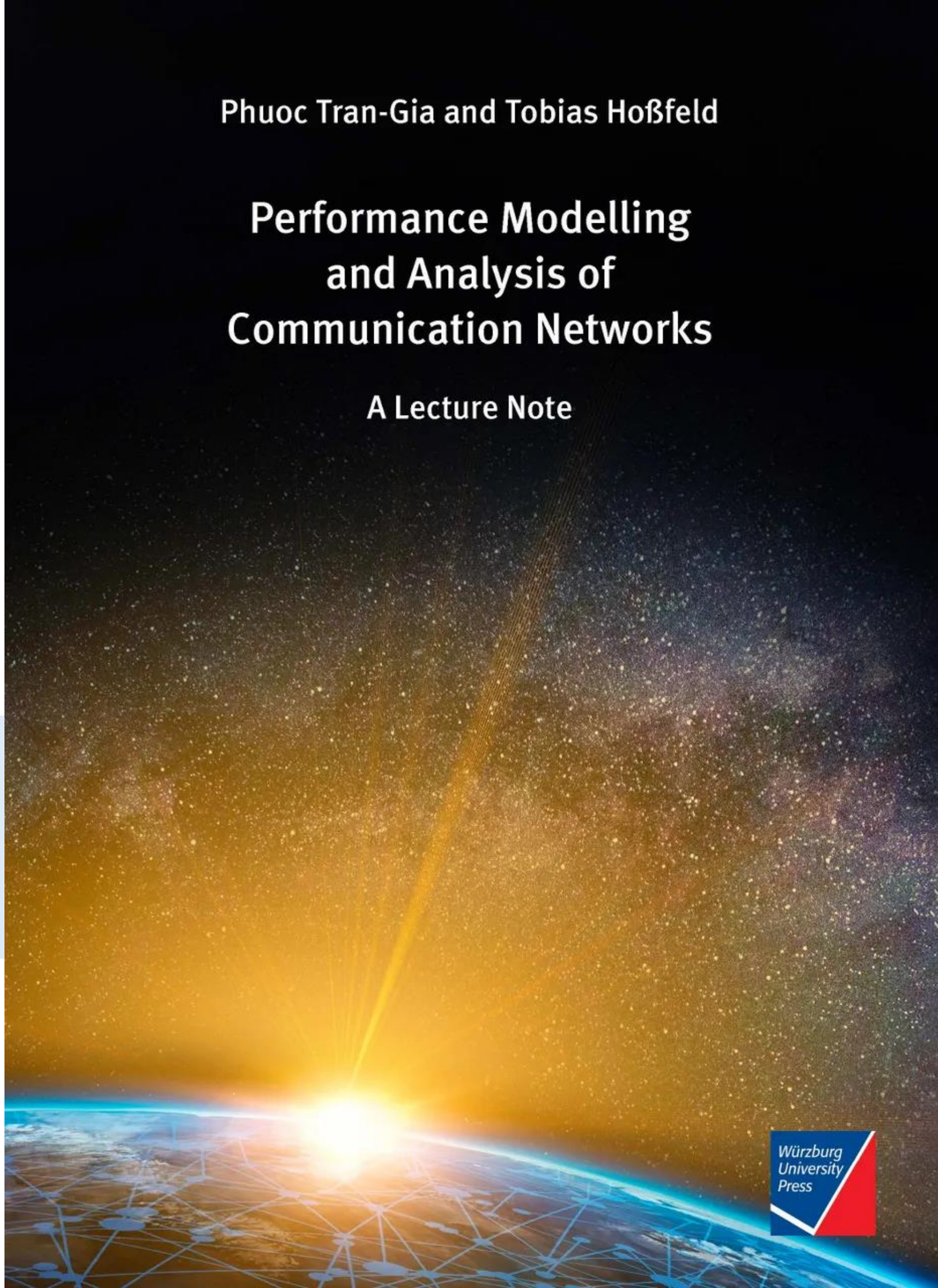
The book must be cited and the disclaimer attached when using lectures slides or scripts.

Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/



Chapter 5

5 Analysis of Non-Markovian Systems

Characteristics of GI/GI/1 Delay Systems

- **Stability condition** $\rho = \frac{\mathrm{E}[B]}{\mathrm{E}[A]} = \lambda \mathrm{E}[B] < 1$

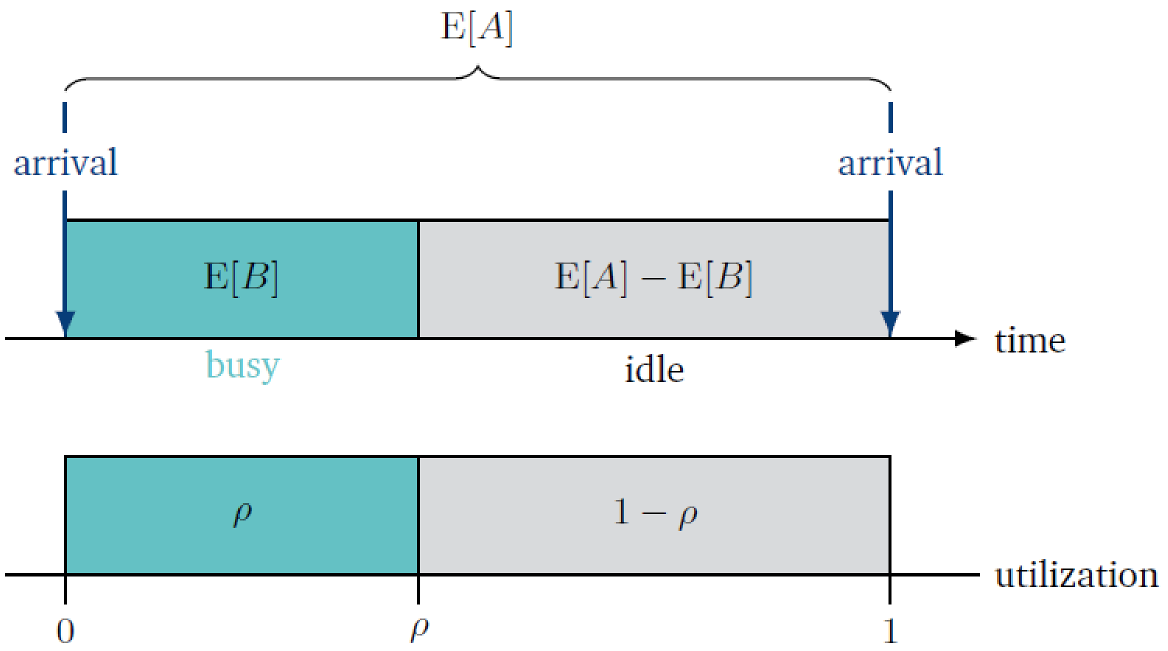
- **Utilization** of the server $\rho = \lambda E[B]$

- Probability that **server is idle** =
  server is empty at arbitrary point in time

  $$x(0) = P(X=0) = 1-\rho = \frac{\mathrm{E}[A]-\mathrm{E}[B]}{\mathrm{E}[A]}$$

- System state from perspective of arrivals

  $$x_A(0) = P(X_A=0) = \frac{(1-\rho)\mathrm{E}[A]}{\mathrm{E}[I]} = \frac{\mathrm{E}[A]-\mathrm{E}[B]}{\mathrm{E}[I]}$$



Characteristics of GI/GI/1 Delay Systems (f.)

- **Waiting probabilit** $p_w = 1 - x_A(0) = \dfrac{\mathrm{E}[I] - (\mathrm{E}[A] - \mathrm{E}[B])}{\mathrm{E}[I]}$

- Mean number of customers in the system due to Little's law

  $\mathrm{E}[X] = \lambda(\mathrm{E}[W] + \mathrm{E}[B])$

- **Mean waiting time** $\mathrm{E}[W] = \mathrm{E}[X] \cdot \mathrm{E}[A] - \mathrm{E}[B]$

- Note: There are no simple equations available for $x_A(0), E[I], E[X]$

- Solution of Lindley integral equation required to derive those characteristics

$$W_{n+1} = \max(W_n + B_n - A_n, 0)$$

$$W = \max(W + B - A, 0) = \max(W + C, 0)$$

derivation of waiting times for discrete-time GI/GI/1 queue

Lindley Integral Equation for GI/GI/1 Systems

- For a GI/GI/1 system under stationary conditions, the following functional relationship for the waiting time distribution function are obtained $W(t)$: **Lindley integral equation**

$$W(t) = \begin{cases} 0 & t < 0 \\ W(t) * c(t) & t \geq 0 \end{cases}$$

where

$$c(t) = b(t) * a(-t)$$

- **System function** $C = B - A$ (r.v.) contains all parameters of stochastic process of GI/GI/1 queue

- Probability density function

$$w(t) = \begin{cases} 0 & t < 0 \\ \delta(t) \int_{-\infty}^{0^+} \Big(w(u) * c(u)\Big) du & t = 0 \\ w(t) * c(t) & t > 0 \end{cases}$$

Compact notation $w(t) = \pi_0\Big(w(t) * c(t)\Big)$

with $\pi_0\Big(f(t)\Big) = \begin{cases} 0 & t < 0 \\ \delta(t) \int_{-\infty}^{0^+} f(u) du & t = 0 \\ f(t) & t > 0 \end{cases}$

Kingman's Approximation of Mean Waiting Times

- Kingman provides an **approximation** for the mean waiting time: **Kingman's formula**

$$\mathrm{E}[W] \approx \left(\frac{\rho}{1-\rho}\right)\left(\frac{c_A^2+c_B^2}{2}\right)\mathrm{E}[B] \stackrel{\text{def}}{=} \widetilde{W}$$

- For a Poisson process, Kingman's formula is exact $\mathrm{E}[W] = \left(\frac{\rho}{1-\rho}\right)\left(\frac{1+c_B^2}{2}\right)\mathrm{E}[B] = \widetilde{W}$

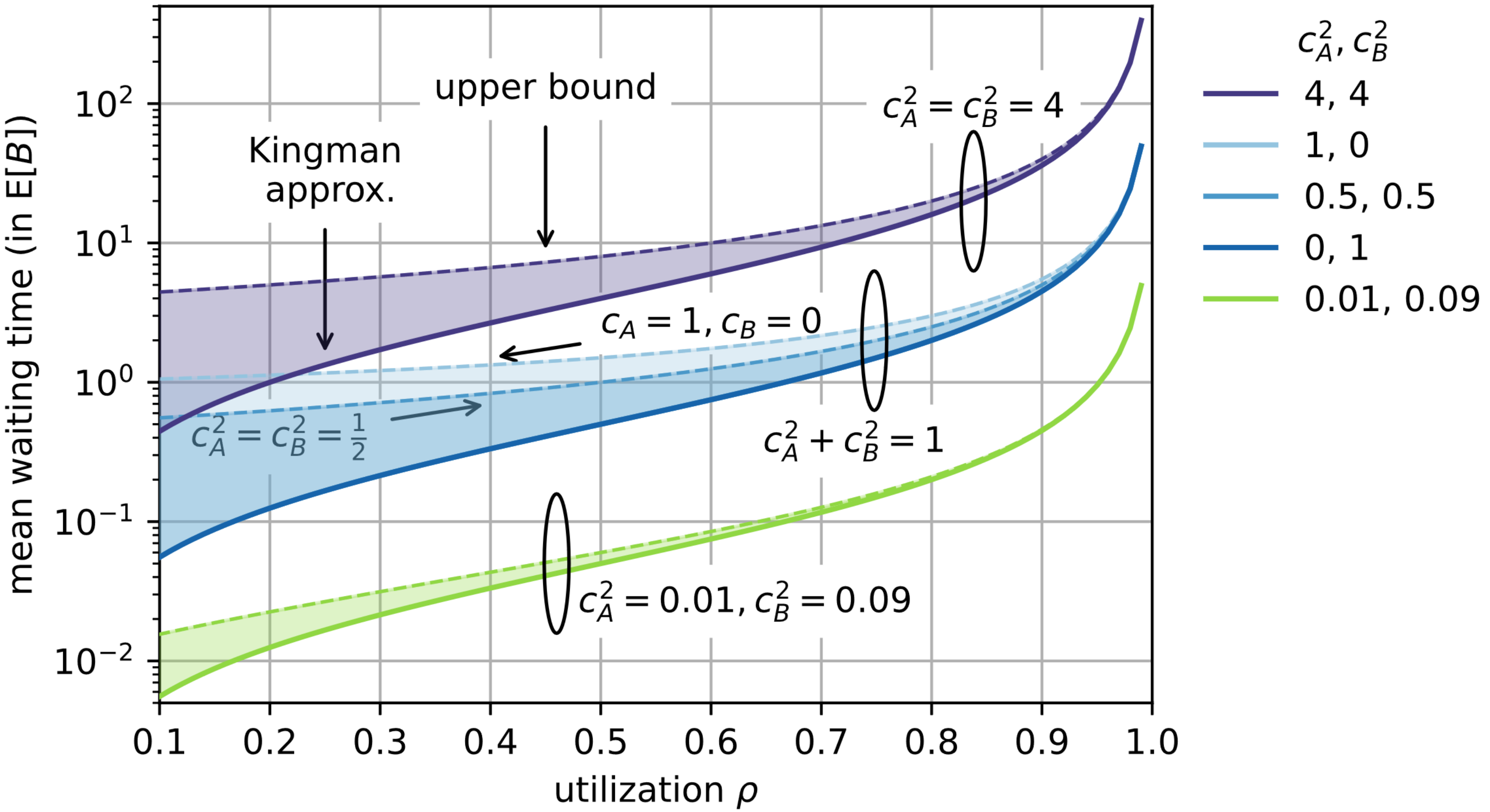
- **Tighter upper bound** of the mean waiting time is provided by Daley

$$\mathrm{E}[W] \leq \frac{(2-\rho)c_A^2+\rho c_B^2}{2(1-\rho)} \cdot \mathrm{E}[B] \stackrel{\text{def}}{=} \widehat{U}$$

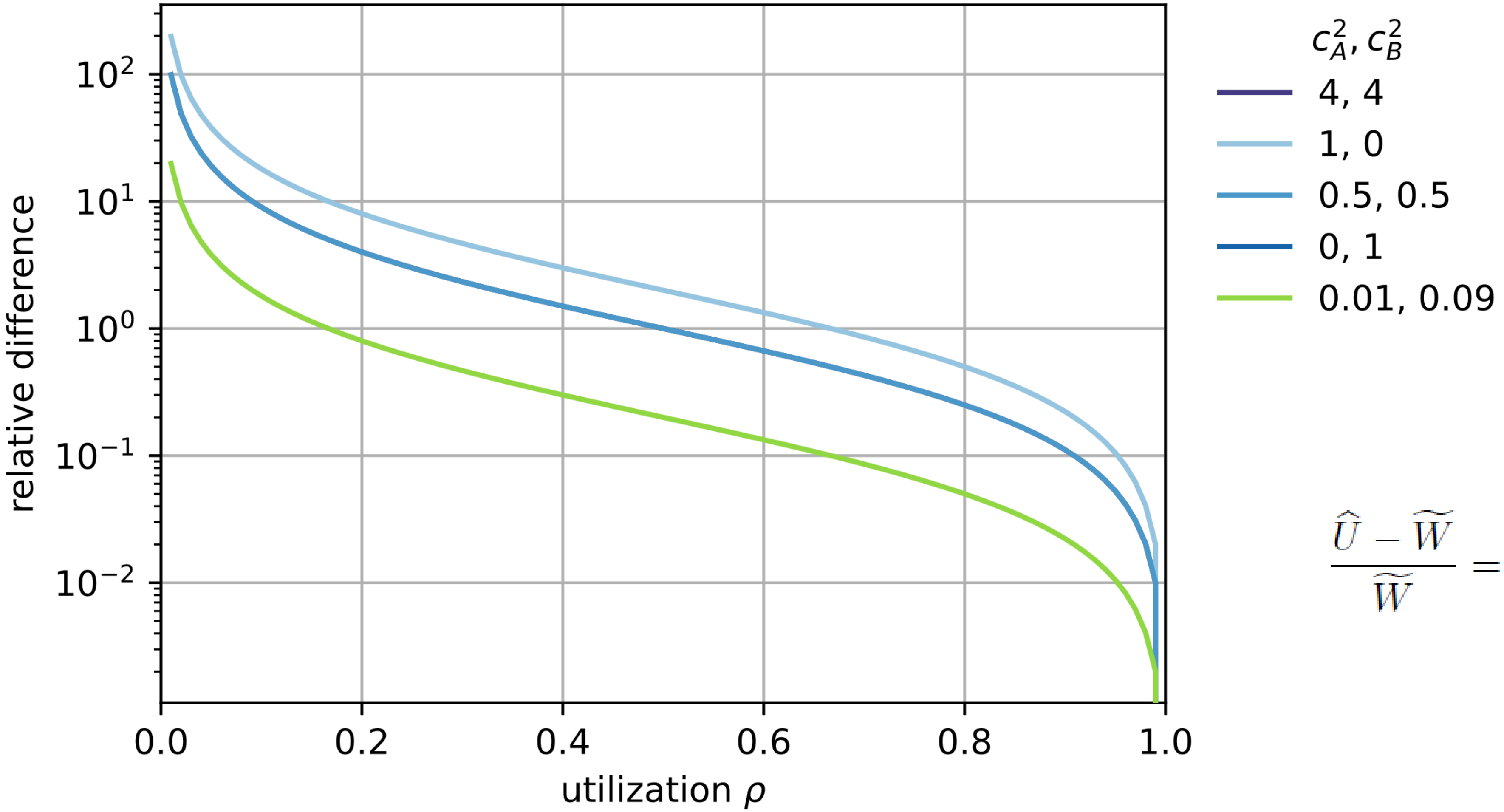
- Difference between upper bound and approximated mean $\widehat{U} - \widetilde{W} = \mathrm{E}[B] \cdot c_A^2$

- Relative difference $\frac{\widehat{U}-\widetilde{W}}{\widetilde{W}} = \frac{2c_A^2}{c_A^2+c_B^2} \cdot \left(\frac{1-\rho}{\rho}\right)$

Kingman’s Approximation and Tight Upper Bounds



Relative Difference



$$\frac{\widehat{U} - \widetilde{W}}{\widetilde{W}} = \frac{2c_A^2}{c_A^2 + c_B^2} \cdot \left(\frac{1-\rho}{\rho}\right)$$