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Institute of Computer Science
Chair of Communication Networks
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Chapter 4.2

Delay System M/M/n

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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The book must be cited and the disclaimer attached when using lectures slides or scripts.

Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/

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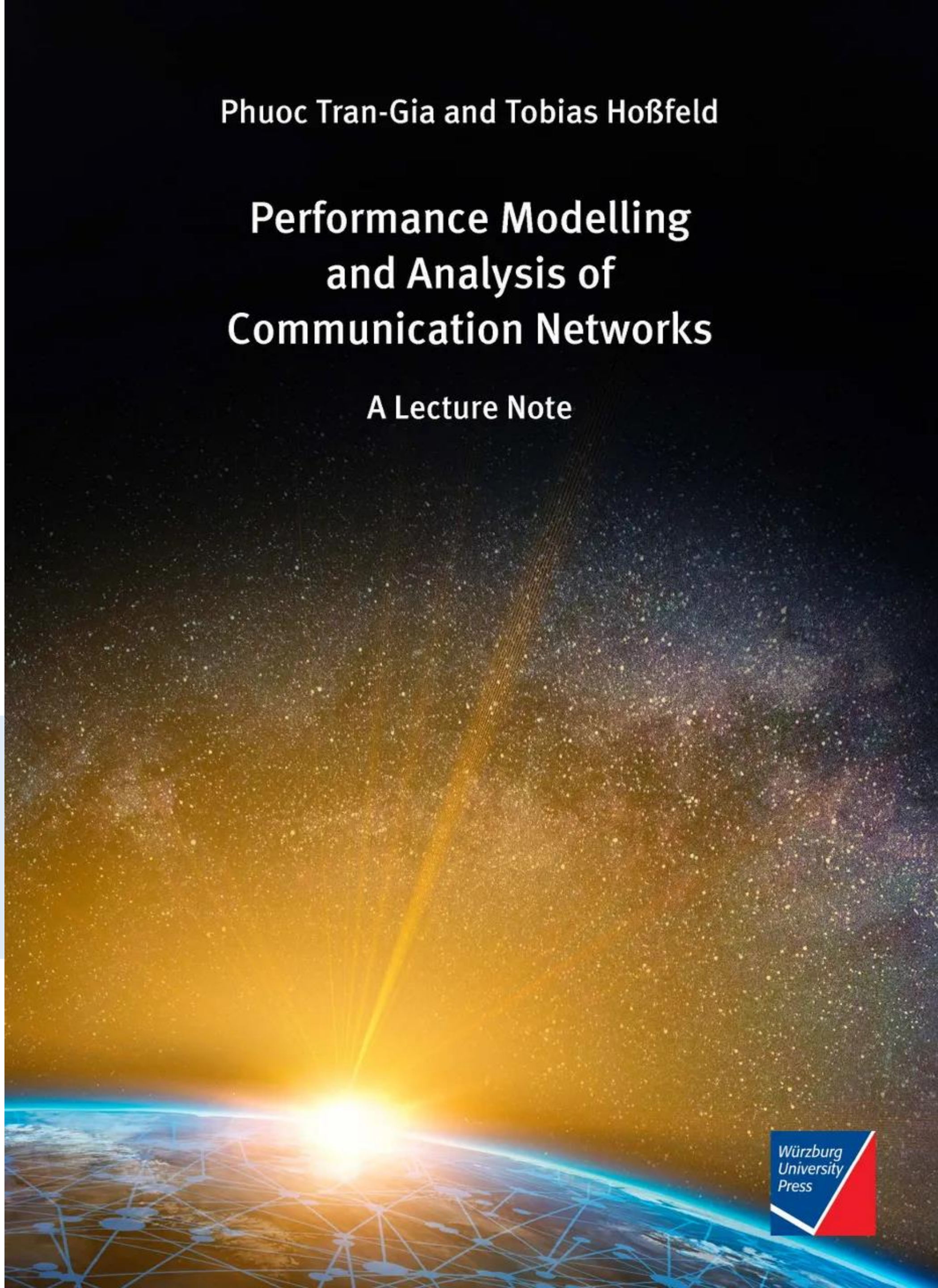


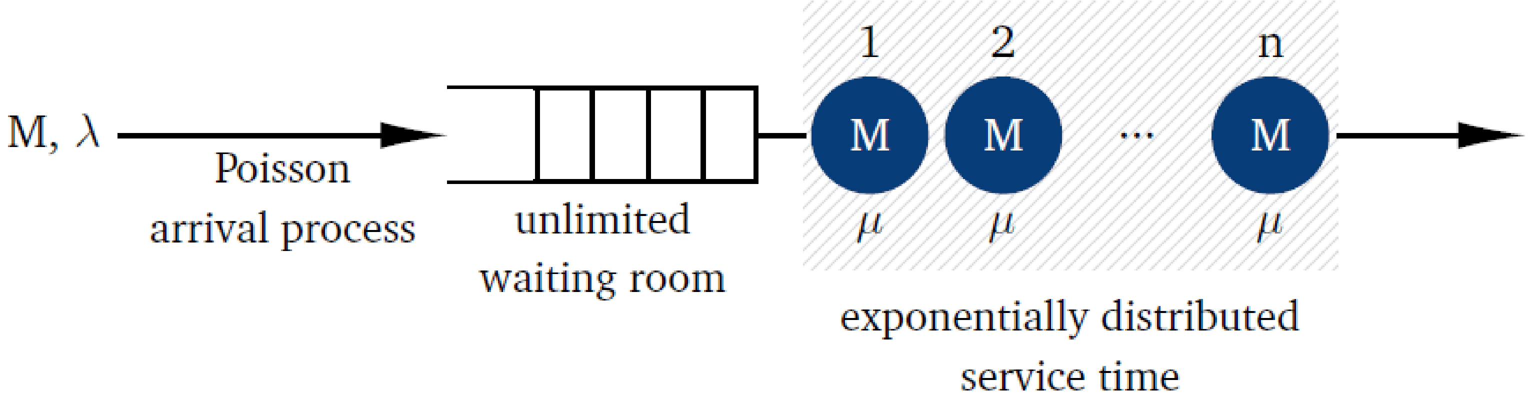
Chapter 4

4 Analysis of Markovian Systems

Model Structure and Parameters

Stability condition

Delay System M/M/n



- Interarrival time A with arrival rate λ
- Service time B with service rate μ
- Offered traffic $a = \frac{\lambda}{\mu}$ in pseudo-unit Erlang [Erl]

$$A(t) = P(A \le t) = 1 - e^{-\lambda t}, \quad E[A] = \frac{1}{\lambda},$$

$$B(t) = P(B \le t) = 1 - e^{-\mu t}, \quad E[B] = \frac{1}{\mu}.$$

- Pure delay system
  - number of waiting places is assumed to be unlimited
  - arriving customer finding upon arrival all servers occupied will join the queue until a server becomes available

Utilization and Stability Condition

- **Offered traffic** is identical to mean number of occupied servers $E[X]$

$$a = \frac{\lambda}{\mu} = \lambda E[B]$$

- **Utilization** or occupancy of a single server is the mean offered traffic per server

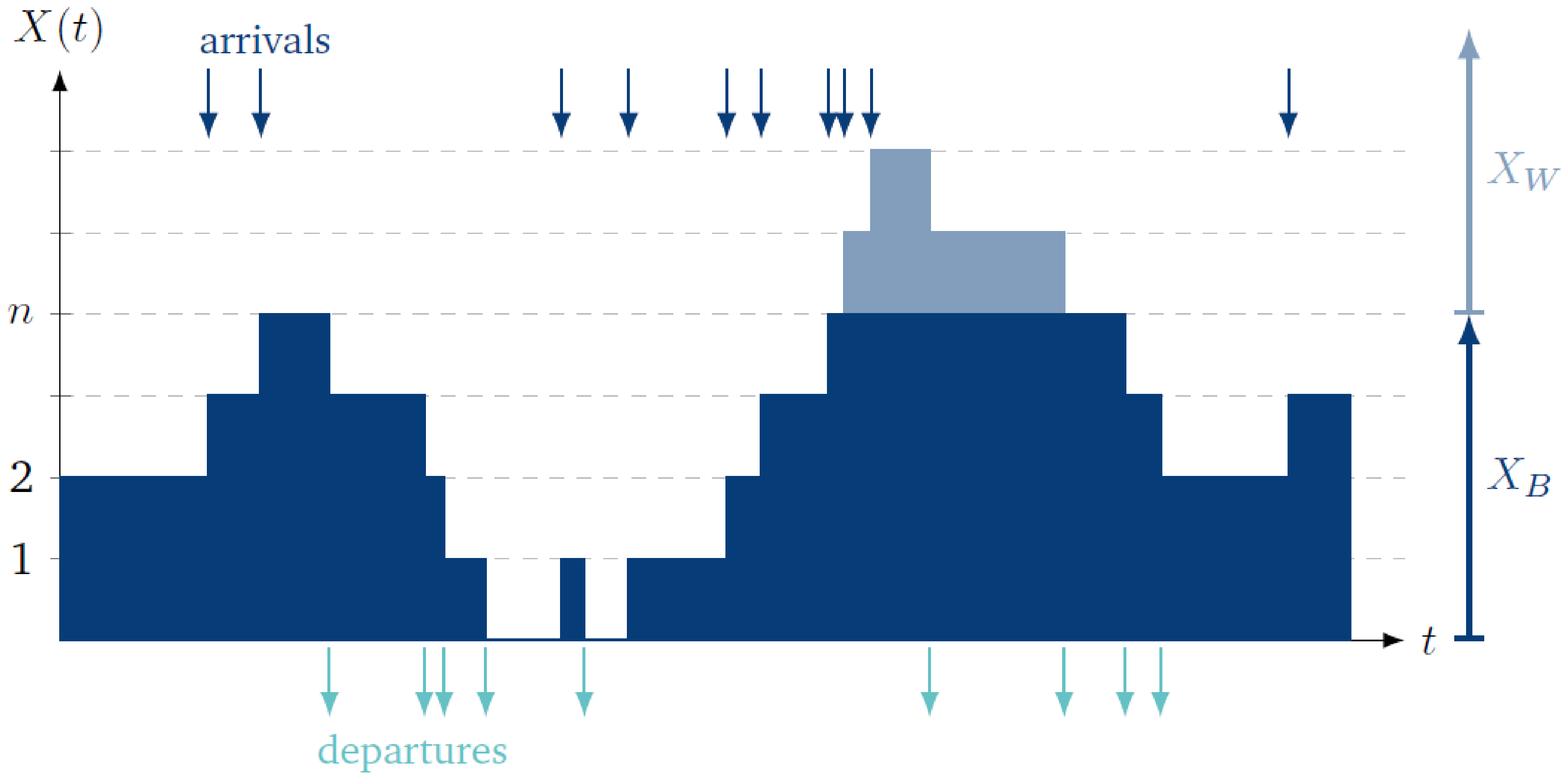
$$\rho = \frac{a}{n} = \frac{\lambda}{\mu n}$$

- **Stability condition**
  - during single service duration, $a = \lambda E[B]$ customers arrive on average
  - at most n customers can be served during single service duration: $\lambda E[B] < n$
  - Stable system requires $a < n$ or

$$\rho = \frac{a}{n} < 1$$

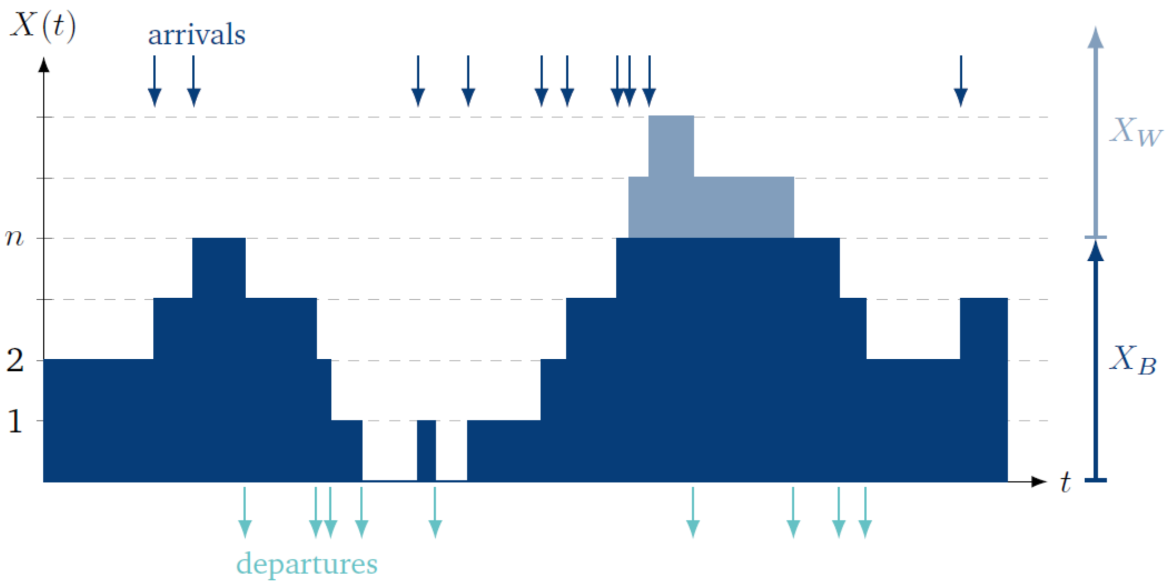
State Process and State Probabilities

State Process of M/M/n Delay System



State Process of M/M/n Delay System

- Complete description of stochastic process $X(t)$ at time t instead of $\{X_B(t), X_W(t)\}$



waiting customers $X_W(t)$

$X_W(t) = 0$ if $X_B(t) < n$

customers in service $X_B(t)$

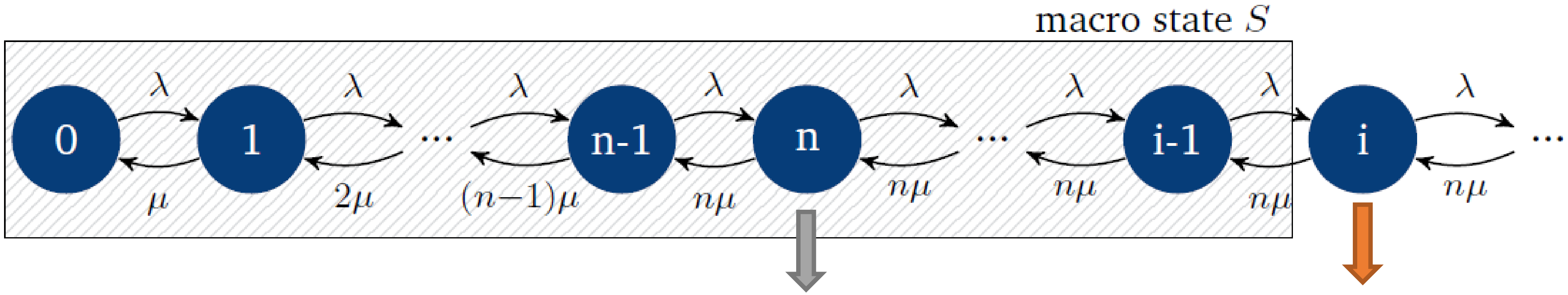
number of customers in system $X(t)$

$$X(t) = \begin{cases} X_B(t) & \text{for} \quad X_B(t) < n \quad (X_W(t) = 0) \\ X_B(t) + X_W(t) & \text{for} \quad X_B(t) = n \end{cases}$$

steady state probability

$$x(i) = P(X(t) = i) = P(X = i), \quad i = 0, 1, \ldots$$

State Transition Diagram

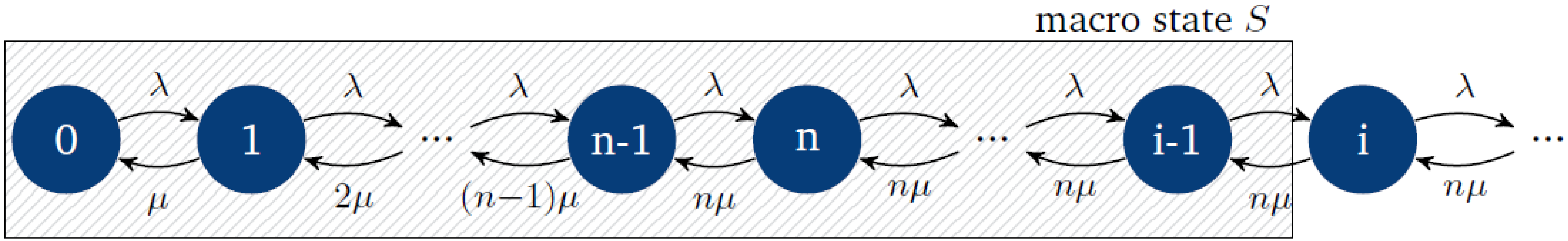


all servers occupied

(micro) state $[X = i]$
customers in the system

- Customer arrival: $[X\ =\ i] \rightarrow [X\ =\ i+1]$ with rate λ for $i = 0,1,\dots,\infty$ (Poisson process)
- Customer departure or service termination: $[X\ =\ i] \rightarrow [X\ =\ i-1]$ with rate $i\mu$
  - service time of one of the i customers ends ($i = 1,2,\dots,n$)
  - service rate for $X > n$: $n\mu$

Macro State Equations



- Differentiate macro states

$$\lambda x(i-1) = i\mu x(i), \qquad i = 1,\ldots,n,$$

$$\lambda x(i-1) = n\mu x(i), \qquad i = n+1,\ldots,$$

- Normalization condition

$$\sum_{i=0}^{\infty} x(i) = 1$$

Macro State Equations: Solution

Macro State Equations: Solution

- Successive iteration yields the solution

$$x(i) = \begin{cases} x(0)\dfrac{a^i}{i!} & i = 0,1,\ldots,n \\ x(0)\dfrac{a^n}{n!}\left(\dfrac{a}{n}\right)^{i-n} = x(n)\rho^{i-n} & i > n \end{cases}$$

- Probability of empty system using normalization condition

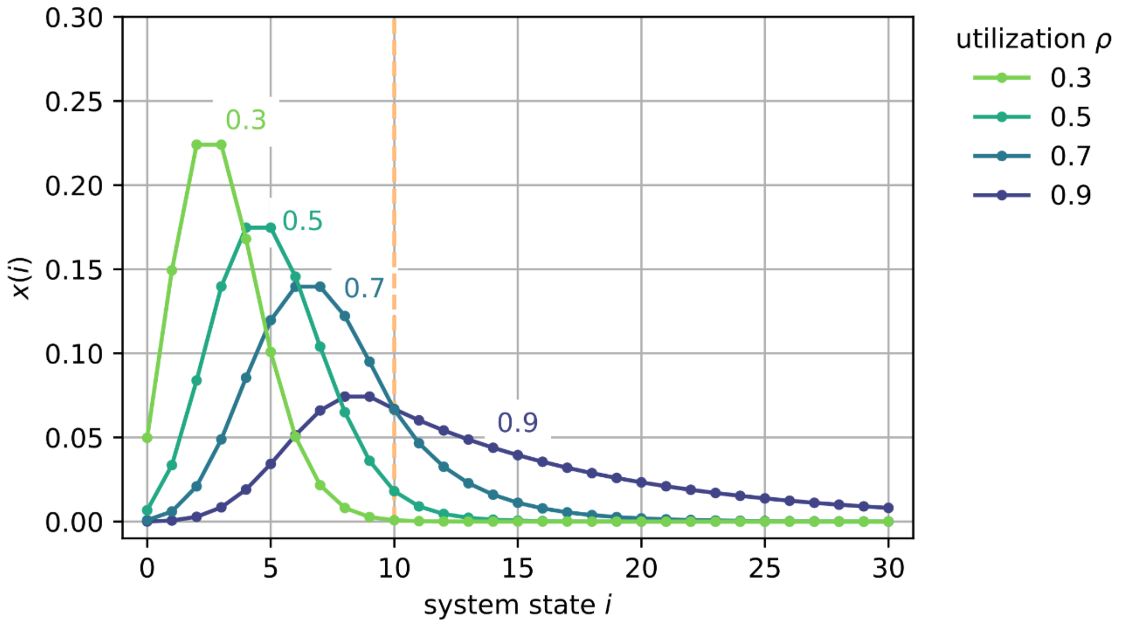
$$x(0) = \left(\sum_{k=0}^{n-1}\frac{a^k}{k!} + \frac{a^n}{n!}\sum_{k=0}^{\infty}\rho^k\right)^{-1}$$

- For stable systems $a < n$

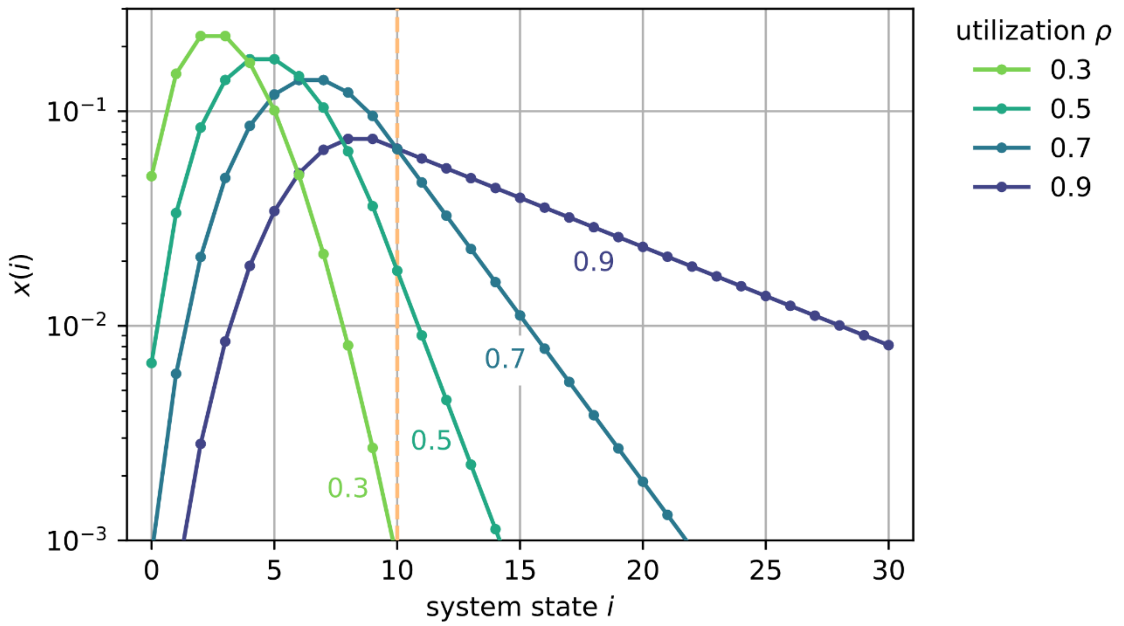
$$x(0) = \left(\sum_{k=0}^{n-1}\frac{a^k}{k!} + \frac{a^n}{n!}\cdot\frac{1}{1-\rho}\right)^{-1}$$

Steady State Distribution

- Example
  - M/M/n delay system for n=10



- Geometric tail for $i > n$
  - $x(i) = x(n) \cdot \rho^{i-n}$
  - logarithmic scale: linear for $i > n$



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Other System Characteristics

Erlang-C formula, waiting probability, mean waiting times

Waiting Probability

- Probability that an arriving customer sees all server busy

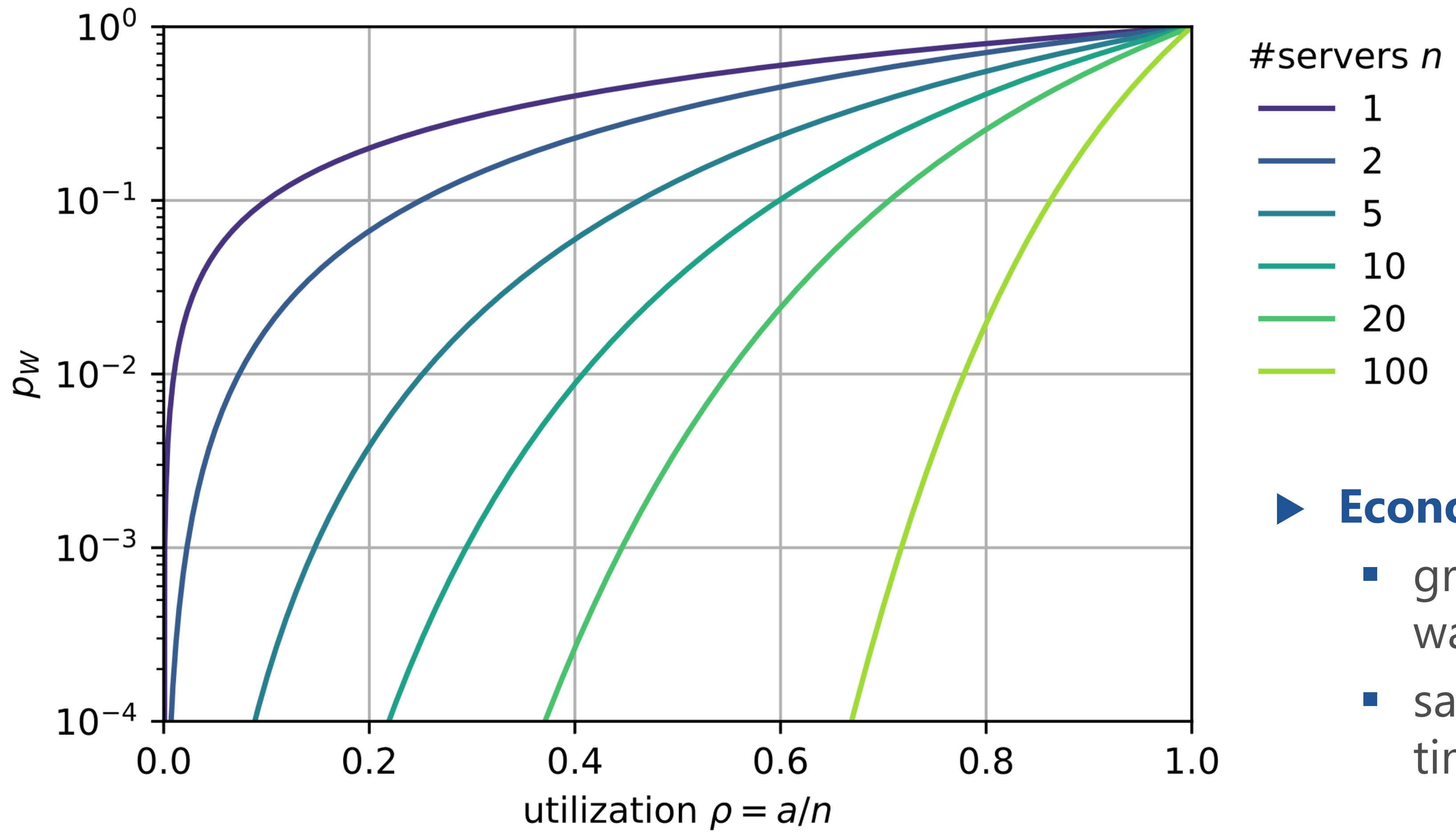
$$p_W = \sum_{i=n}^{\infty} x(i) = x(n)\sum_{i=0}^{\infty}\rho^i = x(n)\frac{1}{1-\rho}$$

- **Erlang-C formula**

$$p_W = \frac{\frac{a^n}{n!}\cdot\frac{1}{1-\rho}}{\sum_{i=0}^{n-1}\frac{a^i}{i!} + \frac{a^n}{n!}\cdot\frac{1}{1-\rho}}$$

- Example: M/M/1 delay system
  - $x(0) = 1-\rho$
  - $x(i) = (1-\rho)\cdot\rho^i$
  - $p_W = a = \rho$

Waiting Probability: Illustration



- **Economy of scale** in delay systems
  - grouping of servers least to lower waiting probabilities for same ρ
  - same result for mean waiting times (see later)

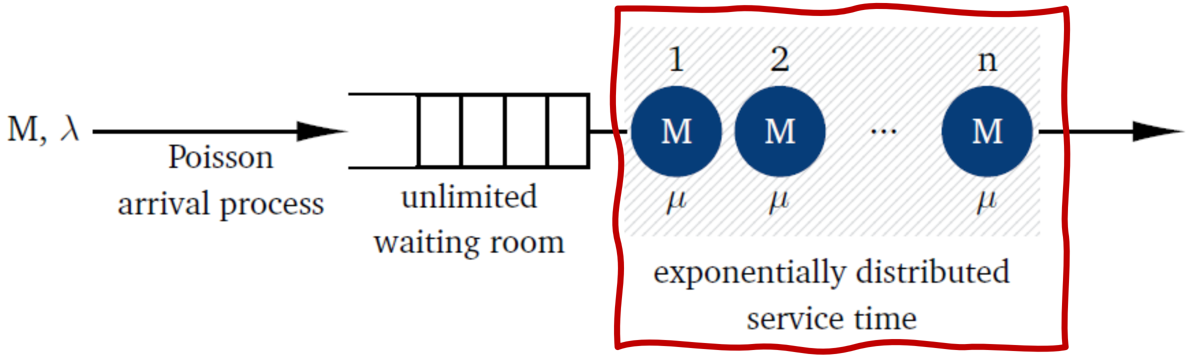
Carried Traffic

- Carried traffic Y is the mean number of occupied servers $E[X_B]$

$$Y = E[X_B] = \sum_{i=0}^{n-1} i \cdot x(i) + n \sum_{i=n}^{\infty} x(i) = a$$

- Using Little's theorem
  - mean arrival rate in the system: λ
  - mean sojourn time in the system: $E[B]$
  - mean number of customers in system: $E[X_B] = Y$

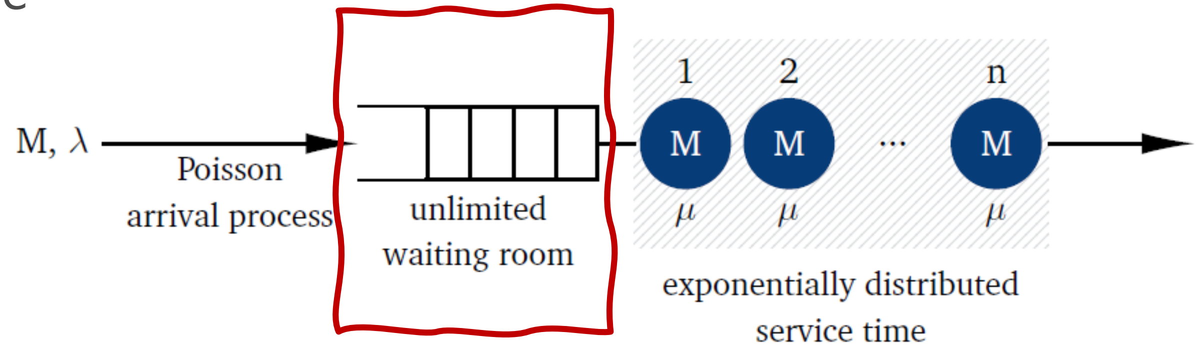
  - Little's law: $$Y = \frac{\lambda}{\mu} = a$$



Mean Queue Length

- Mean number of customers in the waiting space

$$\begin{aligned}\Omega &= E[X_W] \\ &= \sum_{i=n}^{\infty}(i-n)\cdot x(i) = \sum_{i=n}^{\infty}(i-n)\, x(n)\, \rho^{i-n} \\ &= x(n)\sum_{i=0}^{\infty} i\, \rho^i = x(n)\cdot\frac{\rho}{(1-\rho)^2} = x(0)\,\frac{a^n}{n!}\cdot\frac{\rho}{(1-\rho)^2}\end{aligned}$$

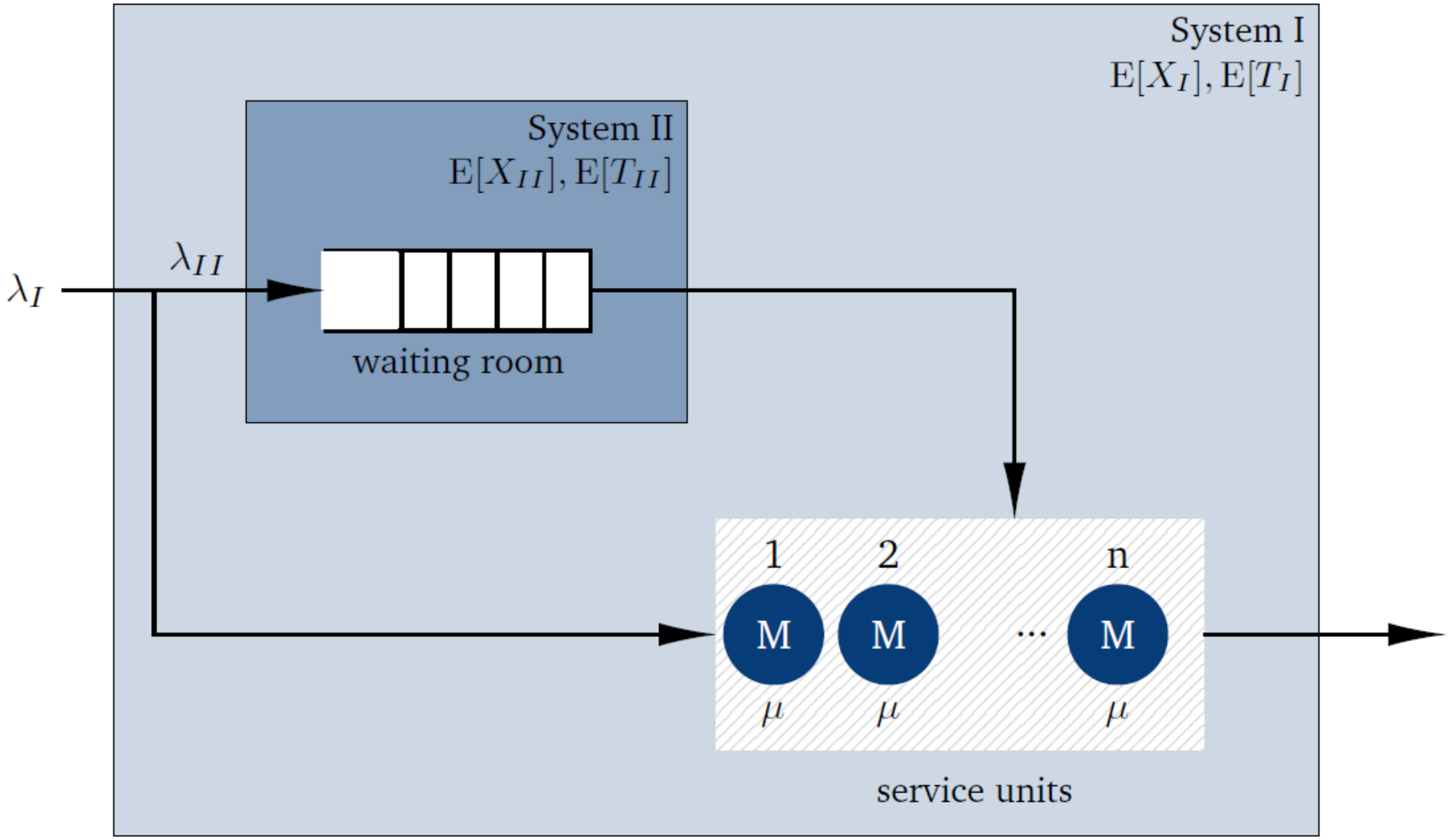


- Using waiting probability p_W

$$\Omega = \underbrace{\underbrace{x(0)\,\frac{a^n}{n!}}_{x(n)}\,\frac{1}{1\text{-}\rho}}_{p_W}\,\frac{\rho}{1\text{-}\rho} = p_W\,\frac{\rho}{1\text{-}\rho}$$

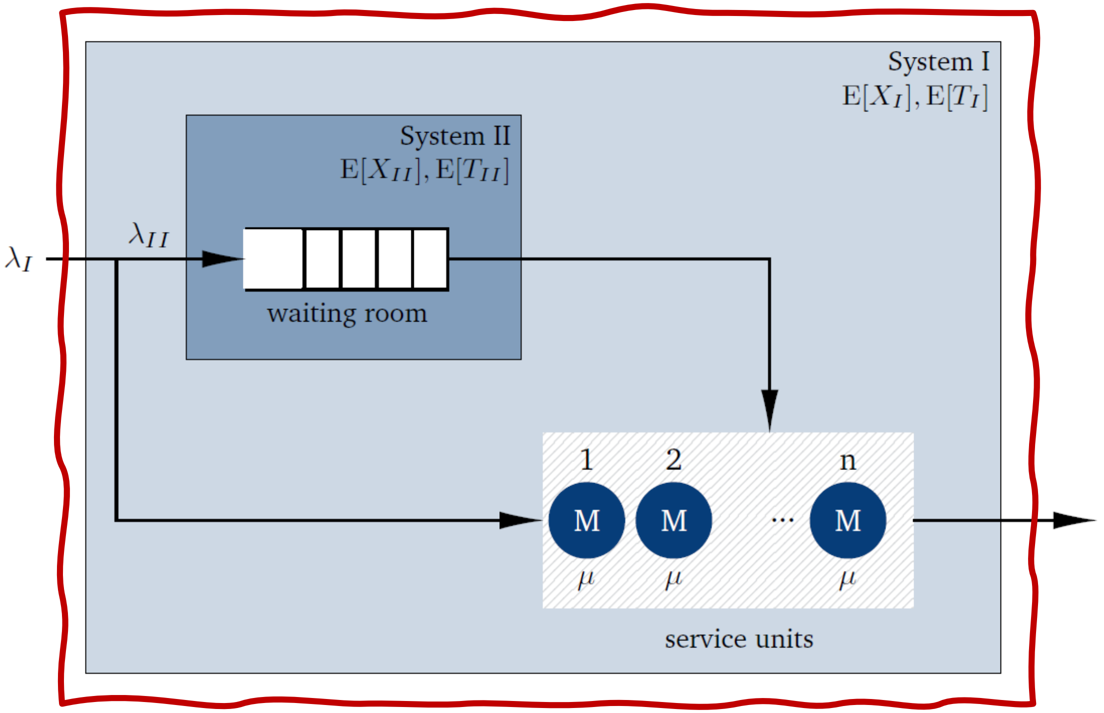
Mean Waiting Time

- Mean waiting times of all customers (system I) and of waiting customers (system II)



Mean Waiting Time of All Customers (System I)

- System I (all customers): entire M/M/n system
  - mean arrival rate λ_I
    $\lambda_I = \lambda$
  - mean number of customers in system $E[X_I]$
    $E[X_I] = E[X_W] + E[X_B] = \Omega + Y$
  - mean sojourn time in system $E[T_I]$
    $E[T_I] = E[W] + E[B]$

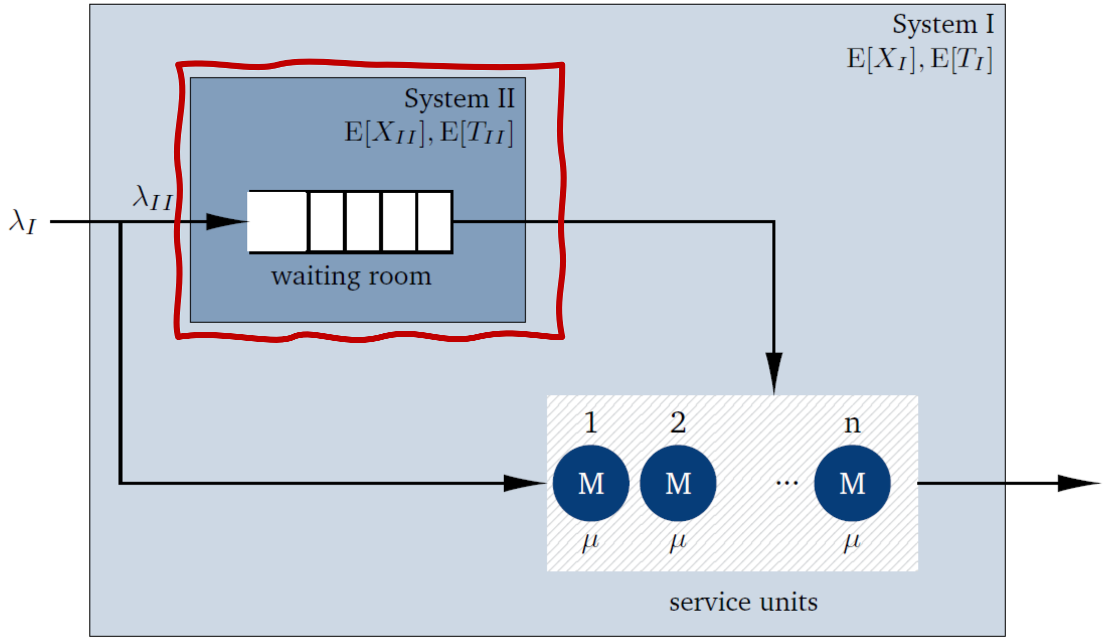


- **Little's law**

$$\lambda_I\, E[T_I] = E[X_I] \quad \Rightarrow \quad \lambda\, E[W] + \underbrace{\lambda\, E[B]}_{Y} = \Omega + Y \quad \Rightarrow \quad \boxed{E[W] = \frac{\Omega}{\lambda}}$$

Mean Waiting Time of Waiting Customers (System II)

- System II (only waiting customers): waiting queue of M/M/n system
  - mean arrival rate λ_{II}: arrival rate of waiting customers $\lambda_{II} = \lambda \cdot p_W$
  - mean number of customers in system $E[X_{II}]$: mean queue length $E[X_{II}] = \Omega = p_W \frac{\rho}{1-\rho}$
  - mean sojourn time in system $E[T_{II}]$: mean waiting time of waiting customers $E[T_{II}] = E[W_1]$

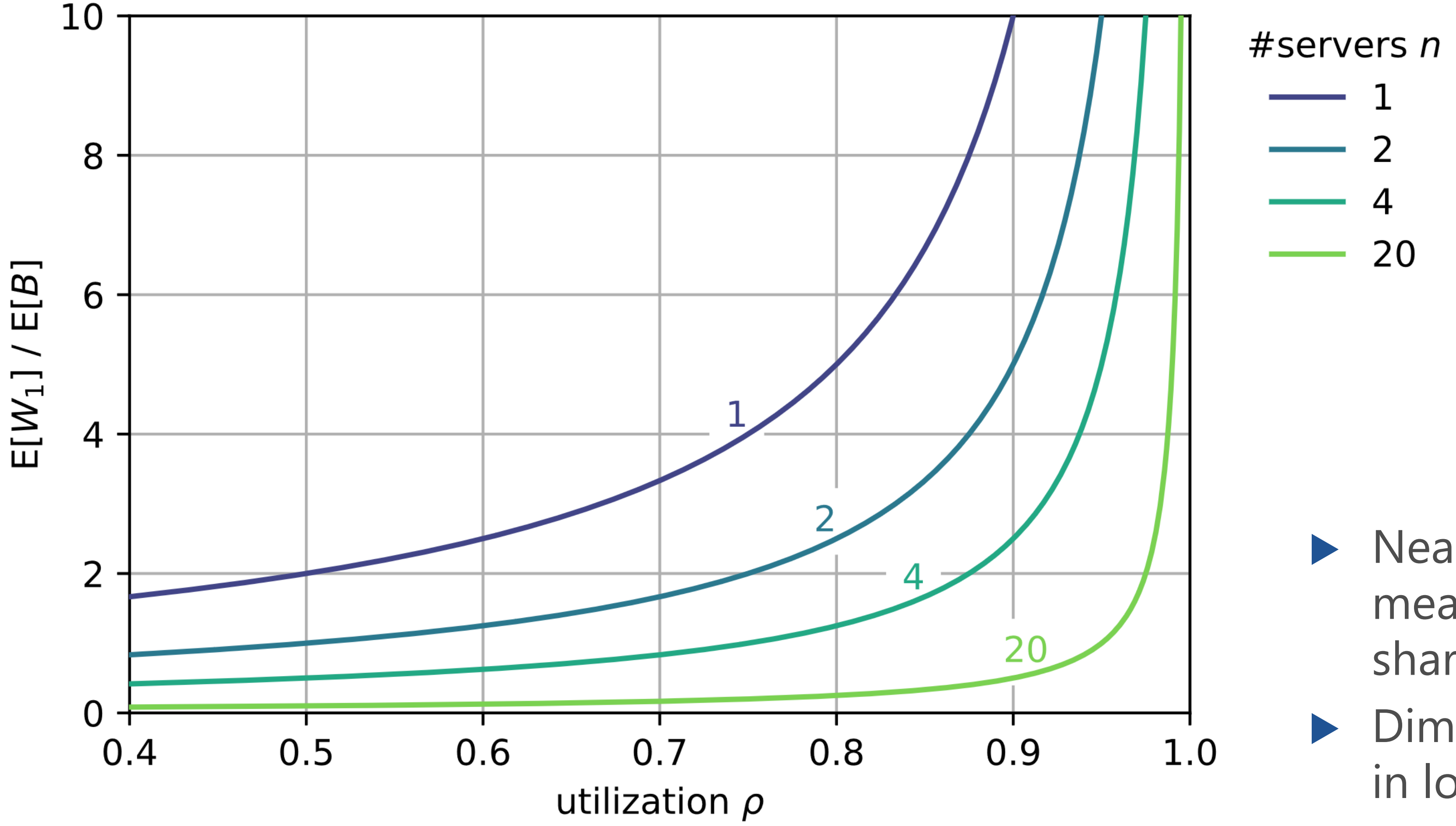


- **Little's law**

$$\lambda_{II} E[T_{II}] = E[X_{II}] \quad \Rightarrow \quad \boxed{E[W_1] = \frac{\Omega}{\lambda \cdot p_W} = \frac{1}{\lambda} \cdot \frac{\rho}{1-\rho}} \quad \Rightarrow \quad E[W_1] = \frac{E[W]}{p_W}$$

$$E[W] = \frac{\Omega}{\lambda}$$

Mean Waiting Time of Waiting Customers



- Near stability boundary $\rho = 1$, mean waiting time increases sharply
- Dimensioning the operation point in lower utilization range

DELAY DISTRIBUTION

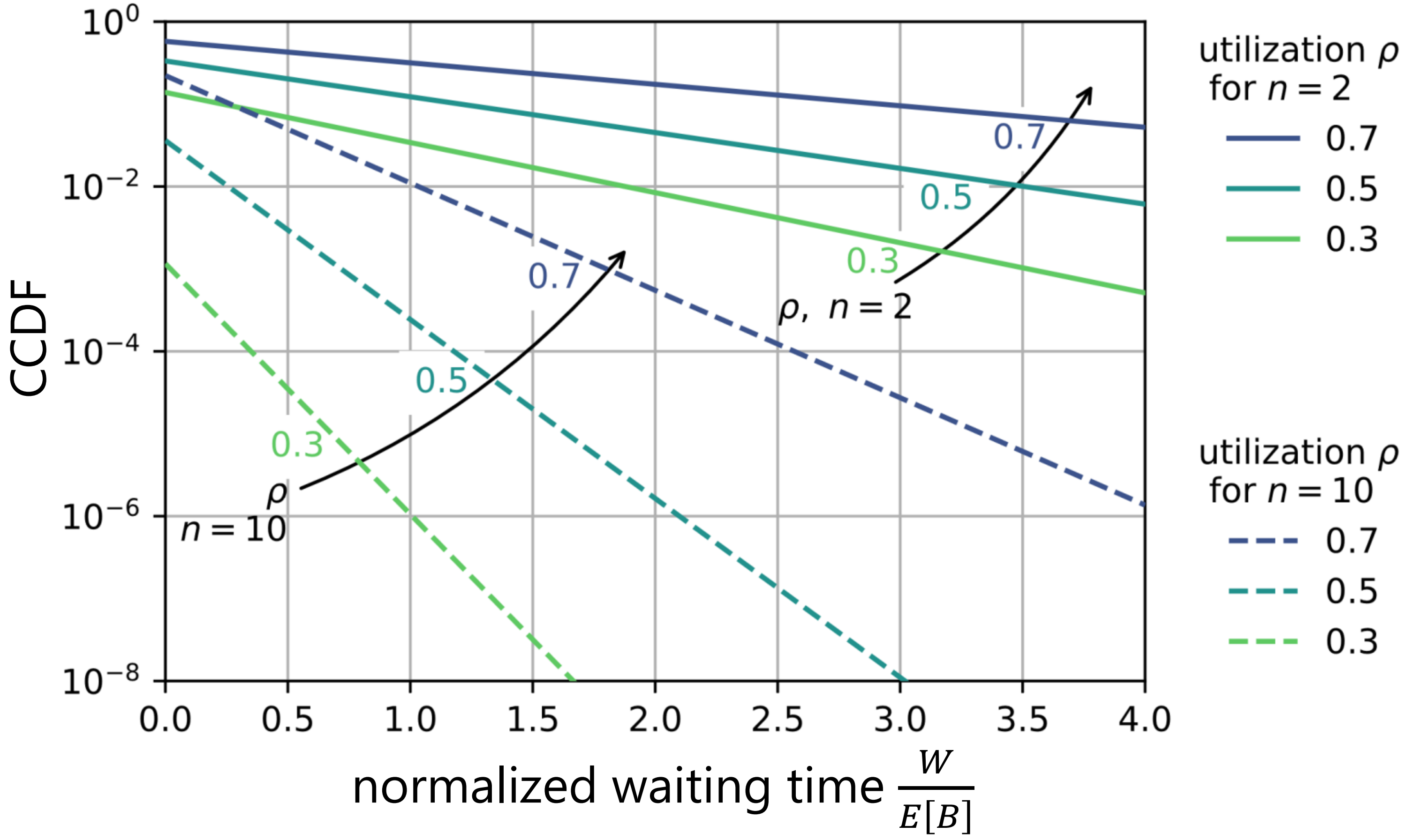
Waiting time distribution

Waiting Time Distribution for All Customers

- Random variable W is the waiting time for all customers

$$W(t) = 1 - p_W \cdot e^{-(1-\rho)n\mu t} = \begin{cases} 0 & t < 0, \\ 1 - p_W & t = 0, \\ 1 - p_W \cdot e^{-(1-\rho)n\mu t} & t > 0. \end{cases}$$

- Waiting probability is $1 - W(0) = p_W$



Waiting Time Distribution: Derivation

Waiting Time Distribution: Derivation (f.)

Lecture

EXAMPLES

Waiting time distribution, economy of scale

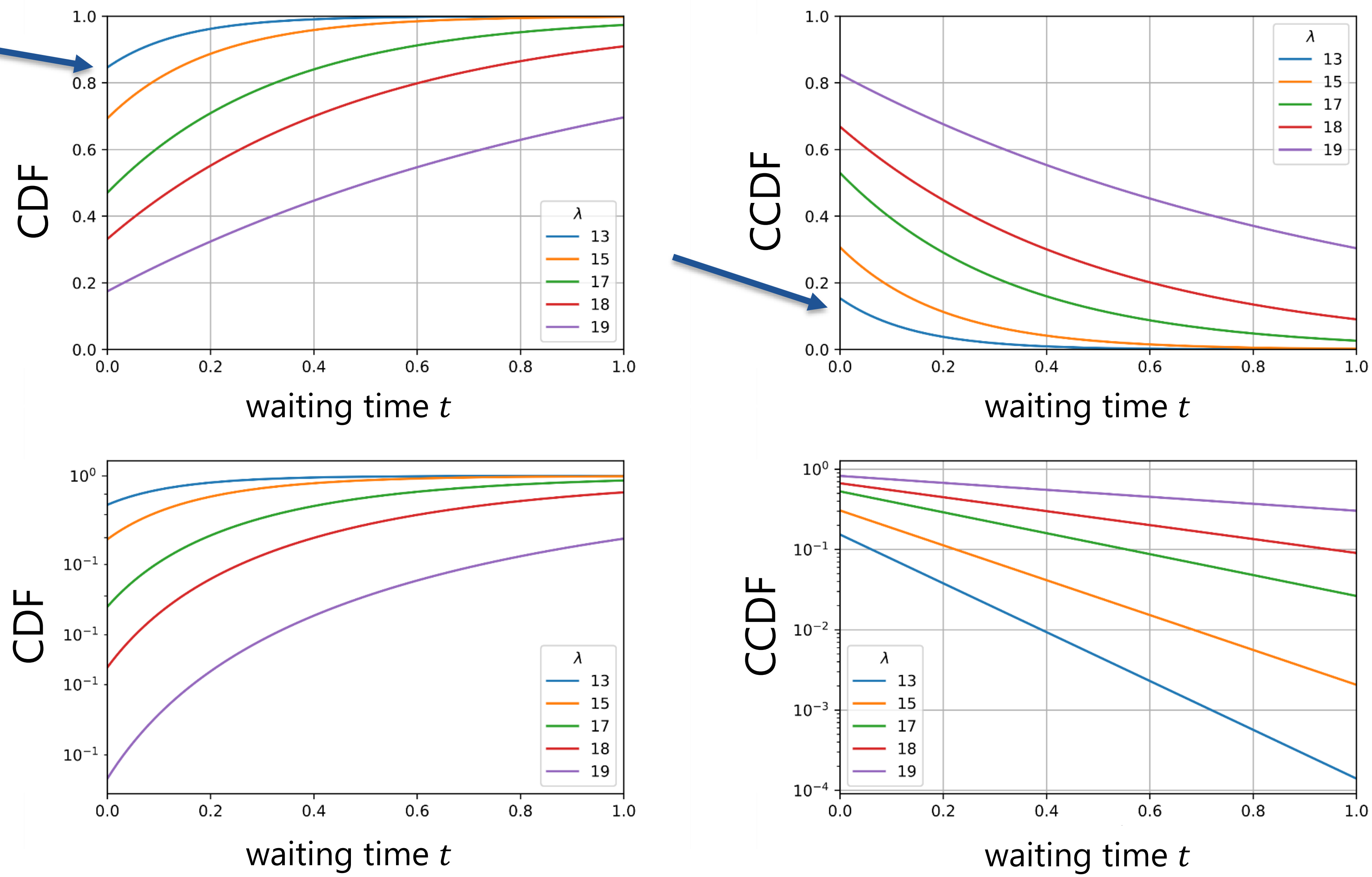
Example: Waiting Time Distributions M/M/n

$1 - p_W$

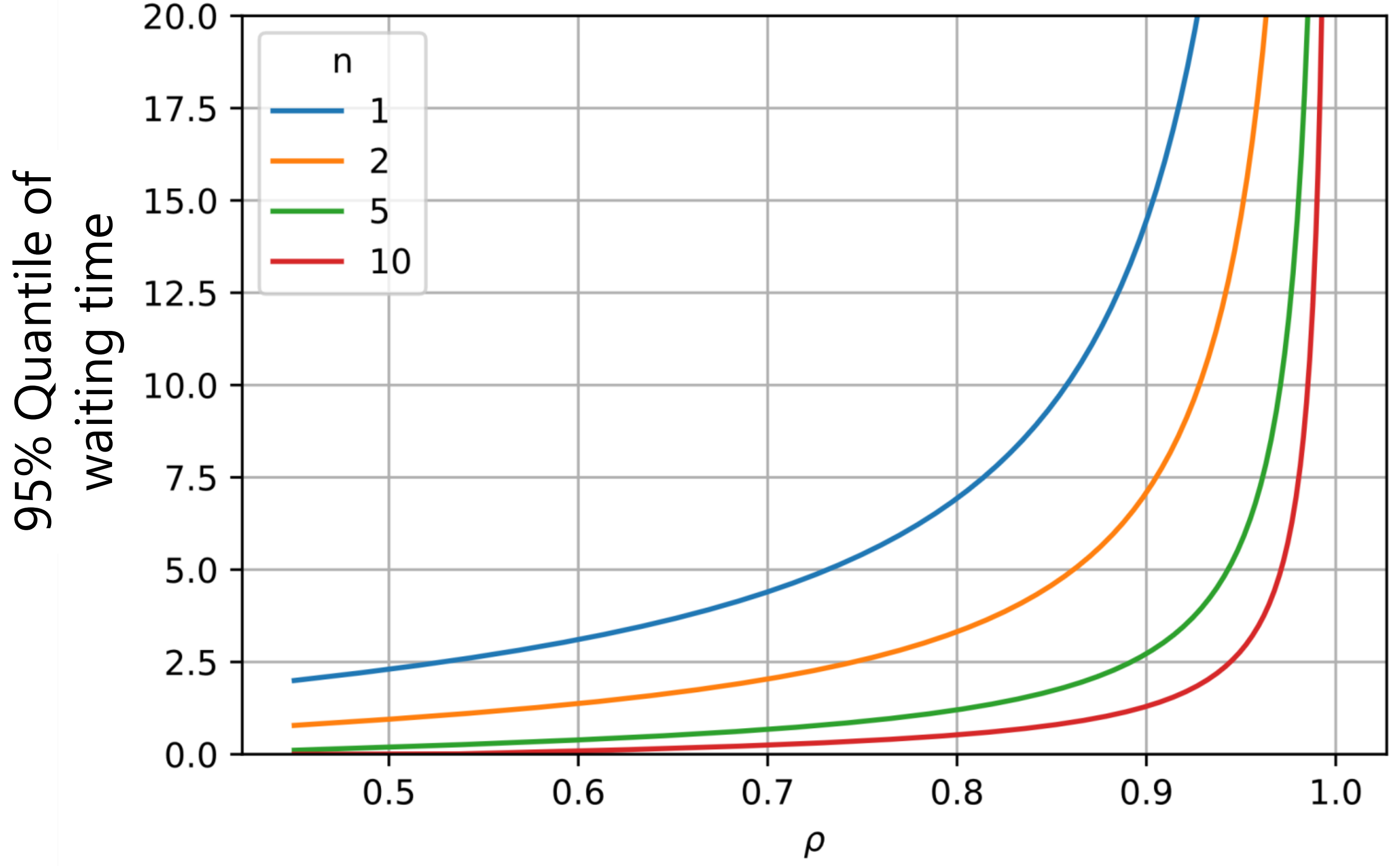
linear scale

parameter
$\mu = 2, n = 10$

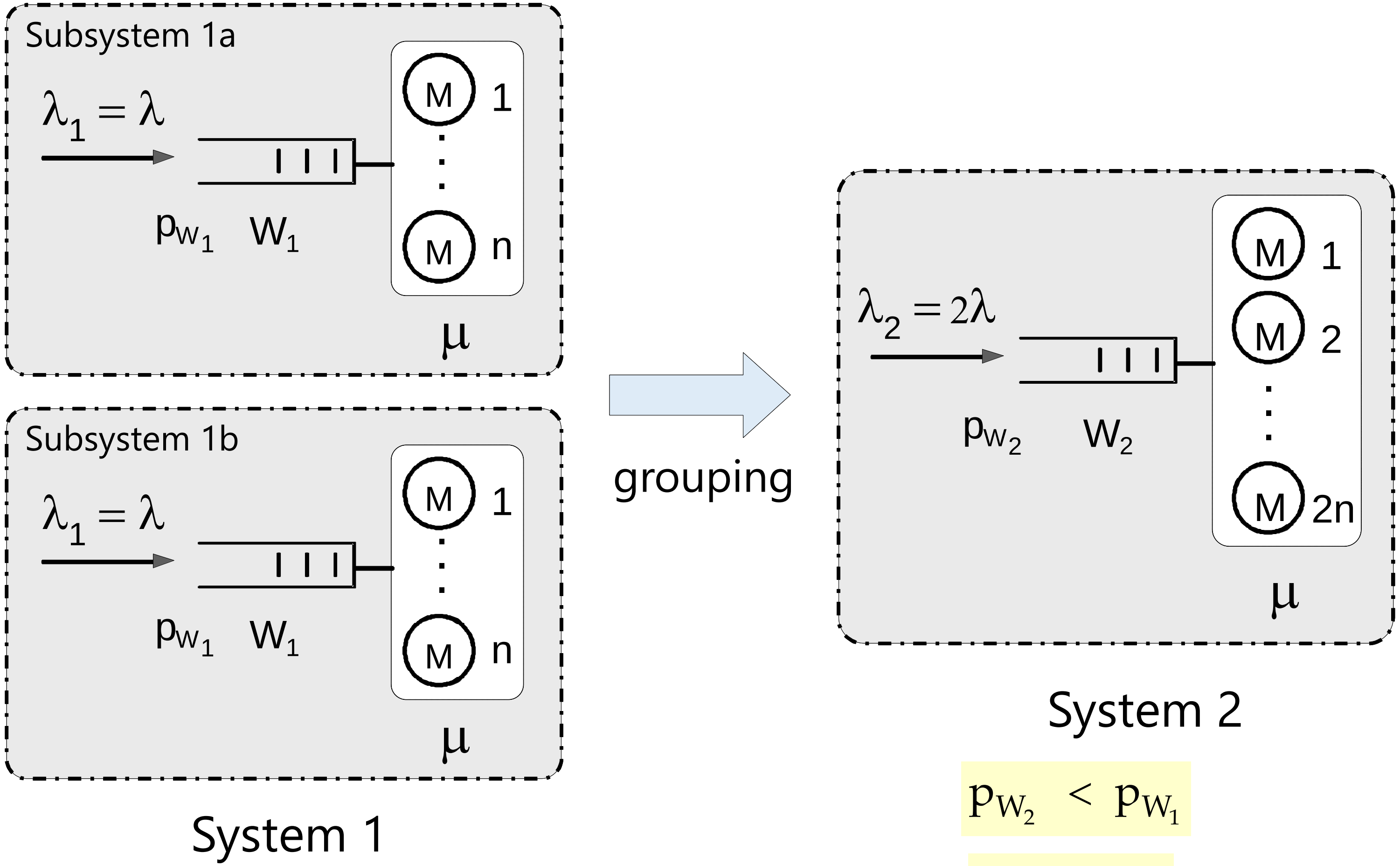
logarithmic scale



Quantiles of Waiting Times



Economy of Scale for Delay Systems



$p_{W_2} < p_{W_1}$

$W_2 < W_1$