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Chapter 2.1

Little's Theorem and General Results

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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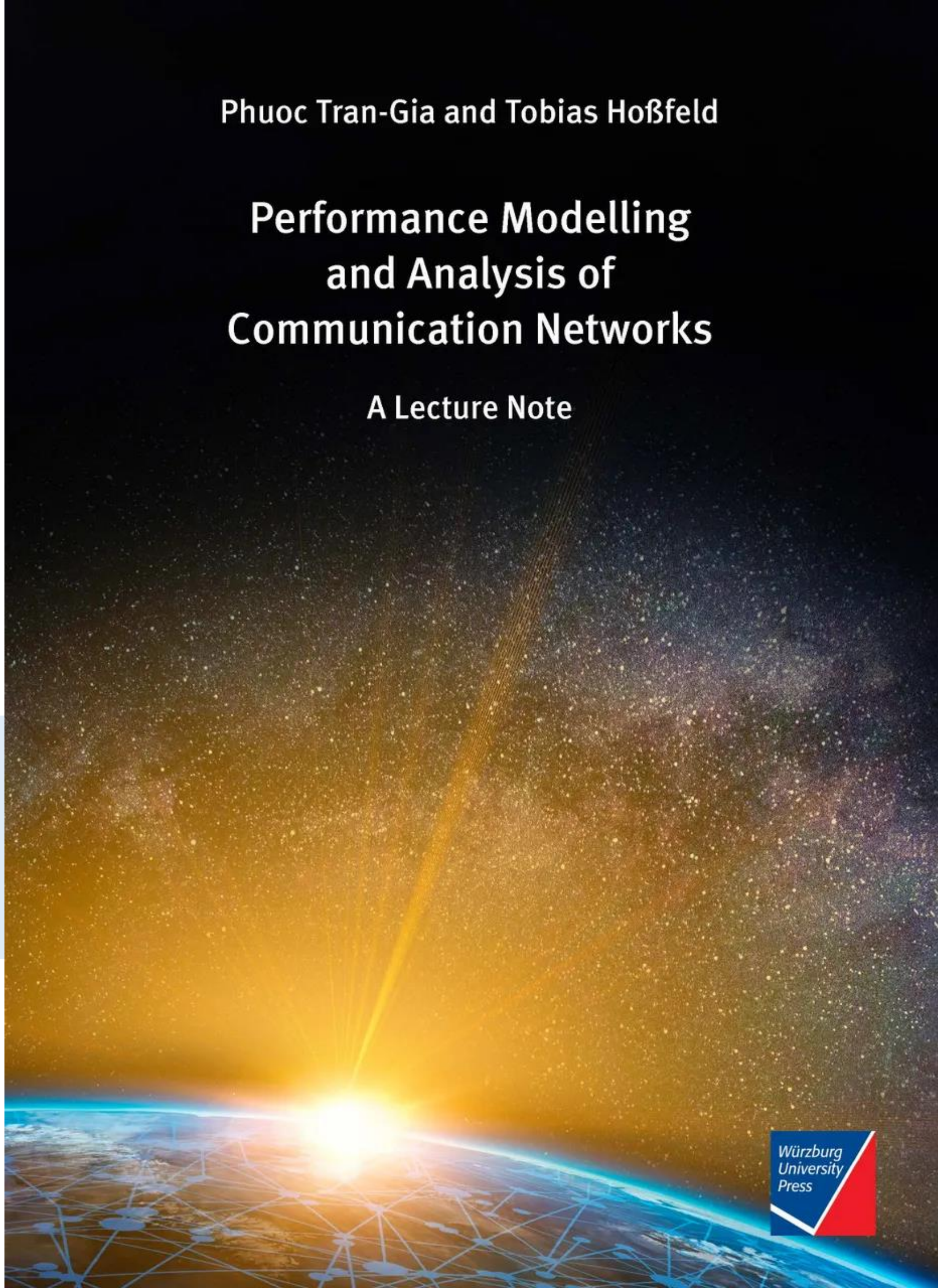
Website to download book, exercises, slides and scripts:
https://modeling.systems/



Chapter 2

2 Fundamentals and Prerequisites

Little‘s Theorem

Little‘s law or Little‘s result

Operational Laws

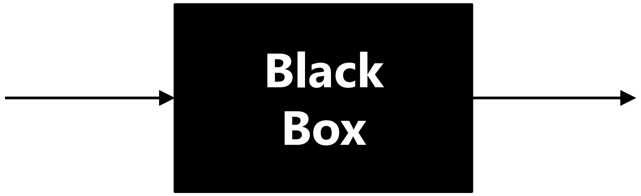
- System is considered as a black box
- Relation between quantities which do not require assumptions on the distribution of arrival times or service times
- “Operational” means directly measurable during operation
- Assumptions that can be verified during operation (in measurements).
  - For example: Is the number of arrivals equal to the number of completions?

Operational Quantities

- Quantities that can be measured directly in a finite observation period

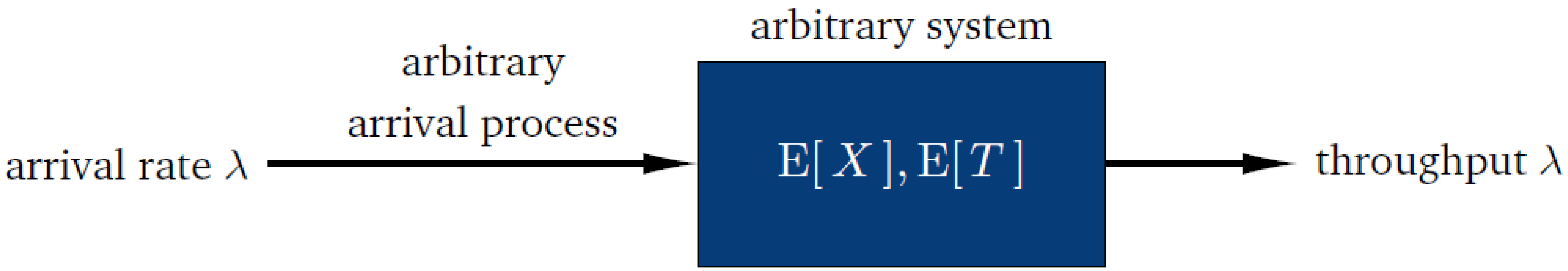
- t = observation period
- $A(t)$ = number of arrivals during (0,t)
- $D(t)$ = number of departures during (0,t)
- $B(t)$ = busy time during (0,t)

- λ_t = Arrival rate during (0,t)
- C_t = Throughput during (0,t)
- U_t = Utilization during (0,t)
- S_t = Mean service time during (0,t)



System

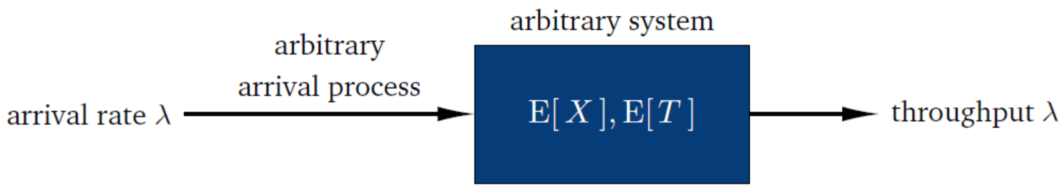
Little's Formula



$$\textbf{Little's Formula} \quad \lambda \cdot E[T] = E[X]$$

- λ mean arrival rate of jobs or customers
- $E[T]$ mean sojourn time of a job or a customer in the system
- $E[X]$ mean number of jobs or customers in the system

Assumptions of Little's Theorem



- General black box system
- No assumptions about
  - distribution of arrival or service process
  - operating discipline (FIFO, LIFO, …)
- Little's law is not valid when the system generates or destroys work
  - e.g. due to a failure in a router, some packets are not forwarded and are not departing from the system or may initiate additional (signaling) traffic within the system.
- Only required assumption
  - two of the quantities $\lambda, E[X], E[T]$ exist and are finite
  - third quantity follows
- Theorem can be applied to *any stable* system and to any *part* of the system.
- Most famous operational law: helpful for measurements and analysis!

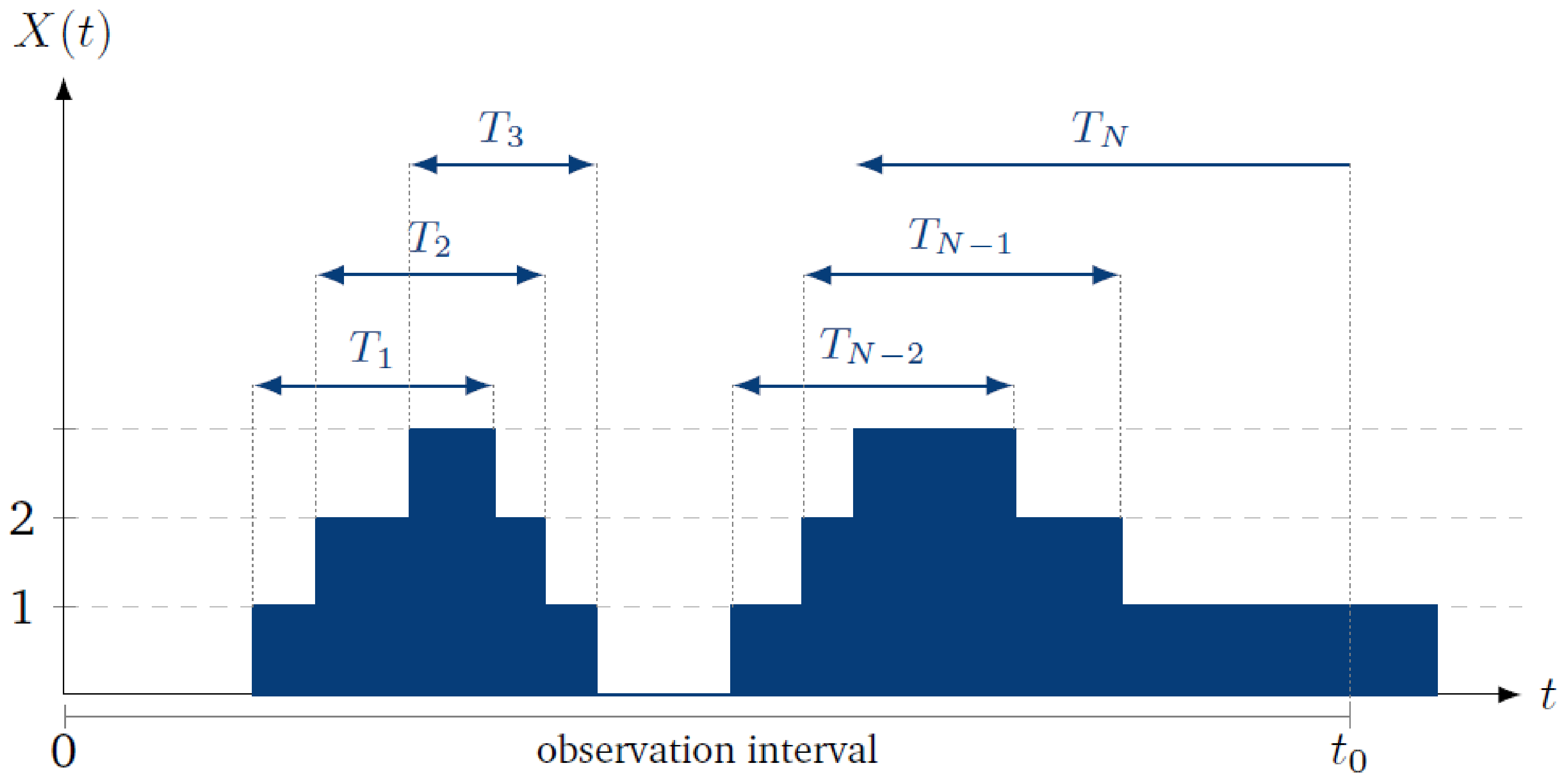
Simple Example of Little‘s Law

- Example 1: Waiting room at the doctor's
  - Arrival rate 10 patients/hour
  - Mean waiting time: 0.5 hours
  - Mean number of patients?

- Example 2: Waiting room at the doctor's
  - Arrival rate 10 patients/hour
  - Mean waiting time: 1.2 hours
  - Mean number of patients?

Proof of Little's Theorem

- During observation interval of length t_0: N arrivals of customers with sojourn times T_i.
- Number of customers at time t is $X(t)$.



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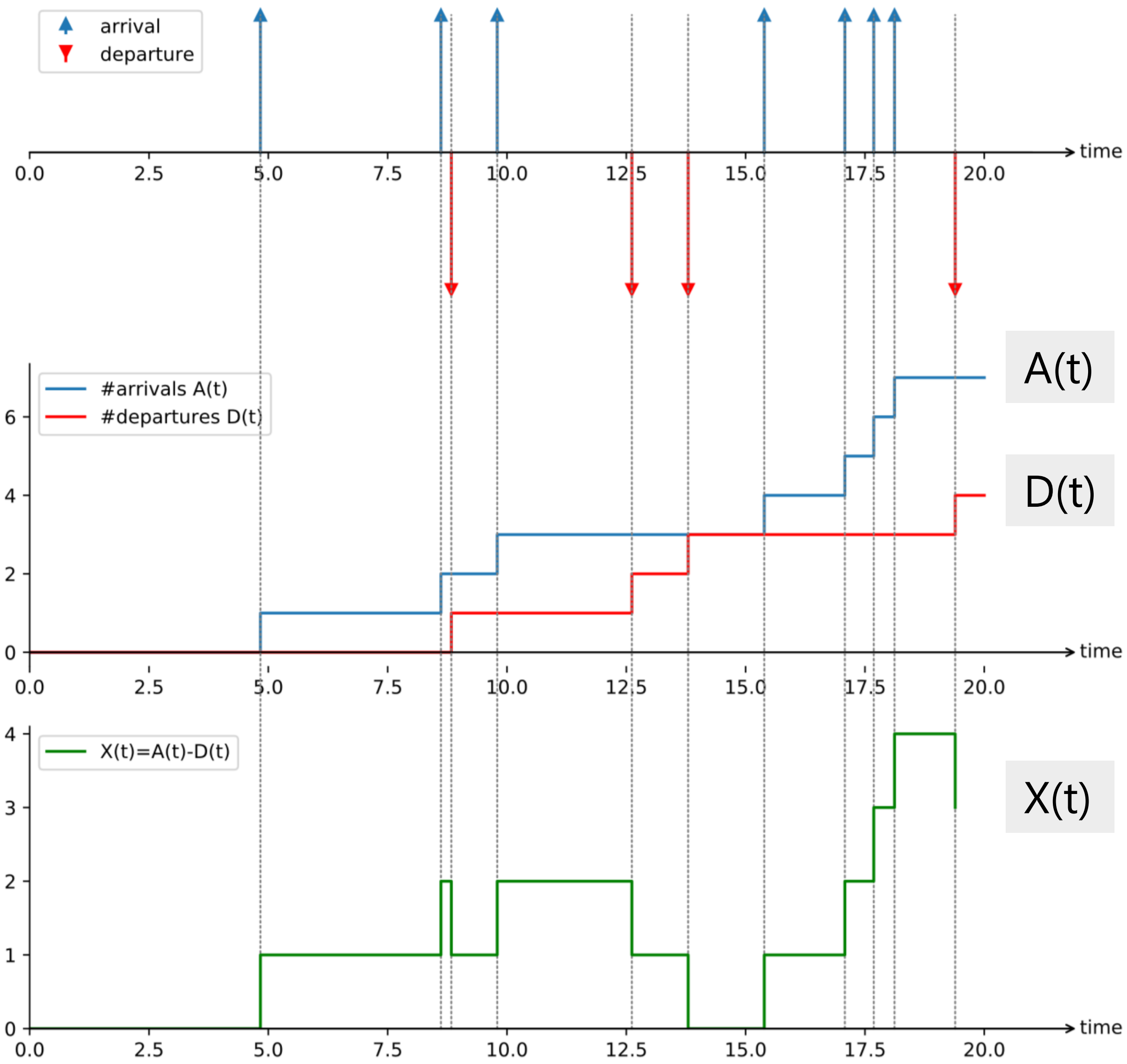
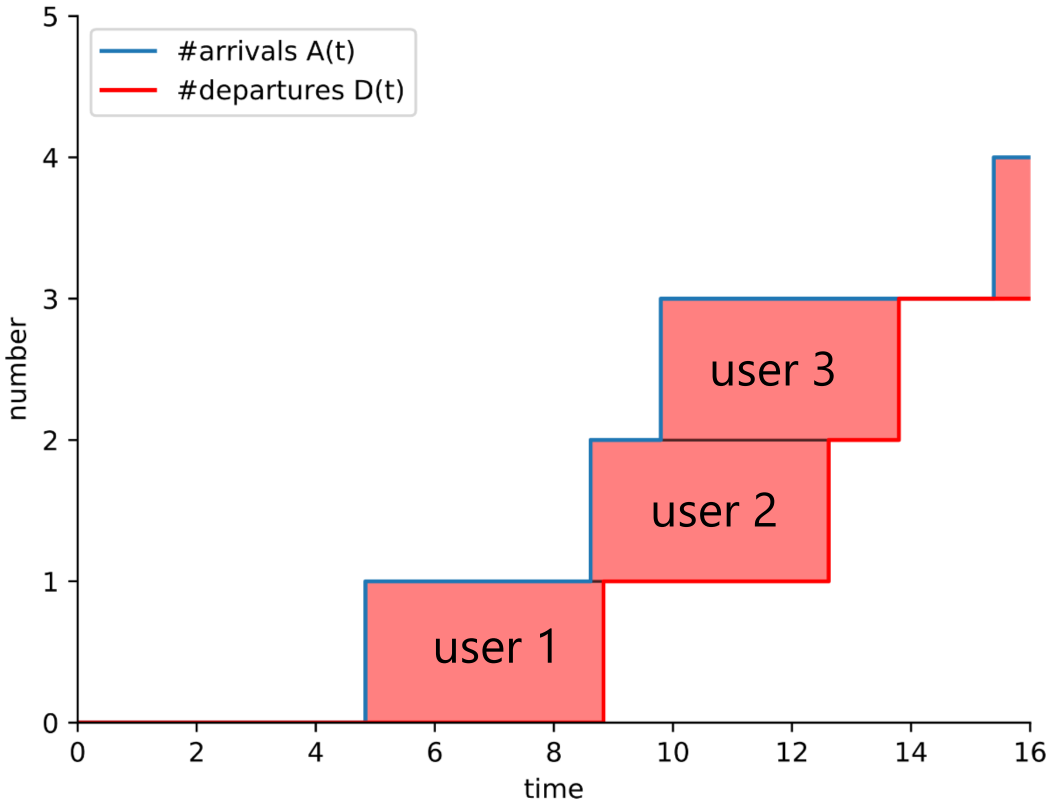
Little's Law: Visualization (M/D/1)



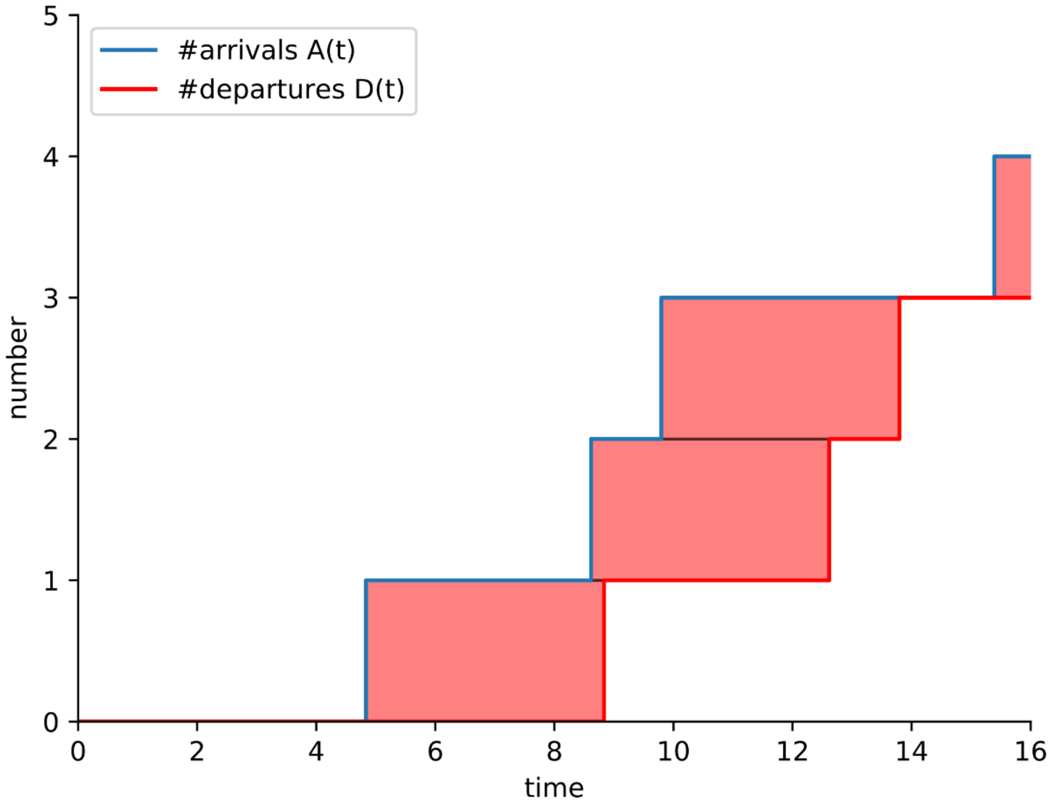
Accumulated Sojourn Time

- Area between $A(t)$ and $D(t)$: $\int_0^t X(\tau)d\tau$
- Accumulated sojourn time across all users



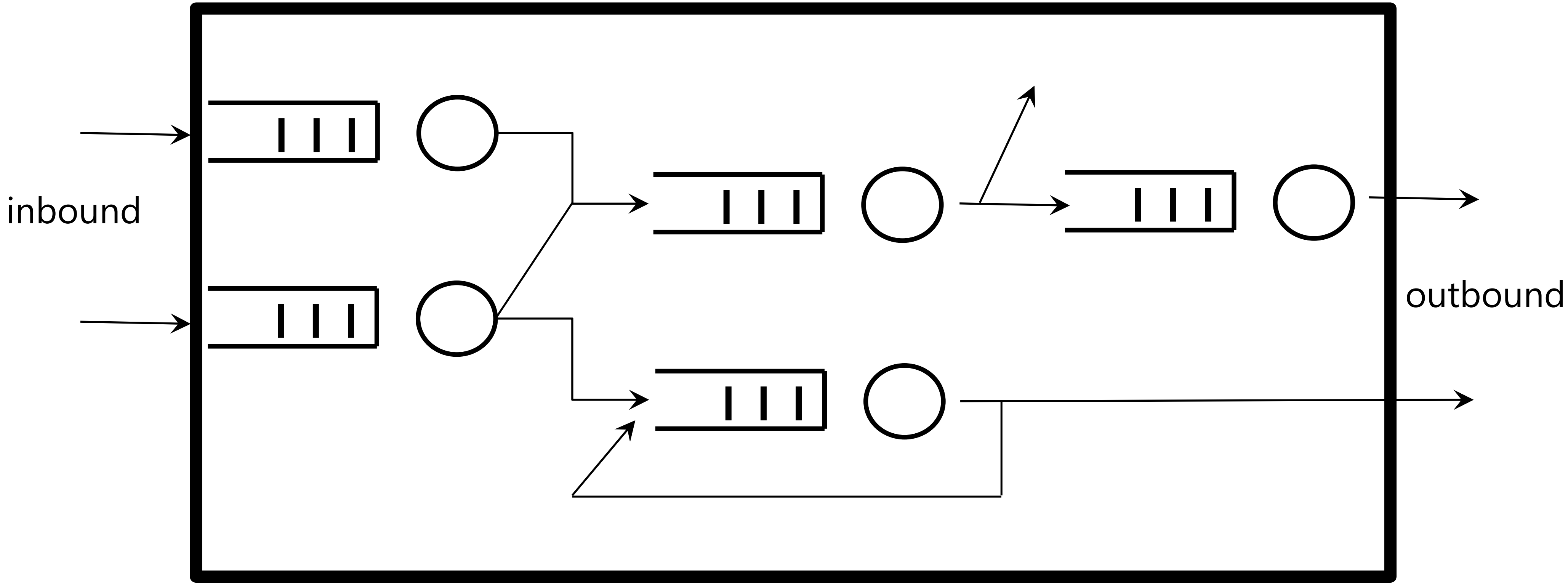
Queueing Discipline

- Do we observe the same mean waiting times for FIFO (first-in first-out) and LIFO (last-in first-out)?



Little‘s Law: Network of Queues

- Theorem can be applied to *any stable* system and to any *part* of the system.

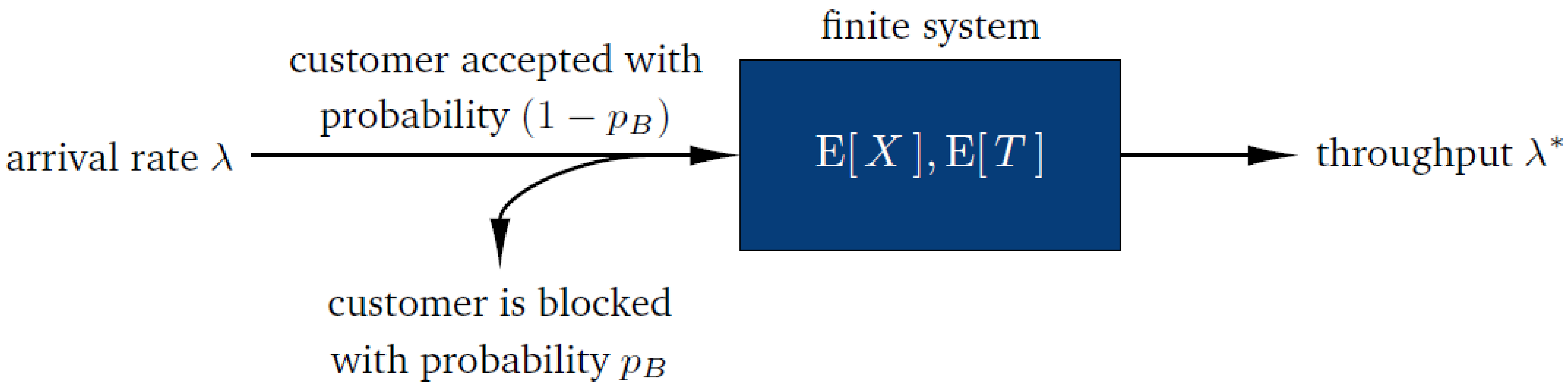


LITTLE'S THEOREM:
FINITE SYSTEM WITH BLOCKING

Loss system GI/GI/n-S

Finite System with Blocking

- Loss system GI/GI/n-S



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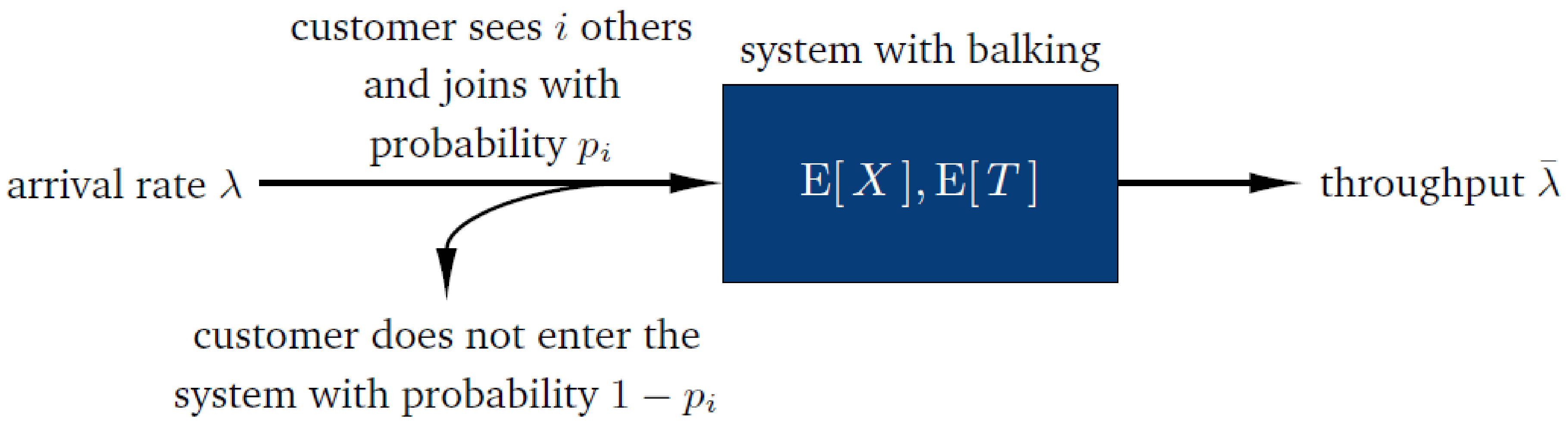
Whiteboard

LITTLE'S THEOREM: QUEUEING SYTEM WITH BALKING

Arriving customers refusing to enter the queue

Queueing System with Balking

- Customer arrives and sees the system in state i which means i other customers in system.
- With probability $1 - p_i$ the arriving customer refuses to enter the queue (balking)
- Example: waiting lines in a supermarket



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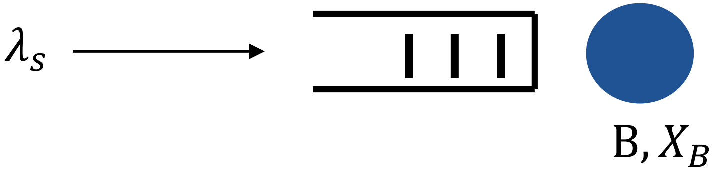
Utilization Law

Utilization of a server

The Utilization Law

- Applying Little's theorem to a server of a queueing system leads to the utilization law.

- Consider average number $E[X_B]$ of customers at a **single server** (i.e. not in waiting queue)
  - identical to **utilization** of the server: fraction of the time the server is busy
  - arrival rate (of accepted) customers at the server: λ_s
  - average service time of a customer: $\mathrm{E}[B]$



- Little's law yields the utilization law: $E[X_B] = \lambda_s \cdot E[B] < 1$

Delay System GI/GI/1-∞

- Consider the single server delay system with infinite waiting room: GI/GI/1-∞

- Utilization law leads to the probability that the system is empty

$$x(0) = P(X = 0) = 1 - E[X_B] = 1 - \lambda_s \cdot E[B]$$

- Example: IoT gateway
  - The measured throughput is 125 packets per second.
  - Each packet requires a forwarding time of 0.002 seconds.
  - What is the utilization?
  - What is the probability that the gateway is idle?

Serendipity…

A Note of Personal History (Little)

How did a sensible young PhD like me get involved in a crazy field like this? From 1957–1962, I taught operations research at the Case Institute of Technology in Cleveland (now Case Western Reserve University). I was asked to teach a course on queuing. OK. Initially I used my own notes, but when Morse (1958) came out, I used his book extensively. Queuing was taken by most of the OR graduate students and, indeed, one of these, Ron Wolff, went on to become a first class queuing theorist (Wolff 1989). One year we were at the point when we had done the basic Poisson-exponential queue and moved through multi-server queues, and some other general cases. I remarked, as many before and after me probably have (and Morse does), that the often reappearing formula $L = \lambda W$ seemed very general. In addition I gave the heuristic proof that is essentially Fig. 5.2 at the beginning of this chapter. After class I was talking to a number of students and one of them (Sid Hess) asked, "How hard would it be to prove it in general?" On the spur of the moment, I obligingly said, "I guess it shouldn't be too hard." Famous last words. Sid replied, "Then you should do it!"

The remark stuck in my mind and I started to think about the question from time to time. Clearly there was something fundamental going on, since, when you

Little, John DC, and Stephen C. Graves. "Little's law." *Building Intuition*. Springer US, 2008. 81-100.

General Results for Delay Systems

GI/GI/n delay systems

Notation

- Notation of random variables
  - interarrival time A
  - waiting time W
  - number of customers in queue X_W
  - service time B
  - number of customers in service X_B

- W, X_W
- B, X_B

- Notation of rates
  - arrival rate $\lambda = \frac{1}{E[A]}$ of customers
  - service rate $\mu = \frac{1}{E[B]}$ of a single server

- Notation of load
  - offered load $a = \lambda \cdot E[B]$
  - normalized offered load $\rho = \frac{a}{n} = \frac{\lambda}{n\mu}$

General Results for GI/GI/n Delay Systems

- **Sojourn time** or **response time:** $T = W + B$
- Number of customers in the system: $X = X_W + X_B$

- **Stability condition** for delay systems: $a = \lambda \cdot E[B] < n$ or $\rho = a/n < 1$
- *Note*: Finite buffer systems GI/GI/n-S are always stable due to blocking.

- Mean number of **busy servers**: $E[X_B] = \lambda \cdot E[B] = a$

- **Utilization**, i.e. fraction of time each server is busy: $\rho = a/n$

Loss Formula

GI/GI/n-S loss systems

Loss Formula for GI/GI/n-S Loss Systems

- Consider a single server in GI/GI/n-S system with ρ_s being the mean offered load of the server
- Mean arrival rate at the considered server is $\lambda_s = \lambda / n$
- It is: $\rho_s = \frac{\lambda \cdot E[B]}{n} = \lambda_s \cdot E[B]$

- Assuming that an arbitrarily chosen server is idle with probability ϕ_s

- Blocking probability that an arbitrary customer is blocked at that server

loss formula $p_B = 1 - \frac{1-\phi_s}{\rho_s}$

- GI/GI/1-∞ delay system?