

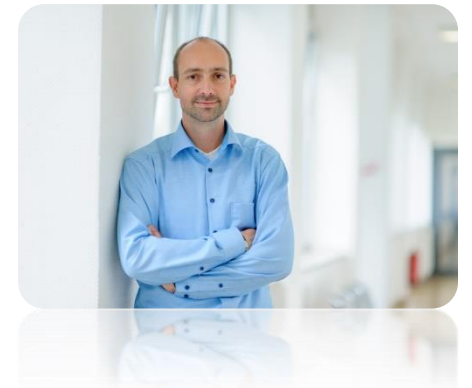
Chapter 3.5

Markov State Process

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022



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*Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication
Networks - A Lecture Note. Würzburg University Press.
<https://doi.org/10.25972/WUP-978-3-95826-153-2>*

Website to download book, exercises, slides and scripts:
<https://modeling.systems/>

Chapter 3

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DEFINITION OF CONTINUOUS-TIME MARKOV CHAIN

Continuous-Time Markov Chain (CTMC)

► Continuous-time Markov process

- Stochastic process $\{X(t), t \geq 0\}$ with Markov property

► Continuous-time Markov chain (CMTC) is defined by

- discrete state space S is finite or countable; e.g. number of customers in system
- transition rates $q_{ij} \geq 0$ for $i \neq j$ and $i, j \in S$
- initial state $X(0)$, i.e. probability distribution of initial state

► Probability $x(i, t) = P(X(t) = i)$ that the system is in state $[X = i]$ at time t

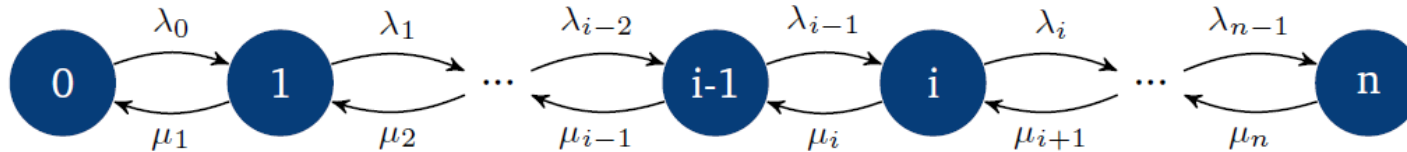
► State vector $\mathbf{X}(t) = (x(0, t), x(1, t), \dots)$

► Definition of **rate matrix** \mathbf{Q} with $q_{ii} = -\sum_{i \neq j} q_{ij}$

- allows compact notation (Kolmogorov equations.)
- row-wise sums of \mathbf{Q} are 0

Illustration of CTMC: Example

- ▶ State space $S = \{0, 1, 2, \dots, n\}$



- ▶ Transition rates

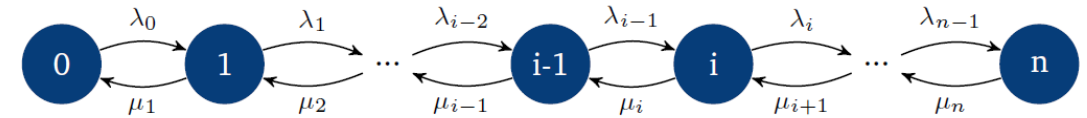
- $q_{i,i+1} = \lambda_i$ for $i = 0, 1, \dots, n-1$
- $q_{i,i-1} = \mu_i$ for $i = 1, \dots, n$
- otherwise: $q_{i,j} = 0$ for $i \neq j$
- $q_{ii} = -\sum_{i \neq j} q_{i,j}$

- ▶ Transition matrix

$$Q = \begin{pmatrix} q_{00} & q_{01} & \dots & q_{0j} & \dots \\ q_{10} & q_{11} & \dots & q_{1j} & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ q_{j0} & q_{j1} & \dots & q_{jj} & \dots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}$$

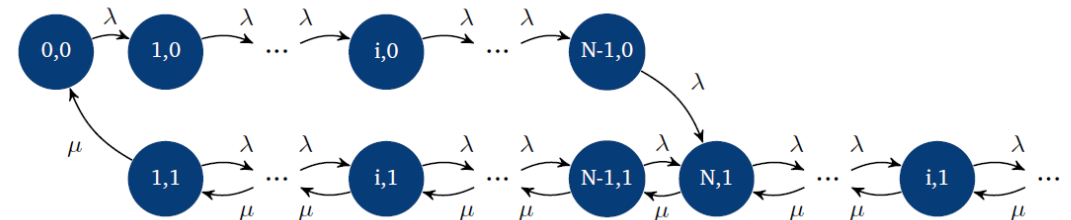
Transition Behavior of CTMC

- ▶ System remains in state i for time T_i
 - exponentially distributed with rate $q_i = -q_{ii} > 0$
 - time to change from i to j : $T_{ij} \sim \text{EXP}(q_{ij})$
 - $T_i = \min_{i \neq j} \{T_{ij}\} \sim \text{EXP}(q_i)$



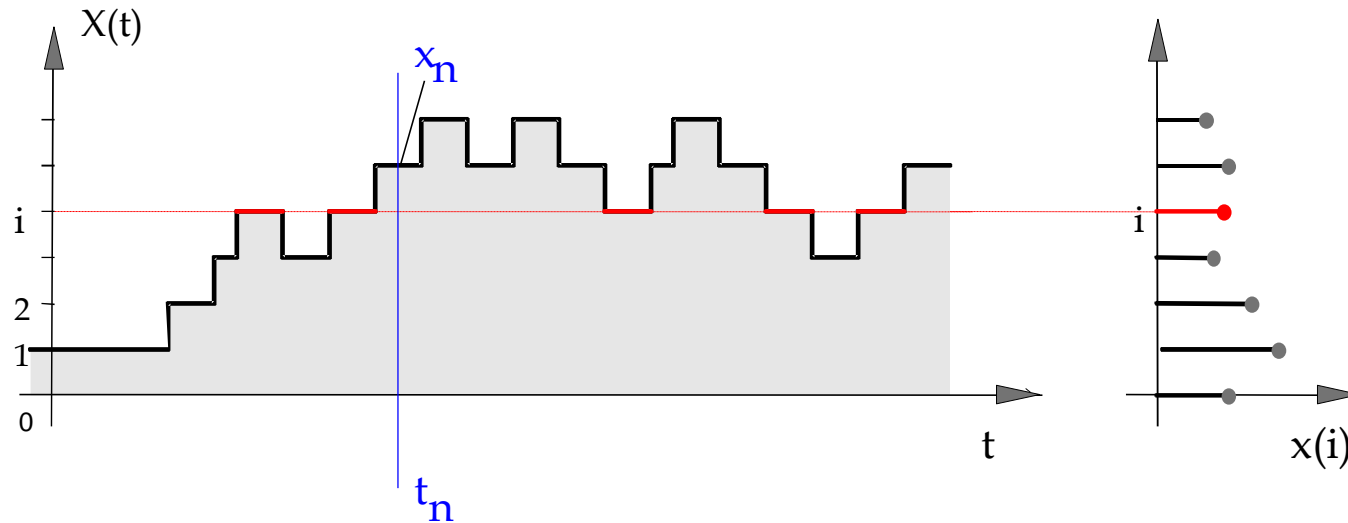
- ▶ When process leaves state i , the state j is reached with probability

- $p_{ij} = \frac{q_{ij}}{q_i}$

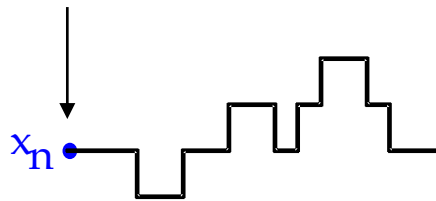


Analysis of Continuous-time Markov Chain (CMTC)

independent outside observer at time t_n sees system state probability x_n



state $[X(t_n) = x_n]$



future development of process only depends on state x_n

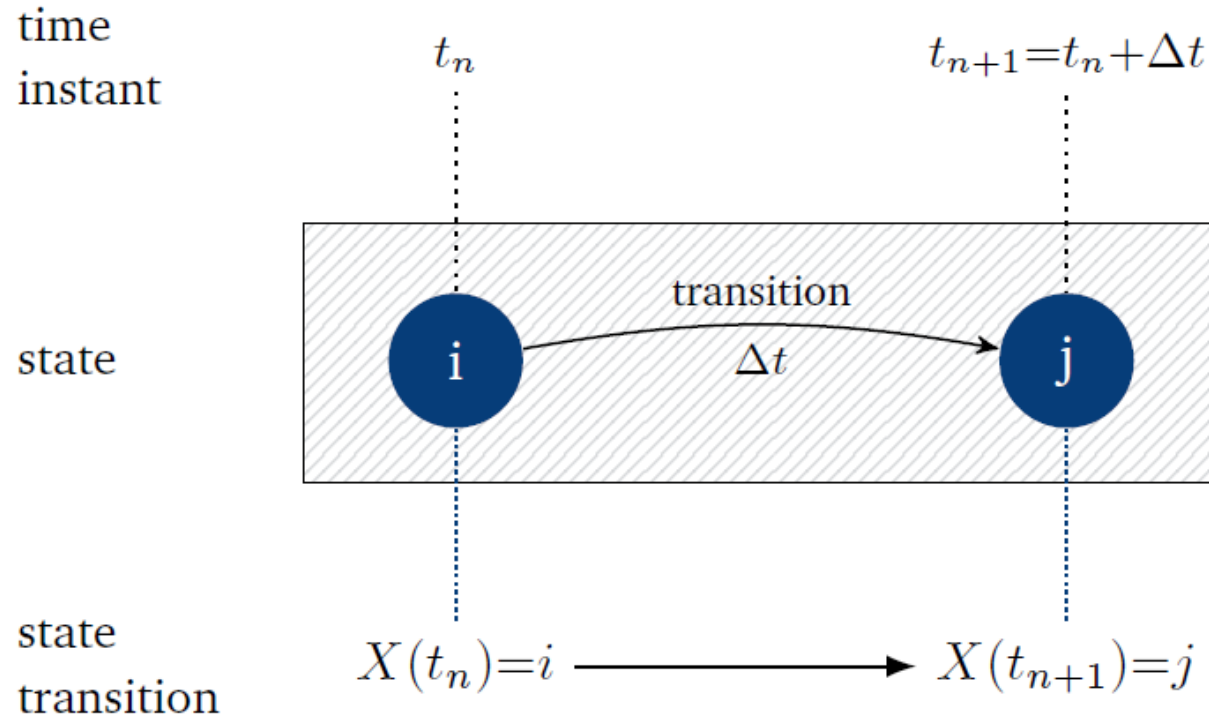
Overview on the Analysis

- ▶ Markov state process and memoryless property
- ▶ **Homogeneous systems**: transition probabilities independent of time instant
- ▶ **Chapman-Kolmogorov equation** for state transition probabilities: $\mathcal{P}(t+\Delta t) = \mathcal{P}(t) \cdot \mathcal{P}(\Delta t)$
 - Kolmogorov forward equation for **probability densities** $\lim_{\Delta t \rightarrow 0} \mathcal{P}(t+\Delta t) \longrightarrow \frac{d\mathcal{P}(t)}{dx} = \mathcal{P}(t) \cdot Q$
 - Kolmogorov forward equation for **state probabilities** $\mathcal{X}(t) = \mathcal{X}(0) \cdot \mathcal{P}(t) \longrightarrow \frac{d\mathcal{X}(t)}{dx} = \mathcal{X}(t) \cdot Q$
- ▶ **Stationary system** $\lim_{t \rightarrow \infty} \longrightarrow \frac{\partial}{\partial t} x(j, t) = 0 \quad \mathcal{X} \cdot Q = 0$
- ▶ Example: **birth-and-death process**

TRANSITION BEHAVIOR OF MARKOVIAN STATE PROCESSES

Chapman-Kolmogorov equation

Transition Behavior of Markovian State Processes



Transition Probability

- ▶ State transition $i \rightarrow j$ during interval $\Delta t = t_{n+1} - t_n$ occurs with probability

$$p_{ij}(t_n, t_{n+1}) = P(X(t_{n+1})=j \mid X(t_n)=i)$$

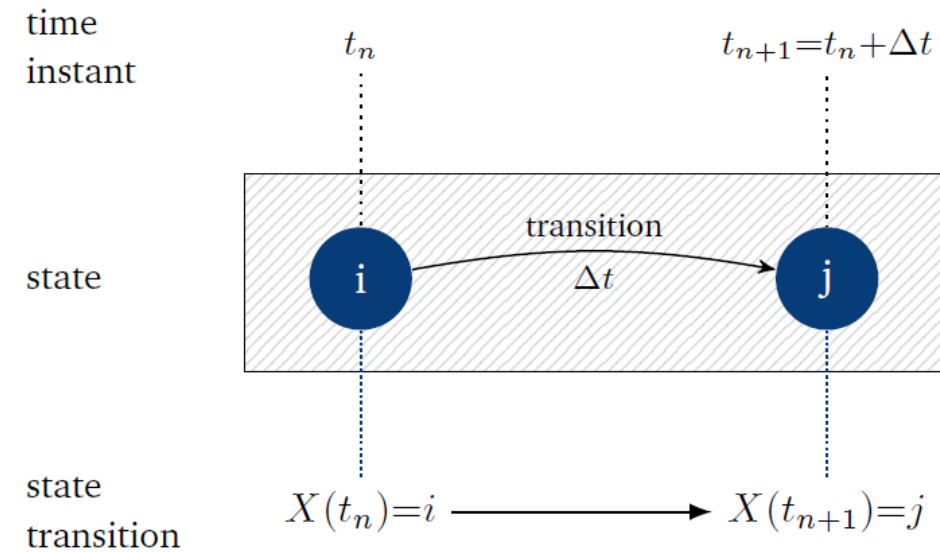
- ▶ **Time-homogeneous state process**

- transition behavior is identical for each process point in time
- transition probability is independent of the observation instant

$$p_{ij}(t_n, t_{n+1}) = p_{ij}(t_{n+1} - t_n) = p_{ij}(\Delta t)$$

- standardization condition for all i

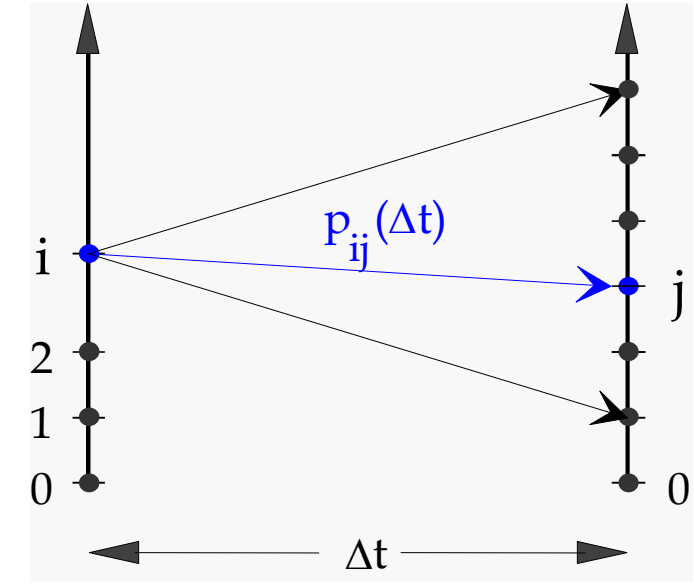
$$\sum_j p_{ij}(\Delta t) = 1, \quad \Delta t \geq 0$$



Transition Matrix

- ▶ Transition probabilities $\{p_{ij}(\Delta t), i, j = 0, 1, \dots\}$ form the transition matrix

$$\mathcal{P}(\Delta t) = \begin{pmatrix} p_{00}(\Delta t) & p_{01}(\Delta t) & \dots & p_{0j}(\Delta t) & \dots \\ p_{10}(\Delta t) & p_{11}(\Delta t) & \dots & p_{1j}(\Delta t) & \dots \\ \vdots & \vdots & & \vdots & \\ p_{i0}(\Delta t) & p_{i1}(\Delta t) & \dots & p_{ij}(\Delta t) & \dots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

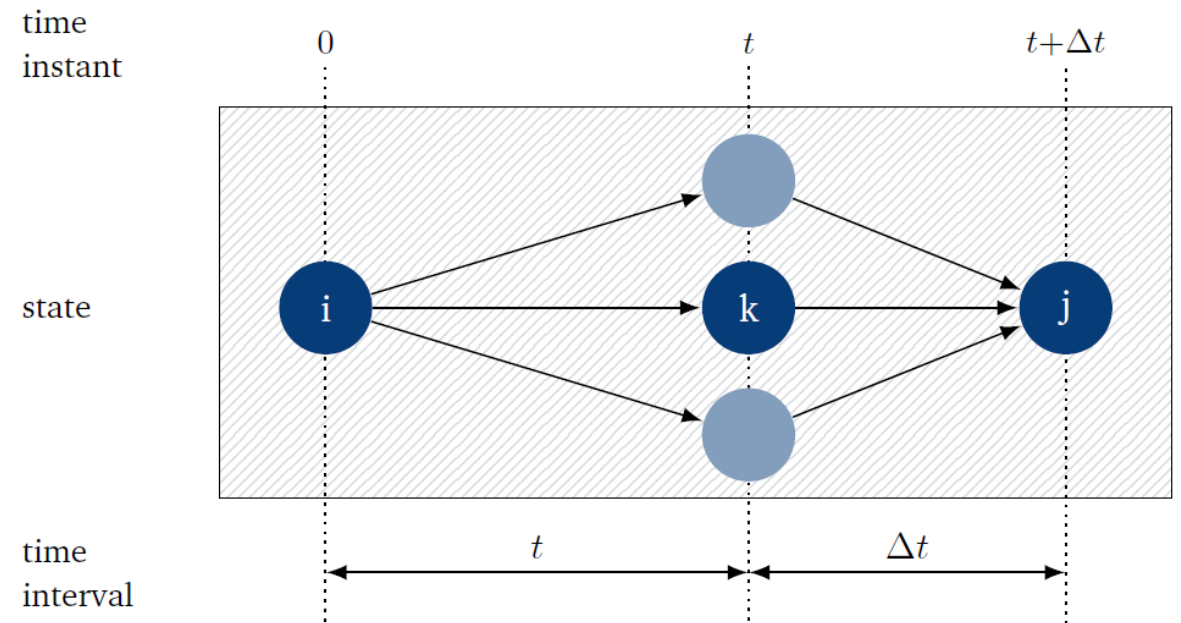


Chapman-Kolmogorov Equation

$$\mathcal{P}(t + \Delta t) = \mathcal{P}(t) \cdot \mathcal{P}(\Delta t)$$

or

$$p_{ij}(t + \Delta t) = \sum_k p_{ik}(t) p_{kj}(\Delta t)$$



Chapman-Kolmogorov Equation (f.)

Lecture

KOLMOGOROV FORWARD EQUATION FOR TRANSITION PROBABILITIES

Kolmogorov Forward Equation for Transition Probabilities

- ▶ Chapman-Kolmogorov equation

$$p_{ij}(t + \Delta t) = \sum_k p_{ik}(t) p_{kj}(\Delta t)$$

- ▶ can be formulated as

$$p_{ij}(t + \Delta t) = \sum_{k \neq j} p_{ik}(t) p_{kj}(\Delta t) + p_{ij}(t) p_{jj}(\Delta t)$$

$$\frac{p_{ij}(t + \Delta t) - p_{ij}(t)}{\Delta t} = \sum_{k \neq j} p_{ik}(t) \cdot \frac{p_{kj}(\Delta t)}{\Delta t} - p_{ij}(t) \cdot \frac{1 - p_{jj}(\Delta t)}{\Delta t}$$

- ▶ Next: limiting process $\Delta t \rightarrow 0$

Kolmogorov Forward Equation: Limiting Process

$$\frac{p_{ij}(t + \Delta t) - p_{ij}(t)}{\Delta t} = \sum_{k \neq j} p_{ik}(t) \cdot \frac{p_{kj}(\Delta t)}{\Delta t} - p_{ij}(t) \cdot \frac{1 - p_{jj}(\Delta t)}{\Delta t}$$

► Limiting process $\Delta t \rightarrow 0$

$$\lim_{\Delta t \rightarrow 0} \frac{p_{ij}(t + \Delta t) - p_{ij}(t)}{\Delta t} = \frac{d}{dt} p_{ij}(t)$$

first derivative of transition probabilities $p_{ij}(t)$ at time t

$$\lim_{\Delta t \rightarrow 0} \frac{p_{kj}(\Delta t)}{\Delta t} = q_{kj}, \quad k \neq j$$

transition probability density for the transition $k \rightarrow j$

$$\lim_{\Delta t \rightarrow 0} \frac{1 - p_{jj}(\Delta t)}{\Delta t} = q_j = \sum_{k \neq j} q_{jk}$$

transition probability density for leaving the state j

► **Kolmogorov forward equation for transition probabilities**

$$\frac{d}{dt} p_{ij}(t) = \sum_{k \neq j} q_{kj} p_{ik}(t) - q_j p_{ij}(t)$$

Kolmogorov Forward Equation: Matrix Notation

- ▶ Kolmogorov forward equation for transition probabilities

$$\frac{d}{dt} p_{ij}(t) = \sum_{k \neq j} q_{kj} p_{ik}(t) - q_j p_{ij}(t)$$

- ▶ Matrix for transition probability densities is defined

$$Q = \begin{pmatrix} q_{00} & q_{01} & \cdots & q_{0j} & \cdots \\ q_{10} & q_{11} & \cdots & q_{1j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ q_{j0} & q_{j1} & \cdots & q_{jj} & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix} \quad \begin{array}{l} \text{with } \sum_k q_{jk} = 0 \quad \text{row-wise sum is 0} \\ \text{and } q_{jj} = -\sum_{k \neq j} q_{jk} = -q_j \quad \text{probability density to remain in state } j \end{array}$$



$$\frac{d\mathcal{P}(t)}{dt} = \mathcal{P}(t) \cdot Q$$

- ▶ Note:

- „ $-q_j$ “ does not imply that there is a negative rate.
- solely notation for the rate to stay in state

Solution of Kolmogorov Forward Equation

- ▶ Kolmogorov forward equation for transition probabilities in matrix notation

$$\frac{d\mathcal{P}(t)}{dt} = \mathcal{P}(t) \cdot Q$$

- ▶ Solution requires the computation of the matrix exponential of the matrix $t \cdot Q$

$$\mathcal{P}(t) = e^{tQ} = \sum_{k=0}^{\infty} \frac{(tQ)^k}{k!}$$

- ▶ See notebook script "3.5 [Markov processes](https://modeling.systems/): nonstationary and stationary analysis [\[ipynb\]](#)"
<https://modeling.systems/>

STATE EQUATIONS AND STATE PROBABILITIES

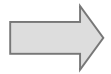
Kolmogorov Forward Equation for State Probabilities

Kolmogorov Forward Equation for State Probabilities

- Probability $x(j, t)$ for the system to be in state j at time t .

$x(j, t) = P(X(t) = j)$ process is in state j at time t

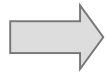
$x(i, 0)$ initial state at time $t = 0$



$$x(j, t) = \sum_i P(X(t) = j | X(0) = i) \cdot P(X(0) = i) = \sum_i x(i, 0) \cdot p_{ij}(t)$$

- Kolmogorov forward equation **for transition probabilities**

$$\frac{d}{dt} p_{ij}(t) = \sum_{k \neq j} q_{kj} p_{ik}(t) - q_j p_{ij}(t)$$



$$\sum_i \frac{d}{dt} p_{ij}(t) x(i, 0) = \sum_{k \neq j} q_{kj} \sum_i (p_{ik}(t) x(i, 0)) - \sum_i (q_j p_{ij}(t) x(i, 0))$$

- Kolmogorov forward equation **for state probabilities**

$$\frac{\partial}{\partial t} x(j, t) = \sum_{k \neq j} q_{kj} x(k, t) - q_j x(j, t), \quad j = 0, 1, \dots, \quad \left(\sum_j x(j, t) = 1 \right)$$

Kolmogorov Forward Eq. for State Probabilities: Matrix Notation

- ▶ Kolmogorov forward equation **for state probabilities** is system of differential equations

$$\frac{\partial}{\partial t} x(j, t) = \sum_{k \neq j} q_{kj} x(k, t) - q_j x(j, t), \quad j = 0, 1, \dots,$$

- ▶ State probabilities as vector $\mathbf{X}(t) = (x(0, t), x(1, t), \dots, x(j, t), \dots)$

- ▶ Initial state and transition probability matrix $\mathbf{X}(t) = \mathbf{X}(0) \cdot \mathcal{P}(t)$

- ▶ **Matrix notation** of Kolmogorov forward equation for state probabilities

$$\frac{d}{dt} \mathbf{X}(t) = \mathbf{X}(t) \cdot \mathcal{Q}$$

Stationary State Probabilities

- ▶ State process is **stationary** if
 - system state stops changing (statistical equilibrium)
 - state probability no longer depends on time t

$$\frac{d}{dt} P(X(t) = j) = \frac{\partial}{\partial t} x(j, t) = 0$$

- ▶ Stationary state probability

$$x(j) = \lim_{t \rightarrow \infty} P(X(t) = j)$$

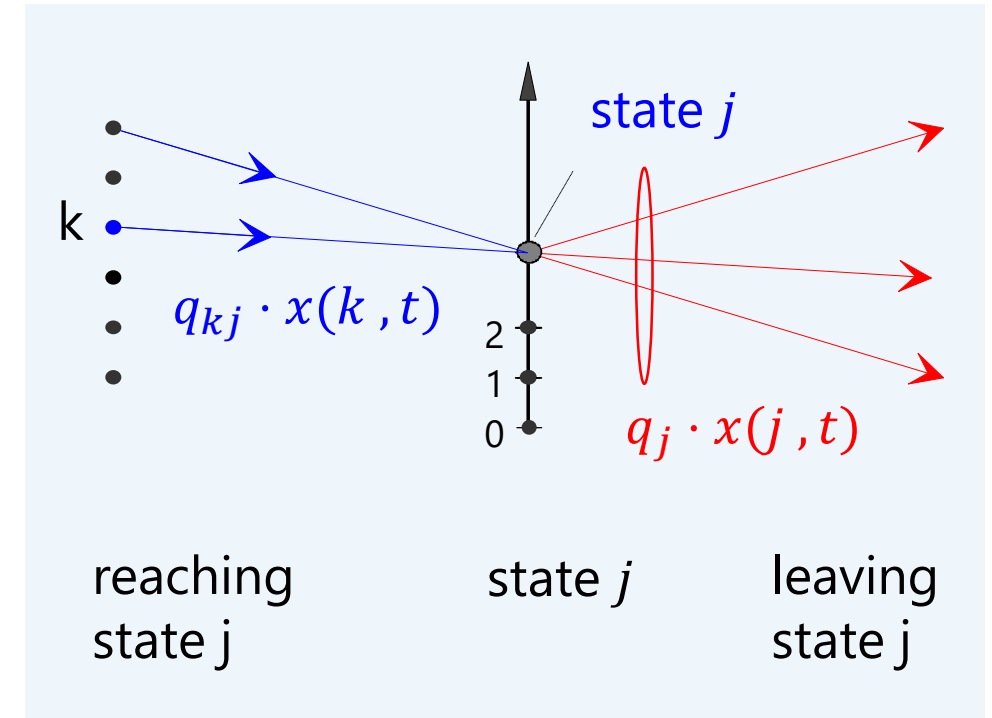
- ▶ Results in stationary equation system

$$\frac{\partial}{\partial t} x(j, t) = \sum_{k \neq j} q_{kj} x(k, t) - q_j x(j, t), \quad j = 0, 1, \dots,$$



$$q_j x(j) = \sum_{k \neq j} q_{kj} \cdot x(k), \quad j = 0, 1, \dots,$$

$$\sum_j x(j) = 1.$$

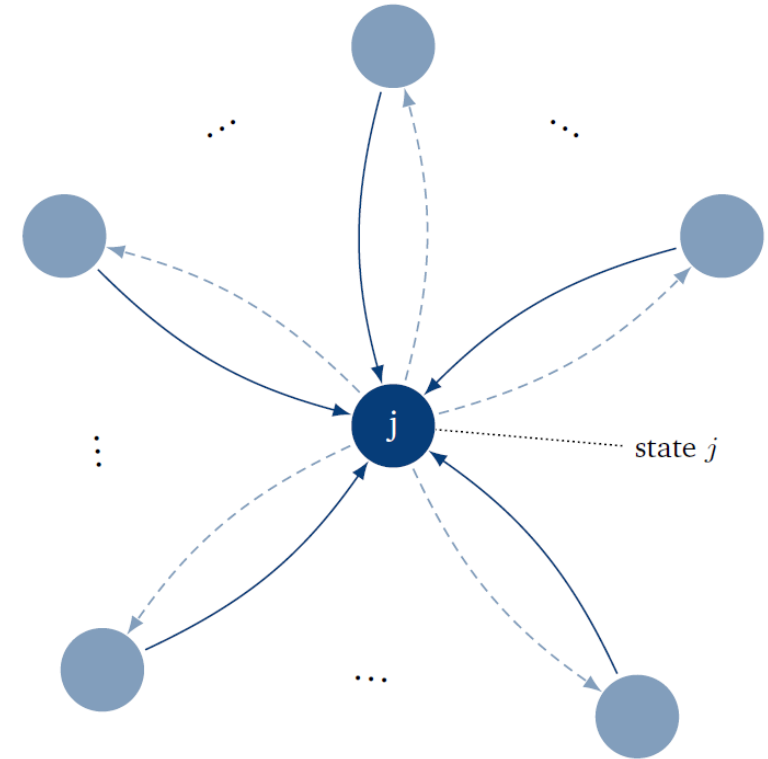


Principle of Maintaining Statistical Equilibrium

$$\sum_j x(j) = 1$$
$$q_j x(j) = \sum_{k \neq j} q_{kj} x(k)$$

↑
rate for
leaving
state j

↑
rate for
reaching
state j



► Principle of maintaining statistical equilibrium

- Stationary system: flows of weighted probability densities for reaching and leaving a state must be in equilibrium, i.e., they are the same
- state probability no longer changes in time

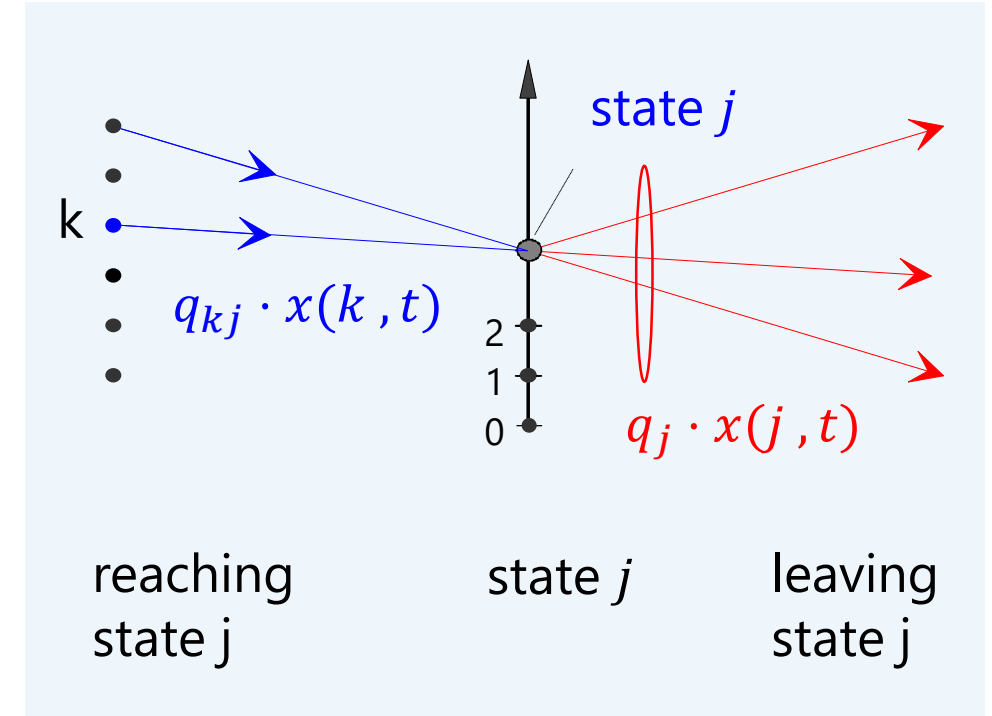
Stationary Equation System: Matrix Notation

$$\sum_j x(j) = 1$$

$$q_j x(j) = \sum_{k \neq j} q_{kj} x(k)$$

↑
↑

rate for leaving state j = rate for reaching state j



► Stationary state probability vector

$$\mathcal{X} = \{x(0), x(1), \dots, x(j), \dots\}$$

► **State equation system**

$$\mathcal{X} \cdot Q = 0$$

$$\left(\sum_j x(j) = 1 \quad \text{or} \quad \mathcal{X} e = 1 \right)$$

Linear Dependency of State Equations

- ▶ Consider a finite state space $\{0,1,2, \dots, N\}$
- ▶ Stationary system of equations

$$\sum_{j=1}^N x(j) = 1$$
$$q_j x(j) = \sum_{k \neq j} q_{kj} x(k), \quad j=0,1,\dots,N$$

- $(N + 2)$ equations for $(N + 1)$ unknowns (state probabilities)
- Any arbitrarily chosen equation can be omitted due to linear dependency to solve the equation system

Macro States

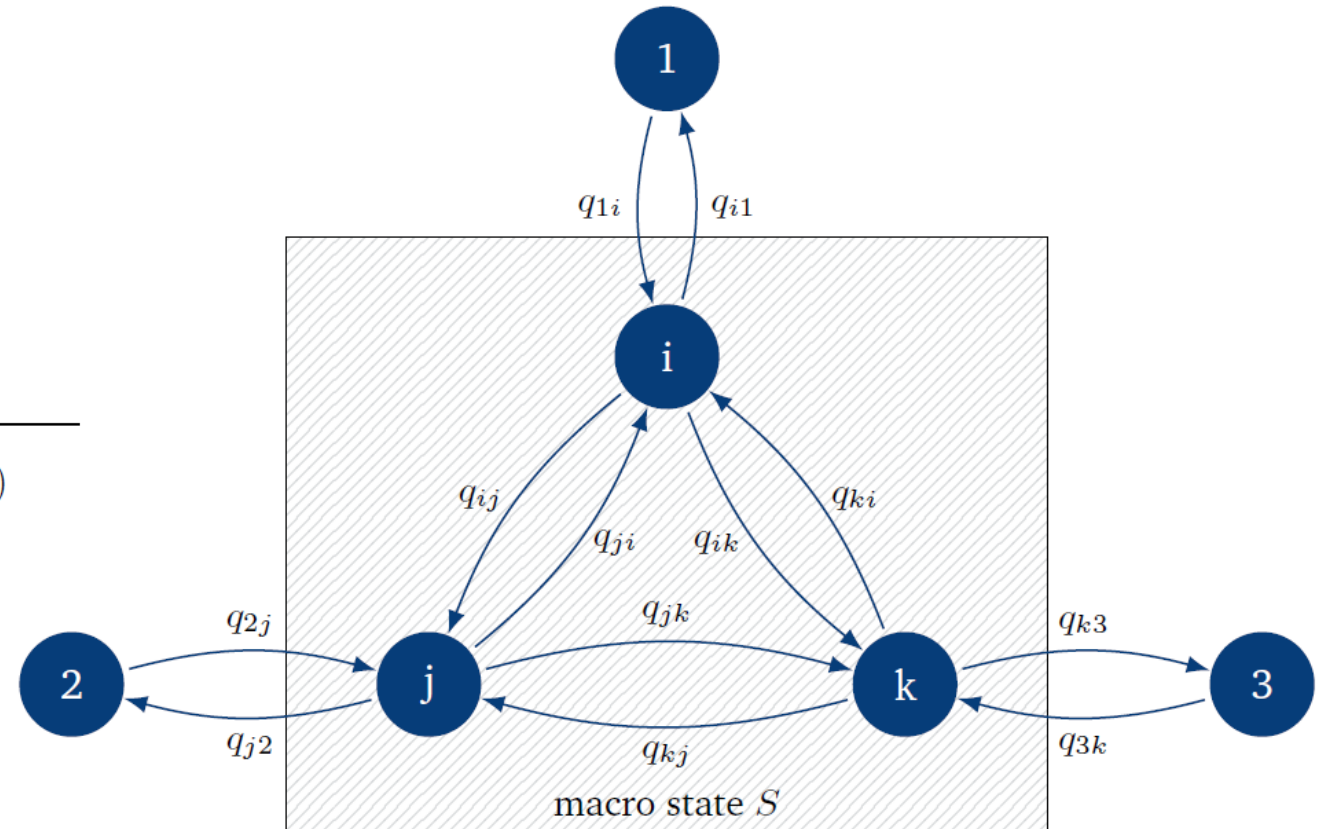
- ▶ Single state that cannot be further decomposed is also called a **micro state**
- ▶ Combination of any set of micro states leads to a **macro state**

- ▶ Example

$$+ \begin{cases} (q_{i1} + q_{ij} + q_{ik})x(i) = q_{1i}x(1) + q_{ji}x(j) + q_{ki}x(k) \\ (q_{j2} + q_{jk} + q_{ji})x(j) = q_{2j}x(2) + q_{kj}x(k) + q_{ij}x(i) \\ (q_{k3} + q_{ki} + q_{kj})x(k) = q_{3k}x(3) + q_{ik}x(i) + q_{jk}x(j) \end{cases}$$

$$q_{i1}x(i) + q_{j2}x(j) + q_{k3}x(k) = q_{1i}x(1) + q_{2j}x(2) + q_{3k}x(3)$$

- ▶ Appropriate choice of macro states often provides simpler system of equations for computing the (micro) state probabilities



Global Equilibrium Equation

- State equation for an arbitrary macro state

$$\underbrace{\sum_{j \in S, u \notin S} q_{ju} x(j)}_{\text{weighted rates for leaving the macro state } S} = \underbrace{\sum_{u \notin S, j \in S} q_{uj} x(u)}_{\text{weighted rates for reaching the macro state } S}$$

- Relates transition probability densities between a macro state and rest in the state space
- Global equilibrium equations are also referred to as full or **global balance equations**

Example: Stationary State Equations

Example: Stationary Macro State Equations

Lecture

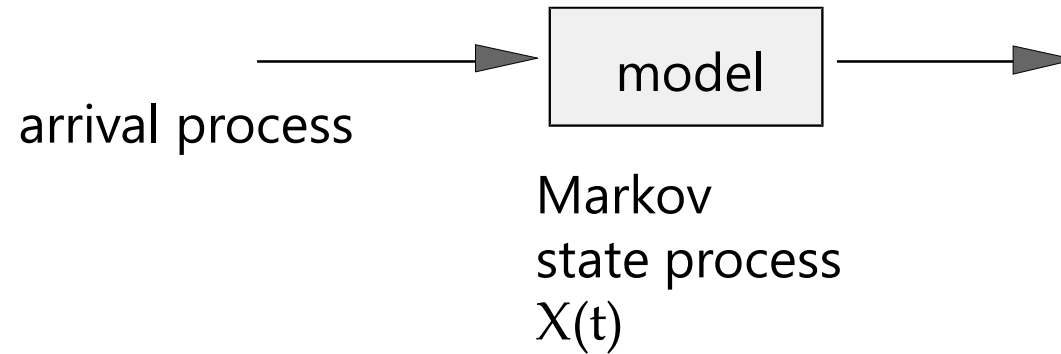
Summary: Stationary Equation System

Lecture

EXAMPLES OF TRANSITION PROBABILITY DENSITIES

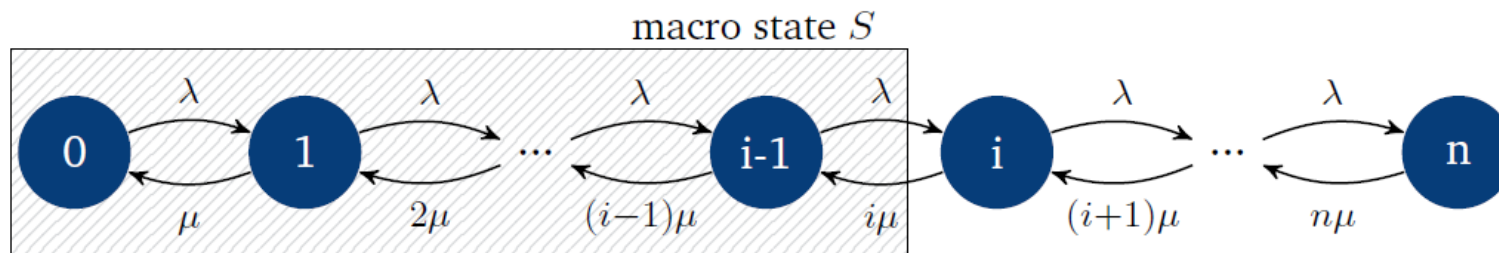
Transition Probability Densities

- What is the transition probability density of a Markov state process?

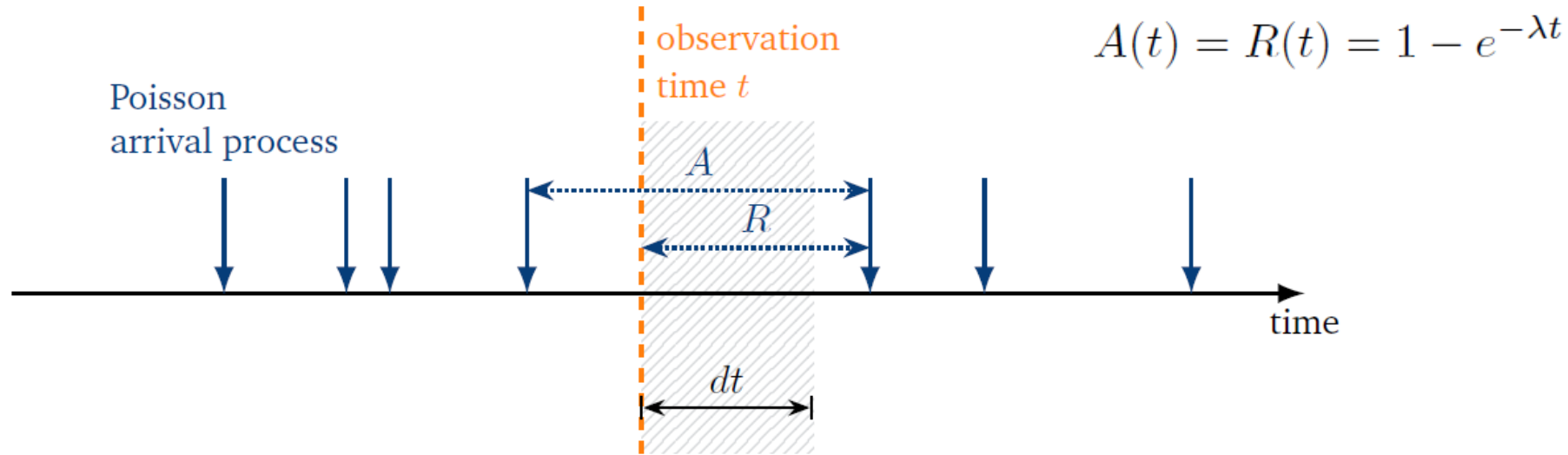


- Example: M/M/n-0

- transition rates of a Poisson process? $q_{i,i+1} = \lambda$.



Transition Probability Density of Poisson Arrival Process



- Transition $i \rightarrow i + 1$: arriving customer is accepted

$$\begin{aligned}
 q_{i,i+1} &= \lim_{dt \rightarrow 0} \frac{p_{i,i+1}(dt)}{dt} \\
 &= \lim_{dt \rightarrow 0} \frac{P(R \leq dt)}{dt} = \lim_{dt \rightarrow 0} \frac{1 - e^{-\lambda dt}}{dt} \stackrel{\text{L'Hospital}}{=} \lambda
 \end{aligned}$$

$$p_{i,i+1}(dt) = P(X(t + dt) = i + 1 | X(t) = i)$$

Transition Probability Density for Exponential Service Time

- ▶ k servers with exponential service times

$$R(t) = B(t) = 1 - e^{-\mu t}$$

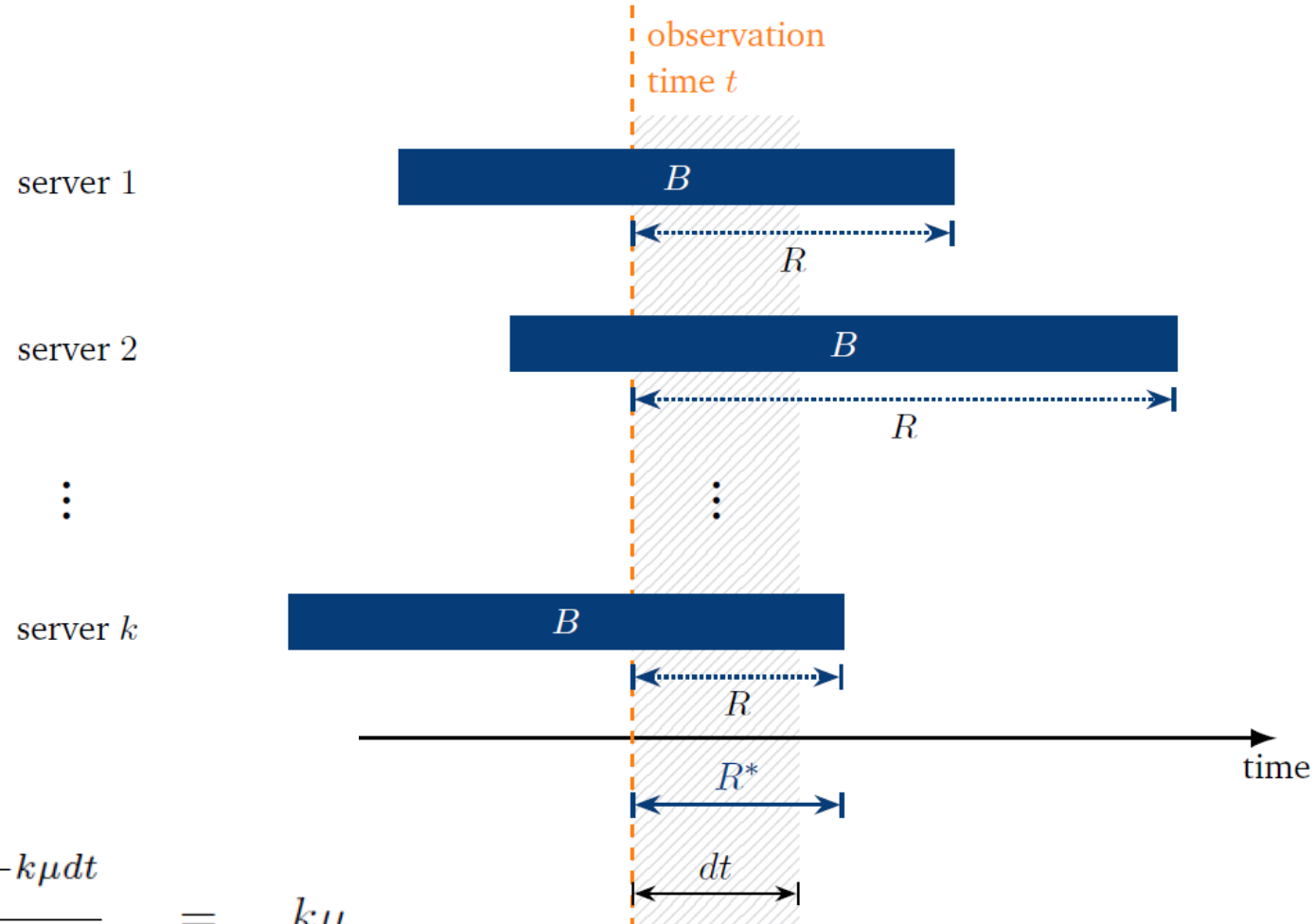
- ▶ Interval R^* until next service termination

$$R^* = \min\{\underbrace{R, \dots, R}_{k\text{-times}}\}$$

$$R^*(t) = 1 - \prod_{i=1}^k (1 - R(t)) = 1 - e^{-k\mu t}$$

- ▶ Transition $k \rightarrow k - 1$

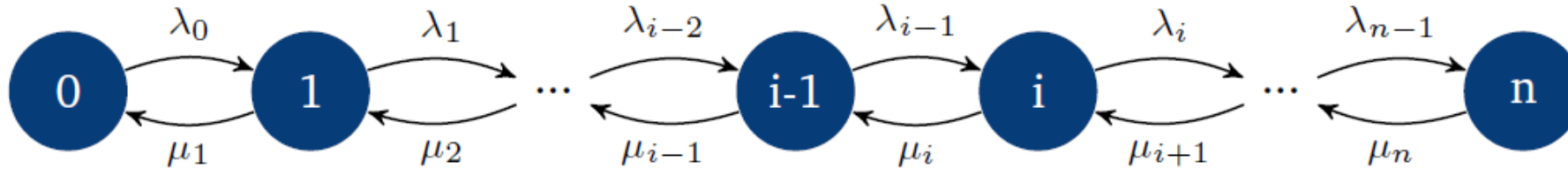
$$\begin{aligned} q_{k,k-1} &= \lim_{dt \rightarrow 0} \frac{p_{k,k-1}(dt)}{dt} \\ &= \lim_{dt \rightarrow 0} \frac{P(R^* \leq dt)}{dt} = \lim_{dt \rightarrow 0} \frac{1 - e^{-k\mu dt}}{dt} \stackrel{\text{L'Hospital}}{=} k\mu \end{aligned}$$



BIRTH-AND-DEATH PROCESSES

Birth-and-Death Processes

- ▶ Markov processes in which only transitions between neighboring states occur
- ▶ State transition diagram of a finite-state, one-dimensional BD process



$$q_{ij} = \begin{cases} \lambda_i & i = 0, 1, \dots, n-1, \quad j = i+1 & \text{birth rate} \\ \mu_i & i = 1, 2, \dots, n, \quad j = i-1 & \text{death rate} \\ 0 & \text{otherwise} \end{cases}$$

- ▶ Pure birth process: all $\mu_i = 0$
 - equilibrium: $x(n) = 1, x(i) = 0$ otherwise
- ▶ Pure death process: all $\lambda_i = 0$
 - equilibrium: $x(0) = 1, x(i) = 0$ otherwise

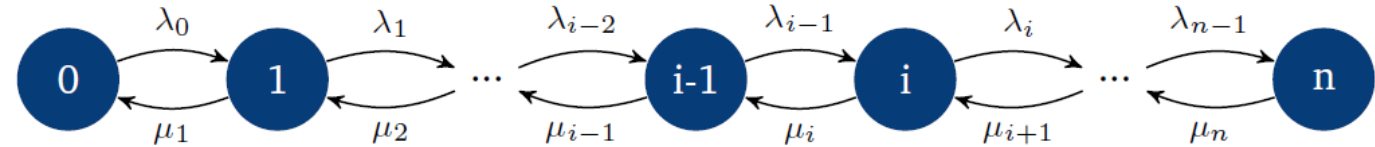
Example: No BD Process

Non-stationary Birth-and-Death Processes

- ▶ Transient (time-dependent) state probabilities of the BD process

$$x(i, t) = P(X(t) = i)$$

$$\Rightarrow \left\{ \begin{array}{l} \frac{\partial}{\partial t} x(0, t) = -\lambda_0 x(0, t) + \mu_1 x(1, t) \\ \frac{\partial}{\partial t} x(i, t) = -(\lambda_i + \mu_i) x(i, t) + \lambda_{i-1} x(i-1, t) + \mu_{i+1} x(i+1, t), \quad i = 1, \dots, n-1 \\ \frac{\partial}{\partial t} x(n, t) = -\mu_n x(n, t) + \lambda_{n-1} x(n-1, t) \end{array} \right.$$



- ▶ Solution of this differential equation system with the initial conditions $\{x(i, 0), i = 0, \dots, n\}$

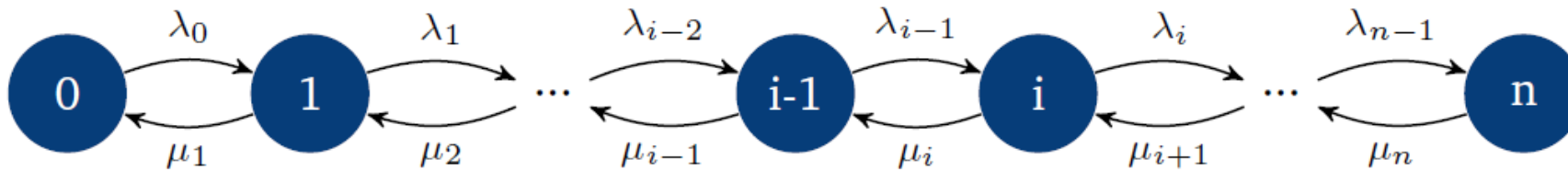
$$\mathbf{X}(t) = (x(0, t), x(1, t), \dots, x(n, t))$$

- ▶ Example: Poisson Process as pure birth process

Example: Poisson Process as Pure Birth Process

Stationary Birth-and-Death Processes

- Equation system for the micro states of the state space

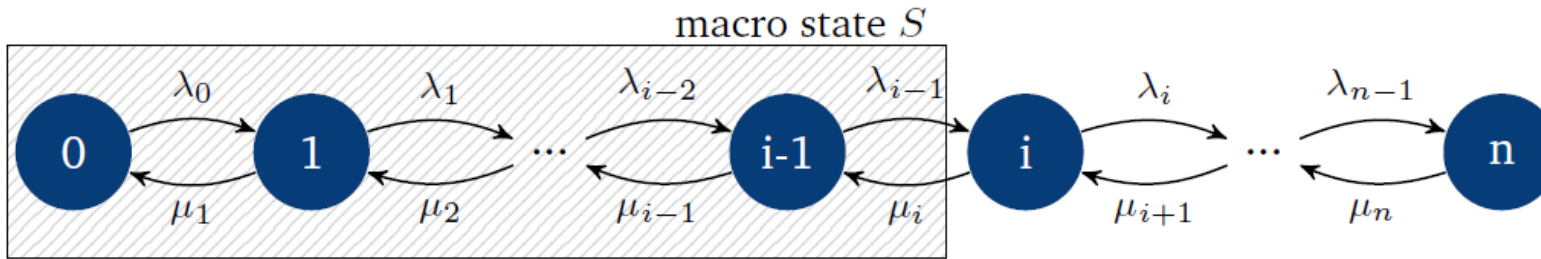


$$\left. \begin{array}{ll} 0 : & \lambda_0 x(0) = \mu_1 x(1) \\ i \in (1, n-1) : & (\lambda_i + \mu_i) x(i) = \lambda_{i-1} x(i-1) + \mu_{i+1} x(i+1), \\ n : & \lambda_{n-1} x(n-1) = \mu_n x(n) \end{array} \right\}$$

because of linear dependency, one of these equations can be omitted

$$\sum_{i=0}^n x(i) = 1$$

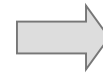
Macro State Equation System



- State equation for macro state S

$$i \in (1, n) \quad \lambda_{i-1} x(i-1) = \mu_i x(i),$$

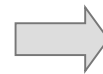
$$\left(\sum_{i=0}^n x(i) = 1 \right)$$



$$x(i) = x(0) \cdot \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=1}^i \mu_k}, \quad i = 1, 2, \dots, n$$

- Normalization yields $x(0)$

$$1 = \sum_{i=0}^n x(i) = x(0) + x(0) \sum_{i=1}^n \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=1}^i \mu_k}$$



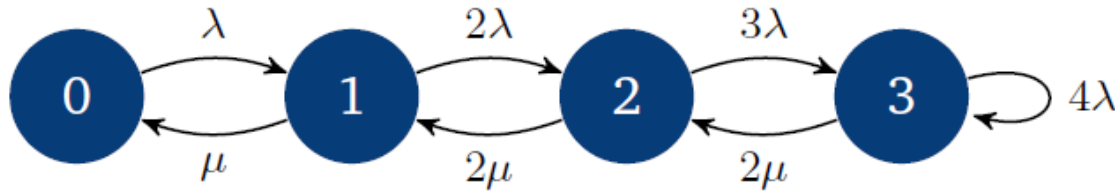
$$x(0) = \left(1 + \sum_{i=1}^n \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=1}^i \mu_k} \right)^{-1}$$

Solving the Macro State Equation System

Lecture

Example: Delay-Loss System with State-dependent Rates

- ▶ Delay-loss system $M(x)/M/2-1$ as an example for birth-and-death process



- ▶ Customers arrive with state-dependent arrival rates
 - not a Poisson process
 - PASTA property is not valid: $x_A(i) \neq x^*(i)$
- ▶ State equation system yields arbitrary-time state probabilities $x^*(i)$
 - e.g. blocking probability requires $x_A(i)$ for arriving customers
 - strong law of large numbers for Markov chains can be applied

Strong Law of Large Numbers for Markov Chains

- ▶ Expected number of arrivals when the system is in state $[X = i]$ in interval of length T is $n_A(i, T)$
- ▶ Expected total number $n_A(T)$ of arrivals

- ▶ **Strong law of large numbers for Markov chains**

$$x_A(i) = \lim_{T \rightarrow \infty} \frac{n_A(i, T)}{n_A(T)} = \frac{\lambda_i \cdot x(i)}{\sum_k \lambda_k \cdot x(k)}$$

- ▶ For birth-and-death process, as in the example, $x_A(i)$ and characteristics like blocking probability or waiting probability can be derived accordingly

