Chapter 3.2

Renewal Processes

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

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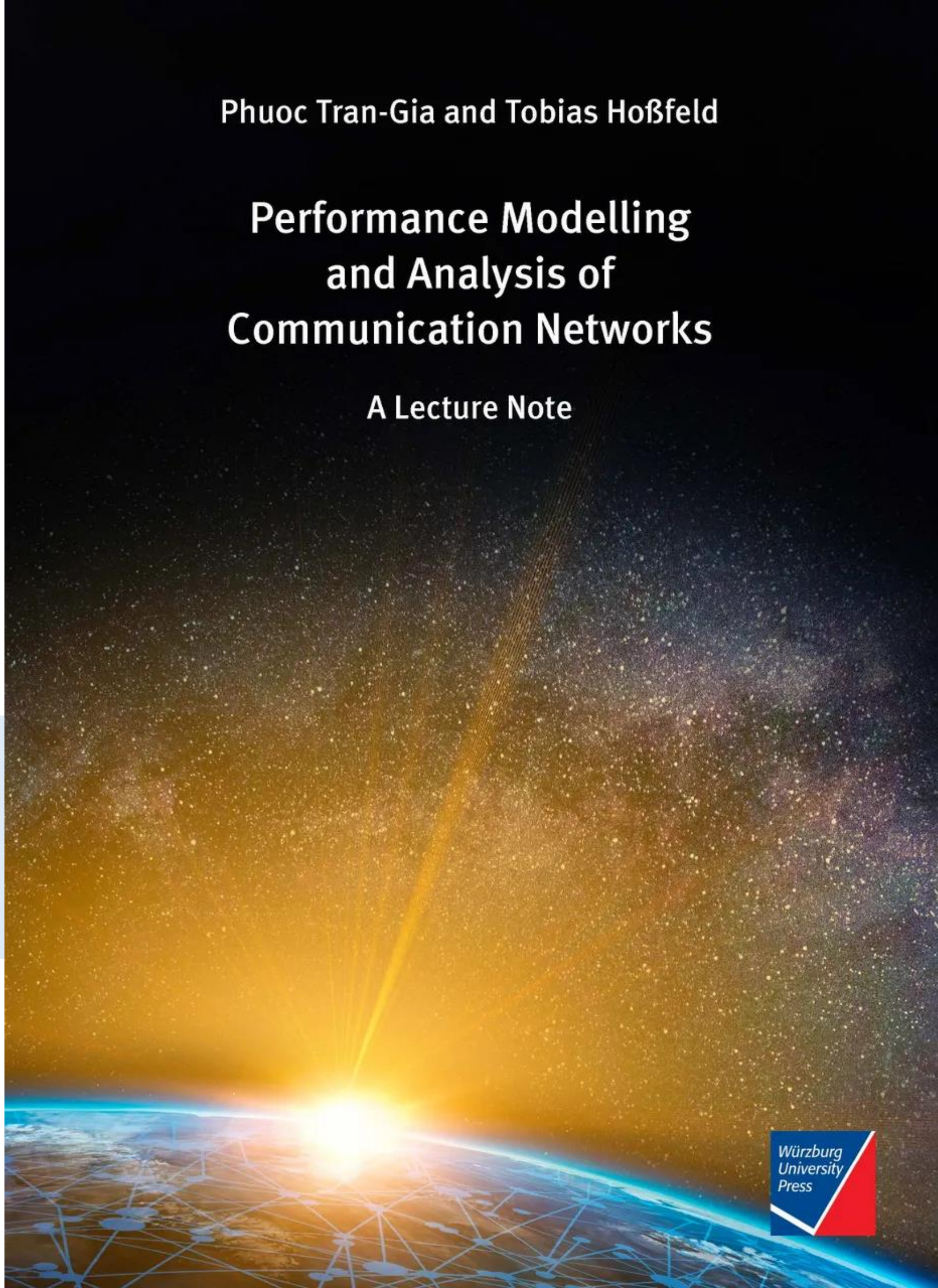
Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/



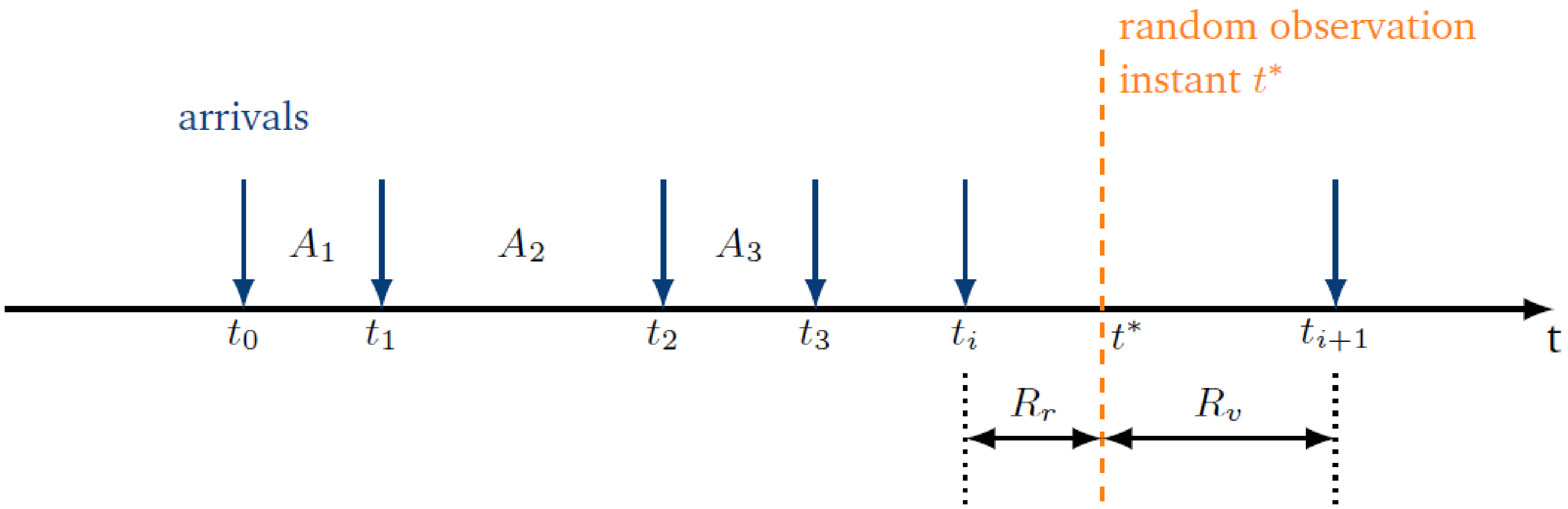
Chapter 3

3 Elementary Random Processes

Point Process and Renewal Process

Point Process

- Finite or infinite sequence of random points in time or occurrences (arrival times of customers)
- Interarrival times A_i between customer arrivals t_{i-1} and t_i may follow different distributions A_i



- Arrival process described by interarrival times or number of arrivals in intervals

Renewal Process

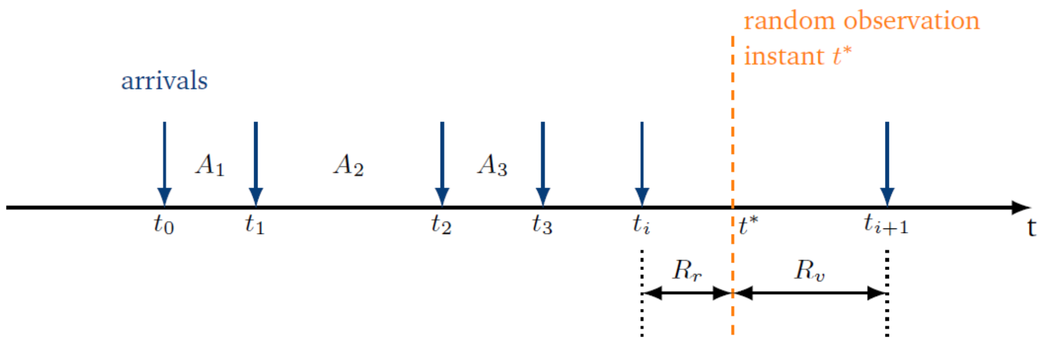
- Renewal process is a special point process

- **Renewal process**: interarrival times are independently and identically distributed (iid)

$$A_i(t) = A(t) \quad \text{for all } i$$

- **Modified renewal process**: first interval deviates from other intervals

$$A_1(t) \;\text{ and }\; A_i(t) = A(t) \;\text{ for } i = 2,3,\ldots$$



Recurrence Time

- Interarrival time A of a renewal process with mean interarrival time $E[A]$

- Total number $N(t)$ of arrival events in the interval $[0;t]$ is counted for a renewal process
- Stochastic process $\{N(t), t\}$ is referred to as **counting process**

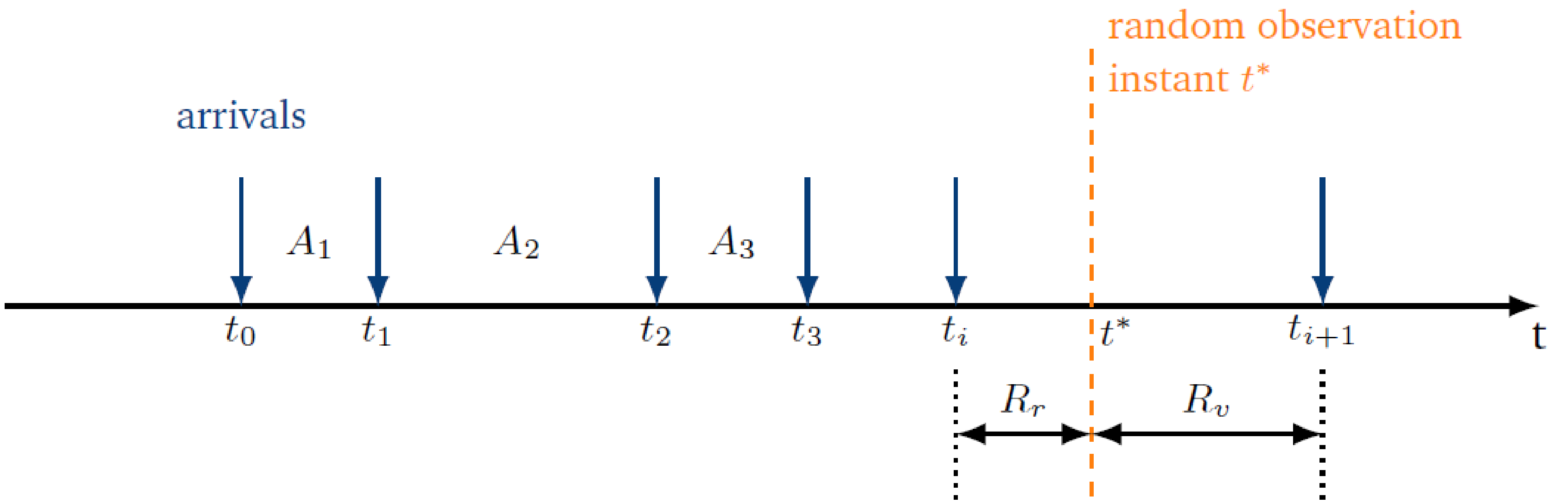
- **Strong law for renewal processes:** With probability 1, the following relation between the arrival rate $\frac{N(t)}{t}$ and the mean interarrival time is observed

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{E[A]}$$

- We typically use $\lambda = \lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{E[A]}$

Recurrence Time

- Process is observed by an independent outside observer at time t^*
- R_v **forward recurrence time**; interval from the observation time to the next arrival;
- R_r **backward recurrence time**; interval from the last arrival to the observation time.



- Random observation time can be at any time: forward and backward recurrence time have same statistical properties ➔ **recurrence time** is analyzed for given iat A

Analysis of Recurrence Time

Cumulative Distribution Function of Recurrence Time

- **Recurrence time** R with $\lambda = \frac{1}{E[A]}$ and interarrival time A

$$r(t) = \frac{1}{E[A]}\,(1 - A(t)) = \lambda\,(1 - A(t))$$

Laplace transform (LT)

$$\Phi_R(s) = \frac{\lambda}{s}(1 - \Phi_A(s))$$

Note:
$\int_0^\infty 1 - A(t)\,dt = E[A]$

- Note
  - $a(t)$ or $A(t)$ known → $r(t)$ can be derived
  - $r(t)$ **and** $E[A]$ known → $a(t)$ can be derived

Recurrence Time: Proof

Recurrence Time: Transform Domain

Lecture

Moments of Recurrence Time

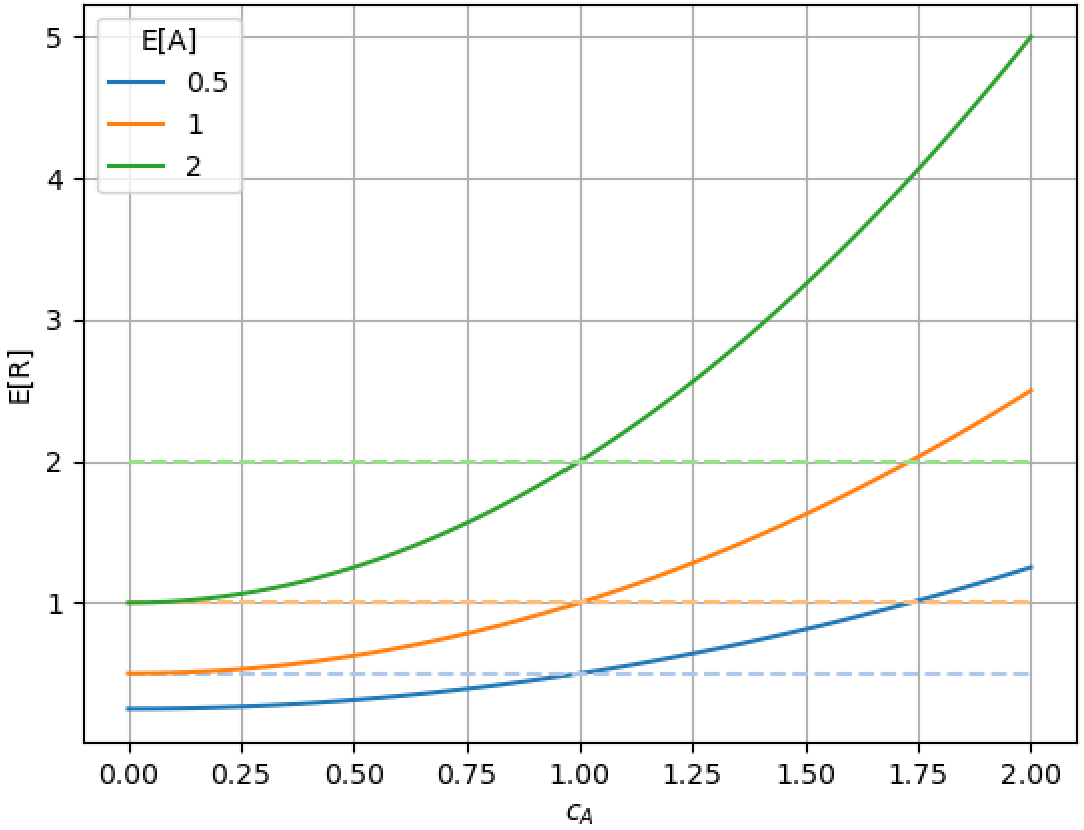
- Ordinary moments of recurrence time

$$E\left[R^k\right] = \frac{E\left[A^{k+1}\right]}{(k+1)\cdot E[A]}.$$

- Mean recurrence time

$$E[R] = \frac{E\left[A^2\right]}{2\,E[A]} = \frac{c_A^2+1}{2}\cdot E[A]$$

$c_A < 1$: $E[R] < E[A]$
$c_A = 1$: $E[R] = E[A]$
$c_A > 1$: $E[R] > E[A]$



- Counterintuitive: $E[R] > E[A]$
- Longer intervals are more frequently encountered by observer
- Larger intervals contribute more often to recurrence time

Moments of Recurrence Time: Proof

Lecture