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Performance Modeling and Analysis of Communication Networks

A Lecture Note

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3 Elementary Random Processes

Question and Answer

Question 3.1. A system state process $X(t)$ is considered at times $t_1, \dots, t_n$ with the initial state $X(t_0)$. The system state $X(t)$ at time t is a random variable. What does the following equation express?

$$P(X(t_{n+1} = x_{n+1}|X(t_n) = x_n, \dots, X(t_0) = x_0) = P(X(t_{n+1} = x_{n+1}|X(t_n) = x_n)$$

a. PASTA property

b. Memoryless property

c. Correlated state process

d. Independence of initial state

Question 3.2. A stochastic process $X(t)$ is considered. Is the steady state distribution depending of the initial state $X(t_0)$?

a. Yes, if system is overloaded.

b. Yes, if system is not overloaded.

c. Yes, there are some cases.

d. No.

Question 3.3. Markov processes $X(t)$ are memoryless. Is $X(t_{n+1}$ independent of $X(t_n)$?

a. Yes, independent, but correlated.

b. Yes, independent and uncorrelated.

c. No, dependent, but uncorrelated.

d. No, dependent, and maybe correlated.

Question 3.4. Can a stochastic process also be state-continuous?

a. Yes, but only for discrete-time systems.

b. Yes, also for discrete-time systems.

c. No, if system is discrete-time.

d. No, if system is continuous.

Question 3.5. In which of the systems is PASTA valid? The arrival rates are λ_i and the service rates are μ_i in state $[X = i]$.

a. M/M/n-0 with $\lambda_i = \lambda$ and $\mu_i = \mu$.

b. $M^{[2]}$/M/n-0 with $\lambda_i = \lambda$ and $\mu_i = \mu$.

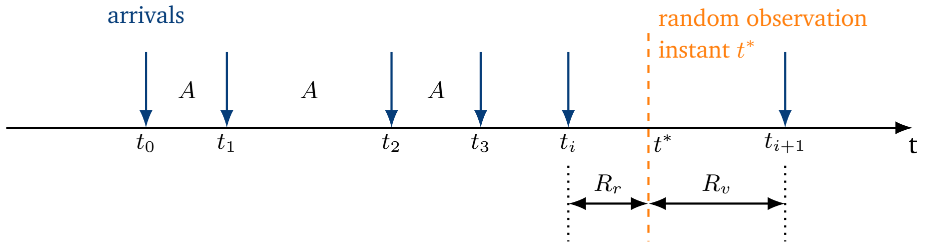
c. M/$M^{[2]}$/n-0 with $\lambda_i = \lambda$ and $\mu_i = \mu$.

d. M/M/n-0 with varying λ_i and $\mu_i = \mu$.

e. M/M/n-0 with $\lambda_i = \lambda$ and varying μ_i.

f. M/GI/1-0 with $\lambda_i = \lambda$ and $\mu_i = \mu$.

Question 3.6. An arrival process with interarrival time A is considered. The forward and backward recurrence time of a random observer is R_v and R_r, respectively. Which of the following equations are correct?



a. $A = R_r + R_v$

b. $A = R_v$

c. $A = R_r$

d. $R_v = R_r$

Question 3.7. Consider an arrival process with interarrival time A and recurrence time R. It is $\mathrm{E}[\,R\,] = \mathrm{E}[\,A\,]$. Is the arrival process memoryless?

a. Yes.

b. No.

Question 3.8. Consider the superposition of n renewal processes. Which conditions are required for the Palm-Khintchine theorem?

a. Processes follow the same distribution.

b. Independence of renewal processes.

c. Overall load is finite.

d. No single process dominates.

Question 3.9. Consider an arrival process with interarrival times $A_i \sim \mathrm{EXP}(i \cdot \lambda)$ with i reflecting the i-th arrival of the process ($i = 1, 2, \dots$). Which statements are correct?

a. This is a renewal process.

b. This is a point process.

c. The process is memoryless.

d. The process is not memoryless.

Question 3.10. What is the mean recurrence time of an arrival process with periodic arrivals?

a. $\mathrm{E}[\,R\,] = 0$

b. $\mathrm{E}[\,R\,] = \mathrm{E}[\,A\,]$

c. $\mathrm{E}[\,R\,] = 1/2$

d. $\mathrm{E}[\,R\,] = \mathrm{E}[\,A\,]/2$

Question 3.11. Which dimension has the rate matrix of an M/M/3 loss system?

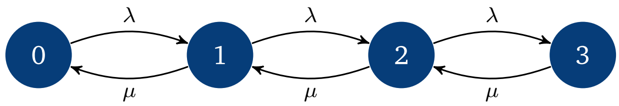
a. 3×1

b. 1×4

c. 3×3

d. 4×4

Question 3.12. What is the first row of the the rate matrix Q the Markovian system given by the following state transition diagram?



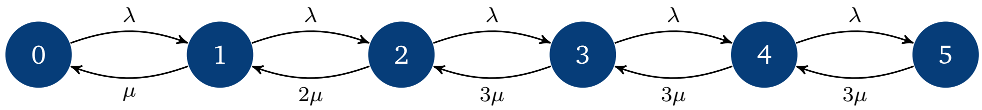
a. $(0 \quad \lambda \quad 0 \quad 0)$

b. $(\mu \quad 0 \quad \lambda \quad 0)$

c. $(\lambda \quad 0 \quad 0 \quad 0)$

d. $(-\lambda \quad \lambda \quad 0 \quad 0)$

Question 3.13. What is Kendall's notation for the Markovian system with the following state transition diagram?



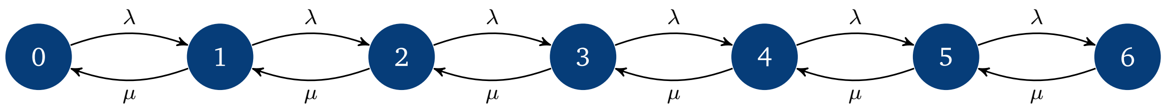
a. M/M/2-3

b. M/M/3-2

c. M/M$^{[3]}$/1-2

d. M/M/5-3

Question 3.14. Which entry is on the diagonal of the rate matrix Q?



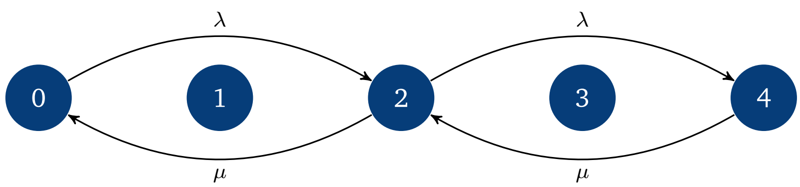
a. $-\lambda - \mu$

b. $\lambda + \mu$

c. $\lambda - \mu$

d. $\mu - \lambda$

Question 3.15. Which features does the system represented by this state transition diagram have? Several answers may be correct.



a. Batch arrivals

b. Bulk service

c. Birth-and-death process

d. Single server

Exercises and Problems

Problem 3.1. Index of Dispersion The so-called *index of dispersion (for intervals)* is used to describe the *burstiness* of arrival processes[1]. The index of dispersion for the random variables $X_1, \ldots, X_n$ is defined as follows:

$$J_n = \frac{\mathrm{VAR}[\,X_1 + \cdots + X_n\,]}{n(\mathrm{E}[\,X_1\,])^2}\ ,$$

where X_n denotes the n -th interarrival time. The covariance of two random numbers X and Y is regarded as a measure of the dependency between X and Y. With the help of the covariance, the variance of a sum of random variables can be expressed as follows:

$$\mathrm{VAR}[\,X_1 + \cdots + X_n\,] = n \cdot \mathrm{VAR}[\,X\,] + 2\sum_{j=1}^{n-1}\sum_{k=1}^{j} \mathrm{COV}[\,X_j, X_{j+k}\,]\ .$$

3.1.1. What are the indices of dispersion J_1 and J_n of a Poisson process?

3.1.2. An expression for J_n depending on the autocorrelation coefficient ϱ_n is to be derived. The autocorrelation coefficient ϱ_n is defined as follows:

$$\varrho_n = \frac{\mathrm{COV}[\,X_1, X_n\,]}{\mathrm{STD}[\,X_1\,] \cdot \mathrm{STD}[\,X_2\,]}.$$

3.1.3. An arrival process is now considered. The interarrival time A follows a hyperexponential distribution with two phases (H_2 distribution). With probability p_1 the phase 1 with arrival rate λ_1 is taken; with probability $p_2 = 1 - p_1$ phase 2 is taken with arrival rate λ_2. For a constant mean value $\mathrm{E}[\,A\,]$, the index of dispersion J_1 is to be calculated depending on p_1 and λ_1, where $\lambda_1 > \frac{p_1}{\mathrm{E}[\,A\,]}$.
What is the limit value of J_1 for the following parameter settings?

(a) $p_1 \to 1$ with $\lambda_1 = \alpha$.

(b) $\lambda_1 \to \infty$ with $p_1 = \alpha$.

(c) $p_1 \to 0$ and $\lambda_1 \to \frac{p_1}{\mathrm{E}[\,A\,]}$.

Problem 3.2. Memoryless Property of the Exponential Distribution The exponential distribution is the only continuous distribution that has the property of *memorylessness (Markov property)*. This property means that in the case of an exponentially distributed random variable A the so-called residual random variable $A_x = A - x$, $x > 0$ follows the same exponential distribution as A under the condition $A > x$. The cumulative distribution function is $A(t) = 1 - e^{\lambda t}$. It has to be shown that the following equation holds:

$$P(A_x \leq t \,|\, A_x > 0) = P(A \leq t).$$

[1] **indexDispersion**

Problem 3.3. Poisson Process and Renewal Function The number of events of a point process in an observation interval $(0, t)$ is denoted by $X(t)$. The random variable for the time until the k-th event occurs is $A^{(k)}$ with the associated cumulative distribution function $A^{(k)}(t)$.

3.3.1. Show that the probability $P(X(t) = k)$ for exactly k events in the observation interval is given by $A^{(k)}(t) - A^{(k+1)}(t)$.

3.3.2. What is the renewal function $H(t) = \mathrm{E}[\, X(t)\,]$ as a function of $A^{(k)}(t)$?

3.3.3. Derive the following expression for the Laplace transform of $H(t)$. $\Phi_{A,k}(s)$ denotes the Laplace transform of the distribution A^K with the PDF $a^{(k)}(t)$.

$$\Phi_H(s) = \frac{1}{s} \cdot \frac{\Phi_{A,k}(s)}{1 - \Phi_{A,k}(s)} \; .$$

Problem 3.4. Hyperexponential Distribution with Symmetry Assumption An arrival process is considered with interarrival times following a second order hyperexponential distribution: $A \sim H_2$. With probability α_1 and α_2, the interarrival time is A_1 and A_2, respectively. The random variables A_1 and A_2 follow an exponential distribution with rate λ_1 and λ_2, respectively. The symmetry assumption is considered, i.e. $\alpha_1 \cdot \mathrm{E}[\, A_1\,] = \alpha_2 \cdot \mathrm{E}[\, A_2\,]$.

3.4.1. What is the CDF $R(t)$ of the forward recurrence time of the arrival process?

3.4.2. What is the probability that $\mathrm{E}[\, R\,] \leq 0.1\,\mathrm{s}$ for $\mathrm{E}[\, A_1\,] = 1\,\mathrm{s}$ and $\mathrm{E}[\, A_2\,] = 0.01\,\mathrm{s}$?

3.4.3. How do you have to choose the ratio $a = \frac{EA_2}{\mathrm{E}[\, A_1\,]}$ so that the expected value of the recurrence time $\mathrm{E}[\, R\,]$ is greater than the expected value of the interarrival time $\mathrm{E}[\, A\,]$?

Problem 3.5. Poisson Process and Erlang$_k$ In the following, the number of arrival events of a Poisson process in an observation interval $(t; t + \tau]$ of the length τ is to be calculated.

3.5.1. The random variable $A^{(k)}$ describes the time until k arrivals have occurred. How is $A^{(k)}$ distributed? Derive the CDF explicitly by induction and calculate the corresponding PDF.

3.5.2. Calculate the probability $A^{\tau}_{min}(k)$ that at least k arrivals occur within the observation interval.

3.5.3. Derive from the results above the probability of exactly k arrivals in the observation interval. What distribution do you get?

Problem 3.6. Birth-and-Death Process The steady state probabilities $x(i)$ of a birth-and-death process $X(t)$ can be determined with the help of the following system of equations:

$$\begin{aligned}
x(0) \cdot \lambda &= x(1) \cdot \mu \\
x(1) \cdot (\lambda + \mu) &= x(0) \cdot \lambda + x(2) \cdot \mu \\
&\vdots \\
x(i) \cdot (\lambda + \mu) &= x(i-1) \cdot \lambda + x(i+1) \cdot \mu \\
&\vdots
\end{aligned}$$

The parameters λ and μ are constant and satisfy $\lambda < \mu$. The individual state probabilities can be determined from the relationships given above with the help of the generating function $X_{GF}(z) = \sum_{i=0}^{\infty} x(i) \cdot z^i$.

3.6.1. Show that $X^*_{GF}(z) = \dfrac{x(0) \cdot \mu}{\mu - \lambda z}$.

3.6.2. Calculate the probability $x(0)$ from $X^*_{GF}(z)$ and specify the generating function $X_{GF}(z)$ independently of $x(0)$.

Problem 3.7. Sojourn Time in State for BDP A birth-and-death process with n states and the following transition probability densities (rates) is given:

$$q_{ij} = \begin{cases} \lambda_i & i = 0, 1, \ldots, n-1, \quad j = i+1, \quad \text{birth rate,} \\ \mu_i & i = 1, 2, \ldots, n, \qquad\quad j = i-1, \quad \text{death rate,} \\ 0 & \text{otherwise .} \end{cases}$$

The sojourn time of a birth-and-death process in state i is defined as the period of time in which the state i of the process remains unchanged. This corresponds exactly to the interval between two immediately successive state transitions. What is the distribution of the sojourn time in the states $i = 1, \ldots, n$? What is the mean value of the sojourn time in state i?

Problem 3.8. Paradox of Recurrence Time An independent observer looks at a Poisson process and does not know the mean value $\mathrm{E}[\,A\,]$ of the time between two events. The observer determines the mean value of the forward recurrence time and the mean value of the recurrence time to be $\mathrm{E}[\,R_v\,] = \mathrm{E}[\,A\,]$ and $\mathrm{E}[\,R_r\,] = \mathrm{E}[\,A\,]$, respectively. Then, the observer concludes that the mean time between two events is $2 \cdot \mathrm{E}[\,A\,]$. How do you explain this paradox?

Problem 3.9. Recurrence Time Show that the exponential distribution is the only continuous distribution for which the cumulative distribution function of the interarrival time $A(t)$ equals the cumulative distribution function of its forward or backward recurrence time $R(t)$.

☞ *Hint:* $\Phi_R(s) = \dfrac{\lambda}{s} \cdot (1 - \Phi_A(s))$

Problem 3.10. General Renewal Process The number of events of a renewal process in an observation interval $(0,t)$ is denoted by $N(t)$. The random variables for the time up to the occurrence of the k -th event is $A^{(k)}$ with the associated distribution function $A^{(k)}(t)$.

3.10.1. Show that the probability $P(N(t)=k)$ for exactly k events in the observation interval is given by $A^{(k)}(t) - A^{(k+1)}(t)$.

3.10.2. What is the renewal function $H(t) = \mathrm{E}[\,N(t)\,]$ as a function of $A^{(k)}(t)$?

3.10.3. Derive a simple expression for the Laplace transform of $H(t)$.

Problem 3.11. Thinning a Poisson Process A Poisson process with arrival rate λ is considered. Which distribution do you get for the interarrival times,

3.11.1. if only every k-th arrival is considered?

3.11.2. if an arrival event is only considered with probability $\frac{1}{k}$?

☞ *Hint:* Use the Laplace transform to prove this.

Problem 3.12. Properties of the Poisson Process Show the following two properties of a Poisson process: superposition and thinning of Poisson processes.

3.12.1. We consider K independent sources which are independently generating requests. The interarrival time of requests are exponentially distributed for all sources, where source k has an arrival rate of λ_k ($k = 1, \ldots, K$). These K arrival streams are now combined into a single one (superposition). Show that the superposition of the the K independent Poisson processes follows a Poisson process with rate $\lambda = \sum_k \lambda_k$.

3.12.2. A Poisson arrival process with rate λ is divided into K separate processes in such a way that a request with probability p_k is assigned to the sub-process k for $k = 1, \ldots, K$ and $\sum_k p_i = 1$. Show that every sub-process k is a Poisson process with rate $p_k\lambda$.

Problem 3.13. Four States BDP We consider a birth-and-death process. The state space includes the states $\{0,1,2,3\}$. The transition probability densities are given as follows:

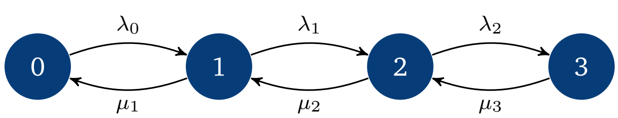


Figure 3.1: Transition state diagram of a birth-and-death process with four states.

Calculate the steady state probabilities $x(i) = P(X = i)$ using the macro state approach for the following cases.

3.13.1. $\lambda_1 = 0$. The remaining rates are not vanishing: $\lambda_0, \lambda_2 > 0$, $\forall i = 1,2,3 : \mu_i > 0$.

3.13.2. $\mu_2 = 0$. The remaining rates are not vanishing: $\mu_1, \mu_3 > 0$, $\forall i = 0,1,2 : \lambda_i > 0$.

Problem 3.14. State-dependent Processing We consider a machine with n service units. We assume a Markovian system and use a Markov state process for the analysis of the system. The requests arrive at the rate λ. They are rejected if no service unit is free to process the incoming request. As a special feature of this system, the performance of the service units depends on the number of requests that are currently being processed. If i service units are already occupied, the mean service time of a service unit is $\mathrm{E}[\,B_i\,] = \frac{1}{\mu \cdot f(i)}$.

3.14.1. Specify the state transition diagram of the Markovian system.

3.14.2. Calculate the steady state probabilities depending on λ, μ and $f(i)$.

3.14.3. Calculate the blocking probability p_B of incoming requests.

3.14.4. Calculate the average rate $lambda^*$ of accepted requests.

3.14.5. Calculate the mean number of customers in the system using the steady state distribution.

3.14.6. Use Little's law to calculate the average sojourn time $E[B^*]$ of a customer as a multiple of $[\frac{1}{\mu}]$.

3.14.7. Complete the following tasks for the different functions $f_\kappa(i),\ \kappa = 1, \ldots, 5$. Draw $f(i)$ for $i = 1, 2$ as a graph. Interpret the service units in terms of their performance. Draw the blocking probability of the system for $\frac{\lambda}{\mu} \in [0.1; 1.5]$. Draw the mean sojourn time for $\frac{\lambda}{\mu} \in [0.1; 1.5]$. Explain the behavior of the curves.

$$f_1(i) = \frac{1}{1 - \frac{i-1}{n}}\ , \quad f_2(i) = \frac{1}{1 - \frac{i-1}{2n}}\ , \quad f_3(i) = 1\ , \quad f_4(i) = \frac{1}{1 + \frac{i-1}{2n}}\ , \quad f_5(i) = \frac{1}{1 + \frac{i-1}{n}}\ .$$