Chapter 2.3

Transform Methods

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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The book must be cited and the disclaimer attached when using lectures slides or scripts.

Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/

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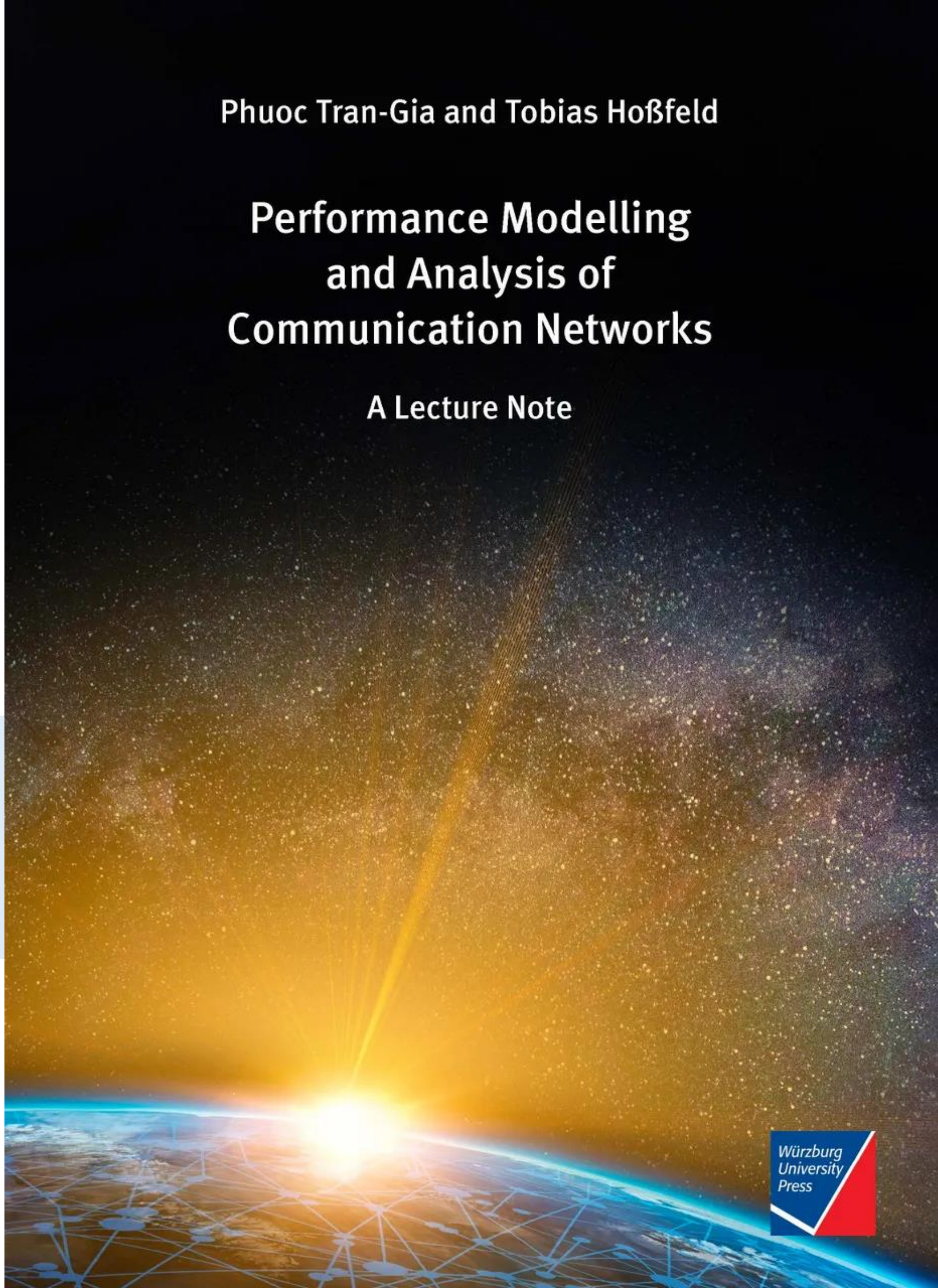


Chapter 2

2 Fundamentals and Prerequisites

2.1 Little's Theorem and General Results
- 2.1.1 Little's Law in Finite Systems with Blocking
- 2.1.2 Example: Multiclass Systems
- 2.1.3 Example: Balking
- 2.1.4 The Utilization Law
- 2.1.5 Assumptions and Limits of Little's Law
- 2.1.6 General Results for GI/GI/n Delay Systems
- 2.1.7 Loss Formula for GI/GI/n-S Loss Systems

2.2 Probabilities and Random Variables
- 2.2.1 Random Experiments and Probabilities
- 2.2.2 Other Terms and Properties
- 2.2.3 Random Variable, Distribution, Distribution Function
- 2.2.4 Expected Value and Moments
- 2.2.5 Functions of Random Variables and Inequalities
- 2.2.6 Functions of Two Random Variables

2.3 Transform Methods
- 2.3.1 Generating Function
- 2.3.2 Laplace and Laplace-Stieltjes Transforms

2.4 Some Important Distributions
- 2.4.1 Discrete Distributions
- 2.4.2 Continuous Distributions
- 2.4.3 Relationship between Continuous and Discrete Distribution

Transform Methods

Time Domain	Transform Domain
Function	Transform (or transformed function)

Transformation:

discrete r.v.	PMF	generating function
	PMF	Z-transform
continuous r.v.		
	PDF,	Laplace transform of a(t),
	CDF	Laplace-Stieltjes-transform of A(t)

Generating Function

Definition, example, convolution theorem

(Probability) Generating Function (GF)

- Given a discrete random variable X with the distribution $x(i)$ for $i = 0,1,\ldots$

- Probability generating function (for short, **generating function GF**) for X is defined as

$$X_{GF}(z) = GF\{x(i)\} = \sum_{i=0}^{\infty} x(i) \cdot z^i = E[z^X]$$

  where z is a complex variable.

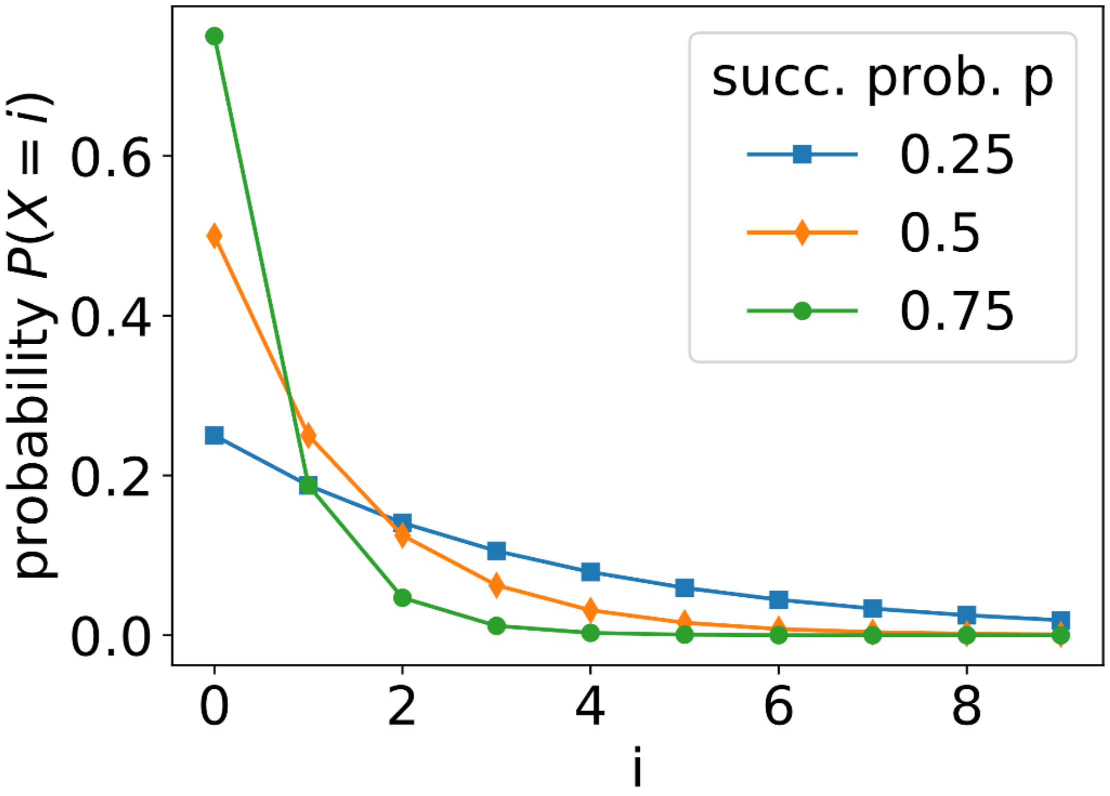
- $X_{GF}(z)$ converges within and on the unit circle $|z| \leq 1$.

- **Inverse transform of GF**

$$x(i) = GF^{-1}\{X_{GF}(z)\} = \frac{1}{i!} \cdot \left. \frac{d^i X_{GF}(z)}{dz^i} \right|_{z=0}$$

Example: Geometric Distribution

Lecture



Properties of the Generating Function

$$X_{GF}(z) = \sum_{i=0}^{\infty} x(i) \cdot z^i$$

- Moments of r.v. X can directly be determined in transform domain

$$X_{GF}(1) = \sum_{i=0}^{\infty} x(i) \cdot 1^i = 1$$

$$X'_{GF}(1) = \left.\frac{dX_{GF}(z)}{dz}\right|_{z=1} = \left.\sum_{i=0}^{\infty} x(i) \cdot i \cdot z^{i-1}\right|_{z=1} = E[X]$$

$$X''_{GF}(1) = \left.\frac{d^2X_{GF}(z)}{dz^2}\right|_{z=1} = E[X^2] - E[X]$$

$$VAR[X] = E[X^2] - E[X]^2 = X''_{GF}(1) + X'_{GF}(1) - X'_{GF}(1)^2$$

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Example: Generating Function

Lecture

Convolution Theorem for Discrete Random Variables

- Sum of two statistically independent discrete random variables X_1 and X_2

$$X = X_1 + X_2$$

- **Convolution theorem**

$$\begin{array}{ccccc} x(i) & = & x_1(i) & * & x_2(i) \\ \circ & & \circ & & \circ \\ \vert & & \vert & & \vert \\ \bullet & & \bullet & & \bullet \end{array}$$

$$X_{GF}(z) = X_{1,GF}(z) \cdot X_{2,GF}(z)$$

Convolution Theorem for Discrete Random Variables

Lecture

Example: Poisson Distribution

Lecture

Laplace- and Laplace-Stieltjes-Transformation

Definition, example, convolution theorem

Laplace and Laplace-Stieltjes Transforms

- Non-negative continuous random variable A
  - cumulative distribution function $A(t)$
  - probability density function $a(t) = \frac{dA(t)}{dt}$

- Definition
  - for a complex variable s with non-negative real part $Re(s) \geq 0$

$$\Phi_A(s) = LST\{A(t)\} = \int_0^\infty e^{-st}\, dA(t)$$

Laplace-Stieltjes transform

$$= LT\{a(t)\} = \int_0^\infty e^{-st} a(t)\, dt$$

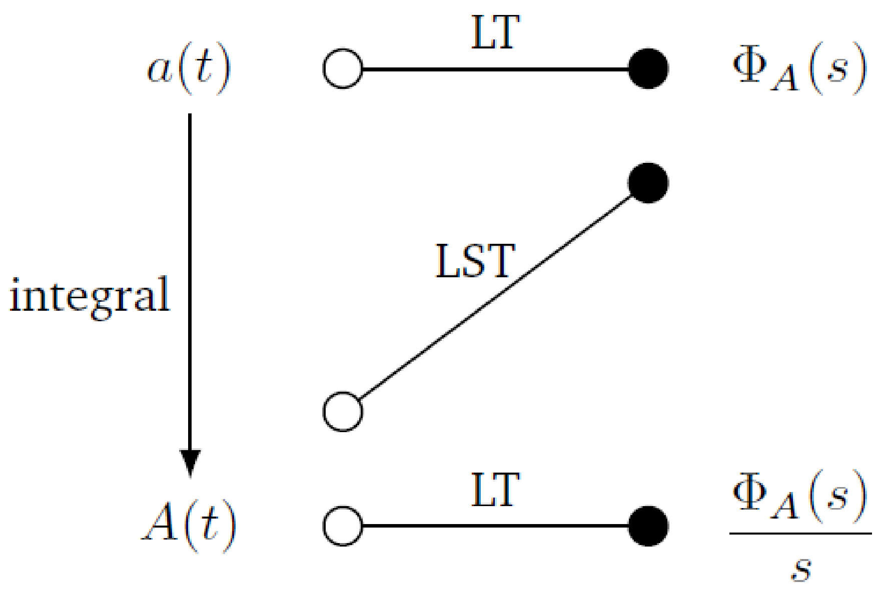
Laplace transform

$$= E[e^{-sA}]$$

Example: Exponential Distribution

Lecture

Relation between Laplace and Laplace-Stieltjes Transforms



Properties of Laplace Transform

- Computation of Moments using Laplace Transform

$$E[A^k] = \int_0^\infty t^k \, a(t) \, dt = (-1)^k \cdot \frac{d^k}{ds^k} \Phi_A(s) \bigg|_{s=0}$$

- Laplace Transforms of Integral and Derivative

$$A(t) \quad \circ\!\!\overset{LT}{-\!\!-\!\!-}\!\!\bullet \quad \frac{1}{s} \Phi_A(s)$$

$$\frac{d}{dt} a(t) \quad \circ\!\!\overset{LT}{-\!\!-\!\!-}\!\!\bullet \quad s \, \Phi_A(s) - a(0)$$

- Laplace Transform Limit Theorems

$$\lim_{t \to 0} a(t) = \lim_{s \to \infty} s \cdot \Phi_A(s)$$

$$\lim_{t \to \infty} a(t) = \lim_{s \to 0} s \cdot \Phi_A(s)$$

Laplace Transform and Convolution Operation

- Non-negative, statistically independent r.v.s A_1 and A_2
  - with PDFs $a_1(t)$ and $a_2(t)$
  - and Laplace transforms $\Phi_{A_1}(s)$ and $\Phi_{A_2}(s)$

- The sum $A = A_1 + A_2$ has the following PDF

$$a(t) = a_1(t) * a_2(t) = \int_{\tau=0}^{t} a_1(\tau) \cdot a_2(t-\tau)\, d\tau$$

- **Continuous convolution theorem:** Laplace transform of $A = A_1 + A_2$

$$\begin{array}{ccccc} a(t) & = & a_1(t) & * & a_2(t) \\ \big\updownarrow \text{LT} & & \big\updownarrow \text{LT} & & \big\updownarrow \text{LT} \\ \boxed{\Phi_A(s)} & = & \Phi_{A_1}(s) & \cdot & \Phi_{A_2}(s) \end{array}$$