Institute of Computer Science
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Chapter 3.3

Poisson Processes

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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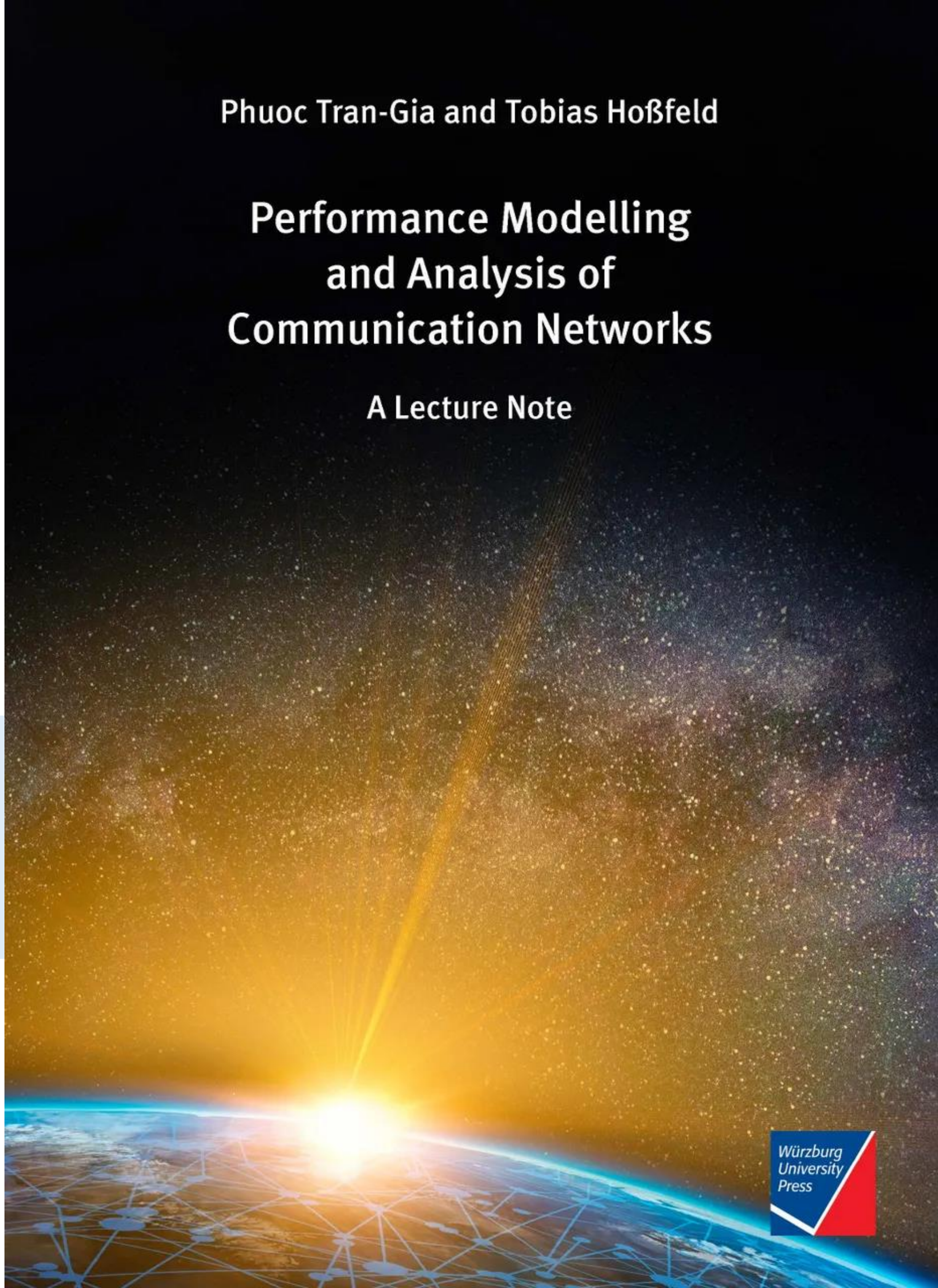
The book must be cited and the disclaimer attached when using lectures slides or scripts.

Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/



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Chapter 3

Definition of a Poisson Process

Poisson Process as Counting Process

- **Counting process** $\{N(t), t \geq 0\}$ is a Poisson process with rate λ if being fulfilled
  - 1. $N(0) = 0$.
  - 2. N(t) has independent increments.
  - 3. N(t) has stationary increments.

- **Independent increments**
  - Number of arrivals during non-overlapping time intervals are independent r.v.s.
  - Consider time series: $0 \leq t_0 < t_1 < \cdots < t_n$
  - $N(t_0), N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots$ are independent
- **Stationary increments**
  - Number of arrivals between t and $t + \tau$ depends only on the length of the interval τ, not on the starting point t.

- For Poisson process: $N(t) \sim \mathrm{POIS}(\lambda t)$ and $N(t+\tau) - N(t) = N(\tau)$

Poisson Process as Renewal Process

- Poisson process with rate λ
  - renewal process with interarrival times following an exponential distribution
  - $A \sim \mathrm{EXP}(\lambda)$
  - CDF $A(t) = 1 - e^{-\lambda t}$
  - PDF $a(t) = \lambda e^{-\lambda t}$
  - Mean $\mathrm{E[A]} = \frac{1}{\lambda}$

- Both definitions of a Poisson process (counting process, renewal process) are equivalent

Properties of a Poisson Process

Memoryless property, recurrence time, PASTA property

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Memoryless Property for Poisson Process

- Memoryless property of a positive random variable A for every $t \geq 0$ and $s \geq 0$

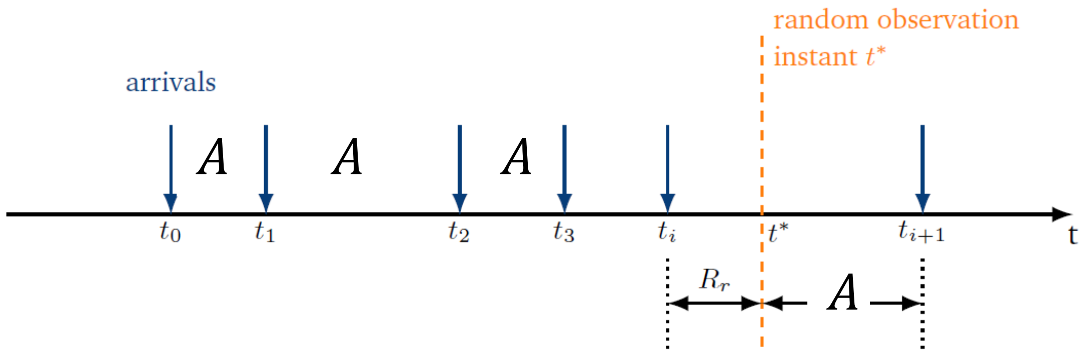
$$P(A > s + t | A > s) = P(A > t)$$

- Exponential distribution has memoryless property
  $P(A > s + t | A > s) = e^{-\lambda t} = P(A > t)$

- **Recurrence time of Poisson process**

$$r(t) = \lambda(1 - A(t)) = \lambda e^{-\lambda t} = a(t)$$

$$R(t) = A(t)$$



  - Interarrival time and recurrence time follow exponential distribution with rate λ
  - Process develops completely independently of its past at any observation time
  - Poisson process is memoryless or has the Markov property

PASTA Property

- PASTA = „Poisson Arrival Sees Time Average"

- X^* number of customers in system at a random time t^* (r.v.)
- X_A number of customers in system as observed by an arriving customer (r.v.)

- **PASTA property**: probability to see i customers is identical for random and arriving customers

$$x^*(i) = x_A(i) \ , \quad i = 0,1,\ldots$$

- PASTA property is often utilized
  - e.g., blocking probabilities of arriving customers in M/M/n loss systems
  - e.g., derivation of system state probabilities $x^*(i)$ for M/GI/1 delay sytems
- PASTA requires Poisson arrival process

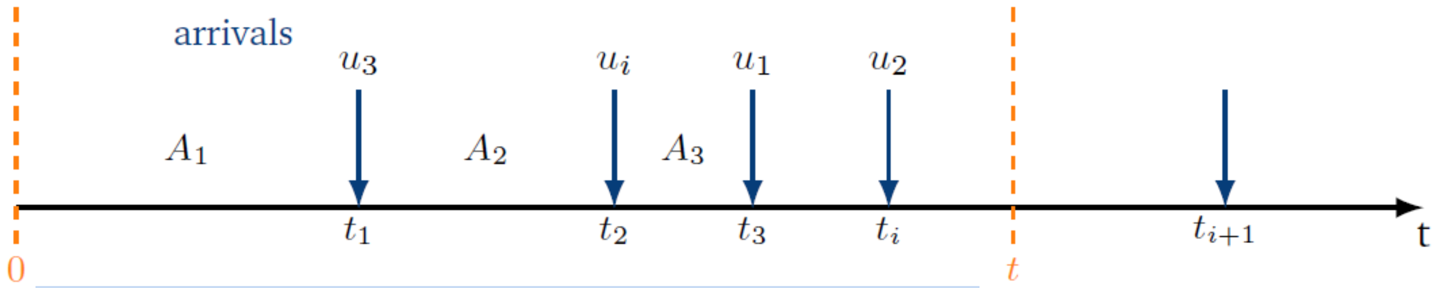
PASTA Property

Counter-example: PASTA

- Simple counter-example that PASTA is not valid for any arrival process
  - D/D/1 queue with $E[A] = 2$ and $E[B] = 1$
  - $x^*(i) \neq x_A(i)$

Arrivals in Fixed Interval

- Number $N(t)$ of Poisson arrivals in a fixed interval of length t with arrival rate λ



$$N(t) \sim \mathrm{POIS}(\lambda t) \text{ with } E[N(t)] = \frac{1}{\lambda t}$$

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

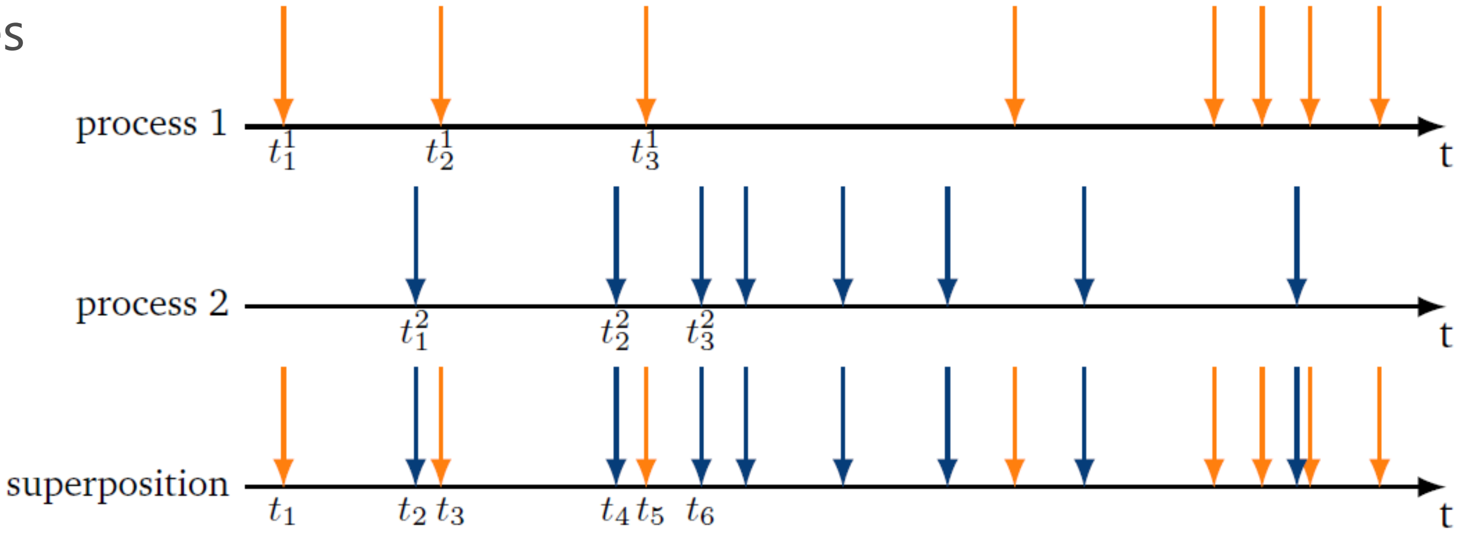
- Uniform distribution of Poisson arrivals in fixed interval

$$u_i \sim U(0,t) \quad \text{for } i = 1,2, \dots, n$$

Merging and Splitting of Poisson Processes

Merging of Poisson Processes

- Superposition of Poisson processes remains a Poisson process
  - Process 1: $A_1 \sim \mathrm{EXP}(\lambda_1)$
  - Process 2: $A_2 \sim \mathrm{EXP}(\lambda_2)$



- Superposition $A \sim \mathrm{EXP}(\lambda_1 + \lambda_2)$

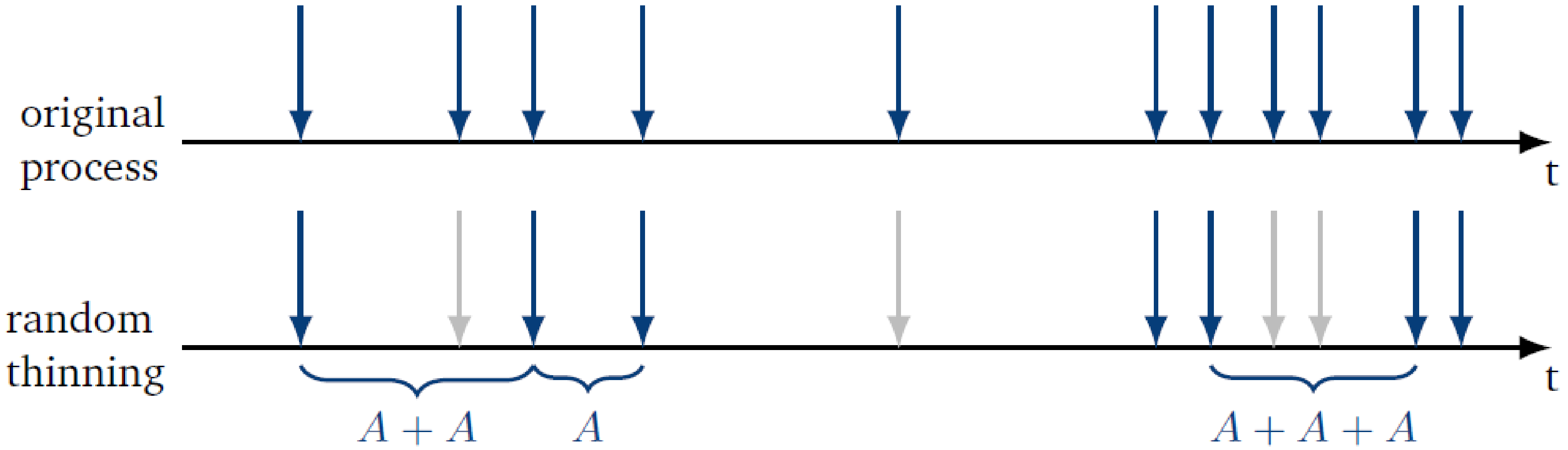
- Proof: random observer sees minimum of recurrence times of both processes
  - recurrence time is exponentially distributed with same rate
  - $A = \min(A_1, A_2)$ with CDF $A(t) = 1 - \big(1 - A_1(t)\big) \cdot \big(1 - A_2(t)\big) = 1 - e^{-(\lambda_1+\lambda_2)t}$
  - $A = \min(A_1, A_2) \sim \mathrm{EXP}(\lambda_1 + \lambda_2)$

- In general: superposition of n Poisson processes

$$A \sim \mathrm{EXP}\left(\sum_{i=1}^{n} \lambda_i\right)$$

Random Thinning of Poisson Processes

- Poisson process with rate λ
- With probability p, an arrival event is considered in the thinned process
- With probability $1-p$, an arrival event is not considered (gray arrows)

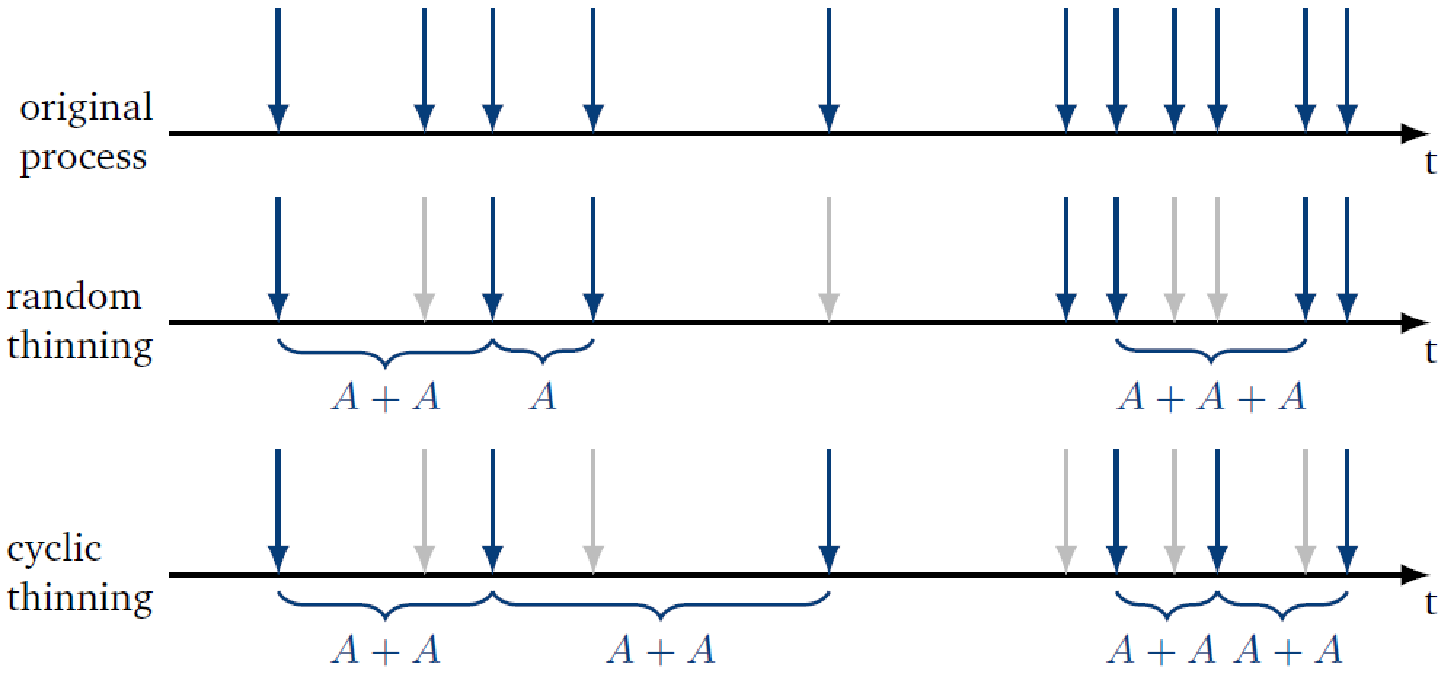


- Thinned process is a Poisson process with rate $p \cdot \lambda$

Random Thinning of Poisson Processes

Cyclic Thinning of Poisson Processes

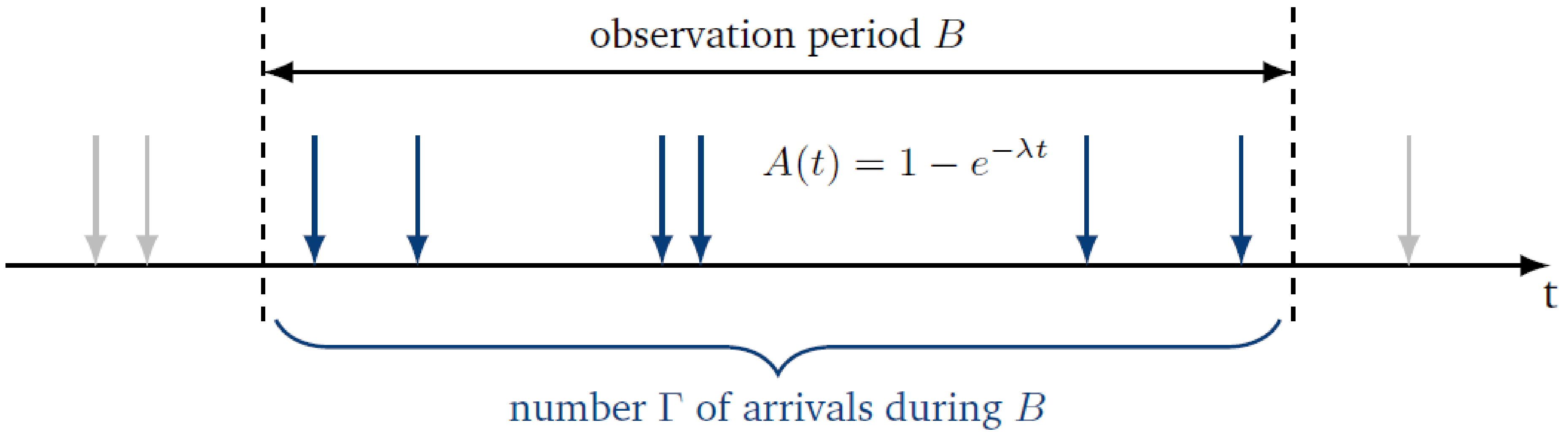
- Poisson process with rate λ
- Cyclic or deterministic thinning
  - e.g., every second arrival is skipped
  - interarrival times follows an Erlang distribution, e.g., with $k = 2$ and λ



Poisson Arrivals during an Arbitrarily Distributed Interval

Arbitrarily Distributed Observation Interval

- Length B of observation window is a r.v. with Laplace transform $\Phi_B(s)$
- Number Γ of arrivals during B is of interest, e.g., analysis of M/GI/1 delay system



- **Generating function** of the number of arrivals $\Gamma_{GF}(z)$ depends on Laplace transform $\Phi_B(s)$

$$\Gamma_{GF}(z) = \Phi_B\big(\lambda(1-z)\big)$$

Proof

Lecture