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Chapter 5.5

Model with Batch Service and Threshold Control

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/

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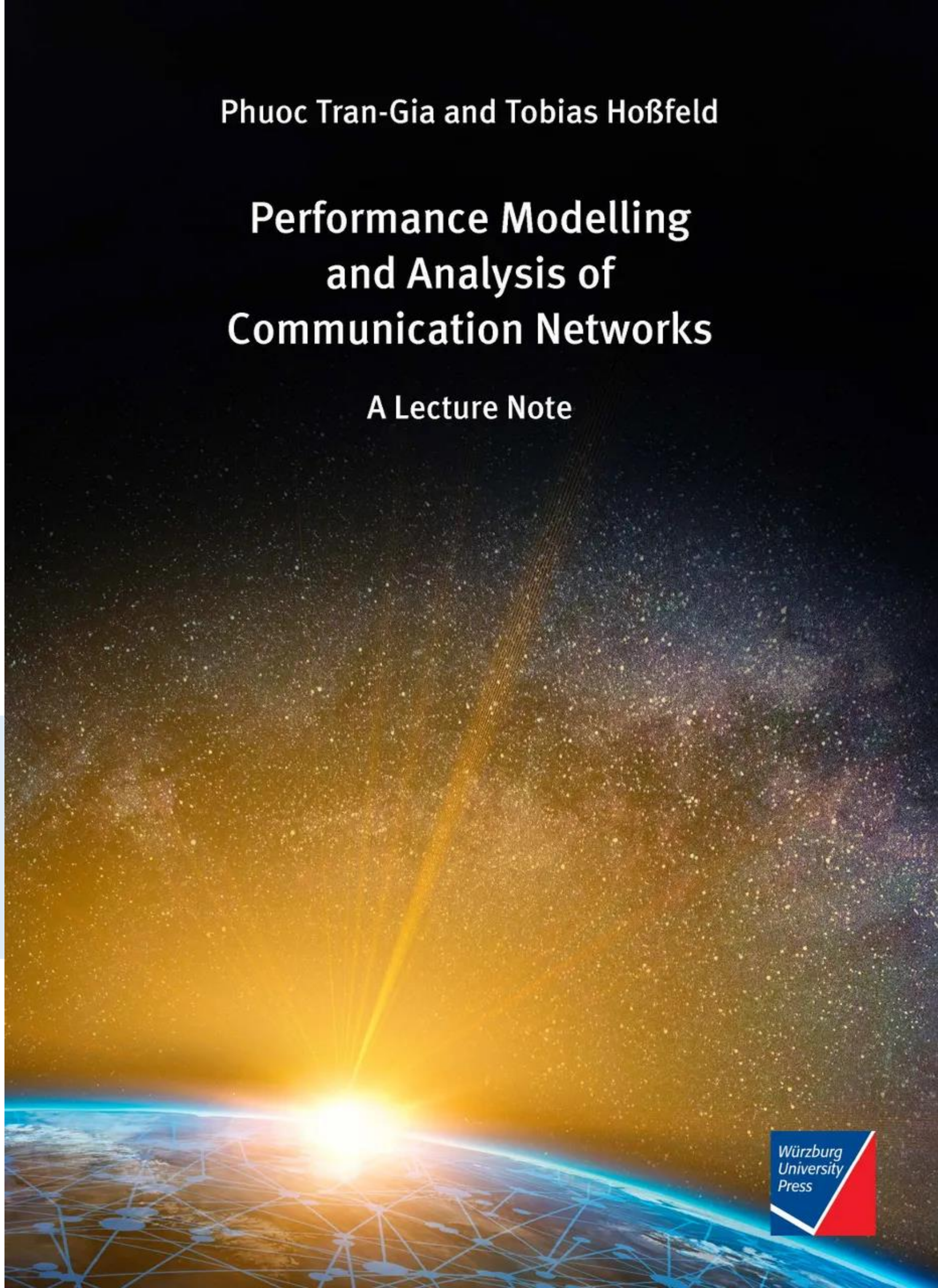
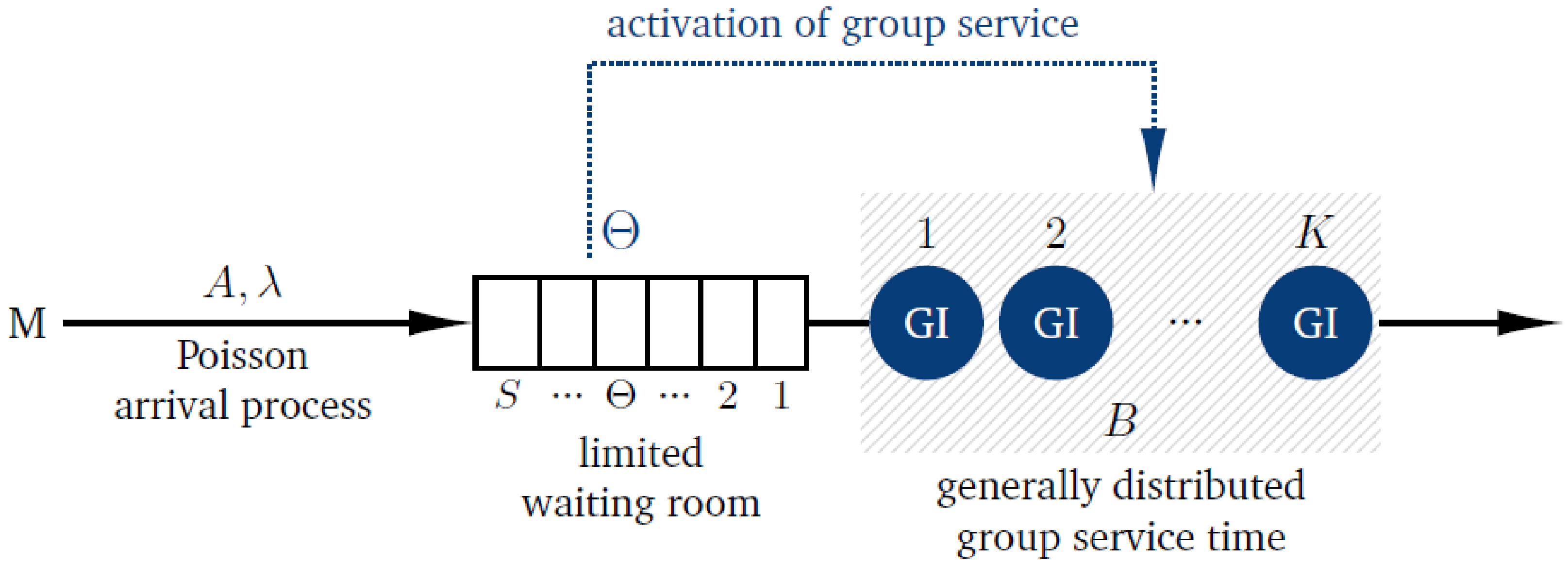


Chapter 5

5 Analysis of Non-Markovian Systems

Model with Batch Service and Threshold Control

$M/GI^{[\Theta,K]}/1\text{-}S$



$$A(t) = P(A \le t) = 1 - e^{-\lambda t}, \quad E[A] = \frac{1}{\lambda}$$

- Poisson arrival process with arrival rate λ
- Limited waiting room with S waiting places
- **Group service unit** with K service places and generally distributed service time B
- **Starting threshold** Θ with the following **trigger mechanism** at the end of an operation
  - server is loaded and started immediately if at least Θ customers are waiting to be processed
  - If there are fewer than Θ customers in the queue, the system waits until the starting threshold Θ is reached, before the next operation begins.

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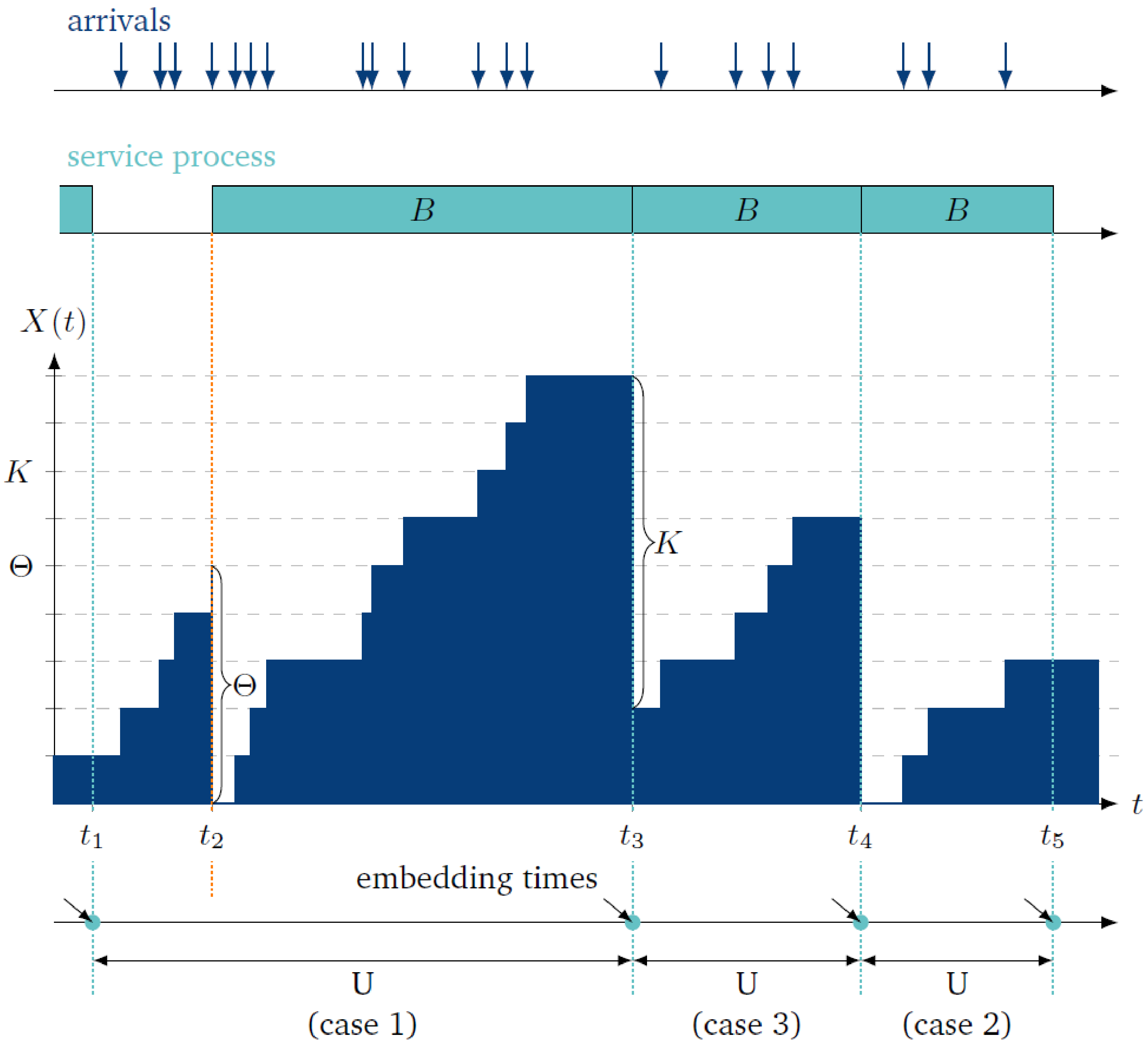
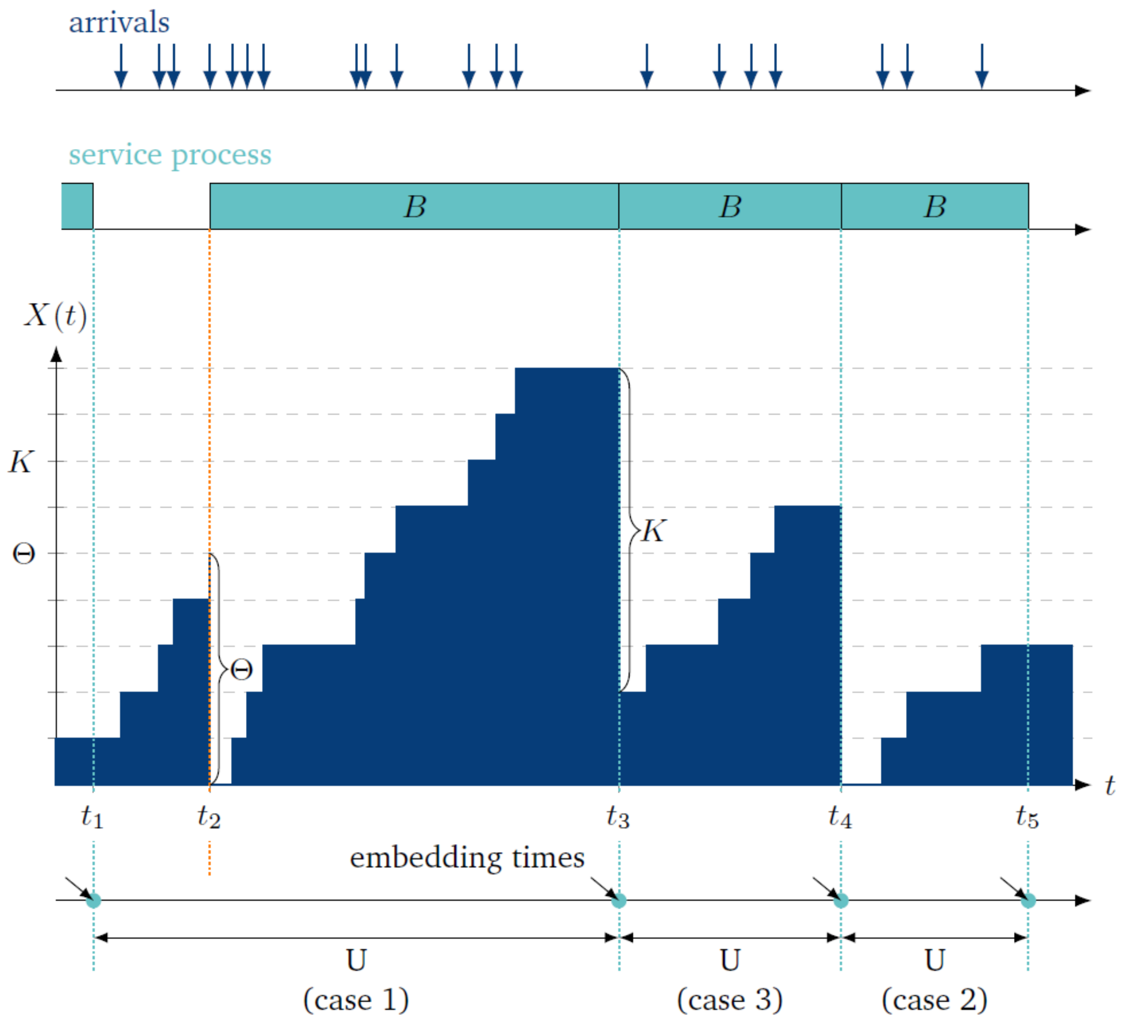
State Process

- R.V $X(t)$ is the number of **customers in the queue** of the M/GI$^{[\Theta,K]}$/1-S queue
- System state is $X(t)$
- Note: $X(t)$ is not reflecting the total number of customers in the system



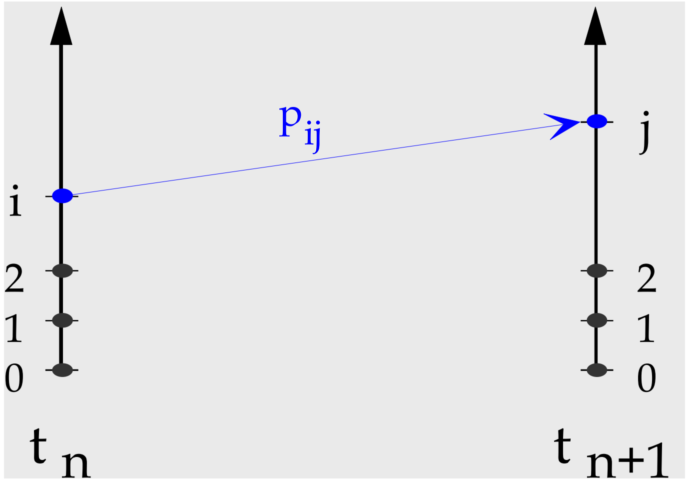
Different Cases in the State Process

- Case 1 (time t_1): $\mathrm{i} < \Theta$
  - minimum number of customers to start the server is not yet reached
  - further $\Theta - i$ customers still have to arrive before service starts
- Case 2 (time t_4): $\Theta \leq \mathrm{i} \leq \mathrm{K}$
  - more than Θ customers in the queue
  - new service period will be started in which all i customers are served (group service)
- Case 3 (time t_4): $\mathrm{K} < \mathrm{i} \leq \mathrm{S}$
  - immediately after service end, the next K requests from queue are served
  - remaining customers will stay in the queue



Markov Chain and State Transition

- Embedded Markov Chain
  - service time is the only non-Markovian model component
  - embedding point **immediately before service ends** (deliberately chosen here due to purposeful analysis)
- Note that we chose regeneration points instants immediately after service ends for M/GI/1 delay system



- System state $X(t_n)$ at embedding time t_n immediately before service end
- State probability $x(j,n) = P(X(t_n) = j)$ at embedding time t_n
- Transition probability

$$p_{ij} = P\left(X(t_{n+1}) = j \middle| X(t_n) = i\right)$$

State Transition Probabilities

- Random variable Γ is the number of customers arriving during service time B
- Case 1 (time t_1): $i < \Theta$

$$p_{ij} = \gamma(j)\,, \qquad j = 0, \ldots, S-1\,,$$

$$p_{iS} = \sum_{k=S}^{\infty} \gamma(k)\,, \quad j = S\,.$$

- Case 2 (time t_4): $\Theta \le i \le K$

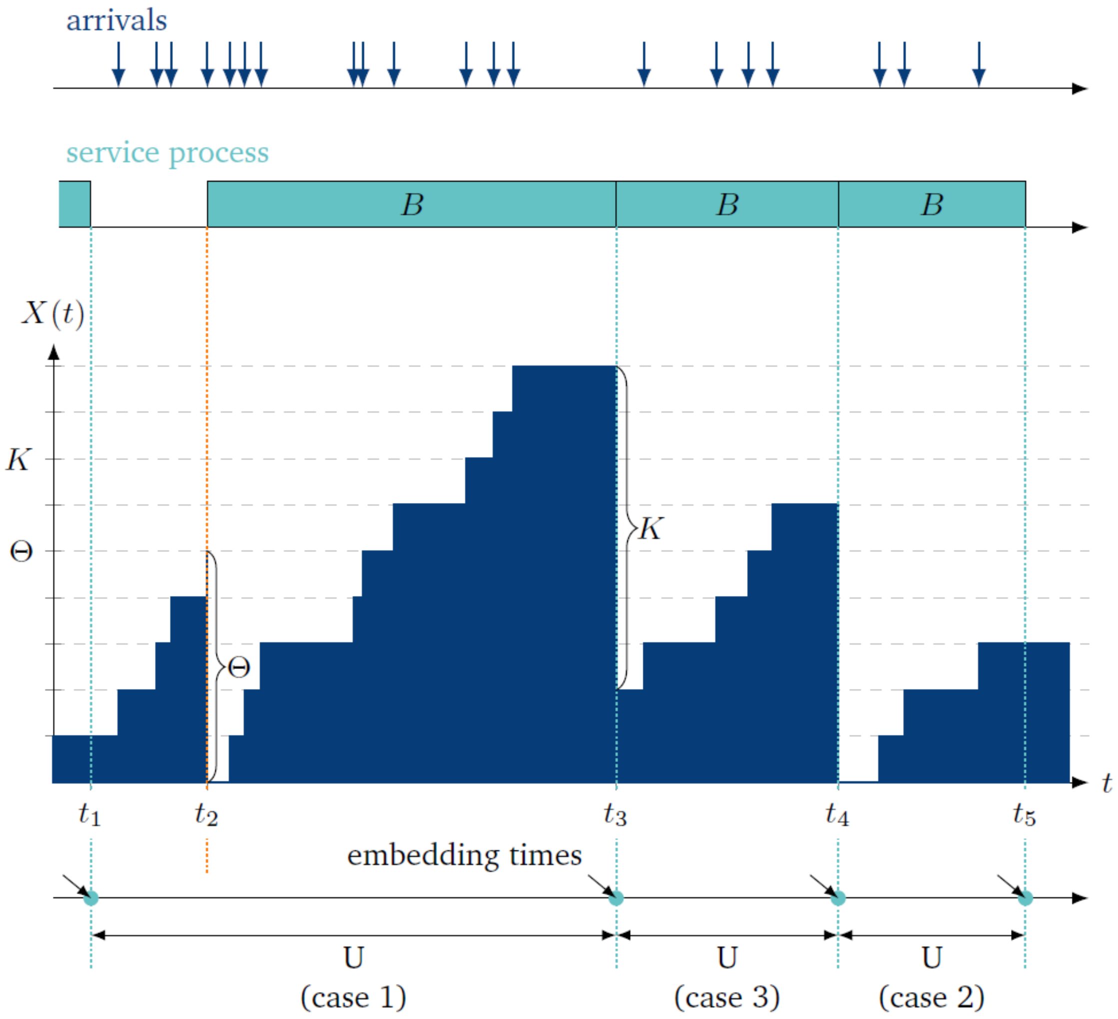
$$p_{ij} = \gamma(j)\,, \qquad j = 0, \ldots, S-1\,,$$

$$p_{iS} = \sum_{k=S}^{\infty} \gamma(k)\,, \quad j = S\,.$$

- Case 3 (time t_4): $K < i \le S$

$$p_{ij} = \gamma(j - i + K)\,, \qquad j = 0, \ldots, S-1\,,$$

$$p_{iS} = \sum_{k=S-i+K}^{\infty} \gamma(k)\,, \qquad j = S\,.$$



$$p_{ij} = P\left(X(t_{n+1}) = j \,\middle|\, X(t_n) = i\right)$$

State Transition Matrix

- Random variable Γ is the number of customers arriving during service time B
- Case 1 (time t_1): $i < \Theta$

$$p_{ij} = \gamma(j) \,, \qquad j = 0, \ldots, S-1 \,,$$

$$p_{iS} = \sum_{k=S}^{\infty} \gamma(k) \,, \quad j = S \,.$$

- Case 2 (time t_4): $\Theta \le i \le K$

$$p_{ij} = \gamma(j) \,, \qquad j = 0, \ldots, S-1 \,,$$

$$p_{iS} = \sum_{k=S}^{\infty} \gamma(k) \,, \quad j = S \,.$$

- Case 3 (time t_4): $K < i \le S$

$$p_{ij} = \gamma(j - i + K) \,, \qquad j = 0, \ldots, S-1 \,,$$

$$p_{iS} = \sum_{k=S-i+K}^{\infty} \gamma(k) \,, \qquad j = S \,.$$

$$\mathcal{P} = \begin{array}{cccccc} 0 & 1 & 2 & & S-1 & S \end{array}$$

$$\begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(S-1) & \sum_{k=S}^{\infty}\gamma(k) \\ \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(S-1) & \sum_{k=S}^{\infty}\gamma(k) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma(0) & \gamma(1) & \gamma(2) & \ldots & \gamma(S-1) & \sum_{k=S}^{\infty}\gamma(k) \\ 0 & \gamma(0) & \gamma(1) & \ldots & \gamma(S-2) & \sum_{k=S-1}^{\infty}\gamma(k) \\ 0 & 0 & \gamma(0) & \ldots & \gamma(S-3) & \sum_{k=S-2}^{\infty}\gamma(k) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma(K-1) & \sum_{k=K}^{\infty}\gamma(k) \end{pmatrix}$$

State Probabilities and System Characteristics

- At each embedding time t_n, state probability vector

$$\mathbf{X}_n = \left(x(0,n), x(1,n), \ldots\right) ,$$
$$x(i,n) = P(X(t_n) = i) .$$

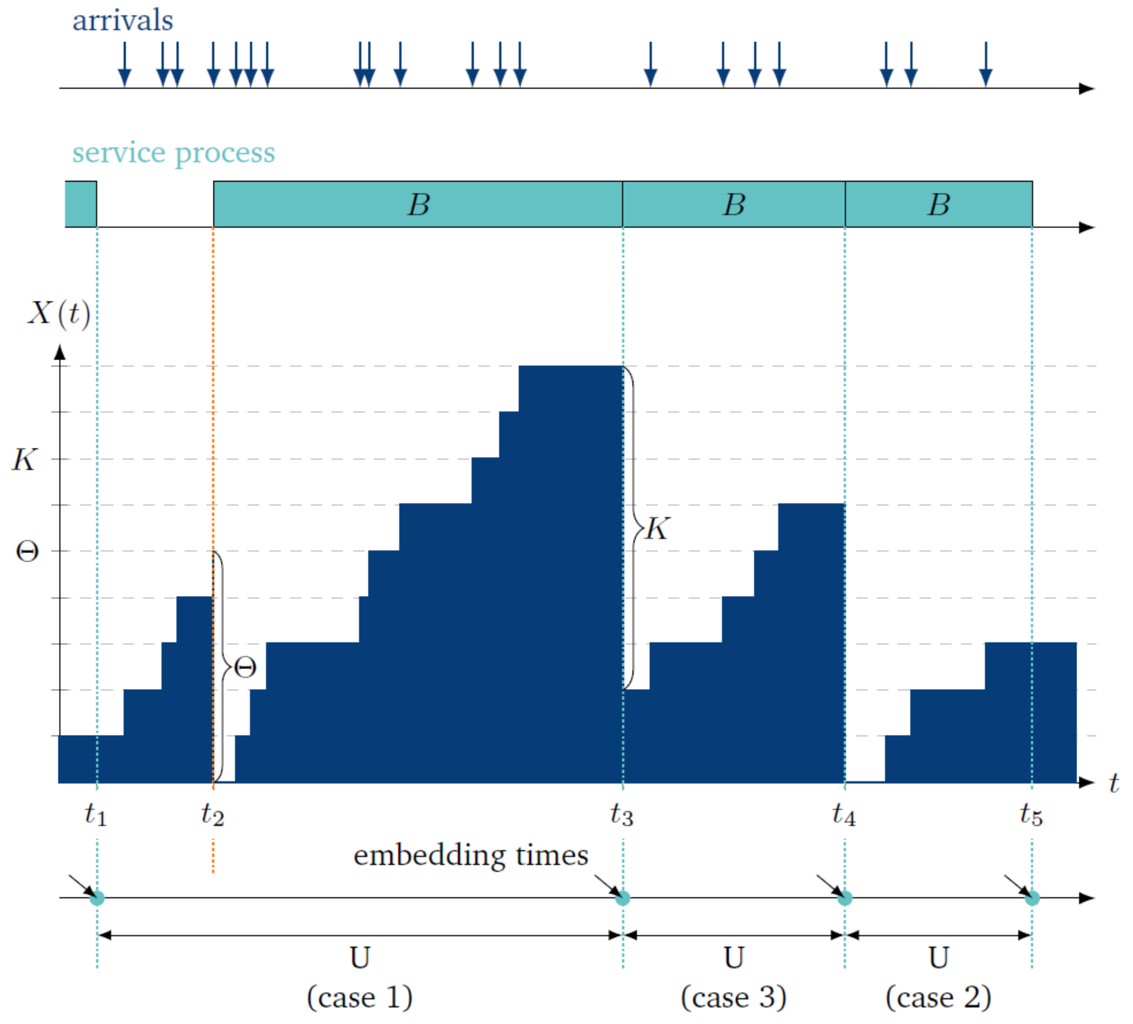
- Non-stationary analysis

$$\mathbf{X}_{n+1} = \mathbf{X}_n \cdot \mathcal{P}$$

- Steady state for $n \to \infty$

$$\mathbf{X}_{n+1} = \mathbf{X}_n = \mathbf{X} , \qquad \mathbf{X} = \mathbf{X} \cdot \mathcal{P}.$$

- Numerical Solution:
  left eigenvector of transition matrix $\boldsymbol{P}$
  or using power method



State Probabilities at Arbitrary Observation Times

- State probability vector at arbitrary times $\mathbf{X}^* = (x^*(0), x^*(1), \ldots, x^*(S))$
- Define r.v. $X_y^*(i) = P(X^* = i, Y = y)$
  - with X^* the number of customers in queue at an arbitrary time
  - while Y indicates if the service is active and serving a batch ($Y = 1$) or idle ($Y = 0$)
- Relation to state probabilities at embedding times is

$$x_0^*(i) = \frac{\sum_{j=0}^{i} x(j)}{\lambda \mathrm{E}[\,B\,] + \sum_{k=0}^{i} (\Theta - i) x(k)}, \quad 0 \le i \le \Theta - 1$$

$$x_1^*(i) = \frac{\sum_{j=i+1}^{\min(K+i,S)} x(j)}{\lambda \mathrm{E}[\,B\,] + \sum_{k=0}^{i} (\Theta - i) x(k)}, \quad 0 \le i \le S - 1$$

$$x_1^*(S) = 1 - \sum_{i=0}^{\Theta-1} x_0^*(i) - \sum_{i=0}^{S-1} x_1^*(i)$$

$$x^*(i) = x_0^*(i) + x_1^*(i).$$

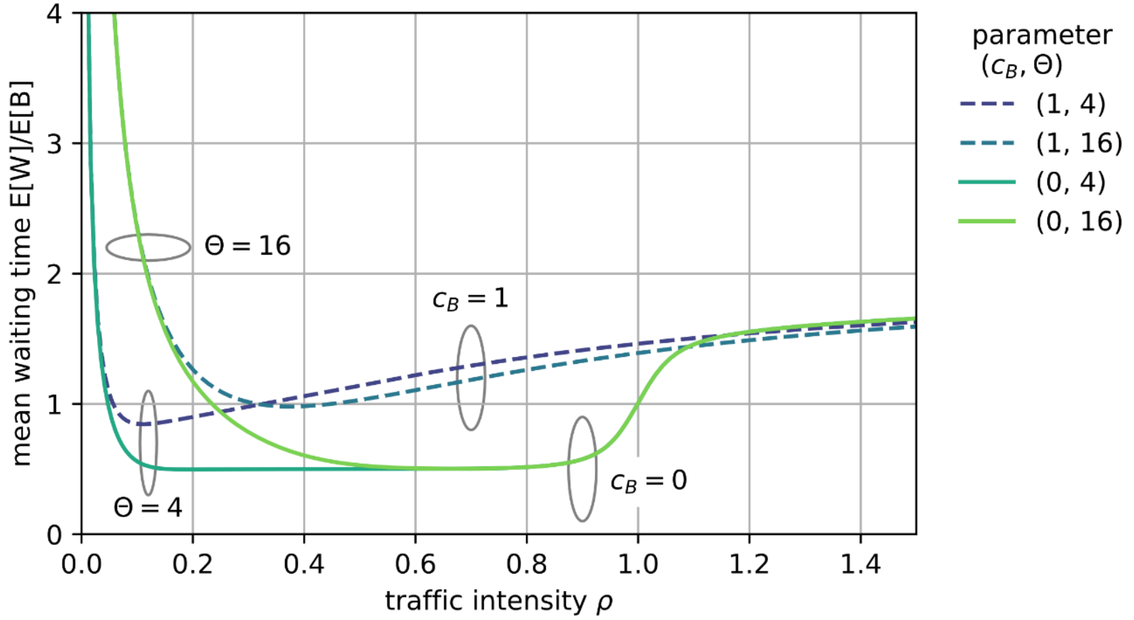
blocking probability $p_B = x^*(S)$

mean waiting time $\mathrm{E}[\,W\,] = \frac{\mathrm{E}[\,X^*\,]}{\lambda(1 - p_B)}$

$$\mathrm{E}[\,X^*\,] = \sum_{k=0}^{S} k \cdot x^*(k)$$

Numerical Results

- At lower traffic intensity ρ the waiting time is long, since the system has to wait for at least Θ customers to arrive before a service period can start
- At higher traffic intensity, the mean waiting time depends rather more on the type of service (c_B)



- Optimum parameter Θ depends strongly on system parameters (c_B, ρ)
- Thus, each system should be configured invidually