

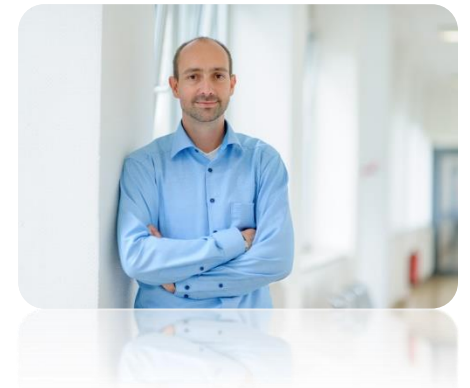
Chapter 5.1

Discrete-Time Markov Chain

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022



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*Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication
Networks - A Lecture Note. Würzburg University Press.
<https://doi.org/10.25972/WUP-978-3-95826-153-2>*

Website to download book, exercises, slides and scripts:
<https://modeling.systems/>

Chapter 5

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Discrete-Time Markov Chain (DTMC)

► Discrete-time Markov process

- Stochastic process $\{X(t), t > 0\}$ has Markov property at discrete (not necessarily equidistant) points in time $\{t_n, n = 0, 1, \dots\}$
- Sequence of r.v.s $\{X(t_0), X(t_1), \dots\}$: $X(t_{n+1})$ only depends on current $X(t_n)$ (Markov property)

► Discrete-time Markov chain

- finite or countable state space S : discrete state space, e.g., analysis of M/GI/1 or GI/M/1
- transition probability $p_{ij} \geq 0$ for $i \neq j$ and $i, j \in S$
- initial state $X(0)$

► Fundamental: **Markov property**

- $P(X(t_{n+1}) = x_{n+1} | X(t_n) = x_n, \dots, X(t_0) = x_0) = P(X(t_{n+1}) = x_{n+1} | X(t_n) = x_n)$
- **transition probability**: $p_{ij} = P(X(t_{n+1}) = x_{n+1} | X(t_n) = x_n)$
- transition probability matrix (stochastic matrix): $\mathbf{P} = \{p_{ij}\}$

Continuous-Time Markov Chain (CTMC)



▶ Continuous-time Markov process

- Stochastic process $\{X(t), t \geq 0\}$ with Markov property

▶ Continuous-time Markov chain (CTMC) is defined by

- discrete state space S is finite or countable; e.g. number of customers in system
- transition rates $q_{ij} \geq 0$ for $i \neq j$ and $i, j \in S$
- initial state $X(0)$, i.e. probability distribution of initial state

▶ Probability $x(i, t) = P(X(t) = i)$ that the system is in state $[X = i]$ at time t

▶ State vector $\mathbf{X}(t) = (x(t, 0), x(t, 1), \dots)$

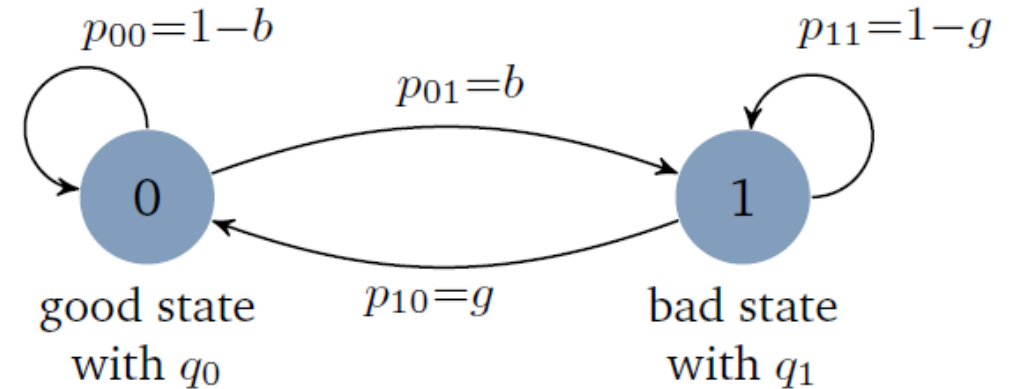
▶ Definition of **rate matrix** \mathbf{Q} with $q_{ii} = -\sum_{i \neq j} q_{ij}$

- allows compact notation (Kolmogorov equations.)
- row-wise sums of \mathbf{Q} are 0

Example: Bursty Channel (Gilbert-Elliot Model)

- ▶ Gilbert-Elliot model has two different states
 - good state [$X = 0$] with low packet loss probability q_0
 - bad state [$X = 1$] with a high packet loss probability q_1
- ▶ DTMC
 - probability b : change from good to bad
 - probability g : change from bad to good

- ▶ Transition diagram with transition probabilities

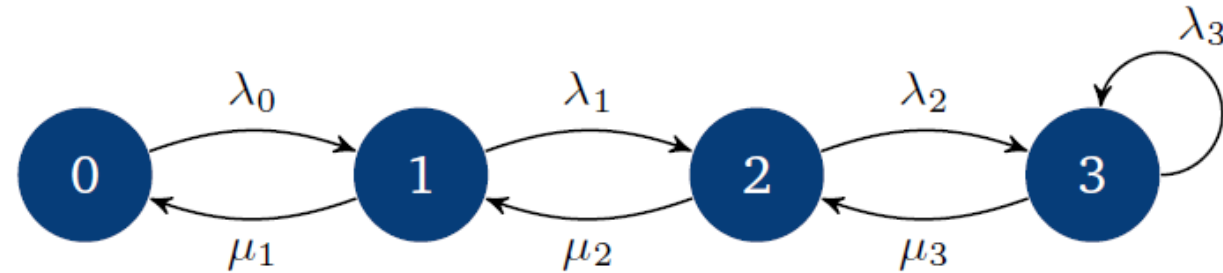


- ▶ Transition matrix

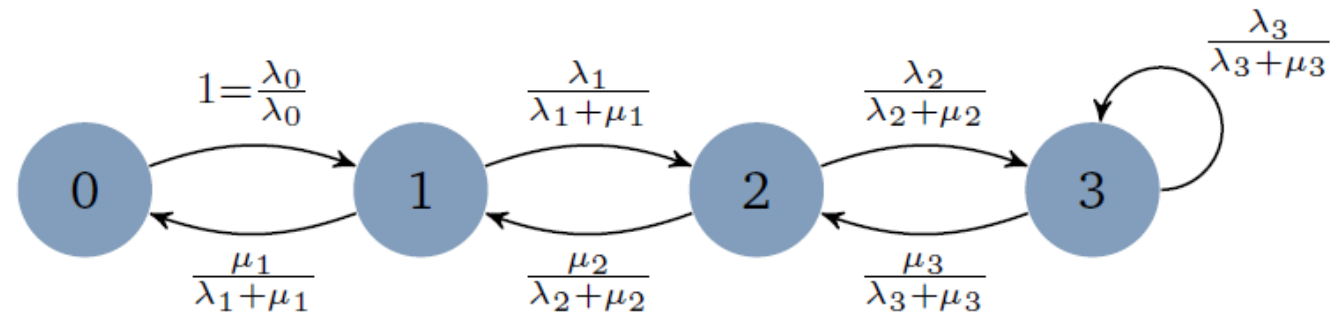
$$\mathcal{P} = \begin{pmatrix} 1 - b & b \\ g & 1 - g \end{pmatrix}$$

DTMC of a CTMC

$$p_{ij} = \frac{q_{ij}}{q_i} = \frac{q_{ij}}{\sum_{i \neq j} q_{ij}}$$



(a) Continuous-time Markov chain (CTMC) with transition rates.



(b) Embedded discrete-time Markov chain (DTMC) with transition probabilities.

Chapter 5.2

Method of Embedded Markov Chain

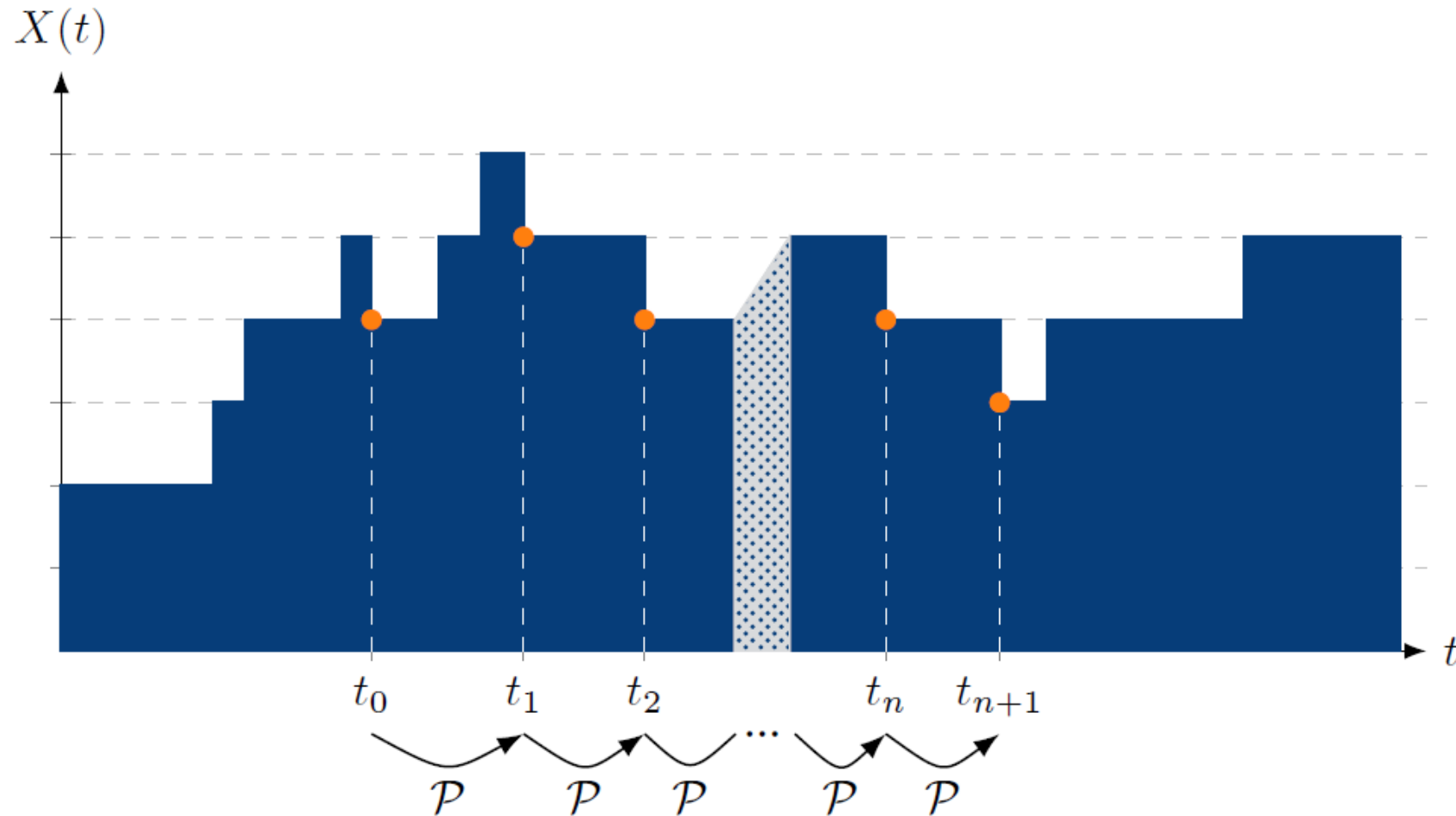
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Regeneration Points of the Embedded Markov Chain



Method of Embedded Markov Chain

- **State probabilities** at embedding times

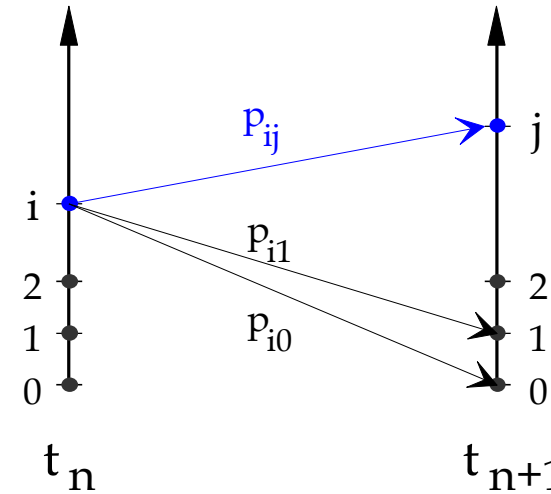
$$X_n = \{x(i,n), i = 0, 1, \dots\}$$

$$x(i,n) = P(X(t_n) = i)$$

- **Transition probability matrix** \mathcal{P}

$$\mathcal{P} = \{p_{ij}\}$$

$$p_{ij} = P(X(t_{n+1}) = j | X(t_n) = i), \quad i, j = 0, 1, \dots$$



$$\sum_j p_{ij} = 1$$

Method of Embedded Markov Chain: Matrix Form

- ▶ At each embedding time t_n , state probability vector

$$\mathbf{X}_n = (x(0, n), x(1, n), \dots) ,$$
$$x(i, n) = P(X(t_n) = i) .$$

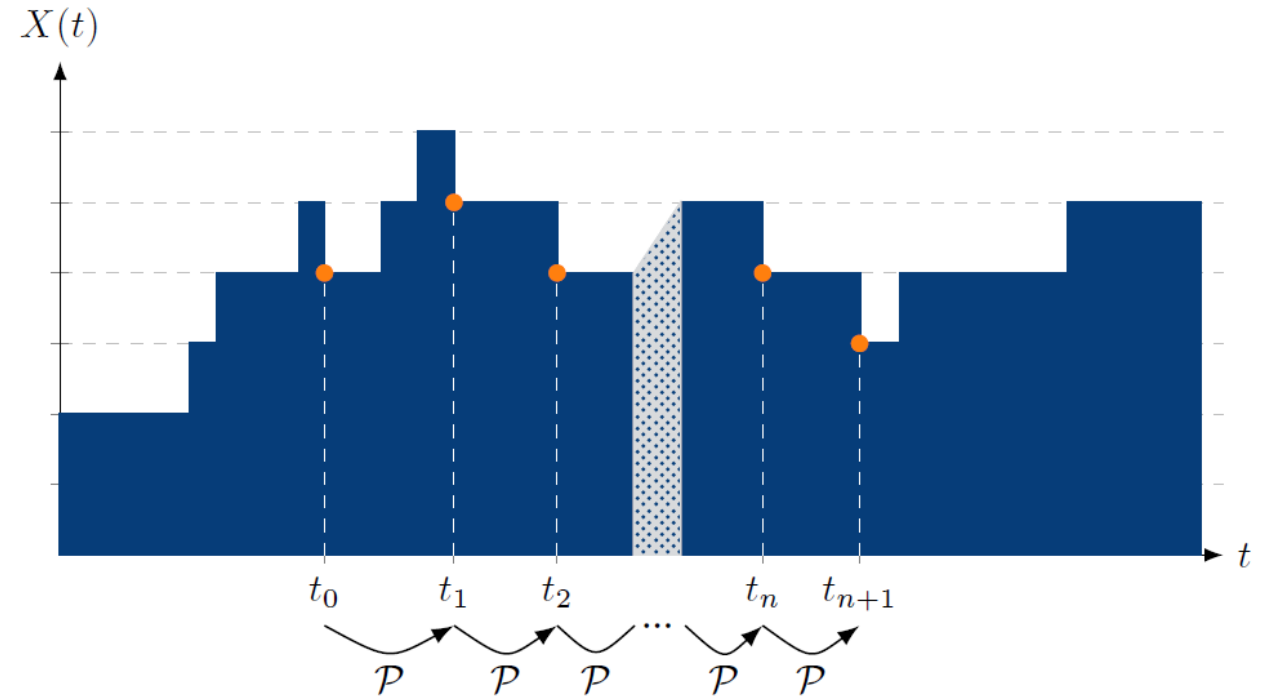
- ▶ Non-stationary analysis

$$\mathbf{X}_{n+1} = \mathbf{X}_n \cdot \mathcal{P}$$

- ▶ Steady state for $n \rightarrow \infty$

$$\mathbf{X}_{n+1} = \mathbf{X}_n = \mathbf{X} , \quad \mathbf{X} = \mathbf{X} \cdot \mathcal{P} .$$

stationary probability vector of the embedded Markov chain is
left-eigenvector of the transition probability matrix with eigenvalue 1



POWER METHOD

Numerical solution

Power Method

- ▶ Numerical robust method for non-stationary analysis of embedded Markov chains
- ▶ An implementation of the power method is provided at <https://modeling.systems>

```
def powerMethod(X0, P, stopFunction):  
    Z = P.shape[0] # P is a quadratic matrix of size Z x Z  
    X_old = numpy.zeros(Z)  
    X1 = X0  
    while stopFunction(X1, X_old): # test if steady state is reached  
        X_old = X1  
        X1 = X_old @ P # compute Xn+1 = Xn*P via matrix multiplication  
    return X1
```

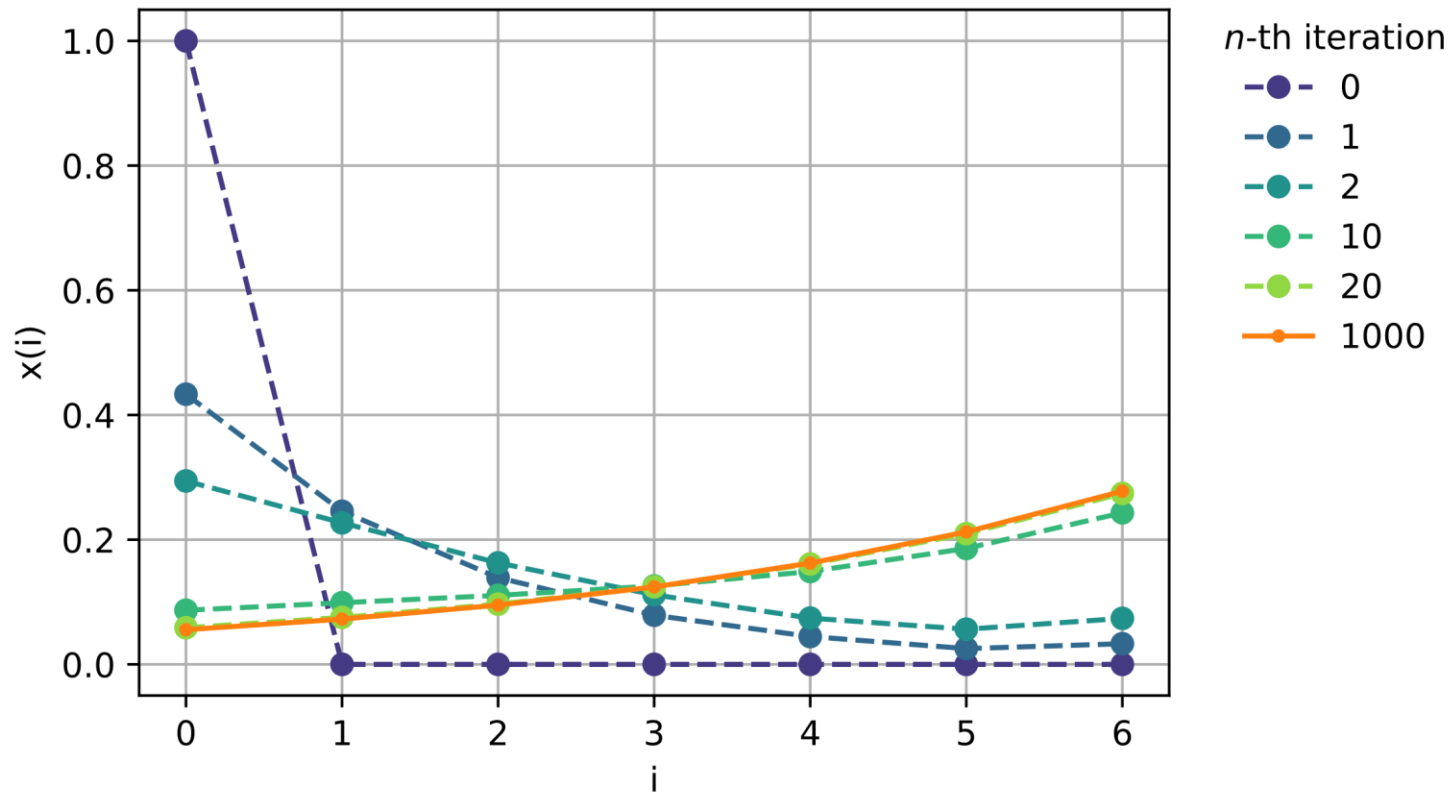
```
def stopFunction_mean(X1, X_old, epsilon=1e-6):  
    i = numpy.arange(len(X1)) # with i = (0, 1, ..., len(X1)-1)  
    EX_old = X_old @ i # expected value E[X_n]  
    EX1 = X1 @ i # expected value E[X_n+1]  
    return abs(EX_old-EX1) > epsilon
```

```
Xmax = 6 # number of states 0, 1, ..., Xmax  
X0 = numpy.zeros(Xmax+1)  
X0[0] = 1 # initialization: empty system  
X = powerMethod(X0, P, stopFunction_mean)
```

$$|E[X(t_{n+1})] - E[X(t_n)]| < \varepsilon = 10^{-6}$$

Illustration of Power Method

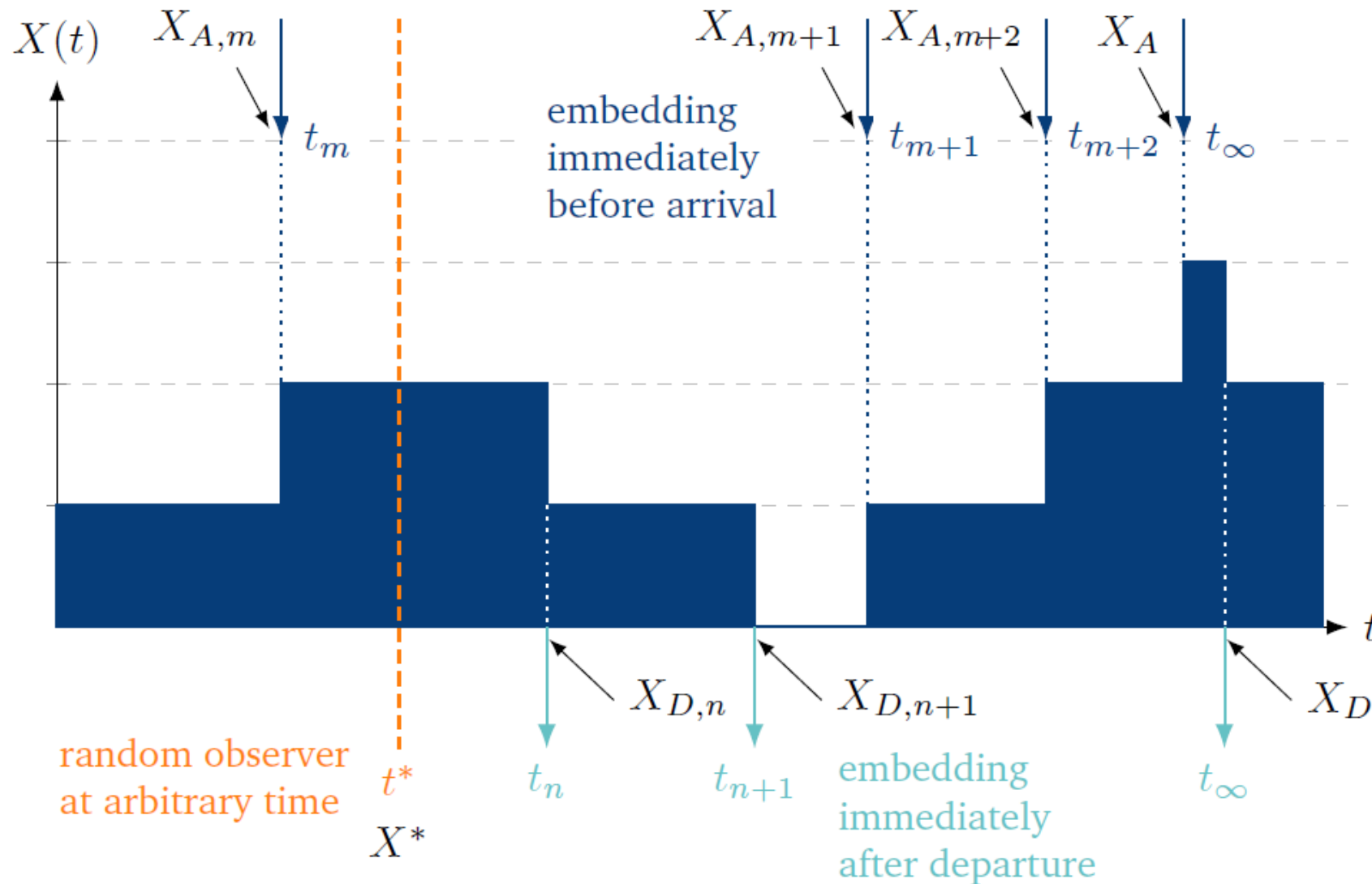
- Analysis of M/GI/1-5 quickly converges



NOTION OF EMBEDDING TIMES

Kleinrock's (Burke's) result

Notion of Embedding Times



Variables of System State at Embedding Times

- ▶ X_A system state as seen by an arriving customer immediately before or after an arrival
- ▶ X_D system state as seen by a departing customer immediately before or after service end
- ▶ X^* system state as seen by a random observer, i.e., at an arbitrary time
- ▶ X system state at defined embedding times

- ▶ Examples for embedding times
 - $X = X_D$ with embedding immediately after departure for M/GI/1
 - $X = X_A$ with embedding immediately before arrival for GI/M/1
 - $X = X^*$ for birth-and-death processes

- ▶ In general: $X^* \neq X_A$
- ▶ PASTA property: $X^* = X_A$

- ▶ In general: $X_A \neq X_D$
- ▶ Kleinrock's (Burke's) result shows for which systems $X_A = X_D$

Kleinrock's (Burke's) Result

- ▶ System state can change at most by +1 or −1. State distribution as seen by an arriving customer will be the same as that seen by a departing customer.

$$x_A(i) = x_D(i) \text{ for } i = 0, 1, \dots$$

- ▶ Example: analysis of M/GI/1 queue utilizes this result
- ▶ Note: result cannot be used for queues with batch arrivals or batch services

Kleinrock's (Burke's) Result: Derivation

