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Institute of Computer Science
Chair of Communication Networks
Prof. Dr. Tobias Hoßfeld

# Chapter 2.4

# Some Important Distributions

**Performance Evaluation of the Internet of Things (IoT)**

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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**The book must be cited and the disclaimer attached when using lectures slides or scripts.**

*Tran-Gia, P. & Hossfeld, T. (2021).*
*Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.*
*https://doi.org/10.25972/WUP-978-3-95826-153-2*

Website to download book, exercises, slides and scripts:
https://modeling.systems/

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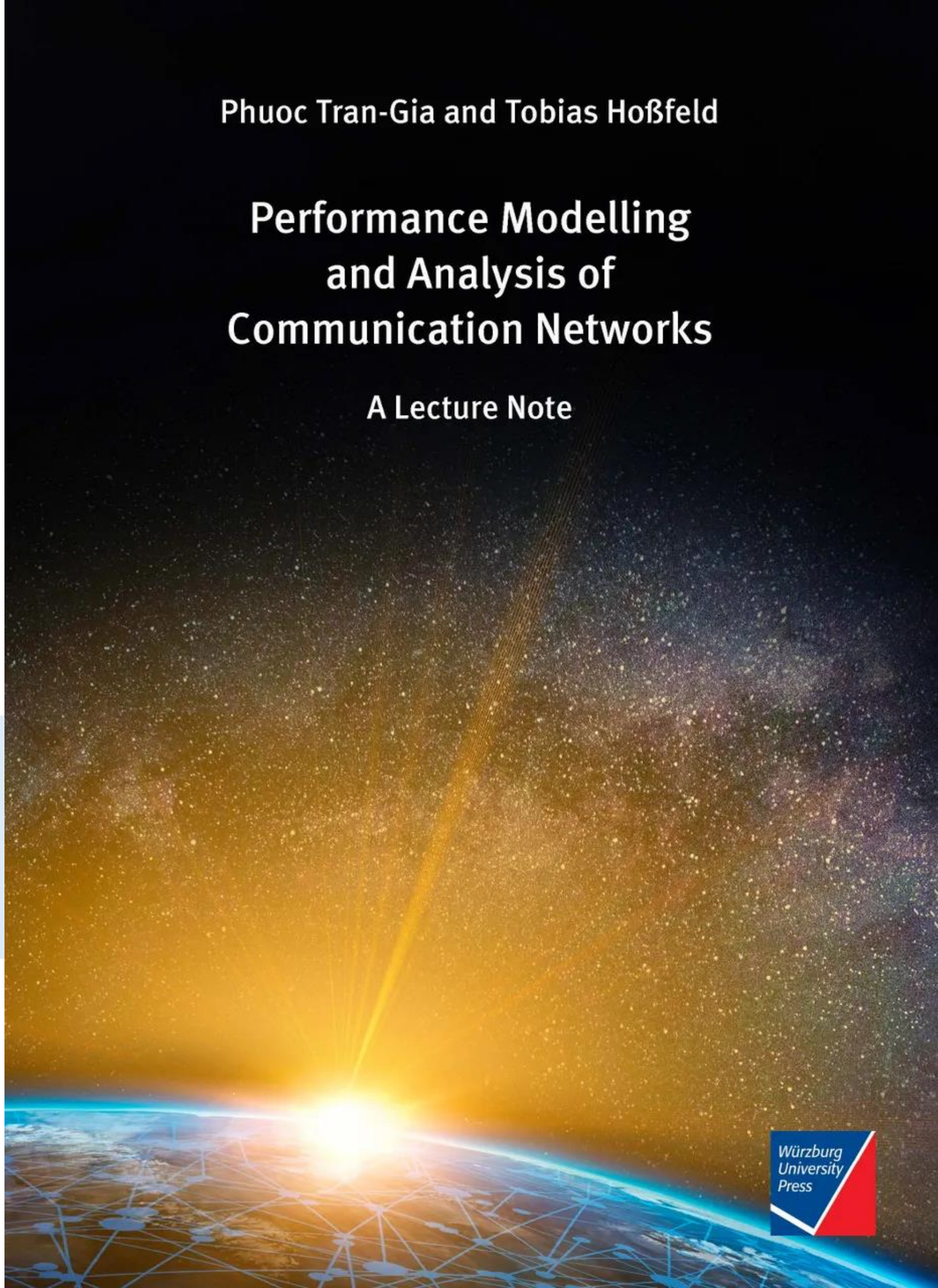


# Chapter 2

## 2 Fundamentals and Prerequisites

# Discrete Distributions

PMF, CDF, transform (generating function)

# Bernoulli Experiment and Distribution (BER)

- Bernoulli experiment $X$ with two possible outcomes
  - success with probability $p$
  - failure with probability $1 - p$

- **Bernoulli distribution**

$$X \sim \mathrm{BER}(p) \quad \text{with} \quad 0 \le p \le 1$$

$$x(i) = P(X = i) = \begin{cases} 1 - p & , \; i = 0 \quad \text{(failure)} \\ p & , \; i = 1 \quad \text{(success)} \end{cases}$$

$$\mathrm{E}[\,X\,] = p\,, \qquad c_X = \sqrt{\frac{1-p}{p}}$$

$$X_{GF}(z) = (1-p) + pz\;.$$

# Binomial Distribution (BINOM)

- Sum of $N$ statistically independent Bernoulli experiments with success probability $p$
  - $\binom{N}{i}$ patterns of $i$ successes
  - Each pattern occurs with probability $p^i(1-p)^{N-i}$

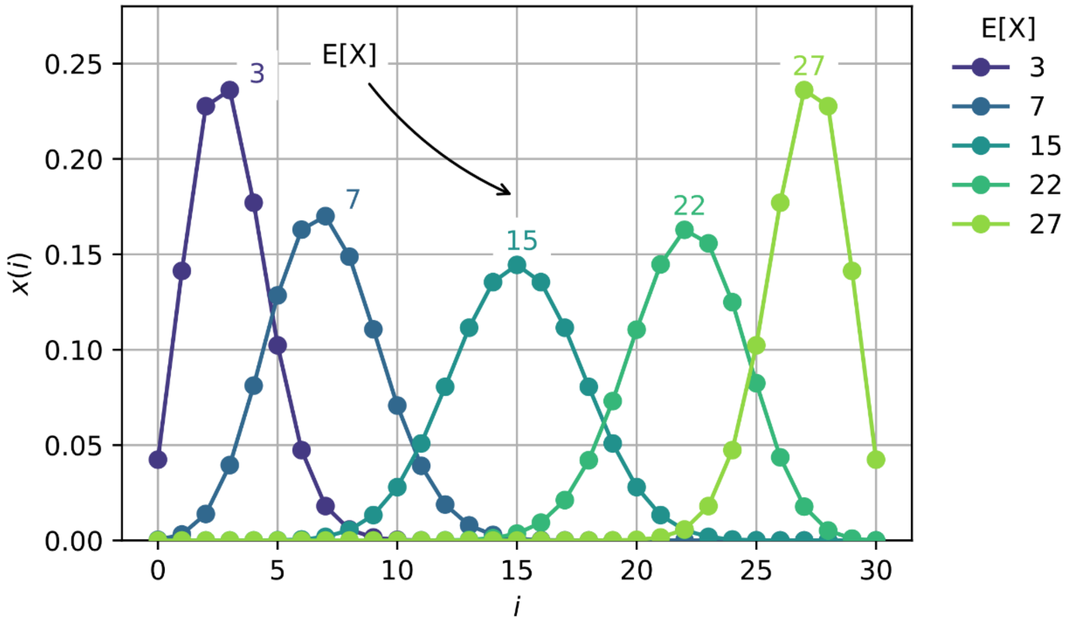
## Binomial distribution

$$X \sim \mathrm{BINOM}(N,p) \ \text{ with } \ 0 \le p \le 1 \,, \ N \in \mathbb{N}^+$$

$$x(i) = \binom{N}{i} p^i \, (1-p)^{N-i} \,, \quad i = 0, 1, \ldots, N$$

$$\mathrm{E}[\,X\,] = Np \,, \qquad c_X = \sqrt{\frac{1-p}{Np}}$$

$$X_{GF}(z) = \Big((1-p) + pz\Big)^N$$



# Geometric Distribution (GEOM)

- Series of Bernoulli experiments with success probability $p$ until first "success" outcome
- Number $X$ of failures before first success follows geometric distribution

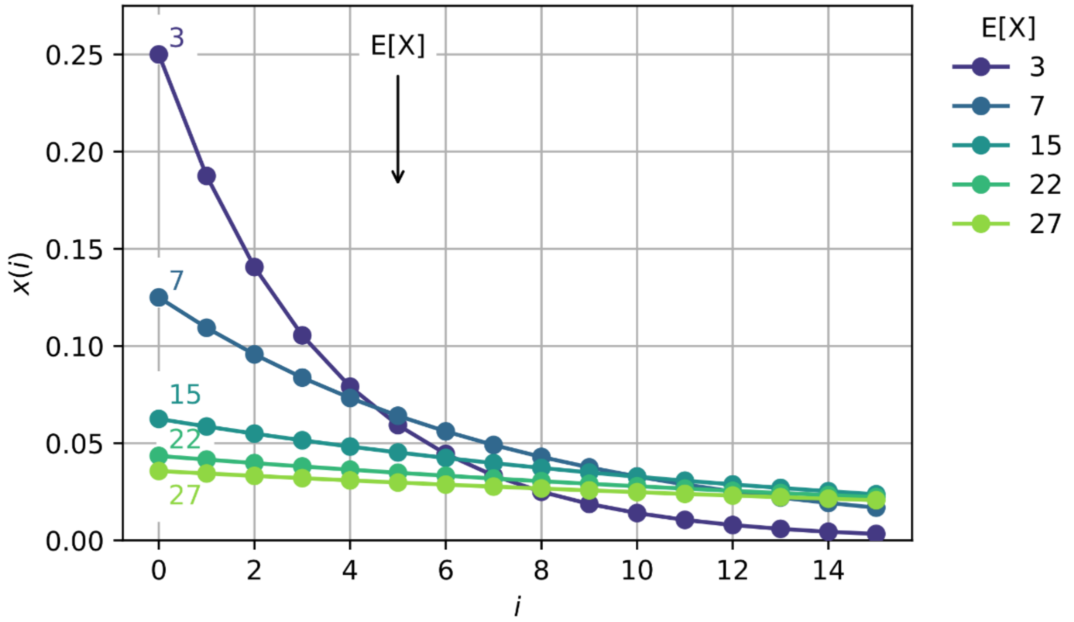
- **Geometric distribution**

$$X \sim \mathrm{GEOM}(p) \;\; \text{with} \;\; 0 \le p \le 1$$

$$x(i) = (1-p)^i \cdot p\,, \quad i = 0, 1, \ldots$$

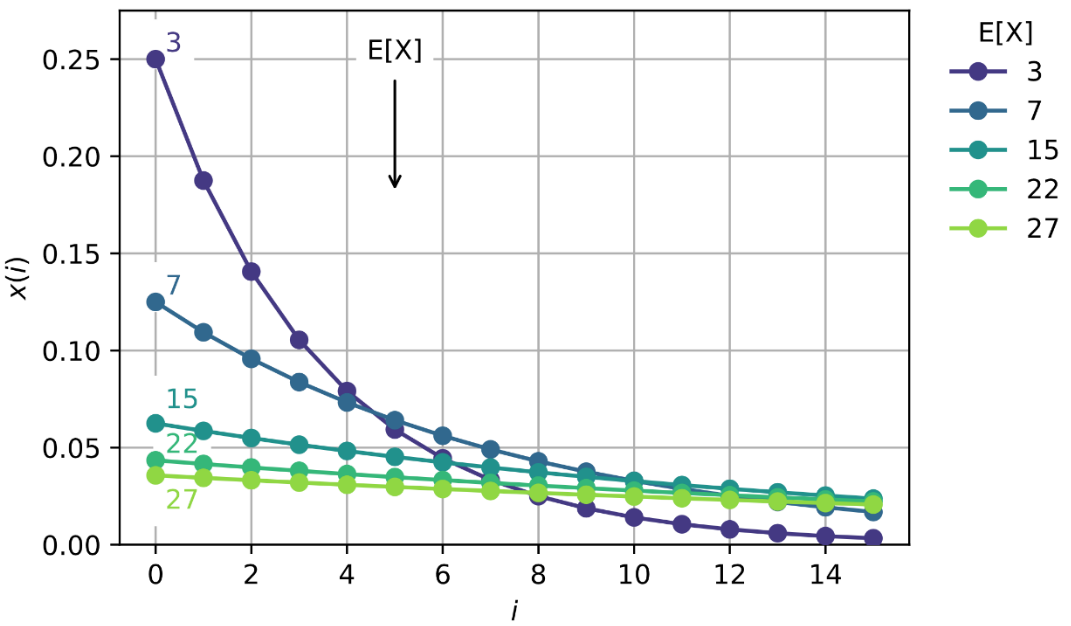
$$\mathrm{E}[\,X\,] = \frac{1-p}{p}\,, \qquad c_X = \frac{1}{\sqrt{1-p}}$$

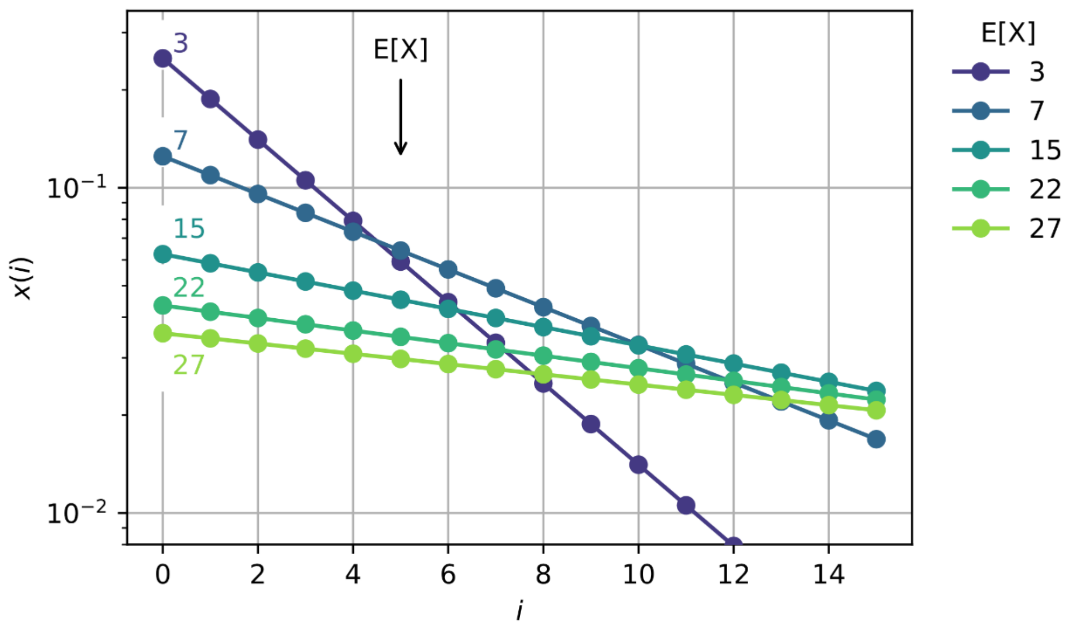
$$X_{GF}(z) = \frac{p}{1-z+pz}$$



# Geometric Distribution: Logarithmic Scale

$$x(0) = (1-p)^0 p = p$$



# Variants of the Geometric Distribution

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# Example: Bit Errors and Packet Errors

- Data packet with $n$ bit
- Bit error probability $p_b$

- Number of bit errors in packet: $X \sim BINOM(n, p_b)$

- Packet error probability: $p_{\mathrm{P}} = 1 - x(0) = 1 - (1 - p_b)^n \neq n \cdot p_b$

- Approximation for $p_b \ll 1$: $p_{\mathrm{P}} \approx 1 - (1 - n \cdot p_b) = n \cdot p_b$

- Number of failed transmission attempts until successful transmission:

$$x(i) = p_P^i (1 - p_P), \qquad i = 0, 1, 2, \ldots$$

# Negative Binomial Distribution (NEGBIN)

- Number $X$ of failures in a sequence of iid. Bernoulli trials
  - with success probability $p$
  - before a specified (real valued) number of successes $y$ occurs

- **Negative binomial distribution**

$$X \sim \mathrm{NEGBIN}(y,p) \;\text{ with }\; 0 \le p \le 1\,,\; y > 0$$

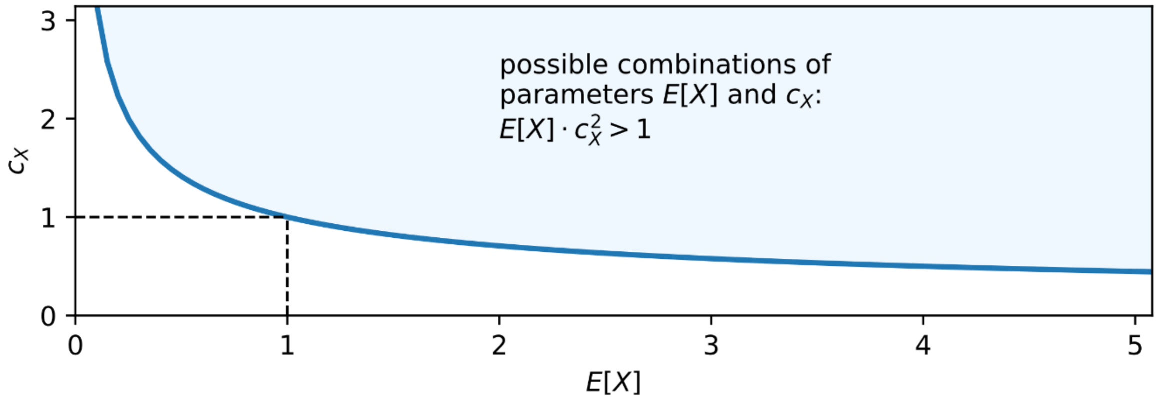
$$x(i) = \binom{y+i-1}{i} p^y (1-p)^i\,, \quad i = 0, 1, \ldots$$

$$\mathrm{E}[\,X\,] = \frac{y(1-p)}{p}\,, \qquad c_X = \frac{1}{\sqrt{y(1-p)}}$$

$$X_{GF}(z) = \left(\frac{p}{1-z+pz}\right)^y$$

For given mean and coefficient of variation

$$p = \frac{1}{\mathrm{E}[\,X\,]\cdot c_X^2}\,, \qquad y = \frac{\mathrm{E}[\,X\,]}{\mathrm{E}[\,X\,]\cdot c_X^2 - 1}$$



# Poisson Distribution (POIS)

- Number $X$ of arrival events
  - occurring in a fixed interval of time $\Delta t$
  - if these events randomly occur with a mean rate $\lambda$.
- Mean number of arrivals in interval: $y = \lambda \cdot \Delta t$
- Important in queueing theory: Poisson distribution and Poisson process (see Ch. 3.3)

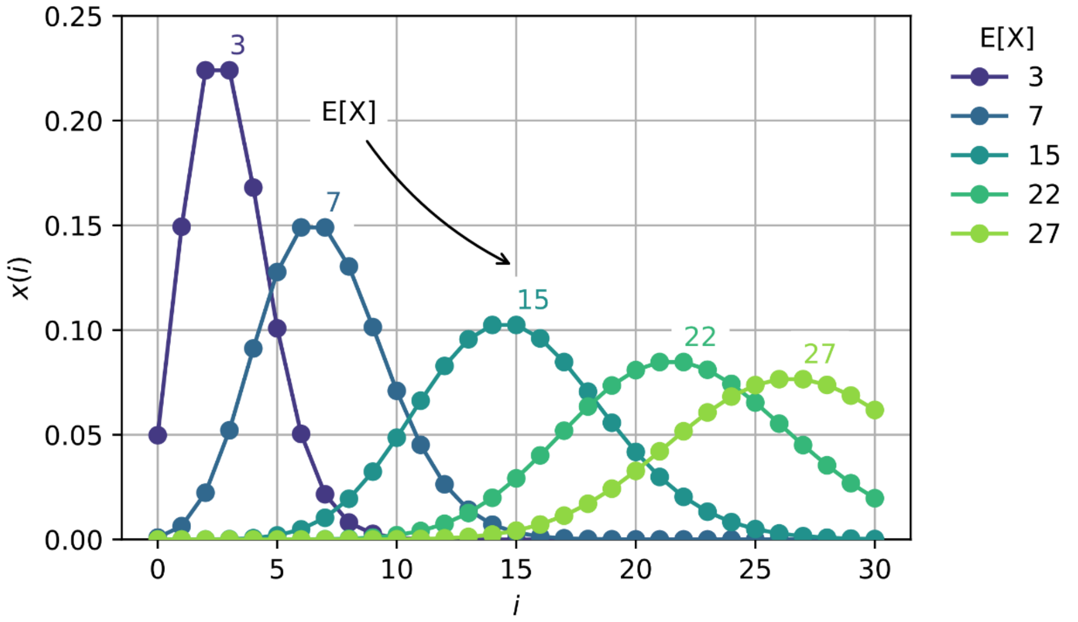
- **Poisson distribution**

$$X \sim \mathrm{POIS}(y) \quad \text{with} \quad y > 0$$

$$x(i) = \frac{y^i}{i!} e^{-y} , \quad i = 0, 1, \ldots$$

$$\mathrm{E}[X] = y , \qquad c_X = \frac{1}{\sqrt{y}}$$

$$X_{GF}(z) = e^{-y(1-z)}$$

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# Poisson Distribution and Poisson Process

Lecture

# CONTINUOUS DISTRIBUTIONS

PDF, CDF, Laplace transform

# Deterministic Distribution (D)

- For systematic reasons, deterministic distribution is introduced
  - r.v. A takes a constant value $t_0$ (no randomness)
  - CDF is a shifted step function

- **Deterministic distribution**

$$A \sim \mathrm{D}(t_0) \quad \text{with} \quad t_0 \in \mathbb{R}$$

$$A(t) = \begin{cases} 0\,, & t < t_0 \\ 1\,, & t \geq t_0 \end{cases}$$

$$a(t) = \delta(t - t_0)$$

$$\mathrm{E}[\,A\,] = t_0\,, \qquad c_A = 0$$

$$\Phi_A(s) = \mathrm{LST}\{\,A(t)\,\} = e^{-st_0}$$

# Deterministic Distribution (D): CDF and PDF

# Dirac Delta Function

# Negative Exponential Distribution (EXP)

- Very important distribution in queueing theory
  - interarrival times $A$ of Poisson process with arrival rate $\lambda$
  - Markov property of exponential distribution

- **Exponential distribution**

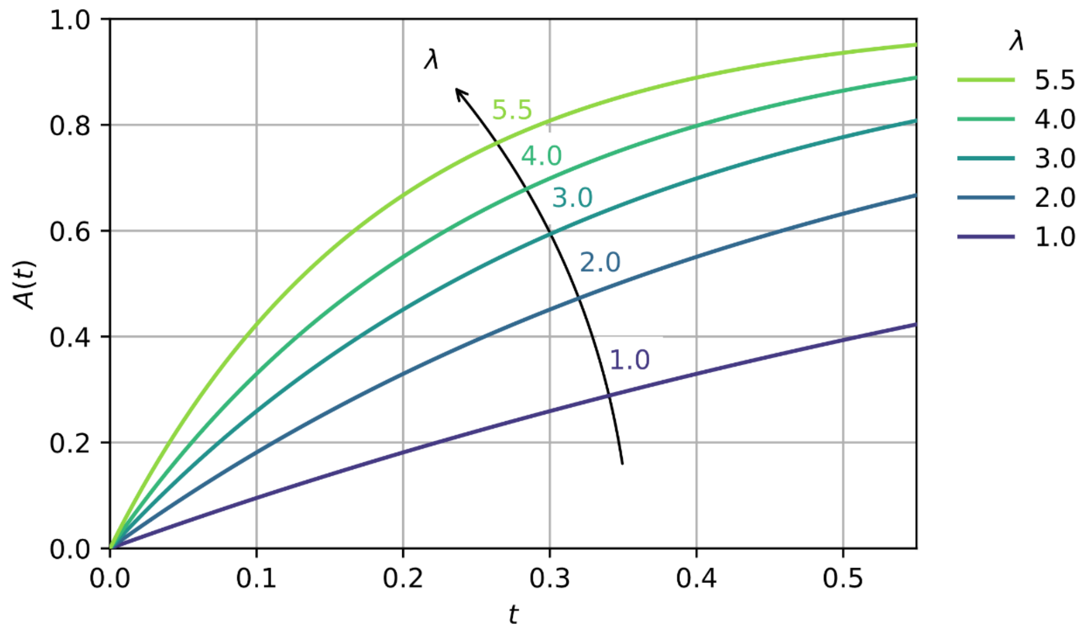
$$A \sim \mathrm{EXP}(\lambda) \quad \text{with} \quad \lambda > 0$$

$$A(t) = 1 - e^{-\lambda t}, \quad t \geq 0$$

$$a(t) = \lambda e^{-\lambda t}$$

$$\mathrm{E}[A] = \frac{1}{\lambda}, \qquad c_A = 1$$

$$\Phi_A(s) = \frac{\lambda}{\lambda + s}$$

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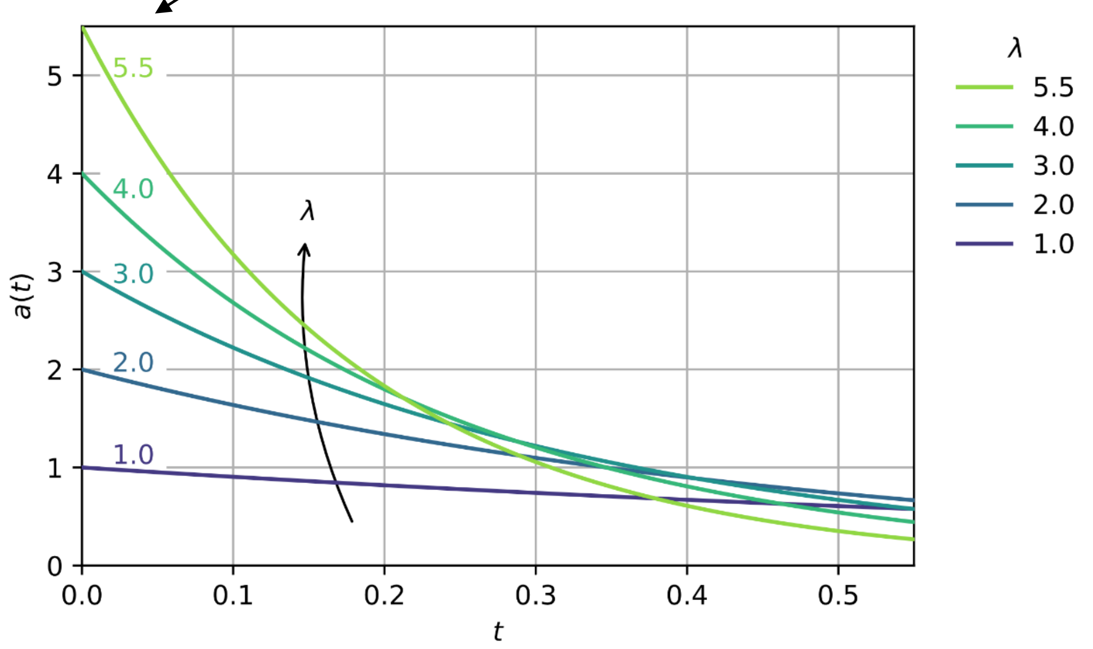
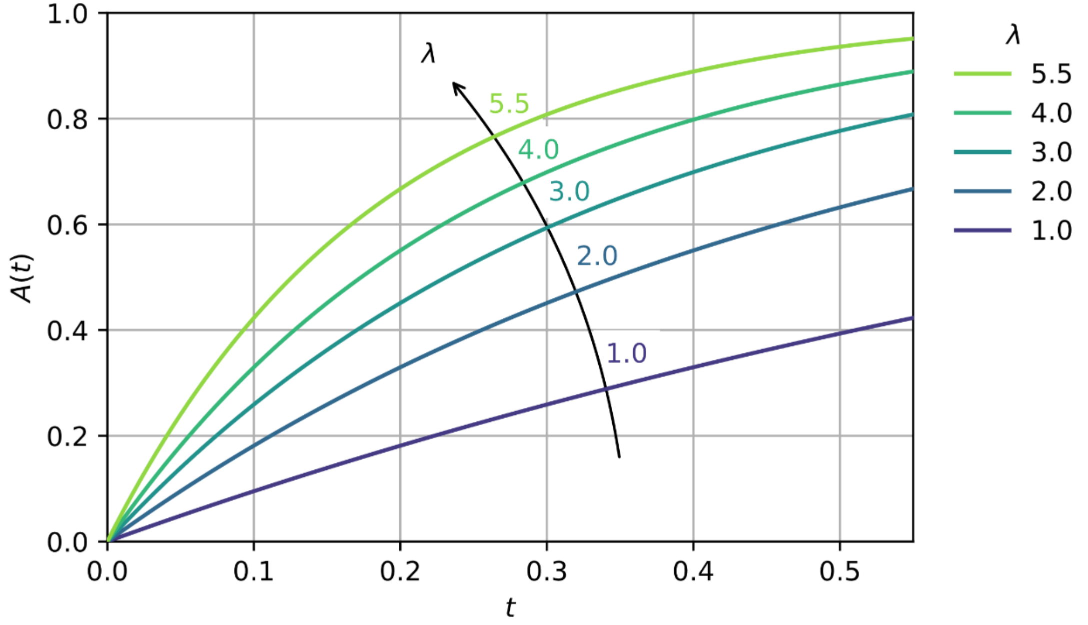
# Exponential Distribution: PDF and CDF

$a(0) = \lambda e^{-\lambda \cdot 0} = \lambda$

$A(0) = 1 - e^{-\lambda \cdot 0} = 0$



# Erlang-k Distribution ($E_k$)

- Sum of $k$ exponentially distributed r.v.s, each with parameter $\lambda$: $A = A_1 + A_2 + \cdots + A_k$
- $A_i$ are independent and identically distributed (iid): $A_i(t) = 1 - e^{-\lambda t}$ for $i = 1,2,\ldots,k$

- **Erlang-k distribution**

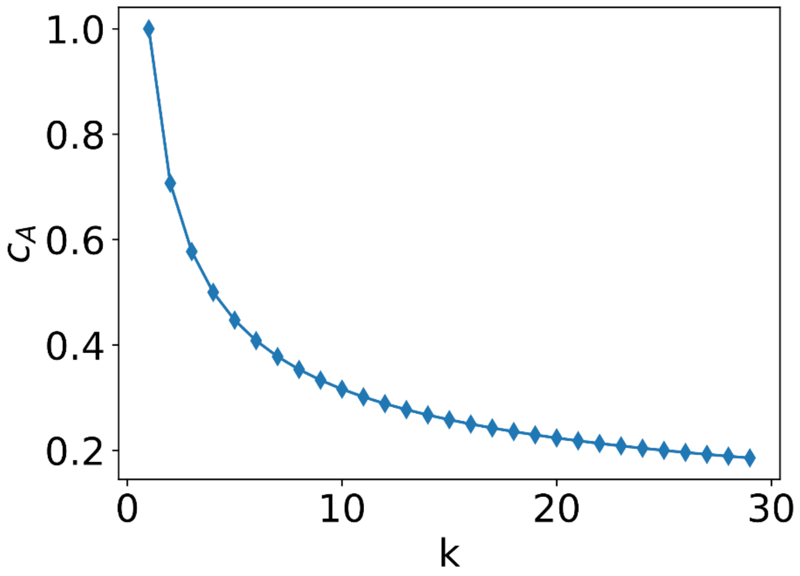
$$A \sim \mathrm{E}_k(\lambda) \quad \text{with} \quad \lambda > 0\,, \quad k \in \mathbb{N}^+$$

$$A(t) = 1 - \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}\,, \qquad t \geq 0$$

$$a(t) = \frac{\lambda(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}$$

$$\mathrm{E}[A] = \frac{k}{\lambda}\,, \qquad c_A = \frac{1}{\sqrt{k}}$$

$$\Phi_A(s) = \left(\frac{\lambda}{\lambda + s}\right)^k$$




For given mean $E[A]$ and coefficient of variation $0 < c_A \leq 1$:

$$k = \frac{1}{c_A^2} \qquad \lambda = \frac{1}{E[A]c_A^2}$$

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# Laplace Transform of Erlang-k Distribution

Lecture

# Hyperexponential Distribution ($H_k$)

- Selection between $k$ exponentially distributed r.v.s with parameters $\lambda_i$ with probability $p_i$
  - with vector $\boldsymbol{\Lambda} = (\lambda_1, \dots, \lambda_k)$ and $\boldsymbol{p} = (p_1, \dots, p_k)$

$$A \sim \mathrm{H}_k(\boldsymbol{\Lambda}, \mathbf{p}) \,, \quad \sum_{i=1}^{k} p_i = 1$$

$$A(t) = \sum_{i=1}^{k} p_i (1 - e^{-\lambda_i t}) = 1 - \sum_{i=1}^{k} p_i e^{-\lambda_i t} \,, \quad t \geq 0 \,,$$

$$a(t) = \sum_{i=1}^{k} p_i \lambda_i e^{-\lambda_i t} \,,$$

$$\mathrm{E}[A] = \sum_{i=1}^{k} \frac{p_i}{\lambda_i} \,, \qquad c_A = \sqrt{2 \left( \sum_{i=1}^{k} \frac{p_i}{\lambda_i^2} \right) \Big/ \left( \sum_{i=1}^{k} \frac{p_i}{\lambda_i} \right)^2 - 1} \,,$$

Note: It is $c_A \geq 1$

$$\Phi_A(s) = \sum_{i=1}^{k} p_i \frac{\lambda_i}{\lambda_i + s} \,.$$

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# Hyperexponential Distribution of Second Order ($H_2$)

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# Substitute Distributions

Lecture

# Uniform Distribution (U)

- Continuous r.v. $A$ follows a uniform distribution in the interval $[a; b]$

- **Uniform distribution**

$$A \sim U(a, b) \quad \text{with } a < b$$

$$A(t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le b \\ 1, & t > b \end{cases}$$

$$a(t) = \frac{1}{b-a}, \quad a \le t \le b$$

$$\mathrm{E}[A] = \frac{a+b}{2}, \qquad c_A = \frac{1}{\sqrt{3}} \cdot \frac{b-a}{a+b}$$

$$\Phi_A(s) = \frac{e^{-sa} - e^{-sb}}{s(b-a)}$$

# Mixture Distribution (MIX)

- $k$ independent r.v.s $A_1, \dots, A_k$ are selected with probability $p_i$

$$A = \begin{cases} A_1 \text{ with } p_1 \\ \dots \\ A_k \text{ with } p_k \end{cases}$$

- **Mixture distribution**

$$A \sim \mathrm{MIX}\big((A_1, \dots, A_k), (p_1, \dots, p_k)\big)$$

$$a(t) = \sum_{i=1}^{k} p_i \cdot a_i(t)\ , \qquad A(t) = \sum_{i=1}^{k} p_i \cdot A_i(t)$$

$$\mathrm{E}[\,A\,] = \sum_{i=1}^{k} p_i \cdot \mathrm{E}[\,A_i\,]\ , \quad \mathrm{E}[\,A^n\,] = \sum_{i=1}^{k} p_i \cdot \mathrm{E}[\,A_i^n\,]$$

$$\Phi_A(s) = \int_0^{\infty} e^{-st} a(t)\ dt = \int_0^{\infty} e^{-st} \sum_{i=1}^{k} p_i \cdot a_i(t)\ dt = \sum_{i=1}^{k} p_i \cdot \Phi_{A_i}(s)$$

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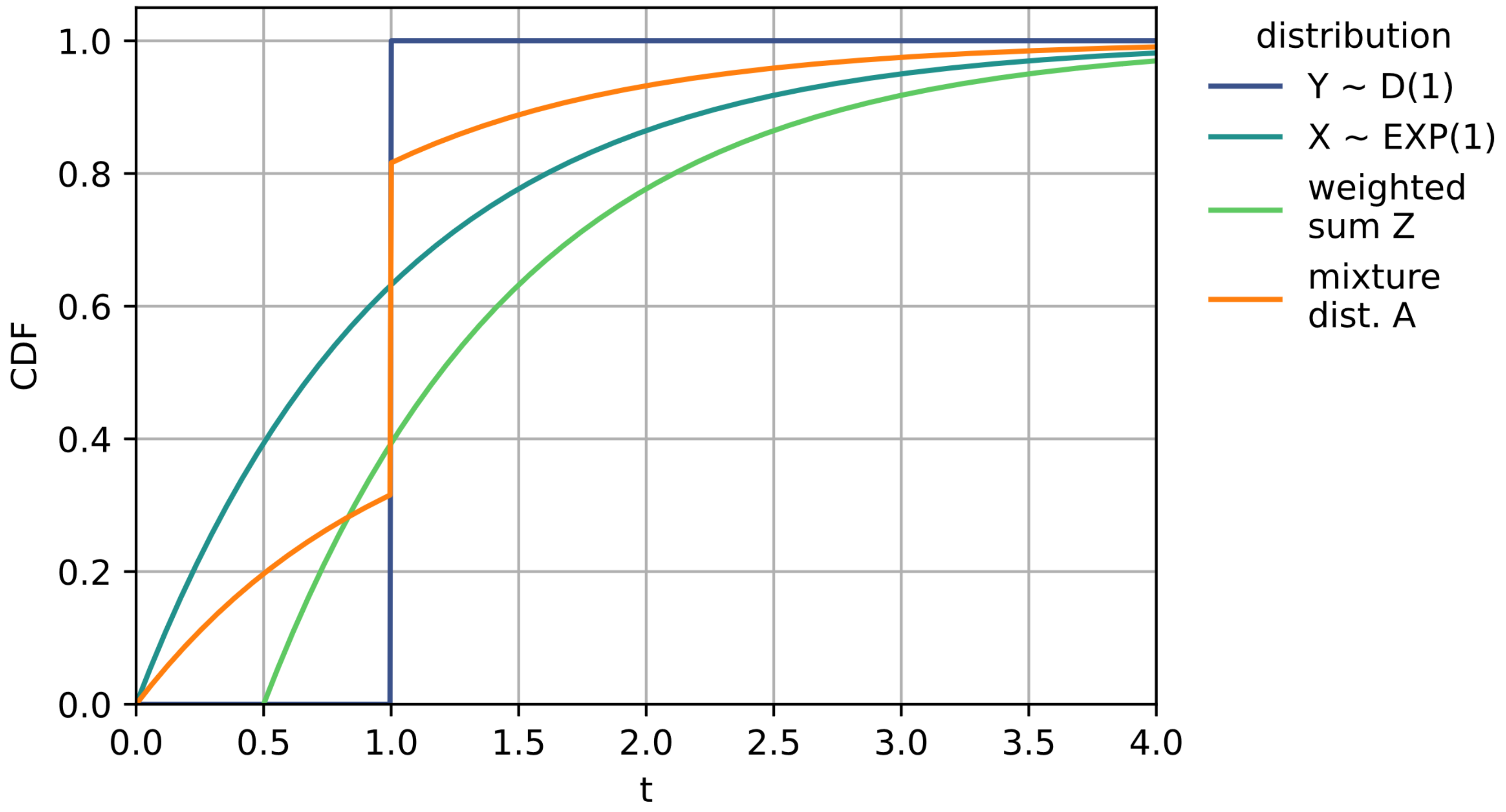
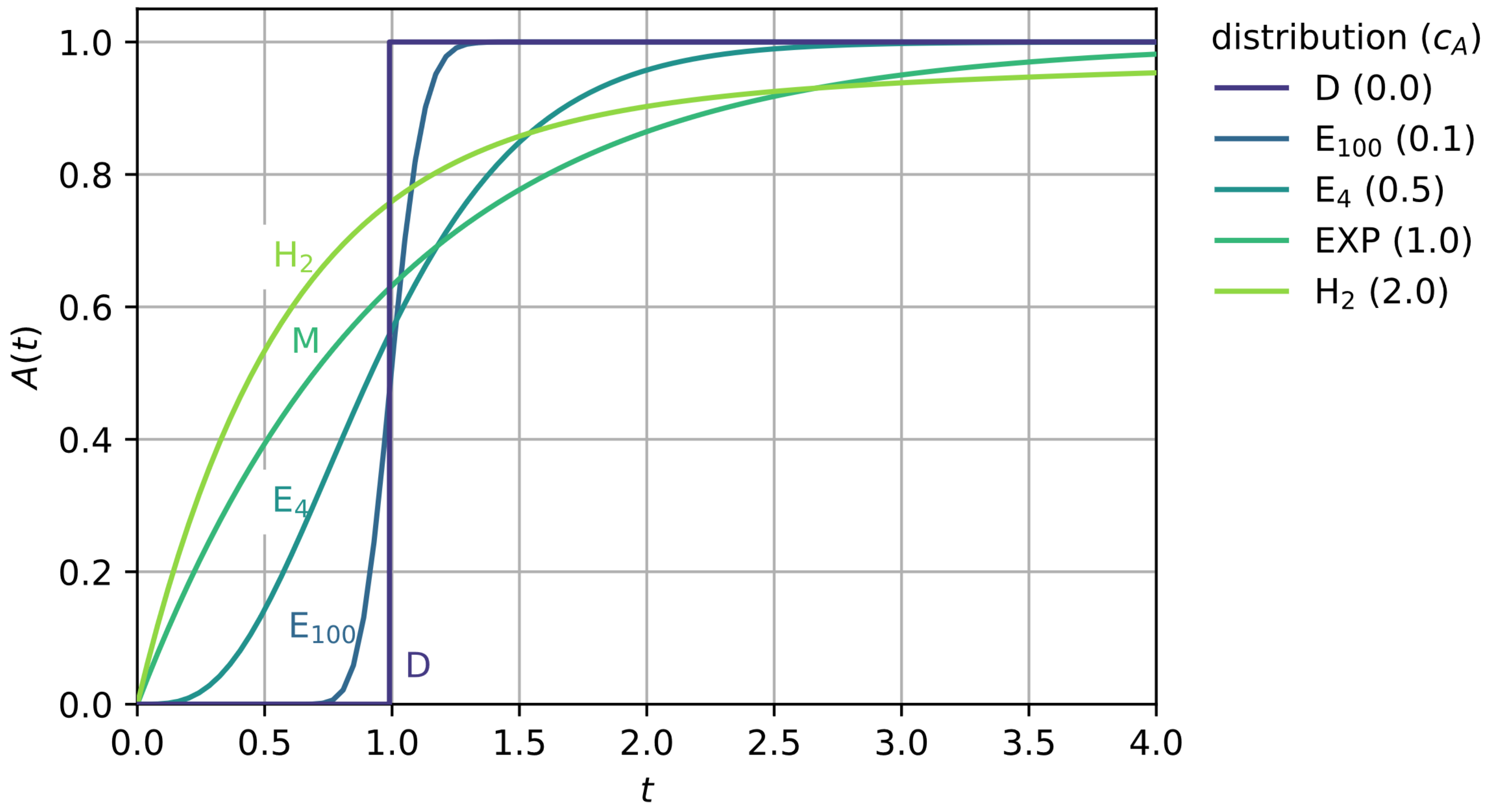
# Mixture Distribution: Visualization



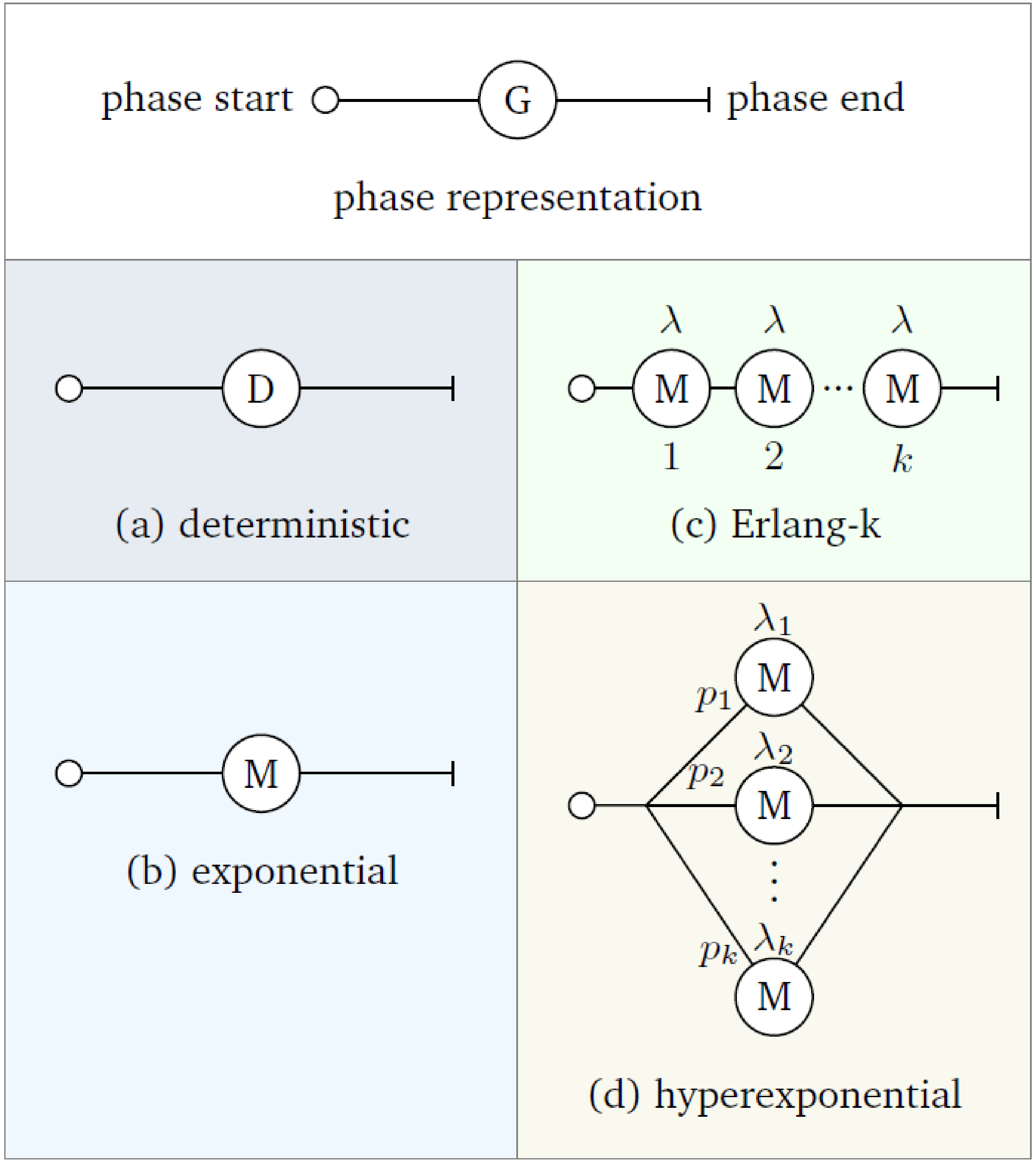
# Comparison of Relevant Continuous Distributions



# IMPORT RELATIONSHIPS

# Continuous Distributions

- Phase representation
  - visual representation of common distributions
  - indicates relationship of composed distributions



# Discrete Distributions

- **Bernoulli distribution**
  - with success probability p

- **Binomial distribution**
  - number of success of N Bernoulli trials

- **Geometric distribution**
  - number of failures until first success in series of iid. Bernoulli trials

- **Negative binomial distribution**
  - number of failures in a sequence of iid. Bernoulli trials with success probability $p$ before a specified number of successes $y$ occurs

# Poisson Arrivals during fixed Interval

- Number of Poisson arrivals during fixed interval follows Poisson distribution
- Interarrival times follow exponential distribution: Poisson process