

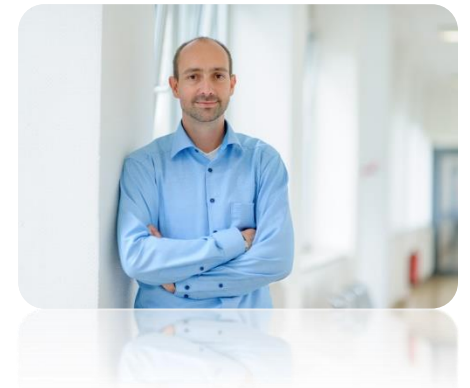
## Chapter 3.2

# Renewal Processes

### **Performance Evaluation of the Internet of Things (IoT)**

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022



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*Tran-Gia, P. & Hossfeld, T. (2021).  
Performance Modeling and Analysis of Communication  
Networks - A Lecture Note. Würzburg University Press.  
<https://doi.org/10.25972/WUP-978-3-95826-153-2>*

Website to download book, exercises, slides and scripts:  
<https://modeling.systems/>

# Chapter 3

## 3 Elementary Random Processes

### 3.1 Stochastic Processes

#### 3.1.1 Definition

#### 3.1.2 Markov Processes

#### 3.1.3 Elementary Processes in Performance Models

### 3.2 Renewal Processes

#### 3.2.1 Definition

#### 3.2.2 Analysis of Recurrence Time

### 3.3 Poisson Process

#### 3.3.1 Definition of a Poisson Process

#### 3.3.2 Properties of the Poisson Process

#### 3.3.3 Poisson Arrivals during Arbitrarily Distributed Interval

### 3.4 Superposition of Independent Renewal Processes

#### 3.4.1 Superposition of Poisson Processes

#### 3.4.2 Palm-Khintchine Theorem

### 3.5 Markov State Process

#### 3.5.1 Definition of Continuous-Time Markov Chain

#### 3.5.2 Transition Behavior of Markovian State Processes

#### 3.5.3 State Equations and State Probabilities

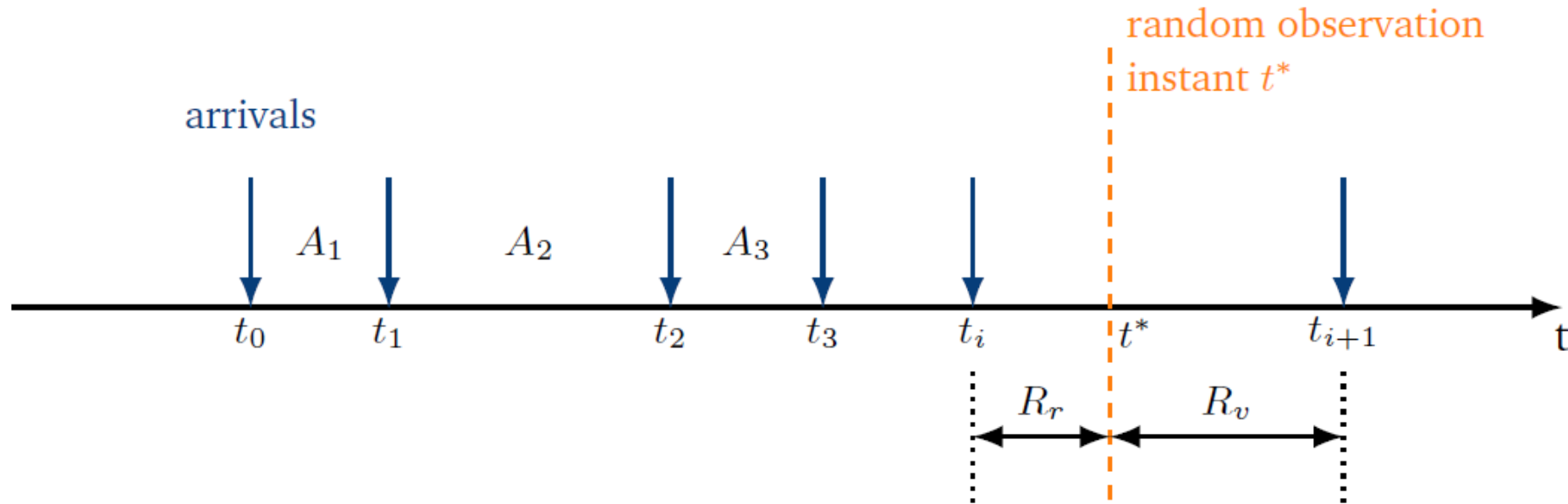
#### 3.5.4 Examples of Transition Probability Densities

#### 3.5.5 Birth-and-Death Processes

# POINT PROCESS AND RENEWAL PROCESS

# Point Process

- ▶ Finite or infinite sequence of random points in time or occurrences (arrival times of customers)
- ▶ Interarrival times  $A_i$  between customer arrivals  $t_{i-1}$  and  $t_i$  may follow different distributions  $A_i$



- ▶ Arrival process described by interarrival times or number of arrivals in intervals

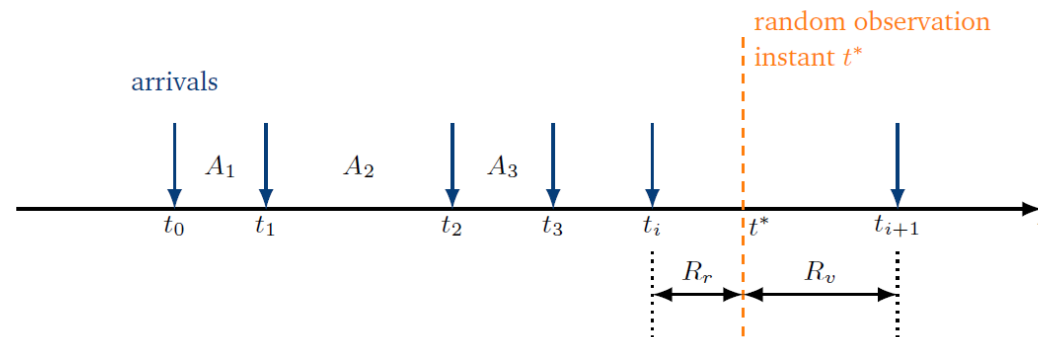
# Renewal Process

- ▶ Renewal process is a special point process
- ▶ **Renewal process:** interarrival times are independently and identically distributed (iid)

$$A_i(t) = A(t) \text{ for all } i$$

- ▶ **Modified renewal process:** first interval deviates from other intervals

$$A_1(t) \text{ and } A_i(t) = A(t) \text{ for } i = 2, 3, \dots$$



# Recurrence Time

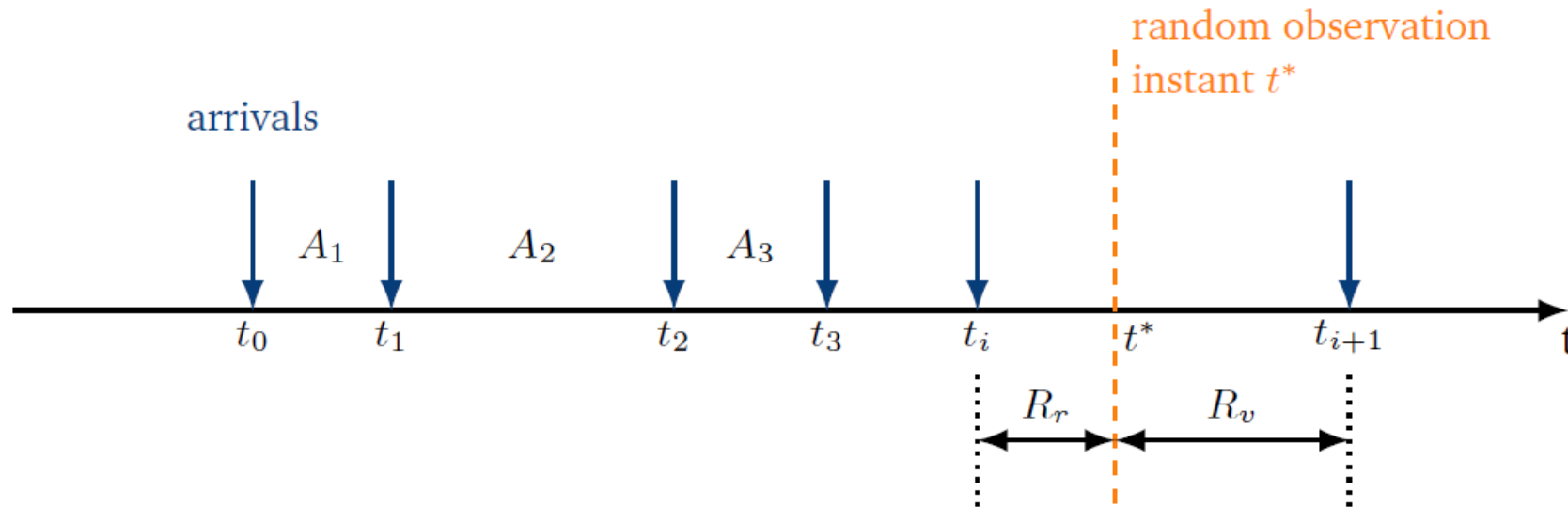
- ▶ Interarrival time  $A$  of a renewal process with mean interarrival time  $E[A]$
- ▶ Total number  $N(t)$  of arrival events in the interval  $[0; t]$  is counted for a renewal process
- ▶ Stochastic process  $\{N(t), t\}$  is referred to as **counting process**
- ▶ **Strong law for renewal processes:** With probability 1, the following relation between the arrival rate  $\frac{N(t)}{t}$  and the mean interarrival time is observed

$$\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{E[A]}$$

- ▶ We typically use  $\lambda = \lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{E[A]}$

# Recurrence Time

- ▶ Process is observed by an independent outside observer at time  $t^*$
- ▶  $R_v$  **forward recurrence time**; interval from the observation time to the next arrival;
- ▶  $R_r$  **backward recurrence time**; interval from the last arrival to the observation time.



- ▶ Random observation time can be at any time: forward and backward recurrence time have same statistical properties → **recurrence time** is analyzed for given iat  $A$



# ANALYSIS OF RECURRENCE TIME

# Cumulative Distribution Function of Recurrence Time

- **Recurrence time**  $R$  with  $\lambda = \frac{1}{E[A]}$  and interarrival time  $A$

$$r(t) = \frac{1}{E[A]} (1 - A(t)) = \lambda (1 - A(t))$$

Note:

$$\int_0^{\infty} 1 - A(t) dt = E[A]$$

○  
| Laplace transform (LT)  
●

$$\Phi_R(s) = \frac{\lambda}{s} (1 - \Phi_A(s))$$

- Note

- $a(t)$  or  $A(t)$  known  $\rightarrow r(t)$  can be derived
- $r(t)$  **and**  $E[A]$  known  $\rightarrow a(t)$  can be derived

# Recurrence Time: Proof

Lecture

# Recurrence Time: Transform Domain

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# Moments of Recurrence Time

- Ordinary moments of recurrence time

$$E[R^k] = \frac{E[A^{k+1}]}{(k+1) \cdot E[A]}.$$

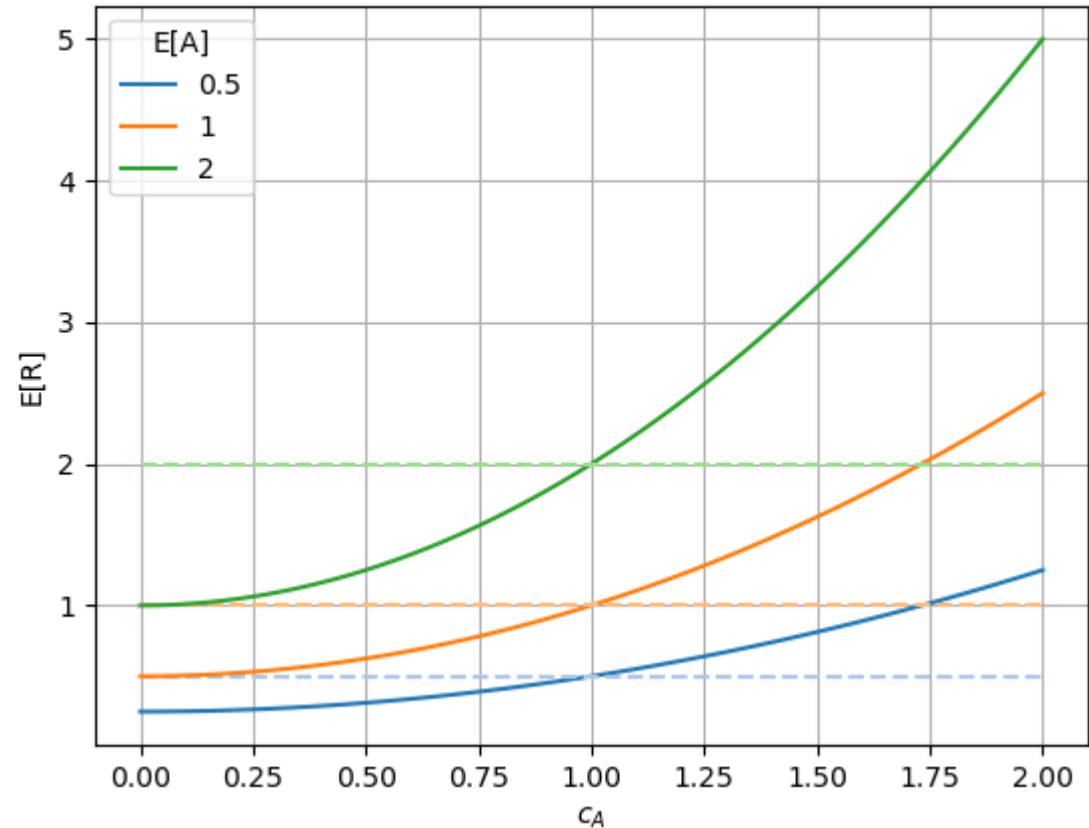
- Mean recurrence time

$$E[R] = \frac{E[A^2]}{2 E[A]} = \frac{c_A^2 + 1}{2} \cdot E[A]$$

$$c_A < 1: \quad E[R] < E[A]$$

$$c_A = 1: \quad E[R] = E[A]$$

$$c_A > 1: \quad E[R] > E[A]$$



- Counterintuitive:  $E[R] > E[A]$
- Longer intervals are more frequently encountered by observer
- Larger intervals contribute more often to recurrence time

# Moments of Recurrence Time: Proof

Lecture