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Institute of Computer Science
Chair of Communication Networks
Prof. Dr. Tobias Hoßfeld

Chapter 4.1

Loss System M/M/n

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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The book must be cited and the disclaimer attached when using lectures slides or scripts.

Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/

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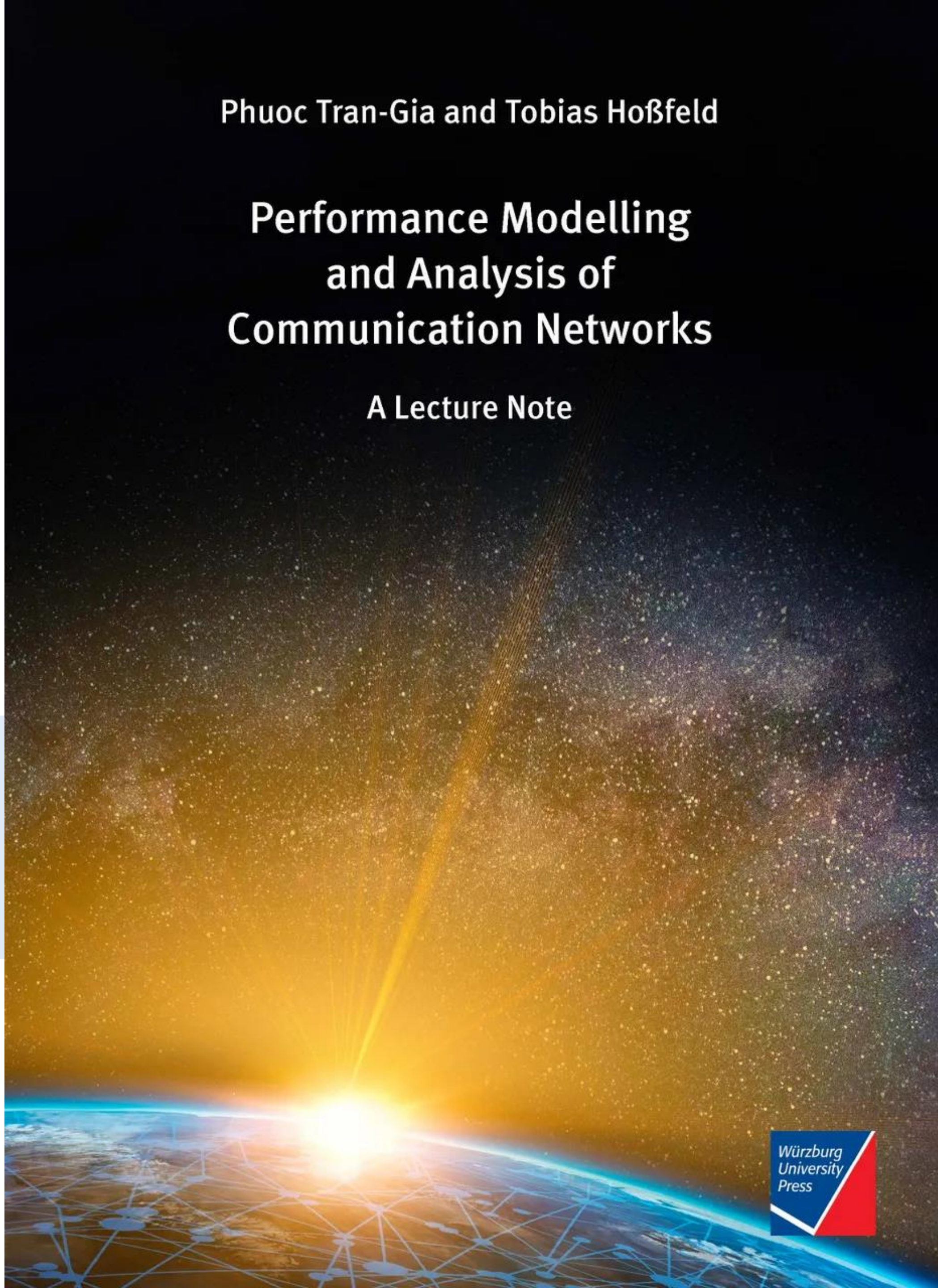
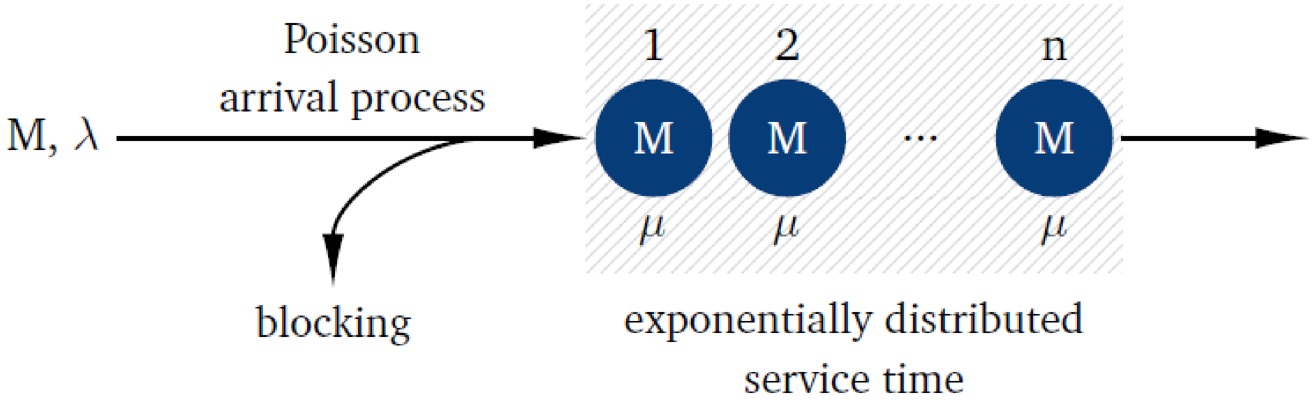


Chapter 4

4 Analysis of Markovian Systems

Model Structure and Parameters

Loss System M/M/n



- Interarrival time A with arrival rate λ
- Service time B with service rate μ
- Offered traffic $a = \frac{\lambda}{\mu}$ in pseudo-unit Erlang [Erl]

$$A(t) = P(A \leq t) = 1 - e^{-\lambda t}, \quad E[A] = \frac{1}{\lambda},$$

$$B(t) = P(B \leq t) = 1 - e^{-\mu t}, \quad E[B] = \frac{1}{\mu}.$$

- Pure loss or blocking operation mode
  - arriving customer finding all servers occupied upon arrival will be blocked
  - blocked customers leave system: no further impact on the system state process

State Process and State Probabilities

Erlang formula for loss systems

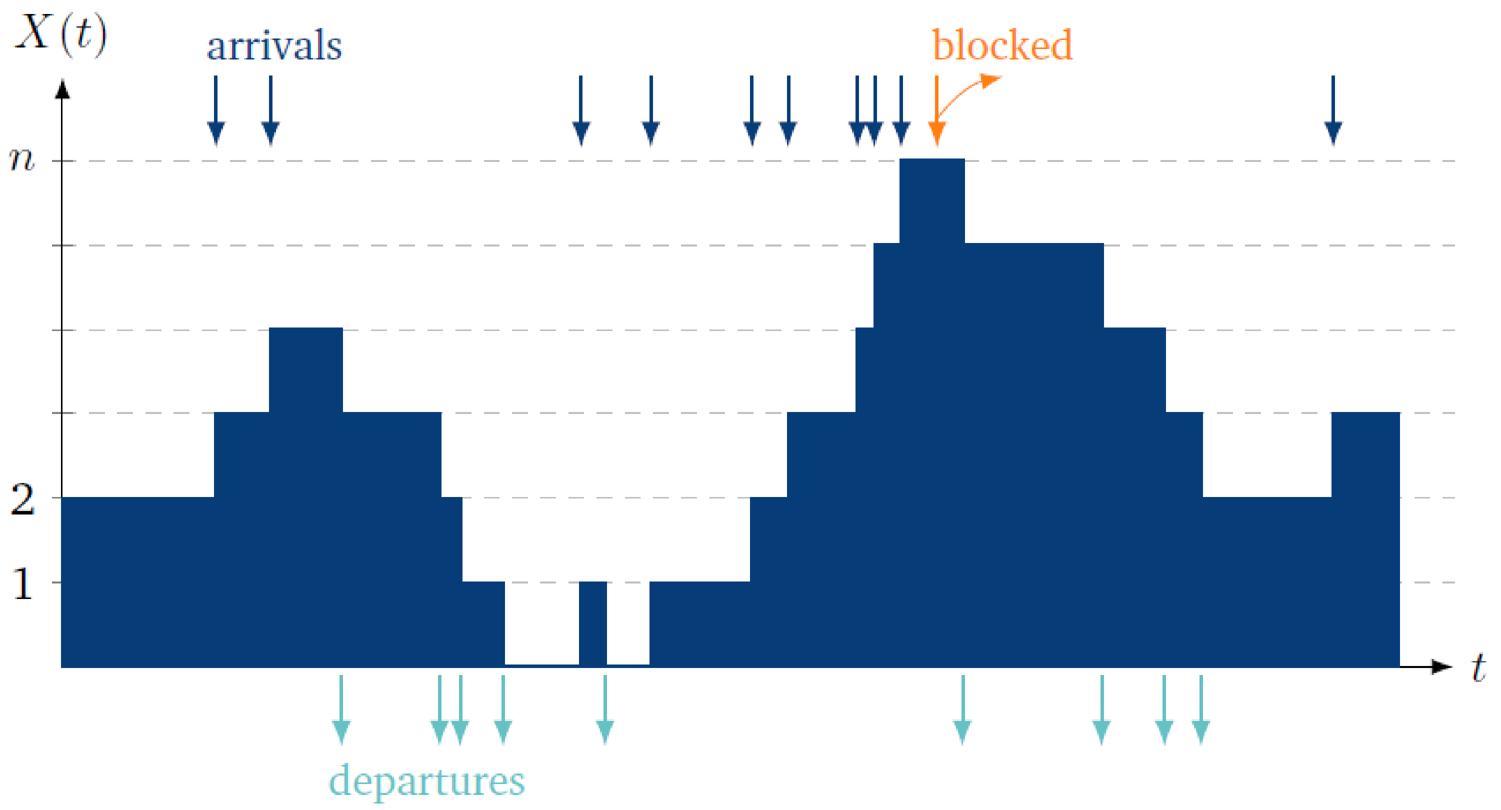
State Process of M/M/n Loss System

- $X(t)$ number of customers or busy servers in system at time t

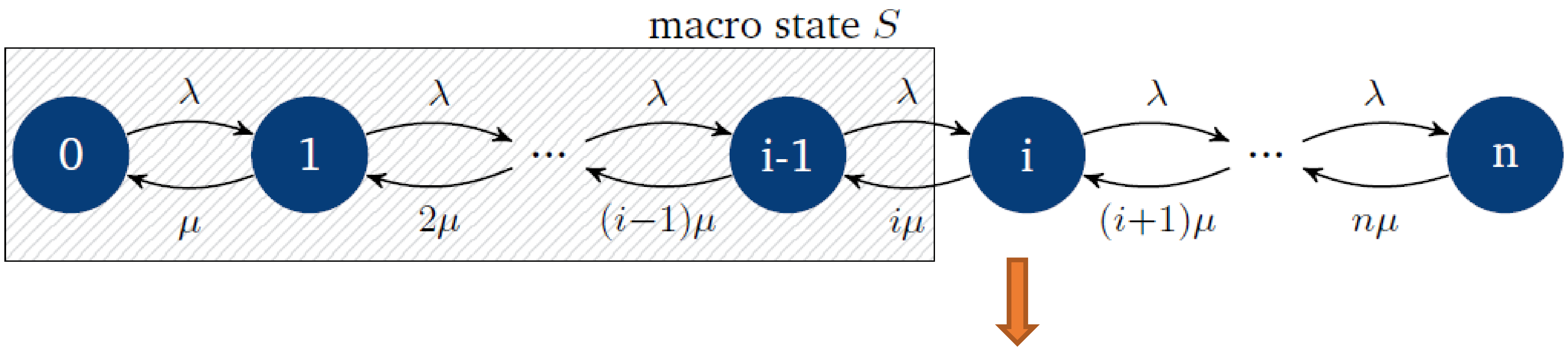
- Stationary system
  $\lim_{t\to\infty} X(t) = X$

- Steady-state probabilities (in statistical equilibrium)

$x(i) = P(X=i) \quad , i=0,1,\ldots$



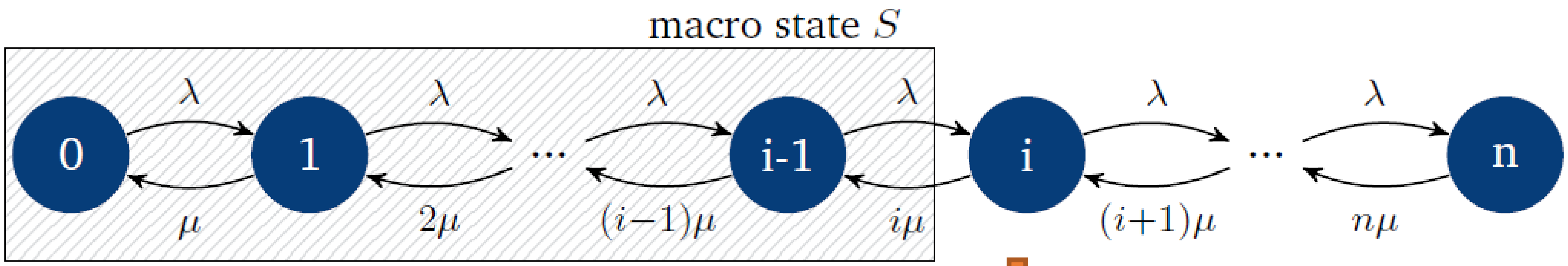
State Transition Diagram



(micro) state $[X = i]$
customers in the system

- Customer arrival: $[X = i] \to [X = i+1]$ with rate λ
  - if arriving customer is accepted $(i = 0,1,\dots,n-1)$
- Customer departure or service termination: $[X = i] \to [X = i-1]$ with rate $i\mu$
  - service time of one of the i customers ends $(i = 1,2,\dots,n)$

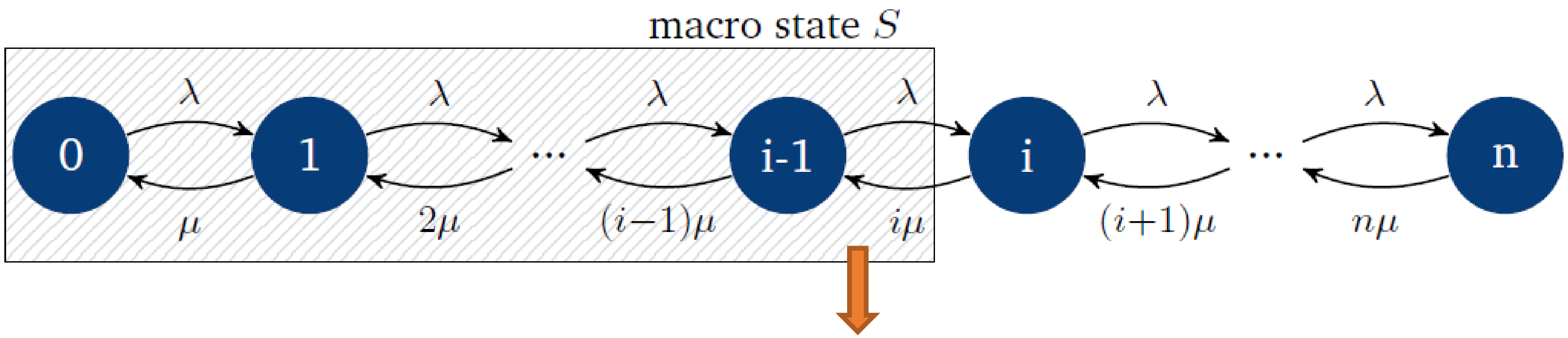
Micro State Equations



- Rate for **leaving** state [X = i] $x(i) \cdot \lambda + x(i) \cdot i \cdot \mu$
- Rate for **reaching** state [X = i] $x(i-1) \cdot \lambda + x(i+1) \cdot (i+1)\,\mu$
- **Micro state equation** for [X = i] $x(i)\,\lambda + x(i)\,i\mu = x(i-1)\,\lambda + x(i+1)(i+1)\mu$

- Special cases for [X = 0] and [X = n] $x(0)\lambda = x(1)\mu$ and $x(n-1)\lambda = x(n)\mu$
- Normalization condition $\sum_{i=1}^{n} x(i) = 1$

Macro State Equations

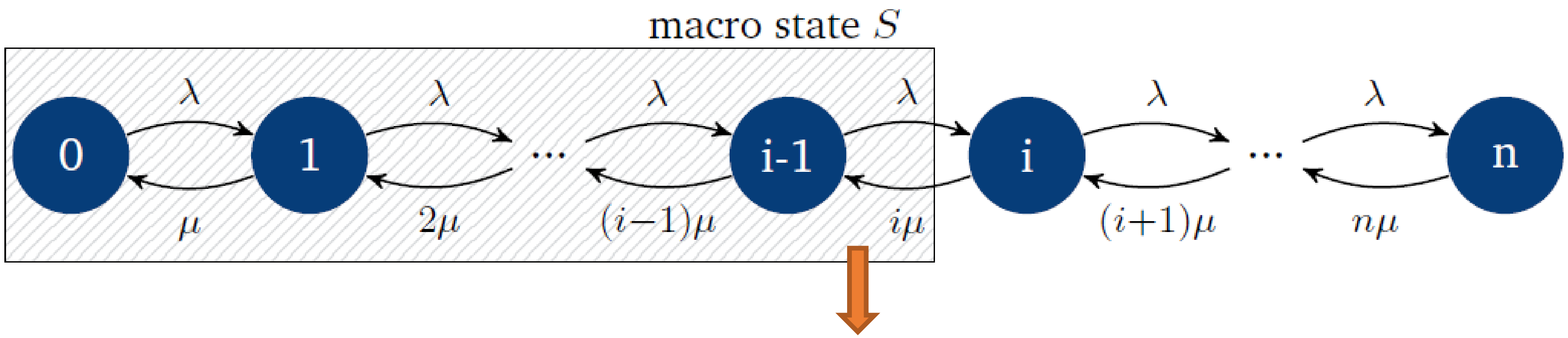


- Macro state $S = \{0,1,\dots,i-1\}$ for $i = 1,2,\dots,n$

- **Macro state equations**

$$\begin{cases} \lambda\, x(i-1) = i\,\mu\, x(i), & i = 1, 2, .., n \\ \sum_{i=0}^{n} x(i) = 1 & \end{cases}$$

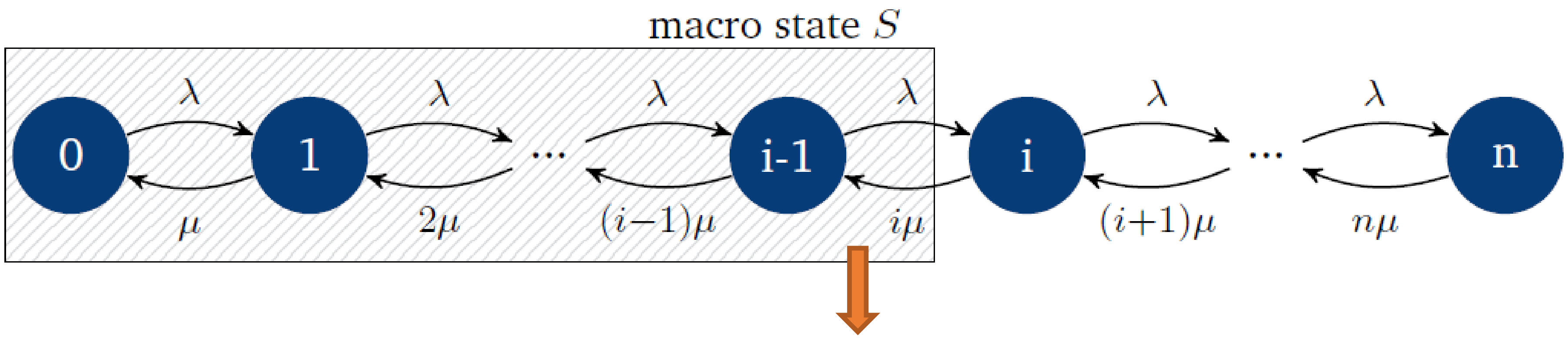
Macro State Equations: Solution



Macro state equations

$$\begin{cases} \lambda\, x(i-1) = i\, \mu\, x(i)\,, \qquad i = 1, 2, .., n \\ \sum_{i=0}^{n} x(i) = 1 \end{cases}$$

Macro State Equations and Erlang Formula for Loss Systems



- Macro state $S = \{0,1,\dots,i-1\}$ for $i = 1,2,\dots,n$

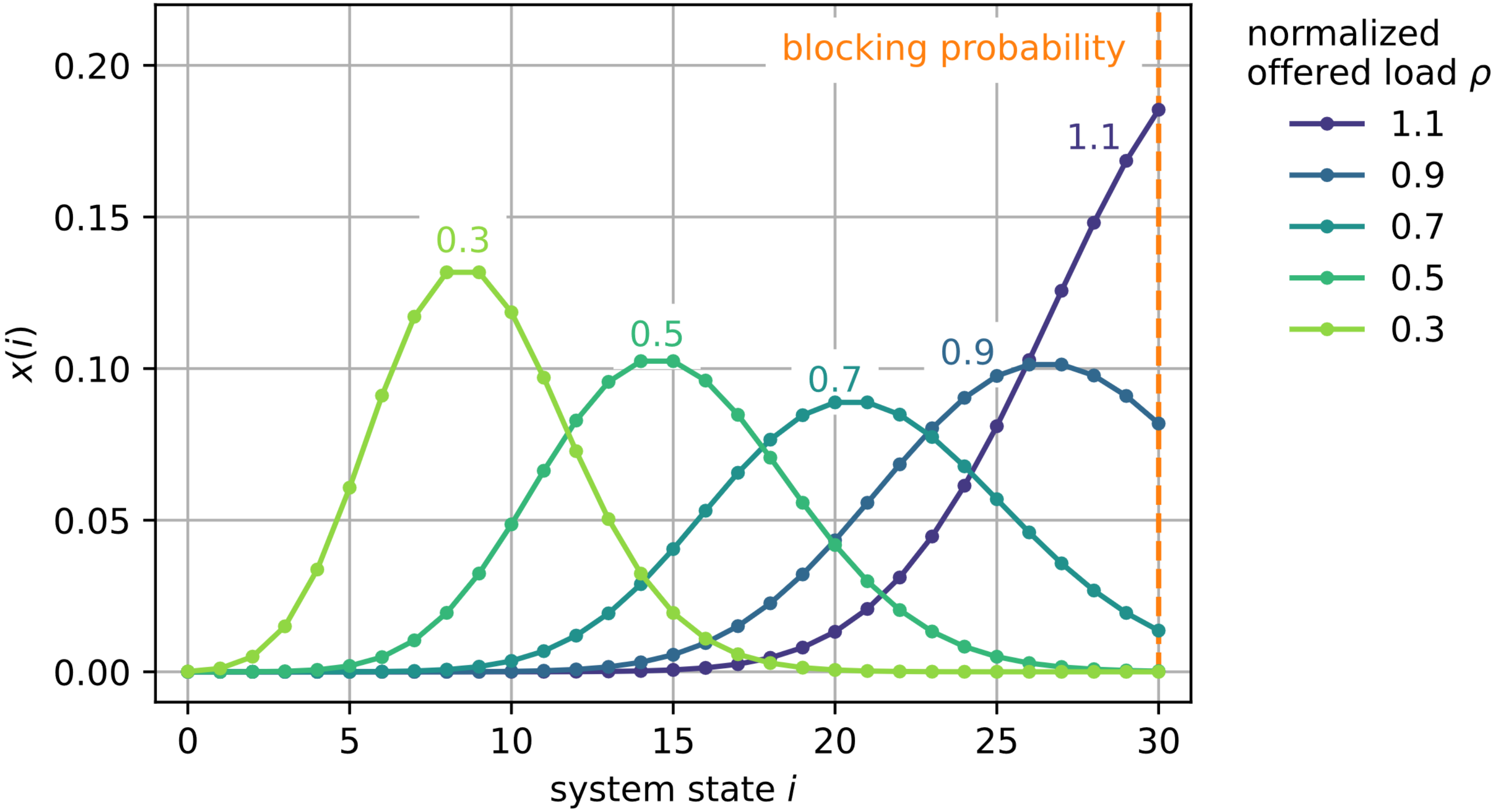
- Macro state equations
$$\begin{cases} \lambda\, x(i-1) = i\,\mu\, x(i)\,, & i = 1,2,\dots,n \\ \sum_{i=0}^{n} x(i) = 1 & \end{cases}$$

- **Erlang formula for loss systems**
  with offered traffic $a = \frac{\lambda}{\mu}$

$$x(i) = \frac{\frac{a^i}{i!}}{\sum_{k=0}^{n} \frac{a^k}{k!}}$$

Steady State Distribution

- M/M/30-0 with normalized offered load $\rho = \frac{a}{n} = \frac{\lambda}{n\mu}$



PASTA property

$x_A(i) = x(i)$

Blocking probability

$p_B = x_A(n) = x(n)$

State Probability at Arbitrary and Arrival Times

- State probability at **arrival time** $x_A(i)$
  - arriving customers sees i other customers in the system

- State probability at **arbitrary time** $x^*(i)$
  - stationary system is in state i at arbitrary time
  - time-averaged steady state probability
  - analysis of Markov state processes leads to $x(i) = x^*(i)$

- Due to memoryless property of the Poisson arrival process
  - $x_A(i) \; = \; x(i)\,,\; i = 0, 1, \dots, n$
  - **PASTA property**: Poisson arrivals see time averages

Other System Characteristics

Blocking probability (Erlang-B formula), carried traffic, utilization

System Characteristics

- Erlang-B formula
  - blocking probability
  - PASTA property utilized $x_A(i) = x(i)$

$$p_B = x_A(n) = x(n) = \frac{\frac{a^n}{n!}}{\sum_{k=0}^{n} \frac{a^k}{k!}}$$

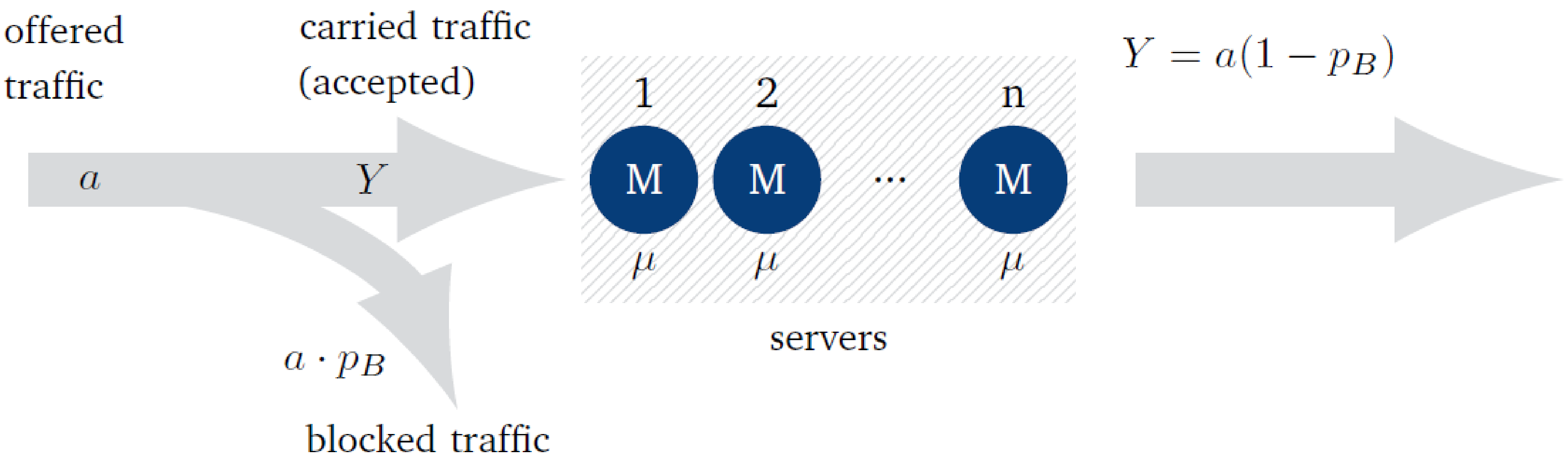
- Carried traffic Y
  - mean number of occupied servers $Y = E[X]$
  - derived using Little's law and traffic flows

$$Y = \sum_{i=0}^{n} i\, x(i) = \lambda\,(1 - p_B)\,\frac{1}{\mu} = a\,(1 - p_B)$$

- Utilization of a single server
  - offered traffic $a = \frac{\lambda}{\mu}$
  - normalized offered traffic $\rho = \frac{a}{n}$

$$\frac{Y}{n} = \rho \cdot (1 - p_B)$$

Traffic Flows in M/M/n Loss System



- Little's law $E[X] = \lambda' \cdot E[T]$
  - arrival rate $\lambda(1 - p_B) = \lambda'$
  - mean sojourn time $E[B] = E[T]$
  - mean number of customers $Y = E[X] = a \cdot (1 - p_B)$

GENERALIZATION TO M/GI/N-0

Insensitivity or robustness property

Insensitivity or Robustness Property

- Erlang formula provides steady state probabilities for M/M/n loss system

- Generalization to M/GI/n-0
  - Erlang formula is also valid for loss systems M/GI/n
  - proof in literature, e.g. by Syski

- **Insensitivity or robustness property**
  - extends applicability of Erlang formula and Erlang-B formula considerably in practice

$$x(i) = \frac{\frac{a^i}{i!}}{\sum_{k=0}^{n} \frac{a^k}{k!}}$$

$$p_B = x_A(n) = x(n) = \frac{\frac{a^n}{n!}}{\sum_{k=0}^{n} \frac{a^k}{k!}}$$

Example: M/GI/1-0

Lecture

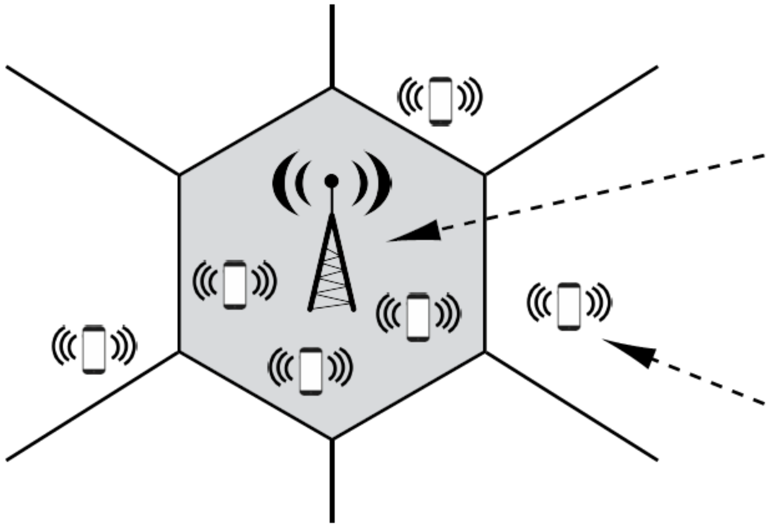
MODELING EXAMPLES AND APPLICATIONS

Dimensioning, economy of scale

Dimensioning of Systems

- Trunk size in telephone networks
  - best known and earliest application of M/GI/n loss system
  - dimension number of trunk groups

- GSM cell with n communication channels
  - dimension number of channels

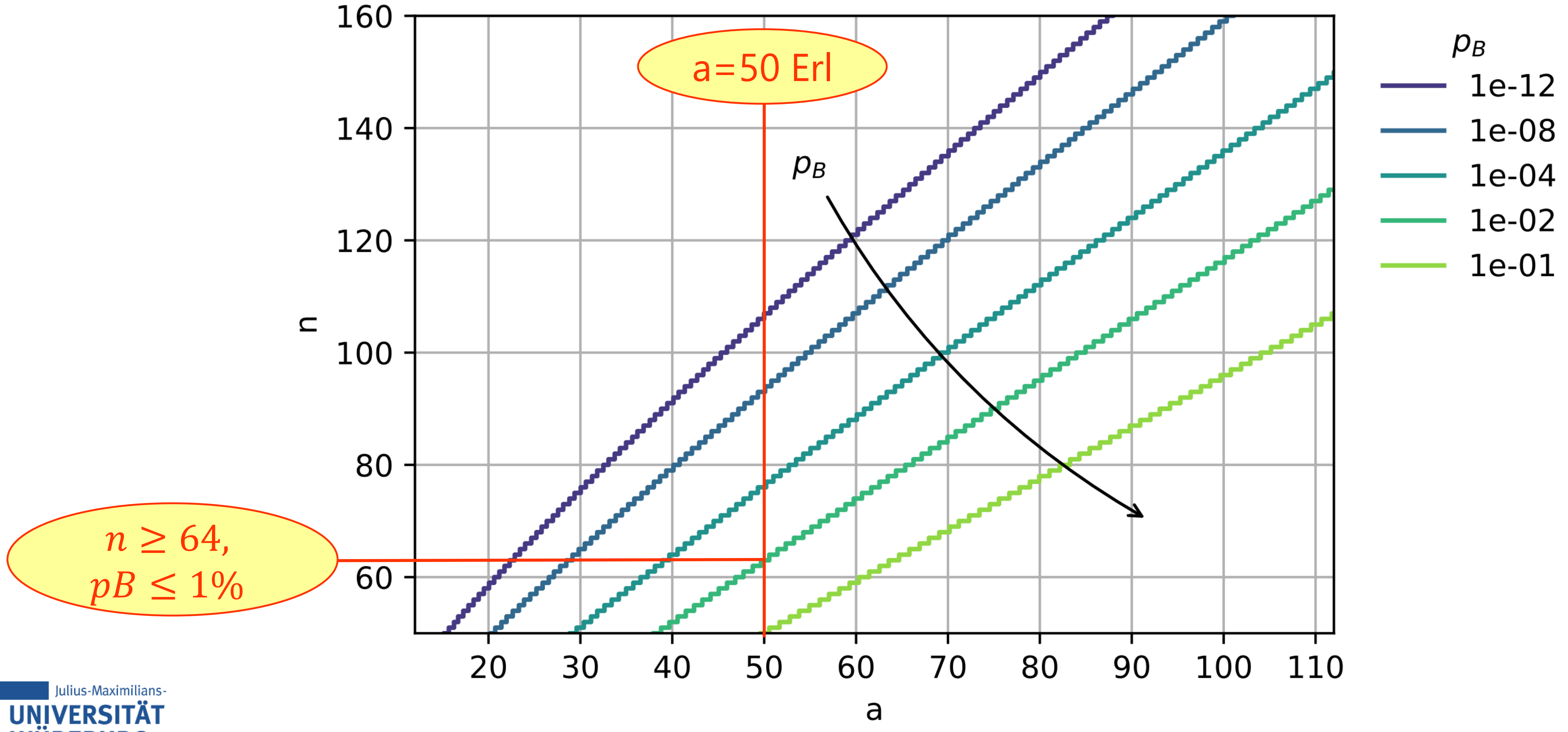


- DHCP server in ISP environment
  - dimension number of IP addresses

- **Dimensioning** means
  - finding number n of servers such that the
  - blocking probability is below a predefined value according to Service Level Agreements (SLAs) $p_B < \epsilon$

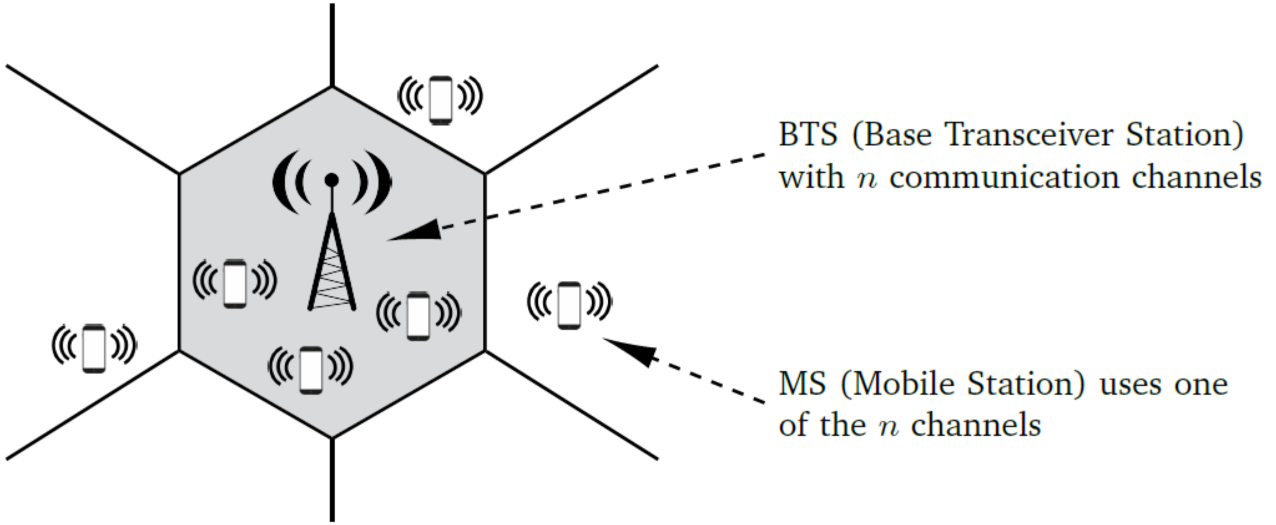
Numerical Solution: Dimensioning

- Efficient implementation of Erlang-B formula: see script "4.1 M/M/n-0 loss system: Erlang-B formula" https://modeling.systems/



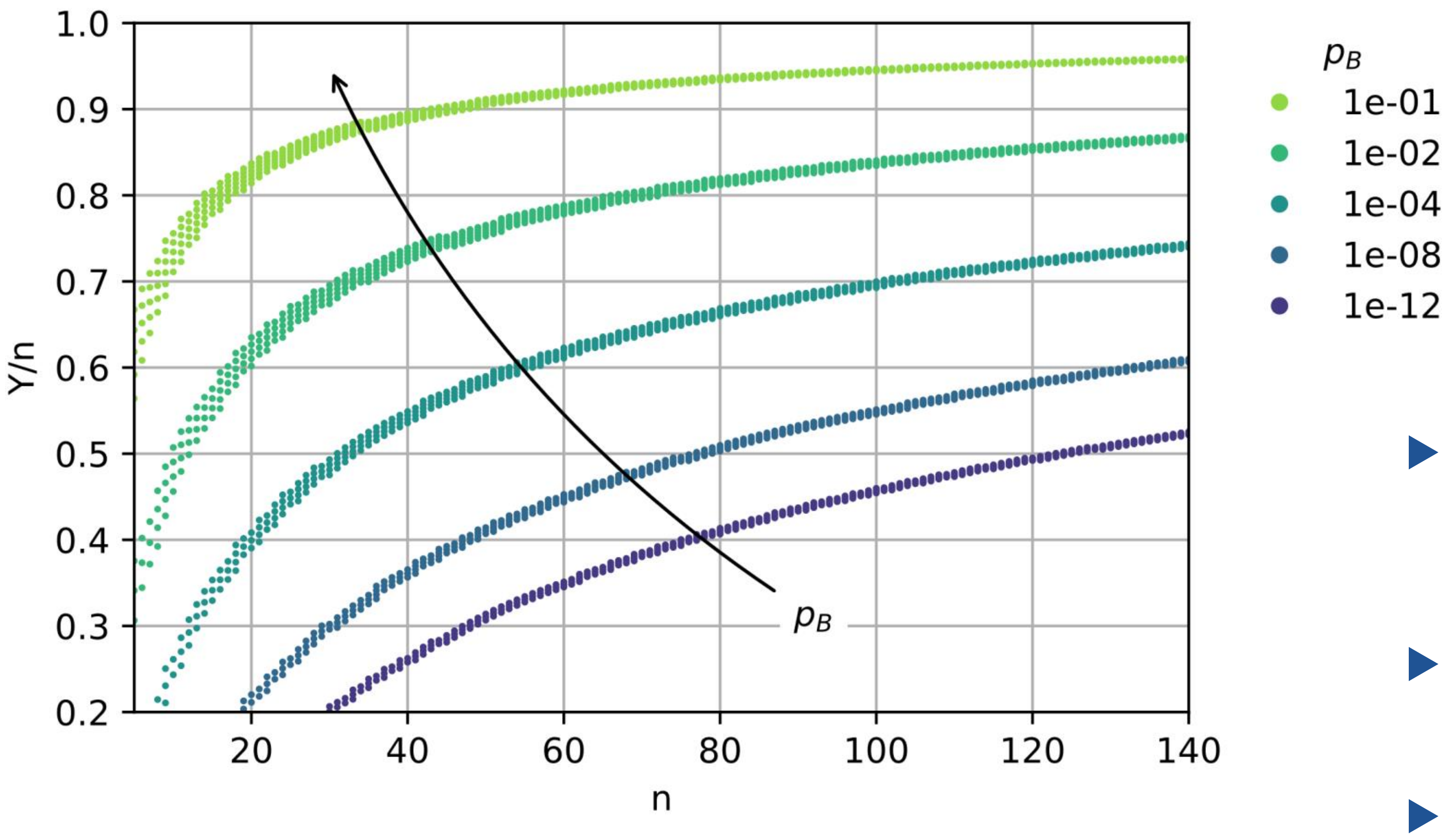
Example: GSM Cell

- n number of channels
- B call duration
  with $E[B] = 90$ s
- λ arrival rate of Poisson process
  with $\lambda = 1$ call/s
- a offered load
  $a = 90$ Erl



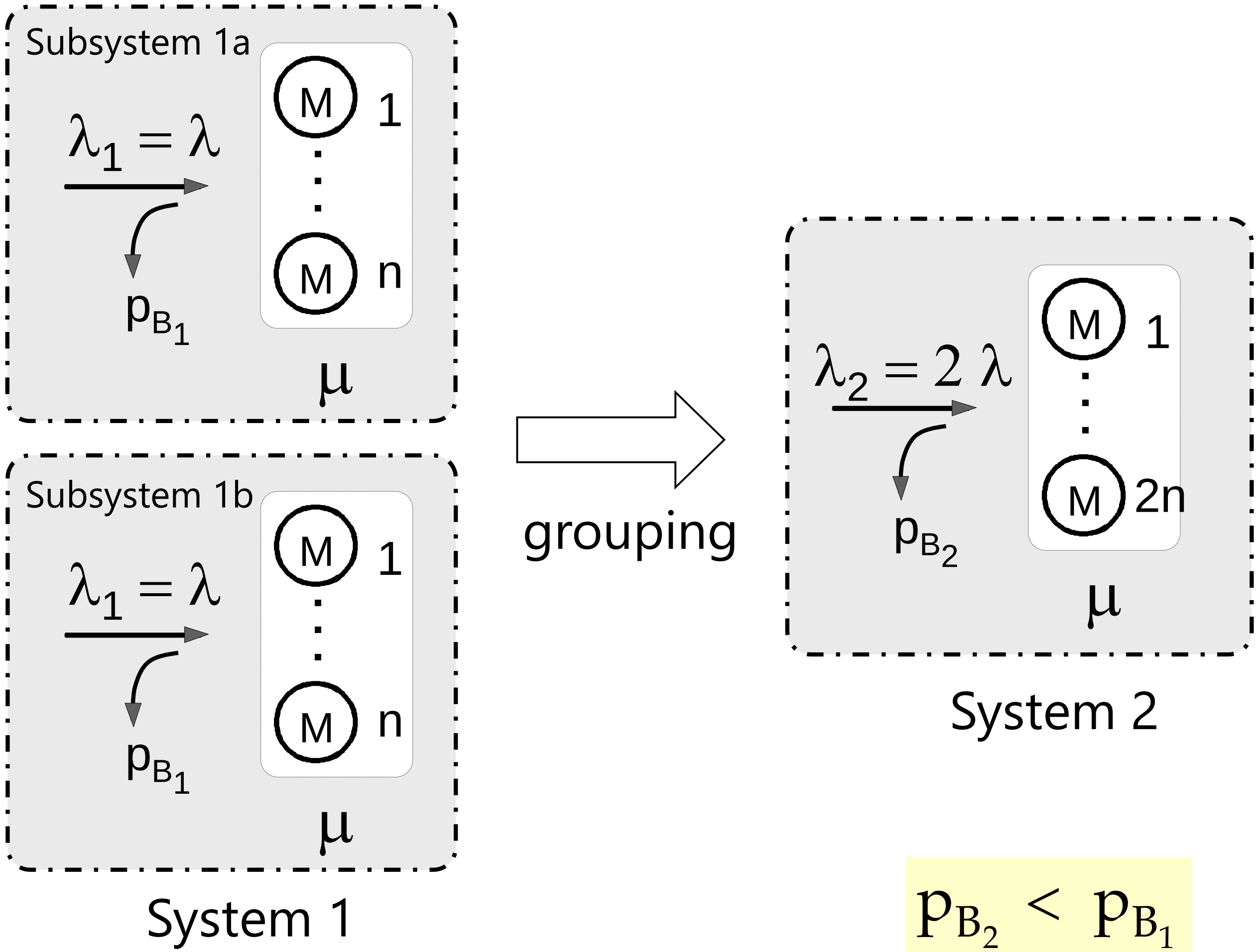
- Goal: Quality of Service (QoS) threshold $p_B \leq 10^{-3}$
- Result: $n = 117$ channels required

Economy of Scale



- The server utilization $\frac{Y}{n}$ increases with number of servers, when we keep required QoS (p_B) constant
- Larger trunk sizes are more economical
- Increase of factor $\frac{Y}{n}$ corresponds to the **economy of scale**.
- Increase is limited (flatter shape)

Economy of Scale in Loss Systems



$$p_{B_2} < p_{B_1}$$

Example: Several Cells

Lecture