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Chapter 6

Discrete-Time Analysis of GI/GI/1

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022

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The book must be cited and the disclaimer attached when using lectures slides or scripts.

Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication Networks - A Lecture Note. Würzburg University Press.
https://doi.org/10.25972/WUP-978-3-95826-153-2

Website to download book, exercises, slides and scripts:
https://modeling.systems/

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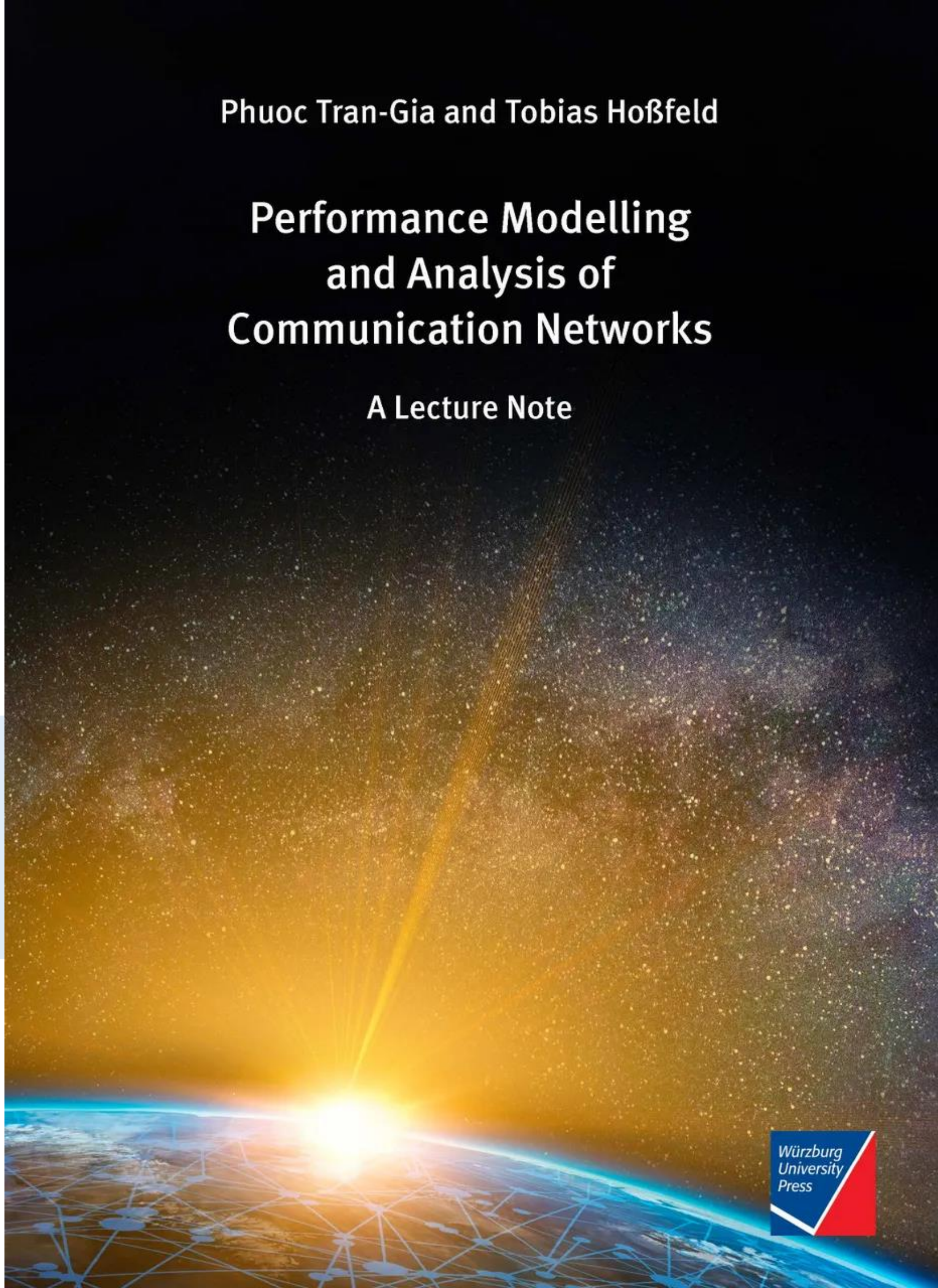
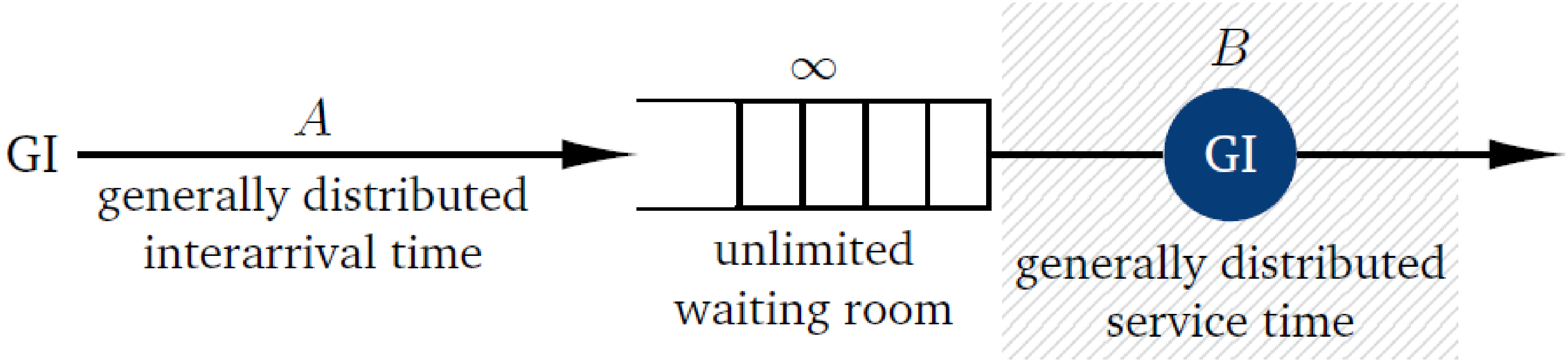


Discrete-Time Delay System GI/GI/1



- Interarrival time A is generally distributed
- Service time B is generally distributed

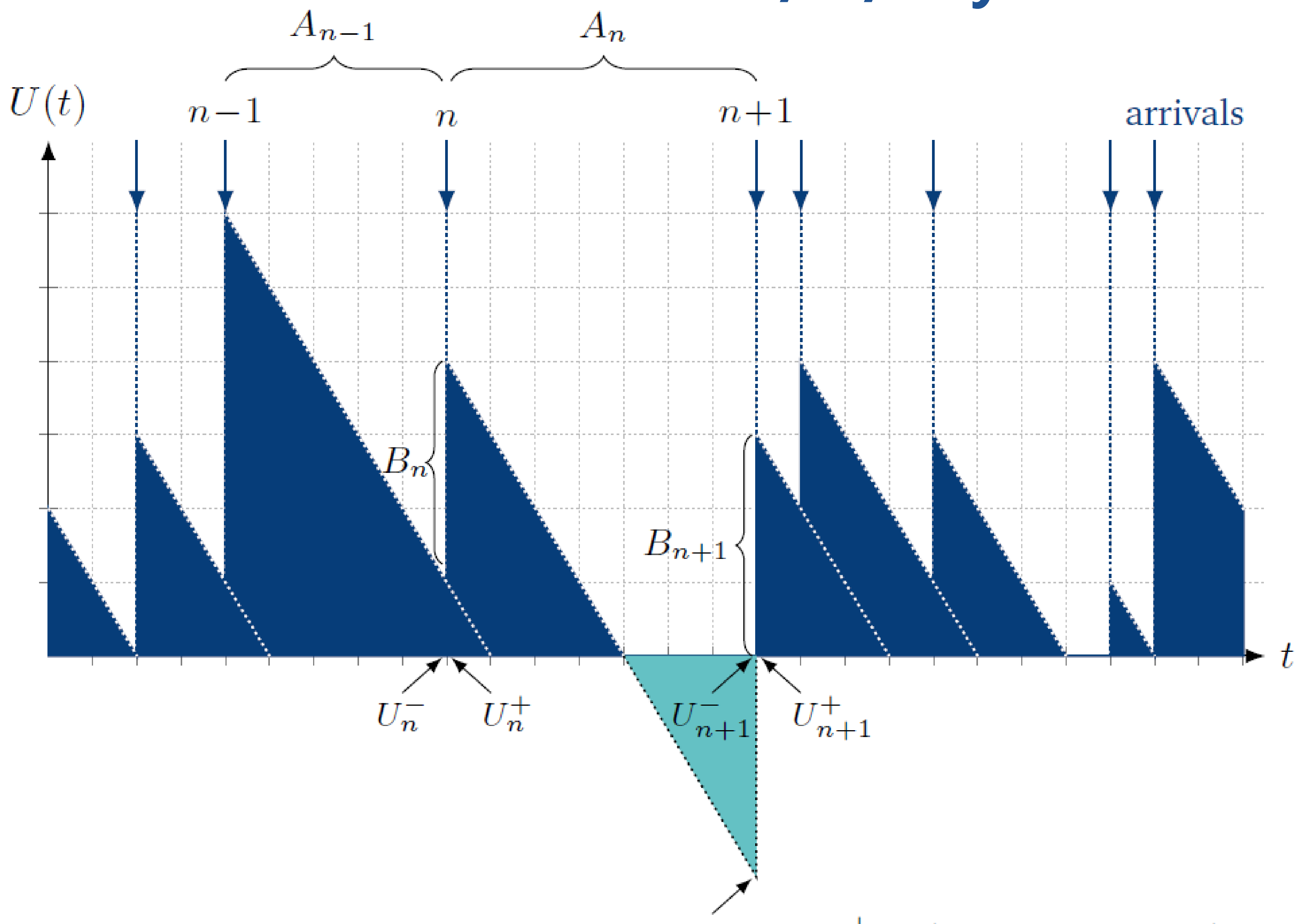
$$a(k) = P(A = k \cdot \Delta t) \quad k = 0, 1, \ldots \;,$$
$$b(k) = P(B = k \cdot \Delta t) \quad k = 0, 1, \ldots \;.$$

- Offered traffic a identical to server utilization ρ : $\rho = \dfrac{\mathrm{E}[\,B\,]}{\mathrm{E}[\,A\,]}$

- Pure delay system: number of waiting places is assumed to be unlimited
- FIFO queue: first-in first-out queuing discipline

- **Stability condition** $\rho < 1$

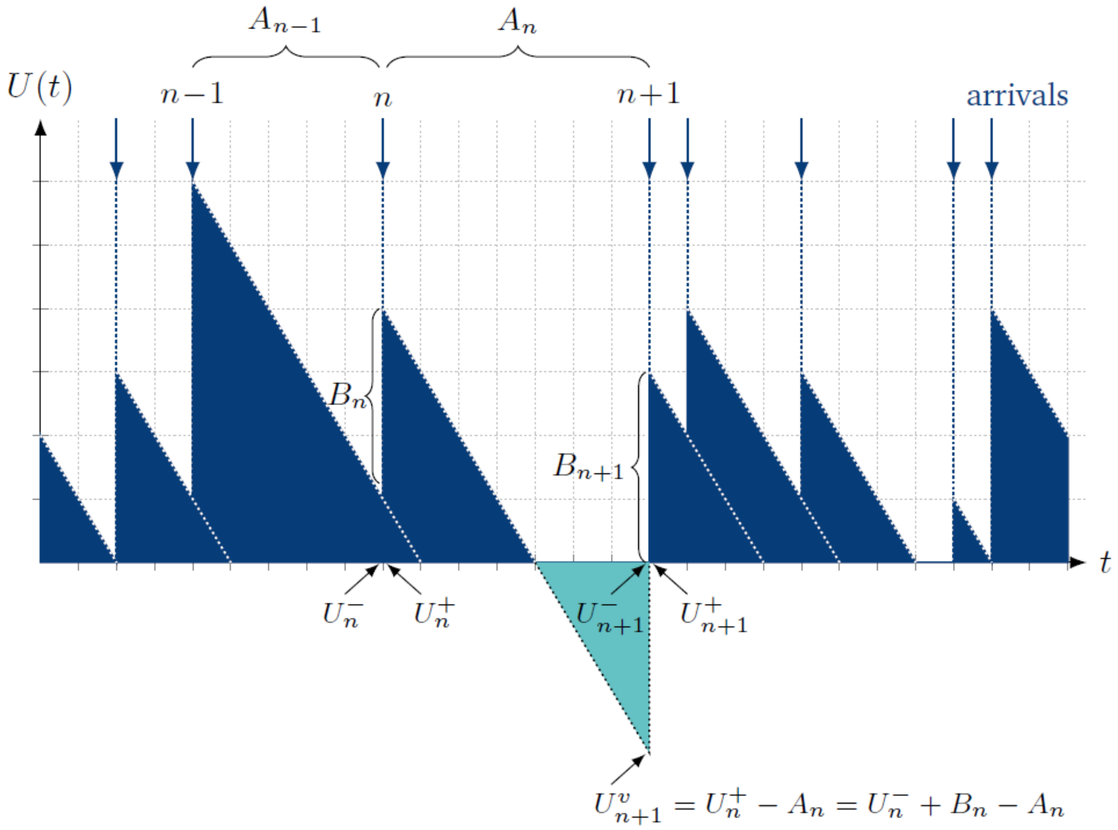
Sample Path of Unfinished Work in GI/GI/1 System



$$U_{n+1}^v = U_n^+ - A_n = U_n^- + B_n - A_n$$

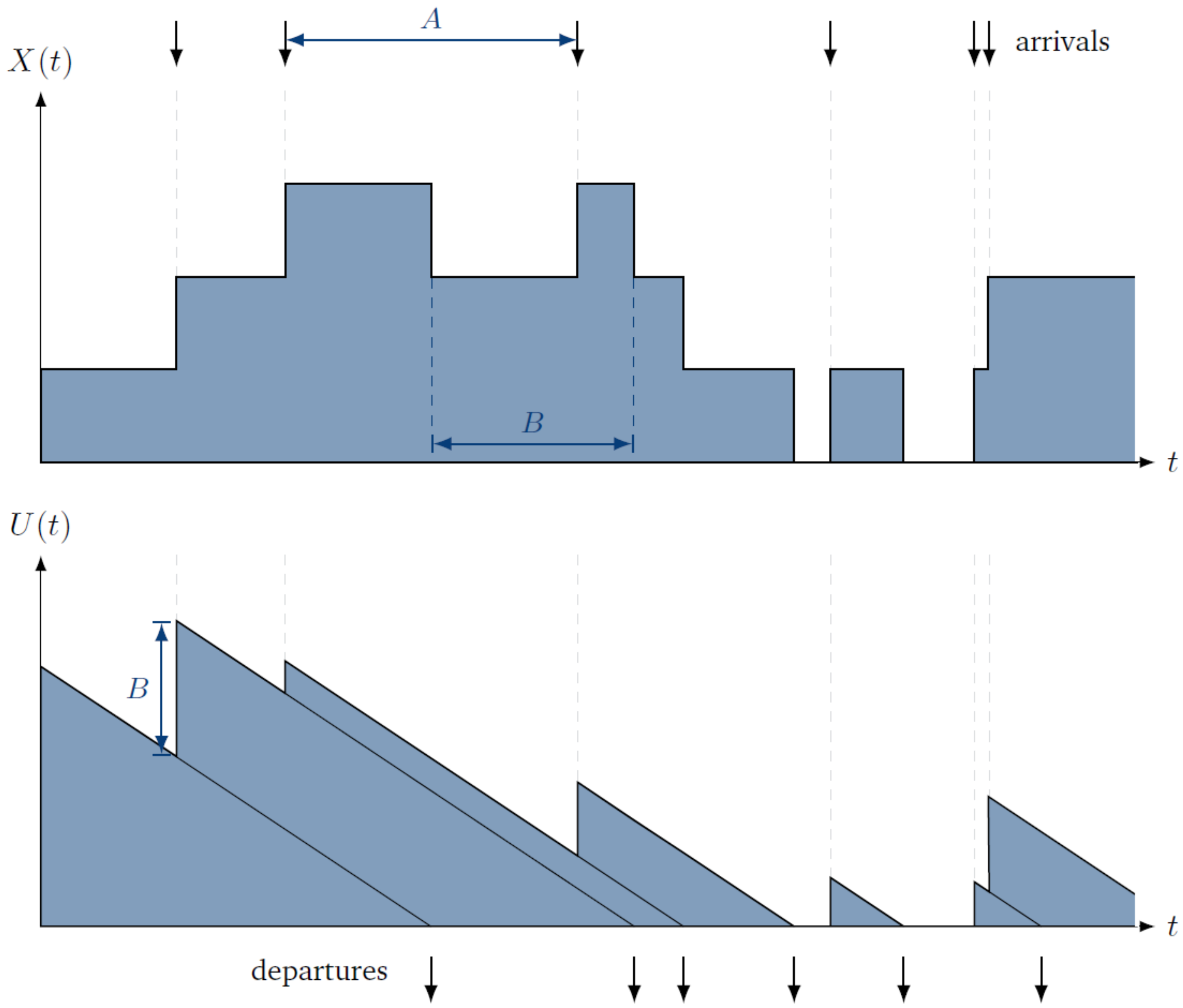
Notation of (Discrete) Random Variables

- A_n interarrival time between customer n and customer $n+1$
- B_n service time of customer n
- U_n^- **unfinished work** immediately **before** arrival of customer n
- U_n^+ **unfinished work** immediately **after** arrival of customer n
- U_{n+1}^v **virtual unfinished work** immediately before the arrival of customer $n+1$ (can assume negative values)



GI/GI/1 State Process

- Embedding point: immediately before arrival of customer n
- State process $\{X(t_n), n\}$
  - no Markov property since unfinished work is unknown at arrival instant
- Unfinished work $\{U(t_n), n\}$
  - Markov property at regeneration points
  - Markov process $\{U(t_n), n\}$
  - Note: not a Markov chain since $U(t)$ is continuous



Analysis in Time Domain

- Unfinished work U_{n+1}^- is derived from U_n^- using Markov property in following steps

$$U_n^- \rightarrow U_n^+ \rightarrow U_{n+1}^v \rightarrow U_{n+1}^-$$

- 1. $U_n^- \rightarrow U_n^+$

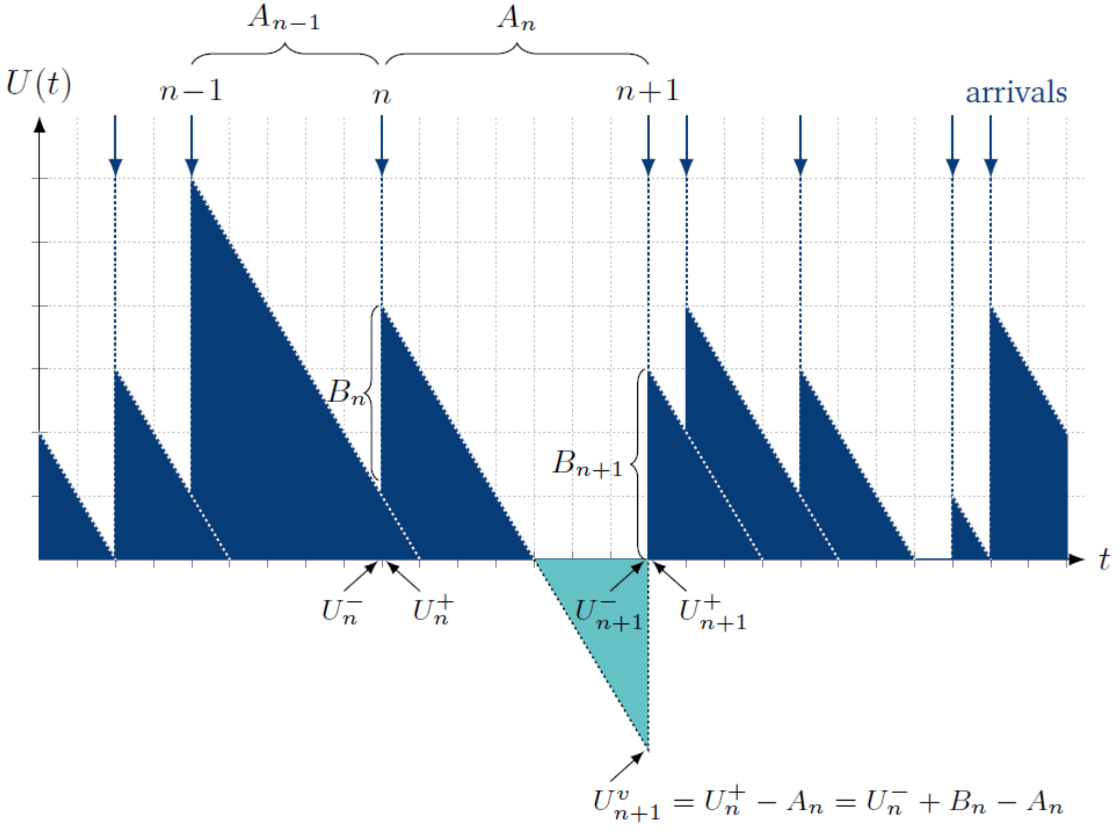
$$U_n^+ = U_n^- + B_n \ ,$$
$$u_n^+(k) = u_n^-(k) * b_n(k)$$

- 2. $U_n^+ \rightarrow U_{n+1}^v$

$$U_{n+1}^v = U_n^+ - A_n \ ,$$
$$u_{n+1}^v(k) = u_n^+(k) * a_n(-k)$$

- 3. $U_{n+1}^v \rightarrow U_{n+1}^-$

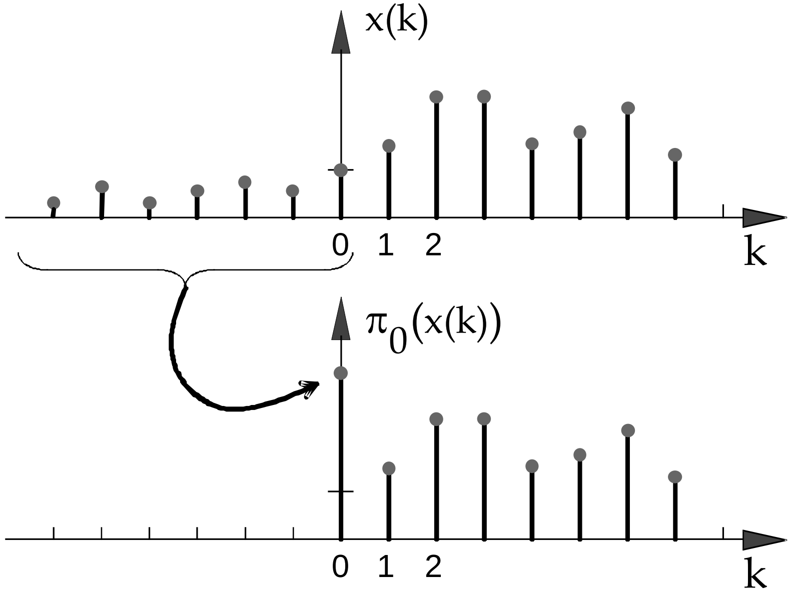
$$U_{n+1}^- = \max(0,\ U_{n+1}^v)$$
$$u_{n+1}^-(k) = \pi_0\Big(u_{n+1}^v(k)\Big) \ .$$



π Operator for Discrete Random Variables

$$\pi_m\big(x(k)\big) = \begin{cases} 0 & k < m \\ \sum_{i=-\infty}^{m} x(i) & k = m \\ x(k) & k > m \end{cases}$$

- For $m = 0$ and discrete r.v. X
  - $Y = \max(0, X)$
  - $y(k) = \pi_0\big(x(k)\big)$



Analysis in Time Domain using π Operator

1. $u_n^+(k) = u_n^-(k) * b_n(k)$ **2.** $u_{n+1}^v(k) = u_n^+(k) * a_n(-k)$ **3.** $u_{n+1}^-(k) = \pi_0\left(u_{n+1}^v(k)\right)$

- Putting together these equations $u_{n+1}^-(k) = \pi_0(u_n^-(k) * \underbrace{a_n(-k) * b_n(k)}_{c_n(k)})$
- With the discrete-time system function $C_n = B_n - A_n$

$$u_{n+1}^-(k) = \pi_0(u_n^-(k) * c_n(k))$$

- FIFO scheduling: waiting time of customer n is unfinished work before arrival $W_n = U_n^-$
- **Discrete Lindley recursion for non-stationary GI/GI/1 delay systems**

$$W_{n+1} = \max(0, W_{n+1} + B_n - A_n) = \max(0, W_{n+1} + C_n)$$
$$w_{n+1}(k) = \pi_0\left(w_n(k) * c_n(k)\right)$$

Note: C_n may be different for every customer, e.g. to analyse overload

Lindley Recursion for GI/GI/1 Delay Systems

- Discrete Lindley recursion for **non-stationary GI/GI/1** delay systems

$$W_{n+1} = \max(0, W_{n+1} + B_n - A_n) = \max(0, W_{n+1} + C_n)$$
$$w_{n+1}(k) = \pi_0\left(w_n(k) * c_n(k)\right)$$

- Discrete Lindley recursion for **stationary GI/GI/1** delay systems: $W = \lim_{n\to\infty} W_n$

$$W = \max(0, W + C)$$
$$w(k) = \pi_0\left(w(k) * c(k)\right)$$

- Waiting time as cumulative distribution function

$$W(k) = \sum_{i=0}^{k} w(i) = \sum_{i=-\infty}^{k} c(i) * w(i) = \sum_{i=-\infty}^{k} \sum_{j=-\infty}^{\infty} c(j)\cdot w(i-j) = \sum_{j=-\infty}^{\infty} c(j) \sum_{i=-\infty}^{k} w(i-j) = \sum_{j=-\infty}^{\infty} c(j)\cdot W(i-j), \quad k = 0,1,\ldots$$

- **Discrete-time Lindley equation**

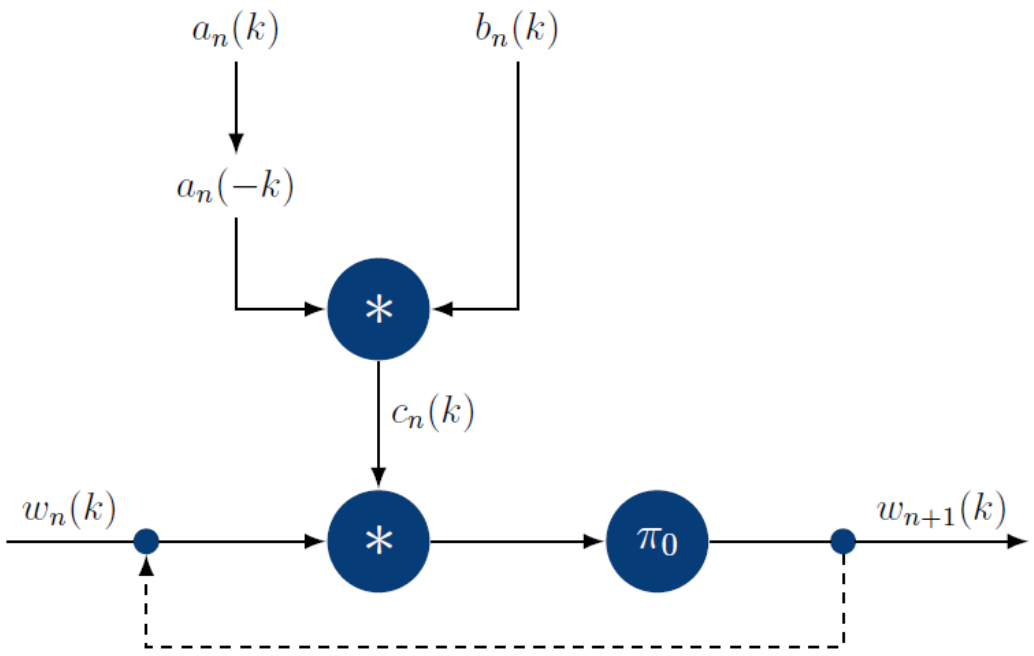
$$W(k) = \begin{cases} 0 & k < 0 \\ c(k) * W(k) & k \geq 0 \end{cases}$$

Computational Diagram for the Analysis of GI/GI/1

- General form of Lindely's integral equation in discrete-time domain

$$w_{n+1}(k) = \pi_0\big(w_n(k) * a_n(-k) * b_n(k)\big) = \pi_0\big(w_n(k) * c_n(k)\big)$$

- Algorithm for computation depicted as **computational diagram**



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Home Scripts Exercises Lecture Addendum Book

Scripts

An implementation of the models in the book is available as interactive notebooks. The scripts will help students to better understand the impact of parameters
on performance characteristics, will avoid common pitfalls in the implementation,
and provide means for numerical robust and efficient implementations for researchers in the domain.

Chapter 1: Introduction +

Chapter 2: Fundamentals +

Chapter 3: Stochastic Processes +

Chapter 4: Markovian Systems +

Chapter 5: Non-Markovian Systems +

Chapter 6: Discrete-Time Analysis —

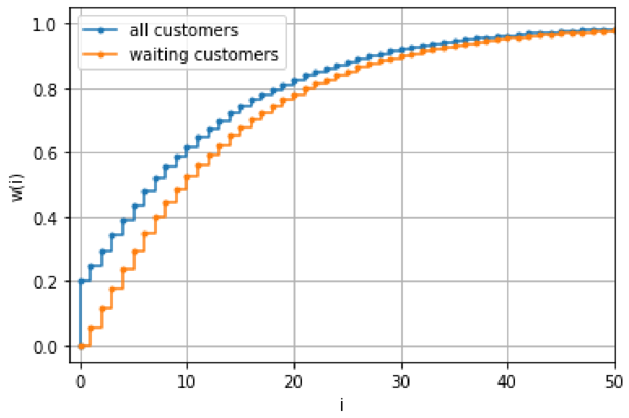
- Module DiscreteTimeAnalysis: documentation [download]
- 6.1 Recurrence time distribution [ipynb]
- 6.2 Z-Transform, DFT and convolution [ipynb]
- 6.3 GEOM(1)/GI/1 waiting time [ipynb]
- 6.4 GI/GI/1 waiting time [ipynb]
- 6.4.8 Idle time distribution [ipynb]
- 6.4 Discretized M/GI/1 queue [ipynb]

Chapter 7: Applications +

https://modeling.systems/

$W_{n+1} = \max(W_n + C, 0) \quad \text{with } C = B - A$

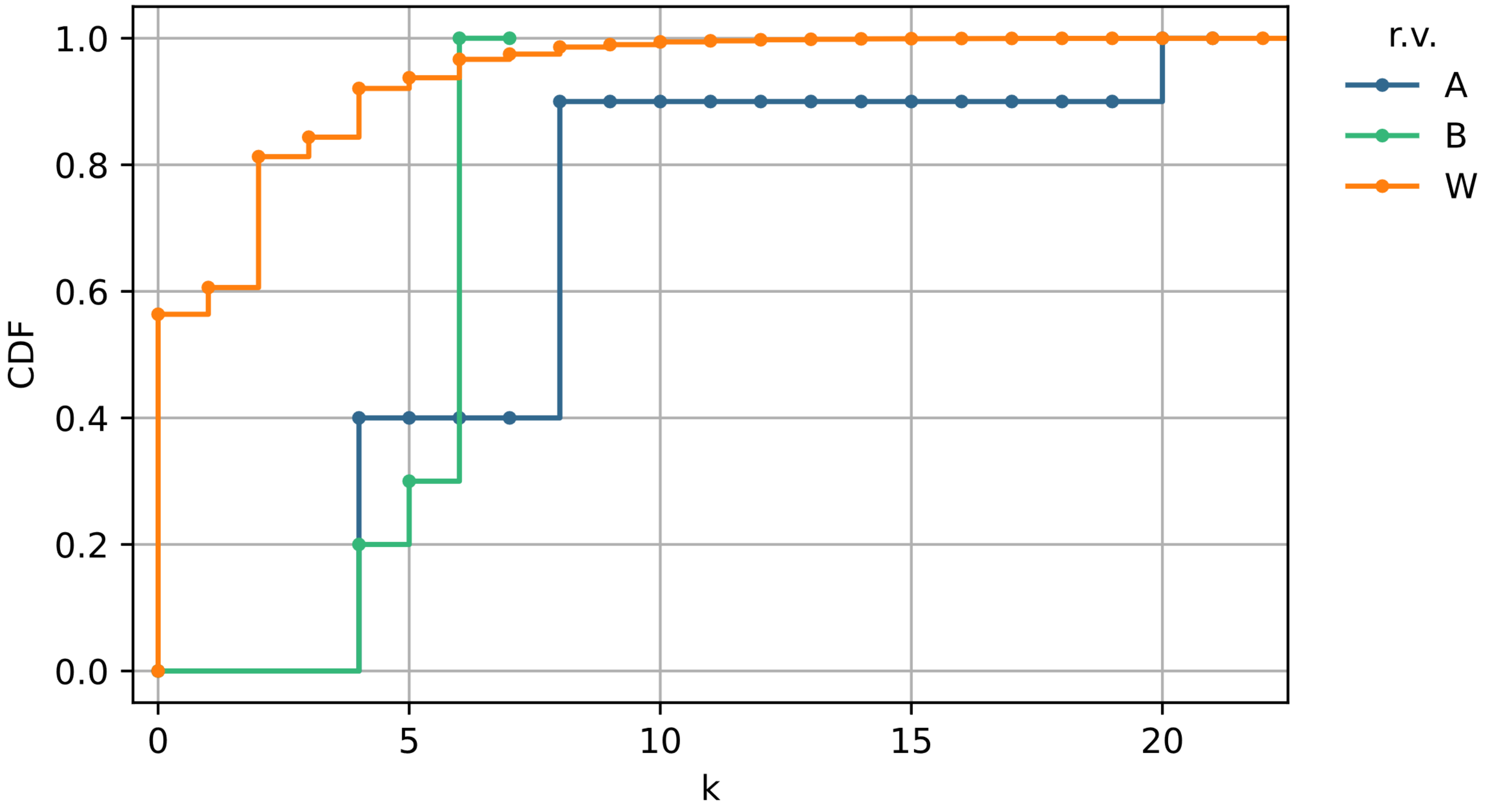
```
Wn1 = DET(0) # empty system
Wn = DET(1)  # just for initialization

# power method
while Wn != Wn1: # comparison based on means of the distributions
    Wn = Wn1
    Wn1 = max( Wn+C ,0)

Wn1.plotCDF(label='all customers')

condition = lambda i: i>0
W_waiting = Wn1 | condition # conditional r.v. of waiting customers
W_waiting.plotCDF(label='waiting customers')

plt.grid(which='major')
plt.xlim([-1, 50])
plt.xlabel('i')
plt.ylabel('w(i)')
plt.legend();
```



Numerical Example 1

$$a(4) = 0.4, \quad a(8) = 0.5, \quad a(20) = 0.1, \quad a(k) = 0 \text{ otherwise,}$$
$$b(4) = 0.2, \quad b(5) = 0.1, \quad b(6) = 0.7, \quad b(k) = 0 \text{ otherwise.}$$

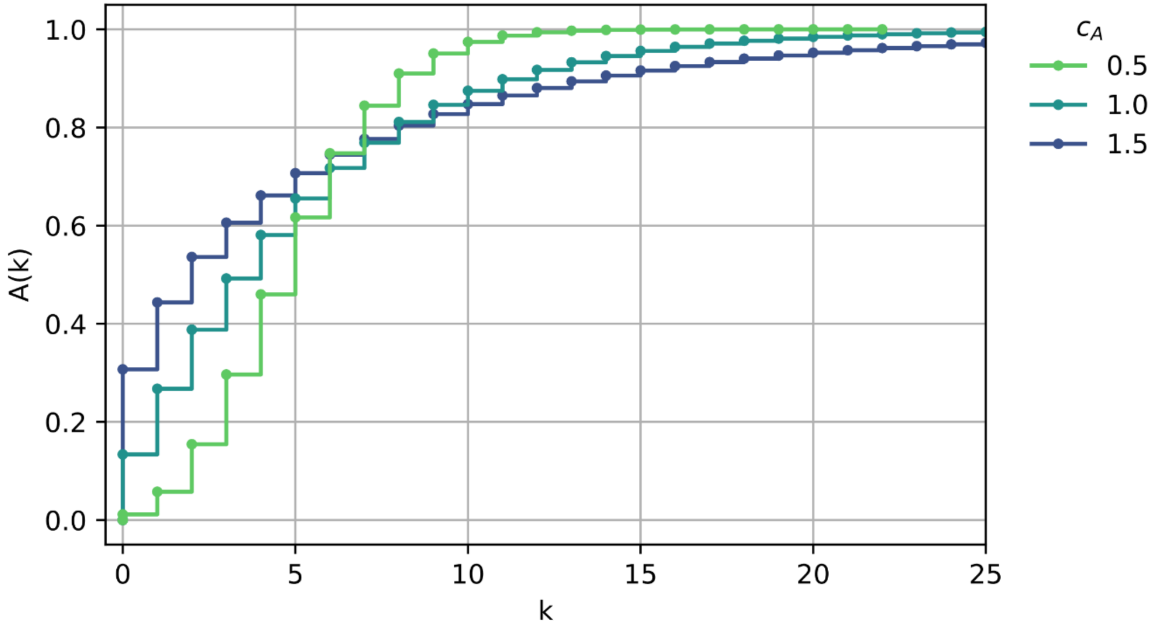
$$\mathrm{E}[\,A\,] = 7.6, \quad \mathrm{E}[\,B\,] = 5.5, \quad \rho = 0.724, \quad c_A = 0.598, \quad c_B = 0.147.$$

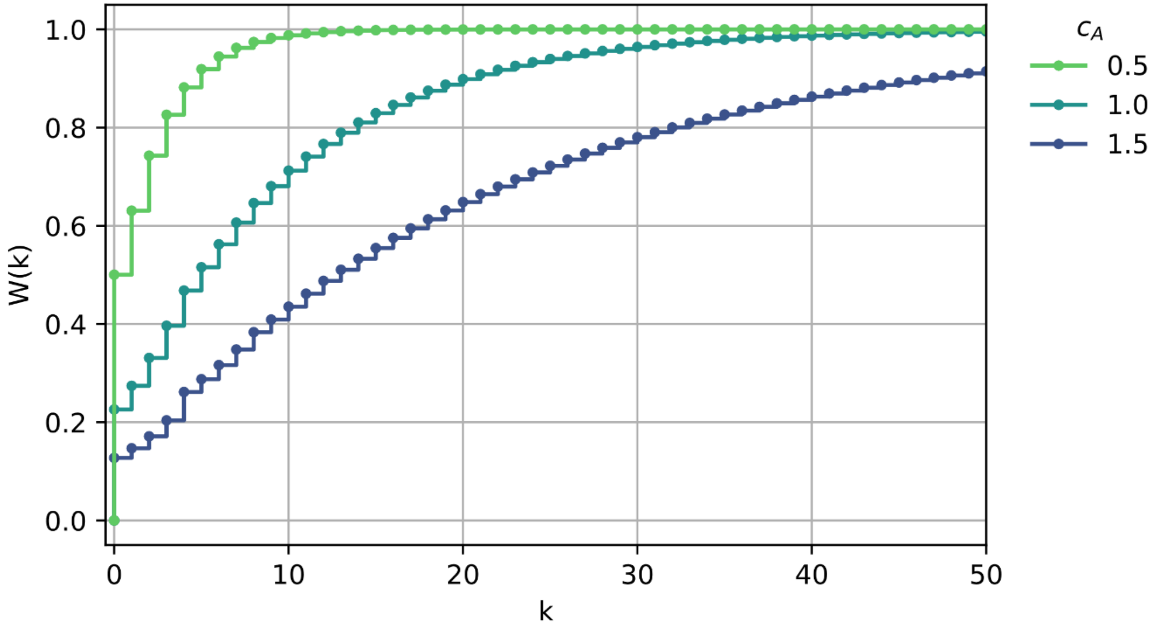


Numerical Example 2

- Discrete-time NEGBIN/D/1 delay queue
  - $E[A] = 5$
  - $E[B] = 4$

- System load
  - $\rho = 0.8$

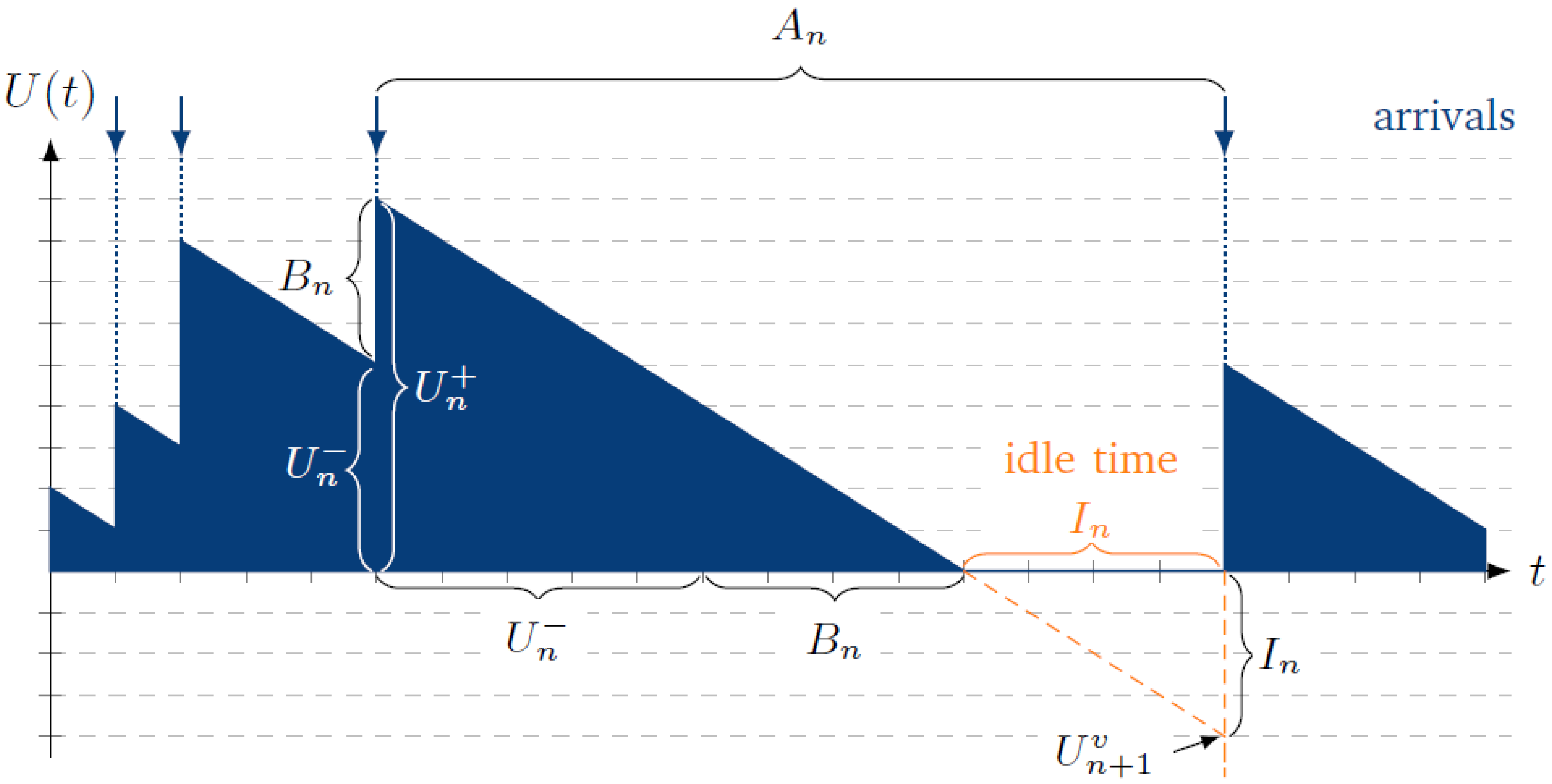




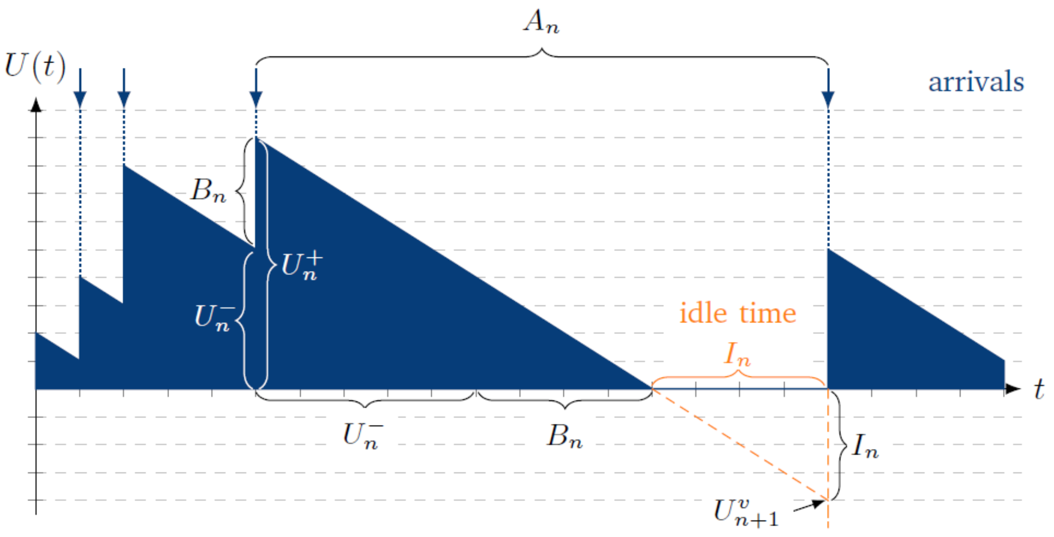
IDLE TIMES AND INTERDEPARTURE TIMES

Derivation of Idle Time for GI/GI/1 Delay System

- Idle time I



Idle Time Distribution



- Analysis of idle times

$$U_{n+1}^v = U_n^+ - A_n = U_n^- + B_n - A_n$$
$$Y_n = -U_{n+1}^v = A_n - (U_n^- + B_n)$$
$$I_n = Y_n | Y_n > 0$$
$$i_n(k) = \begin{cases} \dfrac{a_n(k) * u_n(-k) * b_n(-k)}{P(Y_n > 0)} & \text{for } k > 0 \\ 0 & \text{otherwise} \end{cases}$$

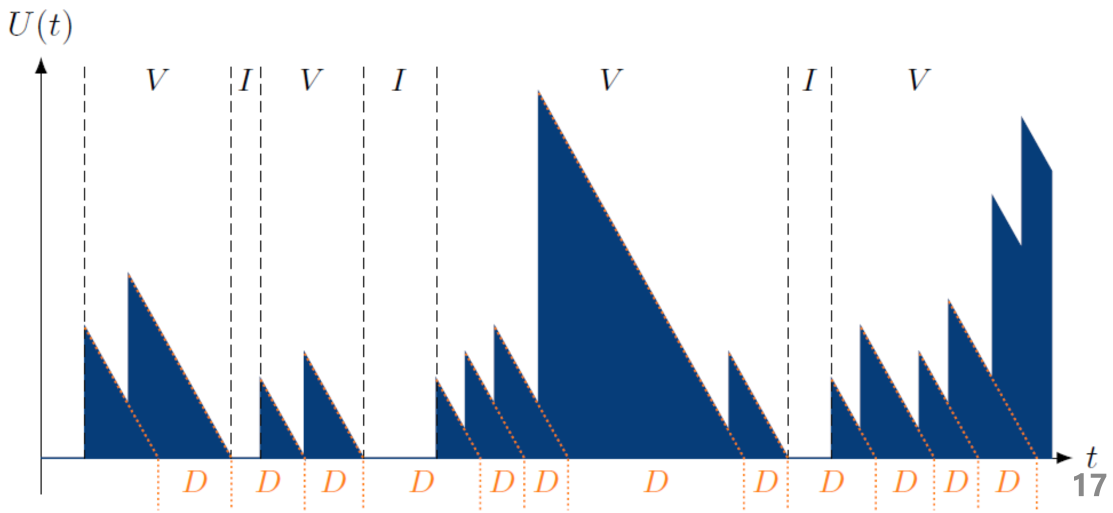
- Probability that a departure happens at the end of a busy period

$$p_I = \frac{1}{n_d^v} = \frac{\mathrm{E}[V] + \mathrm{E}[I]}{\mathrm{E}[A]} = \frac{\mathrm{E}[B]}{\mathrm{E}[V]}$$

- Interdeparture time

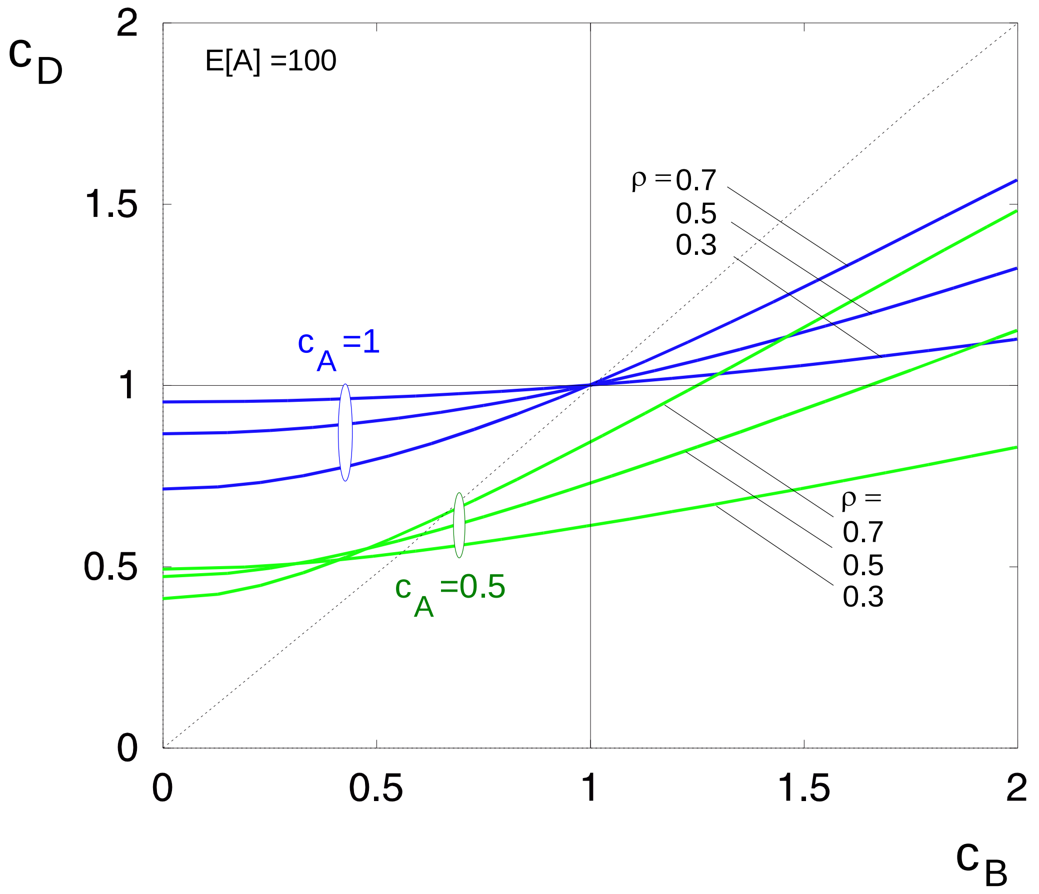
$$D_n = \begin{cases} I_n + B_n & \text{with probability } p_I \\ B_n & \text{with probability } 1 - p_I \end{cases}$$

$$d_n(k) = p_I(i_n(k) * b_n(k)) + (1 - p_I) b_n(k)$$



Output Process: Variation of Interdeparture Times

- Coefficient of variation c_D of interdeparture times



CHARACTERISTICS OF GI/GI/1

Kingman‘s approximation

Characteristics of GI/GI/1 Delay Systems

- **Stability condition** $\rho = \frac{\mathrm{E}[\,B\,]}{\mathrm{E}[\,A\,]} = \lambda \mathrm{E}[\,B\,] < 1$

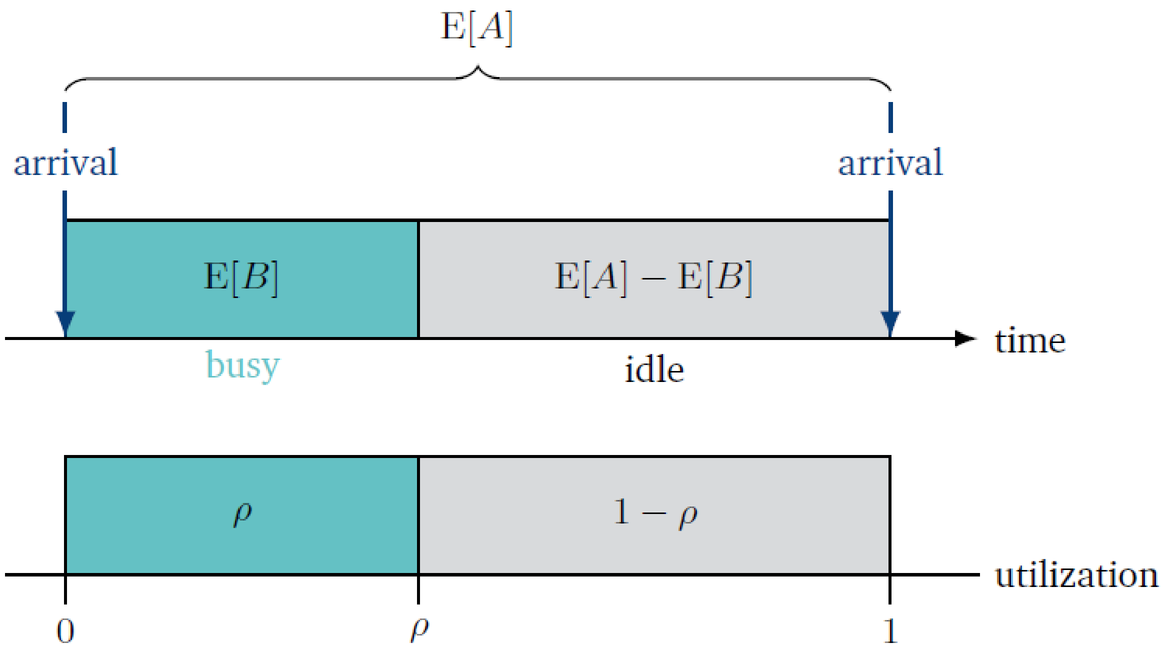
- **Utilization** of the server $\rho = \lambda E[B]$

- Probability that **server is idle** =
  server is empty at arbitrary point in time

  $$x(0) = P(X = 0) = 1 - \rho = \frac{\mathrm{E}[\,A\,] - \mathrm{E}[\,B\,]}{\mathrm{E}[\,A\,]}$$

- System state from perspective of arrivals

  $$x_A(0) = P(X_A = 0) = \frac{(1-\rho)\mathrm{E}[\,A\,]}{\mathrm{E}[\,I\,]} = \frac{\mathrm{E}[\,A\,] - \mathrm{E}[\,B\,]}{\mathrm{E}[\,I\,]}$$



Characteristics of GI/GI/1 Delay Systems (f.)

- **Waiting probability** $p_w = 1 - x_A(0) = \dfrac{\mathrm{E}[I] - (\mathrm{E}[A] - \mathrm{E}[B])}{\mathrm{E}[I]}$

- Mean number of customers in the system due to Little's law

  $$\mathrm{E}[X] = \lambda(\mathrm{E}[W] + \mathrm{E}[B])$$

- **Mean waiting time** $\mathrm{E}[W] = \mathrm{E}[X] \cdot \mathrm{E}[A] - \mathrm{E}[B]$

- Note: There are no simple equations available for $x_A(0), E[I], E[X]$

- Solution of Lindley integral equation required to derive those characteristics

$$W_{n+1} = \max(W_n + B_n - A_n, 0)$$

$$W = \max(W + B - A, 0) = \max(W + C, 0)$$

derivation of waiting times for discrete-time GI/GI/1 queue

Lindley Integral Equation for Continuous GI/GI/1 Systems

- For a GI/GI/1 system under stationary conditions, the following functional relationship for the waiting time distribution function are obtained $W(t)$: **Lindley integral equation**

$$W(t) = \begin{cases} 0 & t < 0 \\ W(t) * c(t) & t \geq 0 \end{cases}$$

where

$$c(t) = b(t) * a(-t)$$

- **System function** $C = B - A$ (r.v.) contains all parameters of stochastic process of GI/GI/1 queue

- Probability density function

$$w(t) = \begin{cases} 0 & t < 0 \\ \delta(t) \int_{-\infty}^{0^+} \Big(w(u) * c(u) \Big) du & t = 0 \\ w(t) * c(t) & t > 0 \end{cases}$$

Compact notation $w(t) = \pi_0\Big(w(t) * c(t) \Big)$

with $\pi_0\Big(f(t) \Big) = \begin{cases} 0 & t < 0 \\ \delta(t) \int_{-\infty}^{0^+} f(u) du & t = 0 \\ f(t) & t > 0 \end{cases}$

Kingman's Approximation of Mean Waiting Times

- Kingman provides an **approximation** for the mean waiting time: **Kingman's formula**

$$\mathrm{E}[\,W\,] \approx \left(\frac{\rho}{1-\rho}\right)\left(\frac{c_A^2+c_B^2}{2}\right)\mathrm{E}[\,B\,] \stackrel{\text{def}}{=} \widetilde{W}$$

- For a Poisson process, Kingman's formula is exact $\mathrm{E}[\,W\,] = \left(\frac{\rho}{1-\rho}\right)\left(\frac{1+c_B^2}{2}\right)\mathrm{E}[\,B\,] = \widetilde{W}$

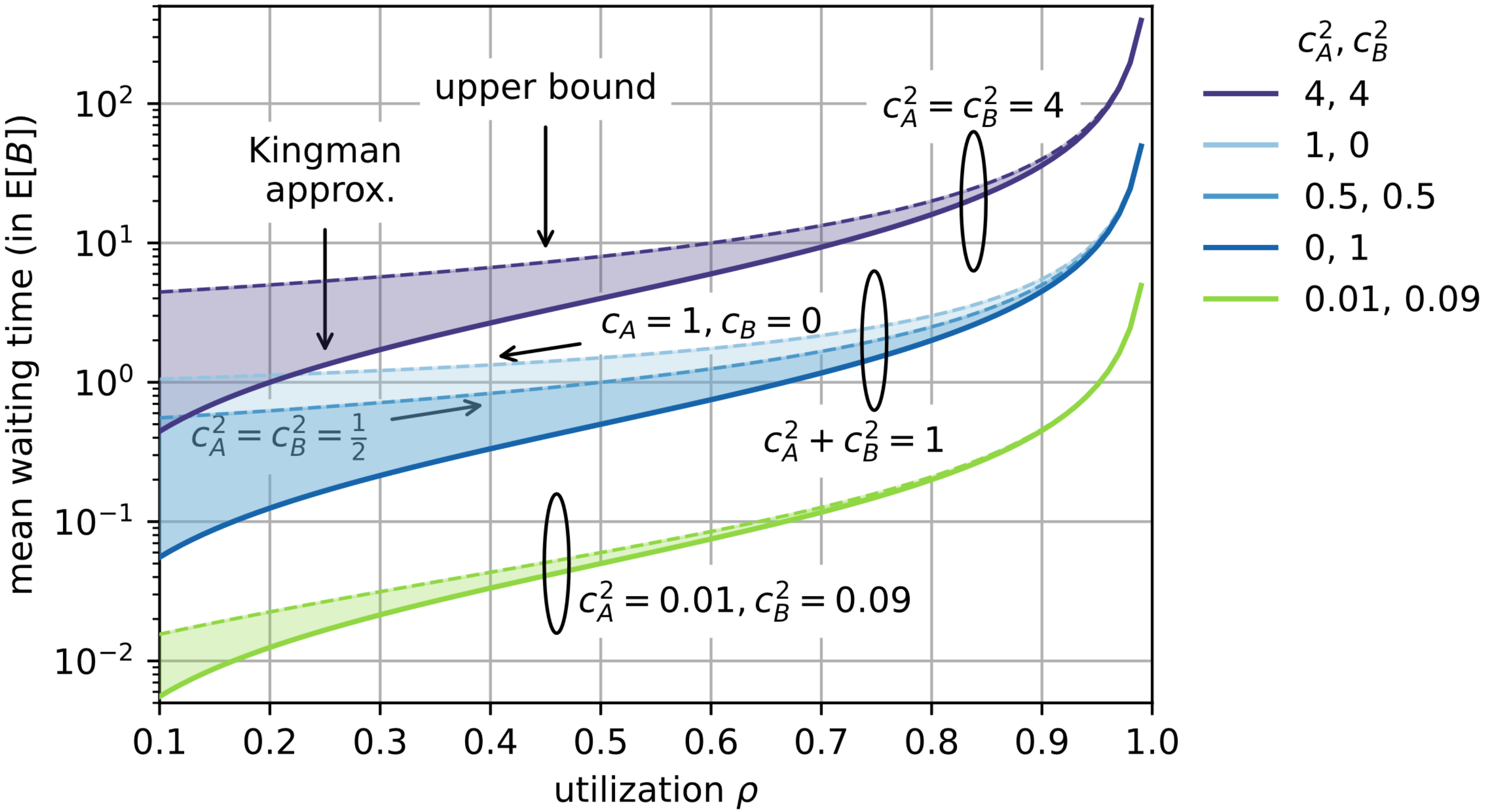
- **Tighter upper bound** of the mean waiting time is provided by Daley

$$\mathrm{E}[\,W\,] \leq \frac{(2-\rho)c_A^2+\rho c_B^2}{2(1-\rho)} \cdot \mathrm{E}[\,B\,] \stackrel{\text{def}}{=} \widehat{U}$$

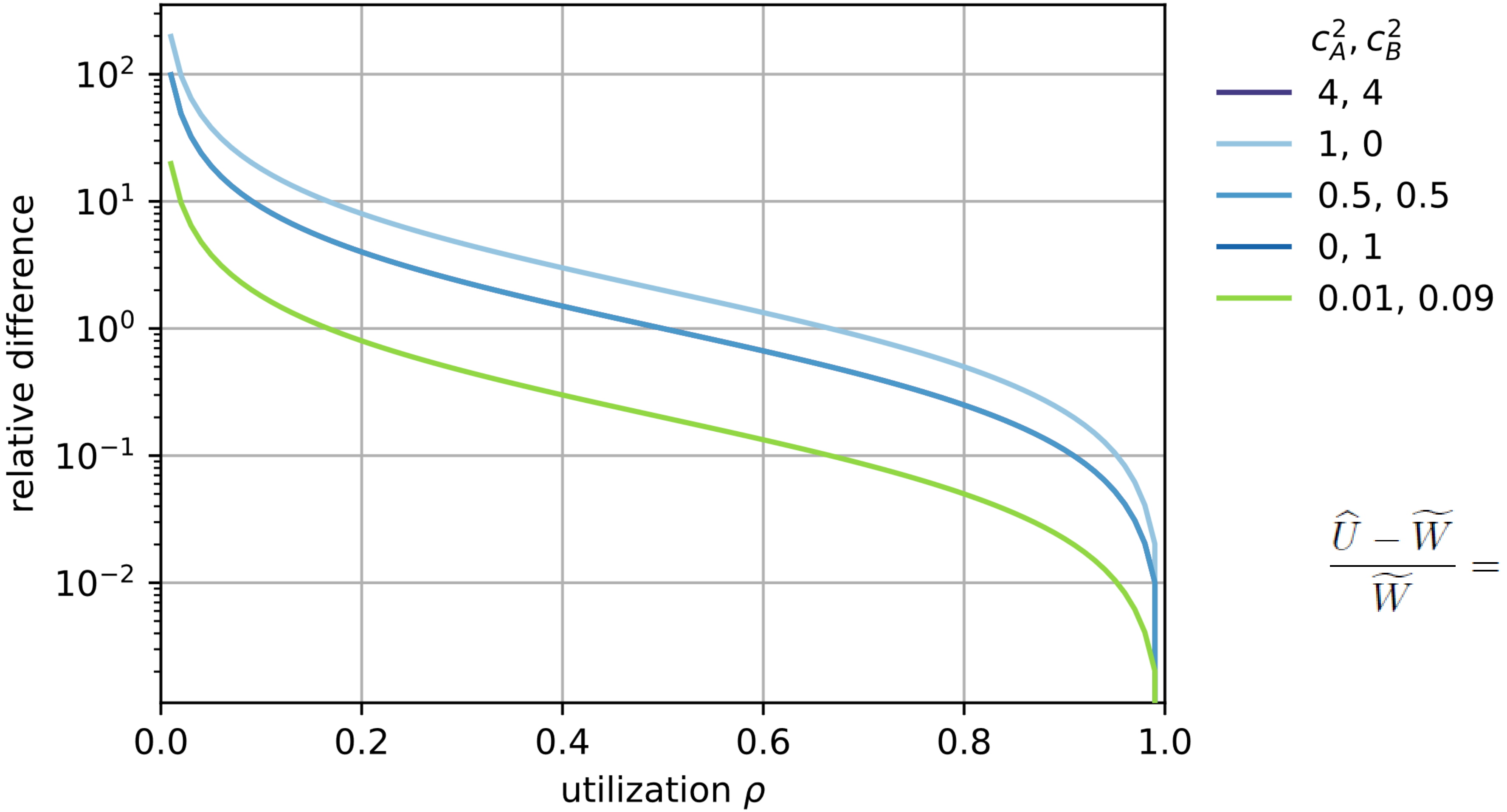
- Difference between upper bound and approximated mean $\widehat{U}-\widetilde{W} = \mathrm{E}[\,B\,]\cdot c_A^2$

- Relative difference $\dfrac{\widehat{U}-\widetilde{W}}{\widetilde{W}} = \dfrac{2c_A^2}{c_A^2+c_B^2}\cdot\left(\dfrac{1-\rho}{\rho}\right)$

Kingman's Approximation and Tight Upper Bounds



Relative Difference



$$\frac{\widehat{U} - \widetilde{W}}{\widetilde{W}} = \frac{2c_A^2}{c_A^2 + c_B^2} \cdot \left(\frac{1-\rho}{\rho} \right)$$