

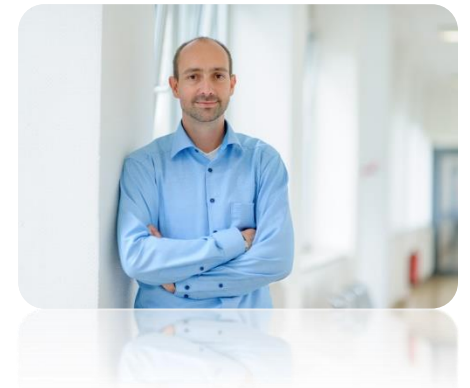
Chapter 5.2

Delay System M/GI/1

Performance Evaluation of the Internet of Things (IoT)

Module Course: Performance Evaluation of Distributed Systems

Prof. Tobias Hoßfeld, Summer Semester 2022



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*Tran-Gia, P. & Hossfeld, T. (2021).
Performance Modeling and Analysis of Communication
Networks - A Lecture Note. Würzburg University Press.
<https://doi.org/10.25972/WUP-978-3-95826-153-2>*

Website to download book, exercises, slides and scripts:
<https://modeling.systems/>

Chapter 5

5 Analysis of Non-Markovian Systems

5.1 Discrete-Time Markov Chain

5.2 Method of Embedded Markov Chain

5.2.1 Power Method for Numerical Derivation

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5.5.1 Model Structure and Parameters

5.5.2 Markov Chain and State Transition

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5.6 Results for Continuous-Time GI/GI/1 Delay Systems

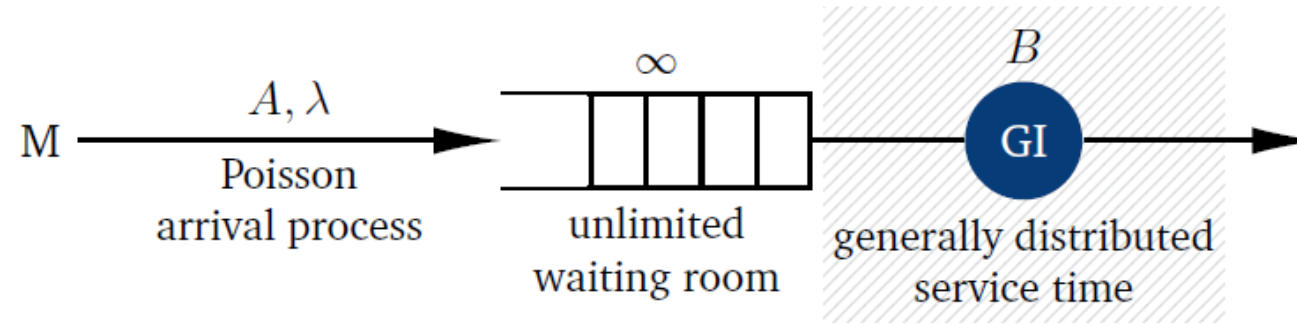
5.6.1 Characteristics of GI/GI/1 Delay Systems

5.6.2 Lindley Integral Eq. GI/GI/1 Systems

5.6.3 Kingman's Approximation of Mean Waiting Times

MODEL STRUCTURE AND PARAMETERS

Delay System M/GI/1

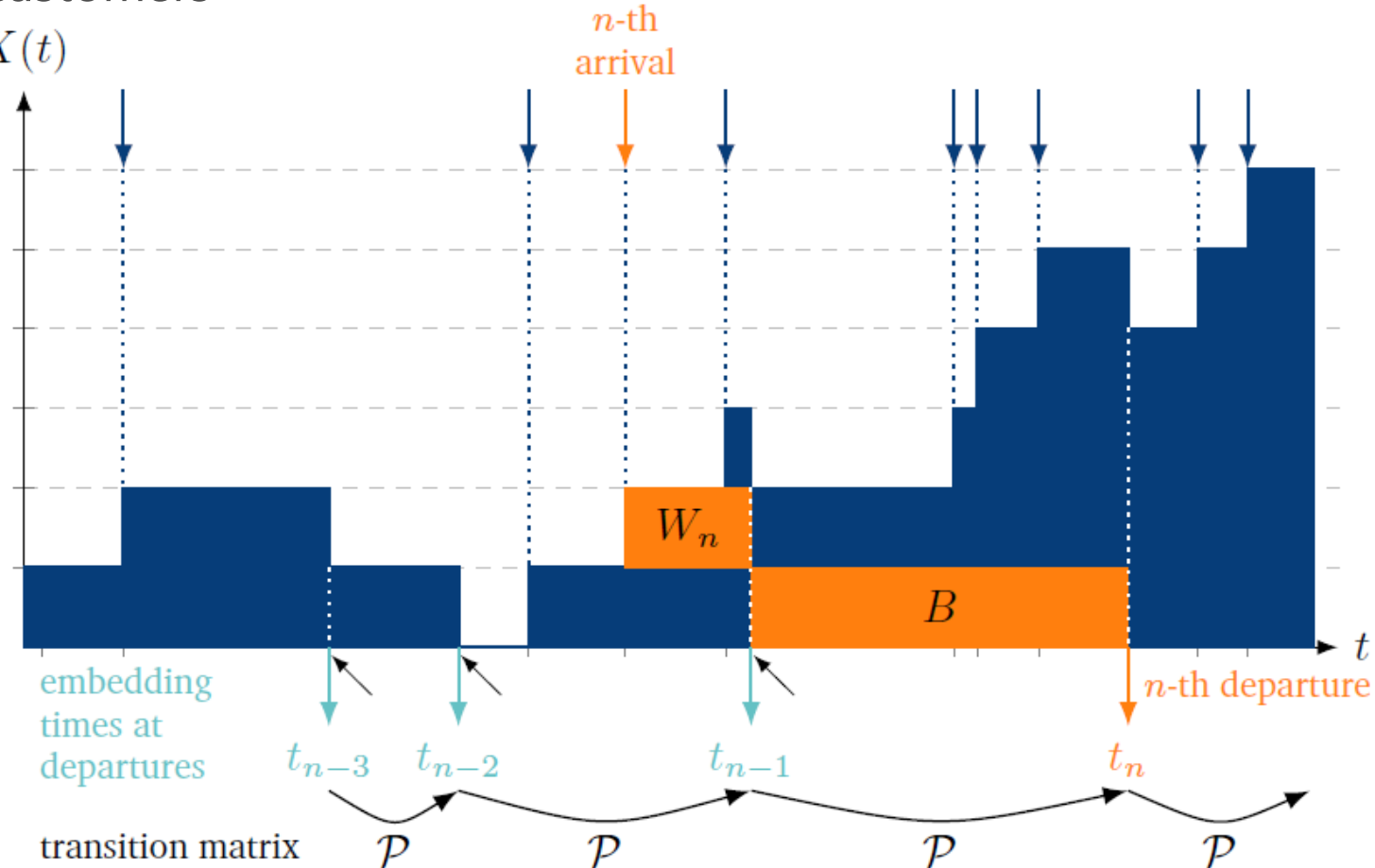


- ▶ Interarrival time A with arrival rate λ
- ▶ Service time B with service rate μ is generally distributed
- ▶ Offered traffic a identical to server utilization ρ : $\rho = a = \frac{\lambda}{\mu}$ in pseudo-unit Erlang [Erl]
- ▶ Pure delay system: number of waiting places is assumed to be unlimited
- ▶ FIFO queue: first-in first-out queuing discipline
- ▶ **Stability condition** $\rho < 1$

$$A(t) = P(A \leq t) = 1 - e^{-\lambda t}, \quad E[A] = \frac{1}{\lambda}$$

State Process of M/GI/1 (FIFO)

number of customers
at time t : $X(t)$



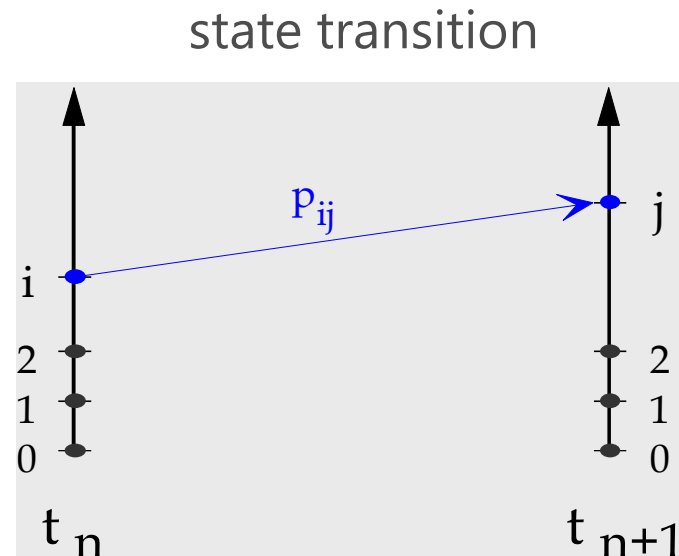
Markov chain: $\{X(t_0), X(t_1), \dots, X(t_n), X(t_{n+1}), \dots\}$

MARKOV CHAIN AND STATE TRANSITION

M/GI/1: Embedded Markov Chain

- ▶ Service process is the only non-Markovian model component (not memoryless)
- ▶ State process becomes memoryless at instances of service ends
 - regeneration points: immediately before or immediately after
 - easier for the analysis of M/GI/1: **immediately after service ends**
- ▶ **System state at embedding time** t_n : $X(t_n) = X_D(t_n)$
- ▶ State probability at time t_n : $x(j, n) = P(X(t_n) = j)$
- ▶ **State transition probability**

$$p_{ij} = P(X(t_{n+1}) = j | X(t_n) = i)$$



Number of Arrivals During Service Time

- ▶ Random variable Γ for the number of arrivals during a service duration B

- distribution $\gamma(i) = P(\Gamma = i)$

- generating function (GF transform of Γ)

- mean value $E[\Gamma] = \left. \frac{d\Gamma_{GF}(z)}{dz} \right|_{z=1} = \lambda E[B] = \rho.$

$$\begin{aligned}\Gamma_{GF}(z) &= \sum_{j=0}^{\infty} \gamma(j) z^j = \sum_{j=0}^{\infty} \int_0^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} b(t) dt z^j \\ &\stackrel{(3.20)}{=} \Phi_B(s) \Big|_{s=\lambda(1-z)} = \Phi_B(\lambda(1-z)).\end{aligned}$$

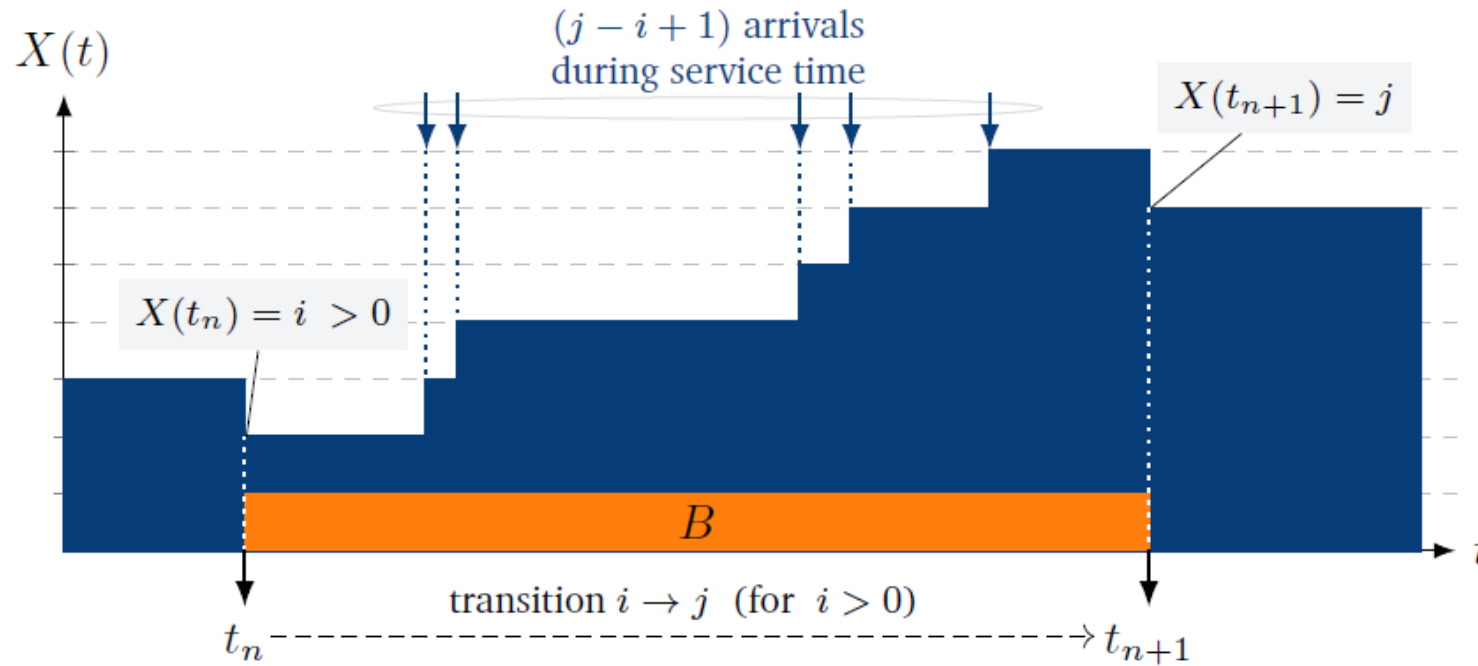
Ch. 3.3 Poisson arrivals during an arbitrarily distributed Interval

- ▶ Example: Poisson arrivals during deterministic service time

- $B \sim D(t_0)$ and arrival rate λ
- Γ follows a Poisson distribution: $\Gamma \sim POIS(\lambda \cdot t_0)$

State Transition Probability (Case a)

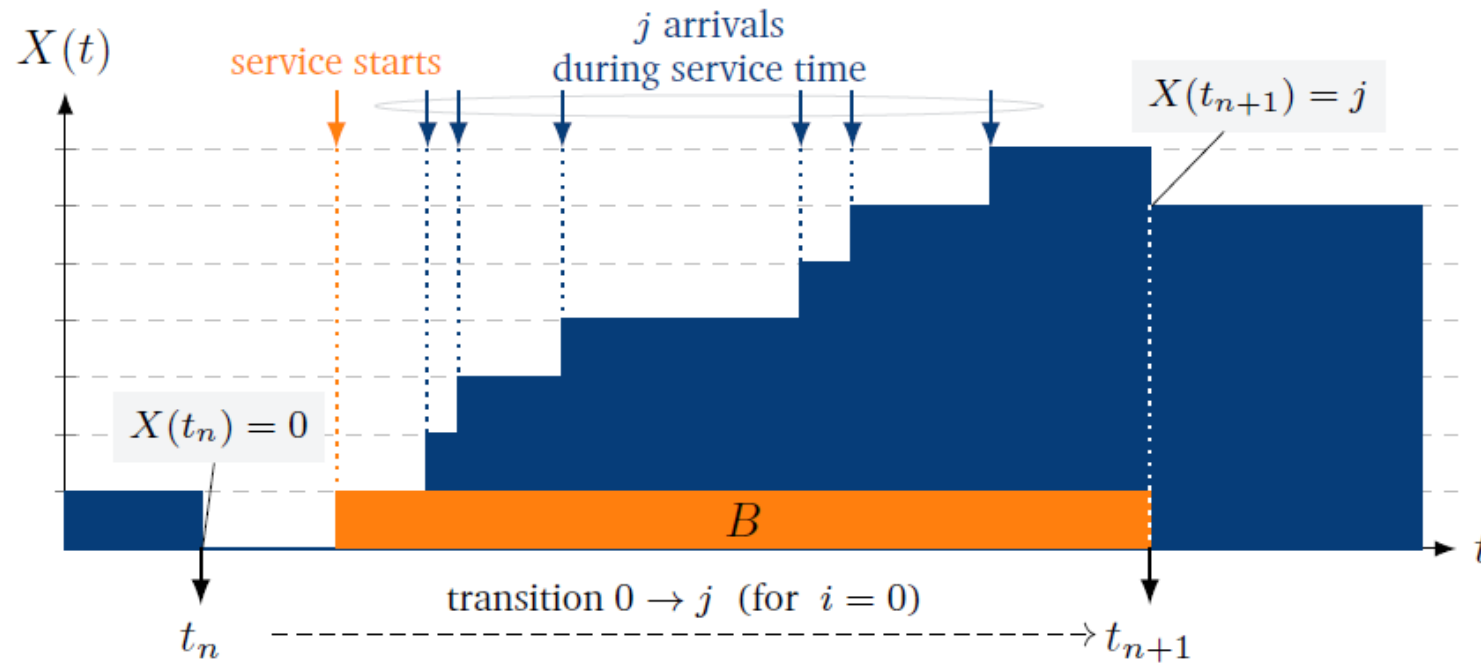
- For $i > 0$: $p_{ij} = \gamma(j - i + 1)$, $i = 1, \dots$, $j = i - 1, i, \dots$



(a) Transition $[X(t_n) = i] \rightarrow [X(t_{n+1}) = j]$ for $(i > 0)$.

State Transition Probability (Case b)

- For $i = 0$: $p_{0j} = \gamma(j)$, $j = 0, \dots$



(b) Transition $[X(t_n) = 0] \rightarrow [X(t_{n+1}) = j]$ for $(i = 0)$.

State Transition Matrix

- ▶ Case a: $i > 0$ $p_{ij} = \gamma(j - i + 1), \quad i = 1, \dots, \quad j = i - 1, i, \dots$
- ▶ Case b: $i = 0$ $p_{0j} = \gamma(j), \quad j = 0, \dots$
- ▶ State transition matrix

$$\mathcal{P} = \{p_{ij}\} = \begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \cdots \\ \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \cdots \\ 0 & \gamma(0) & \gamma(1) & \gamma(2) & \cdots \\ 0 & 0 & \gamma(0) & \gamma(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

STATE EQUATION AND STATE PROBABILITIES

General State Transition Equation

- ▶ State probabilities at the regeneration point t_n

$$X_n = \{x(0,n), x(1,n), \dots, x(j,n), \dots\}$$

$$x(j,n) = P(X(t_n) = j), \quad j = 0, 1, \dots$$

- ▶ **General state transition equation**

$$\begin{cases} X_n \cdot \mathcal{P} = X_{n+1} \\ x(j,n+1) = x(0,n) \gamma(j) + \sum_{i=1}^{j+1} x(i,n) \cdot \gamma(j-i+1), \quad j=0,1,\dots \end{cases}$$

case b: i=0 *case a: i>0*

$$\mathcal{P} = \{p_{ij}\} = \begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \dots \\ \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \dots \\ 0 & \gamma(0) & \gamma(1) & \gamma(2) & \dots \\ 0 & 0 & \gamma(0) & \gamma(1) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- ▶ **Non-stationary analysis**

- With start vector X_0 , future time-dependent state probability vectors can be derived

$$X_1, \dots, X_n, X_{n+1}$$

Stationary Analysis

- In statistical equilibrium

$$X_n = X_{n+1} = \dots = X$$

$$X = \{x(0), x(1), \dots, x(j), \dots\}$$

- Stationary state transition equation

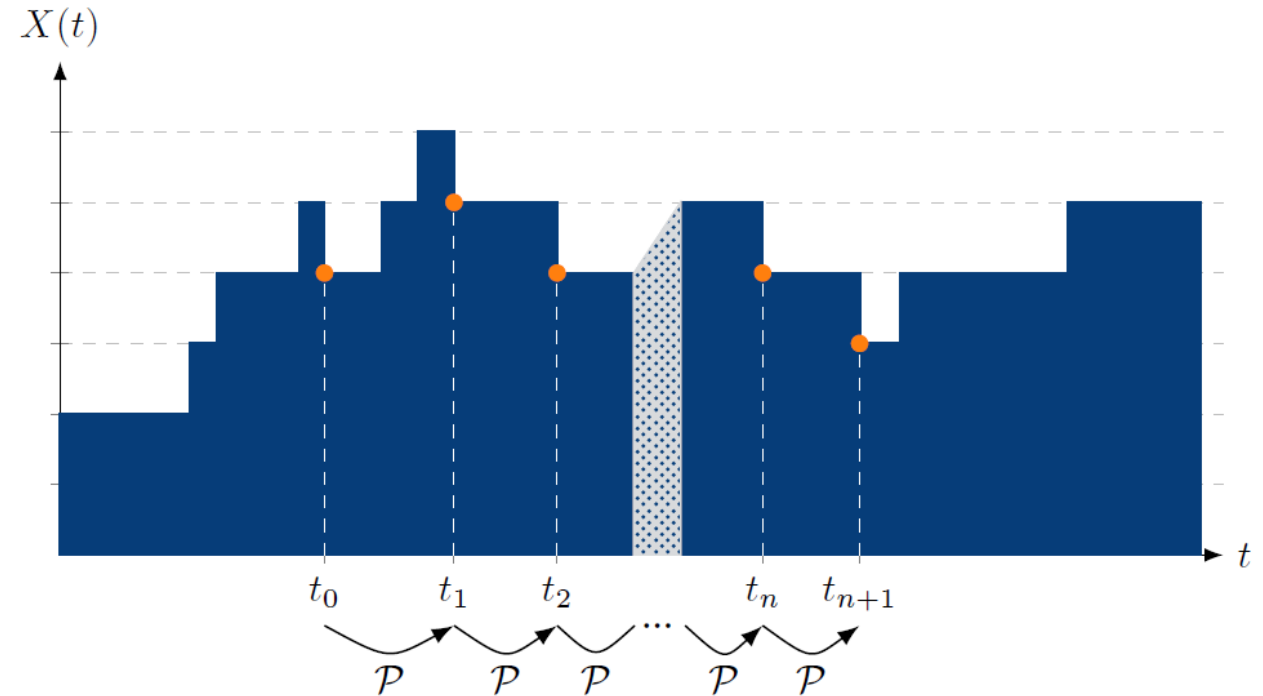
$$X \cdot \mathcal{P} = X$$

- Components of state probability vector

$$x(j) = x(0) \gamma(j) + \sum_{i=1}^{j+1} x(i) \cdot \gamma(j-i+1), \quad j=0,1,\dots$$

case b: $i=0$

case a: $i>0$



M/GI/1: Analysis of States

► Analysis using generating function $X_{GF}(z) = \frac{(1 - \rho)(1 - z)\Gamma_{GF}(z)}{\Gamma_{GF}(z) - z}$

- Generating function of number of arrivals Γ during random service time

$$\begin{aligned}\Gamma_{GF}(z) &= \sum_{j=0}^{\infty} \gamma(j) z^j = \sum_{j=0}^{\infty} \int_0^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} b(t) dt z^j \\ &\stackrel{(3.20)}{=} \Phi_B(s) \Big|_{s = \lambda(1 - z)} = \Phi_B(\lambda(1 - z)).\end{aligned}$$

- **Pollaczek-Khintchine formula** for system state

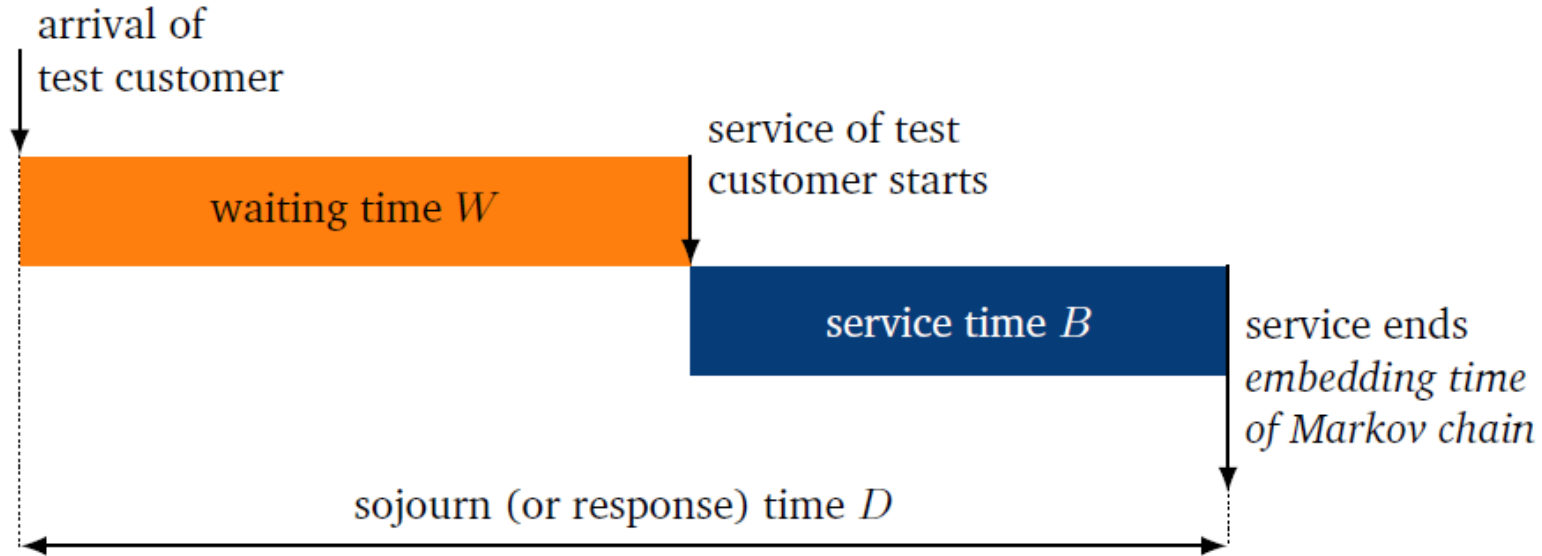
$$X_{GF}(z) = \frac{(1 - \rho)(1 - z)\Phi_B(\lambda(1 - z))}{\Phi_B(\lambda(1 - z)) - z}$$

M/GI/1: Analysis of States (Proof)

Lecture

DELAY DISTRIBUTION

Sojourn Time of Customer



$$D = W + B$$

$$d(t) = w(t) * b(t) \Rightarrow \Phi_D(s) = \Phi_W(s) \cdot \Phi_B(s)$$

Key Idea to Derive Delay Distribution

- Interpretations of the state probabilities at embedding times (after service ends)

$$\begin{aligned}x(k) &= P(\text{test customer left behind } X = k \text{ customers in system}) \\ &= P(k \text{ arrivals during the sojourn time of test customer})\end{aligned}$$

Delay Distribution: Analysis

- ▶ Number of Poisson arrivals X during sojourn time D

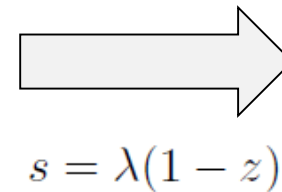
$$X_{GF}(z) = \sum_{k=0}^{\infty} x(k) z^k = \sum_{k=0}^{\infty} \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} d(t) dt z^k$$

$$\stackrel{(3.20)}{=} \Phi_D(s) \Big|_{s=\lambda(1-z)} = \Phi_D(\lambda(1-z)).$$

- ▶ Already derived: Pollaczek-Khintchine formula for system state

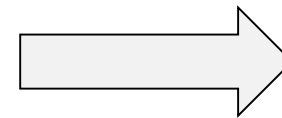
$$X_{GF}(z) = \frac{(1-\rho)(1-z)\Phi_B(\lambda(1-z))}{\Phi_B(\lambda(1-z)) - z}$$

- ▶ Finally $\Phi_D(\lambda(1-z)) = \frac{(1-\rho)(1-z)\Phi_B(\lambda(1-z))}{\Phi_B(\lambda(1-z)) - z}$



$$\Phi_D(s) = \frac{s(1-\rho)}{s - \lambda + \lambda\Phi_B(s)} \Phi_B(s)$$

- ▶ **Pollaczek-Khintchine formula for waiting time**



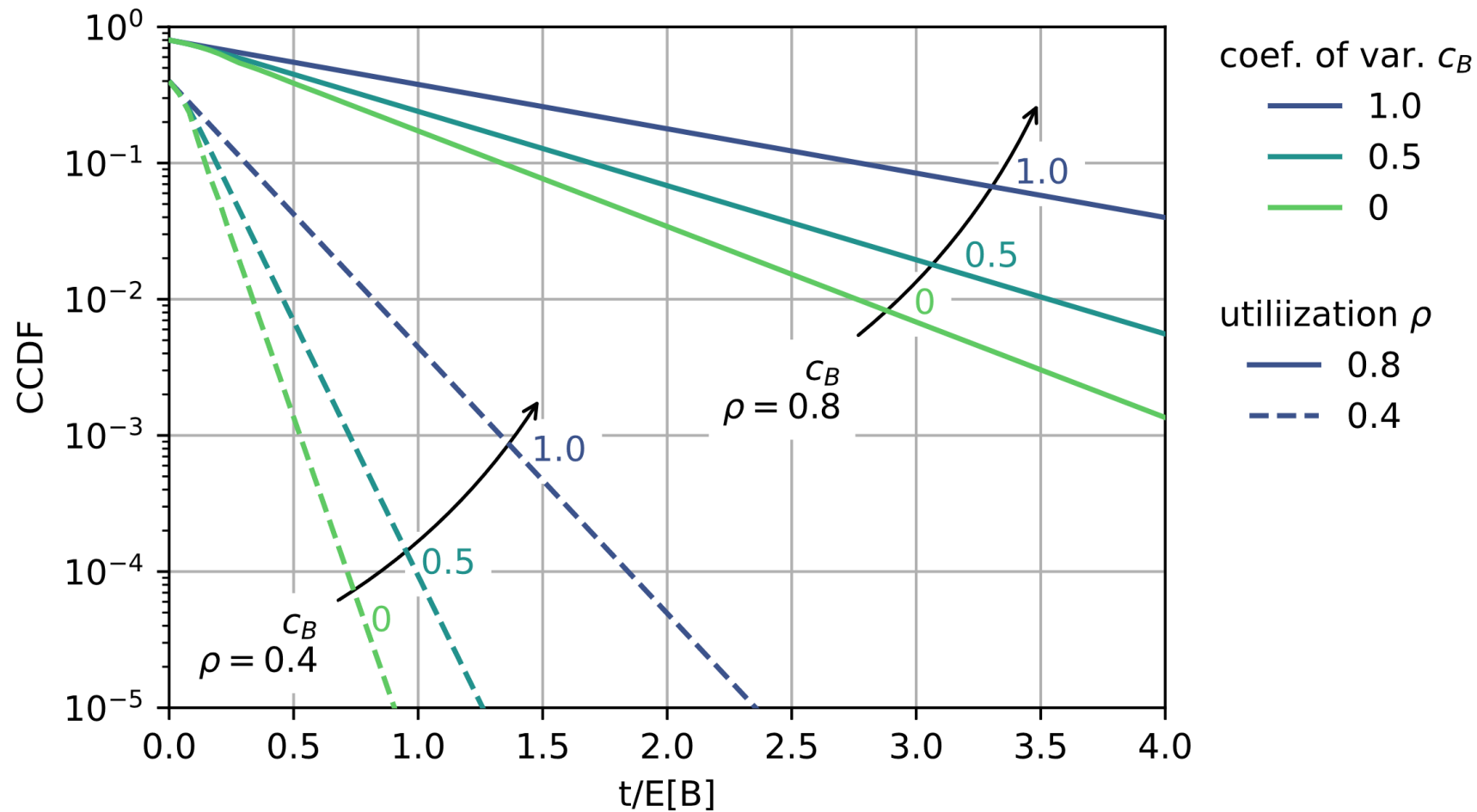
$$\Phi_W(s) = \frac{s(1-\rho)}{s - \lambda + \lambda\Phi_B(s)}.$$

$$\Phi_D(s) = \Phi_W(s) \cdot \Phi_B(s)$$

Convolution Theorem for Continuous R.V.s

Lecture

CCDF of the Waiting Time



OTHER SYSTEM CHARACTERISTICS

Waiting probability, mean waiting times, higher moments of waiting time

Waiting Probability

- ▶ Initial value theorem of Laplace transform applied

$$w(t) \xrightarrow{\text{LT}} \Phi_W(s) \quad \Rightarrow \quad W(t) \xrightarrow{\text{LT}} \frac{\Phi_W(s)}{s}$$

$$P(W=0) = \lim_{t \rightarrow 0} W(t) = \lim_{s \rightarrow \infty} s \cdot \frac{\Phi_W(s)}{s}$$

$$\Phi_W(s) = \frac{s(1-\rho)}{s-\lambda + \lambda \Phi_B(s)}$$

- ▶ **Waiting probability**

$$p_W = P(W > 0) = 1 - P(W = 0) = 1 - W(t)|_{t \rightarrow 0} = \rho.$$

Mean Waiting Times

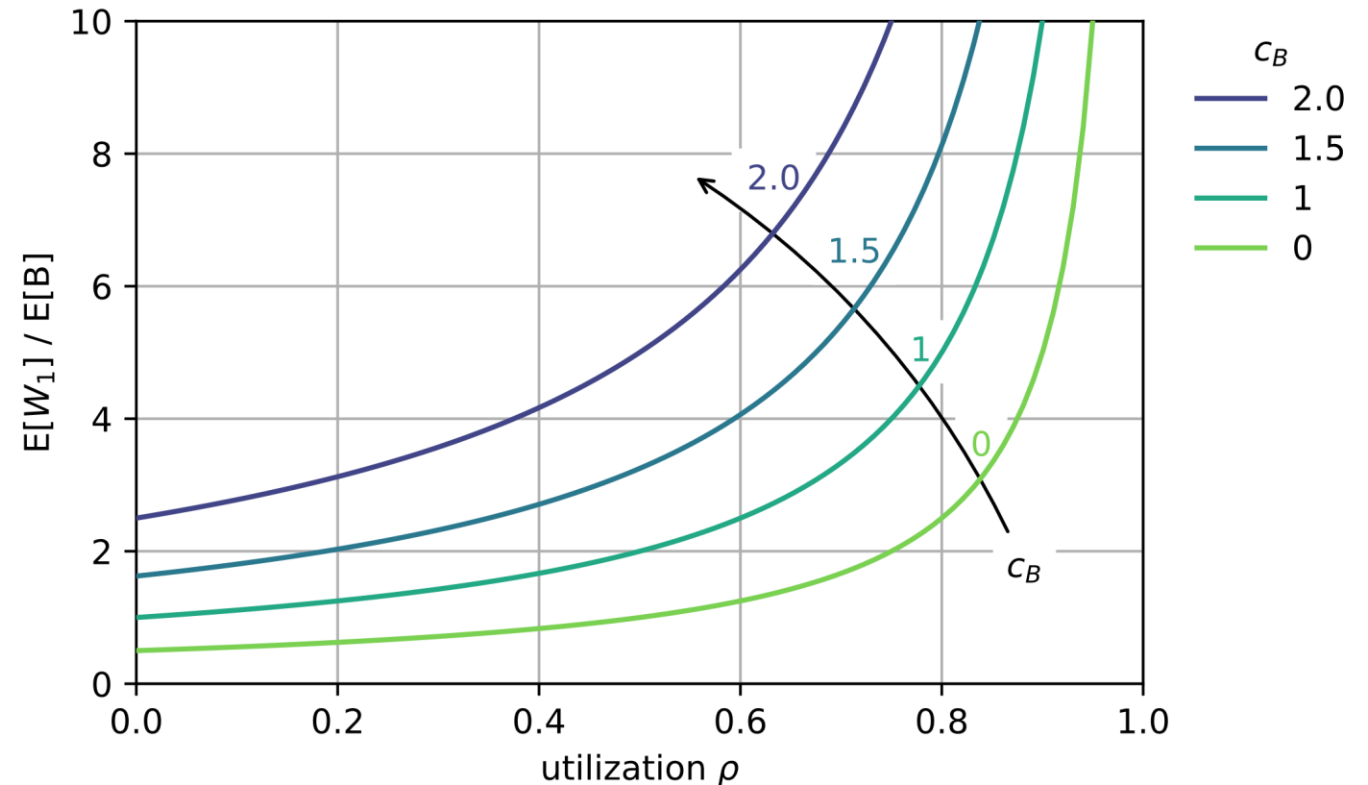
- ▶ Mean waiting time of **all customers** $E[W]$

$$E[W] = E[B] \cdot \frac{\rho (1 + c_B^2)}{2(1 - \rho)} = \frac{\lambda E[B^2]}{2(1 - \rho)}$$

- ▶ Mean waiting time of **waiting customers** $E[W_1]$

$$E[W_1] = \frac{E[W]}{p_W} = E[B] \cdot \frac{1 + c_B^2}{2(1 - \rho)}$$

- ▶ Waiting time W is a mixture distribution
 - $W = 0$ with probability $1 - p_W$
 - $W = W_1$ with probability p_W



Higher Moments of Waiting Time

► Takács recursion formula

$$E[W^k] = \frac{\lambda}{1-\rho} \sum_{i=1}^k \binom{k}{i} \frac{E[B^{i+1}]}{i+1} E[W^{k-i}],$$
$$E[W^0] = 1.$$

► Especially for the two first moments

$$E[W] = \frac{\lambda E[B^2]}{2(1-\rho)}$$

$$E[W^2] = 2 E[W]^2 + \frac{\lambda E[B^3]}{3(1-\rho)}$$

STATE PROBABILITIES AT ARBITRARY TIME

State Probabilities at Arbitrary Time

- ▶ Embedded Markov chain: $x(i) = x_D(i)$
 - state probabilities given in the Pollaczek-Khintchine formula hold at regeneration points of the Markov chain
 - Markov chain: embedded immediately after service ends
- ▶ **Kleinrock's (Burke's) result:** $x_D(i) = x_A(i)$
 - M/GI/1 system state can change at most by +1 or -1
- ▶ **PASTA property:** $x^*(i) = x_A(i)$
 - arrival process is a Poisson process
- ▶ In summary: $x^*(i) = x_A(i) = x_D(i) = x(i)$
 - E.g., $x^*(0) = 1 - \rho$ (utilization law) and $p_W = 1 - x_A(0) = \rho$ (waiting probability)