SAT Lattice model

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definition 1. *Lattice (Poset definition)*
Let $(L, R, 1, 0)$ be a set L together with a relation R, such that:

1. $(a, a) \in R$
2. *if* $(a, b) \in R$ *and* $(b, a) \in R$ *iff* $a = b$
3. *if* $(a, b) \in R$ *and* $(b, c) \in R$ *then* $(a, c) \in R$

Then L is a lattice if it follows the following universal properties:

1. *If for all pairs $(a, b) \in L^2$ there exists an element meet in L, such that $(a, meet) \in R$, $(b, meet) \in R$ and if $c \in R$ such that $(a, c) \in R$, $(b, c) \in R$, then $(meet, c) \in R$.*
2. *If for all pairs $(a, b) \in L^2$ there exists an element join in L, such that $(join, a) \in R$, $(join, b) \in R$ and if $c \in R$ such that $(c, a) \in R$, $(c, b) \in R$, then $(c, c = join) \in R$.*
3. *There is a special element 1 such that $(1, a) \in R$ for all $a \in L$. Dually, there is a special element 0, such that $(a, 0) \in R$ for all $a \in R$.*

We usually write $a \leq b$, instead of $(a, b) \in R$.

definition 2. *Lattice (Algebraic definition)*
Let $(L, \wedge, \vee, 1, 0)$ be a set L together with two binary operations $\wedge$ and $\vee$, that satisfy the following axioms:

1. *Commutative: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$.*
2. *Associative: $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$*
3. *Absorption: $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.*
4. *Idempotent: $a \wedge a = a = a \vee a$ for all $a \in L$*
5. *Infima and suprema: there exists special elements 1 and 0 such that $a \wedge 1 = a$ and $a \vee 0 = a$ for all $a \in L$.*

definition 3. *Let $(L, \leq_L)$ be a lattice. We say a is covered by b, written $a <: b$ if $a \leq_L b$ and for all $c \leq_L b$ then $a \leq_L c$.*

definition 4. *Geometric lattice*
A Lattice $(L, <_L)$

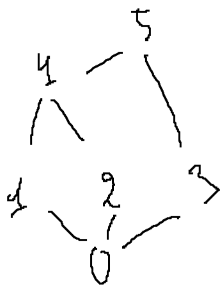
1. *(Semimodular Law) For all $a, b \in L$, if $a \wedge b <: a$ then $b <: a \vee b$*

2. *(Atomistic) Every element is the meet of atoms, where an atom is an element that covers* 0*. That is, for all* $x \in L$, $x = y_1 \vee y_2 \vee \ldots \vee y_k$ *where* $0 <: y_i$ *for all* $i \in [k]$.

Being semimodulat is the same as:

1. *(Mac Lane) For all* $a, b, c \in L$ *such that* $b \wedge c <_L a <_L c <_L b \vee a$ *there is an element* d *such that* $b \wedge c <_L d \leq_L b$ *and* $a = (a \vee d) \wedge c$

The problem with the SAT model is that the lattices are not semimodular, but they seem to be atomistic. It does generate lattices correctly (at least it generates the correct number for small examples), ignoring the extra conditions for geometric lattices. It is also possible that the pictures are not accurate.



$a \wedge b <: a$

$\Rightarrow$

$b <: a \vee b$

$3 \wedge 4 = 5 <: 4$

$4 <: 3 \vee 4 = 0$

is false!