SAT Lattice model

Delyan Kirov

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definition 1. *Lattice (Poset definition)*
Let $(L, R, 1, 0)$ *be a set* L *together with a relation* R*, such that:*

1. $(a, a) \in R$
2. *if* $(a, b) \in R$ *and* $(b, a) \in R$ *iff* $a = b$
3. *if* $(a, b) \in R$ *and* $(b, c) \in R$ *then* $(a, c) \in R$

Then L *is a lattice if it follows the following universal properties:*

1. *If for all pairs* $(a, b) \in L^2$ *there exists an element meet in* L*, such that* $(a, meet) \in R$*,* $(b, meet) \in R$ *and if* $c \in R$ *such that* $(a, c) \in R$*,* $(b, c) \in R$*, then* $(meet, c) \in R$*.*
2. *If for all pairs* $(a, b) \in L^2$ *there exists an element join in* L*, such that* $(join, a) \in R$*,* $(join, b) \in R$ *and if* $c \in R$ *such that* $(c, a) \in R$*,* $(c, b) \in R$*, then* $(c, c = join) \in R$*.*
3. *There is a special element* 1 *such that* $(1, a) \in R$ *for all* $a \in L$*. Dually, there is a special element* 0*, such that* $(a, 0) \in R$ *for all* $a \in R$*.*

We usually write $a \leq b$*, instead of* $(a, b) \in R$*.*

definition 2. *Lattice (Algebraic definition)*
Let $(L, \wedge, \vee, 1, 0)$ *be a set* L *together with two binary operations* $\wedge$ *and* $\vee$*, that satisfy the following axioms:*

1. *Commutative:* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$*.*
2. *Associative:* $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *and* $(a \vee b) \vee c = a \vee (b \vee c)$
3. *Absorption:* $a \wedge (a \vee b) = a$ *and* $a \vee (a \wedge b) = a$*.*
4. *Idempotent:* $a \wedge a = a = a \vee a$ *for all* $a \in L$
5. *Infima and suprema: there exists special elements* 1 *and* 0 *such that* $a \wedge 1 = a$ *and* $a \vee 0 = a$ *for all* $a \in L$*.*

definition 3. *Let* $(L, \leq_L)$ *be a lattice. We say* a *is covered by* b*, written* $a <: b$ *if* $a \leq_L b$ *and for all* $c \leq_L b$ *then* $a \leq_L c$*.*

An element that covers 0 *is called an atom.*

definition 4. *Let* $L(\vee, \wedge, 0, 1)$ *be a lattice. An element* $(1 \neq a \neq 0) \in L$ *is called join irreducible (similarly meet irreducible) if* $a = b \vee c$ *implies* $b = a$ *or* $c = a$ *or both (similarly* $a = b \wedge c$ *implies*

$b = a$ or $c = a$ or both).

This definition is similar to prime numbers. Note also that $a = b \vee c$ implies $a \geq_L b$ and $a \geq_L c$.

theorem 1. *Every nonzero element in a finite lattice L can be written as the join of finitely many join-irreducibles.*

Proof: Pick an element $k \in L$, which is not 0 and is join irreducible. Then since there must be a path from an atom to k then $k = k \vee a$ where a is some atom. Thus affirming the claim.

Pick an element $k \in L$, which is not 0 and is not join irreducible. Then there exist join-reducible $k_1, k_2 \in L$ such that $k = k_1 \vee k_2$. Similarly $k_1 = k_3 \vee k_4$ and so on. However, notice that $k_i = k_j \vee k_l$ implies that $k_i \geq_L k_j$ and $k_i \geq_L k_l$. So this process must end, that is, there exists $t \in N$, such that k_t cannot be written as the join of join-reducible elements. So k can indeed be written as the join of join-irreducible elements.

definition 5. *A lattice L is called atomistic if all elements can be written as the join of atoms. That is for all $b \in L$, there exists a subset of atoms A such that $b = a_1 \vee a_2 \vee ... \vee a_{|A|}$, where $a_i \in A$.*

theorem 2. *Let L be a lattice. The following are equivalent:*

1. *L is atomistic*
2. *All join irreducibles are atoms*
3. *every element $k \in L$ which is not zero and not an atom can be written as $k = a \vee b$, where $a \neq k$ and $b \neq k$.*

Proof: (1) iff (2) is obvious. Indeed if all join irreducibles are atoms, then every non zero element can be written as a join of atoms, so the lattice is atomistic. Similarly, if the lattice is atomistic, then if an element is join-irreducible, it must be an atom.

(2) iff (3). Indeed, assume (2). Then if ($k = a \vee b$ implies $k = a$ or $k = b$) then k must be an atom. Therefore the claim follows. Now assume (3). Let k be join-irreducible. Then k is either zero, or it's an atom. This finishes the proof.

definition 6. *Geometric lattice*
A Lattice $(L, <_L)$ is geometric if:

1. *(Semimodular Law) For all $a, b \in L$, if $a \wedge b <: a$ then $b <: a \vee b$*
2. *(Atomistic) Every element is the meet of atoms, where an atom is an element that covers 0. That is, for all $x \in L$, $x = y_1 \vee y_2 \vee ... \vee y_k$ where $0 <: y_i$ for all $i \in [k]$.*

Being semimodulat is the same as:

1. *(Mac Lane) For all $a, b, c \in L$ such that $b \wedge c <_L a <_L c <_L b \vee a$ there is an element d such that $b \wedge c <_L d \leq_L b$ and $a = (a \vee d) \wedge c$*