

# A UNIFICATION OF PERMUTATION PATTERNS RELATED TO SCHUBERT VARIETIES

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**ABSTRACT.** We obtain new connections between permutation patterns and singularities of Schubert varieties, by giving a new characterization of Gorenstein varieties in terms of so called *bivincular patterns*. These are generalizations of classical patterns where conditions are placed on the location of an occurrence in a permutation, as well as on the values in the occurrence. This clarifies what happens when the requirement of smoothness is weakened to factoriality and further to Gorensteinness, extending work of Bousquet-Mélou and Butler (2007), and Woo and Yong (2006). We also show how *mesh patterns*, introduced by Brändén and Claesson (2011), subsume many other types of patterns and define an extension of them called *marked mesh patterns*. We use these new patterns to further simplify the description of Gorenstein Schubert varieties and give a new description of Schubert varieties that are *defined by inclusions*, introduced by Gasharov and Reiner (2002). We also give a description of 123-hexagon avoiding permutations, introduced by Billey and Warrington (2001), Dumont permutations and cycles in terms of marked mesh patterns.

## 1. INTRODUCTION

In this paper we exhibit new connections between permutation patterns and singularities of Schubert varieties  $X_\pi$  in the complete flag variety  $\text{Flags}(\mathbb{C}^n)$ , by giving a new characterization of Gorenstein varieties in terms of which *bivincular patterns* the permutation  $\pi$  avoids. Bivincular patterns, defined by Bousquet-Mélou, Claesson, Dukes and Kitaev [6], are generalizations of classical patterns where conditions are placed on the location of an occurrence in a permutation, as well as on the values in the occurrence. This clarifies what happens when the requirement of smoothness is weakened to factoriality and further to Gorensteinness, extending work of Bousquet-Mélou and Butler [5], and Woo and Yong [26]. We also prove results that translate some known patterns in the literature into bivincular patterns. In particular we will give a characterization of the Baxter permutations.

Table 1 summarizes the main results in the paper related to bivincular patterns. The first line in the table is due to Ryan [21], Wolper [25] and Lakshmibai and Sandhya [18] and says that a Schubert variety  $X_\pi$  is non-singular (or smooth) if and only if  $\pi$  avoids the patterns 1324 and 2143. Note that some authors use a different convention for the correspondence between permutations and Schubert varieties, which results in the reversal of the permutations. These authors would then use the patterns 4231, 3412 to identify the smooth Schubert varieties. Saying that the variety  $X_\pi$  is non-singular means that every local ring is regular.

A weakening of this condition is the requirement that every local ring only be a unique factorization domain; a variety satisfying this is a *factorial* variety. Bousquet-Mélou and Butler [5] proved a conjecture stated by Woo and Yong (personal communication) that factorial Schubert varieties are those that correspond to permutations avoiding 1324 and bar-avoiding  $21\bar{3}54$ . In the terminology of Woo

TABLE 1. Connections between singularity properties and bivincular patterns

$X_\pi$ is	The permutation $\pi$ avoids the patterns		
smooth	2143 and 1324		
factorial	<u>2</u> 143 and 1324		
Gorenstein	$\begin{smallmatrix} 12345 \\ 31524 \end{smallmatrix}$ , $\begin{smallmatrix} 12345 \\ 24153 \end{smallmatrix}$	and associated Grassmannians avoid two bivincular pattern families	

and Yong [26] the bar-avoidance of the latter pattern corresponds to avoiding 2143 with Bruhat restriction ( $1 \leftrightarrow 4$ ), or equivalently, interval avoiding [2413, 2143] in the terminology of Woo and Yong [27]. However, as remarked by Steingrímsson [22], bar-avoiding 21 $\bar{3}$ 54 is equivalent to avoiding the vincular pattern  $\underline{2143}$ . See Theorem 7 for the details.

A further weakening is to only require that the local rings of  $X_\pi$  be Gorenstein local rings, in which case we say that  $X_\pi$  is a *Gorenstein* variety. Woo and Yong [26] showed that  $X_\pi$  is Gorenstein if and only if it avoids two patterns with two Bruhat restrictions each, as well as satisfying a certain condition on descents. We will translate their results into avoidance of bivincular patterns; see Theorem 19.

We also show how *mesh patterns*, introduced by Brändén and Claesson [7], subsume many other types of patterns, such as interval patterns defined by Woo and Yong [27], and define an extension of them called *marked mesh patterns*. We use these new patterns to further simplify the description of Gorenstein Schubert varieties (see Theorem 27) and give a new description of Schubert varieties that are *defined by inclusions*, introduced by Gasharov and Reiner [11] (see Theorem 26). We also give a description of 123-hexagon avoiding permutations, introduced by Billey and Warrington [3], in terms of the avoidance of 123 and one marked mesh pattern (see Proposition 28). Finally, in Example 20, we describe Dumont permutations [10] (of the first and second kind) and cycles with marked mesh patterns.

## 2. THREE TYPES OF PATTERN AVOIDANCE

Here we recall definitions of different types of patterns. We will use one-line notation for all permutations, *e.g.*, write  $\pi = 312$  for the permutation in  $S_3$  that satisfies  $\pi(1) = 3$ ,  $\pi(2) = 1$  and  $\pi(3) = 2$ .

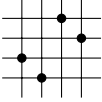
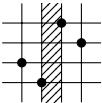
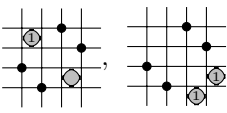
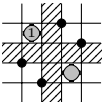
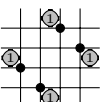
The three types correspond to:

- Bivincular patterns, subsuming vincular patterns and classical patterns.
- Barred patterns.
- Bruhat-restricted patterns.

**2.1. Bivincular patterns.** We denote the symmetric group on  $n$  letters by  $S_n$  and refer to its elements as *permutations*. We write permutations as words  $\pi = a_1a_2 \cdots a_n$ , where the letters are distinct and come from the set  $\{1, 2, \dots, n\}$ . A *pattern*  $p$  is also a permutation, but we are interested in when a pattern is *contained* in a permutation  $\pi$  as described below.

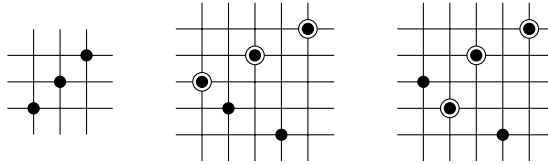
An *occurrence* (or *embedding*) of a pattern  $p$  in a permutation  $\pi$  is classically defined as a subsequence in  $\pi$ , of the same length as  $p$ , whose letters are in the same relative order (with respect to size) as those in  $p$ . For example, the pattern 123 corresponds to an increasing subsequence of three letters in a permutation. If we use the notation  $1_\pi$  to denote the first,  $2_\pi$  for the second and  $3_\pi$  for the third letter in an occurrence, then we are simply requiring that  $1_\pi < 2_\pi < 3_\pi$ . If a permutation has no occurrence of a pattern  $p$  we say that  $\pi$  *avoids*  $p$ .

TABLE 2. Connections between singularity properties and marked mesh patterns

$X_\pi$ is	The permutation $\pi$ avoids the patterns
smooth	 and 1324
factorial	 and 1324
defined by inclusions	 and 1324
Gorenstein	 and associated Grassmannians avoid two mesh pattern families
123-hexagon av.	

**Example 1.** The permutation 32415 contains two occurrences of the pattern 123 corresponding to the sub-words 345 and 245. It avoids the pattern 132.

The occurrence of a pattern in a permutation  $\pi$  can also be defined as a subset of the diagram  $G(p) = \{(i, \pi(i)) \mid 1 \leq i \leq n\}$ , that “looks like” the diagram of the pattern. Below is the diagram of the pattern 123 and two copies of the diagram of the permutation 32415 where we have indicated the two occurrences of the pattern by circling the dots.

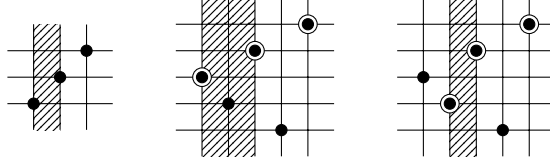


In a *vincular pattern* two adjacent letters may or may not be underlined. If they are underlined it means that the corresponding letters in the permutation  $\pi$  must be adjacent.

**Example 2.** The permutation 32415 contains one occurrence of the pattern 123 corresponding to the sub-word 245. It avoids the pattern 123. The permutation  $\pi = 324615$  has one occurrence of the pattern 2143, namely the sub-word 3265, but no occurrence of 2143, since 2 and 6 are not adjacent in  $\pi$ .

It is also convenient to consider vincular patterns as certain types of diagrams. We use dark vertical strips between dots that are required to be adjacent in the pattern. Notice that only the second occurrence of the classical pattern 123 satisfies the requirements of the vincular pattern, since in the former the dot corresponding

to 2 in  $\pi$  lies in the forbidden strip.

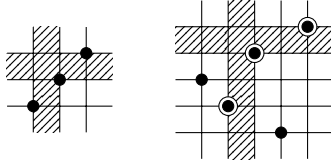


These types of patterns have been studied sporadically for a very long time but were not defined in full generality until Babson and Steingrímsson [1].

This notion was generalized further in Bousquet-Mélou et al [6]: In a *bivincular pattern* we are also allowed to place restrictions on the values that occur in an embedding of a pattern. We use two-line notation to describe these patterns. If there is a line over the letters  $i, i+1$  in the top row, it means that the corresponding letters in an occurrence must be adjacent in values. This is best described by an example:

**Example 3.** An occurrence of the pattern  $\frac{1\overline{23}}{123}$  in a permutation  $\pi$  is an increasing subsequence of three letters, such that the third one is larger than the second by exactly 1, or more simply,  $3_\pi = 2_\pi + 1$ . The permutation 32415 contains two occurrence of this bivincular pattern corresponding to the sub-words 345 and 245. The second one is also an occurrence of  $\frac{1\overline{23}}{\underline{123}}$ . The permutation avoids the bivincular pattern  $\frac{1\overline{23}}{\underline{123}}$ .

By also using horizontal strips we are able to draw diagrams of bivincular patterns. Below is the diagram of  $\frac{1\overline{23}}{\underline{123}}$  together with one occurrence of it.



We will also use the notation of [6] to write bivincular patterns: A bivincular pattern consists of a triple  $(p, X, Y)$  where  $p$  is a permutation in  $S_k$  and  $X, Y$  are subsets of  $\llbracket 0, k \rrbracket = \{0, \dots, k\}$ . With this notation an occurrence of a bivincular pattern in a permutation  $\pi = \pi_1 \dots \pi_n$  in  $S_n$  is a subsequence  $\pi_{i_1} \dots \pi_{i_k}$  such that the letters in the subsequence are in the same relative order as the letters of  $p$  and

- for all  $x$  in  $X$ ,  $i_{x+1} = i_x + 1$ ; and
- for all  $y$  in  $Y$ ,  $j_{y+1} = j_y + 1$ , where  $\{\pi_{i_1}, \dots, \pi_{i_k}\} = \{j_1, \dots, j_k\}$  and  $j_1 < j_2 < \dots < j_k$ .

By convention we put  $i_0 = 0 = j_0$  and  $i_{k+1} = n + 1 = j_{k+1}$ .

**Example 4.** We can translate all of the patterns we have discussed above into this notation:

$$\begin{aligned} 123 &= (123, \emptyset, \emptyset), & 132 &= (132, \emptyset, \emptyset), & \underline{123} &= (123, \{1\}, \emptyset), \\ \underline{123} &= (123, \{2\}, \emptyset), & 2143 &= (2143, \emptyset, \emptyset), & 2\underline{143} &= (2143, \{2\}, \emptyset), \\ \overline{123} &= (123, \emptyset, \{1\}), & \overline{123} &= (123, \emptyset, \{1, 2\}), & \overline{123} &= (123, \{2\}, \{1, 2\}). \end{aligned}$$

We have not considered the case when 0 or  $k$  are elements of  $X$  or  $Y$ , as we will not need those cases. We just remark that if  $0 \in X$  then an occurrence of  $(p, X, Y)$

must start at the beginning of a permutation  $\pi$ , in other words,  $\pi_{i_1} = \pi_1$ . The other cases are similar.

The bivincular patterns behave well with respect to the operations reverse, complement and inverse: Given a bivincular pattern  $(p, X, Y)$  we define

$$\begin{aligned}(p, X, Y)^r &= (p^r, k - X, Y), & (p, X, Y)^c &= (p^c, X, k - Y), \\ (p, X, Y)^i &= (p^i, Y, X),\end{aligned}$$

where  $p^r$  is the usual reverse of the permutation  $p$ ,  $p^c$  is the usual complement of the permutation  $p$ , and  $p^i$  is the usual inverse of the permutation  $p$ . Here  $k - M = \{k - m \mid m \in M\}$ . In [6] the following is proved.

**Lemma 5.** *Let  $a$  denote one of the operations above, or a composition of them. Then a permutation  $\pi$  avoids the bivincular pattern  $p$  if and only if the permutation  $\pi^a$  avoids the bivincular pattern  $p^a$ .*

**2.2. Barred patterns.** We will only consider a single pattern of this type, but the general definition is easily inferred from this special case. We say that a permutation  $\pi$  *avoids the barred pattern*  $21\bar{3}54$  if  $\pi$  avoids the pattern  $2143$  (corresponding to the unbarred elements) *except* where that pattern is a part of the pattern  $21354$ . This notation for barred patterns was introduced by West [24]. It turns out that avoiding this barred pattern is equivalent to avoiding the vincular pattern  $\underline{2143}$ ; see section 3. See also section 4 on how to write barred patterns as mesh patterns.

**Example 6.** *The permutation  $\pi = 4257613$  avoids the barred pattern  $21\bar{3}54$  since the unique occurrence of  $2143$ , as the sub-word  $4276$ , is contained in the sub-word  $42576$  which is an occurrence of  $21354$ . Note that it also avoids  $\underline{2143}$ .*

**2.3. Bruhat-restricted patterns.** We recall the definition of Bruhat-restricted patterns from Woo and Yong [26]. First we need the *Bruhat order* on permutations in  $S_n$ , defined as follows: Given integers  $i < j$  in  $\llbracket 1, n \rrbracket$  and a permutation  $\pi \in S_n$  we define  $\pi(i \leftrightarrow j)$  as the permutation that we get from  $\pi$  by swapping  $\pi(i)$  and  $\pi(j)$ . For example  $24153(1 \leftrightarrow 4) = 54123$ . We then say that  $\pi(i \leftrightarrow j)$  *covers*  $\pi$  if  $\pi(i) < \pi(j)$  and for every  $k$  with  $i < k < j$  we have either  $\pi(k) < \pi(i)$  or  $\pi(k) > \pi(j)$ . We then define the Bruhat order as the transitive closure of the above covering relation. This definition should be compared to the construction of the graph  $G_\pi$  in subsection 3.1. We see that in our example above that  $24153(1 \leftrightarrow 4)$  does not cover  $24153$  since we have  $\pi(2) = 4$ . Now, given a pattern  $p$  with a set of transpositions  $\mathcal{T} = \{(i_\ell \leftrightarrow j_\ell)\}$  we say that a permutation  $\pi$  *contains*  $(p, \mathcal{T})$ , or that  $\pi$  *contains the Bruhat-restricted pattern*  $p$  (if  $\mathcal{T}$  is understood from the context), if there is an embedding of  $p$  in  $\pi$  such that if any of the transpositions in  $\mathcal{T}$  are carried out on the embedding the resulting permutation covers  $\pi$ .

We should note that Bruhat-restricted patterns were further generalized to *intervals of patterns* in Woo and Yong [27]. We delay the discussion of this type of pattern avoidance until section 4, where we also introduce *mesh patterns* and show that an interval pattern is a special case of a mesh pattern.

In the next section we will show how these three types of patterns are related to one another.

### 3. CONNECTIONS BETWEEN THE THREE TYPES

**3.1. Factorial Schubert varieties and forest-like permutations.** Bousquet-Mélou and Butler [5] defined and studied *forest-like* permutations. Here we recall their definition: Given a permutation  $\pi$  in  $S_n$ , construct a graph  $G_\pi$  on the vertex set  $\{1, 2, \dots, n\}$  by joining  $i$  and  $j$  if

- (1)  $i < j$  and  $\pi(i) < \pi(j)$ ; and

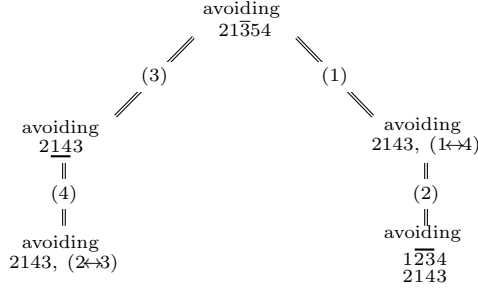


FIGURE 1. The barred pattern  $21\bar{3}54$  gives a connection between two bivincular patterns. The labels on the edges correspond to the enumerated list below

- (2) there is no  $k$  such that  $i < k < j$  and  $\pi(i) < \pi(k) < \pi(j)$ .

The permutation  $\pi$  is *forest-like* if the graph  $G_\pi$  is a forest. In light of the definition of Bruhat covering above we see that the vertices  $i$  and  $j$  are connected in the graph of  $G_\pi$  if and only if  $\pi(i \leftrightarrow j)$  covers  $\pi$ .

They then show that a permutation is forest-like if and only if it avoids the classical pattern 1324 and the barred pattern  $p_{\text{bar}} = 21\bar{3}54$ . This barred pattern can be described in terms of Bruhat-restricted embeddings and in terms of bivincular patterns, as we now show.

- (1) Bousquet-Mélou and Butler [5] remark that forest-like permutations  $\pi$  correspond to factorial Schubert varieties  $X_\pi$  and avoiding the barred pattern is equivalent to avoiding  $p_{\text{Br}} = 2143$  with Bruhat restriction  $(1 \leftrightarrow 4)$ . This last part is easily verified.
- (2) Avoiding  $p_{\text{Br}} = 2143$  with Bruhat restriction  $(1 \leftrightarrow 4)$  is equivalent to avoiding the bivincular pattern  $p_{\text{bi}} = \frac{1\bar{2}34}{2143}$ , as we will now show:

Assume  $\pi$  contains the bivincular pattern  $p_{\text{bi}}$ , so we can find an embedding of it in  $\pi$  such that  $3_\pi = 2_\pi + 1$ . This embedding clearly satisfies the Bruhat restriction.

Now assume that  $\pi$  has an embedding of  $p_{\text{Br}}$ . If  $3_\pi = 2_\pi + 1$  we are done. Otherwise  $2_\pi + 1$  is either to the right of  $3_\pi$  or to the left of  $2_\pi$  (because of the Bruhat restriction). In the first case change  $3_\pi$  to  $2_\pi + 1$  and we are done. In the second case replace  $2_\pi$  with  $2_\pi + 1$ , thus reducing the distance in values to  $3_\pi$ , then repeat.

- (3) The barred pattern  $p_{\text{bar}} = 21\bar{3}54$  has another connection to bivincular patterns: avoiding it is equivalent to avoiding the bivincular pattern  $q_{\text{bi}} = \frac{21\bar{4}3}{2143}$ , as remarked in the survey by Steingrímsson [22].
- (4) We can translate this into Bruhat-restricted embeddings as well: Avoiding the bivincular pattern  $q_{\text{bi}} = \frac{21\bar{4}3}{2143}$  is equivalent to avoiding  $q_{\text{Br}} = 2143$  with Bruhat restriction  $(2 \leftrightarrow 3)$ :

Assume  $\pi$  has an embedding of  $q_{\text{Br}}$ . If  $1_\pi$  and  $4_\pi$  are adjacent then we are done. Otherwise look at the letter to the right of  $1_\pi$ . If this letter is larger than  $4_w$  we can replace  $4_w$  by it and we are done. Otherwise this letter must be less than  $4_w$ , which implies by the Bruhat restriction, that it must also be less than  $1_w$ . In this case we replace  $1_w$  by this letter, and repeat.

Now assume  $\pi$  has an embedding of the bivincular pattern  $q_{\text{bi}}$ . If  $1_\pi$  and  $4_\pi$  are adjacent we are done. Otherwise look at the letter to the right of  $1_\pi$ . This letter is either smaller than  $1_\pi$  or larger than  $4_\pi$ . In the first case,

replace  $1_\pi$  with this letter; in the second case, replace  $4_\pi$  with this letter. Then repeat if necessary.

The above argument will be generalized in Proposition 14, but this special case gives us:

**Theorem 7.** *[5],[22] Let  $\pi \in S_n$ . The Schubert variety  $X_\pi$  is factorial if and only if  $\pi$  avoids the patterns  $\underline{2143}$  and  $1324$ .*

From the equivalence of the patterns in Figure 1 we also get that a permutation  $\pi$  avoids the bivincular pattern

$$\underline{2143} = (2143, \{2\}, \emptyset)$$

if and only if it avoids

$$\frac{1\overline{23}4}{2143} = (2143, \emptyset, \{2\}).$$

We will prove this without going through the barred pattern, and then generalize the proof, but first of all we should note that these bivincular patterns are inverses of one another, and that will simplify the proof.

Assume  $\pi$  contains  $\frac{1\overline{23}4}{2143}$ . If  $1_\pi$  and  $4_\pi$  are adjacent in  $\pi$  we are done. Otherwise consider the element immediately to the right of  $1_\pi$ . If this element is less than  $2_\pi$  then replace  $1_\pi$  by it and we will have reduced the distance between  $1_\pi$  and  $4_\pi$ . If this element is larger than  $2_\pi$  it must also be larger than  $3_\pi$ , since  $3_\pi = 2_\pi + 1$ , so replace  $4_\pi$  by it. This will (immediately, or after several steps) produce an occurrence of  $\underline{2143}$ .

Now assume  $\pi$  contains  $\underline{2143}$ . Then  $\pi^i$  contains the inverse pattern

$$(\underline{2143})^i = \frac{1\overline{23}4}{2143}.$$

Then by the above,  $\pi^i$  contains  $\underline{2143}$ , so  $\pi = (\pi^i)^i$  contains  $(\underline{2143})^i = \frac{1\overline{23}4}{2143}$ .

This generalizes to:

**Proposition 8.** *Let  $p$  be the pattern*

$$\cdots \underline{1k} \cdots = (\cdots 1k \cdots, \{j\}, \emptyset)$$

*in  $S_k$ , where  $j = p^i(1)$  is the index of 1 in  $p$ . A permutation  $\pi$  in  $S_n$  that avoids the pattern  $p$  must also avoid the bivincular pattern*

$$\frac{1\overline{2 \cdots \cdots k}}{\cdots 1k \cdots} = (\cdots 1k \cdots, \emptyset, \{2, 3, \dots, k-2\}).$$

*Proof.* Assume a permutation  $\pi$  contains the latter pattern in the proposition. If  $1_\pi$  and  $k_\pi$  are adjacent in  $\pi$  we are done. Otherwise consider the element immediately to the right of  $1_\pi$ . If this element is larger than  $(k-1)_\pi$  we replace  $k_\pi$  by it and are done. Otherwise this element must be less than  $(k-1)_\pi$  and therefore less than  $2_\pi$ , so we can replace  $1_\pi$  by it, and repeat.  $\square$

By applying the reverse to everything in Proposition 8 we get:

**Corollary 9.** *Let  $p$  be the pattern*

$$\cdots \underline{k1} \cdots = (\cdots k1 \cdots, \{j\}, \emptyset)$$

*in  $S_k$ , where  $j = p^i(k)$  is the index of  $k$  in  $p$ . A permutation  $\pi$  in  $S_n$  that avoids the pattern  $p$  must also avoid the bivincular pattern*

$$\frac{1\overline{2 \cdots \cdots k}}{\cdots k1 \cdots} = (\cdots k1 \cdots, \emptyset, \{2, 3, \dots, k-2\}).$$

By repeatedly applying the operations of inverse, reverse and complement we can generate six other corollaries. We will not need them here.

**Example 10.** *Let's look at some simple applications:*

- (1) Consider the bivincular pattern  $p_1 = 3\underline{1}42$ . Proposition 8 shows that a permutation  $\pi$  that avoids  $p_1$  must also avoid  $\frac{12\overline{3}4}{3142}$ . In fact, the converse can be shown to be true, by taking inverses and applying the proposition. We will say more about the pattern  $p_1$  in Example 11.
- (2) Consider the bivincular pattern  $p_2 = 3\underline{1}524$ . The proposition shows that a permutation  $\pi$  that avoids  $p_2$  must also avoid  $\frac{12\overline{3}45}{31524}$ . We will say more about the pattern  $p_2$  in subsection 3.2.

**Example 11.** *The Baxter permutations were originally defined and studied in relation to the “commuting function conjecture” of Dyer, see Baxter [2], and were enumerated by Chung, Graham, Hoggatt and Kleiman [8]. Gire [12] showed that these permutations can also be described as those avoiding the barred patterns  $41\overline{3}52$  and  $25\overline{1}34$ . It was then pointed out by Ouchterlony [20] that this is equivalent to avoiding the vincular patterns  $3\underline{1}42$  and  $24\underline{1}3$ .*

*Similarly to what we did above we can show that the Baxter permutations can also be characterized as those avoiding the bivincular patterns  $\frac{12\overline{3}4}{3142}$  and  $\frac{12\overline{3}4}{2413}$ , and this is essentially a translation of the description in [8] into bivincular patterns.*

Finally, here is an example that shows the converse of Proposition 8 is not true.

**Example 12.** *The permutation  $\pi = 423165$  avoids the pattern  $\frac{12\overline{3}45}{23154}$  but contains the pattern  $23\underline{1}54$ , as the sub-word  $23165$ .*

**3.2. Gorenstein Schubert varieties in terms of bivincular patterns.** Woo and Yong [26] classify those permutations  $\pi$  that correspond to Gorenstein Schubert varieties  $X_\pi$ . They do this using embeddings of patterns with Bruhat restrictions, which we have described above, and with a certain condition on the associated Grassmannian permutations of  $\pi$ , which we will describe presently:

First, a *descent* in a permutation  $\pi$  is an integer  $d$  such that  $\pi(d) > \pi(d+1)$ , or equivalently, the index of the first letter in an occurrence of the pattern  $\underline{21}$ . A *Grassmannian permutation* is a permutation with a unique descent. Given any permutation  $\pi$  we can associate a Grassmannian permutation to each of its descents as follows: Given a particular descent  $d$  of  $\pi$  we construct the sub-word  $\gamma_d(\pi)$  by concatenating the right-to-left minima of the segment strictly to the left of  $d+1$  with the left-to-right maxima of the segment strictly to the right of  $d$ . More intuitively we start with the descent  $\pi(d)\pi(d+1)$  and enlarge it to the left by adding decreasing elements without creating another descent and similarly enlarge it to the right by adding increasing elements without creating another descent. We then denote the *flattening* (or *standardization*) of  $\gamma_d(\pi)$  by  $\tilde{\gamma}_d(\pi)$ , which is the unique permutation whose letters are in the same relative order as  $\gamma_d(\pi)$ .

**Example 13.** *Consider the permutation  $\pi = 11|6|12|94153728|10$  where we have used the symbol  $|$  to separate two-digit numbers from other numbers. For the descent at  $d = 4$  we get  $\gamma_4(\pi) = 694578|10$  and  $\tilde{\gamma}_4(\pi) = 3612457$ .*

Now, given a Grassmannian permutation  $\rho$  in  $S_n$  with its unique descent at  $d$  we construct its *associated partition*  $\lambda(\rho)$  as the partition inside a bounding box  $d \times (n-d)$ , with  $d$  rows and  $n-d$  columns, whose lower border is the lattice path that starts at the lower left corner of the bounding box and whose  $i$ -th step, for



$i \in \llbracket 1, n \rrbracket$ , is vertical if  $i$  is weakly to the left of the position  $d$ , and horizontal otherwise. A corner of the lattice path is called an *inner corner* if it corresponds to a right turn on the path, otherwise it is called an *outer corner*. We are interested in the *inner corner distances* of this partition, that is, for every inner corner we add its distance from the left side and the distance from the top of the bounding box. If all these inner corner distances are the same then the inner corners all lie on the same anti-diagonal.

In Theorem 1 of Woo and Yong [26] they show that a permutation  $\pi \in S_n$  corresponds to a Gorenstein Schubert variety  $X_\pi$  if and only if

- (1) for each descent  $d$  of  $\pi$ ,  $\lambda(\tilde{\gamma}_d(\pi))$  has all of its inner corners on the same anti-diagonal; and
- (2) the permutation  $\pi$  avoids both 31524 and 24153 with Bruhat restrictions  $\{(1 \leftrightarrow 5), (2 \leftrightarrow 3)\}$  and  $\{(1 \leftrightarrow 5), (3 \leftrightarrow 4)\}$ , respectively.

Let's take a closer look at condition 2: Proposition 14 below shows that avoiding 31524 with Bruhat restrictions  $\{(1 \leftrightarrow 5), (2 \leftrightarrow 3)\}$  is equivalent to avoiding the bivincular pattern

$$\frac{12\overline{345}}{3\underline{15}24} = (31524, \{2\}, \{3\}).$$

Similarly, avoiding 24153 with Bruhat restrictions  $\{(1 \leftrightarrow 5), (3 \leftrightarrow 4)\}$  is equivalent to avoiding the bivincular pattern

$$\frac{12\overline{345}}{24\underline{15}3} = (24153, \{3\}, \{2\}).$$

**Proposition 14.** (1) Let  $p$  be the pattern

$$\cdots 1k \cdots$$

in  $S_k$ . Let  $j = p^i(1)$  be the index of 1 in  $p$ . A permutation  $\pi$  in  $S_n$  avoids  $p$  with Bruhat restriction  $(j \leftrightarrow j+1)$  if and only if  $\pi$  avoids the vincular pattern

$$\cdots \underline{1k} \cdots = (\cdots 1k \cdots, \{j\}, \emptyset).$$

- (2) Let  $\ell \in \llbracket 1, k-1 \rrbracket$  and  $p$  be the pattern

$$\ell \cdots (\ell+1)$$

in  $S_k$ . A permutation  $\pi$  in  $S_n$  avoids  $p$  with Bruhat restriction  $(1 \leftrightarrow k)$  if and only if  $\pi$  avoids the bivincular pattern

$$\frac{1 \cdots \overline{\ell \ell+1} \cdots k}{\ell \cdots \cdots \ell+1} = (\ell \cdots (\ell+1), \emptyset, \{\ell\}).$$

*Proof.* We consider each case separately.

- (1) Assume  $\pi$  contains the vincular pattern mentioned. Then it clearly also contains an embedding satisfying the Bruhat restriction.

Conversely assume  $\pi$  contains an embedding satisfying the Bruhat restriction. If  $1_\pi$  and  $k_\pi$  are adjacent then we are done. Otherwise look at the element immediately to the right of  $1_\pi$ . This element must be either larger than  $k_\pi$ , in which case we can replace  $k_\pi$  by it and are done, or smaller, in which case we replace  $1_\pi$  by it, and repeat.

- (2) Assume  $\pi$  contains the bivincular pattern mentioned. Then it clearly also contains an embedding satisfying the Bruhat restriction.

Conversely assume  $\pi$  contains an embedding satisfying the Bruhat restriction. If  $(\ell+1)_\pi = \ell_\pi + 1$  then we are done. Otherwise consider the element  $\ell_\pi + 1$ . It must either be to the right of  $(\ell+1)_\pi$  or to the left of  $\ell_\pi$ . In the first case we can replace  $(\ell+1)_\pi$  by  $\ell_\pi + 1$  and be done. In the second case replace  $\ell_\pi$  with  $\ell_\pi + 1$  and repeat.  $\square$

As a consequence we get:

**Corollary 15.** *A permutation  $\pi$  in  $S_n$  avoids*

$$\cdots 1k \cdots, (j \leftrightarrow j+1),$$

*where  $j$  is the index of 1, if and only if the inverse  $\pi^{-1}$  avoids*

$$j \cdots (j+1), (1 \leftrightarrow k).$$

Note that we could have proved the statement of the corollary without going through bivincular patterns and then used that to prove part 2 of Proposition 14, as part 2 is the inverse statement of the statement in part 1.

Translating condition 1 of Theorem 1 of Woo and Yong [26] into bivincular patterns is a bit more work. The failure of this condition is easily seen to be equivalent to some partition  $\lambda$  of an associated Grassmannian permutation  $\tilde{\gamma}_d(\pi)$  having an outer corner that is either “too wide” or “too deep”. More precisely, given a Grassmannian permutation  $\rho$  and an outer corner of  $\lambda(\rho)$ , we say that it is *too wide* if the distance upward from it to the next inner corner is smaller than the distance to the left from it to the next inner corner. Conversely we say that an outer corner is *too deep* if the distance upward from it to the next inner corner is larger than the distance to the left from it to the next inner corner. We say that an outer corner is *unbalanced* if it is either too wide or too deep. We say that an outer corner is *balanced* if it is not *unbalanced*.

If a permutation has an associated Grassmannian permutation with an outer corner that is too wide we say that the permutation itself is *too wide* and similarly for *too deep*. If the permutation is either too wide or too deep we say that it is *unbalanced*, otherwise it is *balanced*. It is time to see some examples.

**Example 16.** *See Figure 2 for drawings of the partitions below.*

- (1) *Consider the permutation  $\rho = 14235$  with a unique descent at  $d = 2$ . It corresponds to the partition  $(2) \subseteq 2 \times 3$  and has just one outer corner. This outer corner is too wide.*
- (2) *Consider the permutation  $\rho = 13425$  with a unique descent at  $d = 3$ . It corresponds to the partition  $(1, 1) \subseteq 3 \times 2$  and has just one outer corner. This outer corner is too deep.*
- (3) *Consider the permutation  $\rho = 134892567|10$  with a unique descent at  $d = 5$ . It corresponds to the partition  $(4, 4, 1, 1) \subseteq 5 \times 5$  and has two outer corners. The first outer corner is too deep and the second is too wide.*
- (4) *Consider the permutation  $\rho = 13672458$  with a unique descent at  $d = 4$ . It corresponds to the partition  $(3, 3, 1) \subseteq 4 \times 4$  and has two outer corners that are both balanced.*

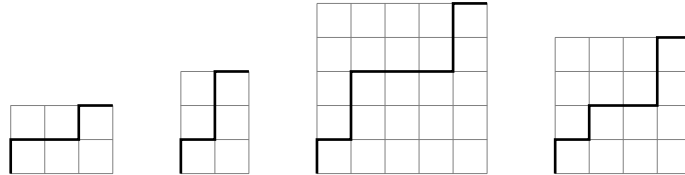


FIGURE 2. The associated partitions of the permutations in Example 16

We now show how these properties of Grassmannian permutations can be detected with bivincular patterns.

**Lemma 17.** *Let  $\rho$  be a Grassmannian permutation.*

- (1) *The permutation  $\rho$  is too wide if and only if it contains at least one of the bivincular patterns from the infinite family*

$$\mathcal{F} = \left\{ \overline{12345}, \overline{1234567}, \overline{123456789}, \dots \right\}.$$

*The general member of this family is of the form*

$$\overline{12 \cdots \cdots k}, \\ 1 \ell + 1 \cdots 2 \cdots \ell k,$$

*where  $\ell = (k + 1)/2$ , and  $k$  is odd.*

- (2) *The permutation  $\rho$  is too deep if and only if it contains at least one of the bivincular patterns from the infinite family*

$$\mathcal{G} = \left\{ \overline{12345}, \overline{1234567}, \overline{123456789}, \dots \right\}.$$

*The general member of this family is of the form*

$$\overline{12 \cdots \cdots k}, \\ 1 \ell + 1 \cdots 2 \cdots \ell k,$$

*where  $\ell = (k - 1)/2$ , and  $k$  is odd.*

Note that these two infinite families can be obtained from one another by reverse complement.

*Proof.* We only consider part 1, as part 2 is proved analogously. Assume that  $\rho$  is a Grassmannian permutation that is too wide, so it has an outer corner that is too wide. Let  $\ell$  be the distance from this outer corner to the next inner corner above. Then the distance from this outer corner to the next inner corner to the left is at least  $\ell + 1$ . This allows us to construct an increasing sequence  $t$  of length  $\ell$  in  $\rho$ , starting at a distance at least two to the right of the descent. We can also choose  $t$  so that every element in it is adjacent both in location and values. Similarly we can construct an increasing sequence  $s$  of length  $\ell$  in  $\rho$ , located strictly to the left of the descent. We can also choose  $s$  so that every element in it is adjacent both in location and values. This produces the required member of the family  $\mathcal{F}$ .

Conversely, assume  $\rho$  contains the  $i$ -th member of the family  $\mathcal{F}$ , the pattern

$$\overline{12 \cdots \cdots k}, \\ 1 \ell + 1 \cdots 2 \cdots \ell k,$$

where  $k = 2i + 3$ . Then the occurrence of the pattern corresponds to an outer corner in the partition of  $\rho$  of width  $\ell - 1$  and depth  $\ell - 2$ .  $\square$

We have now shown that:

**Proposition 18.** *A permutation  $\pi$  is balanced if and only if every associated Grassmannian permutation avoids every bivincular pattern in the two infinite families  $\mathcal{F}$  and  $\mathcal{G}$  in Lemma 17.*

This gives us:

**Theorem 19.** *Let  $\pi \in S_n$ . The Schubert variety  $X_\pi$  is Gorenstein if and only if*

- (1)  *$\pi$  is balanced; and*
- (2) *the permutation  $\pi$  avoids the bivincular patterns*

$$\overline{12345} \quad \text{and} \quad \overline{12345}, \\ \underline{31524} \quad \text{and} \quad \underline{24153}.$$

With these descriptions of factorial and Gorenstein varieties it is simple to show that smoothness implies factoriality, which implies Gorensteinness: If a variety is not factorial it contains either  $21\overline{4}3$  or  $1324$ , so it must contain either  $2143$  and  $1324$  and is therefore not smooth. If a variety is not Gorenstein then at least one of the following are true,

- (1)  $\pi$  has an associated Grassmannian permutation that contains one of the bivincular patterns in the infinite families  $\mathcal{F}$  and  $\mathcal{G}$  so it also contains  $1324$  and is therefore not factorial.
- (2)  $\pi$  contains  $\begin{smallmatrix} 12345 \\ 31524 \end{smallmatrix}$  or  $\begin{smallmatrix} 12345 \\ 24153 \end{smallmatrix}$ , so it also contains  $21\overline{4}3$  and is therefore not factorial.

#### 4. MESH PATTERNS AND MARKED MESH PATTERNS

**4.1. Mesh patterns.** Brändén and Claesson [7] introduced a new type of pattern called a mesh pattern and showed they generalize bivincular patterns and (most) barred patterns. Here we recall their definition: A *mesh pattern* is a pair  $(p, R)$  where  $p$  is a permutation of rank  $k$  and  $R$  is a subset of the square  $\llbracket 0, k \rrbracket \times \llbracket 0, k \rrbracket$ . An occurrence of that pattern in a permutation  $\pi$  is first of all an occurrence of  $p$  in  $\pi$  in the usual sense, that is, a subset of the diagram  $G(\pi) = \{(i, \pi(i)) \mid 1 \leq i \leq n\}$ . This occurrence must also satisfy the restrictions determined by  $R$ , that is, there are order-preserving injections  $\alpha, \beta : \llbracket 1, k \rrbracket \rightarrow \llbracket 1, n \rrbracket$  such that if  $(i, j) \in R$  then  $R_{ij} \cap G(\pi)$  is empty, where

$$R_{ij} = \llbracket \alpha(i) + 1, \alpha(i + 1) - 1 \rrbracket \times \llbracket \beta(j) + 1, \beta(j + 1) - 1 \rrbracket;$$

with  $\alpha(0) = 0 = \beta(0)$  and  $\alpha(k + 1) = n + 1 = \beta(k + 1)$ . It is best to unwind this formal definition with a few examples.

**Example 20.** (1) *The mesh pattern  $(21, \{(1, 0), (1, 1), (1, 2)\})$  can be depicted as follows:*

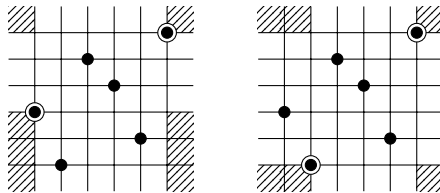


*An occurrence of this mesh pattern in a permutation is an inversion (an occurrence of the classical pattern 21) with the additional requirement that there is nothing in between the two elements in the occurrence. We usually refer to this pattern as the vincular pattern  $\underline{21}$ , that is, a descent.*

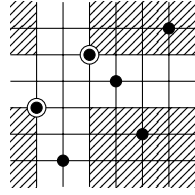
- (2) *Now consider the more complicated mesh pattern below.*



*There are two occurrences of this mesh pattern in the permutation  $\pi = 315426$ , shown below.*

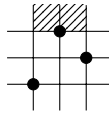


Every other occurrence of the classical pattern 12 in  $\pi$ , e.g.,



fails to satisfy the requirements given, since some of the shaded areas will have a dot in them.

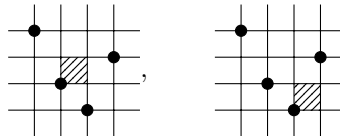
- (3) Dukes and Reifegerste [9] defined certified non-inversions as occurrences of 132 that are neither part of 1432 nor 1342. Equivalently these are occurrences of the mesh pattern below.



Brändén and Claesson [7] showed that a barred pattern with only one barred letter can be written as a mesh pattern<sup>1</sup>. The procedure is as follows: If  $\pi(i)$  is the only barred letter in a barred pattern  $\pi$  then the corresponding mesh pattern is  $(\pi', \{(i-1, \pi(i)-1)\})$  where  $\pi'$  is the standardization of  $\pi$  after the removal of  $\pi(i)$ . For example

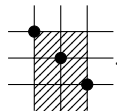
$$1\bar{2}3 = \begin{array}{|c|c|c|} \hline & & \\ \hline & \bullet & \\ \hline & & \\ \hline \end{array}.$$

More general barred patterns can be also be translated into mesh patterns as long as the barred letters are neither adjacent in locations nor in values. The procedure is essentially the same as above, so for example the barred pattern 6341 $\bar{2}$ 5 is contained in a permutation  $\pi$  if and only if at least one of the mesh patterns



is contained in  $\pi$ .

It is possible to classify simsun permutations by the avoidance of mesh patterns as follows: A permutation in  $S_n$  is *simsun* if it contains no double descent ( $\underline{321}$ ) in any of its restrictions to the interval  $[1, k]$  for some  $k \leq n$ . For example the permutation 452613 is not simsun since if we restrict it to  $[1, 4]$  it becomes 4213 which contains a double descent. It is now almost trivial to check that a permutation is simsun if and only if it avoids the mesh pattern



This was also noticed independently and simultaneously by Brändén and Claesson [7].

It is easy to see that bivincular patterns are special cases of mesh patterns. Adjacency conditions on positions in the bivincular pattern become vertical strips, while adjacency conditions on values become horizontal strips;  $R$  is then the union

<sup>1</sup>This had been noticed earlier as well, see unpublished work of Isaiah Lankham.

of all the strips. For example the bivincular pattern  $\frac{12\overline{34}5}{3\underline{15}24}$  from Theorem 19 corresponds to the mesh pattern



We have seen in Proposition 14 that some Bruhat-restricted patterns can be turned into bivincular patterns. We now show that any Bruhat-restricted pattern can be turned into a mesh pattern.

Given a pattern  $p$  with one Bruhat restriction  $(a \leftrightarrow b)$  first note that this means that  $a < b$  and  $p(a) < p(b)$ . Then a permutation contains  $p$  with the restriction if and only if it contains the mesh pattern  $(p, R)$ , where  $R$  consists of all the squares in the region with corners  $(a, \pi(a))$ ,  $(b, \pi(a))$ ,  $(b, \pi(b))$ ,  $(a, \pi(b))$ . For example the pattern  $31524$ ,  $(1 \leftrightarrow 5)$  corresponds to



Given a pattern  $p$  with multiple Bruhat restrictions we superpose the mesh patterns we get for each individual restriction. For example the pattern  $31524$ ,  $(1 \leftrightarrow 5)$ ,  $(3 \leftrightarrow 4)$ , which is one of the patterns that determines whether a permutation is Gorenstein or not, corresponds to



Recall that we had already shown (Proposition 14) that this Bruhat-restricted pattern corresponds to the bivincular pattern (1). It is easy to see directly that the mesh pattern (3) is equivalent to the bivincular pattern (1), in terms of being contained/avoided by a permutation.

It is now possible to translate Theorem 19 into mesh patterns, and completely get rid of the middle step of considering the Grassmannian subpermutations. But this was essentially done in Woo and Yong [27] using interval patterns, which we now show to be special cases of mesh patterns.

**4.2. Interval patterns.** Woo and Yong [27] defined the avoidance of *interval patterns* as a generalization of Bruhat-restricted patterns. We recall the definition here, with the modification that we reverse the usual Bruhat order on  $S_n$ . We do this so the definition can be directly compared with the definition of Bruhat-restricted avoidance. The (*reversed*) *Bruhat order* on  $S_n$  is the partial order defined by  $\rho < \pi$  if  $\pi$  can be obtained from  $\rho$  by composing with a transposition and  $\pi$  has more non-inversions than  $\rho$ . Recall that a *non-inversion* is an occurrence of the classical pattern 12; we let  $\ell(\pi)$  denote the number of non-inversions in  $\pi$ . Now we say that a permutation  $\pi$  *contains the interval*  $[p, q]$  if there exists a permutation  $\rho \leq \pi$  and a common embedding of  $p$  into  $\rho$  and  $q$  into  $\pi$  such that the entries outside of the embedding agree and the posets  $[p, q]$ ,  $[\rho, \pi]$  are isomorphic.

**Example 21.** The interval pattern  $[41523, 31524]$  corresponds to the Bruhat-restricted pattern  $31524, (1 \leftrightarrow 5)$ , shown as a mesh pattern above, (2); and  $[45123, 31524]$  to  $31524, (1 \leftrightarrow 5), (3 \leftrightarrow 4)$  also shown above, (3).

To show that any interval pattern can be turned into a mesh pattern we need a preliminary definition: Given a permutation  $\pi$  of rank  $n$  and integers  $j, k \in \llbracket 1, n+1 \rrbracket$ , we define a new permutation  $\pi \oplus_j k$  of rank  $n+1$  as follows:

$$(\pi \oplus_j k)(\ell) = \begin{cases} \pi(\ell) & \text{if } \ell < j \text{ and } \pi(\ell) < k, \\ \pi(\ell) + 1 & \text{if } \ell < j \text{ and } \pi(\ell) \geq k, \\ k & \text{if } \ell = j, \\ \pi(\ell - 1) & \text{if } \ell \geq j \text{ and } \pi(\ell) < k, \\ \pi(\ell - 1) + 1 & \text{if } \ell \geq j \text{ and } \pi(\ell) \geq k. \end{cases}$$

For example  $34125 \oplus_3 4 = 354126$ .

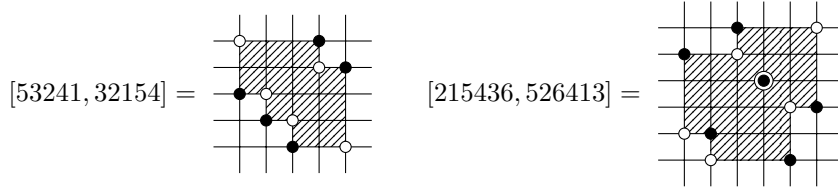
**Lemma 22.** A permutation  $\pi$  contains an interval pattern  $[p, q]$  if and only if it contains the mesh pattern  $(q, R)$  where  $R$  consists of boxes  $(i, j)$  such that

$$\ell(q) - \ell(p) \neq \ell(q \oplus_i j) - \ell(p \oplus_i j)$$

*Proof.* This lemma is a direct corollary of Lemma 2.1 in Woo and Yong [27] which states that an embedding  $\Phi$  of  $[p, q]$  into  $[\rho, \pi]$  is an interval pattern embedding if and only if  $\ell(q) - \ell(p) = \ell(\pi) - \ell(\rho)$ .  $\square$

It should be noted that although the general definition of a mesh pattern did not exist many authors had drawn diagrams analogous to the diagrams we have been drawing for mesh patterns, see e.g., [4], [5].

**Example 23.** To realize the interval  $[53241, 32154]$  as a mesh pattern we start by drawing  $53241$  with white dots and  $32154$  with black dots into the same diagram and consider the boxes  $(i, j)$  that satisfy the condition in the lemma above.



In Theorem 6.6 of Woo and Yong [27] they show that a permutation  $\pi \in S_n$  corresponds to a Gorenstein Schubert variety  $X_\pi$  if and only if  $\pi$  avoids intervals of the form

- (1)  $g_{a,b} = [(a+2) \cdots (a+b+2)1 \cdots a(a+1), 1(a+2) \cdots (a+b+1)2 \cdots a(a+1)(a+b+2)]$  for all integers  $a, b > 0$  such that  $a \neq b$ ,
- (2)  $h_{a,b} = [(a+4) \cdots (a+b+4)(a+2)(a+3)1 \cdots (a+1), (a+2)(a+4) \cdots (a+b+3)1(a+b+4)2 \cdots (a+1)(a+2)]$  for all integers  $a, b \geq 0$  such that either  $a > 0$  or  $b > 0$ .

See Figure 3 for some patterns appearing in these two lists.

**4.3. Marked mesh patterns, DBI- and Gorenstein varieties.** We introduce a generalization of mesh patterns which we call *marked mesh patterns* and use them to give an alternative description of Schubert varieties defined by inclusions, Gorenstein Schubert varieties, 123-hexagon avoiding permutations, Dumont permutations and cycles in permutations.

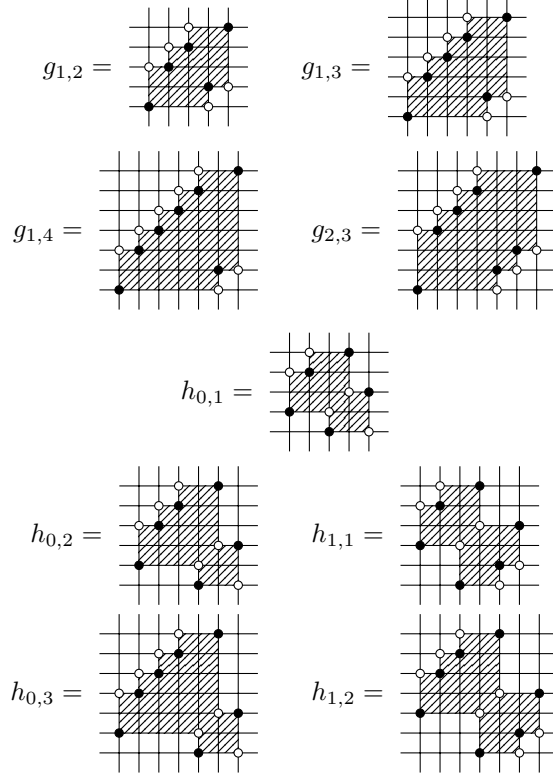


FIGURE 3. The first few patterns in Theorem 6.6 of Woo and Yong [27] shown as mesh patterns (patterns that are the reverse complement of a pattern that has already appeared are omitted)

**Definition 24.** A *marked mesh pattern*  $(p, \mathcal{C})$  of rank  $k$  consists of a classical pattern  $p$  of rank  $k$  and a collection  $\mathcal{C}$  which contains pairs  $(C, \square j)$  where  $C$  is a subset of the square  $\llbracket 0, k \rrbracket \times \llbracket 0, k \rrbracket$ ,  $j$  is a non-negative integer and  $\square$  is one of the symbols  $\leq, =, \geq$ . An occurrence of  $(p, \mathcal{C})$  in a permutation  $\pi$  is first of all an occurrence of  $p$  in  $\pi$  in the usual sense, that is, a subset of the diagram  $G(\pi) = \{(i, \pi(i)) \mid 1 \leq i \leq n\}$ . This occurrence must also satisfy the restrictions determined by  $\mathcal{C}$ , that is, there are order-preserving injections  $\alpha, \beta : \llbracket 1, k \rrbracket \rightarrow \llbracket 1, n \rrbracket$  such that for each pair  $(C, \square j)$  we have

$$\#(C' \cap G(\pi)) \square j,$$

where

$$C' = \bigcup_{(i,j) \in C} R_{ij}.$$

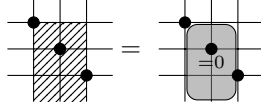
As above,  $R_{ij} = \llbracket \alpha(i) + 1, \alpha(i + 1) - 1 \rrbracket \times \llbracket \beta(j) + 1, \beta(j + 1) - 1 \rrbracket$ , with  $\alpha(0) = 0 = \beta(0)$  and  $\alpha(k + 1) = n + 1 = \beta(k + 1)$ .

Since regions of the form  $(C, \geq j)$  are the most common we also write them more simply as  $(C, j)$ .

**Example 25.** (1) Every mesh pattern  $(p, R)$  is an example of a marked mesh pattern, we just define  $\mathcal{C} = \{(R, =0)\}$ . For example, here is the mesh pattern that identifies the simsun permutations, written as a marked mesh



pattern.

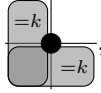


- (2) Consider the marked mesh pattern

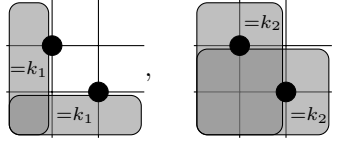


If a permutation  $\pi$  contains it, it has an occurrence of the classical pattern 12 where there is at least one element  $x$  in the permutation with the property that  $1_\pi < x < 2_\pi$ . This is equivalent to saying that  $\pi$  contains at least one of the classical patterns 213, 123, 132.

- (3) A fixed point of a permutation is an occurrence of the marked mesh pattern

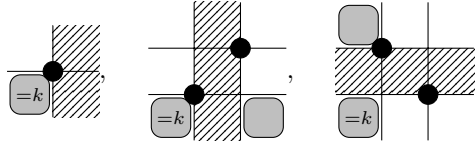


for some integer  $k \geq 0$ . This generalizes to occurrences of cycles in a permutation. For example, a 2-cycle is an occurrence the marked mesh pattern  $(21, C)$  where  $C$  consists of the four marked regions below

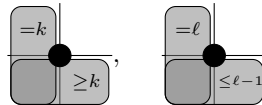


for some integers  $k_2 > k_1 \geq 0$ . These types of patterns can also be extended to include unions of cycles and thus subsume the patterns defined in McGovern [19].

- (4) Dumont permutations of the first kind [10] are permutations of even rank with the property that every even integer is followed by a smaller integer and every odd integer is either the last entry in the permutation or is followed by a larger integer. Therefore a permutation is a Dumont permutation of the first kind if and only if it avoids the marked mesh patterns

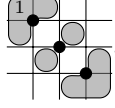


where  $k$  is an odd integer. Note that in the second pattern there is a single marked region  $(\{(0,0), (2,0)\}, =k)$ , consisting of two separated boxes. Similarly for the third pattern. Dumont permutations of the second kind are also defined in [10]. They can also be defined by the avoidance of the marked mesh patterns



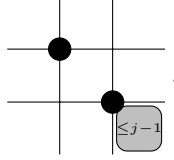
where  $k$  is an odd integer and  $\ell$  is an even integer.

- (5) *Green and Losonczy [13] defined freely braided permutations as those permutations avoiding the classical patterns 3421, 4231, 4312, 4321. Equivalently, these are the permutations avoiding the marked mesh pattern*



*marked with a single region consisting of  $(2, 0)$ ,  $(3, 0)$ ,  $(1, 1)$ ,  $(3, 1)$ ,  $(0, 2)$ ,  $(2, 2)$ ,  $(3, 0)$  and  $(3, 1)$ .*

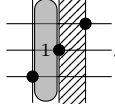
- (6) *Labelle, Leroux, Pergola and Pinzani [17] defined an inversion of the  $j$ -th kind in a permutation  $\pi$  to be a pair of elements  $\pi(s) > \pi(t)$ , with  $s < t$  and such that there do not exist  $j$  distinct indices  $t+1 \leq t_1, t_2, \dots, t_j \leq n$  such that  $\pi(t) > \pi(t_i)$  for  $i = 1, \dots, j$ . Alternatively, an inversion of the  $j$ -th kind is an occurrence of the marked mesh pattern*



- (7) *Kitaev, see e.g. [16], introduced partially ordered patterns (POP) as a generalization of vincular patterns. Some POPs can be written as marked mesh patterns, e.g., an occurrence of the POP 121 in a permutation means the occurrence of either 231 or 132. Therefore*

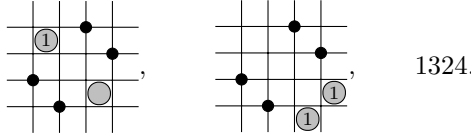
$$121 = \begin{array}{c} \bullet \\ | \\ \textcircled{1} \textcircled{1} \end{array}.$$

- (8) *Hou and Mansour [14] studied permutations avoiding  $1 \square \underline{23}$  which are permutations avoiding the marked mesh pattern*



Gasharov and Reiner [11] defined Schubert varieties *defined by inclusions* (or just *DBI*) and characterized them with pattern avoidance of the patterns 24153, 31524, 426153 and 1324. We now show how the first three of these patterns can be represented as two marked mesh patterns.

**Theorem 26.** *Let  $\pi \in S_n$ . The Schubert variety  $X_\pi$  is defined by inclusions if and only if the permutation  $\pi$  avoids the patterns*

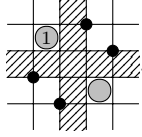


Where it should be noted that the first marked mesh pattern is marked with a single region,  $\{(1, 3), (3, 1)\}$ , consisting of two boxes, and the number of dots in this region is at least 1.

*Proof.* For the first marked mesh pattern note that  $2143 \oplus_2 4 = 24153$  and  $2143 \oplus_4 2 = 31524$ . For the second marked mesh pattern note that  $(2143 \oplus_4 1) \oplus_6 4 = 426153$ .  $\square$

We can also use these patterns to describe Gorenstein Schubert varieties:

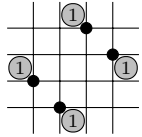
**Theorem 27.** *Let  $\pi \in S_n$ . The Schubert variety  $X_\pi$  is Gorenstein if and only if it is balanced and avoids the pattern*



*Proof.* Similar to the proof of Theorem 26.  $\square$

In [3] Billey and Warrington introduced 123-*hexagon avoiding* permutations as permutations avoiding the classical patterns 123, 53281764, 53218764, 43281765, 43218765.<sup>2</sup> We now show how these four patterns can then be combined into one marked mesh pattern.

**Proposition 28.** *A permutation  $\pi$  is 123-hexagon avoiding if and only if it avoids 123 and the marked mesh pattern*



*Proof.* The “if” part is easily verified. Now assume  $\pi$  contains the marked mesh pattern. Let  $x, y, z, w$  correspond, respectively, to elements in the marked regions, read clockwise and starting at the top. Let us assume that  $\pi^i(x) < \pi^i(z)$  and  $y < w$ , as the other cases are similar. Then  $\pi$  contains the pattern 53281764.  $\square$

Billey and Warrington also showed that 123-hexagon avoiding permutations can be characterized by the avoidance of 123 and the avoidance of a hexagon in the *heap* of the permutation. See [3].

Tenner [23] studied<sup>3</sup> permutations avoiding 123 and 2143 and showed that a permutation  $\pi$  avoids these two patterns if and only if it is *boolean* in the sense that the principal order ideal in strong Bruhat order  $B(\pi)$  is *Boolean* (that is, isomorphic to the Boolean poset  $B_r$  of subsets of  $[r]$  for some  $r$ ). So we immediately get that a Boolean permutation is 123-hexagon avoiding.

The author is working with a coauthor on determining which patterns control local complete intersections. The two marked mesh patterns that appear for Schubert varieties defined by inclusions appear in the description along with one other marked mesh pattern.

We end with a diagram in Figure 4 that shows which pattern definitions subsume which.

There are still other definitions of permutation patterns such as *grid patterns*, defined by Huczynska and Vatter [15]. I do not know where they fit into the hierarchy in Figure 4.

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<sup>2</sup>Actually they introduced 321-*hexagon avoiding* permutations as permutations avoiding the reverse of the patterns listed here. We consider the reversed definition to be compatible with what has appeared above.

<sup>3</sup>Actually, permutations that avoid the reverse of these patterns were considered.

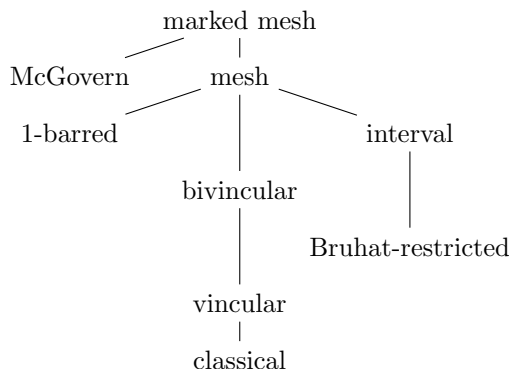


FIGURE 4. A diagram showing a hierarchy of permutation pattern definitions. 1-barred refers to barred patterns with one bar

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