Sárközy's theorem

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Introduction

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- As well as 2 elements which have a difference of 1 less than a prime number.
- The theorem was first conjured by Laszlo Lovasz but was proved by Sarkozy and Furstenberg in the late 70s.

Theorem

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Let $R_{sark}(N)$ be the maximum size of a set $1, 2, ..., N$ such that it has no perfect square differences. If $X \subseteq R_{sark}(N)$ and $\mid X \mid > R_{sark}(N)$ then, the subset X will contain 2 elements which differ by a square number.

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- Similarly, if $R_{sark}(N)$ has no 1 less than a prime differences then, X will contain 2 elements which differ by 1 less than a prime number.

Data analysis

Let us look at the example interval $[1, 10]$

Interval $[1, 10]$

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

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Greedy algorithm

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Proof

1, 3, 5, 7, 9 and 2, 4, 6, 8, 10

```
sark = function(vector)
{
vectorlen = length(vector)
bi = f2(veclen) #defines binomial coef.
count = 0 #demention of matrix
for (i in 2:veclen)
{ #create matrices
  vectordiff = seq(1:i)
  combo = combn(1:k1, i, simplify = TRUE)
  veccombo = as.vector(combo)
  diff = bi[k1-i+1] #create matrices
  for (n in 1:diff)
  {
    for(m in 1:i)
    { #go thru matrix
      count = count+1
      vecdiff[m] = veccombo[a]}
    if (squares(vecdiff) != 0){
      sarkn = squares(vecdiff)}}
  count = 0}
return(sarkn)} #return the number of subsets
```

Matrices

$$\left|\begin{array}{c|c|c|c|c|c|c|c|c|c} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 3 & 4 & 5 & 3 & 4 & 5 & 3 & 5 & 5 \end{array}\right|$$

$$\left|\begin{array}{c|c|c|c|c|c|c|c|c|c} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 & 3 & 4 & 3 & 3 & 4 & 4 \\ 3 & 4 & 5 & 4 & 5 & 5 & 4 & 5 & 5 & 5 \end{array}\right|$$

$$\left|\begin{array}{c|c|c|c|c} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 4 & 4 & 4 \\ 4 & 5 & 5 & 5 & 5 \end{array}\right|$$

$$\left|\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\right|$$

Figure of the Sark function

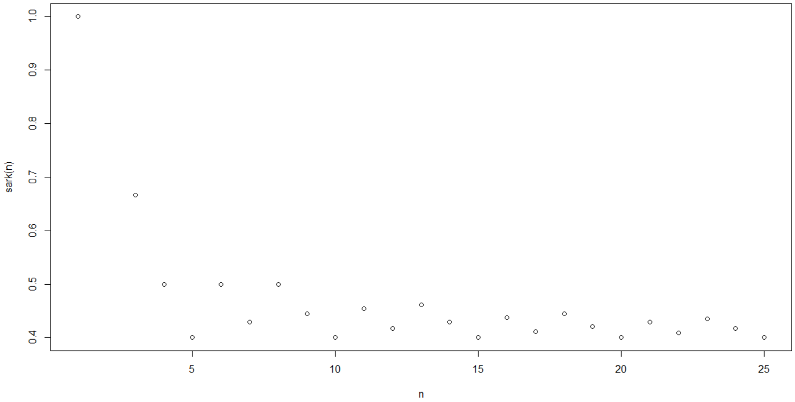

Figure: sark function

prime-1 squence

Definition of prime-1 seqence: $\mathbb{S}$ is a set of natural numbers with the property that no two numbers in $\mathbb{S}$ differ by prime number-1.

example

- Set U$\in$\{1,2,3...10\}

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- Bound 3$\leq$ size of prime-1 sequence$\leq$10
- In Furstenberg Sarkozy theorem,it states the natural density of square sequence is 0.
  Prime-1 sequence shares similar property.

similarity

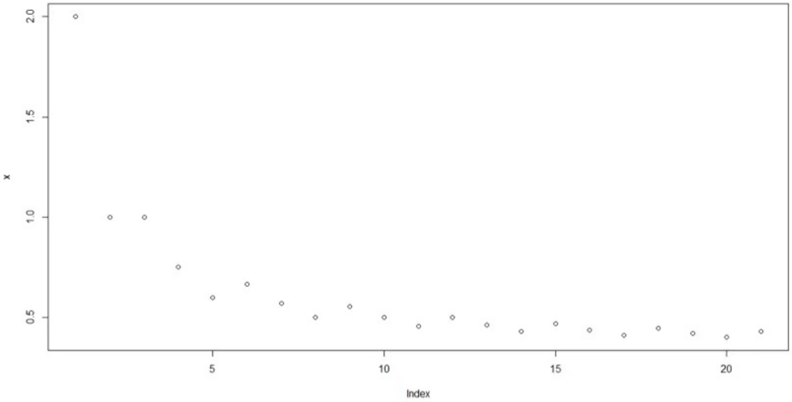

Figure: 1

Notation

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1. **The big O notation:**

$$f(n) = O(g(n)) \iff \exists k > 0 \exists n_0 \forall n > n_0 : |f(n)| \leq kg(n)$$

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When we write Ω_ϵ or O_ϵ, this implies that the constant k is dependent on ϵ.

Lower bound

What lower bound are we looking for?

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The smallest number m, dependent on the size of A, such that for any A, the largest square-difference-free subset of A is at least m.

$$A = \{1, 2, 3, ..., n\}$$

Lower Bound

Lower bound in the literature:

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$$O(n^{1/2} \log^k n),$$

Lower Bound

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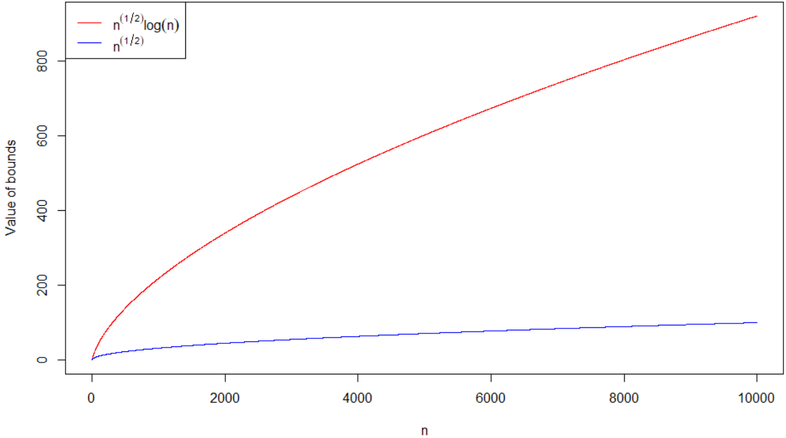
$$O(n^{1/2} \log^k n),$$

$$O_\epsilon(n^{1/2+\epsilon})$$

Lower Bound



$n^{(1/2)}\log(n)$
$n^{(1/2)}$
Value of bounds
0
200
400
600
800
0
2000
4000
6000
8000
10000
n

Lower Bound

Bounds created using Imre Z. Ruzsa construction:

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$$\Omega(n^{(1+\log_{65} 7)/2}) \approx n^{0.733077}$$

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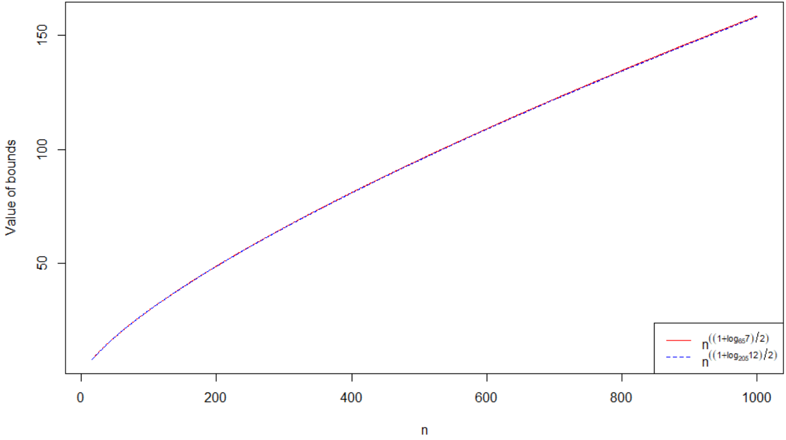
$$\Omega(n^{(1+\log_{205} 12)/2}) \approx n^{0.733412}$$

Lower Bound



Value of bounds
50
100
150
0
200
400
600
800
1000
n
$n^{((1+\log_{65}7)/2)}$
$n^{((1+\log_{205}12)/2)}$

Lower Bound

Conjectured lower bound:

Lower Bound

Conjectured lower bound:

$$\Omega_\epsilon(n^{1-\epsilon})$$

Lower Bound

Conjectured lower bound:

$$\Omega_\epsilon(n^{1-\epsilon})$$

$$n^{0.75}$$

Upper bound

The upper bound can be calculated using

$$O(n/\log(n)^{\frac{1}{4}\log(\log(\log(\log(n))))}) \tag{1}$$

- A majority of the proofs for this use Fourier analysis or Ergodic theory
- Not required to prove a basic form of the theorem, that all square-difference-free sets have zero density

Upper bound

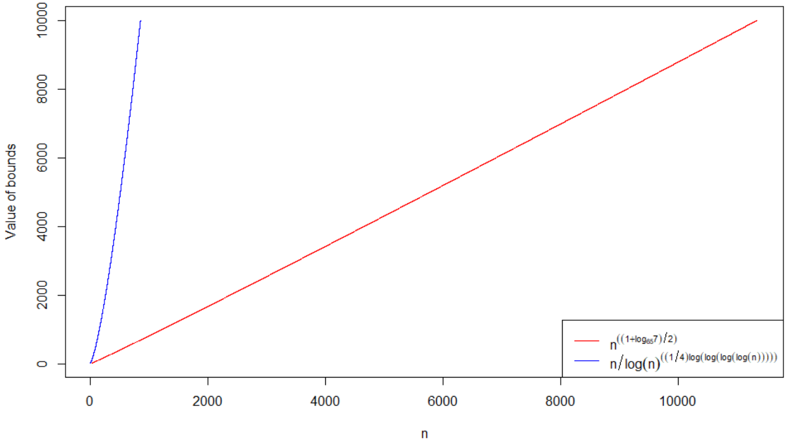


Figure: Comparison of upper and lower bound for n between 16 and 10,000. (Our first $n = 16$, because $\log(0)$ is not defined, so we have to choose n sufficiently large to be able to graph this functions.)

Bibliography

TEX, LATEX, and Beamer

- Donald E. Knuth. *The TEXbook*. Addison-Wesley, 1984.
- Leslie Lamport. *LATEX: A Document Preparation System*. Addison-Wesley, 1986.
- M. Goossens, F. Mittelbach, and A. Samarin. *The LATEX Companion*. Addison-Wesley, 1994.
- Till Tantau. *User's Guide to the Beamer Class Version 3.01*. Available at `http://latex-beamer.sourceforge.net`.
- A. Mertz and W. Slough. Edited by B. Beeton and K. Berry. *Beamer by example* In TUGboat, Vol. 26, No. 1., pp. 68-73.

Bibliography

- Eisner, Tanja; Farkas, Bálint; Haase, Markus; Nagel, Rainer (2015), "20.5 The Furstenberg–Sárközy Theorem", Operator Theoretic Aspects of Ergodic Theory, Graduate Texts in Mathematics, 272, Cham, Switzerland: Springer, pp. 455–457
- https://en.wikipedia.org/wiki/Big_O_notation
- https://en.wikipedia.org/wiki/Furstenberg%E2%80%93S%C3%A1rk%C3%B6zy_theorem
- https://github.com/delyan-kirov/Sarkozy-s-function.git
- https://terrytao.wordpress.com/2013/02/28/a-fourier-free-proof-of-the-furstenberg-sarkozy-theorem/
- https://arxiv.org/pdf/0804.4892.pdf

Bibliography

- https://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i1p32/pdf
- Nathanson, Melvyn B. (2000). Elementary Methods in Number Theory. Graduate Texts in Mathematics. 195. Springer-Verlag. ISBN 978-0387989129. Zbl 0953.11002.