UNIVERSITY OF OKLAHOMA
Honors College

THE DEVELOPMENT OF INTERACTIVE APPLICATIONS TO ASSIST WITH A LINEAR ALGEBRAIC CURRICULUM

Exploring Linear Independence, Gauss-Jordan Elimination, and a Change of Basis with GeoGebra

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Bachelor of Science in Mathematics

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Abstract

We design three interactive mathematical tools for the purpose of assisting in the communication of linear algebra topics, with particular emphasis on forging connections between spatial and symbolic understandings of those topics. More

specifically, tools are created using the *GeoGebra* environment for conveying ideas about linear independence, Gauss-Jordan elimination, and change of basis.

Introduction

The primary purpose of this paper is to discuss the development of interactive tools to assist in the teaching of undergraduate first-year linear algebra topics. Specifically, these tasks are designed to help students make linear algebraic connections under David Tall's "Three Worlds of Mathematics" framework, particularly the "embodied" and "symbolic" worlds. Three topics were selected: linear independence, Gauss-Jordan elimination, and change of basis; for each of these, desired outcomes were formulated and applications subsequently created for use with the *GeoGebra* online program.

Background

Interactive mathematical programs have seen regular use in classrooms since at least the late 1980s (Olive, 1992). One of the earliest mathematic visualization software environments that saw use educationally was the Geometric Supposer series, which allowed students to step-by-step generate shapes using a menuing system (Yerushalmy & Houde, 1986). By 1991, programs such as the *Geometer's Sketchpad* (Jackiw, 1991) allowed users to dynamically transform geometric representations through dragging on-screen figures; it and other programs with similar functionality are today known as Dynamic Geometry Software or Dynamic Geometry Environments (DGEs).

As well as the expected effectiveness for elementary geometric concepts, DGEs have also been found to be effective in assisting learning in a variety of areas, including abstract algebra (Hockman, 2005) and calculus (Hoffkamp, 2004). DGEs additionally have been shown to make students think about eigentheoretic problems in new ways (Tabaghi & Sinclair, 2013). Interactive applications in other formats have also been found to be beneficial teaching linear algebraic concepts (Dogan-Dunlap, Hall, & Paso, 2004).

While it is worth noting that some research indicates that interactive applications have little effect on learning outcomes when used in certain ways (Cavanaugh, Gillan Bosnick, Hess, & Scott, 2008; Petrov, Gyudzhenov, & Tuparova, 2015), the previously-mentioned research should make it clear that there are many situations where such applications can be useful.

Additionally, regarding linear algebra specifically, research has shown that students benefit from expanding their concept images of ideas. For example, Dogan-Dunlap finds that "geometric representations in the presence of algebraic and arithmetic modes appear to help learners begin to consider the diverse representational aspects of a concept flexibly" (Dogan-Dunlap, 2010). DGEs offer students the opportunity to probe conclusions drawn from their concept images in real-time, and in the case of DGEs like *GeoGebra* which also feature Computer Algebra Systems (CASs), the opportunity to connect algebraic ideas with geometric ideas.

Relevant Technologies and Frameworks

These applications were developed for use by instructors at the University of Oklahoma as part of their research into teaching linear algebra using Tall's "Three Worlds of Mathematics" framework (see Stewart et al. 2019a and 2019b). Specifically, we hoped to facilitate inter-world thinking between the "embodied" and "symbolic" worlds, which for the purposes of this paper roughly comprise geometric and symbolic representations of concepts, respectively (Tall 2013).

The specific DGE we chose for our interactive applications was *GeoGebra*. *GeoGebra* offers several advantages over competing software: it is an in-browser application, so it will run on any computer or phone with sufficient web technology support regardless of architecture; it runs without extra software, so students are not required to download anything on top of the webpages themselves; and its programs are modifiable, so that the technology can be released to the public to build upon or change.

Previous Work

Multiple authors have created interactive applications to assist in the learning of linear algebraic concepts, including those covered. James D. Factor of Alverno College has presented multiple times on creating such applications. Some of the following applications were designed to mimic functionality in either those applications available from his page at https://www.geogebra.org/u/james+factor#materials/created, or those mentioned in previous presentations of his (Factor 2014), although some of the versions in question have since been replaced with newer work.

His work, however, was not explicitly suitable for our needs for a couple of interrelated reasons. Factor's work is not released to the public for editing, so any additional changes we wished to make necessitated that we develop our own software; additionally, it had more information than desired in one instance. In a second, it was in three dimensions when we instead wanted it to be in two dimensions. In a third, we wished to add functionality to it that did not yet exist.

Other work that we drew upon when developing our applications was the Inquiry-Oriented Linear Algebra (IOLA) group's student and instructor materials (Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013), which we aimed to make our software compatible with.

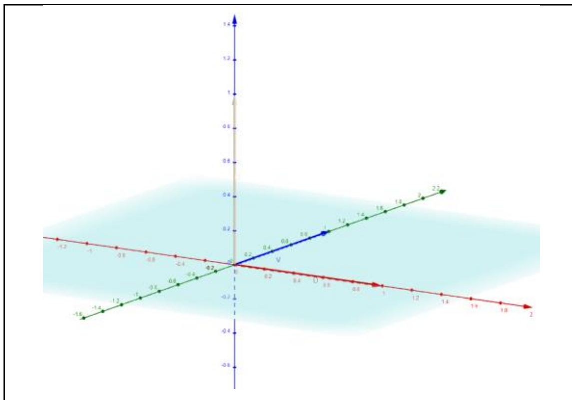
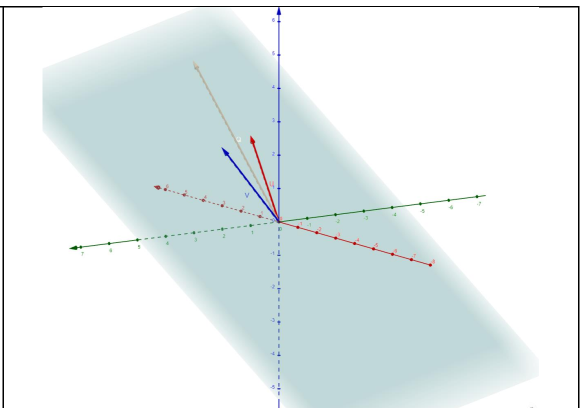
Application: Linear Independence

Mathematical Theory

Linear independence and dependence play an important role in linear algebra. Namely, a set of vectors is considered *linearly independent* if no element of such set can be formed through a linear combination of the other elements of the set; such a set that does not satisfy this property is said to be *linearly dependent*.

Formally, this has that a set of vectors $S = \{v_1, v_2 \dots v_n\}$ has the property that, for all $k \in \mathbb{N}, 1 \leq k \leq n$, the situation $x_1 v_1 + x_2 v_2 + \cdots + x_{k-1} v_{k-1} + x_{k+1} v_{k+1} + \cdots + x_n v_n$ is not satisfied for any collection of xs.

Geometrically, this corresponds to whether you can, from the origin, move, including negatively, along the directions some of the vectors point to reach some point p that can be reached by going directly from the origin along some other, unused vector's direction. More pointedly, this is precisely the situation when a "linear combination" of some of the vectors (moving some distance along each of their directions, i.e. moving along a scaled version of each of them) allows you to reach an unrelated vector.

	
The tan vector, (0, 0, 1), is linearly independent of the red and blue vectors, (1, 0, 0) and (0, 1, 0), as no linear combination $a \cdot (1,0,0) + b \cdot (0,1,0)$ (that is, any combination on the plane shaded in blue) describes the tan vector.	The tan vector, (5, -0.3, 4), is linearly dependent on both of the red and blue vectors, (3, -1, 2) and (2, 0.7, 2), as the linear combination $1 \cdot (3, -1,\ 2) + 1 \cdot (2, 0.7, 2)$ describes the tan vector.

Application Overview

Upon opening the application, the user is presented with three screens. On the upper-left one, checkboxes allow users to turn each of 6 vector displays on and off, and sliders to the right allow them to manually scale three of the vectors (henceforth U, V, and Q). The bottom-left screen shows a 2D representation of U, V, and the plane that they span, as well as any of the generated vectors on that plane. The right-hand subscreen displays a 3D representation of all vectors toggled to be visible.

Pedagogical Decisions

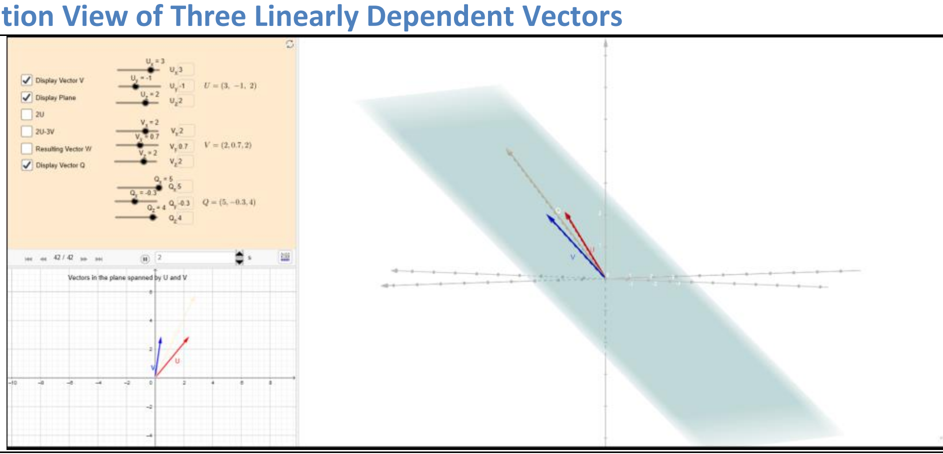
The purpose of this application was to have students realize that a third vector Q is linearly dependent to vectors U and V if and only if it lies on the plane spanned by them. To that end, we devoted an entire subsection of the screen to displaying the plane spanned by U and V. The vector Q only appears on it if it's linearly dependent on U and V (and, consequently, visually intersecting the plane spanned by U and V on the 3D subscreen).

A deliberate choice was made to not rescale the spanned plane to be in terms of U and V as basis vectors, as displaying them instead in terms of the standard basis (0, 1) and (1, 0) means that when students rescale the elements in the upper-left quadrant using the sliders, the vectors scale accordingly on *both* the 3D representation and the 2D spanned plane representation. This was done based on informal feedback that rescaling the vectors made people "feel less like all three representations of the vectors were the same".

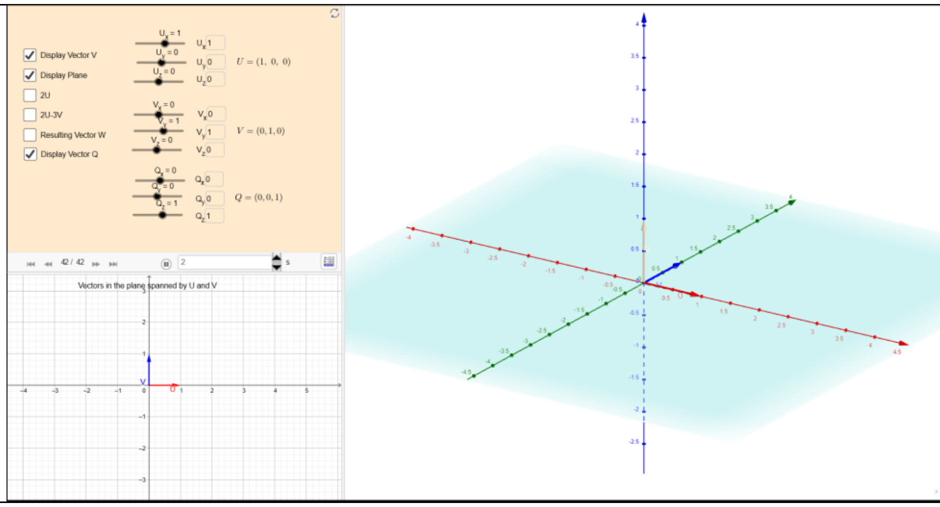
A checkbox was provided for a default vector W that represents the sum of scaling U by two and linearly combining it with V scaled by -3; this was done in order to give users an easy visual representation of the effect of linearly combining two vectors, and to show them that such a vector would show up on the plane spanned by U and V (that is, the lower-left quadrant).

Application Images

Application View of Three Linearly Dependent Vectors

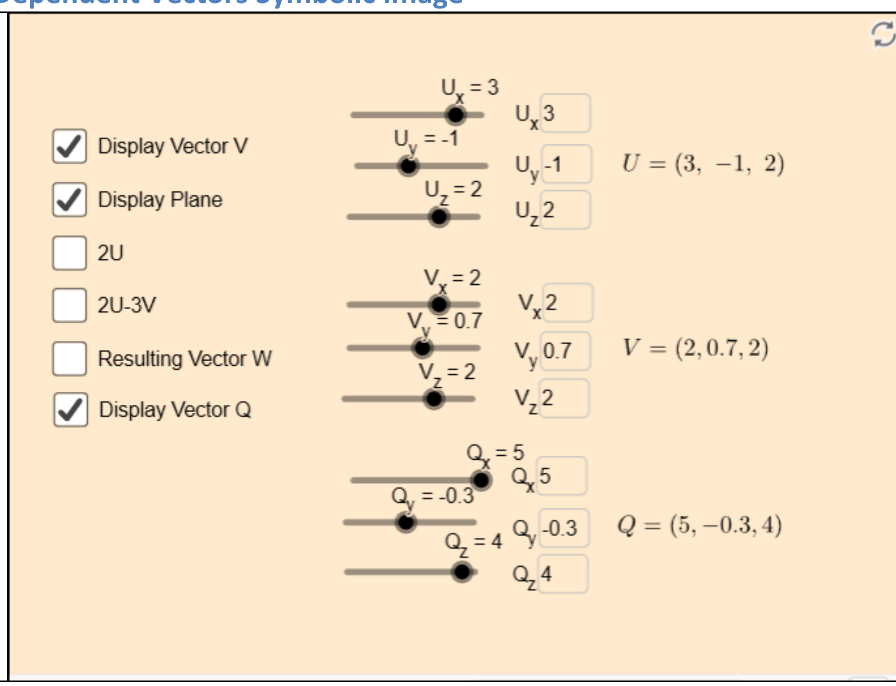


Application View of Three LD vectors
(U=(3 -1 2), V= (2 0.7 2), Q = (5 -0.3 4))



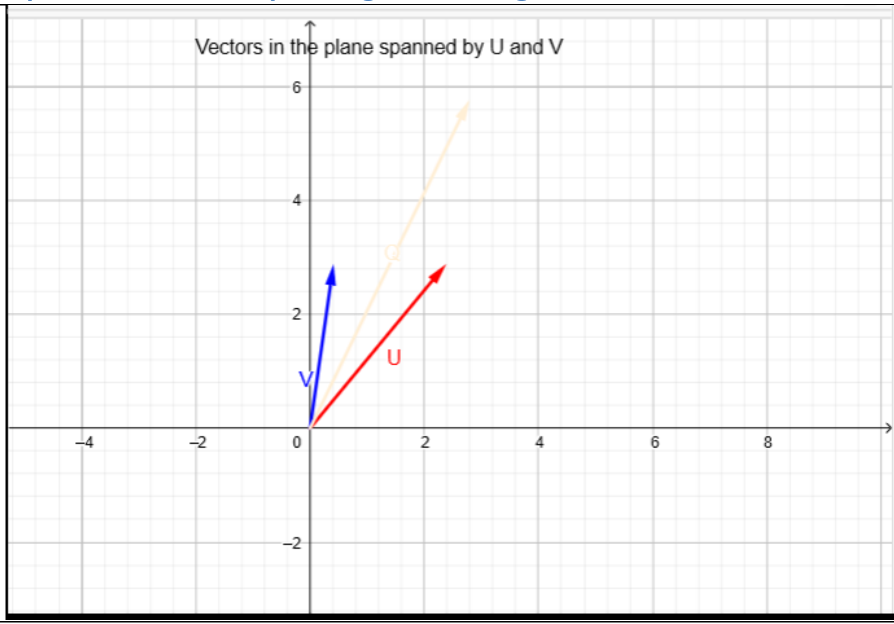
Application View of Three LI vectors
(U=(1 0 0), V= (0 1 0), Q = (0 0 1))

Linearly Dependent Vectors Symbolic Image



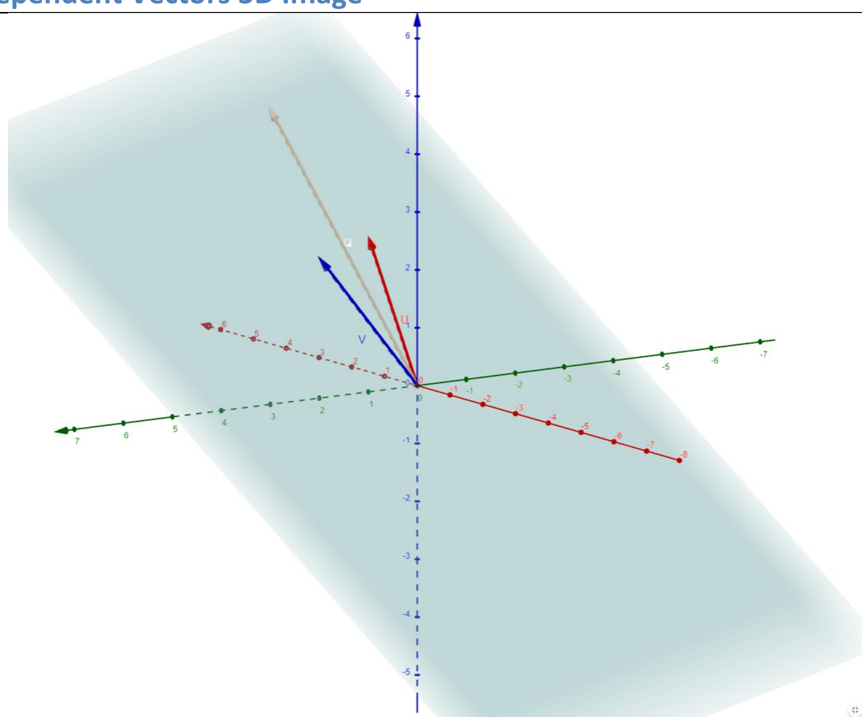
Symbolic View of Three LD Vectors (U=(3 -1 2), V= (2 0.7 2), Q = (5 -0.3 4))

Linearly Dependent Vectors Spanning Plane Image



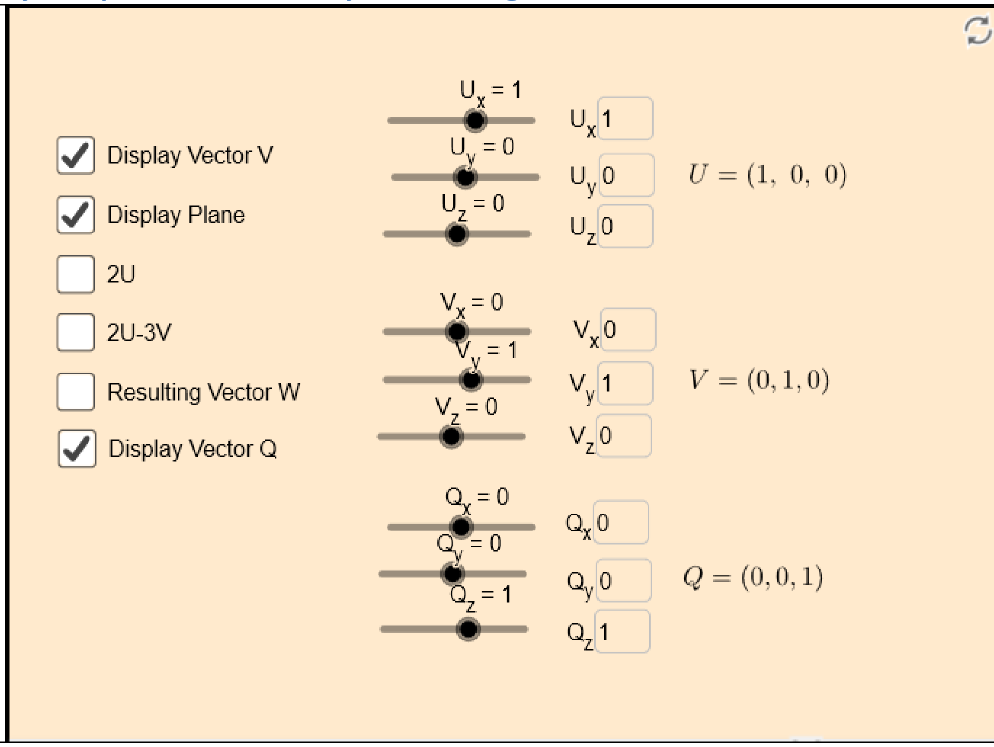
2D View of Three LD Vectors from Plane Spanned by U and V
(U=(3 -1 2), V= (2 0.7 2), Q = (5 -0.3 4))

Linearly Dependent Vectors 3D Image



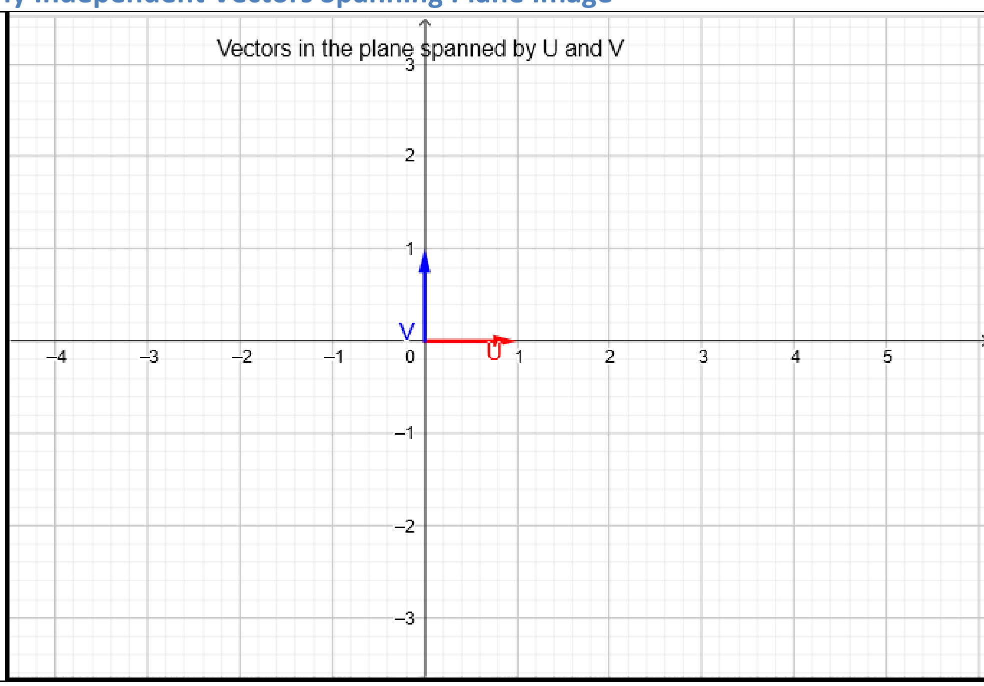
3D View of Three LD Vectors (U=(3 -1 2), V= (2 0.7 2), Q = (5 -0.3 4))

Linearly Independent Vectors Symbolic Image



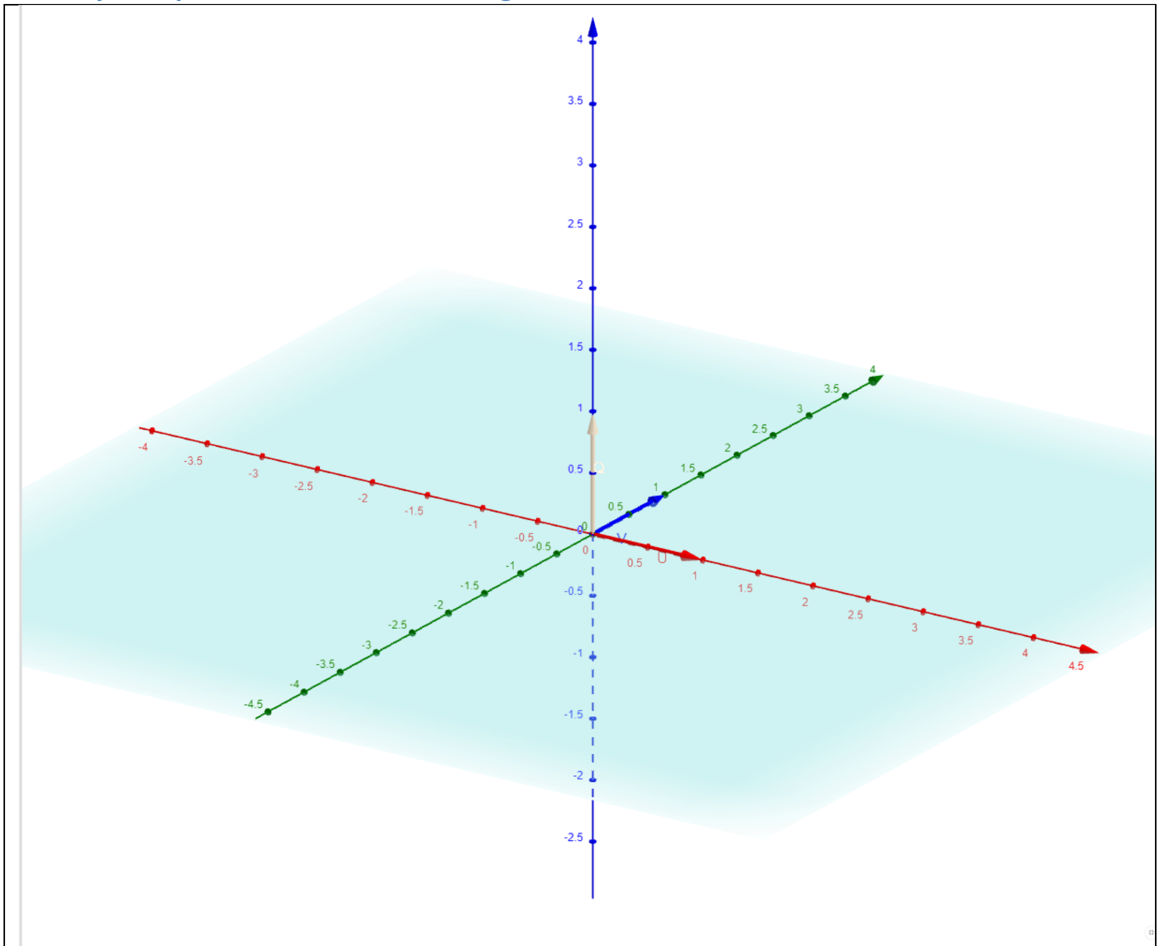
Symbolic View of Three LI Vectors (U=(1 0 0), V= (0 1 0), Q = (0 0 1))

Linearly Independent Vectors Spanning Plane Image



2D View of Three LI Vectors from Plane Spanned by V and U
(U=(1 0 0), V= (0 1 0), Q = (0 0 1))

Linearly Independent Vectors 3D-Image



3D View of Three LI Vectors (U=(1 0 0), V= (0 1 0), Q = (0 0 1))

Application: Gauss-Jordan Elimination

Mathematical Theory

Gaussian/Gauss-Jordan elimination is the process of using elementary row operations to put a matrix into row-echelon (Gaussian Elimination) or reduced row-echelon (Gauss-Jordan elimination) forms. The two terms are sometimes used interchangeably, as once the process of Gaussian elimination is established, the remaining steps to reach reduced row-echelon form are rather trivial.

More concretely, Gaussian elimination, given a matrix/system of equations (every system of linear equations can be represented as a matrix, so we will use the two terms somewhat interchangeably here), transforms it into an upper triangular matrix, meaning that the leftmost nonzero element of each row is, column-wise, strictly to the left of the column of the leftmost nonzero element of any of the lower rows. Additionally, all all-zero rows are below any non-all-zero rows (or, equivalently, one can define the column of the leftmost nonzero element for all-zero rows to be $1 + the\_number\_of\_columns$, in which case the previous property is sufficient). Gauss-Jordan elimination takes it a step further, requiring that all elements to the right of each leftmost nonzero element be zero.

The one caveat to the above description is that Gauss-Jordan elimination is done entirely with what are called *elementary row operations*. Specifically, these include

- Swapping one row with another row
- Scaling one row by a nonzero number
- Adding one row to another (in combination with the previous operation, this is, in effect, adding the multiple of one row to another)

By using these elementary row operations to perform Gauss-Jordan elimination, many useful effects can be achieved, such as finding a matrix's determinant, inverse (assuming the matrix is invertible), or rank.

Representing the equations as matrices, we can also plot the original equations and gain intuition as to what performing Gaussian and Gauss-Jordan elimination means graphically.

Application Overview

When the program is launched, users see the screen segmented into two parts: on the left are symbolic representations of the equations/matrices and options for performing elementary row operations, while on the right are graphical representations of those equations. An "edit equations directly" button also allows users to manually input values for the matrix entries, and two "preset" buttons at the bottom of the application let users generate a pair of consistent examples.

Pedagogical Goals

Row-swapping and row-scaling do nothing to the resulting graphs, so in actuality, the graphical representation of matrices represents the impact of adding one row/multiples of one row to another.

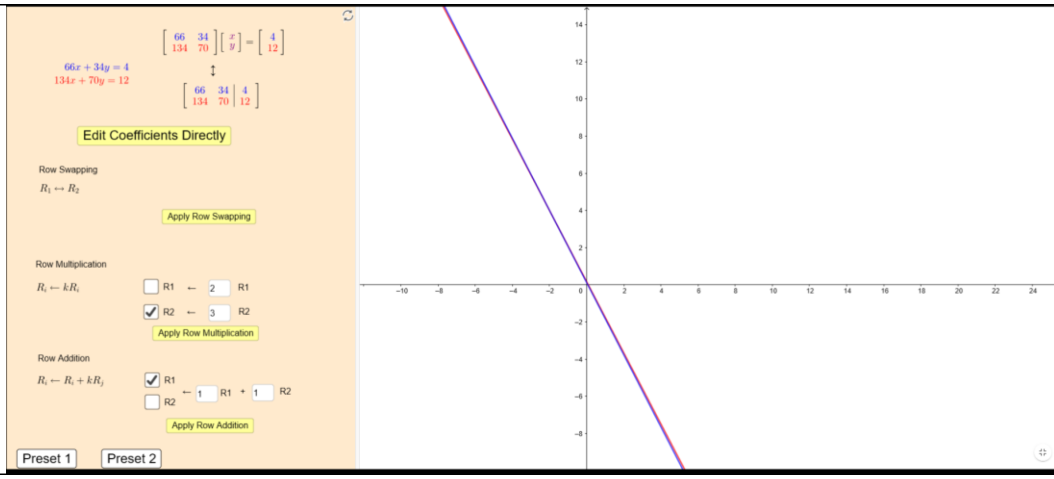
Specifically, performing Gaussian elimination puts the plane represented by each equation in parallel with more and more of the basis vectors. The final steps of Gauss-Jordan elimination go back up the line of equations and put each plane in parallel with all of the basis vectors but one (unique to each equation), with which the representative plane is perpendicular. In short, Gauss-Jordan elimination reorients the planes so that each plane is perpendicular to a unique basis vector while preserving the planes' points of intersection (that is, the equations' solutions).

Additional pedagogical goals included bolstering students' understanding that augmented matrices can represent systems of equations. To assist with this, the color of each row's symbolic representation on the left corresponds with the color of the drawn plane on the right, and the equation is represented as a raw system of equations, as a matrix equation, and as an augmented matrix.

We also attempt to keep the interface as simple as possible, as, when considering some other interactive Gauss-Jordan elimination examples, extraneous cognitive load when determining how to use the application was an issue. To that end, an early decision was made to restrict the application to two dimensions, as well.

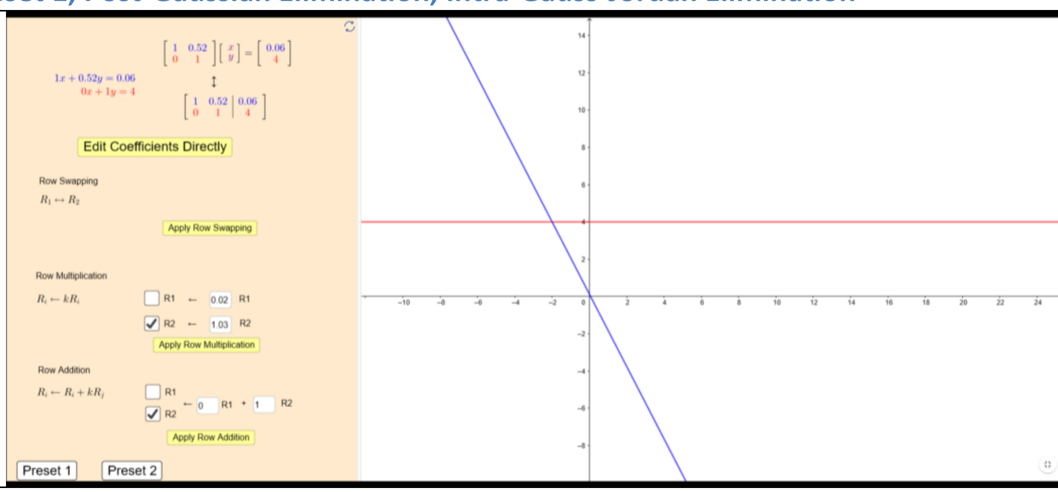
Application Pictures

Preset 1, Pre-Gaussian Elimination



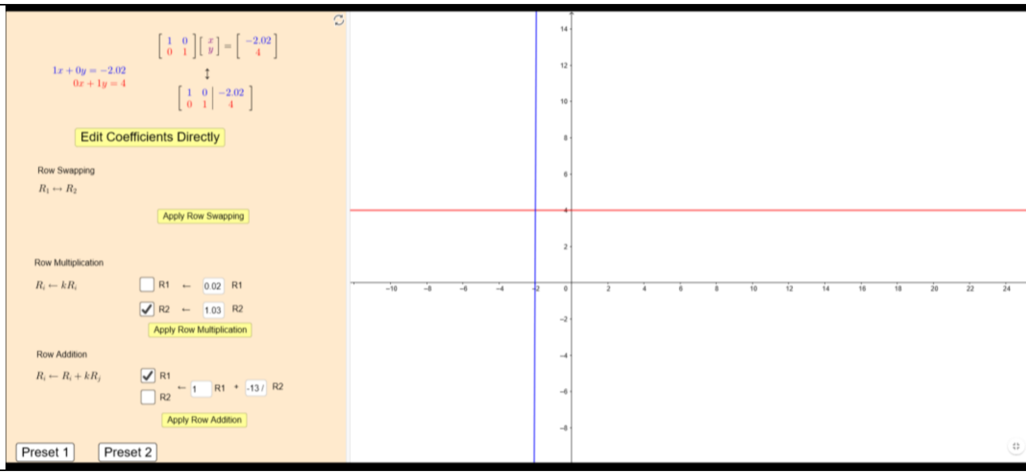
Preset 1, Pre Gaussian Elimination

Preset 1, Post-Gaussian Elimination, Intra-Gauss-Jordan Elimination



Preset 1, Post-Gaussian Elimination, Intra-Gauss-Jordan Elimination

Preset 1, Post-Gauss-Jordan Elimination



Preset 1, Post-Gauss-Jordan Elimination

Application: Change of Basis

Mathematical Theory

A *basis* in linear algebra for a vector space is a set of linearly independent vectors whose span contains that vector space. Vectors within that vector space can therefore be represented in terms of those basis vectors. Changes of basis are incredibly useful in all sorts of fields, so a deep conceptual understanding is desirable.

Application Overview

The application itself is split into two sections: a symbolic view and a graphical view. The symbolic view displays the change of basis as a set of basis vectors.

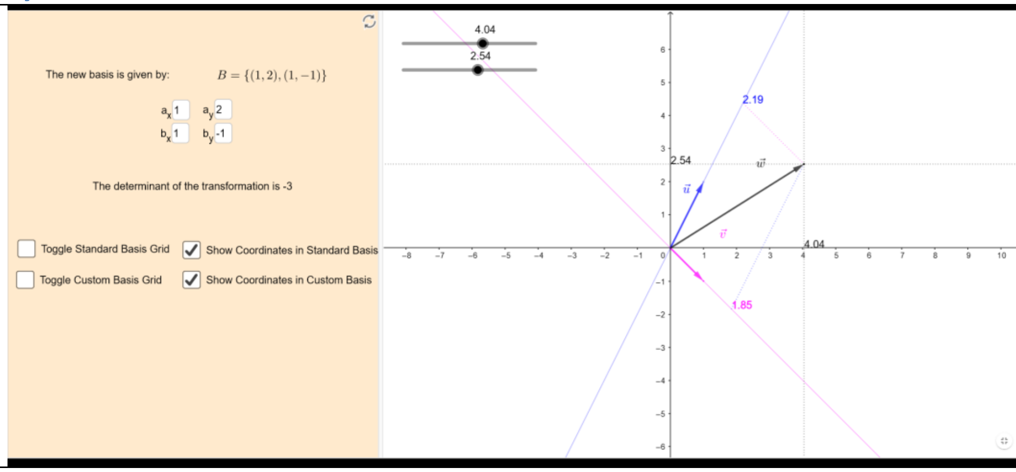
Pedagogical Goals

Geometrically, a change of basis merely involves changing the constituent vectors from which to construct a chosen vector. In short, from a learner's perspective, it merely involves changing the gridlines. As such, we allow for users to toggle the gridlines for the standard basis and the user-specified basis. The gridlines for the user-specified basis are color-coded on the comprising vectors, as well.

As an additional goal, we include the determinant of the basis, with the hope that the users will recognize that the sign on the determinant depends on the orientation of their new basis vectors with respect to the standard basis.

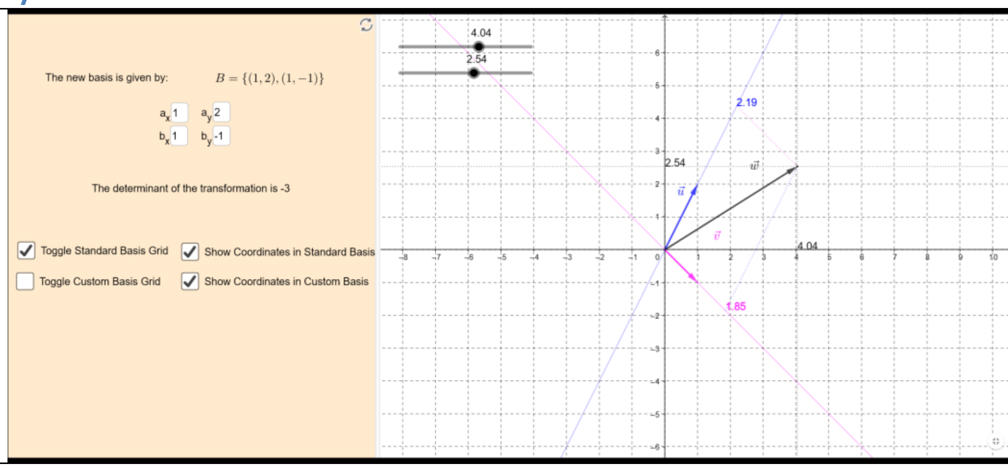
Application Pictures

Display without Gridlines



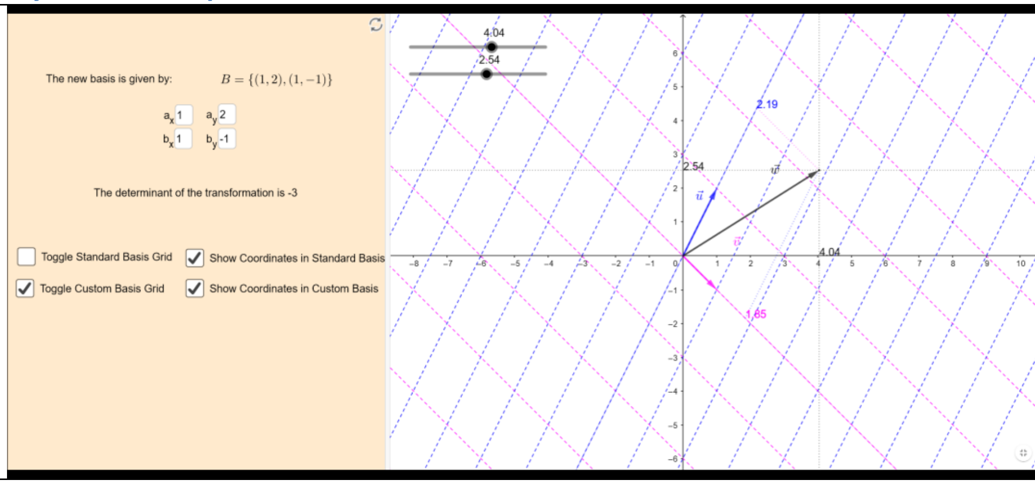
Default Vector Display, with Coefficients for the Standard Basis and for the User-Specified Basis

Display with Standard Basis Gridlines



Vector Display with Standard Basis Gridlines

Display with User-Specified Basis Gridlines



Vector Display with User-Specified Basis Gridlines

Future Work

For future work, we plan on implementing more presets into the applications. Additionally, GeoGebra is a bit slow for some of the applications; consequently, we would like to improve the performance of the interactives by writing them in JavaScript (which will be harder for most people to make changes to, but the performance gains make it worth considering). Finally, we would like to include animated, worked-out preset problems with audio instruction, so that users have more guidance when using the applications.

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