Calc 1.5 Notes

David DeMark

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Chapter 1

Euler's Formula & Limits I

1.0 Euler's Formula

Goal. Use Euler's Formula (Theorem 1.0.3) to derive any number of trig identities.

Note that this isn't actually part of the standard pre-calc curriculum, but it is an immensely powerful tool which simplifies down quite a bit of pre-calc—frankly, it's almost offensive that this isn't taught during pre-calc. We may be able to prove Theorem 1.0.3 using Taylor series some time in August, but for now we'll need to just trust Euler on it.

We'll be using the following fact over and over throughout this section:

Fact 1.0.1. For any (real, complex, or otherwise!) a, b, c, we have that $a^b * a^c = a^{b+c}$.

Definition 1.0.2. Recall the imaginary number $i := \sqrt{-1}$.

Theorem 1.0.3. *For any real number θ,*

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Caution 1.0.3.1. This is only true in radians! And is a compelling reason never to use degrees for much of anything...

Fact 1.0.4. We let $w = a + ib$ and $z = c + id$ be complex numbers (that is $w, z \in \mathbb{C}$) with a, b, c, d all real numbers (that is, $a, b, c, d \in \mathbb{R}$). Then, $w = z$ if and only if $a = c$ and $b = d$.

This is a somewhat obvious fact, but it will prove extremely useful—basically, if we can find two different ways to write the same complex number, we can then conclude that the real parts and the imaginary parts must be the same (because it's the same number!). We'll use this fact to show quite a few trig identities.

Example 1.0.5. Let θ be any real number. Find formulas for $\sin(2\theta)$ and $\cos(2\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.

Response. We use Euler's Formula (Theorem 1.0.3) to write

$$e^{i(2\theta)} = \cos(2\theta) + i\sin(2\theta).$$

However, we also have by properties of exponentiation that $e^{i(2\theta)} = e^{2(i\theta)} = (e^{i\theta})^2$, and note that we already know how to expand out $e^{i\theta}$ (once again by Theorem 1.0.3). That is, we can write

$$\begin{aligned} e^{i(2\theta)} &= (e^{i\theta})^2 \\ &= (\cos(\theta) + i\sin(\theta))^2 \\ &= \cos^2(\theta) + 2i\cos(\theta)\sin(\theta) + (i\sin(\theta))^2 \\ &= (\cos^2\theta - \sin^2\theta) + i(2\cos(\theta)\sin(\theta)) \end{aligned}$$

Now, we've written the same complex number two different ways:

$$\begin{aligned} e^{i(2\theta)} &= \cos(2\theta) + i\sin(2\theta) \\ &= (\cos^2(\theta) - \sin^2(\theta)) + i(2\cos(\theta)\sin(\theta)) \end{aligned}$$

Thus, we can use Fact 1.0.4 to now write that $\cos(2\theta) = cos^2(\theta) - \sin^2(\theta)$, and $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$. □

Exercise 1. Simplify the expression $e^{i\pi} + 1$

Exercise 2. Use the same techniques as in Example 1.0.5 to find formulas for $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Exercise 3. Let θ_1 and θ_2 be real numbers. Use the same techniques as in Example 1.0.5 to find formulas for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$ in terms of $\sin(\theta_1)$, $\sin(\theta_2)$, $\cos(\theta_1)$, and $\cos(\theta_2)$.

Exercise 4. We know that in general, for any x,

$$e^x * e^{-x} = 1. \tag{1.1}$$

This is still true if x is imaginary (or complex for that matter). Let $x = i\theta$. Reinterpret equation (1.1) in terms of $\sin(\theta)$ and $\cos(\theta)$. What familiar trigonometric identity do we recover? (Hint: what symmetries do cos and sin have?)

1.1 Limits I: Formal Definition & First Properties

See: Stewart, §2.4.

Goal. Understand limits in terms of formal properties, prove basic facts regarding them.

Next Week. Use these properties to do a bunch of more concrete limit computations, possibly start out derivatives.

Definition 1.1.1. Let f be a function on the real numbers, and let a and L be real numbers. We say $\lim_{x\to a} f(x) = L$ if for any real number $\epsilon > 0$, there exists some real number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Remark 1.1.1.1. ϵ is the Greek letter *epsilon.* δ is the Greek letter *delta.*

Remark 1.1.1.2. Let's unpack that definition a bit. First, keep in mind that we should think of $|x - a|$ as the *distance* between x and a. When we say "for any ϵ, there exists a δ," that means that if *we are given* some number ϵ, then (theoretically) we can find some number δ with the property that if x is "less than δ away from a," then $f(x)$ is "less than ϵ away from L."

To illustrate this more concretely, let's add in some literal illustrations. Suppose we have the curve $y = f(x)$ as in figure 1.1, and suppose we are considering the statement $\lim_{x\to a} f(x) = L$.

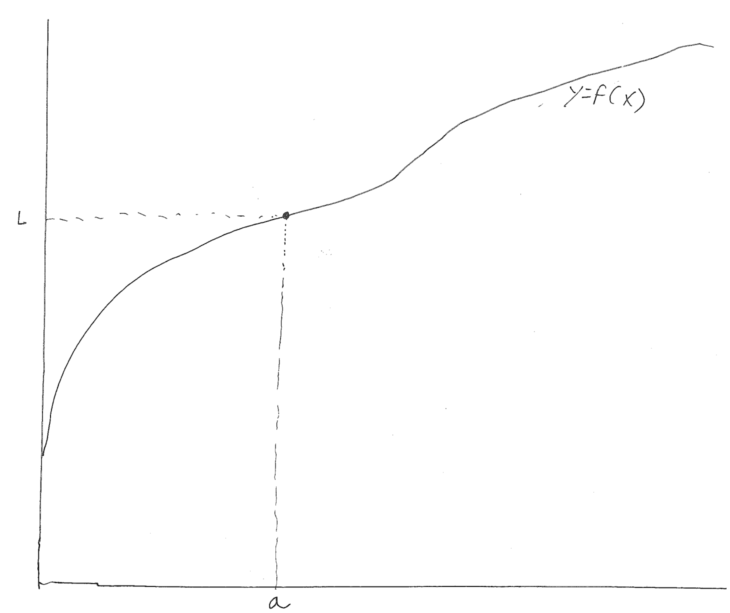

Figure 1.1: The graph of $y = f(x)$ with $x = a$ and $y = L$ drawn in.

Now, suppose we are given some real number $\epsilon > 0$. ϵ corresponds to some region around L on the y-axis (the region where $|y - L| < \epsilon$). That region is drawn in red in figure 1.2

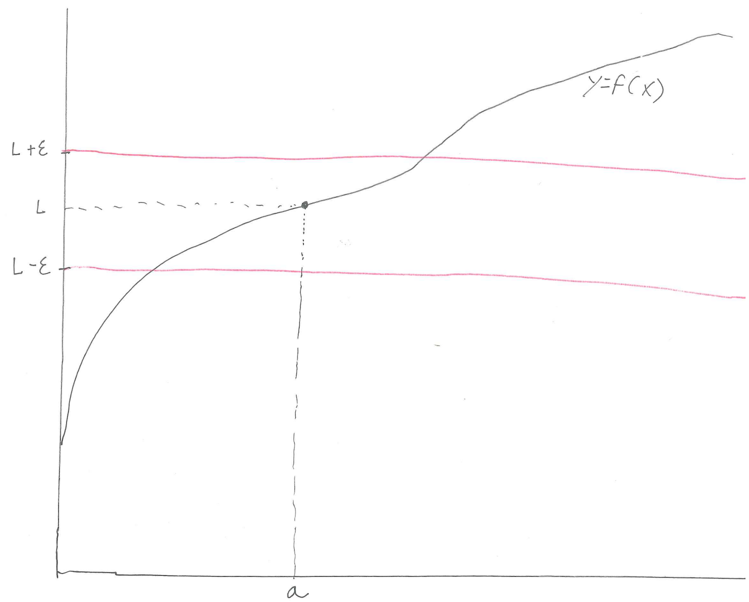

Figure 1.2: The graph of $y = f(x)$ with the vertical region between $L - \epsilon$ and $L + \epsilon$ drawn in.

Given such an ϵ, we need to find some horizontal region around $x = a$ such that for any x in that region, $f(x)$ is "close enough" to L. We can see from the graph that such a region indeed exists—note the part of the curve which sits between the two red lines. We choose some δ such that everything between $a - \delta$ and $a + \delta$ (that is, all x with $|x - a| < \delta$) has its corresponding point on the curve between $L - \epsilon$ and $L + \epsilon$. Such a δ and the corresponding region around $x = a$ is drawn in figure 1.3.

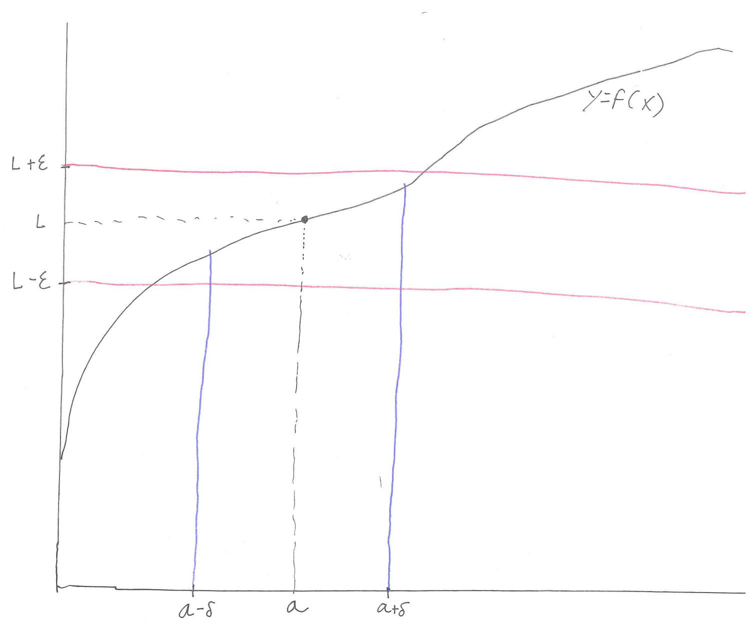

Figure 1.3: The graph of $y = f(x)$, now with a horizontal region around $x = a$.

Note that we could have chosen a different δ—in particular, any smaller δ would work. Also note that we would not be done after just finding this one δ—we must find such a δ for *any* ϵ, no matter how small—in other words, we must repeat this process infinitely many times! As that would take all day (as well as all of the other days), in practice we almost always do this abstractly (for some *general* ϵ) instead of case-by-case, as in Example 1.1.2.

Example 1.1.2. Use Definition 1.1.1 to show that $\lim_{x\to 2} 4x - 3 = 5$.

Response. Let x be arbitrary (that is, let it be any real number). Let's try writing out $f(x) - f(2)$.

$$\begin{aligned} f(x) - f(2) &= (4x - 3) - (4(2) - 3) \\ &= 4x - 3 - 4(2) + 3 \\ &= 4(x - 2) \end{aligned}$$

Now, suppose we are given some $\epsilon > 0$. We need to choose a δ such that if $|x - 2| < \delta$, then $|f(x) - f(2)| < \epsilon$. But as we just saw, as long as we pick $0 < \delta < \epsilon/4$, we get that if $|x - 2| < \delta$, then $|f(x) - f(2)| = |4(x - 2)| = 4|x - 2| < 4\delta < 4(\epsilon/4) = \epsilon$. So indeed, $\lim_{x\to 2}(4x - 3) = 5$. □

Exercise 5. Use the techniques of Example 1.1.2 to show that

$$\lim_{x\to 3} 5x - 6 = 9$$

Exercise 6. Or if that's too boring, use the techniques of Example 1.1.2 to show that

$$\lim_{x\to a} mx + b = ma + b$$

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1.1.1 Limits at/to infinity

Definition 1.1.3. We say that $\lim_{x\to a} f(x) = \infty$ if for any real number N, there exists a real number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > N$.

Remark 1.1.3.1. Basically, the way this differs from 1.1.1 is defining "closeness to infinity." That is, we say that f "approaches infinity" when for any number N (no matter how large!) there is a region surrounding $x = a$ where $f(x) > N$ (replacing the condition $|f(x) - L| < \epsilon$).

Definition 1.1.4. We say that $\lim_{x\to\infty} f(x) = L$ if for any $\epsilon > 0$, there exists some real number R such that for all $x > R$, $|f(x) - L| < \epsilon$

Remark 1.1.4.1. Once again, let's unpack this a bit. Suppose we are considering the graph of $y = f(x)$ as in figure 1.4 and wish to show that $\lim_{x\to\infty} f(x) = L$.

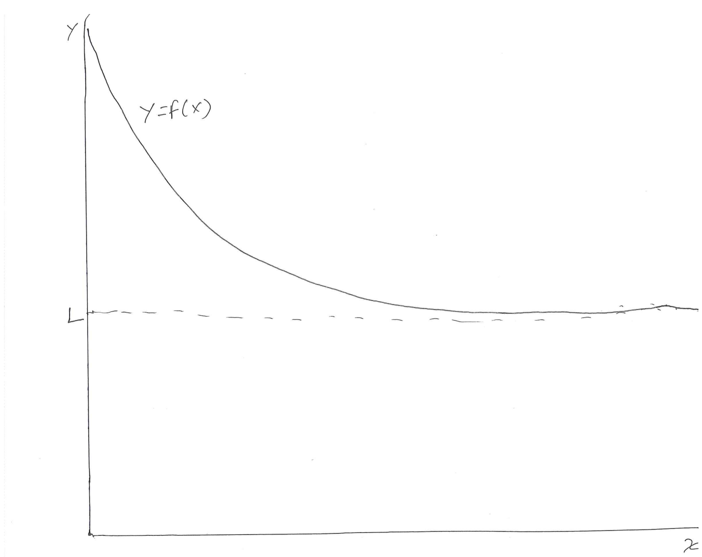

Figure 1.4: The graph of $y = f(x)$ with $y = L$ drawn in.

Now suppose we are once again given some $\epsilon > 0$. In Figure 1.5, we draw in the region on the y-axis of all points y with $|y - L| < \epsilon$ in red.

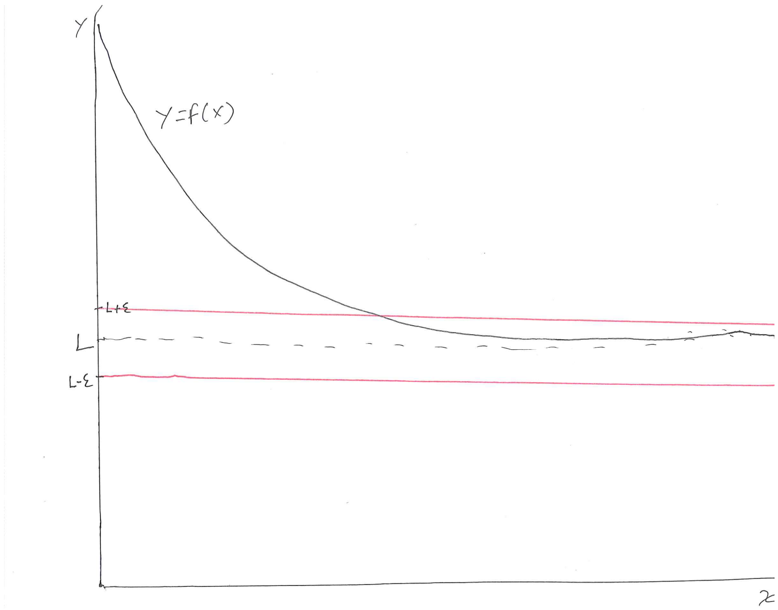

Figure 1.5: The graph of $y = f(x)$ with $y = L$ and the surrounding region of radius ϵ drawn in.

We need to find some R such that for all x to the right of R, the graph of the function stays within the two red lines. However, we may indeed find such an R (assuming the function isn't too badly behaved to the right of our frame). This R is inserted in figure 1.6.

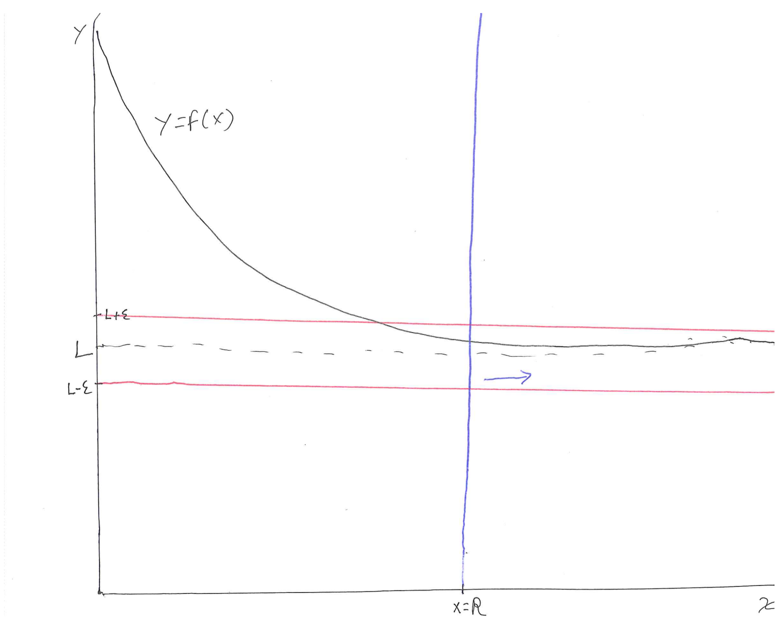

Figure 1.6: The graph of $y = f(x)$ with $y = L$ drawn in.

Of course, once more, we wouldn't actually be done once we had found this x-coordinate R—we must do so for *any* possible ϵ, no matter how small.

Exercise 7. Rewrite Definitions 1.1.3 and 1.1.4 replacing ∞ with $-\infty$ and making the appropriate other adjustments.

Exercise 8. Combine definitions 1.1.4 and 1.1.3 in order to define what it means to say that $\lim_{x\to\infty} f(x) = \infty$. (Hint: read ahead to the next example if you're having trouble). You don't have to write it out, but also convince yourself that you'd be able to do the same for $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Example 1.1.5. Show that $\lim_{x\to\infty} x = \infty$.

Response. We need to show that for any real N, there exists a number R such that if $x > R$, then $f(x) > N$. Let N be arbitrary. Let $R \geq N$. Then, for any $x > R$, $f(x) = x > R \geq N$. Thus, $\lim_{x\to\infty} x = \infty$. □

Exercise 9. Show that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$.

Exercise 10. Show that $\lim\limits_{x\to 0} \frac{1}{x^2} = \infty$.

1.1.2 Supplemental: Example from 6/15/18 meeting

Example 1.1.6. *Show that* $\lim\limits_{x\to 2} x^2 = 4$.

Response. We want to find some $\delta(\epsilon)$ (which we will sort of think of as a function of ϵ) for *any* given $\epsilon > 0$ satisfying Definition 1.1.1. We will find our δ by working backwards, then work forwards to show that our δ works.

A key fact for us is that $f(x) = x^2$ is an *increasing* function for $x > 0$. We can use this! We want that $|x^2 - 4| < \epsilon$ whenever $0 < |x - 2| < \delta$. This first condition can be restated as

$$4 - \epsilon < x^2 < 4 + \epsilon$$

and the second can be restated as

$$2 - \delta < x < 2 + \delta$$

(take some time to prove that to yourself). We can use the fact that $f(x)$ is increasing in the region we care about to find some δ such that for any $x > 2 - \delta$, $f(x) > 4 - \epsilon$. To find such a δ, we solve the equation $(2 - \delta)^2 = 4 - \epsilon$ for δ. This simplifies to $\delta^2 - 4\delta + \epsilon = 0$, which has solutions $2 \pm \sqrt{4 - \epsilon}$. We take the smaller positive solution $2 - \sqrt{4 - \epsilon}$ (convince yourself this is really the smaller solution and that it's really positive). Now, if $\delta < 2 - \sqrt{4 - \epsilon}$, we (because $f(x)$ is increasing!) have that $(2 - \delta)^2 > (2 - (2 - \sqrt{4 - \epsilon}))^2 = (\sqrt{4 - \epsilon})^2 = 4 - \epsilon$, as desired.

Now let's work backwards to find a similar maximal δ on the other side. We want to find some δ such that whenever $x < 2 + \delta$, then $x^2 < 4 + \epsilon$. Let's solve $(2 + \delta)^2 = 4 + \epsilon$. This simplifies to $\delta^2 + 4\delta - \epsilon = 0$ which has solutions $-2 \pm \sqrt{4 + \epsilon}$. We take the only positive solution $-2 + \sqrt{4 + \epsilon}$. Note now that whenever $\delta < -2 + \sqrt{4 + \epsilon}$, we have that $(2 + \delta)^2 < (2 + (-2 + \sqrt{4 + \epsilon}))^2 = 4 + \epsilon$, as desired.

To recap, we've found one positive number $2 - \sqrt{4 - \epsilon}$ such that when $\delta < -\sqrt{4 - \epsilon}$, we have that for $x > 2 - \delta$, $x^2 > 4 - \epsilon$. We've also found another positive number $-2 + \sqrt{4 + \epsilon}$ such that when $\delta < -2 + \sqrt{4 + \epsilon}$ and $x < 2 + \delta$, we have that $x^2 < 4 + \epsilon$. We need both of these conditions to be true simultaneously. Thus, we let

$$0 < \delta < \min\{(2 - \sqrt{4 - \epsilon}), (-2 + \sqrt{4 + \epsilon})\}$$

and have indeed that both conditions can be achieved simultaneously. This completes our proof. □

Exercise 11. Show that $\lim\limits_{x\to 3} x^2 = 9$

Exercise 12. Show that $\lim\limits_{x\to 3} x^2 - 2x + 1 = 4$

Chapter 2

Polynomial Long Division & Limits II

Before we begin, a warm-up exercise (stolen from Stephen Abbott's *Understanding Analysis*).

Exercise 13. Describe what we would have to demonstrate in order to disprove each of the following statements.

1. At every college in the United States, there is a student who is at least seven feet tall.
2. For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.
3. There exists a college in the United States where every student is at least six feet tall.

2.0 Polynomial Long Division

Theorem 2.0.1. *Let $f(x)$ and $g(x)$ be polynomials with coefficients in $\mathbb{R}$ (the real numbers) or $\mathbb{Q}$ (the rational numbers), and let $\deg f(x) \geq \deg g(x)$. Then, there exist polynomials $q(x)$, $r(x)$ such that*

$$f(x) = q(x)g(x) + r(x)$$

where $\deg r(x) < \deg g(x)$.

Rather than prove this theorem (well, I suppose it's a proof of sorts), let's dive straight in to how to find $q(x)$ and $r(x)$.

We work from the following framework. At the ith step of our process, we write

$$f(x) = q_i(x)g(x) + r_i(x). \tag{2.1}$$

We will iteratively define q_i and r_i until we get an r_i with degree strictly less than that of g. Set $q_0(x) = 0$ and set $r_0(x) = f(x)$ (check that equation (2.1) is actually satisfied by these!). We suppose $\deg g(x) = m$ and $\deg r_i(x) = n$ with $n \geq m$. Write $g(x) = a_m x^m + \dots$ and $r_i(x) = b_n x^n + \dots$. Then, set $q_{i+1}(x) = q_i(x) + \frac{b_n}{a_m} x^{n-m}$ and set $r_{i+1}(x) = r_i(x) - \frac{b_n}{a_m} x^{n-m} g(x)$. If $\deg r_{i+1}(x) < \deg g(x)$, we are done and can let $q(x) = q_{i+1}(x)$ and $r(x) = r_{i+1}(x)$. Otherwise, go back to the top and find q_{i+2} and r_{i+2}.

Example 2.0.2. Let $f(x) = x^7 - 3x^5 + x^4 - x + 7$ and $g(x) = x^2 - 3x + 1$. Write $f(x) = q(x)g(x) + r(x)$ with $\deg r(x) < \deg g(x) = 2$.

Response. We begin with $r_0(x) = f(x) = x^7 - 3x^5 + x^4 - x + 7$ and $q_0(x) = 0$. Here, $m = 2$ and $n = 7$, with our $a_m = b_n = 1$. Thus, our new $q_1(x)$ is $x^{n-m} = x^5$. We then compute that $x^5 g(x) = x^7 - 3x^6 + x^5$, and so $r_1(x) = r_0(x) - x^5 g(x) = x^7 - 3x^5 + x^4 - x + 7 - (x^7 - 3x^6 + x^5) = 3x^6 - 4x^5 + x^4 - x + 7$.

Now, $m = 2$ still and $n = 6$, with our new $a_m = 3$. Thus, $q_2(x) = q_1(x) + 3x^{6-2} = x^5 + 3x^4$, and $r_2(x) = r_1(x) - 3x^4 g(x) = 5x^5 - 2x^4 - x + 7$.

We continue the process and find:

$$\begin{aligned}
q_3(x) &= x^5 + 3x^4 + 5x^3 & r_3(x) &= 13x^4 - 5x^3 - x + 7 \\
q_4(x) &= x^5 + 3x^4 + 5x^3 + 13x^2 & r_4(x) &= 34x^3 - 13x^2 - x + 7 \\
q_5(x) &= x^5 + 3x^4 + 5x^3 + 13x^2 + 34x & r_5(x) &= 89x^2 - 35x + 7 \\
q_6(x) &= x^5 + 3x^4 + 5x^3 + 13x^2 + 34x + 89 & r_6(x) &= 232x - 82
\end{aligned}$$

We finally have that $r_6(x)$ has a lower degree than g! Thus, we have our q and our r and may write

$$f(x) = (x^5 + 3x^4 + 5x^3 + 13x^2 + 34x + 89)g(x) + (232x - 82)$$

$\square$

Example 2.0.3. Write $f(x)/g(x)$ in the form of a polynomial plus a rational function with the rational function's numerator having a lesser degree than the denominator.

Response. We have that

$$\frac{f(x)}{g(x)} = \frac{x^7 - 3x^5 + x^4 - x + 7}{x^2 - 3x + 1} = \frac{(x^5 + 3x^4 + 5x^3 + 13x^2 + 34x + 89)(x^2 - 3x + 1) + (232x - 82)}{x^2 - 3x + 1}$$

We may separate out the second fraction and cancel like terms:

$$\begin{aligned}\frac{f(x)}{g(x)} &= \frac{(x^5 + 3x^4 + 5x^3 + 13x^2 + 34x + 89)(x^2 - 3x + 1)}{x^2 - 3x + 1} + \frac{232x - 82}{x^2 - 3x + 1} \\ &= x^5 + 3x^4 + 5x^3 + 13x^2 + 34x + 89 + \frac{232x - 82}{x^2 - 3x + 1}\end{aligned}$$

$\square$

Exercise 14. Given f and g write f in the form $q(x)g(x) + r(x)$ where $\deg r(x) < \deg g(x)$.

a) $f(x) = x^3 - 2x^2 + 1$, $g(x) = x - 2$

b) $f(x) = x^3 - 5x^2 + 8x - 4$, $g(x) = x - 1$

c) $f(x) = 5x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1$, $g(x) = x^3 + 2x - 1$

Exercise 15. Write each of the following rational functions in the form of a polynomial plus a rational function with the rational function's numerator having a lesser degree than the denominator.

a) $\dfrac{x^2 - 2x + 1}{x^2 + 2x + 1}$

b) $\dfrac{x^3}{x - 2}$

c) $\dfrac{x^4 - 2x + 6}{x^2 - 2}$

2.1 Limits II: More Limit Laws, Limits of Sequences, and Continuity

2.1.1 More Limit Laws

The following fact is extremely important in all types of mathematical analysis

Foundationally Important Fact 2.1.1. The *triangle identity* states that for any $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Exercise 16. Come up with an explanation for the name "triangle identity"—it may help to recall how we measure distance in the real numbers.

Theorem 2.1.2. *For $a \in \mathbb{R}$ or $a = \pm\infty$, the following hold provided $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist and are not $\pm\infty$.*

(a) $\lim_{x\to a} cf(x) = c\left(\lim_{x\to a} f(x)\right)$ *for all* $c \in \mathbb{R}$.

(b) $\lim_{x\to a} (f(x) + g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right)$

(c) $\lim_{x\to a} (f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

(d) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *provided* $\lim_{x\to a} g(x) \neq 0$.

Exercise 17. Prove part (a) of Theorem 2.1.2 (possibly after reading the proof to part (b)). Be sure to handle the case $a = \pm\infty$ (although it's probably only necessary to do $a = +\infty$—convince yourself that's enough).

Proof of theorem 2.1.2(b). We suppose that $a \neq \pm\infty$ and let $L_f := \lim_{x\to a} f(x)$ and $L_g := \lim_{x\to a} g(x)$. We wish to show that $\lim_{x\to a} f(x) + g(x) = (L_f + L_g)$ By definition, for any $\epsilon > 0$ there is some $\delta_f > 0$ and $\delta_g > 0$ such that if $0 < |x-a| < \delta_f$ then $|f(x) - L_f| < \epsilon$ and if $0 < |x-a| < \delta_g$ then $|g(x) - L_g| < \epsilon$. Let $\epsilon > 0$ be arbitrary and set $\hat{\epsilon} = \epsilon/2$. Then (as we just said!), there's some δ_f, $\delta_g > 0$ such that if $0 < |x-a| < \delta_f$ then $|f(x) - L_f| < \hat{\epsilon}$ and if $0 < |x-a| < \delta_g$ then $|g(x) - L_g| < \hat{\epsilon}$. Now, set $\delta := \min\{\delta_f, \delta_g\} > 0$. Notice that if $0 < |x-a| < \delta$, then in particular, $0 < |x-a| < \delta_f$ and $0 < |x-a| < \delta_g$. Thus, for any x with $0 < |x-a| < \delta$, we have $|f(x) - L_f| < \hat{\epsilon}$ and $|g(x) - L_g| < \hat{\epsilon}$. Thus, we have that for such an x, $|f(x) - L_f| + |g(x) - L_g| < 2\hat{\epsilon} = \epsilon$. But note that by Foundationally Important Fact 2.1.1, $|f(x) - L_f| + |g(x) - L_g| \geq |(f(x) - L_f) + (g(x) - L_g)| = |(f(x) + g(x)) - (L_f + L_g)|$. Hence, for arbitrary $\epsilon > 0$, we have established that there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $|(f(x) + g(x)) - (L_f + L_g)| < \epsilon$, so $\lim_{x\to a}(f(x) + g(x)) = L_f + L_g$ as desired. The case $a = \pm\infty$ is left to the reader. □

Exercise 18. Read over the proof of part (b) of Theorem 2.1.2. Take notes of the key ideas if you wish but do *not* copy the proof down wholesale (or if you do, put it out of sight and away). Then, rewrite the proof in your own words.

Exercise 19. Finish the proof of theorem 2.1.2(b) (i.e. deal with the case $a = \pm\infty$).

We shall not prove parts (c) or (d) to avoid unnecessarily complicated arguments for the time being...

Theorem 2.1.2 allows us to compute many limits we may previously have been stymied by. For instance, we can use the theorem alone to prove the following (easy) fact.

Exercise 20. Let $f(x)$ be a polynomial, and $a \neq \infty$. Then, $\lim_{x\to a} f(x) = f(a)$.

We now recall the definition of continuity

Definition 2.1.3. We say that $f(x)$ *is continuous at* $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

Now, exercise 20 implies the following fact:

Corollary 2.1.4. *Polynomials are continuous.*

Next, we'll show one of our key tools in proving limits, in the form of a few exercises.

Lemma 2.1.5 (Squeeze Theorem: Prelude)**.** *If there exists some interval* $I = (b, c)$ *with* $b < a < c$ *such that* $f(x) \geq g(x)$ *on* I*, then if* $L_f = \lim_{x\to a} f(x)$ *and* $L_g = \lim_{x\to a} g(x)$ *exist, we have that* $L_f \geq L_g$.

Exercise 21. Prove Lemma 2.1.5.

Hint 21.1. Recall how proof by contradiction works. We want to show a statement is true, so we *assume* that it is *false* instead, then show that our new situation is impossible. Here, try assuming that $f \geq g$ on I, but $L_f < L_g$. Then, show that in fact there has to be some $x \in I$ such that $g(x) > f(x)$.

Hint 21.2. The proof should come down to "choosing the right ϵ."

Hint 21.3. If you're struggling, feel free to ask me for more hints!

Exercise 22. Show the following result using Exercise 21 and Definition

Corollary 2.1.6 (Squeeze Theorem, final version)**.** *We suppose there is some interval* $I = (b, c)$ *with* $b < a < c$ *such that* $f(x) \geq g(x) \geq h(x)$ *for all* $x \in I$*. Then, if* $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$*, we have that* $\lim_{x\to a} g(x)$ *exists and* $\lim_{x\to a} g(x) = L$.

Hint 22.1. Try assuming first that $\lim_{x\to a} g(x)$ exists, and then show that it's equal to L. The hard part is showing that it exists.

2.1.2 Limits of Sequences

See: Stewart, §11.1.

Definition 2.1.7. A *sequence* is an ordered set of numbers indexed by the integers, ususally denoted $\{x_n\}_{n=1}^{\infty}$ or just $\{x_n\}$.

Example 2.1.8. As a first example, suppose we set the rule $x_n = 1/n$. Then, our sequence will be $1, 1/2, 1/3, 1/4, \ldots$.

As the chapter heading has probably tipped you off to, we can take limits of sequences just as for functions.

Definition 2.1.9. For $L \neq \pm\infty$, we say that $\{x_n\}$ has a limit at L (usually denoted $\lim\limits_{n\to\infty} x_n = L$ or $x_n \to L$ as $n \to \infty$ or simply $x_n \to L$) if for any $\epsilon > 0$, there is some N such that for any $m > N$, $|x_m - L| < \epsilon$. We say that $\lim\limits_{n\to\infty} x_n = \infty$ if for any real number R there is some N such that for any $m > N$, $x_m > R$.

Be sure to compare Definition 2.1.9 to Definitions 1.1.4 and 1.1.3!!