Math 8272 Homework

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So with Eric's preamble, \prime is re-def'd to be \mathfrak {p}, which causes the apostrophe to be interpreted as "to the mathfrak p." is there a way around this? or are we okay with letting mathfrak p be referred to as \pfr?

4.)

Proposition (Eisenbud Ex. 9.2). There exists an infinite-dimensional Noetherian ring.

Proof. We let...

- k be any field,
- $R = k[x_1, x_2, \ldots]$,
- $d : \mathbb{N}_0 \to \mathbb{N}$ a strictly increasing function with first difference function $\delta : \mathbb{N} \to \mathbb{N}$ defined by $s(m) = d(m) - d(m-1)$ such that $d(0) = 1$ and s is strictly increasing as well,
- $P_m = \left\langle x_{d(m-1)}, x_{d(m-1)+1}, \ldots, x_{d(m)} \right\rangle$ for $m \geq 1$,
- U be the multiplicative system $\left(\bigcup_{m=1}^{\infty} P_m\right)^c$,
- and S be the ring $U^{-1}R$.

We shall now show that $\dim S = \infty$, but S is Noetherian. We break this argument down into a series of claims.

Proposition 4.A (Eisenbud, Ex. 3.14)**.** *The maximal ideals of S are precisely the ideals P_m.*

Proof of Proposition 4.A. We let I be a proper ideal of S (noting that necessarily, $I \subset \bigcup_{m=1}^{\infty} P_m$) and $0 \neq f \in I$ an arbitrary element. We let $\mathcal{A}_f := \{P_{i_1}, \ldots, P_{i_n}\} := \{P_i : P_i \text{ contains a monomial of } f\}$. We let $g \neq f$ be another arbitrary element of I and suppose for the sake of contradiction that g has some monomial term $g^{\mathfrak{p}}$ such that $g^{\mathfrak{p}} \notin \bigcup_{j=1}^{n} P_{i_j}$. Then, $f + g$ has a nonzero coefficient for $g^{\mathfrak{p}}$. As each P_m is a monomial ideal and hence contains all monomials of each of its elements, we now have that for any $P_{i_k} \in \mathcal{A}_f$, $f+g \notin P_{i_k}$. However, by an identical argument, for any $P_j \ni g^{\mathfrak{p}}$, $f+g \notin P_j$, since f necessarily has monomial terms not in P_j. Returning to the monomial ideal argument, we have now shown that $f+g \notin \bigcup_{m=1}^{\infty} P_m$, thus inducing a contradiction. Thus, for any ideal $I \subset S$, we have that $I \subset \bigcup_{k=1}^{N} P_{j_N}$ for some finite $\{j_1, \dots, J_N\}$. Prime avoidance then implies $I \subset P_M$ for some $M \in \mathbb{N}$. As it is the case that $P_m \not\subseteq P_{m^{\mathfrak{p}}}$ for $m \neq m^{\mathfrak{p}}$, this completes our proof. ■

Next, as suggested by the text, we prove Eisenbud's lemma 9.4.

Lemma 4.B (Eisenbud, Lemma 9.4)**.** *Let Q be a ring with the properties (i) for any maximal $\mathfrak{m} \subset Q$, $Q_{\mathfrak{m}}$ is Noetherian and (ii) each element $s \in Q$ is contained in finitely many maximal ideals. Then, Q is Noetherian.*

Proof of Lemma 4.B. We suppose for the sake of contradiction that there exists an infinite chain of ideals $0 = I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \dots$ in Q. We then define the function $N : \text{Max-Spec}(Q) \to \mathbb{N}_0$ by $\mathfrak{m} \mapsto \min\{n \ : \ I_n \not\subseteq \mathfrak{m}\}$. As each $Q_{\mathfrak{m}}$ is Noetherian, we must have that $N(\mathfrak{m})$ exists and is finite. We also define the choice function $C : \mathbb{N}_0 \to \text{Max-Spec}(Q)$ which assigns to each $n \in \mathbb{N}_0$ some $\mathfrak{m} \in \text{Max-Spec}(Q)$ such that $I_n \subset \mathfrak{m}$. As each ideal of a ring must be contained in a maximal ideal by Zorn's lemma, there exists some well-defined such C. We observe that $C(N(\mathfrak{m})) \neq \mathfrak{m}$ for any $\mathfrak{m} \in \text{Max-Spec}(Q)$ as $I_{N(\mathfrak{m})} \not\subset \mathfrak{m}$ by construction. We also observe that $n \leq N(C(n))$, as $I_m \subset C(n)$ for any $m \leq n$ but $I_{N(C(n))} \not\subseteq C(n)$ similarly by construction. We now iteratively define a sequence of distinct maximal ideals $\{\mathfrak{m}_1, \mathfrak{m}_2, \dots\}$ by letting $\mathfrak{m}_1 := C(1)$ and for $i > 1$, $\mathfrak{m}_i := C(N(\mathfrak{m}_{i-1}))$. As $N \circ C$ has been shown to be a strictly increasing function, we have by well-ordering that for any n, there exists a J such that $I_n \subset \mathfrak{m}_j$ for all $j > J$. However, then $I_n \subset \bigcap_{j=J}^{\infty} \mathfrak{m}_j$, contradicting our assumptions on Q. ■

Corollary 4.C. *S is Noetherian*

□

Problem 5: (I'm not quite sure how our formatting works). Krull dimension satisfies the first half of axiom D1, and also the axiom D2. In other words,

$$\dim R = \sup_{P \subset R \text{prime}} \dim R_P$$

and if I is a nilpotent ideal, then $\dim R = \dim R/I$.

Proof: If P is a prime ideal of R, let $P_0 \subset \dots \subset P_n$ be a chain of primes in R_P. If ϕ is the natural map from $R \to R_P$, then Proposition 2.2 of Eisenbud tells us that $P_i = \phi^{-1}(P_i)R_P$. The ideal $\phi^{-1}(P_i) \subset R$ is prime because if the complement of $\phi^{-1}(P_i)$ weren't multiplicatively closed, then the map ϕ would tell us that the complement of P_i was also not multiplicatively closed. In addition, if $P_i \subsetneq P_j$, then $\phi^{-1}(P_i)R_P \subsetneq \phi^{-1}(P_j)R_P$, so $\phi^{-1}(P_i) \subsetneq \phi^{-1}(P_j)$. Therefore, any chain of primes in R_P lifts to an equal length chain in R.

On the other hand, let $P_1, P_2, \ldots$ be a sequence of primes in R such that $\dim P_i \to \dim R$. This is possible because for a finite chain with minimal prime Q, $\dim Q$ is the length of that chain, and for an infinite chain, by taking smaller and smaller primes in the chain, we get such a sequence. If $P_i \subset Q_{i1} \subset Q_{i2} \ldots$ is a chain in R starting with P_i (i.e. a chain corresponding to one in R/P_i), then it will be a chain of the same length in R_{P_i}. Thus we have that $\dim R_{P_i} \geq \dim P_i$, so $\sup_{P \subset R \text{prime}} \dim R_P \geq \dim R$.

Now, if I is nilpotent, we have $\dim R \geq \dim R/I$, the fourth isomorphism theorem gives us a correspondence between prime ideals of R/I and prime ideals of R containing I. Now I is contained in the nilradical of R, so it is contained in every prime of R, so chains of primes of R are in one-to-one correspondence with chains of primes of R/I, so $\dim R = \dim R/i$. □

OLD PROBLEMS FOR FORMATTING CHECKS

Theorem 2.7. Show that the universal property of localization is unique up to unique isomorphism; that is, if another $R \to S$ has the same property....

Theorem 2.4. Let $R = k[x]$. Describe as explicitly as possible:

1. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m)$ and $\mathrm{Hom}_R(R/(x^n), R/(x^m))$,
2. $\mathbb{Z}_n \otimes \mathbb{Z}_m$ and $R/(x^n) \otimes R/(x^m)$,
3. $R \otimes_k R$ (describe this as an algebra).

Theorem 3.17. Show that if $k = \mathbb{Z}_2$ then the ideal $(x, y) \subseteq k[x, y]/(x, y)^2$ is the union of three properly smaller ideals.

Let k be any field, and $I_1 = (x), I_2 = (y)$ and $J = (x^2, y)$ ideals in the ring $k[x, y]/(xy, y^2)$. Show that the homogeneous elements of J are contained in $I_1 \cup I_2$, but that $J \not\subseteq I_1, I_2$. (Note that one of the I is prime.)

Theorem 3.6–8. Which monomial ideals are prime? Irreducible? Radical? Primary?

Find an algorithm for computing the radical of a monomial ideal.

Find an algorithm for computing an irreducible decomposition, and thus a primary decomposition, of a monomial ideal.

Theorem 4.7. Show the Jacobsen radical of R is $\{r : 1 + rs$ is a unit for every $s \in R\}$.

Theorem 4.11.

1. Use Nakayama's lemma to show that if R is local and M is finitely generated projective, then M is free. If R is a positively graded ring, with R_0 a field, and M is a finitely generated graded projective, then M is a graded free module.
2. Use Prop 2.10 (contains the snippet $\operatorname{Hom}_S \otimes R \operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_S \otimes_R M, S \otimes_R N)$) to show that a finitely presente module M is projective iff M is locally free in the sense that localization M_P is free over R_P for every maximal ideal of R.

Theorem 4.24. Let R be either of the domains $\mathbb{C}[x, y]/(y^2 - x^3)$ or $\mathbb{C}[x, y](y^2 - x^2(x + 1))$ and let $t = y/x$ an element of the quotient field. Show that in each case, $R[t] = \mathbb{C}[t]$.

Theorem 4.26. Suppose that the additive group of R is a finitely generated abelian group. If P is a maximal ideal of R, show that R/P is a finite field. Show that every prime ideal of R that is not maximal is a minimal prime ideal.

Ryan Comment

Theorem 5.1. Let R be a ring and M be an R-module. Suppose that $\cdots \subseteq M_1 \subseteq M_0 = M$ is a filtration by submodules. Although the map $M \to \operatorname{gr} M$ sending f to $\operatorname{in}(f)$ is not a homomorphism of abelian groups, show that $\operatorname{in}(f) + \operatorname{in}(g)$ is either $\operatorname{in}(f + g)$ or 0.

Moreover, suppose that $M = R$ and the filtration is multiplicative. Show that $\operatorname{in}(f) \operatorname{in}(g)$ is either $\operatorname{in}(fg)$ or 0.

Eric Comment

Theorem 5.8.

1. Let $R = k[x, y]/(x^2 - y^3)$, and let $I = (x, y)$. Show that R is a domain, but $\operatorname{in}(x)^2 = 0$ in $\operatorname{gr}_I(R)$.

2. Let $R = k[t^4, t^5, t^{11}] \subseteq k[t]$, and let $I = (t^4, t^5, t^{11})$. Show that $\mathrm{in}(I)\,\mathrm{in}(t^{11}) = 0$

Andy Comment

Theorem 6.1. Let R be a ring and M an R-module. Show that M is flat iff $\mathrm{Tor}_1(M, N) = 0$ for all R-modules N iff $\mathrm{Tor}(N, M) = 0$ for all R-modules N iff $\mathrm{Tor}_i = 0$ for all R-modules N and all $i > 0$.

David Comment

Theorem 7.11. Let R be Nötherian, and $\mathfrak{m} = (a_1, \dots, a_n)$ be an ideal. Show that

$$\hat{R}_{\mathfrak{m}} \cong R[[x_1, \dots, x_n]]/(x_1 - a_1, \dots, x_n - a_n).$$

Theorem A3.6. Let R be Nötherian and M be any finitely generate R-module.

1. Let P be prime. Show that if $M \to E(R/P)$ is any map [into the injective envelope], then $\ker \alpha$ is a P-primary submodule of M.
2. Show that $E(M)$ is a finite direct sum of indecomposable projectives. Let $M \to E(M) = \oplus E(R/P_i)$, and show that if P is a prime ideal and $M(P)$ is the kernel of the composite map $M \to E(M) \to \oplus_{P_i = P} E(R/P_i)$, then $M(P)$ is P-primary. Show that $0 = \cap M(P)$ is a primary decomposition of zero, and that the set of P that occurs among the P_i above is precisely the set $\mathrm{Ass}(M)$.

Theorem A3.13. Show that if $0 \to N_f \to F \to M \to 0$ and $0 \to N_G \to G \to M \to 0$ are exact with F and G projective, then $N_F \oplus G \cong N_G \oplus F$ and both are $\ker(F \oplus G \to M)$.

Theorem A3.18. Let $(R, \mathfrak{m})$ be a local ring. We say that a free reslution (F_i, φ_i) is minimal if each φ_i has an image contained in $\mathfrak{m}F_{i-1}$. If F as above is a minial free resolution of M and rank $F_i = b_i$, then show that $\operatorname{Tor}_i(R/\mathfrak{m}) \cong (R/\mathfrak{m})^{b_i}$. [The b_i are called Betti numbers of M, in loose analogy with the situation in topology where F is a chain complex.]

Theorem A3.23. If x is not a zero-divisor in a ring R, compute $\operatorname{Ext}^i(R/x, M)$. In particular, compute $\operatorname{Ext}^i(\mathbb{Z}_n, \mathbb{Z}_m)$ for any integers n, m.

Theorem A3.24. Show that a finitely generated abelian group A is free iff $\operatorname{Ext}_{\mathbb{Z}}(a, \mathbb{Z}) = 0$. It was conjectured that this would hold for all groups, but Shelah proved in 1974 that this depends on your set theory.