Free Resolutions Over a Complete Hypersurface (And Friends)

Based on "Homological algebra on a complete intersection, with an application to group representations." by David Eisenbud

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Notation & Motivation

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Such a B is a *complete intersection of codimension n.*
We shall study the structure of B-free resolutions of B-modules, relating these to their liftings to A.

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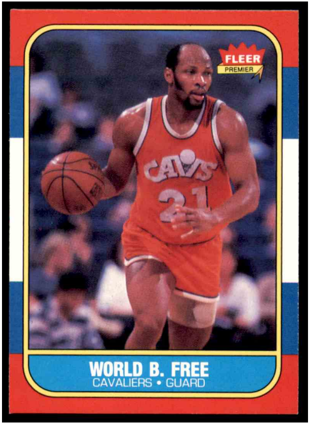

Figure: Another "B-Free" Object.

Why do we care?

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- Classifying maximal CM modules over complete hypersurfaces! (case $n = 1$)

Theorem (6.1)

Let $x \in A$, $d = \dim A$, and $B := A/\langle x \rangle$. Then,

- *For any B-module M and minimal free resolution $\mathbf{F}$, the truncation at F_{d+1} is periodic with period 2.*
- *$\mathbf{F}$ periodic $\iff$ M is a maximal CM module w/o a free summand.*
- *If so, $\mathbf{F}$ is induced a matrix factorization.*

Why do we care?

- Expanding upon a familiar characterization!

Theorem (Auslander-Buchsbaum-Serre)

For $(R, \mathfrak{m})$ local, R is regular $\iff \operatorname{gl\,dim}(R) < \infty$.

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Theorem (6.1)

For $(R, \mathfrak{m})$ local with $\dim R = d$, TFAE:

- *For some $x_1, \ldots, x_{d+1} \in R$, $\mathfrak{m} = \langle x_1, \ldots, x_{d+1} \rangle$ and $\hat{0}$ is unmixed in $\hat{R}^{\mathfrak{m}}$ (i.e. all associated primes of $\hat{0}$ are minimal).*
- *For any f.g. R-module M with minimal free resolution $\mathbf{F}$, the truncation of $\mathbf{F}$ at degree $d+1$ is periodic of period 2.*
- *There exists a free resolution $\mathbf{F} : \cdots \to F_1 \to F_0 \to 0$ of $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ where for some n, $\mathrm{rank} F_n < n$.*

We call such an R an abstract hypersurface.

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Codim n: Set-up

Recall that $B := A/\langle x_1, \ldots, x_n \rangle$. Let M be a B-module, and $\mathbf{F}$ a free resolution of B.

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- $\tilde{F}_i$ be a free A-module with $\operatorname{rank}_A \tilde{F}_i = \operatorname{rank}_B F_i$
- $\tilde{\partial}_i$ denote an arbitrary lifting of (the entries of) ∂_i to A

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$$\tilde{\mathbf{F}} : \ldots \xrightarrow{\tilde{\partial}_3} \tilde{F}_2 \xrightarrow{\tilde{\partial}_2} \tilde{F}_1 \xrightarrow{\tilde{\partial}_1} \tilde{F}_0 \xrightarrow{\tilde{\partial}_0} 0$$

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- Finally, let $t_j = t_j(A, \{x_i\}, \mathbf{F}) : \mathbf{F} \to \mathbf{F}$ be defined by $t_j = B \otimes \tilde{t}_j$.

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- *The t_j commute with each other up to homotopy.*
- *If M, N are B-modules, with B-free resolutions $\mathbf{F} \to M$, $\mathbf{G} \to N$ $t_j(\mathbf{F})$ and $t_j(\mathbf{G})$ induce the same map on $\operatorname{Tor}^B_\bullet(M, N)$.*

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- *The t_j play only a little less well with ring morphisms $\alpha : A \to A'$.*

Interlude: Matrix Factorizations

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DEFINITION

The ordered pair (ψ, ϕ) is a *matrix factorization of x* if