Math 8212 Homework

Ryan Coopergard, David DeMark, Andy Hardt, Eric Stucky

2 May 2018

We've got 10 problems boiz!

Problems
1: done
2: done
3:
4: done
5: done
6:
7: done
8: done
9:
10: done
11:
12: Ryan
13: done
14:
15:
16:
17: Andy
18: Eric
19:
20:
21: done
22: done
23:
24:
25: Ryan
26:
27:
28:
29:
30:
31:
32:

33:
34:
35:

1. (Eisenbud Exercise 9.2)

   Let k be a field.

   (a) Let $f(x, y) \in k[x, y]$ be any polynomial, and consider the "variable" $x' = x - y^n$. Show that $k[x, y] = k[x', y]$, and that if n is sufficiently large, then as a polynomial in x' and y, f is a scalar times a monic polynomial in y. Deduce that $k[x, y]/f$ is integral over its subring $k[x']$. Use this to prove that $\dim k[x, y] = 2$.

   Proof. We show that $k[x, y] = k[x', y]$. It is clear that $k[x', y] \subseteq k[x, y]$. Now $x' + y^n = x$ is in $k[x', y]$, as is y, so $k[x, y] \subseteq k[x', y]$ and we have equality. Let $f \in k[x, y]$ be any polynomial, and let r and s be the highest degree of x and y in f, respectively. Set $n = s + 1$. We claim that in x' and y, $f(x, y)$ is a scalar times a monic polynomial in y. It suffices to show that if $\alpha x^r y^d$ is a term in f with d maximal, then αy^{nr+d} is a monomial in $f(x', y)$ and $nr + d$ is the largest power of y appearing in f. Indeed αy^{nr+d} is a monomial in $f(x', y)$. Note that from any monomial $\beta x^a y^b$ in $f(x, y)$, we have the summand $\beta(x' + y^n)^a y^b$ in $f(x', y)$. The highest y-degree in this summand is $na + b$. If $a < r$, then because $b \leq s$ and $n = s + 1$, we have $na + b \leq nr < nr + d$. If $a = r$, and this polynomial is not $\alpha x^r y^d$, then $b < d$ and $na + b < nr + d$. This proves our claim.

   Now $k[x, y]/(f(x, y)) \cong k[x', y]/(f(x', y))$. Moreover, $k[x', y]/(f(x', y))$ is generated by $1, y, y^2, \ldots, y^{nr+d-1}$ as a $k[x']$-module. Hence $k[x', y]/(f(x', y))$ is integral over $k[x']$, and in particular they both have dimension 1.

   Let $0 \subset p_1 \subset \cdots \subset p_n$ is a maximal chain of distinct prime ideals in $k[x, y]$. Then $p_1 = (f)$ for some $f \in k[x, y]$. This chain gets mapped to the maximal chain of primes $0 \subset p_2 + (f) \subset \cdots p_n + (f)$ in $k[x, y]/(f)$. Because $\dim k[x, y]/(f) = 1$ and the original chain has distinct primes, it follows that $n = 2$ and $p_2 + (f)$ is maximal. This implies that the original chain is of length 2, so $\dim k[x, y] = 2$. □

   (b) Show that the same things are true for $x' = x - ay$ for all but finitely many $a \in k$. (If k is finite, this could be all $a \in k$.)

   Proof. We need only show that for all but finitely many $a \in k$, the polynomial $f(x', y)$ is a scalar times a monic polynomial in y. Suppose f has degree d. After substituting $x = x' + ay$, the highest terms of y will be a sum of terms of the form $a^n \alpha y^d$ for some scalar α and some n, i.e.

   $$f(x', y) = (\alpha_0 a^d + \alpha_1 a^{d-1} + \cdots \alpha_{d-1} a + \alpha_d) y^d + \text{ l.o.t.s.}$$

   for some $\alpha_i \in k$ with $\alpha_i \neq 0$ for at least one i. This is a monic in y if and only if $(\alpha_0 a^d + \alpha_1 a^{d-1} + \cdots \alpha_{d-1} a + \alpha_d) \neq 0$. Allowing a to vary, there are at most d solutions to $(\alpha_0 a^d + \alpha_1 a^{d-1} + \cdots \alpha_{d-1} a + \alpha_d) = 0$, so $f(x', y)$ is monic in y for all but finitely many choices of $a \in k$. The rest follows as in part (a). □

Eisenbud 9.3. Suppose that a ring S is integral over the image of a ring homomorphism $\phi : R \to S$. Show that the Krull dimension of M as an S-module is the same as the Krull dimension of M as an R-module.

We first show that $\dim_R(M) \geq \dim_S(M)$. Suppose that $Q_1 \subset Q_2 \subset \cdots \subset Q_n \subset S/\operatorname{Ann}(M)$ is a maximum-size chain of distinct prime ideals in $S/\operatorname{Ann}(M)$, and consider the chain $P_i = \varphi^{-1}(Q_i)$ in $R/\operatorname{Ann}(M)$. By Corollary 4.18 (Incomparability), the fact that the Q_i are distinct but comparable primes implies that they have distinct intersections with $\phi(R)$, and therefore, the P_i are distinct.

The proof that $\dim_R(M) \leq \dim_S(M)$ is similar in spirit. Suppose that $P_1 \subset P_2 \subset \cdots \subset P_n \subset R/\operatorname{Ann}(M)$ is a maximum-size chain of distinct prime ideals in $R/\operatorname{Ann}(M)$, and consider the chain Q_i guaranteed by Going Up (Proposition 4.15). Then these Q_i must be distinct because they have distinct intersections with $\phi(R)$.

4.)

Proposition (Eisenbud Ex. 9.2). There exists an infinite-dimensional Noetherian ring.

Proof. We let...

- k be any field,
- $R = k[x_1, x_2, \ldots]$,
- $d : \mathbb{N}_0 \to \mathbb{N}_0$ a strictly increasing function with first difference function $\delta : \mathbb{N} \to \mathbb{N}$ defined by $\Delta(m) = d(m) - d(m-1)$ such that $d(0) = 0$ and Δ is strictly increasing as well,
- $P_m = \left\langle x_{d(m-1)+1}, x_{d(m-1)+2}, \ldots, x_{d(m)} \right\rangle$ for $m \geq 1$,
- U be the multiplicative system $\left(\bigcup_{m=1}^{\infty} P_m\right)^c$,
- and S be the ring $U^{-1}R$.

We shall now show that $\dim S = \infty$, but S is Noetherian. We break this argument down into a series of claims.

Proposition 4.A (Eisenbud, Ex. 3.14)**.** *The maximal ideals of S are precisely the ideals P_m.*

Proof of Proposition 4.A. We let I be a proper ideal of S (noting that necessarily, $I \subset \bigcup_{m=1}^{\infty} P_m$) and $0 \neq f \in I$ an arbitrary element. We let $\mathcal{A}_f := \{P_{i_1}, \ldots, P_{i_n}\} := \{P_i \ :$

P_i contains a monomial of $f\}$. We let $g \neq f$ be another arbitrary element of I and suppose for the sake of contradiction that g has some monomial term g' such that $g' \notin \bigcup_{j=1}^n P_{i_j}$. Then, $f+g$ has a nonzero coefficient for g'. As each P_m is a monomial ideal and hence contains all monomials of each of its elements, we now have that for any $P_{i_k} \in \mathcal{A}_f$, $f+g \notin P_{i_k}$. However, by an identical argument, for any $P_j \ni g'$, $f+g \notin P_j$, since f necessarily has monomial terms not in P_j. Returning to the monomial ideal argument, we have now shown that $f+g \notin \bigcup_{m=1}^\infty P_m$, thus inducing a contradiction. Thus, for any ideal $I \subset S$, we have that $I \subset \bigcup_{k=1}^N P_{j_N}$ for some finite $\{j_1, \ldots, J_N\}$. Prime avoidance then implies $I \subset P_M$ for some $M \in \mathbb{N}$. As it is the case that $P_m \not\subseteq P_{m'}$ for $m \neq m'$, this completes our proof. ■

Next, as suggested by the text, we prove Eisenbud's lemma 9.4.

Lemma 4.B (Eisenbud, Lemma 9.4)**.** *Let Q be a ring with the properties (i) for any maximal $\mathfrak{m} \subset Q$, $Q_\mathfrak{m}$ is Noetherian and (ii) each element $s \in Q$ is contained in finitely many maximal ideals. Then, Q is Noetherian.*

Proof of Lemma 4.B. We suppose for the sake of contradiction that there exists an infinite chain of ideals $0 = I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \ldots$ in Q. We then define the function $N : \text{Max-Spec}(Q) \to \mathbb{N}_0$ by $\mathfrak{m} \mapsto \min\{n \ : \ I_n \not\subseteq \mathfrak{m}\}$. As each $Q_\mathfrak{m}$ is Noetherian, we must have that $N(\mathfrak{m})$ exists and is finite. We also define the choice function $C : \mathbb{N}_0 \to \text{Max-Spec}(Q)$ which assigns to each $n \in \mathbb{N}_0$ some $\mathfrak{m} \in \text{Max-Spec}(Q)$ such that $I_n \subset \mathfrak{m}$. As each ideal of a ring must be contained in a maximal ideal by Zorn's lemma, there exists some well-defined such C. We observe that $C(N(\mathfrak{m})) \neq \mathfrak{m}$ for any $\mathfrak{m} \in \text{Max-Spec}(Q)$ as $I_{N(\mathfrak{m})} \not\subset \mathfrak{m}$ by construction. We also observe that $n \leq N(C(n))$, as $I_m \subset C(n)$ for any $m \leq n$ but $I_{N(C(n))} \not\subseteq C(n)$ similarly by construction. We now iteratively define a sequence of distinct maximal ideals $\{\mathfrak{m}_1, \mathfrak{m}_2, \ldots\}$ by letting $\mathfrak{m}_1 := C(1)$ and for $i > 1$, $\mathfrak{m}_i := C(N(\mathfrak{m}_{i-1}))$. As $N \circ C$ has been shown to be a strictly increasing function, we have by well-ordering that for any n, there exists a J such that $I_n \subset \mathfrak{m}_j$ for all $j > J$. However, then $I_n \subset \bigcap_{j=J}^\infty \mathfrak{m}_j$, contradicting our assumptions on Q. ■

Corollary 4.C. *S is Noetherian.*

Proof of Corollary 4.C. We wish to show that S satisfies properties *(i)* and *(ii)* of Lemma 4.B. We let $X = \{x_1, \ldots\}$ and $X_m = \{x_{d(m-1)+1}, \ldots, x_{d(m)}\}$. In order to show that S_{P_m} is Noetherian, we present the following alternative characterization of S_{P_m}: we let $R' = k[X \setminus X_m] = k[x_1, \ldots, x_{d(m-1)}, x_{d(m)+1}, \ldots]$, so that $R = R'[X_m]$. We let $S' = K(R') = k(X \setminus X_m)$. We note that S' is a field and hence Noetherian. Then, by the Hilbert Basis theorem, $Q = S'[X_m] = S'[x_{d(m-1)+1}, \ldots, x_{d(m)}]$ is Noetherian as well. As Noetherianness is preserved by localization, Q_{P_m} is also Noetherian, and as localizations commute and the generators of Q and S can be identified with one another, $Q_{P_m} = S_{P_m}$, so indeed S_{P_m} is Noetherian for any maximal ideal $P_m \subset S$. Now, we show that any $s \in S$ is contained in finitely many maximal ideals. We note that there exists some $u \in S^\times$ such that $us \in R \cap S$, so without loss of generality, we assume $s \in R \cap S$. We let $\{m_1, \ldots, m_k\}$ be distinct integers and note that $\bigcap_{j=1}^k P_{m_j} = P_{m_1} P_{m_2} \ldots P_{m_k}$, which is generated by its homogenous elements of degree (in

R) k. Hence, letting $d = \deg_R s$, we have that $s \notin \bigcap_{j=1}^{d+1} P_{m_j}$ for any set of pairwise distinct $\{m_1, \ldots m_{d+1}\}$. As $\deg_R s$ is well-defined for any $s \in R \cap S$, this shows that s is contained in finitely many maximal ideals, thus showing S to satisfy the hypotheses of Lemma 4.B; our corollary follows immediately from its conclusion. ■

Proposition 4.D. $\dim S = \infty$.

Proof of Proposition 4.D. We note that as there is an inclusion-preserving bijection between prime ideals of the ring R and prime ideals of $S = U^{-1}R$ not meeting U, the ideals $\langle x_{d(m-1)+1}, \ldots, x_{d(m)-r} \rangle \subset P_m$ are prime in S for any integer $0 \leq r < d(m) - d(m-1)$. Hence, for any m, we have a chain of prime ideals of length $\Delta(m)$ given by $0 \subsetneq \langle x_{d(m-1)+1} \rangle \subsetneq \langle x_{d(m-1)+1}, x_{d(m-1)+2} \rangle \subsetneq \ldots \subsetneq P_m$. This shows that $\dim S \geq \sup\{\Delta(m) : m \in \mathbb{N}\}$. As $\Delta(m)$ is a strictly increasing function on the integers by assumption, we have that $\sup\{\Delta(m) : m \in \mathbb{N}\} = \infty$. This completes our proof both of the proposition and of the main theorem. ■

□

Problem 5. Krull dimension satisfies the first half of axiom D1, and also the axiom D2. In other words,

$$\dim R = \sup_{P \subset R \text{prime}} \dim R_P$$

and if I is a nilpotent ideal, then $\dim R = \dim R/I$.

Proof: If P is a prime ideal of R, let $P_0 \subset \ldots \subset P_n$ be a chain of primes in R_P. If ϕ is the natural map from $R \to R_P$, then Proposition 2.2 of Eisenbud tells us that $P_i = \phi^{-1}(P_i)R_P$. The ideal $\phi^{-1}(P_i) \subset R$ is prime because if the complement of $\phi^{-1}(P_i)$ weren't multiplicatively closed, then the map ϕ would tell us that the complement of P_i was also not multiplicatively closed. In addition, if $P_i \subsetneq P_j$, then $\phi^{-1}(P_i)R_P \subsetneq \phi^{-1}(P_j)R_P$, so $\phi^{-1}(P_i) \subsetneq \phi^{-1}(P_j)$. Therefore, any chain of primes in R_P lifts to an equal length chain in R.

On the other hand, let $P_1, P_2, \ldots$ be a sequence of primes in R such that $\dim P_i \to \dim R$. This is possible because for a finite chain with minimal prime Q, $\dim Q$ is the length of that chain, and for an infinite chain, by taking smaller and smaller primes in the chain, we get such a sequence. If $P_i \subset Q_{i1} \subset Q_{i2} \ldots$ is a chain in R starting with P_i (i.e. a chain corresponding to one in R/P_i), then it will be a chain of the same length in R_{P_i}. Thus we have that $\dim R_{P_i} \geq \dim P_i$, so $\sup_{P \subset R\text{prime}} \dim R_P \geq \dim R$.

Now, if I is nilpotent, we have $\dim R \geq \dim R/I$, the fourth isomorphism theorem gives us a correspondence between prime ideals of R/I and prime ideals of R containing I. Now I is contained in the nilradical of R, so it is contained in every prime of R, so chains of primes of R are in one-to-one correspondence with chains of primes of R/I, so $\dim R = \dim R/i$. □

6.)

In this problem, we let R be a Noetherian ring and $\mathfrak{p} \lhd R$ a prime ideal of codimension c.

Proposition (Eisenbud 10.2)**.** *Let $Q \lhd R[x]$ such that $Q \cap R = \mathfrak{p}$]. Then, either* (a) $Q = \mathfrak{p}R[x]$ *in which case* $\operatorname{codim} Q = c$ *or* (b) $Q \supsetneq \mathfrak{p}R[x]$ *in which case* $\operatorname{codim} Q = c + 1$*. We break our response into two parts along those lines.*

a.)

Proposition. $\operatorname{codim}(\mathfrak{p}R[x]) = c$

Proof. We first show $\operatorname{codim} \mathfrak{p}R[x] \geq c$. We let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_c = \mathfrak{p}$ be a chain of primes in R terminating at $\mathfrak{p}$. Then, $\mathfrak{p}_0 R[x] \subsetneq \mathfrak{p}_1 R[x] \subsetneq \cdots \subsetneq \mathfrak{p}_c R[x] = \mathfrak{p}R[x]$ is a chain of primes in $R[x]$ terminating at $\mathfrak{p}R[x]$ so $\operatorname{codim} \mathfrak{p}R[x] \geq c$. We now show that $\operatorname{codim} \mathfrak{p}R[x] \leq c$. By the converse to the principal ideal theorem (Eisenbud 10.5), there exists some $Z = \{z_1, \ldots, z_c\} \subset R$ such that $\mathfrak{p}$ is minimal in the set of primes containing Z. Then, via the ring extension $R \subset R[x]$, we have that any prime of $R[x]$ containing Z must contain $\mathfrak{p} \subset R$ as well via minimality of $\mathfrak{p}$ over Z. As $\mathfrak{p}R[x]$ is the unique minimal prime ideal of $R[x]$ containing $\mathfrak{p}$, we have that $\mathfrak{p}R[x]$ is minimal over $Z \subset R[x]$. Thus, by the principal ideal theorem (Eisenbud 10.2), we have that $\operatorname{codim} \mathfrak{p}R[x] \leq c$, and hence $\operatorname{codim} \mathfrak{p}R[x] = c$. □

b.)

Proposition. *If* $\mathfrak{p}R[x] \neq Q \lhd R[x]$ *with* $Q \cap R = \mathfrak{p}$*, then* $\operatorname{codim} Q = c + 1$

Before proving the main proposition, we introduce a crucial lemma.

Lemma 6.A. *There is a containment-preserving bijection between prime ideals of $R[x]$ intersecting R in $\mathfrak{p}$ and ideals of $k(\mathfrak{p})[x]$ where $k(\mathfrak{p})$ is the residue field $k(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.*

Proof of Lemma 6.A. By the fourth isomorphism theorem, there is a containment and primality-preserving bijection between ideals of $R[x]$ containing $\mathfrak{p}R[x]$ and ideals of $R[x]/(\mathfrak{p}R[x]) \cong (R/\mathfrak{p})[x]$. Then by Eisenbud's Proposition 2.2a, there is a containment-preserving bijection between prime ideals of $(R/\mathfrak{p})[x]$ not meeting $R/\mathfrak{p} \setminus \{0\} \subset (R/\mathfrak{p})[x]$ and prime ideals of $((R/\mathfrak{p})[x]) \left[(R/\mathfrak{p} \setminus \{0\})^{-1}\right] \cong (R/\mathfrak{p})_{\mathfrak{p}}[x]$. The composition of these two primality-preserving and containment-preserving bijections then gives the desired identification. ■

Proof of main proposition. We induct on $\dim \mathfrak{p}$. To establish a basis, we let $\mathfrak{p}$ be of codimension 0 and $Q \lhd R[x]$ with $Q \cap R = \mathfrak{p}$. We then let $Q_0 \subsetneq \cdots \subsetneq Q_m = Q$ with $m := \operatorname{codim} Q$ be a chain of prime ideals. We let $\mathfrak{q}_i := Q_i \cap R \triangleleft R$ and note necessarily $\mathfrak{q}_i \subseteq \mathfrak{p}$ with $\mathfrak{q}_i$ prime. Thus, $\mathfrak{q}_i = \mathfrak{p}$ by our assumption $\operatorname{codim} \mathfrak{p} = 0$, so by Lemma 6.A $Q_0 \subsetneq \cdots \subsetneq Q_m$ corresponds to a chain of prime ideals $\tilde{Q}_0 \subsetneq \cdots \subsetneq \tilde{Q}_m \triangleleft k(\mathfrak{p})[x]$. As $\dim k(\mathfrak{p})[x] = 1$ by theorem A in §8.2.1 of Eisenbud, we then have that $m \leq 1$ with equality if $Q \neq \mathfrak{p}R[x]$, thus establishing a basis for induction.

For our inductive step, we let $\dim \mathfrak{p} = c$ and let $Q \triangleleft R[x]$ with $Q \cap R = \mathfrak{p}$ and $\dim Q =: m \geq c$. As before, we let $Q_0 \subsetneq \cdots \subsetneq Q_m = Q$ be a chain of prime ideals in $R[x]$ and let $\mathfrak{q}_i := Q_i \cap R$. We consider two cases:

(*Case 1:* $\mathfrak{q}_{m-1} = \mathfrak{p}$): By the lemma, we have that if $\mathfrak{p}R[x] \subseteq Q' \subsetneq Q$ with Q' prime in $R[x]$, we have that $Q' = \mathfrak{p}R[x]$. Thus, we must have that $Q_{m-1} = \mathfrak{p}R[x] \neq Q$ and by maximality of $Q_0 \subsetneq \cdots \subsetneq Q_m$, we have that $m - 1 = c$ and hence $m = c + 1$ as desired.

(*Case 2:* $\mathfrak{q}_{m-1} \subsetneq \mathfrak{p}$): By our inductive hypothesis, we now have that $\dim Q_{m-1} \leq (c-1)+1 = c$. Thus, by maximality of $Q_0 \subsetneq \cdots \subsetneq Q_m$, we have that $c \leq m \leq c+1$ with equality only if $Q \neq \mathfrak{p}R[x]$, as desired.

This completes our induction and thus our proof. □

7.)

(Eisenbud 10.3) Let k be a field. Show that the ring $k[x] \times k[x]$ contains a principal prime ideal of codimension 1, although it is not a domain. (By the argument of Corollary 10.14, there is no such example in a local ring.)

Proof. Consider the ideal $P = \langle (1, x) \rangle$. This is a prime ideal because $(k[x] \times k[x])/P \cong 0 \times k \cong k$, which is a domain.

Suppose $Q = \langle \{f_i, g_i\}_{i=1}^n \rangle$ is another prime ideal strictly contained in P. The f_i must generate a prime ideal of $k[x]$ and the g_i must generate a prime ideal of $k[x]$ contained in (x). Thus the g_i generate all of x or all the g_i are 0. This implies that there is one g_i and it is either 0 or x, and there is one f_i.

Thus there are five types of ideals strictly contained in P: $\langle (1, 0) \rangle, \langle (x - a, 0) \rangle$ for some $a \in k$, $\langle (0, 0) \rangle, \langle (x, x) \rangle$,, and $\langle (0, x) \rangle$. Of these, the only prime ideal is $\langle (1, 0) \rangle$, so $\operatorname{codim} P = 1$. □

8.)

Prompt. *Find a variety X in $\mathbb{A}^3$ which consists of a hyperplane P and a line L perpendicular to the hyperplane such that for any hyperplane parallel to P, $P \cap X$ is of dimension 0.*

Response. We note that the xy-plane in $\mathbb{A}^3$ is the variety of the ideal $\mathfrak{p} = \langle z \rangle \triangleleft R := k[x, y, z]$, while the z axis is the variety of the ideal $\mathfrak{q} = \langle x, y \rangle$. Thus, we may use the inclusion-reversing nature of the bijection given by the Nullstellensatz to compute the union of the two as $X = V(\mathfrak{p} \cap \mathfrak{q}) = V(\mathfrak{p}\mathfrak{q}) = V(xz, yz)$. We denote by J the ideal $\mathfrak{p}\mathfrak{q}$. As we show in problem 10, $\dim J = 2$. We consider the hyperplane Y which is given by $V(z-1)$. Then $Y \cap X$ is given by $V(I)$ where $I := \langle z - 1 \rangle + \langle xz, yz \rangle$. We note that $x = -x(z-1) + xz$ and $y = -y(z-1) + yz$, and thus $I = \langle z - 1, x, y \rangle$ which is well known to be maximal and hence of dimension 0. □

10.)

Prompt. *Consider the ring $R = k[x, y, z]/(xz, yz)$. Show that the ring is 2-dimensional, find an explicit system of parameters, and prove that the ring does not have any regular sequence $f_1, f_2 \subseteq \mathfrak{m}$.*

Response. In what follows, we identify without comment R with the vector space $k[x, y] \oplus zk[z]$ having the appropriate ring structure. Note that R is not local, but it is graded, so we can modify the definitions of systems of parameters and regular sequences for this setting.

We claim that $\langle z \rangle \subset \langle x, z \rangle \subset \mathfrak{m}$ is a chain of distinct prime ideals. Primeness follows because $R/\langle z \rangle \cong k[x, y]$ and $R/\langle x, z \rangle \cong k[y]$ are both domains; the second ideal is not maximal because $k[y]$ is not a field, and the first two ideals are distinct because x annihilates $\langle z \rangle$ but not $\langle x, z \rangle$.

We also claim that $\{x+z, y+z\}$ is a system of parameters, i.e. that $\langle x, y, z \rangle^n \subseteq \langle x + z, y + z \rangle$ for all large n. Our proof proceeds by showing that the inclusion holds already at $n = 2$, using brute force. An arbitrary element of $\mathfrak{m}^2$ can be written in the form

$$\sum_{i \geq 2} A_i x^i + B_i y^i + C_i z^i + \sum_{i,j \geq 1} D_{ij} x^i y^j,$$

where A_i, B_i, C_i, and D_{ij} are all in R. (Note this representation is not unique, and we are not claiming that these four families may be chosen independently.) Then define $E_k = \sum_{i=1}^{k-1} D_{i(k-i)}$ and consider the following element of R:

$$\sum_{i \geq 2} A_i(x+z)^i + B_i(y+z)^i + \sum_{i,j \geq 1} D_{ij}(x+z)^i(y+z)^j + \sum_{k \geq 2}(C_k - A_k - B_k - E_k)z(x+z)^{k-1}.$$

This agrees with the element of $\mathfrak{m}^2$ described above. The coefficients of $x^i y^j$ all come from the first two summation symbols, and in this case they clearly agree. Moreover, the z^i coefficients agree because all the z^i that appear in the first two summations are cancelled out in the third, and then a coefficient of C_k is added. Finally, observe that every term has at least one factor of $x + z$ or $y + z$, and therefore, an arbitrary element of $\mathfrak{m}^2$ is contained in $\langle x + z, y + z \rangle$.

Hence we have shown that the dimension of R is 2. However, we wish to show that R has no regular sequence (f_1, f_2) contained in the maximal ideal. Unwinding the definitions, this means that we need to show that for all nonzero $f_1, f_2 \in \mathfrak{m}$, then either f_1 is a zerodivisor, f_1 divides f_2, or there exist nonzero $\alpha, \beta \in R$ such that $\beta f_1 = \alpha f_2$, but f_1 does not divide α (in R).

The proof requires a bit of casework: We note first that $f_i \in \mathfrak{m}$ means $f_i = p_i(x, y) + q_i(z)$ where p_i and q_i have no constant term. The "typical" case is that p_1, p_2, q_1, and q_2 are all nonzero. In this case, let $\beta = q_2(z)$ and $\alpha = q_1(z)$. We see (by direct computation, if you want) that f_1 does not divide α, but that $\beta f_1 = q_2(z)q_1(z) = f_2\alpha$.

This choice of α and β also works if $p_2 = 0$ but all others are nonzero. Similarly, if $q_2 = 0$ but all others are nonzero, we can choose $\beta = p_2(x, y)$ and $\alpha = p_1(x, y)$, since then $\beta f_1 = p_2p_1 = f_2\alpha$. If either $p_1 = 0$ or $q_1 = 0$ (regardless of p_2 or q_2) then either zf or xf are zero, i.e. f is a zerodivisor. □

12.)

Prompt. *(Eisenbud 10.4) Let a, b be a regular sequence in a domain R, and let $S = R[x]$ be the polynomial ring in one variable over R. Show that $ax - b$ is a prime of S.*

Proof. We will show that the map $\phi : S \to R[1/a]$ given by $\phi(x) = b/a$ has kernel $(ax - b)$. This would imply that $(ax - b)$ is a prime ideal because then $S/(ax - b)$ would be isomorphic to a domain.

Suppose $p(x) \in \ker \phi$. We will show $p(x) \in (ax - b)$ by induction on $\deg p(x)$, starting with $\deg p(x) = 0$. In this case, $p(x) = 0$, so $p(x) \in (ax - b)$.

Now suppose $p(x) = \sum_{i=1}^n r_i x^i \in \ker \phi$. This implies

$$\sum_{i=1}^n r_i (b/a)^i = 0$$

in $R[1/a]$. □

13.)

Prompt. *(Eisenbud 10.6) Here is an example showing that* $\operatorname{codim} PS \leq \operatorname{codim} P$ *may fail when R is not regular. Let $R = k[x, y, s, t]/(xs - yt)$ and let $S = R/(x, y) \cong k[s, t]$. For $P = (s, t) \subseteq R$, prove that* $\operatorname{codim} P = 1$ *but* $\operatorname{codim} PS = 2$.

Proof. We know that $\operatorname{codim} PS = 2$ immediately from Corollary 10.4; and we can show that $\operatorname{codim} P \neq 1$ simply by showing that P is not nilpotent (using, for instance, the graded version of Corollary 10.7). But it is clear that $s^n \in P^n$ is nonzero for any n.

It remains to show that $\operatorname{codim} P \leq 1$; for this we are done if P satisfies the conditions of the prime ideal theorem; i.e. if P is minimal among the primes containing some element of R. Consider all primes Q containing s. Necessarily $yt = xs \in Q$; since Q is prime, either y or t must be in Q. If $t \notin Q$, then $y \in Q$. But $y \notin P$ so these ideals are incomparable. On the other hand, if $t \in Q$ then all of P's generators are in Q, so $P \subseteq Q$. □

17.)

Prompt. *(Eisenbud 13.2) Let G be a finite group acting on a domain T, and let R be the ring of invariants $R = T^G$. Then every element $b \in T$ is integral over R, and in fact over the subring generated by the elementary symmetric functions in the conjugates σb.*

Proof. The i-th elementary symmetric function is

$$e_i = \sum_{\{\sigma_1,\ldots,\sigma_i\}\subset G} (\sigma_1 b)\ldots(\sigma_i b),$$

and clearly this is G-invariant since action by an element of G simply permutes the terms. So indeed $e_i \in R$.

?????????????? □

21.)

Prompt. *Let $I = \langle xz - y^2, yw - z^2, yz - xw\rangle$, and let $f = x^2y^2w^2 - y^4z^2$. Determine whether or not f lies in I.*

Response. We will use the division algorithm using the lex monomial order with the variable order $x > y > z > w$. The leading term of f is $x^2y^2w^2$, and the first term of the generator $yw - z^2$ divides this. $x^2y^2w^2 - y^4z^2 = yw(x^2yw)$, and subtracting $(yw - z^2)(x^2yw)$ from f gives us $f' = x^2yz^2w - y^4z^2$. yw still divides this leading term, so we subtract $(yw - z^2)(x^2z^2)$, and get $f_1 = x^2z^4 - y^4z^2 = f - (yw - z^2)(x^2yw + x^2z^2)$.

Now, the leading term of f_1 is divisible by the leading term of $xz - y^2$, so we subtract $(xz - y^2)(xz^3)$ from f_1, and get $f_1' = xy^2z^3 - y^4z^2$. Then $xz - y^2$ still divides the leading term, so we subtract $(xz - y^2)(y^2z^2)$ from f_1', giving us $f_2 = y^4z^2 - y^4z^2 = 0 = f_1 - (xz - y^2)(xz^3 + y^2z^2)$. Therefore, f is in the ideal, and we can write $f = (yw - z^2)(x^2yw + x^2z^2) + (xz - y^2)(xz^3 + y^2z^2)$. □

22.)

Prompt. *Let $R = \mathbb{Q}[x, y, z]$. Find polynomials, $f, g_1, g_2 \in R$ such that $(f\%g_1)\%g_2 \neq (f\%g_2)\%g_1$.*

Response. We fix the lex monomial order with $x > y > z$. Let $f = x^2y^2 - y$, $g_1 = x^2y - 1$, $g_2 = xy^2 - 1$. Then, $f = yg_1$, so $(f\%g_1) = 0 = (f\%g_1)\%g_2$. We now perform polynomial long division on f/g_2 to determine the remainder r. We have that $LT(f) = xLT(g_2)$, so we let $r_1 = f - xg_2 = -y + x$. Now, no term of r_1 is divisible by the leading term of g_2, so our process terminates and we have that $f = xg_2 + (x - y)$, so $(f\%g_2) = x - y$. Now, the leading term x^2y of g_1 divides no terms of $(f\%g_2)$, so we have $(f\%g_2)\%g_1 = (f\%g_2) = x - y \neq 0 = (f\%g_1)\%g_2$. $\square$