Free Resolutions Over a Complete Hypersurface (And Friends)

Based on "Homological algebra on a complete intersection, with an application to group representations." by David Eisenbud

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MATH 8212 University of Minnesota

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Notation & Motivation

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Such a B is a *complete intersection of codimension n.*
We shall study the structure of B-free resolutions of B-modules, relating these to their liftings to A.

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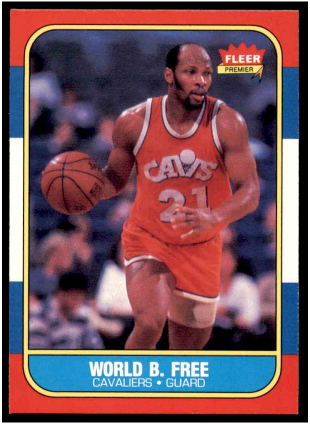

Figure: Another "B-Free" Object.

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For $(R, \mathfrak{m})$ local, R is regular $\iff \operatorname{gl}\dim(R) < \infty$.

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Theorem (Auslander-Buchsbaum-Serre)

For $(R, \mathfrak{m})$ *local, R is regular* $\iff \operatorname{gl}\dim(R) < \infty$.

We will give a similar (albeit more complicated) characterization of codim-1 quotients of regular local rings.

Codim n: Set-up

Recall that $B := A/\langle x_1, \ldots, x_n \rangle$. Let M be a B-module, and $\mathbf{F}$ a free resolution of B.

$$\mathbf{F} : \ldots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$$

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- $\tilde{F}_i$ be a free A-module with $\operatorname{rank}_A \tilde{F}_i = \operatorname{rank}_B F_i$
- $\tilde{\partial}_i$ denote an arbitrary lifting of (the entries of) ∂_i to A

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- Write $\tilde{\partial}_i \otimes \tilde{\partial}_{i+1} = \sum_{j=1}^{n} x_j \tilde{t}_j$ for some transformation $\tilde{t}_j := \tilde{t}_j(A, \{x_i\}, \mathbf{F}) : \tilde{\mathbf{F}} \to \tilde{\mathbf{F}}$.

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- Finally, let $t_j = t_j(A, \{x_i\}, \mathbf{F}) : \mathbf{F} \to \mathbf{F}$ be defined by $t_j = B \otimes \tilde{t}_j$.

The t_j have many nice properties!

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- *The t_j commute with each other up to homotopy.*
- *If M, N are B-modules, with B-free resolutions $\mathbf{F} \to M$, $\mathbf{G} \to N$ $t_j(\mathbf{F})$ and $t_j(\mathbf{G})$ induce the same map on $\mathrm{Tor}_{\bullet}^{B}(M, N)$.*

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- *The t_j play only a little less well with ring morphisms $\alpha : A \to A'$.*

Interlude: Matrix Factorizations

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Theorem (5.5)

Given ϕ, there exists a ψ such that (ψ, ϕ) is a matrix factorization if and only if **each** *of the following hold:*

- $\operatorname{rank} F = \operatorname{rank} G$
- $\det \phi$ *is a non-zero divisor*
- $x \cdot \mathrm{Fitt}_1(\phi) \subset \langle \det \phi \rangle$, *i.e.* x *annihilates* $\mathrm{coker}\phi$ *i.e.* ϕ *is f*

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- Thus, $\mathrm{Fitt}_1(\phi) = \langle W(Y-Z), (Y-Z)^2 \rangle$ and indeed $x \cdot \mathrm{Fitt}_1 \subset \langle \det\phi \rangle$
- ψ is uniquely determined by ϕ: $\psi = \frac{x}{\det\phi}\phi^c = \frac{(Y+Z)}{(Y-Z)}\phi^c$

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- We can construct a minimal B-free resolution $\mathbf{F}(\psi, \phi)$ of $\mathrm{coker}\phi$ with differentials ϕ, ψ.

$$\mathbf{F}(\psi, \phi) : \ldots \xrightarrow{\psi} F \xrightarrow{\phi} G \xrightarrow{\psi} F \xrightarrow{\phi} G \longrightarrow 0$$

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- If $\det_A \phi = x^k u$ with $u \notin \langle x \rangle$, then $\mathrm{rank}_B\, \mathrm{coker}\phi = k$. (5.6)

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3. *If so, $\mathbf{F}$ is induced by a matrix factorization.*

Sketch of Proof of Theorem 6.1

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Lemma (0.1–Well-known)

For B a local ring and B-free resolution

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1. *If $x_1, \ldots, x_r$ is a B-sequence, then it is a regular sequence on $\mathrm{Im}\partial_k$ for all $k \geq r$.*
2. *If $\mathbf{F}$ is minimal, $\mathrm{Im}\partial_k$ has no free summands for $k \geq 1 + \mathrm{depth} B$*

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- ($2 \implies$) Note that $M = \mathrm{coker}(G \to F)$.

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- Show if $\mathbf{F}(\phi, \psi)$ is not minimal, M contains a free summand.

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- Show that for any periodic resolution $\mathbf{F}$ over B, $t(A, x, \mathbf{F})$ can be chosen to be the identity!

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For $(R, \mathfrak{m})$ *local with* $\dim R = d$*, TFAE:*

- *For some* $x_1, \ldots, x_{d+1} \in R$, $\mathfrak{m} = \langle x_1, \ldots, x_{d+1} \rangle$ *and* $\hat{0}$ *is unmixed in* $\hat{R}^{\mathfrak{m}}$ *(i.e. all associated primes of* $\hat{0}$ *are minimal).*
- *For any f.g. R-module M with minimal free resolution* $\mathbf{F}$*, the truncation of* $\mathbf{F}$ *at degree* $d+1$ *is periodic of period 2.*
- *There exists a free resolution* $\mathbf{F} : \cdots \to F_1 \to F_0 \to 0$ *of* $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ *where for some* n, $\mathrm{rank} F_n < n$*.*

We call such an R an abstract hypersurface.

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Thank you for listening!