Free Resolutions Over a Complete Hypersurface (And Friends)

Based on "Homological algebra on a complete intersection, with an application to group representations." by David Eisenbud

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Notation & Motivation

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We shall study the structure of B-free resolutions of B-modules, relating these to their liftings to A.

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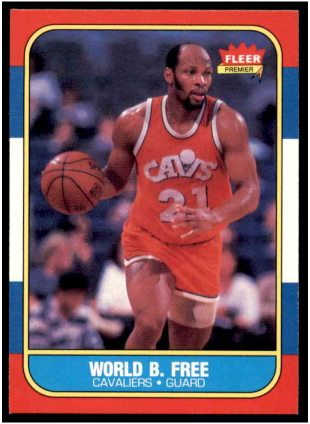

Figure: Another "B-Free" Object.

Why do we care?

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- Classifying maximal CM modules over complete hypersurfaces! (case $n = 1$)

Theorem (6.1)

Let $x \in A$, $d = \dim A$, and $B := A/\langle x \rangle$. Then,

- *For any B-module M and minimal free resolution $\mathbf{F}$, the truncation at F_{d+1} is periodic with period 2.*
- *$\mathbf{F}$ periodic $\iff$ M is a maximal CM module w/o a free summand.*
- *If so, $\mathbf{F}$ is induced a matrix factorization.*

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- Expanding upon a familiar characterization!

Theorem (Auslander-Buchsbaum-Serre)

For $(R, \mathfrak{m})$ local, R is regular $\iff \operatorname{gl}\dim(R) < \infty$.

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For $(R, \mathfrak{m})$ local with $\dim R = d$, TFAE:

- *For some $x_1, \ldots, x_{d+1} \in R$, $\mathfrak{m} = \langle x_1, \ldots, x_{d+1} \rangle$ and $\hat{0}$ is unmixed in $\hat{R}^{\mathfrak{m}}$ (i.e. all associated primes of $\hat{0}$ are minimal).*
- *For any f.g. R-module M with minimal free resolution $\mathbf{F}$, the truncation of $\mathbf{F}$ at degree $d+1$ is periodic of period 2.*
- *There exists a free resolution $\mathbf{F} : \cdots \to F_1 \to F_0 \to 0$ of $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ where for some n, $\mathrm{rank} F_n < n$.*

We call such an R an abstract hypersurface.

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Codim n: Set-up

Recall that $B := A/\langle x_1, \ldots, x_n \rangle$. Let M be a B-module, and $\mathbf{F}$ a free resolution of B.

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- $\tilde{F}_i$ be a free A-module with $\operatorname{rank}_A \tilde{F}_i = \operatorname{rank}_B F_i$
- $\tilde{\partial}_i$ denote an arbitrary lifting of (the entries of) ∂_i to A

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- Write $\tilde{\partial}_i \otimes \tilde{\partial}_{i+1} = \sum_{j=1}^{n} x_j \tilde{t}_j$ for some transformation $\tilde{t}_j := \tilde{t}_j(A, \{x_i\}, \mathbf{F}) : \tilde{\mathbf{F}} \to \tilde{\mathbf{F}}$.

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- Finally, let $t_j = t_j(A, \{x_i\}, \mathbf{F}) : \mathbf{F} \to \mathbf{F}$ be defined by $t_j = B \otimes \tilde{t}_j$.

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- *The t_j commute with each other up to homotopy.*
- *If M, N are B-modules, with B-free resolutions $\mathbf{F} \to M$, $\mathbf{G} \to N$ $t_j(\mathbf{F})$ and $t_j(\mathbf{G})$ induce the same map on $\operatorname{Tor}^B_\bullet(M, N)$.*

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- *The t_j play only a little less well with ring morphisms $\alpha : A \to A'$.*

Interlude: Matrix Factorizations

Let $x \in A$ and $\phi : F \to G$ a morphism of A-modules.

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Definition

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Theorem (5.5)

Given ϕ, there exists a ψ such that (ψ, ϕ) is a matrix factorization if and only if **each** *of the following hold:*

- rank $F =$ rank G
- $\det \phi$ *is a non-zero divisor*
- $x \cdot \mathrm{Fitt}_1(\phi) \subset \langle \det \phi \rangle$, *i.e. x annihilates* $\mathrm{coker}\phi$ *i.e. ϕ is f*

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- Thus, $\mathrm{Fitt}_1(\phi) = \langle W(Y - Z), (Y - Z)^2 \rangle$ and indeed $x \cdot \mathrm{Fitt}_1 \subset \langle \det \phi \rangle$
- ψ is uniquely determined by ϕ: $\psi = \frac{x}{\det \phi} \phi^c = \frac{(Y+Z)}{(Y-Z)} \phi^c$

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Let $x \in A$ be a nonzero divisor and (ψ, ϕ) a matrix factorization of x.

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