Free Resolutions Over a Complete Hypersurface (And Friends)

Based on "Homological algebra on a complete intersection, with an application to group representations." by David Eisenbud

David DeMark

MATH 8212 University of Minnesota

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Such a B is a *complete intersection of codimension n.*
We shall study the structure of B-free resolutions of B-modules, relating these to their liftings to A.

Clarification: B-Free

We do **NOT** mean...

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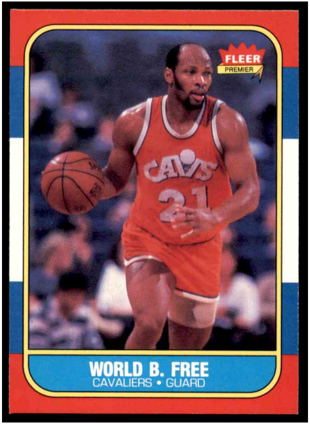

Figure: Another "B-Free" Object.

Why do we care?

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- Classifying maximal CM modules over complete hypersurfaces! (case $n = 1$)

Theorem (6.1)

Let $x \in A$, $d = \dim A$, and $B := A/\langle x \rangle$. Then,

- *For any B-module M and minimal free resolution $\mathbf{F}$, the truncation at F_{d+1} is periodic with period 2.*
- *$\mathbf{F}$ periodic $\iff$ M is a maximal CM module w/o a free summand.*
- *If so, $\mathbf{F}$ is induced a matrix factorization.*

Why do we care?

- Generalizing Auslander-Buchsbaum and Serre!

Theorem (Auslander-Buchsbaum-Serre)

For $(R, \mathfrak{m})$ local, R is regular $\iff \operatorname{gl}\dim(R) < \infty$.

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Theorem (6.1)

For $(R, \mathfrak{m})$ local with $\dim R = d$, TFAE:

- *For some $x_1, \ldots, x_{d+1} \in R$, $\mathfrak{m} = \langle x_1, \ldots, x_{d+1} \rangle$ and $\hat{0}$ is unmixed in $\hat{R}^{\mathfrak{m}}$ (i.e. all associated primes of $\hat{0}$ are minimal).*
- *For any f.g. R-module M with minimal free resolution $\mathbf{F}$, the truncation of $\mathbf{F}$ at degree $d + 1$ is periodic of period 2.*
- *There exists a free resolution $\mathbf{F} : \cdots \to F_1 \to F_0 \to 0$ of $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ where for some n, $\mathrm{rank} F_n < n$.*

We call such an R an abstract hypersurface.

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Set-up

Let M be a B-module, and $\mathbf{F}$ a free resolution of B.

$$\mathbf{F} : \ldots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$$

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Let $\tilde{\partial}_i$ denote an arbitrary lifting of