MATH 8254 Homework I

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1.)

3.)

Prompt. *Let S be a scheme. Construct a universal (w/r/t S) morphism $\Phi : \mathbb{P}^m_S \to \mathbb{P}^n_S$ such that when $S = \operatorname{Spec} k$ for k a field, the corresponding morphism on k-points*

(i) maps $[x_0 : \ldots : x_m] \mapsto [x_0 : \ldots : x_m : 0 : \ldots : 0]$ for any $n \geq m$.

(ii) maps $[x_0 : \ldots : x_m] \mapsto [M_{\mathbf{a}}(x) : \mathbf{a} \in C_m(d)]$ where $C_m(d)$ is the set of (ordered) compositions of d with $m+1$ parts (allowing parts of size zero) for $n = \binom{m+d}{d}$ and $M_{\mathbf{a}}(x) = x_0^{a_0} \ldots x_m^{a_m}$ for $\mathbf{a} = (a_0, \ldots, a_m)$.

Response. *(i)* We let $\mathbb{P}^m := \mathbb{P}^m_{\mathbb{Z}}$ be constructed by gluing schemes $u_i := \operatorname{Spec} \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_m}{x_i}]$ with gluing morphisms φ_{ij} as is standard and $\mathbb{P}^. := \mathbb{P}^n_{\mathbb{Z}}$ be constructed in much the same way from $v_i := \operatorname{Spec} \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$ with gluing morphisms ψ_{ij}. We construct the following morphism $\phi : \mathbb{P}^m \to \mathbb{P}^n$ by defining $\phi_i : u_i \to \mathbb{P}^n$ for each $i = 0, \ldots, m$ then checking intersections to show the maps ϕ_i commute with the gluing maps φ_{ij} on u_{ij}. We define ϕ_i by mapping u_i to v_i via the affine scheme morphism induced by the ring morphsim $\phi_i^\# : \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \to \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_m}{x_i}]$ given by

$$\frac{x_\ell}{x_i} \mapsto \begin{cases} \frac{x_\ell}{x_i} & \ell \leq m \\ 0 & \text{else} \end{cases}$$

We symmetrize the rings $\mathcal{O}_{u_i}(u_i)$ by letting $R_m := \mathbb{Z}[x_0, \ldots, x_m, x_0^{-1}, \ldots, x_m^{-1}]_0$, that is the degree-0 component of the ring of Laurent polynomials in $\mathbb{Z}$. Then we may identify $\mathcal{O}_{u_i}(u_i) = \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] = \mathbb{Z}[x_0, \ldots, x_n, x_i^{-1}]_0$ with its isomorphic image as a subring of R_m for all i. We do the same for the ring $R_n := \mathbb{Z}[x_0, \ldots, x_n, x_0^{-1}, \ldots, x_n^{-1}]_0$ w/r/t $\mathcal{O}_{v_j}(v_j) = \mathbb{Z}[x_0, \ldots, x_n, x_j^{-1}]_0$. Then, $u_{ij} = u_{ji} = \mathbb{Z}[x_0, \ldots, x_m, x_i^{-1}, x_j^{-1}]_0$ with φ_{ij} the identity morphism. Now, $\phi_i|_{u_{ij}} : u_{ij} \to v_{ij}$ and $\phi_j|_{u_{ij}} : u_{ij} \to v_{ij}$ are clearly both the maps inherited from $\phi_{ij}^\# : \mathcal{O}_{v_{ij}}(v_{ij}) = \mathbb{Z}[x_0, \ldots, x_n, x_i^{-1}, x_j^{-1}]_0 \to \mathbb{Z}[x_0, \ldots, x_m, x_i^{-1}, x_j^{-1}]_0 = \mathcal{O}_{u_{ij}}(u_{ij})$ defined by

$$\frac{x_\ell}{x_i} \mapsto \begin{cases} \frac{x_\ell}{x_i} & \ell \leq m \\ 0 & \text{else} \end{cases} \qquad\qquad \frac{x_\ell}{x_j} \mapsto \begin{cases} \frac{x_\ell}{x_j} & \ell \leq m \\ 0 & \text{else} \end{cases}$$

and indeed it is trivial that $\phi_{ij} = \phi_{ji} \circ \varphi_{ij}$ as φ_{ij} is simply the identity. Thus, we have indeed defined a morphism $\phi : \mathbb{P}^m \to \mathbb{P}^n$.

Letting S be a scheme, we let $\mathbb{P}^k_S = \mathbb{P}^k \times_{\mathbb{Z}} S$ and have (from problem 7)[1] that there is a natural[2] bijection $\operatorname{Mor}_{\mathbf{Sch}}(\mathbb{P}^m_S, \mathbb{P}^n_S) \equiv \operatorname{Mor}_{\mathbf{Sch}}(\mathbb{P}^m_S, \mathbb{P}^k) \times \operatorname{Mor}_{\mathbf{Sch}}(\mathbb{P}^m_S, S)$. We consider Φ, the map given by (ϕ, id_S) in the case that $S = \operatorname{Spec} k$ and restrict to $u_i \times_{\mathbb{Z}} S \cong \operatorname{Spec} \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_m}{x_i}] \otimes_{\mathbb{Z}} k \cong \operatorname{Spec} k[\frac{x_0}{x_i}, \ldots, \frac{x_m}{x_i}] \subset \mathbb{P}^m_S$. Then, $\Phi|_{u_i \times_{\mathbb{Z}} S} : u_i \times_{\mathbb{Z}} S \to v_i \times_{\mathbb{Z}} S$ corresponds to the ring morphism $\Phi|_{u_i \times_{\mathbb{Z}} S}{}^\# : k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \to k[\frac{x_0}{x_i}, \ldots, \frac{x_m}{x_i}]$ by

$$\frac{x_\ell}{x_i} \mapsto \begin{cases} \frac{x_\ell}{x_i} & \ell \leq m \\ 0 & \text{else} \end{cases}$$

So for any k-point $\gamma : \operatorname{Spec} k \to \mathbb{P}^m_k$ over S corresponding to $[a_0 : \ldots : a_m]$ where $a_j = \gamma_i(\frac{x_j}{x_i})$ for some i such that $\operatorname{Im}(\gamma) \subset u_i$ as topological spaces when $j \neq i$ and $a_i = 1$, we have that the k-point of $\mathbb{P}^n_k$ given by $\Phi \circ \gamma$

[1] ooh fun a forward reference!

[2] Is this a correct usage of the word natural? I'm not sure—all I mean by it is that the bijection is induced by the universal property of the product.

corresponds to the composition of ring maps $k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \to k[\frac{x_0}{x_i}, \ldots, \frac{x_m}{x_i}] \to k$ by

$$\frac{x_\ell}{x_i} \mapsto \begin{cases} \frac{x_\ell}{x_i} & \ell \leq m \\ 0 & else \end{cases} \mapsto \begin{cases} \gamma_i(\frac{x_\ell}{x_i}) = a_i & \ell \leq m \\ 0 & else \end{cases}$$

and hence the induced map on k-points is given by $[a_0 : \ldots : a_m] \mapsto [a_0 : \ldots : a_m : 0 : \ldots : 0]$

(ii) Much of our construction here is identical to that as precedes and as such, we shall skip over many of the technical details and emphasize the differences in the construction. We again define a map $\phi : \mathbb{P}^m \to \mathbb{P}^n$ where $n = \binom{m+d}{d}$ by defining its restrictions to u_i and checking that each ϕ_i agrees on the intersection. We define distinguished compositions $\mathbf{a}^{(i)} = (0, \ldots, 0, d, 0, \ldots,)$ such that $M_{\mathbf{a}^{(i)}}(x)$ and let $\phi_i : u_i \to v_{\mathbf{a}^{(i)}}$ be defined by the ring map $\phi_i^{\#} : \mathbb{Z}[\frac{x_{\mathbf{b}}}{x_{\mathbf{a}^{(i)}}} : \mathbf{b} \in C_m(d)] \to \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_m}{x_i}]$ given by

$$\frac{x_{\mathbf{b}}}{x_{\mathbf{a}^{(i)}}} \mapsto M_{\mathbf{b}} \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_m}{x_i} \right)$$

Then, identifying $\mathcal{O}_{u_i}(u_i)$ with its isomorphic image in R_m as in the previous problem, this corresponds to $M_{\mathbf{b}}(x_0, \ldots, x_m)/x_i^d = M_{\mathbf{b}}(x)/M_{\mathbf{a}^{(i)}}(x)$. Now, by the symmetry of our definition, it is clear that our maps ϕ_i, ϕ_j agree on intersections, defining a morphism $\phi : \mathbb{P}^m \to \mathbb{P}^n$.

We let Φ be as before with respect to our new morphism ϕ and let $\gamma : \operatorname{Spec} k \to \mathbb{P}^m_k$ be a k-point corresponding to $[a_1 : \ldots : a_m]$. We let i be such that a_i is nonzero and take the equivalence class with $a_i = 1$ so $a_j = \gamma_j(\frac{x_j}{x_i})$. Then $\operatorname{Im}(\gamma) \subset u_i$ as topological spaces. Then, $\Phi \circ \gamma$ is a k-point of $\mathbb{P}^n_k$, corresponding to the composition of ring maps $k[\frac{x_{\mathbf{b}}}{x_{\mathbf{a}^{(i)}}} : \mathbf{b} \in C_m(d)] \to k[\frac{x_0}{x_i}, \ldots, \frac{x_m}{x_i}] \to k$ by

$$\frac{x_{\mathbf{b}}}{x_{\mathbf{a}^{(i)}}} \mapsto M_{\mathbf{b}} \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_m}{x_i} \right) \mapsto M_{\mathbf{b}}(a_1, \ldots, a_m) \tag{1}$$

Thus in general, the induced map of k-points is $[a_1 : \ldots : a_m] \mapsto [M_{\mathbf{b}}(\mathbf{a}) : \mathbf{b} \in C_m(d)]$.

□

4.)

Proposition. *For a field K, $\mathbb{P}^2_K$ is not necessarily scheme-isomorphic to $\mathbb{P}^1_K \times_K \mathbb{P}^1_K$.*

Proof. We follow the suggestion of the hint and count $\mathbb{F}_p$-points of $\mathbb{P}^2_{\mathbb{F}_P}$ and of $\mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}$.

Proposition 4.A. $|\mathbb{P}^2_{\mathbb{F}_P}(\mathbb{F}_P)| = p^2 + p + 1$

Proof of Proposition 4.A. We let $u_i = \operatorname{Spec} \mathbb{F}_p[\frac{x_\ell}{x_i} : \ell = 0, 1, 2] \cong \mathbb{A}^2_{\mathbb{F}_P}$ with $u_{ij} = D(\frac{x_j}{x_i}) \subset u_i$ and $u_{ijk} = D(\frac{x_k}{x_i}) \subset u_{ij}$ and let $\mathbb{P}^2_{\mathbb{F}_P}$ be the resulting scheme from gluing the u_i along the u_{ij}, as is standard. We recall that a $\mathbb{F}_P$-point α of $\mathbb{P}^2_{\mathbb{F}_P}$ is by definition a (scheme-) morphism $\operatorname{Spec} \mathbb{F}_P \to \mathbb{P}^2_{\mathbb{F}_P}$ over $\operatorname{Spec} \mathbb{F}_P$ as in Figure 1.

$$\begin{tikzcd} \operatorname{Spec} \mathbb{F}_P \arrow[rr, "\alpha"] \arrow[dr] & & \mathbb{P}^2_{\mathbb{F}_P} \arrow[dl] \\ & \operatorname{Spec} \mathbb{F}_P & \end{tikzcd}$$

Figure 1: A $\mathbb{F}_p$-point α of $\mathbb{P}^2_{\mathbb{F}_P}$.

We note that $\operatorname{Spec} \mathbb{F}_p$ is a single point as a topological space. Hence, as a map of topological spaces $\operatorname{Im}\alpha \subset u_i$ for some u_i. Thus, for any scheme morphism over $\mathbb{F}_p$ $\alpha : \operatorname{Spec} \mathbb{F}_p \to \mathbb{P}^2_{\mathbb{F}_P}$, α restricts to a scheme morphism over $\mathbb{F}_p$ $\alpha' : \operatorname{Spec} \mathbb{F}_p \to u_i \cong \mathbb{A}^2_{\mathbb{F}_P}$, resulting in the diagram of Figure 2. We may then apply the principal of inclusion-exclusion to count

$$\left|\mathbb{P}^2_{\mathbb{F}_P}(\mathbb{F}_P)\right| = \sum_{i=0}^{2} |u_i(\mathbb{F}_P)| - \sum_{0 \leq i < j \leq 2} |u_{ij}(\mathbb{F}_P)| + |u_{012}(\mathbb{F}_P)|.$$

By the symmetry of the definitions of u_i and u_{ij}, we fix $i < j$ and have that

$$\left|\mathbb{P}^2_{\mathbb{F}_P}(\mathbb{F}_P)\right| = 3|u_i(\mathbb{F}_P)| - 3|u_{ij}(\mathbb{F}_P)| + |u_{012}(\mathbb{F}_P)|. \tag{2}$$

$$\begin{tikzcd} \operatorname{Spec}\mathbb{F}_P \arrow[rr, "\alpha'"] \arrow[dr] & & u_i \arrow[dl] \\ & \operatorname{Spec}\mathbb{F}_P & \end{tikzcd}$$

Figure 2: Restricting $\alpha \in \mathbb{P}^2_{\mathbb{F}_P}(\mathbb{F}_P)$ to $\alpha' \in u_i(\mathbb{F}_P)$

$$\begin{tikzcd} \mathbb{F}_P & & \mathbb{F}_P[x,y] \arrow[ll, "\alpha'"'] \\ & \mathbb{F}_P \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Figure 3: Restricting $\alpha \in \mathbb{P}^2_{\mathbb{F}_P}(\mathbb{F}_P)$ to $\alpha' \in u_i(\mathbb{F}_P)$

Now, as each of the schemes in Figure 2 are affine, we may use the canonical bijection between affine scheme morphisms and ring morphisms to biject $u_i(\mathbb{F}_P)$ to morphisms over the ring $\mathbb{F}_P$ as in Figure 3.

We note that as a ring morphism is necessarily unital and ${\mathbb{F}_P}^+$ is cyclic, there is at most one ring morphism $\mathbb{F}_P \to R$ for any ring R and in particular exactly one for $R = \mathbb{F}_P, \mathbb{F}_P[x,y]$. Thus, for *any* ring morphism $\alpha' : \mathbb{F}_P[x,y] \to \mathbb{F}_P$, the diagram of Figure 3 commutes. Hence, $u_i(\mathbb{F}_P)$ is in bijection with $\operatorname{Mor}_{\mathbf{Ring}}(\mathbb{F}_P[x,y], \mathbb{F}_P)$. As any morphism $\alpha' \in \operatorname{Mor}_{\mathbf{Ring}}(\mathbb{F}_P[x,y], \mathbb{F}_P)$ is defined by two elements $\alpha'(x), \alpha'(y) \in \mathbb{F}_P$, we have that $|u_i(\mathbb{F}_P)| = p^2$.

We may use identical techniques to biject $u_{ij}(\mathbb{F}_P)$ with $\operatorname{Mor}_{\mathbf{Ring}}(\mathcal{O}_{u_{ij}}(u_{ij}), \mathbb{F}_p)$. As $u_{ij} = D(\frac{x_j}{x_i}) \subset u_i$, we have that $\mathcal{O}_{u_{ij}}(u_{ij}) \cong \mathbb{F}_P[x,y]_x$. By the universal property of localization, we have that $\operatorname{Mor}_{\mathbf{Ring}}(\mathcal{O}_{u_{ij}}(u_{ij}), \mathbb{F}_p)$ is in bijection with $\{\alpha \in \operatorname{Mor}_{\mathbf{Ring}}(\mathbb{F}_P[x,y], \mathbb{F}_P) \ : \ \alpha(x) \in {\mathbb{F}_P}^\times\}$, which is in turn in bijection with ${\mathbb{F}_P}^\times \times \mathbb{F}_P$. Thus, $|u_{ij}(\mathbb{F}_P)| = p(p-1)$.

Finally, identical techniques again may be employed to biject $u_{012}(\mathbb{F}_P)$ with $\operatorname{Mor}_{\mathbf{Ring}}(\mathcal{O}_{u_{012}}(u_{012}), \mathbb{F}_p)$. As $\mathcal{O}_{u_{012}}(u_{012}) \cong \mathbb{F}_P[x,y]_{x,y}$, we may use the same universal property argument to see $\operatorname{Mor}_{\mathbf{Ring}}(\mathcal{O}_{u_{012}}(u_{012}), \mathbb{F}_p)$ is in bijection with $({\mathbb{F}_P}^\times)^2$. Thus, $|u_{012}(\mathbb{F}_P)| = p(p-1)$.

We substitute our values for $|u_{\vec{i}}(\mathbb{F}_P)|$ into (2) to yield our final count for $\mathbb{P}^2_{\mathbb{F}_P}(\mathbb{F}_P)$:

$$\begin{aligned} \left|\mathbb{P}^2_{\mathbb{F}_P}(\mathbb{F}_P)\right| &= 3|u_i(\mathbb{F}_P)| - 3|u_{ij}(\mathbb{F}_P)| + |u_{012}(\mathbb{F}_P)| \\ &= 3p^2 - 3p(p-1) + (p-1)^2 = p^2 + p + 1, \end{aligned}$$

as desired. ■

Proposition 4.B. $\left|\left(\mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}\right)(\mathbb{F}_P)\right| = \left|\operatorname{Mor}_{\mathbf{Sch}}(\operatorname{Spec}\mathbb{F}_P, \mathbb{P}^1_{\mathbb{F}_P})\right|^2$

Proof of Proposition 4.B. We claim $\left(\mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}\right)(\mathbb{F}_P)$ is in bijection with $\mathbb{P}^1_{\mathbb{F}_P}(\mathbb{F}_P)^2$. Indeed, we note that $\left(\mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}\right)(\mathbb{F}_P)$ is in bijection with commuting diagrams of the form of Figure 4. As we have noted, there is only one scheme morphism

$$\begin{tikzcd} \operatorname{Spec}\mathbb{F}_P \arrow[rr, "\alpha"] \arrow[dr] & & \mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P} \arrow[dl] \\ & \operatorname{Spec}\mathbb{F}_P & \end{tikzcd}$$

Figure 4: A $\mathbb{F}_P$-point α of $\mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}$.

$\operatorname{Spec}\mathbb{F}_P \to \operatorname{Spec}\mathbb{F}_P$, so this diagram commutes automatically for any α. Hence, $\left(\mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}\right)(\mathbb{F}_P)$ is in bijection with $\operatorname{Mor}_{\mathbf{Sch}}\left(\operatorname{Spec}\mathbb{F}_P, \mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}\right)$. We consider the diagram of figure 5.

We have from the universal property of $\mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}$ that $\operatorname{Mor}_{\mathbf{Sch}}(\operatorname{Spec}\mathbb{F}_P, \mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P})$ is in bijection with pairs (α, β) such that the outer square in the diagram of figure 5 commutes. However, as there is only one Scheme morphism $\operatorname{Spec}\mathbb{F}_P \to \operatorname{Spec}\mathbb{F}_P$, this square commutes automatically for any $(\alpha, \beta) \in \operatorname{Mor}_{\mathbf{Sch}}(\operatorname{Spec}\mathbb{F}_P, \mathbb{P}^1_{\mathbb{F}_P})$. This proves our proposition. ■

Corollary 4.C. *There are no p such that* $|\mathbb{P}^2_{\mathbb{F}_P}(\mathbb{F}_P)| = \left|\left(\mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}\right)(\mathbb{F}_P)\right|$.

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{F}_P \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& \mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P} \arrow[r] \arrow[d] & \mathbb{P}^1_{\mathbb{F}_P} \arrow[d] \\
& \mathbb{P}^1_{\mathbb{F}_P} \arrow[r] & \operatorname{Spec}\mathbb{F}_P
\end{tikzcd}$$

Figure 5: A diagram illustrating maps $\operatorname{Spec}\mathbb{F}_P \to \mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}$

Proof of Corollary 4.C. We have from Proposition 4.B that $\left|\left(\mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}\right)(\mathbb{F}_P)\right| = \left|\mathrm{Mor}_{\mathbf{Sch}}(\operatorname{Spec}\mathbb{F}_P, \mathbb{P}^1_{\mathbb{F}_P})\right|^2$. As $\left|\mathrm{Mor}_{\mathbf{Sch}}(\operatorname{Spec}\mathbb{F}_P, \mathbb{P}^1_{\mathbb{F}_P})\right|$ is an integer, this implies $\left|\left(\mathbb{P}^1_{\mathbb{F}_P} \times_{\mathbb{F}_P} \mathbb{P}^1_{\mathbb{F}_P}\right)(\mathbb{F}_P)\right|$ is a perfect square. However, from Proposition 4.A, we have that $p^2 < \left|\mathbb{P}^2_{\mathbb{F}_P}(\mathbb{F}_P)\right| = p^2 + p + 1 < (p+1)^2$ for any $p > 0$ and hence $\left|\mathbb{P}^2_{\mathbb{F}_P}(\mathbb{F}_P)\right|$ is not a perfect square for any prime p. $\blacksquare$

$\square$