MATH 8254 Homework V

David DeMark

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2.)

Proposition. *We let X be a* ***ringed***[1] *space, $\mathscr{F}$ a sheaf of Abelian groups on X, and $\mathfrak{u}$ an open covering on X with the property that for any finite $\{U_{i_1}, \dots, U_{i_p}\} \subset \mathfrak{u}$, $H^k\left(\bigcap_{j=1}^p U_{i_j}, \mathscr{F}|_{\bigcap_{j=1}^p U_{i_j}}\right) = 0$ for all $k > 0$. Then, the natural maps $H^i(\mathfrak{u}, \mathscr{F}) \to H^i(X, \mathscr{F})$ are isomorphisms for any $i \geq 0$.*

Proof. The case $i = 0$ is ensured by [Har77, p. III.4.1]. As sheaves of Abelian groups on X have enough injectives by [Har77, p. III.2.3], we may embed $\mathscr{F} \hookrightarrow \mathscr{G}$ where $\mathscr{G}$ is injective, inducing exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0. \tag{1}$$

We let V be an arbitrary finite intersection of sets in U. By hypothesis, $H^i(V, \mathscr{F}|_V) = 0$ for all $i > 0$. In particular, $H^1(V, \mathscr{F}|_V) = 0$ so the long exact sequence for the right-derived functor cohomology then gives an exact sequence of Abelian groups

$$0 \to \Gamma(\mathscr{F}|_V, V) \to \Gamma(\mathscr{G}|_V, V) \to \Gamma(\mathscr{H}|_V, V) \to 0$$

functorial in V as it is induced by a sequence of sheaves (1). Thus, taking products along all p-fold intersections in V, by a basic result on (finite) products of exact sequences, we yield an exact sequence

$$0 \longrightarrow C^p(\mathfrak{u}, \mathscr{F}) \longrightarrow C^p(\mathfrak{u}, \mathscr{G}) \longrightarrow C^p(\mathfrak{u}, \mathscr{H}) \longrightarrow 0$$

And by functoriality, we have that this extends to an exact sequence of complexes

$$0 \longrightarrow C^\bullet(\mathfrak{u}, \mathscr{F}) \longrightarrow C^\bullet(\mathfrak{u}, \mathscr{G}) \longrightarrow C^\bullet(\mathfrak{u}, \mathscr{H}) \longrightarrow 0 \tag{2}$$

As $\mathscr{G}$ is injective, its higher Čech cohomology disappears by [Har77, pp. III.2.4-5]. Thus, the long exact sequence for homology induced by (2) gives short exact sequences

$$0 \longrightarrow H^0(\mathfrak{u}, \mathscr{F}) \longrightarrow H^0(\mathfrak{u}, \mathscr{G}) \longrightarrow H^0(\mathfrak{u}, \mathscr{H}) \longrightarrow H^1(\mathfrak{u}, \mathscr{F}) \longrightarrow 0 \tag{3}$$

and for $p > 0$

$$0 \longrightarrow H^p(\mathfrak{u}, \mathscr{H}) \longrightarrow H^{p+1}(\mathfrak{u}, \mathscr{F}) \longrightarrow 0 \tag{4}$$

The natural map $H(\mathfrak{u}, \cdot) \to H(X, \cdot)$ now gives a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathfrak{u}, \mathscr{F}) \arrow[r] \arrow[d] & H^0(\mathfrak{u}, \mathscr{G}) \arrow[r] \arrow[d] & H^0(\mathfrak{u}, \mathscr{H}) \arrow[r] \arrow[d] & H^1(\mathfrak{u}, \mathscr{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(X, \mathscr{F}) \arrow[r] & H^0(X, \mathscr{G}) \arrow[r] & H^0(X, \mathscr{H}) \arrow[r] & H^1(X, \mathscr{F}) \arrow[r] & 0
\end{tikzcd} \tag{5}$$

As we know the first three vertical morphisms are isomorphisms, it follows $H^1(\mathfrak{u}, \mathscr{F}) \xrightarrow{\sim} H^1(X, \mathscr{F})$. □ $p > 1$

6.)

We let k be a field, $X = \mathbb{P}_k^n$, and $Y \hookrightarrow X$ a closed subscheme which is a complete intersection of dimension $q \geq 1$. We have by problem four on the previous homework that Y corresponds to an ideal $\mathfrak{a} \lhd = k[X_0, \dots, X_n]$ which allows us to give the convenient definition that $\mathfrak{a}$ can be generated by $r := \operatorname{codim} Y = n - q$ elements, i.e. $\mathfrak{a} = \langle y_1, \dots, y_r \rangle$.

[1] I'm sure the proposition is true in general, but there is one snag I have not resolved without this weakening. See next footnote for where this is relevant.

a.)

Proposition. *For all $n \in \mathbb{Z}$, the natural map*

$$H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

is surjective.

Proof. We induct on $r = \operatorname{codim} Y$. In the case $r = 0$, then $Y = X$ and $H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is the identity map. For $r > 0$, we let $Z = V(x_1, \dots, x_{r-1})$. As Z is a complete intersection of codimension $r-1$, we have that $H^0(X, \mathcal{O}_X(n)) \to H^0(Z, \mathcal{O}_Z(n))$ is surjective for all n by inductive hypothesis. It remains to be shown that $H^0(Z, \mathcal{O}_Z(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is surjective. We let $S = k[X_0, \dots, X_n]/\langle x_1, \dots, x_{r-1}\rangle$ so that $Z = \operatorname{Proj} S$ and $Y = \operatorname{Proj} S/\langle x_r \rangle$. We consider the exact sequence of graded S-modules

$$0 \longrightarrow S(n-1) \xrightarrow{\cdot x_r} S(n) \longrightarrow S(n)/\langle x_r \rangle \longrightarrow 0 \tag{6}$$

Then, as association is an exact functor, we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_Z(n-1) \longrightarrow \mathcal{O}_Z(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0 \tag{7}$$

We consider the exact sequence of cohomology induced by (7).

$$0 \longrightarrow H^0(Z, \mathcal{O}_Z(n-1)) \longrightarrow H^0(Z, \mathcal{O}_Z(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)) \longrightarrow H^1(Z, \mathcal{O}_Z(n-1)) \longrightarrow \dots \tag{8}$$

From this sequence, we yield that if $H^1(Z, \mathcal{O}_Z(n-1)) = 0$, our desired result follows. We make an ugly forward reference to (d). □

c.)

Proposition. *For all $n \in \mathbb{Z}$ and $0 < i < q$, $H^i(Y, \mathcal{O}_Y(n)) = 0$.*

Proof. We once again induct on r. The case $r = 0$ is exactly [Har77, III.5.1(b)]. For $n > r > 0$, letting Z be as it was before, we once again the sequence (8). $\dots \longrightarrow H^i(Z, \mathcal{O}_Z(n-1)) \longrightarrow H^i(Z, \mathcal{O}_Z(n)) \longrightarrow H^i(Y, \mathcal{O}_Y(n)) \longrightarrow H^{i+1}(Z, \mathcal{O}_Z(n$
By inductive hypothesis, as $\dim Z = q+1$ and $i < q$, we have that $H^i(Z, \mathcal{O}_Z(n)) = H^{i+1}(Z, \mathcal{O}_Z(n-1)) = 0$. Thus, it follows that $H^i(Y, \mathcal{O}_Y(n)) = 0$. □

d.)

Corollary. $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$.

Proof. We have that $H^0(Y, \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y) \neq 0$ by sheaf properties and thus $\dim_k H^0(Y, \mathcal{O}_Y) \geq 1$. We have further more that it is (/may be viewed as) a k-algebra. By (a), there is a surjection $k \to H^0(Y, \mathcal{O}_Y)$ and thus $\dim_k H^0(Y, \mathcal{O}_Y) \leq 1$ so $\dim_k H^0(Y, \mathcal{O}_Y) = 1$. By (c), we have that $\dim_k H^i(Y, \mathcal{O}_Y) = 0$ for all $0 < i < q$. Thus, we compute

$$\chi(\mathcal{O}_Z) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(\mathcal{O}_Y, Y) = 1 + (-1)^q \dim_k H^q(Y, \mathcal{O}_Y).$$

We may then compute $p_a(Y) = (-1)^q(\chi(\mathcal{O}_Y) - 1) = (-1)^{2q} \dim_k H^q(Y, \mathcal{O}_Y) = \dim_k H^q(Y, \mathcal{O}_Y)$. □

References

[Har77] Robin Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.