MATH 8254 Homework VI

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STILL 2-DU 1-3 need work

1.)

We let A be a ring, $\mathfrak{a} \lhd A$, $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/\mathfrak{a}$, and $\mathrm{Bl}_Z X = \operatorname{Proj} A[\mathfrak{a}T]$.

Proposition. *The scheme theoretic preimage $E_Z(Z) := Z \times_X \mathrm{Bl}_Z X$ of Z w/r/t the projection $\mathrm{Bl}_Z X \to X$ is an effective Cartier divisor.*

Proof. ***a.)***

Lemma 1.A. *For any $a \in \mathfrak{a}$, $\Gamma(D_+(aT), \mathcal{O}_{\mathrm{Bl}_Z X}) = (A[\mathfrak{a}T]_{aT})_0$ is isomorphic to the subalgebra $A[a^{-1}\mathfrak{a}] = A \oplus a^{-1}\mathfrak{a} \oplus a^{-2}\mathfrak{a}^2 \oplus \cdots \subset A_a$*

Proof. We note that any element $x \in (A[\mathfrak{a}T]_{aT})_0$ can be written $x = \frac{bT^k}{(aT)^k}$ where $b \in \mathfrak{a}^k$, while any $y \in A[a^{-1}A]$ can be written $\frac{c}{a^\ell}$ where $c \in \mathfrak{a}^\ell$. We construct $\phi : (A[\mathfrak{a}T]_{aT})_0 \to A[a^{-1}A]$ by $\frac{bT^k}{(aT)^k} \mapsto \frac{b}{a^k}$ and wish to show ϕ is an isomorphism. To show that ϕ is well-defined, we suppose $\frac{bT^k}{(aT)^k} = \frac{b'T^\ell}{(aT)^\ell}$. Then, we have that for some r,

$$(aT)^r \left((aT)^\ell bT^k - (aT)^k b'T^\ell\right) = 0$$

in $A[\mathfrak{a}T]$. We view $A[\mathfrak{a}T]$ as a subalgebra of $A[T]$. Then, we have that

$$T^{\ell+k+r}\left(a^{r+\ell}b - a^{r+k}b'\right) = 0$$

As T is a nonzero-divisor in $A[T]$, we have that $a^{r+\ell}b - a^{r+k}b' = 0$ in $A \subset A[T]$. Thus, in A_a, we have that $\frac{b}{a^k} = \frac{b'}{a^\ell}$, as desired, so our ϕ is indeed well-defined. Injectivity and surjectivity then come along easily: to see injectivity, we suppose $\phi(\frac{b}{(aT)^k}) = \phi(\frac{b'}{(aT)^\ell})$, i.e. $\frac{b}{a^k} = \frac{b'}{a^\ell}$. We then have that there is some r such that, in A,

$$a^r(ba^\ell - b'a^k) = 0$$

Then, in $A[T]$, we multiply through by $T^{k+\ell+r}$ to yield

$$T^{k+\ell+r}a^r(ba^\ell - b'a^k) = (aT)^r\left((aT)^\ell bT^k - (aT)^k b'T^\ell\right) = 0$$

Thus, we have that $\frac{bT^k}{(aT)^k} = \frac{b'T^\ell}{(aT)^\ell}$. Finally, surjectivity follows obviously from what we have done so far. If $x = \frac{bT}{a^k} \in A[a^{-1}\mathfrak{a}]$, then, $x = \phi(\frac{bT^k}{(aT)^k})$. ∎

b.)

Proposition 1.B. *The quotient algebra $((A[\mathfrak{a}T]/\mathfrak{a}A[\mathfrak{a}T])_{aT})_0$ may be identified with the quotient of the algebra $A[a^{-1}\mathfrak{a}]$ by the ideal $aA[a^{-1}\mathfrak{a}]$.*

Proof. We note that the graded ideal $\mathfrak{a}A[\mathfrak{a}T]$ may be written as the idrect product $\bigoplus_{i\geq 0}\mathfrak{a}^{i+1}T^i$. Thus, the quotient ring $(A[\mathfrak{a}T]/\mathfrak{a}A[\mathfrak{a}T])$ may be written

$$A[\mathfrak{a}T]/\mathfrak{a}A[\mathfrak{a}T] = \bigoplus_{i\geq 0}(\mathfrak{a}^iT^i)/(\mathfrak{a}^{i+1}T^i) = \bigoplus_{i\geq 0}(\mathfrak{a}^i/\mathfrak{a}^{i+1})T^i.$$

Thus, we can write that

$$(A[\mathfrak{a}T]/\mathfrak{a}A[\mathfrak{a}T])_{aT} = A_{aT} \otimes \left(\bigoplus_{i\geq 0}(\mathfrak{a}^i/\mathfrak{a}^{i+1})T^i\right) = \bigoplus_{i\geq 0}\left((\mathfrak{a}^i/\mathfrak{a}^{i+1})T^i \otimes A_{aT}\right)$$

∎

□

3.)

We let M, N be graded R-modules where R is a positively graded ring and $X := \operatorname{Proj} R$.

Prompt. *Construct a sheaf morphism* $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to (M \otimes_R N)^{\sim}$

Proof. We use the following well-known lemma from commutative algebra:

Lemma 3.A. *If Q/S is an extension of graded rings and A, B are Q-modules (and hence S-modules as well), then there is an induced surjective graded homomorphism $A \otimes_S B \to A \otimes_Q B$.*

From the lemma, we acquire for any $f \in R$ a graded morphism $\overline{\psi}_f : M_f \otimes_{(R_f)_0} N_f \to M_f \otimes_{R_f} N_f$. Then, as $\overline{\psi}_f$ is a graded homomorphism and $(M_f)_0 \otimes_{(R_f)_0} (N_f)_0 \subset (M_f \otimes_{(R_f)_0} N_f)_0$ as modules, we have that $\overline{\psi}_f$ restricts to a morphism $\psi_f : (M_f)_0 \otimes_{(R_f)_0} (N_f)_0 \to (M_f \otimes_{R_f} N_f)_0$. □

b.)

Proposition. *The morphism of the previous part is an isomorphism if R is standardly graded.*

4.)

Proposition. *We let A be a graded ring. For any closed embedding $Z \hookrightarrow \mathbb{P}^n_A$, there is some graded ideal $\mathfrak{a} \lhd A[X_0, \ldots, X_n]$ such that $Z \hookrightarrow \mathbb{P}^n_A$ is the closed embedding corresponding to the morphism of rings $A[X_0, \ldots, X_n] \to A[X_0, \ldots, X_n]/\mathfrak{a}$.*

Proof. By [Har77, p. II.5.9], we have that the ideal sheaf $\mathcal{I}_Z$ corresponding to Z is quasi-coherent. We note that as $A[X_0, \ldots, X_n]$ is finitely generated by $X_0, \ldots, X_n$ over its degree-0 part A, we have that $A[X_0, \ldots X_n]$ fulfills the hypotheses of [Har77, p. II.5.15] and thus $\beta : \Gamma_*(\mathcal{I}_Y)^{\sim} \to \mathcal{I}_Y$ is an isomorphism. Thus, the $\mathfrak{a}$ in the statement of the problem above is in fact the graded module $\Gamma_*(\mathcal{I}_Y)$. □

5.)

a.)

Prompt. *Determine the Čech cohomology of the structure sheaf of X, the affine line with doubled origin with respect to the usual open affine cover $\mathcal{U}$ consisting of two open affine subsets isomorphic to $\mathbb{A}^1_k$.*

Computation. We name our two open affines U_0 and U_1 with each $U_i \cong \operatorname{Spec} k[x] = \mathbb{A}^1_k$. We have that $C^0(\mathcal{U}, \mathcal{O}_X) = \mathcal{O}_X(U_1) \otimes \mathcal{O}_X(U_2) = k[x] \oplus k[x]$. We also have only one nontrivial twofold intersection of the sets in our cover and thus an element of $C^1(\mathcal{U}, \mathcal{O}_X)$ is uniquely determined by its U_{01} component, that is $C^1(\mathcal{U}, \mathcal{O}_X) = \mathcal{O}_X(U_{01}) = k[x, x^{-1}]$. We now compute our only nontrivial differential d^0 on an arbitrary element $(g_0(x), g_1(x)) \in C^0(\mathcal{U}, \mathcal{O}_X)$ and have that $d^0 : (g_0(x), g_1(x)) = g_1(x) - g_0(x)$ as our restriction maps are both the standard localization maps $x \mapsto x$. Thus, $H^0(\mathcal{U}, \mathcal{O}_X) = \ker d^0$, which is the diagonal subring $D(k[x] \oplus k[x]) = \{(g(x), g(x)) : g(x) \in k[x]\} \cong k[x]$. We also have that (as $C^2 = 0$) $H^1(\mathcal{U}, \mathcal{O}_X) = \operatorname{coker} d^0 = k[x, x^{-1}]/k[x]$ as an Abelian group (or module over the ring of global sections $k[x]$). □

b.)

Prompt. *Determine the Čech cohomology of the structure sheaf of $\mathbb{P}^1_k$ with respect to the usual open affine cover $\mathcal{U}$ consisting of two open affine subsets $D_+(x_0), D_+(x_1)$ isomorphic to $\mathbb{A}^1_k$.*

Computation. We keep our setup from before, only changing our restriction map ρ^1_{01} on U_1 to the map $x \mapsto x^{-1}$. We have still that $C^0(\mathcal{U}, \mathcal{O}_X) = \mathcal{O}_X(U_1) \otimes \mathcal{O}_X(U_2) = k[x] \oplus k[x]$ and $C^1(\mathcal{U}, \mathcal{O}_X) = \mathcal{O}_X(U_{01}) = k[x, x^{-1}]$. We now compute our only nontrivial differential d^0 on an arbitrary element $(g_0(x), g_1(x)) \in C^0(\mathcal{U}, \mathcal{O}_X)$ and have that $d^0 : (g_0(x), g_1(x)) = g_1(x^{-1}) - g_0(x)$, with the x^{-1} in the argument of g_1 forced by our restriction map ρ^1_{01}. We note that if g_1 is homogenous of positive degree, then $g_1(x^{-1})$ is homogenous of negative degree, while $g_0(x)$ is of nonnegative degree. Thus, only degree 0 elements are in the kernel of d^0, that is $\ker d^0 = H^0(\mathcal{U}, \mathcal{O}_X) = k$. We claim that d^0 is a surjection. Indeed, $C_1(\mathcal{U}, \mathcal{O}_X)$ is the ring of Laurent polynomials in k; we let $g(x) = a_m x^m + \cdots + a_{-n} x^{-n}$ be an arbitrary element. Then, we let $g_1(x) = \sum_{k=0}^{n} a_{-k} x^k$ and $g_0(x) = \sum_{k=1}^{m} -a_k x^k$ and have that $d^0(g_0, g_1) = g$, thus showing that $H^1(\mathcal{U}, \mathcal{O}_X) = 0$. □

References

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.