MATH 8254 Homework V

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2.)

Proposition. *We let X be a* ***ringed***[1] *space, $\mathscr{F}$ a sheaf of Abelian groups on X, and $\mathfrak{u}$ an open covering on X with the property that for any finite $\{U_{i_1}, \ldots, U_{i_p}\} \subset \mathfrak{u}$, $H^k\left(\bigcap_{j=1}^{p} U_{i_j}, \mathscr{F}|_{\bigcap_{j=1}^{p} U_{i_j}}\right) = 0$ for all $k > 0$. Then, the natural maps $H^i(\mathfrak{u}, \mathscr{F}) \to H^i(X, \mathscr{F})$ are isomorphisms for any $i \geq 0$.*

Proof. The case $i = 0$ is ensured by [**?**, III.4.1]. As sheaves of Abelian groups on X have enough injectives by [**?**, III.2.3], we may embed $\mathscr{F} \hookrightarrow \mathscr{G}$ where $\mathscr{G}$ is injective, inducing exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0. \tag{1}$$

We let V be an arbitrary finite intersection of sets in U. By hypothesis, $H^i(V, \mathscr{F}|_V) = 0$ for all $i > 0$. In particular, $H^1(V, \mathscr{F}|_V) = 0$ so the long exact sequence for the right-derived functor cohomology then gives an exact sequence of Abelian groups

$$0 \to \Gamma(\mathscr{F}|_V, V) \to \Gamma(\mathscr{G}|_V, V) \to \Gamma(\mathscr{H}|_V, V) \to 0$$

functorial in V as it is induced by a sequence of sheaves (1). Thus, taking products along all p-fold intersections in V, by a basic result on (finite) products of exact sequences, we yield an exact sequence

$$0 \longrightarrow C^p(\mathfrak{u}, \mathscr{F}) \longrightarrow C^p(\mathfrak{u}, \mathscr{G}) \longrightarrow C^p(\mathfrak{u}, \mathscr{H}) \longrightarrow 0$$

And by functoriality, we have that this extends to an exact sequence of complexes

$$0 \longrightarrow C^\bullet(\mathfrak{u}, \mathscr{F}) \longrightarrow C^\bullet(\mathfrak{u}, \mathscr{G}) \longrightarrow C^\bullet(\mathfrak{u}, \mathscr{H}) \longrightarrow 0 \tag{2}$$

As $\mathscr{G}$ is injective, its higher Čech cohomology disappears by [**?**, III.2.4-5]. Thus, the long exact sequence for homology induced by (2) gives short exact sequences

$$0 \longrightarrow H^0(\mathfrak{u}, \mathscr{F}) \longrightarrow H^0(\mathfrak{u}, \mathscr{G}) \longrightarrow H^0(\mathfrak{u}, \mathscr{H}) \longrightarrow H^1(\mathfrak{u}, \mathscr{F}) \longrightarrow 0 \tag{3}$$

and for $p > 0$

$$0 \longrightarrow H^p(\mathfrak{u}, \mathscr{H}) \longrightarrow H^{p+1}(\mathfrak{u}, \mathscr{F}) \longrightarrow 0 \tag{4}$$

The natural map $H(\mathfrak{u}, \cdot) \to H(X, \cdot)$ now gives a morphism of exact sequences

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^0(\mathfrak{u}, \mathscr{F}) & \longrightarrow & H^0(\mathfrak{u}, \mathscr{G}) & \longrightarrow & H^0(\mathfrak{u}, \mathscr{H}) & \longrightarrow & H^1(\mathfrak{u}, \mathscr{F}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(X, \mathscr{F}) & \longrightarrow & H^0(X, \mathscr{G}) & \longrightarrow & H^0(X, \mathscr{H}) & \longrightarrow & H^1(X, \mathscr{F}) & \longrightarrow & 0
\end{array} \tag{5}$$

As we know the first three vertical morphisms are isomorphisms, it follows $H^1(\mathfrak{u}, \mathscr{F}) \xrightarrow{\sim} H^1(X, \mathscr{F})$. □ $p > 1$

[1] I'm sure the proposition is true in general, but there is one snag I have not resolved without this weakening. See next footnote for where this is relevant.