MATH 8254 Homework V

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2 (well, 8) April 2018

1.)

a.)

4.)

I'm only convinced that this is even true if

5.)

a.)

Prompt. *Determine the Čech cohomology of the structure sheaf of X, the affine line with doubled origin with respect to the usual open affine cover $\mathcal{U}$ consisting of two open affine subsets isomorphic to $\mathbb{A}^1_k$.*

Computation. We name our two open affines U_0 and U_1 with each $U_i \cong \operatorname{Spec} k[x] = \mathbb{A}^1_k$. We have that $C^0(\mathcal{U}, \mathcal{O}_X) = \mathcal{O}_X(U_1) \otimes \mathcal{O}_X(U_2) = k[x] \oplus k[x]$. We also have only one nontrivial twofold intersection of the sets in our cover and thus an element of $C^1(\mathcal{U}, \mathcal{O}_X)$ is uniquely determined by its U_{01} component, that is $C^1(\mathcal{U}, \mathcal{O}_X) = \mathcal{O}_X(U_{01}) = k[x, x^{-1}]$. We now compute our only nontrivial differential d^0 on an arbitrary element $(g_0(x), g_1(x)) \in C^0(\mathcal{U}, \mathcal{O}_X)$ and have that $d^0 : (g_0(x), g_1(x)) = g_1(x) - g_0(x)$ as our restriction maps are both the standard localization maps $x \mapsto x$. Thus, $H^0(\mathcal{U}, \mathcal{O}_X) = \ker d^0$, which is the diagonal subring $D(k[x] \oplus k[x]) = \{(g(x), g(x)) : g(x) \in k[x]\} \cong k[x]$. We also have that (as $C^2 = 0$) $H^1(\mathcal{U}, \mathcal{O}_X) = \operatorname{coker} d^0 = k[x, x^{-1}]/k[x]$ as an Abelian group (or module over the ring of global sections $k[x]$). □

Prompt. *Determine the Čech cohomology of the structure sheaf of $\mathbb{P}^1_k$ with respect to the usual open affine cover $\mathcal{U}$ consisting of two open affine subsets $D_+(x_0), D_+(x_1)$ isomorphic to $\mathbb{A}^1_k$.*

Computation. We keep our setup from before, only changing our restriction map ρ^1_{01} on U_1 to the map $x \mapsto x^{-1}$. We have still that $C^0(\mathcal{U}, \mathcal{O}_X) = \mathcal{O}_X(U_1) \otimes \mathcal{O}_X(U_2) = k[x] \oplus k[x]$ and $C^1(\mathcal{U}, \mathcal{O}_X) = \mathcal{O}_X(U_{01}) = k[x, x^{-1}]$. We now compute our only nontrivial differential d^0 on an arbitrary element $(g_0(x), g_1(x)) \in C^0(\mathcal{U}, \mathcal{O}_X)$ and have that $d^0 : (g_0(x), g_1(x)) = g_1(x^{-1}) - g_0(x)$, with the x^{-1} in the argument of g_1 forced by our restriction map ρ^1_{01}. We note that if g_1 is homogenous of positive degree, then $g_1(x^{-1})$ is homogenous of negative degree, while $g_0(x)$ is of nonnegative degree. Thus, only degree 0 elements are in the kernel of d^0, that is $\ker d^0 = H^0(\mathcal{U}, \mathcal{O}_X) = k$. We claim that d^0 is a surjection. Indeed, $C_1(\mathcal{U}, \mathcal{O}_X)$ is the ring of Laurent polynomials in k; we let $g(x) = a_m x^m + \cdots + a_{-n} x^{-n}$ be an arbitrary element. Then, we let $g_1(x) = \sum_{k=0}^{n} a_{-k} x^k$ and $g_0(x) = \sum_{k=1}^{m} -a_k x^k$ and have that $d^0(g_0, g_1) = g$, thus showing that $H^1(\mathcal{U}, \mathcal{O}_X) = 0$. □