MATH 8302 Take-Home Final

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1.)

We let $f : S^2 \to T^2$ be any smooth map.

a.)

We recall that by de Rahm's theorem, we have an isomorphism of rings $\Psi : H^*_{\mathrm{dR}}(T^2) \xrightarrow{\sim} H^*(T^2, \mathbb{R})$. As we showed in class last semester (or can be found in Hatcher), $H^*(T^2, \mathbb{R})$ is generated in degree one by $[\alpha], [\beta] \in H^1(T^2)$ under the relations $\langle [\alpha] \cup [\beta] + [\beta] \cup [\alpha], [\alpha]^2, [\beta]^2 \rangle$. In particular, we have that $H^2(T^2, \mathbb{R}) \cong \mathbb{R}\{[\alpha] \cup [\beta]\}$. We let $[\omega_\alpha] = \Psi^{-1}([\alpha])$, and $[\omega_\beta] = \Psi^{-1}([\beta])$. We recall as well that $H^*(S^2, \mathbb{R})$ is generated in degree two by a single element $[\gamma] \in H^2(S^2, \mathbb{R})$ modulo the relation $\langle [\gamma] \cup [\gamma] \rangle$. We let $[\omega_\gamma] \in H^2_{\mathrm{dR}}(S^2)$ be its image under de Rahm's theorem. We finally also recall that thanks to de Rahm's isomorphism of rings, the map f^* induces a graded morphism of $\mathbb{R}$-algebras.

Proposition. *For any closed $\omega \in \Omega^2(T^2)$, $f^*(\omega) \in \Omega^2(S^2)$ is exact.*

Proof. We let $\omega \in \Omega^2(T^2)$ be an arbitrary closed 2-form on T^2. We suppose for the sake of contradiction that $f^*(\omega)$ is not exact, that is $f^*([\omega]) = r[\omega_\gamma]$ for some $0 \neq r \in \mathbb{R}$. Then, we have two cases to consider. If ω is exact, we have that $[\omega] = [0]$ in $H^2(T^2)$. Then, $f^*([0]) \neq [0]$ contradicting that f^* is a ring homomorphism. On the other hand, if ω is not exact, then $[\omega] = q[\omega_\alpha] \wedge [\omega_\beta]$ for some $0 \neq q \in \mathbb{R}$. Then, we have that $f^*(q[\omega_\alpha]) \wedge f^*([\omega_\beta]) = r[\omega_\gamma]$, but as $H^1_{\mathrm{dR}}(S^2) = 0$ and f^* is a *graded* homomorphism, we then have that $[0] \wedge [0] = r[\omega_\gamma]$, contradicting that $r \neq 0$. This completes our proof. $\square$

b.)

Proposition. $\deg f = 0$.

Proof. As a corollary of de Rahm's theorem that we showed in class, for smooth path-connected q-manifolds M, N with smooth $f : M \to N$, $\deg f$ can be calculated as the determinant of $f^* : H^q_{\mathrm{dR}}(N) \to H^q_{\mathrm{dR}}(M)$. As we showed above, $f^* H^2_{\mathrm{dR}}(T^2) \to H^2_{\mathrm{dR}}(S^2)$ is the zero map, and hence $\deg f = 0$. $\square$

2.)

We let $f : \mathbb{C}^2 \to \mathbb{C}$ be defined by $f(z, w) = w^2 - z^3$, and let $X \subseteq \mathbb{C}^2$ be defined by $X := \{(z, w) \mid f(w, z) = c\}$.

a.)

We use the diffeomorphism $\mathbb{C}^2 \xrightarrow{\sim} \mathbb{R}^4$ by $(r + is, x + iy) \mapsto (r, s, x, y)$ and identify $\mathbb{C}^2$ as $\mathbb{R}^4$ as such (and similarly identify $\mathbb{C}$ with $\mathbb{R}^2$). Then, $f : \mathbb{R}^4 \to \mathbb{R}^2$ can be given new coordinates as

$$f(r, s, x, y) = (x^2 - y^2 - r^3 + 3rs^2, 2xy - 3r^2 s + s^3).$$

Then, letting $c = a + ib$, X is defined as $X = X_c := \{(r, s, x, y) \mid f(r, s, x, y) = (a, b)\}$

Proposition. *For any $(a, b) \neq (0, 0)$, X_c is a smooth manifold.*

As f is a smooth map with Euclidean domain and codomain, we may compute $\mathrm{d}f_P$ as the Jacobian of f at $P = (r, s, x, y)$. Doing so yields the following:

$$\mathrm{d}f_P = \begin{bmatrix} 3(s^2 - r^2) & 6rs & 2x & -2y \\ -6rs & 3(s^2 - r^2) & 2y & 2x \end{bmatrix}$$

We have that $c = (a, b) \in \mathbb{R}^2$ is a critical value if for some $P = (r, s, x, y)$ with $f(P) = c$, the derivative $\mathrm{d}f_P$ fails to be surjective, i.e. its maximal minors are identically zero. In particular, if P is a critical point, this forces the relations

$\Delta_{12}(\mathrm{d}f_P) = 9(s^2 - r^2)^2 + 36r^2s^2 = 9(s^2 + r^2)^2 = 0$ and $\Delta_{34}(\mathrm{d}f_P) = 4(x^2 + y^2) = 0$. As r, s, x, y are **real** coordinates, these two relations force $(r, s, x, y) = \vec{0}$. Hence, $\vec{0} \in \mathbb{R}^4$ is the only critical point of f, and $\vec{0} \in \mathbb{R}^2$ its corresponding critical value. By the regular value theorem, for $c = (a, b) \neq (0, 0)$, we have that X_c is a smooth manifold of dimension $\dim \mathbb{R}^4 - \dim \mathbb{R}^2 = 2$.

b.)

We let[1] $g : \mathbb{C}^2 \to \mathbb{C}$ be defined by $g(z, w) = w$. Under our change of coordinates above, this is the map $g : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $g(r, s, x, y) = (x, y)$.

Proposition. *The critical points of g are exactly the plane $r = s = 0$.*

Proof. We have by a proposition on page 24 of Guillemin-Pollack that $T_PX = \ker \mathrm{d}f_P$. Assuming that $P \neq \vec{0}$ (justified by part a), we compute manually that

$$\ker \mathrm{d}f_P \supseteq \mathrm{Span}\left\{ v_1 = \begin{bmatrix} 2x \\ 2y \\ -3(s^2 - r^2) \\ 6rs \end{bmatrix}, v_2 = \begin{bmatrix} 2y \\ -2x \\ 6rs \\ 3(s^2 - r^2) \end{bmatrix} \right\}$$

We note that the linear independence of v_1 and v_2 can be checked by computing the rank of the matrix $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$. Doing so reveals the same maximal minors as $\mathrm{d}f_P$, implying that v_1 and v_2 span the tangent space at P for any $P \in X$ by part a. We take v_1 and v_2 to be a basis for T_PX. Then, the inclusion map $i : T_PX \to T_P\mathbb{R}^4$ is given by $i(v) = \begin{bmatrix} v_1 & v_2 \end{bmatrix} v$. We let h be the extension of g to all of $\mathbb{R}^4$ given coordinate-wise by the same formula as g. Then, we may compute $\mathrm{d}h_P$ as

$$\mathrm{d}h_P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Furthermore, we have that $g = h \circ i$ and thus $\mathrm{d}g_P = \mathrm{d}h_{i(P)}\, \mathrm{d}i_P$. We compute

$$\mathrm{d}g_P = (\mathrm{d}h_{i(P)} \circ \mathrm{d}i_P)(v) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2x & 2y \\ 2y & -2x \\ -3(s^2 - r^2) & 6rs \\ 6rs & 3(s^2 - r^2) \end{bmatrix} v = \begin{bmatrix} -3(s^2 - r^2) & 6rs \\ 6rs & 3(s^2 - r^2) \end{bmatrix} v$$

We have that P is a critical point of g if $\mathrm{d}g_P$ fails to be surjective, i.e. fails to be an isomorphism. We compute $\det \mathrm{d}g_P = 9(s^2 + r^2)^2$ and have that $\det \mathrm{d}g_P = 0 \iff (s, r) = (0, 0)$. As $\det \mathrm{d}g_P = 0$ if and only if P is a critical point, this proves our claim. □

3.)

Proposition. *If ω is a closed 2-form on S^4, then $\omega \wedge \omega$ vanishes somewhere on S^4.*

Proof. We break our proof into a series of claims.

Claim 3.A. *$\omega \wedge \omega$ is closed*

Proof. We write $d(\omega \wedge \omega) = d\omega \wedge \omega + \omega \wedge d\omega = 0 \wedge \omega + \omega \wedge 0 = 0$. ■

Claim 3.B. *$\omega \wedge \omega$ is exact.*

Proof. We recall that de Rahm's isomorphism $\Psi : H^*_{\mathrm{dR}}(S^4) \xrightarrow{\sim} H^*(S^4, \mathbb{R})$ induces a ring structure via the wedge product on $H^*_{\mathrm{dR}}(S^4)$. We have that ω is necessarily exact as it is closed and $H^2(S^4) = 0$. Thus, $[\omega] = [0] \in H^2(S^4)$, so if $[\omega \wedge \omega] \neq [0]$, we have that $0 \neq [\omega \wedge \omega] = [\omega] \wedge [\omega] = [0] \wedge [0] = [0]$, a contradiction. Thus, $[\omega \wedge \omega] = [0]$ indeed, i.e. $\omega \wedge \omega$ is exact. ■

As an immediate corollary, as de Rahm's isomorphism works by integration on $H^4_{\mathrm{dR}}(S^4)$, we have that $\int_{S^4} \omega \wedge \omega = 0$.

Lemma 3.C (Mean value theorem for smooth manfiolds)**.** *If M is a d-manifold and $\sigma \in \Omega^d(M)$ is exact, then there is some point $p \in M$ such that where $\omega = f(x)\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$, $f(p) = 0$.*

Proof. ■

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□

[1] Oops, I'm changing names on you—my bad.

4.)

a.)

b.)

5.)