MATH 8301 Homework XI

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All references to theorems come from Allen Hatcher's *Algebraic Topology* unless otherwise stated.

1.)

We let $p \in \mathbb{R}\mathrm{P}^n$, $D \ni p$ a small open disk, $S := \partial D \cong S^{n-1}$, and $X := \mathbb{R}\mathrm{P}^n \setminus D$.

a.)

Proposition. $X \simeq \mathbb{R}\mathrm{P}^{n-1}$.

Proof. We let $\pi : S^n \to \mathbb{R}\mathrm{P}^n$ be such that $\pi^{-1}(p) = \{\pm(1, 0, \ldots, 0)\}$. We shall show that $S^n \setminus \pi^{-1}(D) \simeq S^{n-1}$ via a homotopy $\tilde{H} : \left(S^n \setminus \pi^{-1}(D)\right) \times [0, 1] \to S^n \setminus \pi^{-1}(D)$ with $\tilde{H}_t(\mathbf{x}) = -\tilde{H}_t(-\mathbf{x})$. We give $\tilde{H}$ explicitly, letting

$$\tilde{H}_t(x_0, \ldots, x_n) = \frac{1}{\|(1-t)x_0, x_1, x_2, \ldots, x_n\|}\left((1-t)x_0, x_1, x_2, \ldots, x_n\right)$$

Then, $\tilde{H}_1(S^n \setminus \pi^{-1}(D)) = S^{n-1} = \{\mathbf{x} \in S^n \;:\; \mathbf{x} = (0, x_1, \ldots, x_n)\}$, and it is immediately clear from construction that $\tilde{H}_t(-\mathbf{x}) = -\tilde{H}_t(\mathbf{x})$. Hence, $\tilde{H}$ factors through the quotient map to $\mathbb{R}\mathrm{P}^n \setminus D$ giving a homotopy equivalence $H : \mathbb{R}\mathrm{P}^n \times [0, 1] \to (S^{n-1}/\sim) \cong \mathbb{R}\mathrm{P}^{n-1}$. $\square$

b.)

Proposition. *The inclusion $\iota : S \to \mathbb{R}\mathrm{P}^n \setminus D$ induces the map $\iota_* : H_*(S) \to H_*(X)$ where $\iota_* : H_{n-1}(S) \cong \mathbb{Z} \to H_{n-1}(X) \cong \mathbb{Z}$ is the multiplication-by-two map if n is even, $\iota_* : H_0(S) \to H_0(S)$ is the identity map, and the 0 map otherwise.*

Proof. That ι_* is the identity map on H_0 is trivial. If n is odd, then $H_{n-1}(X) = 0$, so $\iota_* : H_{n-1}(S) \to H_{n-1}(X)$ is necessarily the 0 map, and if $m \neq 0, n-1$, $H_m(S) = 0$. Thus, the only interesting case is that of $\iota_* : H_{n-1}(S) \to H_{n-1}(X)$ in the case that n is even. Theorem 2.13 gives that since S is a deformation retract of a neighborhood in X, there is a long exact sequence

$$\ldots \longrightarrow \tilde{H}_m(S) \xrightarrow{\iota_*} \tilde{H}_m(X) \longrightarrow \tilde{H}_m(X/D) \longrightarrow \tilde{H}_{m-1}(S) \longrightarrow \ldots$$

We note that $X/D \cong \mathbb{R}\mathrm{P}^n$ and recall from problem 1 that $X \simeq \mathbb{R}\mathrm{P}^{n-1}$. Thus, we in fact have an exact sequence (up to isomorphism)

$$\ldots \longrightarrow \tilde{H}_m(S^{n-1}) \xrightarrow{\iota_*} \tilde{H}_m(\mathbb{R}\mathrm{P}^{n-1}) \longrightarrow \tilde{H}_m(\mathbb{R}\mathrm{P}^n) \longrightarrow \tilde{H}_{m-1}(S^{n-1}) \longrightarrow \ldots$$

As n is even, $\tilde{H}_n(\mathbb{R}\mathrm{P}^n) = 0$ and $\tilde{H}_{n-1}(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2$, giving us the short exact sequence

$$\begin{tikzcd}
\tilde{H}_n(\mathbb{R}\mathrm{P}^n) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[r, "\iota_*"] \arrow[d, "\cong"] & \tilde{H}_{n-1}(\mathbb{R}\mathrm{P}^{n-1}) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_{n-1}(\mathbb{R}\mathrm{P}^n) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_{n-2}(S^{n-1}) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\iota_*"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Thus, by exactness we may conclude that ι_* is the multiplication-by-two map. $\square$

c.)

Proposition.

$$H_m(\mathbb{R}\mathrm{P}^n \# \mathbb{R}\mathrm{P}^n) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & m = n \equiv 1 \mod 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z} & m = n-1 \equiv 1 \mod 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & 0 < m < n,\, m \equiv 1 \mod 2 \\ \mathbb{Z} & m = 0 \end{cases} \tag{1}$$

Proof. We note that $\mathbb{R}\mathrm{P}^n \# \mathbb{R}\mathrm{P}^n$ can be constructed from two copies of X by gluing along a neighborhood of S. Hence, we have the Meyer-Vitetorus (CHECK SPELLING BEFORE TURNING IN) sequence

$$\dots \longrightarrow H_m(S) \longrightarrow H_m(X) \oplus H_m(X) \longrightarrow H_m(\mathbb{R}\mathrm{P}^n \# \mathbb{R}\mathrm{P}^n) \longrightarrow H_{m-1}(S) \longrightarrow \dots$$

□

2.)

a.)

b.)

3.)

N/A (Problem de-assigned)

4.)

a.)

b.)