MATH 8302 Homework III

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1.)

Proposition. *Let X and Y be compact, connected smooth d-manifolds and let $f : X \to Y$ be a smooth submersion. Then, f is a covering map.*

Proof. As f is a submersion, we have that at each point $x \in X$, $\mathrm{d}f_x : TX_x \to TY_{f(x)}$ is surjective, i.e. of rank d. As TX_x is of dimension d, we have that $\mathrm{d}f_x$ is an isomorphism for all $x \in X$. By the inverse function theorem, f is then a local diffeomorphism everywhere. We let $y \in Y$ and claim that $f^{-1}(y)$ is a finite set. As f is a local diffeomorphism, for each $x \in f^{-1}(y)$, there is some open neighborhood $U_x \ni x$ such that $f|_{U_x}$ is a diffeomorphism. Note that necessarily, $f^{-1}(y) \cap U_x = \{x\}$ by bijectivity. We let $x \in V_x \subsetneq U_x$ with V_x open and let $V = (\bigcup_{x \in f^{-1}(y)} V_x)^c$. Then, $\{V\} \bigcup \{U_x\}_{x \in f^{-1}(y)}$ is an open cover of X and hence has a finite subcover by compactness. However, $\bigcup_{x \in f^{-1}(y)} U_x$ is an irredundant union by our observations above and hence $f^{-1}(y)$ is a finite set. As such, we may choose each U_x such that $U_x \cap U_{x'} = \varnothing$ for $x \neq x'$. We let $U = \bigcap_{x \in f^{-1}(y)} f(U_x)$. As U is a finite union of open sets, U is open. Furthermore, as each component of $f^{-1}(U)$ is contained in some set U_x, we have that f is diffeomorphic on each component of $f^{-1}(U)$. Finally, as the U_x are disjoint, we have that the preimages of U are disjoint. Thus, f is a covering map. $\square$

2.)

Notation 2.A. Throughout this problem, we let $\vec{v_i} = \begin{bmatrix} v_{i1} & v_{i2} & \dots & v_{in} \end{bmatrix}^T \in \mathbb{R}^n$ and $\mathbf{v} = (\vec{v_1}, \dots, \vec{v_k}) \in (\mathbb{R}^n)^{\times k}$, with identical notational standards for $\mathbf{w}$ and each $\vec{w_i}$. To emphasize this point, we shall think of $(\mathbb{R}^n)^{\times k}$ as a space of row vectors with each entry a column vector.

a.)

Proposition. *Let $S = \{(\vec{v_1}, \dots, \vec{v_k}) \,:\, \dim\langle \vec{v_1}, \dots, \vec{v_k}\rangle = k\} \subset (\mathbb{R}^n)^{\times k}$. S is an open set.*

Proof. We let $\mathbf{v}$ be an arbitrary multivector satisfying our stated independence condition. We let $V = (v_{ij}) \in \mathrm{Mat}_{n,k} \cong \mathbb{R}^{nk}$. Then, as the $\vec{v_i}$ are linearly independent, we have that $\mathrm{rank}(V) = k$. Thus, there exists some $I = (i_1, \dots, i_k)$ with $1 \leqslant i_1 < \dots < i_k \leqslant n$ such that the minor Δ_I taken on rows $i_1, \dots i_k$ has $\Delta_I(V) \neq 0$. As Δ_I is a polynomial in the entires of V, it is a continuous function. We let $d = \Delta_I(V)$. Then, letting $0 < \epsilon < d$, we have that $\Delta_I^{-1}((d-\epsilon, d+\epsilon))$ is an open set in $(\mathbb{R}^n)^{\times k}$ contained within S. As $\mathbf{v}$ was an arbitrary element of S, we now have that S is an open set. $\square$

b.)

Proposition. *We let $\sigma : (\mathbb{R}\backslash\{0\})^k \times S \to \mathbb{R}^n$ be the map $\sigma : [(t_1, \dots, t_k), (\vec{v_1}, \dots, \vec{v_k})] \mapsto \sum_{i=1}^k t_i \vec{v_i}$. Then, σ is a submersion.*

Proof. We note $Z := (\mathbb{R}\backslash\{0\})^k \times S$ is the product of an open set in $\mathbb{R}^k$ with one in $\mathbb{R}^{n*k}$ and thus $Z \subset \mathbb{R}^{(n+1)k}$ is open. Thus, we may take the entirety of Z to be a coordinate chart equipped with the identity map. Under this parametrization, we have that for some ordering of $[k] \times [n]$ $\{(\ell m)_j\}_{j \in [(n+1)k] \backslash [k]}$,

$$(\mathrm{d}\sigma)_{ij} = \begin{cases} \frac{\partial \sigma_i}{\partial t_j} & j \leqslant k \\ \frac{\partial \sigma_i}{\partial v_{(\ell m)_j}} & j > k \end{cases} = \begin{cases} v_{ji} & j \leqslant k \\ t_{\ell_j} & j > k \,\&\, m_j = i \\ 0 & j > k \,\&\, m_j \neq i \end{cases}$$

Then, in particular, letting I be the set of row-indices corresponding to v_{jj} with j ranging over $[n]$, we have that the submatrix corresponding to I is

$$\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_k \end{bmatrix},$$

and hence $\Delta_I(\mathrm{d}\sigma) = t_1 \dots t_k$. As each $t_i \neq 0$, we have that $\Delta_I(\mathrm{d}\sigma) \neq 0$ and hence $\mathrm{d}\sigma$ has rank n and is indeed surjective. Thus, σ is a submersion. □

c.)

Proposition. *$GL_k(\mathbb{R})$ acts on S by the action*

$$A(v_1, v_2, \dots, v_k) = (\sum_j a_{1j} v_j, \sum_j a_{2j} v_j \dots, \sum_j a_{kj} v_j).$$

The orbits of the $\mathrm{GL}_k(\mathbb{R})$-action on S are in bijection with the k-dimensional linear subspaces of $\mathbb{R}^n$.

Proof. We first construct a map Φ on the set of orbits of the $\mathrm{GL}_k(\mathbb{R})$-action on S to the set of k-dimensional linear subspaces of $\mathbb{R}^n$. We let $\mathbf{v} \in S$. Then, as $\vec{v_1}, \dots, \vec{v_k}$ are linearly independent, they span a k-dimensional linear subspace of $\mathbb{R}^n$. We let $\Phi([\mathbf{v}]) = \mathrm{Span}(\mathbf{v})$. To show this is well-defined, we let $\mathbf{w} = A\mathbf{v}$. Then, $\mathbf{v} = A^{-1}\mathbf{w}$, so each $\vec{v_i}$ may be written as a combination of the $\vec{w_j}$'s and vice versa. Thus $\mathrm{Span}\,\mathbf{v} = \mathrm{Span}\,\mathbf{w}$, so Φ is well-defined.

To construct an inverse to Φ, we let Ψ be any splitting map to Φ. To show that Ψ is indeed an inverse, we suppose $\mathrm{Span}\,\mathbf{v} = \mathrm{Span}\,\mathbf{w}$. Then, we may write $\vec{w_i} = \sum_k b_{ik} v_k$ for some coefficients b_{ik}. Similarly, we may write $\vec{v_i} = \sum_k a_{ik} w_k$. Thus, $AB = 1 = BA$, so $A \in \mathrm{GL}_k(\mathbb{R}^n)$ and $[\mathbf{v}] = [\mathbf{w}]$ so indeed Ψ is surjective and hence an inverse to Φ.

□

d.)

Proposition. *Let X be any submanifold of $\mathbb{R}^n$. Let $Q \subset S$ be the set of $(v_1, \dots, v_k)$ which have span which intersect X transversally. Show that Q is dense in S.*

3.)

Prompt. *For $V \cong \mathbb{R}^n$ and $W \cong \mathbb{R}^m$, we write $f(v_i) = \sum_j a_{ij} w_j$ where the set of $\{v_i\}$ form a basis for V and $\{w_i\}$ one for W. Find the matrix for $\bigwedge^2 f$.*

Response. We have that

$$\begin{aligned} \bigwedge^2 f(v_i \wedge v_k) &= \left(\sum_j a_{ij} w_j\right) \wedge \left(\sum_\ell a_{k\ell} w_\ell\right) \\ &= \sum_{j,k} a_{ij} a_{k\ell} (w_j \wedge w_\ell) \\ &= \sum_{j<k} (a_{ij} a_{k\ell} - a_{i\ell} a_{kj}) (w_j \wedge w_\ell) \end{aligned}$$

Hence, $\left(\bigwedge^2 f\right)_{(i,k)(j,\ell)} = a_{ij} a_{k\ell} - a_{i\ell} a_{kj}$ □

b.)

Proposition. *The map $\bigwedge^2 : \mathrm{Hom}(V, W) \to \mathrm{Hom}(\bigwedge^2 V, \bigwedge^2 W)$ is smooth.*

Proof. $\mathrm{Hom}(V, W) \cong \mathbb{R}^{nm}$ and $\mathrm{Hom}(\bigwedge^2 V, \bigwedge^2 W) \cong \mathbb{R}^{\binom{n}{2}\binom{m}{2}}$ by realizing each element as a matrix for fixed bases of each vector space. Hence, each space is its own coordinate chart. As we have now shown $\bigwedge^2$ to be a polynomial function, it is thus smooth. □