MATH 8301 Homework IV

David DeMark

2 October 2017

1.)

a.)

Prompt. *For $|(V,\mathcal{F})|$ a compact connected surface, find a relation between e, f*

Response. As $|(V,\mathcal{F})|$ is compact, we may assume each edge of each 2-simplex is identified with an edge of a distinct 2-simplex. We let $S = \{(\Delta,\epsilon) \in \mathcal{F}^2 : |\Delta| = 3,\ |\epsilon| = 2,\ \epsilon \subset \Delta\}$. As each 2-simplex Δ_0 has boundary composed of three edges $\epsilon_1, \epsilon_2, \epsilon_3$, we have that $|S| = 3f$. We also have that each edge is boundary to two 2-simplices and thus $|S| = 2e$. This gives us the relation $3f = 2e$ □

b.)

Prompt. *Find formulas for f, e in terms of χ, v.*

Response. We have that $\chi = v - e + f$. Substituting $f = \frac{2}{3}e$, we have $\chi = v - \frac{e}{3} \implies e = 3(v-\chi)$. Now, substituting $e = \frac{3}{2}f$, we have $\frac{3}{2}f = 3(v-\chi) \implies f = 2(v-\chi)$. □

c.)

Proposition. *Show for any triangulation of a compact surface $S = |(V,\mathcal{F})|$ with $\chi(S) = 0$ that $v(S) = v \geq 7$.*

Proof. We have that any triangulation of a surface must have at least one vertex as else the simplicial complex $(V,\mathcal{F})$ is necessarily empty. Furthermore, as the 1-skeleton of $(V,\mathcal{F})$ must be a graph in the classical sense (that is, with no loops or double edges), we have that $3v = e \leq \binom{v}{2}$. Hence, we must have that v satisfies the solution $3v \leq \frac{v^2-v}{2}$, or simplifying, $v^2 - 7v \geq 0$, that is $v \leq 0$ or $v \geq 7$. However, as we have already noted that $v \geq 1$, we must have that $v \geq 7$. □

2.)

a.)

Proposition. *Let P be star-shaped with respect to p. Then, P is contractible.*

Proof. We let $H : P \times [0,1] \to P$ be given by $H(q,t) = (1-t)q + tp$. It is well-known and obvious that H is continuous should it be well-defined, and one may view the definition of star-shaped precisely as $H(q_0, t_0) \in P$ for any $q_0 \in P$, $t_0 \in [0,1]$. Thus, H is well defined and as $H(q,1) = p$ for any $q \in P$, we have that it is a deformation retract and hence a detraction. □

b.)

Proposition. *If P is a polygon star-shaped w/r/t $p \in \mathrm{int}P$, then $f : P \setminus \{p\} \to S^1$ given by $f(q) = \frac{q-p}{|q-p|}$ is a homotopy equivalence.*

Proof. As $p \in \mathrm{int}P$, there exists some $\delta > 0$ such that $\overline{B}_\delta(p) \subset P$. We let $S = \partial\overline{B}_\delta(p)$. We then let $g : S^1 \to P$ be given by $g(x) = \delta x + p$ and note $\mathrm{im}\, g = S$. Then, $fg(x) = \mathrm{id}_{S^1}$, as $fg(x) = f(\delta x + p) = \frac{(\delta x+p)-p}{|(\delta x+p)-p|} = \frac{x}{|x|} = x$ as $|x| = 1$. On the other hand, we have that $gf(q) = g\left(\frac{q-p}{|q-p|}\right) = \delta\frac{(q-p)}{|q-p|} + p$–in particular, as $gf(q)$ is of the form $p + s(q-p)$, we have that for any $\underline{q \notin B_\delta(p)}$, $gf(q)$ is on the line $\overline{pq}$. On the other hand, for $q \in B_\delta(q)$, $gf(q)$ is still colinear to p, q, so we have that $q \in \overline{p(gf(q))}$. Hence, we may take $H : P \times [0,1] \to P$ by $(q,t) \mapsto (1-t)q + tgf(q)$ to be our homotopy given our equivalence □

c.)

Proposition. $T^2 \setminus \{p\} \approxeq S^1 \# S^1$

Proof. We begin with a lemma.

Lemma. *We let X be a topological space, $A \subset X$ and $\sim$ an equivalence relation on X such that for all $x \in X \setminus A$, $[x]$ is a singleton. Then, if $f : X \to A$ is a homotopy equivalence with homotopy inverse the inclusion ι such that $\iota f \approxeq_A \mathrm{id}_X$, $X/\sim$ is in homotopy equivalence with $A/\sim$.*

Proof. We let the natural maps $X \to X/\sim$ and $A \to A/\sim$ be denoted q_X and q_A respectively. We let $H : X \times [0,1] \to X$ give a homotopy relative to A from id_X to ιf. We define $\tilde{f} : X/\sim \to A/\sim$ by $\tilde{f}([x]) = q_A \circ f(x)$, with $\tilde{\iota}$ defined similarly, and $\tilde{H}$ given by $\tilde{H}([x], t) = q_X \circ H(x,t)$. As our hypothesis forces f, ι and $H(-,t)$ to restrict to the identity (composed or precomposed with the inclusion ι when appropriate) on A and $[x]$ is a singleton for $x \in X \setminus A$ we have that each of f, ι and H respect $\sim$ and hence $\tilde{f}, \tilde{\iota}$ and $\tilde{H}$ well defined. We note that $\tilde{f}\tilde{\iota} = \mathrm{id}_{A/\sim}$ as $f\iota = \mathrm{id}_A$, and hence need only show that $\tilde{H}$ gives a homotopy between $\tilde{\iota}\tilde{f}$ and $\mathrm{id}_{X/\sim}$. As $\tilde{H}(-,1) = \mathrm{id}_{X/\sim}$ by construction, we need only show that $\tilde{H}(-,0) = \tilde{\iota}\tilde{f}$. We have that $\tilde{H}(x,0) = (q_X \circ \iota \circ f)(x)$, and note that $q_X \circ \iota = \tilde{\iota} \circ q_A$ as q_X restricts to q_A on A. Further, we defined $\tilde{f}([x]) = q_A \circ f(x)$. Hence, $\tilde{H}(x,0) = \tilde{\iota}\tilde{f}$ and hence gives the desired homotopy. ■

By the lemma, we now need only show that $I^2 \setminus \{p\}$ where I is the unit interval and $p \in \mathrm{int} I^2$, is homotopy equivalent to ∂I^2 relative to ∂I^2. As I^2 is convex, we have that for any $q \in I^2$, the line segment $\overline{pq}$ is colinear to precisely one point $d_q \in \partial I^2$. We let $f(q) = d_q$, that is we let f be the projection-to-boundary map. By convexity, it is clear that f is continuous, and as I^2 is star-shaped with respect to any point of its interior, we have that the linear homotopy H from $\mathrm{id}_{I^2 \setminus \{p\}}$ to f by $(q,t) \mapsto (1-t)q + td_q$ is well-defined and continuous. Further, as $q = d_q$ for any $q \in \partial I^2$, we have that H is a homotopy relative to ∂I^2. Furthermore, as $f\iota = \mathrm{id}_{\partial I^2}$, f is a homotopy equivalence relative to ∂I^2. Thus, letting $\sim$ be the standard toral side-identification, we have that $T^2 \setminus \{[p]\} = I^2 \setminus \{p\}/\sim \approxeq \partial I^2/\sim = S^1 \# S^1$ □