Tropical Grassmannians and the Speyer–Williams Fan

David DeMark

9 October 2017

0 Introduction

Cluster algebras were introduced in the first years of the current century in part to reflect the mutative structure which had been observed in studies of total positivity. Among the most well-known examples of cluster structure arising in applications of total positivity to familiar contexts is that of the Grassmannian $\mathrm{Gr}_{2,n}$, which parametrizes two-dimensional planes through the origin in n-dimensional linear space. As we shall expand upon momentarily, elements of the (2,n)-Grassmannian are distinguished by *Plücker coordinates*, that is, the projective coordinates in $\mathbb{P}^{\binom{n}{2}-1}$ given by the $\binom{n}{2}$ 2×2 minors of a given $2 \times n$ matrix whose rows give a basis for the 2-dimensional plane in question [unfinished]

Oh christ how will we ever reconcile this with §1.1

1 Preliminaries & Basic Definitions

1.1 Grassmannians

Let $\mathbb{K}$ be a field fixed over the duration of this subsection. The *k, n-Grassmannian over* $\mathbb{K}$ (denoted $\mathrm{Gr}_{k,n}(\mathbb{K})$ or just $\mathrm{Gr}_{k,n}$ when $\mathbb{K}$ is clear from context) can be thought of as the k-dimensional subspaces through the origin in the $\mathbb{K}$-vector space $\mathbb{K}^n$. There are several ways of defining the Grassmannian one can refer to, including one [Hat09, §1.2] in which $\mathrm{Gr}_{k,n}$ is actually viewed the set of such k-dimensional subspaces, which is then topologized via a surjection from the Stiefel manifold of k-touples of orthonormal vectors in $\mathbb{K}^n$. Here, we present the definition of the grassmannian most common in algebraic geometry, with the naïve understanding of above as inspiration.

A k-dimensional subspace S of $\mathbb{K}^n$ can be described by a basis, which we represent here as a full-rank $k \times n$ matrix P. This description is not unique however, indeed S is invariant under the action of $GL(S)$. This points to another definition of $\mathrm{Gr}_{k,n}$ as a quotient $GL_k(\mathbb{K}^n)/GL(S)$, which gives $\mathrm{Gr}_{k,n}$ a smooth manifold structure if $GL_k(\mathbb{K}^n)$ may be taken as a lie group. Instead, we consider the $\binom{n}{k}$ $k \times k$ minors of P, which we write as p_T for $T \in \binom{[n]}{k}$. This gives projective coordinates known as *Plücker coordinates* for $\mathrm{Gr}_{k,n}$ in $\mathbb{P}^N_{\mathbb{K}} := \mathbb{P}_{\mathbb{K}}^{\binom{n}{k}-1}$. However, an arbitrary point in $\mathbb{P}^N_{\mathbb{K}}$ need not necessarily correspond to a k-dimensional subspace in $\mathbb{K}^n$, indeed there are homogenous relations in $\mathbb{K}\left[p_T \,:\, T \in \binom{[n]}{k}\right]$

known as *Pücker relations* which the minors of any $k \times n$ matrix satisfy, including for instance the three-term relations

$$p_{T'\cup\{ij\}}p_{T'\cup\{kl\}} - p_{T'\cup\{ik\}}p_{T'\cup\{jl\}} + p_{T'\cup\{il\}}p_{T'\cup\{jk\}}$$

where $T' \in \binom{[n]}{k-2}$ and $i, j, k, l \in [n] \setminus T'$ are distinct. These relations characterize $\mathrm{Gr}_{k,n}$ completely as a variety, which we formalize in the following definition.

Definition 1.1. $\mathrm{Gr}_{k,n}$ is the $k(n-k)$-dimensional projective variety in $\mathbb{P}^N_{\mathbb{K}}$ defined by the ideal $I_{k,n} \subset \mathbb{K}[p_T]$ which is generated by the homogenous Plücker relations.

Moreover, our description of the Grassmannian by Plücker coordinates enables us to define its totally positive part, which shall be our primary concern here. We let $\mathbb{K} = \mathbb{R}$. Then, the totally positive part of $\mathrm{Gr}_{k,n}(\mathbb{R})$, here denoted $\mathrm{Gr}^+_{k,n}(\mathbb{R})$ is the subset of $\mathrm{Gr}_{k,n}(\mathbb{R})$ where (some presentation of) the Plücker coordinates (p_T) are all positive real numbers. Analogously, in $\mathcal{R}$, the totally positive (k,n)-Grassmannian $\mathrm{Gr}^+_{k,n}(\mathcal{R})$ is the subset of $\mathrm{Gr}_{k,n}(\mathcal{R})$ for which the coefficient of the lowest-degree term in each Plücker coordinate is positive. These subvarieties have been of great algebro-geometric and combinatorial interest in recent years.

wait is that the right terminology?

2 Cluster Algebras and the Associahedron

Brief description. Example: Whichever polygon triangulation is type 2,6

2.1 Polyhedral Geometry & Gröbner Bases

In order to properly discuss the Grassmannian, its totally positive part, and the tropicalization of each, we shall give a few definitions from polyhedral geometry which we shall refer to throughout the rest of this paper.

Definition 2.1. A *cone* C is a subset of $\mathbb{R}^n$ such that for any finite set $S \subset C$, all subtraction-free linear combinations of elements in S are elements of C. A *polyhedral cone* is a finitely generated cone C; that is a subset of $\mathbb{R}^d$ with the property that there exists some $\mathbf{s}_1, \ldots, \mathbf{s}_k \in C$ such that for all $\mathbf{x} \in C$, there exist $a_1, \ldots, a_k \in \mathbb{R}^+$ such that $\mathbf{x} = \sum_{i=1}^k a_i \mathbf{s}_i$.

Definition 2.2. A *polyhedral complex* Σ is a collection of polyhedra containing the empty polyhedron such that for any polyhedron $P \in \Sigma$, each face $F \subset \partial P$ is in Σ, and the intersection of any two polyhedra in Σ is a common face of each. The *support* or *underlying point set* of Σ is $|\Sigma| = \bigcup_{P\in\Sigma} P$. We say a polyhedron $P \in \Sigma$ is maximal if it is not a face of another element of Σ. The *dimension* of Σ is the supremum of the dimensions of all elements of Σ, and we say Σ is "pure d-dimensional" if all maximal polyhedra are dimension-d.

Definition 2.3. The *face poset* $\mathcal{P}(\Sigma)$ of a polyhedral complex Σ is the graded poset in which the nodes are polyhedra of Σ and P covers Q if Q is a maximal proper face of

P. $\mathcal{P}(\Sigma)$ is graded by dimension; we refer to each rank level by $\mathcal{P}_i(\Sigma)$ containing all i-dimensional faces of Σ. Two polyhedral complexes are *combinatorially equivalent* if their face posets coincide. Considering the empty face to be dimension -1, the *f-vector* of Σ is $(\#\mathcal{P}_{-1}(\Sigma), \#\mathcal{P}_0(\Sigma), \dots, \#\mathcal{P}_d(\Sigma))$.

Definition 2.4. A *fan F* is a polyhedral complex in which each element is a cone.

We shall also require some basic notions from Gröbner basis theory. This will allow us to discuss Gröbner complexes, a polyhedral complex corresponding to a given homogenous ideal which shall be critical to our discussion of the tropical Grassmannian.

We let $S = \mathbb{K}[x_1, \dots, x_n]$ where $\mathbb{K}$ is the field of *Puiseaux series in* $\mathbb{C}$ *or* $\mathbb{R}$, $\mathcal{C} := \overline{\mathbb{C}(t)} = \bigcup_{n\geq 1} \mathbb{C}((t^{1/n}))$ and $\mathcal{R} := \bigcup_{n \geq 1} \mathbb{R}((t^{1/n}))$ respectively. We equip $\mathcal{R}$ and $\mathcal{C}$ with $\mathrm{val} : \mathbb{K} \to \mathbb{Q}$ where $\mathrm{val}(\sum_{w\in\mathbb{Q}} a_w t^w) = \min_{w\in\mathbb{Q}}\{w \ : \ a_w \neq 0\}$. We note that val *splits,* that is, $\mathrm{val}(t^w) = w$. We denote the residue field of $\mathbb{K}$ by $\Bbbk$ and denote the image of $a \in \mathbb{K}$ in $\Bbbk$ by $\overline{a}$.

Definition 2.5. We let $\mathbf{w} \in \mathbb{R}^n$ be fixed and $f = \sum_{\mathbf{u}\in\mathbb{N}^n} c_u x^{\mathbf{u}} \in S$. We let $W := \mathrm{Trop}(f)(\mathbf{w})$. The *initial segment* of f w/r/t $\mathbf{w}$ is

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{\mathrm{val}(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=W} \overline{c_{\mathbf{u}} t^{-\mathrm{val}(c_{\mathbf{u}})}} x^{\mathbf{u}} \in \Bbbk[x_1, \dots, x_n]$$

Similarly, for an ideal $I = \langle f_1, \dots, f_n \rangle \subset S$, we define the initial segment of I w/r/t/ $\mathbf{w}$ as $\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f_1), \dots, \mathrm{in}_{\mathbf{w}}(f_n) \rangle \subset \Bbbk[x_1, \dots, x_n]$.

We may now define a central concept to tropical algebraic geometry.

Definition 2.6 (Definition/Proposition)**.** We let $I \subset S$ be an ideal and $\mathbf{w} \in \mathbb{R}^n$. We define a set

$$C_I[\mathbf{w}] := \{\mathbf{w}' \in \mathbb{R}^n \ : \ \mathrm{in}_{\mathbf{w}'}(I) = \in_w (I)\}.$$

We denote its closure under the Euclidean topology as $\overline{C_I[\mathbf{w}]}$. $\overline{C_I[\mathbf{w}]}$ is a polyhedron whose lineality space contains $\mathbb{R}\mathbf{1}$. If $\in_{\mathbf{w}} (I)$ is not a monomial ideal, then there is some $\mathbf{w}' \in \mathbb{R}^n$ such that $\overline{C_I[w]}$ is a proper face of $\overline{C_I[\mathbf{w}']}$. We define the Gröbner complex as $\Sigma(I) := \{C_I[\mathbf{w}]\}_{\mathbf{w}\in\mathbb{R}^n}$; it follows from the observation above that $\Sigma(I)$ is a polyhedral complex.

Example 2.7.

Gröbner example for PL(2,6)

2.2 Tropical Varieties and Tropicalization

In general, tropical algebraic geometry is the study of the *tropical semiring* $(\mathbb{R}, \oplus, \odot)$, where $a \oplus b := \min\{a, b\}$ and $a \odot b := a + b$. We often may discuss tropical geometry without explicit reference to $\oplus$ and $\otimes$ via a process called *tropicalization*, which gives a correspondence between a variety and its "tropicalized" counterpart. In this section, we largely follow [MS15, §3].

recall puiseaux....

We let $f(x) = \sum_{\mathbf{u}\in\mathbb{N}^n} c_{\mathbf{u}} x^{\mathbf{u}} \in \mathbb{K}[x]$ We may then define the *tropicalization* of f as a function $\operatorname{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ by

$$\operatorname{trop}(f)(\mathbf{w}) = \min_{u\in\mathbb{N}^{n+1}} \{\operatorname{val}(c_{\mathbf{u}}) + \mathbf{w}\cdot\mathbf{u} \ : \ \mathbf{u} \in \mathbb{N}^n\} \tag{1}$$

Intuitively, what tropicalization “does” is “translate” the coefficients of f to $\mathbb{R}$ with the valuation map and evaluate f at $\mathbf{u}$ with tropical operations substituted in for their classical counterparts. Now, the graph of $\operatorname{Trop}(f)$ over $\mathbb{R}^n$ is a piecewise linear “tent” with “(tent) poles”[1] where $\operatorname{Trop}(f)$ fails to be differentiable, that is to say where the minimum in its definition is achieved at least twice. We define $\mathcal{T}(f)$, the *tropical hypersurface associated to f* as precisely those points in $\mathbb{R}^n$ at which $\operatorname{Trop}(f)$ fails to be differentiable. When $n = 2$, $\mathcal{T}(f)$ is an embedding of a connected graph into $\mathbb{R}^2$, pointing towards the central motif of tropical algebraic geometry: transforming data related to smooth varieties into combinatorial data.

Pushing this idea further, we may define the tropical variety of an ideal I as

$$\mathcal{V}(I) := \bigcap_{f\in I} \mathcal{T}(f).$$

This is the definition which shall serve as our intuitive understanding of tropical varieties. From the fundamental theorem of tropical algebraic geometry, we may also take the definition of $\mathcal{V}(I)$ to be (i) the set of all vectors $\mathbf{w} \in \mathbb{R}^n$ with $\operatorname{in}_{\mathbf{w}}(I) \neq \langle 1\rangle$, or (ii) letting $X = V(I) \subset (\mathbb{K}^*)^n$, the closure of the image of the coordinate-wise application of the val map on X. We now a slightly modified theorem which shall shed light on the structure of the tropical Grassmannian.

Theorem 2.8 (Structure Theorem for Tropical Varieties)**.** *If $V(I)$ is an irreducible d-dimensional variety in $(\mathbb{K}^*)^n$, then $\mathcal{V}(I)$ is the support of a pure dimension-d rational polyhedral complex. In particular, if I has a generating set F with **constant coefficients** in the sense that* $\operatorname{val} a_{\mathbf{u}} = 0$ *for any coefficient $a_{\mathbf{u}}$ of $f \in F$, $\mathcal{V}(I)$ is a pure dimension-d fan in $\mathbb{R}^n$.*

In particular, $\mathcal{V}(I)$ is a subcomplex of the Gröbner complex $\Sigma(I_{\text{proj}})$. We use the third definition of $\mathcal{V}(I)$ to define the totally positive part of a tropical variety as $\mathcal{V}^+(I) = \overline{\operatorname{val}(V(I) \cap (\mathbb{K}^+)}$, that is the closure of the image of the restriction of the valuation map to the totally positive part of the classical variety $V(I)$. Speyer and Williams [SW05] prove that a point $\mathbf{w} \in \mathcal{V}(I)$ lies in $\mathcal{V}^+(I)$ if and only if $\operatorname{in}_{\mathbf{w}}(I)$ contains no nonzero elements of $\mathbb{R}^+[x_1, \ldots, x_n]$.

3 Total Positivity, $\operatorname{Trop}\left(\operatorname{Gr}^+_{k,n}\right)$ and the “Speyer–Williams Fan” $F_{k,n}$

3.1 The Tropical Grassmannian

In this section, we work largely from Speyer and Sturmfels' paper [SS04] of the same title as well as [MS15, §4.3]. Theorem 2.8 implies that $\operatorname{Trop}(\operatorname{Gr}_{k,n}(\mathcal{R}))$ is a pure $k(n-k)$-dimensional

[1] Yes, this is confusing terminology; we use it only colloquially and only right now with the caution that we are *not* referring to poles in the sense of analysis.

polyhedral fan in $\mathcal{R}^N$. Its cones have a common intersection which may be parametrized by the map $\mathrm{Trop}\phi : \mathbb{R}^n \to \mathrm{Trop}(\mathrm{Gr}_{k,n}(\mathcal{R}))$ which takes $(i_1, \dots i_n)$ to the $\binom{n}{k}$-vector which for $K = j_1, j_2, \dots, j_k \in \binom{[n]}{k}$ has K-coordinate $i_{j_1} + i_{j_2} + \dots + i_{j_k}$. $\mathrm{Trop}\phi$ is in fact an injection, and thus its image is a pure n-dimensional cone. We may also consider $\phi : (\mathbb{K}^+)^n \to \mathrm{Gr}_{k,n}(\mathbb{K})$, the de-tropicalization of $\mathrm{Trop}\phi$, which maps $(i_1, \dots i_n)$ to the $\binom{n}{k}$-vector with K-coordinate $i_{j_1} i_{j_2} \dots i_{j_k}$. Consider the $(\mathbb{K}^*)^n$-action on $\mathrm{Gr}_{k,n}(\mathbb{K})$, in which an element $(\lambda_1, \dots, \lambda_n) \in (\mathbb{K}^*)^n$ acts on a $k \times n$ matrix A representing a point $P \in \mathrm{Gr}_{k,n}(\mathbb{K})$ by multiplying each column i by λ_i. This takes the Plücker coordinate p_K to $\left(\prod_{i \in K} \lambda_i\right) p_k$. Then, $\mathrm{Gr}_{k,n}(\mathbb{K})/\phi((\mathbb{K}^*)^n)$ is as a set the orbits of the $(\mathbb{K}^*)^n$-action we have just described. In the next section, we shall discuss a bijective parametrization of this quotient.

3.2 Parametrizing the classical Grassmannian and its quotients

Postnikov has given an explicitly combinatorial parametrization of the Grassmannian [Pos06], which is generalized by Speyer–Williams [SW05] in their study of tropical total positivity, which we follow along with from here. In this Postnikov's method associates the directed graph $\mathrm{Web}_{k,n}$ with $\mathrm{Gr}^+_{k,n}$, where $\mathrm{Web}_{k,n}$ is the directed graph obtained from a $k \times (n-k)$ grid with rows indexed $1, \dots, k$ and columns indexed $(k+1), \dots n$ by adding left- and down-facing arrows to each vertex (as well as sources labeled by $[k]$ along the right side of the grid and sinks labeled by $[n] \setminus [k]$ along the bottom with all labellings increasing clockwise). Each edge is given weighting x_e, and a path (compatible with the orientation of the graph) $e_1 e_2 \dots e_m$ is associated to the monomial $\mathrm{Prod}_p(x) = \prod_{i=1}^m x_{e_i}$. For a set of paths S, we let $\mathrm{Prod}_S(x) = \prod_{p \in S} \mathrm{Prod}_p(x)$. We then let $A_{n,k}$ be the $k \times n$ matrix with entries $a_{ij}(x) = (-1)^{i+1} \sum \mathrm{Prod}_p(x)$ summing over all paths from the source at vertex i to the sink at vertex j. We let $K \in \binom{[n]}{k}$ and let Path(K) be the set of tuples of pairwise vertex-disjoint paths with sinks in K and sources in its complement. An application of the familiar Gessel-Viennot trick for determinental calculations (for an exposition thereof see e.g. [Sta99, §2.7]) yields the following result:

Proposition 3.1. $P_K(x) := p_K(A_{k,n}(x)) = \sum_{S \in \mathrm{Path}(k)} \mathrm{Prod}_S(x)$.

Then, substituting elements of $\mathbb{R}^+$ for the $2k(n-k)$ weight variables x_e gives the Plücker coordinates of element of $\mathrm{Gr}^+_{3,6}(\mathbb{R})$. This gives a map $\Phi_0 : (\mathbb{R}^+)^{2k(n-k)} \to \mathrm{Gr}^+_{3,6}(\mathbb{R})$, which, as it turns out, is surjective but due to obvious dimension concerns, is not injective. Speyer and Williams refine this map by replacing the weighting scheme as follows: rather than weighting by *edges*, we weight by *regions*, which are defined as follows: inner regions are the maximal connected components of the complement of an embedding of $\mathrm{Web}_{k,n}$ into $\mathbb{R}^2$, while outer regions are those which would satisfy the definition of inner region if we added edges between each source/sink i and $i+1$, but are not inner regions. These regions are weighted by monomials z_r in $\{x_e\}$ and $\{x_e^{-1}\}$, with each counterclockwise-oriented edge e bordering region r contributing x_e and each counterclockwise edge f contributing x_f^{-1}. Then, one may check that indeed the path monomials $\mathrm{Prod}_p(x)$ may be viewed as monomials in the z_r, giving a map $\Phi_1 : (\mathbb{R}^+)^{k(n-k)} \to \mathrm{Gr}^+_{k,n}(\mathbb{R})$. Speyer and Williams explicitly construct an inverse map Ψ to Φ_1, showing bijectivity. A key feature of the map Ψ is that each coordinate map (indexing $\mathbb{R}^{k(n-k)}$) by $[k] \times [j]$) $\Psi_{i,j}$ is the ratio of two monomials in the Plücker coordinates

p_K. Thus, the composition $\Psi\Phi_1 : (\mathbb{R}^+)^{k(n-k)} \to (\mathbb{R}^+)^{k(n-k)}$ expresses each region variable x_r as a ratio of monomials in $P_K(x)$. In particular, the multiset formed from the indices of the numerator of x_R and that formed from the denominator coincide if and only if R is an inner region. Thus, the composition of Ψ with the map ϕ of section 3.1 fixes the x_r corresponding to the inner regions, while acting transitively on the x_r corresponding to the outer regions. orbits? This observation gives a bijective map $\Phi_2 : (\mathbb{R}^+)^{(n-k-1)(k-1)} \to \mathrm{Gr}^+_{k,n}(\mathbb{R})/\phi(\mathbb{R}^+)^n$ by lifting $c \in (\mathbb{R}^+)^{(n-k-1)(k-1)}$ to a point $\tilde{c} \in (\mathbb{R}^+)^{k(n-k)}$ agreeing on the coordinates corresponding to the inner regions, applying Φ_1, then going down to the corresponding point in the quotient. Speyer and Williams also prove that the results above still apply when $\mathbb{R}^+$ is replaced with $\mathcal{R}^+$, establishing a bijection between $(\mathcal{R}^+)^{(k-1)(n-k-1)}$ and $\mathrm{Gr}^+_{k,n}(\mathcal{R})/\phi(\mathcal{R}^+)^n$. Then, the tropicalization of Φ_2 gives a surjective map $\mathrm{Trop}\phi_2 : \mathbb{R}^{(k-1)(n-k-1)} \to \mathrm{TropGr}^+_{k,n}/\mathrm{Trop}\phi(\mathbb{R}^+)^n$.

3.3 The Speyer–Williams Fan

We have now built up the machinery to define the central object of Speyer and Williams' study:

Definition 3.2. The Speyer–Williams fan $F_{k,n}$ is the complete fan in $\mathbb{R}^{(k-1)(n-k-1)}$ which has as its maximal cones the domains of linearity for $\mathrm{Trop}\phi_2$.

Speyer and Williams then show the following results:

Theorem 3.3 ([SW05], §5-7)**.**

1. *The fan $F_{2,n}$ is combinatorially equivalent to the* ***Stanley–Pitman*** *fan F_{n-3}, which has structure determined by the set of plane binary trees with $n-1$ leaves*

2. *The fan $F_{3,6}$ has f-vector* $(16, 66, 98, 48)$*, which very nearly coincides with the f-vector* $(16, 66, 100, 50)$ *of the fan normal to the type-D_4 generalized associahedron associated to the cluster algebra of type D_4. The discrepancy in the latter two coordinates can be explained by noting that two of the cones in $F_{3,6}$ are "cones over a bipyramid" which, when subdivided, yield a refined polyhedral complex with f-vector coinciding with that of the fan normal to the type-D_4 generalized associahedron.*

3. *The fan $F_{3,7}$ has f-vector* $(42, 392, 1463, 2583, 2163, 693)$*. There exists a refinement of $F_{3,7}$ which establishes a polyhedral complex with f-vector coinciding with that of the fan normal to the type-E_6 generalized associahedron,* $(42, 399, 1547, 2856, 2499, 833)$.

In later work, Brodsky, Ceballos, and Labbé [BCL17] make the connection between $F_{3,6}$ and type-D_4 cluster algebras more precise by giving an explicit bijection between combinatorial types of tropical planes in tropical projective space $\mathbb{TP}^5$, which are realized by $\mathrm{TropGr}_{3,6}$, and clusters in the cluster algebra of type D_4

Example 3.4.

compute fan of $F_{2,6}$

References

[BCL17] Sarah B. Brodsky, Cesar Ceballos, and Jean-Philippe Labbé. "Cluster algebras of type D_4, tropical planes, and the positive tropical Grassmannian". In: *Beitr. Algebra Geom.* 58.1 (2017), pp. 25–46.

[FWZ16] S. Fomin, L. Williams, and A. Zelevinsky. "Introduction to Cluster Algebras. Chapters 1-3". In: *ArXiv e-prints* (Aug. 2016). arXiv: `1608.05735 [math.CO]`.

[FWZ17] S. Fomin, L. Williams, and A. Zelevinsky. "Introduction to Cluster Algebras. Chapters 4-5". In: *ArXiv e-prints* (July 2017). arXiv: `1707.07190 [math.CO]`.

[Har95] Joe Harris. *Algebraic geometry*. Vol. 133. Graduate Texts in Mathematics. A first course, Corrected reprint of the 1992 original. Springer-Verlag, New York, 1995, pp. xx+328.

[Hat09] Allen Hatcher. "Vector Bundles and K-theory". avaible at `http://www.math.cornell.edu/~hatcher/VBKT/VB.pdf`. 2009.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*. Vol. 161. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015, pp. xii+363.

[Pos06] A. Postnikov. "Total positivity, Grassmannians, and networks". In: *ArXiv Mathematics e-prints* (Sept. 2006). arXiv: `math/0609764`.

[SS04] David Speyer and Bernd Sturmfels. "The tropical Grassmannian". In: *Advances in Geometry* 4.3 (2004), pp. 389–411.

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1999, pp. xii+581.

[SW05] David Speyer and Lauren Williams. "The tropical totally positive Grassmannian". In: *J. Algebraic Combin.* 22.2 (2005), pp. 189–210.