MATH 8301 Homework X

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1.)

We let $X = S^2 \cup D$ where $D = ([-1, 1] \times 0, 0) \subset \mathbb{R}^3$.

a.)

Proposition. *$\pi_1(X, x_0) = \mathbb{Z}$ for any $x_0 \in X$.*

Proof. We first note that X is clearly path connected, ensuring that a statement of the form of ours is well-formed. We let U be the union of D and a small open thickening of a geodesic arc between $(-1, 0, 0)$ and $(1, 0, 0)$, and let $V = S^2$. Then, $U \cap V$ is homeomorphic to a disk and hence simply-connected, U deformation-retracts to S^1 by "pulling in the thickening," and $V = S^2$ is well-known to be simply-connected. Thus, Siefert-von Kampen yields that $\pi_1(X, x_0) = \mathbb{Z}$. □

b.)

Prompt. *Construct geometrically $\tilde{X}$, a simply-connected covering space of X.*

Response. We claim that such a space would be an "infinite necklace" of spheres S^2 laid out on the real line with paths connecting each to its neighbors. To see this, we note that "folding" the necklace such that each sphere is mapped to one, alternating orientation along the chain, yields a space homeomorphic to X via a homeomorphism taking D to a path in $\mathbb{R}^3$ connecting the two poles through the exterior of the unit disk. □

c.)

Prompt. *Enumerate the subgroups of $\pi_1(X, x_0)$*

Response. As $\pi_1(X, x_0) \cong \mathbb{Z}$, we have that its subgroups are in bijection with its elements modulo an equivalence relation unifying each element with its inverse—indeed, each positive element of $\mathbb{Z}$ generates a unique subgroup, and each subgroup of $\mathbb{Z}$ is generated by one element by unique factorization. Thus, all subgroups are of the form $m\mathbb{Z}$ for $m \in \mathbb{Z}^{\geq 0}$, and all proper subgroups are of the form $m\mathbb{Z}$ for $m \in \mathbb{Z}^{>1}$. We note as well that as $\mathbb{Z}$ is Abelian, each subgroup is in a singleton conjugacy class. □

d.)

Prompt. *Compute the set of isomorphism classes of covers of X and describe each as a topological space.*

Response. We have that the isomorphism classes of covering spaces $((\hat{X}, \hat{x_0}), p)$ of X are in bijection with subgroups of $\pi_1(X, x_0)$ via the map $\hat{X} \mapsto p_*(\pi_1(\hat{X}, \hat{x_0}))$. Thus, we wish to associate a covering space for each nonnegative integer m. For $m = 0$, the universal cover of part b) is the covering space in question, and for $m = 1$, the original space X is. For $m > 1$, we claim that such a space would be the "pearl necklace with m beads," that is the space which is the union of m spheres arranged in a circle with antipodal points A and B marked on each and a path between point B on a given sphere and point A on the sphere one "link" clockwise. That this is a covering space is almost immediately clear by mapping each sphere to X such that point A maps to $(-1, 0, 0)$ and point B maps to $(1, 0, 0)$, and we see immediately that as there are m points A that this is an m-sheeted cover. □

2.)

We let $X = \mathbb{R}P^2 \wedge \mathbb{R}P^2$.

a.)

Proposition. $\pi_1(X, x_0) = \mathbb{Z}/2 * \mathbb{Z}/2$.

Proof. Wait haven't we proven in class/aren't we able to use as a theorem that 1) $\pi_1(X \wedge Y, z_0) = \pi_1(X, z_0) * \pi_1(Y, z_0)$ and 2) $\pi_1(\mathbb{R}P^2, x_0) = \mathbb{Z}/2$? Well, regardless, an easy application of SvK clears up all. $\square$

b.)

Proposition. *We let $\tilde{X}$ be the dijoint union of a set of copies of S^2, $\{S_i\}$ indexed by $\mathbb{Z}$ each with marked antipodal points A_i and B_i, quotiented by the equivalence relation $A_i \sim B_{i+1}$. Then, $\tilde{X}$ is a universal cover for X.*

Proof. We note that the quotient map $q : S^2 \to \mathbb{R}P^2$ identifying antipodal points is a universal cover for $\mathbb{R}P^2$, as S^2 is simply connected and the identification of antipodal points ensures that for any open cover for which no set is not contained in some half-sphere, there are two disjoint pre-images of each set which are mapped homeomorphically by q. Then, as our $\tilde{X}$ is clearly simply-connected, there is an obvious covering map which applies that quotient to each S_i, and identifies S_i with S_{i+2}. $\square$

c.)

Proposition. $G := \mathbb{Z}/2 \ltimes \mathbb{Z} \cong \mathbb{Z}/2 * \mathbb{Z}/2$

Proof. We shall prove the proposition by constructing a map $\phi : H := \mathbb{Z}/2 * \mathbb{Z} \to \mathbb{Z}/2 * \mathbb{Z}/2$, then showing the kernel is precisely elements of the form $tmtm$ where $0 \neq t \in \mathbb{Z}/2$ and $m \in \mathbb{Z}$. This would prove the proposition, as $\mathbb{Z}/2 \ltimes \mathbb{Z}$ is precisely the quotient group of $\mathbb{Z}/2 * \mathbb{Z}$ by those elements. We let $\mathbb{Z}/2 * \mathbb{Z}/2$ be generated by order-2 elements t_1, t_2 and let ϕ map on generators $t, 1$ by $t \mapsto t_1$ and $1 \mapsto (t_1 t_2)$. Then, $\phi(tmtm) = t_1(t_1t_2)^m t_1(t_1t_2)^m = t_2(t_1t_2)^{m-1} t_2(t_1t_2)^{m-1} = (t_2t_1)^{m-1}(t_1t_2)^{m-1} = (t_1t_2)^{1-m}(t_1t_2)^{m-1} = 0$, so $\langle tmtm \rangle_{m\in\mathbb{Z}} \subset \ker\phi$. We observe now that all elements of the form $mtmt \in \langle tmtm \rangle_{m\in\mathbb{Z}}$. We then suppose that $W \in \ker\phi$. We note that all elements in H may be written in one of four forms: $F_1 :\ m_1tm_2tm_3t \ldots tm_n$, $F_2 :\ m_1tm_2tm_3t \ldots tm_nt$, $F_3 :\ tm_1tm_2tm_3 \ldots tm_n$, or $F_4 :\ tm_1tm_2tm_3 \ldots tm_nt$ where $m_i \in \mathbb{Z}$. We let the m-length of an element in any of these forms be n, that is, the minimal quantity of elements of $\mathbb{Z}$ necessary to write the element. Then, we have for a word in form F_1 that $\phi(m_1tm_2tm_3t \ldots tm_n) = \phi((m_1 - m_2 + m_3)tm_4 \ldots tm_n)$, thus showing that for any word of length m of form F_1, there is a word in its equivalence class modulo $\langle tmtm \rangle_{m\in\mathbb{Z}}$ with length $n-1$. A similar argument can be given for the other three forms, showing that the only candidates for being a "unexpected" member of the kernel are those of length 0, or 1 which are all of the form tmt, tm, or mt. However, direct inspection shows that none of these are in the kernel of ϕ, showing $\langle tmtm \rangle_{m\in\mathbb{Z}} \supset \ker\phi$. Thus, the uniqueness of quotient groups proves our proposition. $\square$

d.)

Prompt. *The subgroups of G are all of the form $\langle (t, m) \rangle$, $\langle (0, n) \rangle$ for some $m \in \mathbb{Z}$, $n \in \mathbb{Z}^{>0}$ or $\langle (t, m), (0, n) \rangle$ for some $m < n \in \mathbb{Z}^{>0}$. Furthermore, this list is irredundant as m, n vary.*

Proof.

Claim. *All subgroups of G are generated by at most two elements.*

Proof. We first note that $(t, m)(t, n) = (0, m-n)$ and $(0, m-n)(t, m) = (t, n)$. Thus, $\langle (t, m), (t, n) \rangle = \langle (t, m)(0, m-n) \rangle$. Further, by well-known facts about $\mathbb{Z}$, $\langle (0, m), (0, n) \rangle = \langle (0, \gcd(m, n)) \rangle$. Thus, for any subgroup given by a presentation with more than two generators, if two of them are of the form (t, m), we may replace them with one of that form and one with first coordinate zero. Then, we may replace all with first coordinate zero by their common divisor in $\mathbb{Z}$. This process reduces all presentations to two generators. $\blacksquare$

Corollary. *Any subgroup generated minimally by two elements is of the form $\langle (t, m), (0, n) \rangle$*

Finally, what is left to prove is that our list is irredundant. We note that $\langle (t, m) \rangle$ is of order two, and hence cannot be written in any other minimal presentation. Furthermore, those of the form $\langle (0, n) \rangle$ are clearly uniquely presented by well-known facts about $\mathbb{Z}$. Finally, ...oops ran out of time! $\square$

I have been catching up on all of my classes after being sick last week, and the absolutely aggravating menial work required for the rest of this problem seems like a good candidate for the major casualty of my catching up.