MATH 8253 Homework I

David DeMark

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Notation

- For $S \subset A$ where A is a ring, we let the ideal generated by S be denoted $\langle S\rangle$ (at least until we encounter some other notational standard for $\langle\cdot\rangle$ which conflicts).
- For S a set, we let $\mathcal{P}(S)$ denote the power set of S.

1.)

Proposition. *Categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if there exists some functor $F : \mathcal{C} \to \mathcal{D}$ for which $F : \mathrm{Mor}_{\mathcal{C}}(c_1, c_2) \to \mathrm{Mor}_{\mathcal{D}}(F(c_1), F(c_2))$ is a bijection for any $c_1, c_2 \in \mathrm{Obj}\ \mathcal{C}$ and for any $d \in \mathrm{Obj}\ \mathcal{D}$, there is an isomorphism $(\phi_d : d \to F(c)) \in \mathrm{Mor}_{\mathcal{D}}(d, F(c))$ for some $c \in \mathrm{Obj}\ \mathcal{C}$.*

Proof. ($\Longleftarrow$) We suppose such a functor exists. We construct an "inverse functor" $G : \mathcal{D} \to \mathcal{C}$ as such: for any d in $\mathcal{D}$, we have that there exists an isomorphism $\phi_d : d \to F(c_d)$ where $c_d \in \mathcal{C}$. We choose such an isomorphism (the identity morphism when d is in the image of F). We let $G(d) = c_d$. For $h \in \mathrm{Mor}_{\mathcal{D}}(d, d')$, we have that $g = \phi_{d'} \circ h \circ \phi_d^{-1} \in \mathrm{Mor}_{\mathcal{D}}(F(c_d), F(c_{d'}))$. As F is bijective on morphisms, we have that there exists a unique $F^{-1}(g) \in \mathrm{Mor}_{\mathcal{C}}(c_d, c_{d'})$. We let $G(h) = F^{-1}(g)$.

We now show that $GF : \mathcal{C} \to \mathcal{C}$ is naturally equivalent to the identity functor $\mathrm{id}_{\mathcal{C}}$ on $\mathcal{C}$. For $c \in \mathcal{C}$, we have that $GF(c) = c$ as $\phi_{F(c)}$ was chosen to be the identity morphism. Hence, we may let $m_c \in \mathrm{Mor}_{\mathcal{C}}(c, -)$ be the identity morphism on c, id_c (note that, trivially, m_c is an isomorphism). Further, for $f \in \mathrm{Mor}_{\mathcal{C}}(c, c')$, we have that $GF(f) = F^{-1}(\phi_{F(c')} \circ F(f) \circ \phi_{F(c)}^{-1}) = F^{-1}(\mathrm{id}_{c'} \circ F(f) \circ \mathrm{id}_c) = f$. Thus, $f \circ m_c = m_{c'} \circ GF(f) = f$, so GF is indeed naturally equivalent to the identity functor.

We finally show $FG : \mathcal{D} \to \mathcal{D}$ is naturally equivalent to the identity functor $\mathrm{id}_{\mathcal{D}}$—we do so, however, somewhat "backwards"—in particular, we let m_- be a natural transformation *from* the identity functor *to* FG. We let $m_d = \phi_d$ for all $d \in \mathcal{D}$. By construction, m_d is then an isomorphism. We let $f \in \mathrm{Mor}_{\mathcal{D}}(d, d')$ and have that $FG(f) = F(F^{-1}(\phi_{d'} \circ f \circ \phi_d^{-1})) = \phi_{d'} \circ f \circ \phi_d^{-1} \in \mathrm{Mor}_{\mathcal{D}}(F(c_d), F(c_{d'}))$. Then, $FG(f) \circ m_d = \phi_{d'} \circ f$, and $m_{d'} \circ f = \phi_{d'} \circ f$ so our proof of this side of the implication is complete.

($\Longrightarrow$) We suppose that $\mathcal{C}, \mathcal{D}$ are equivalent by $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and wish to show that F fulfills the desired properties. We let m_- be a natural isomorphism from GF to $\mathrm{id}_{\mathcal{C}}$ and n_- a natural isomorphism from FG to $\mathrm{id}_{\mathcal{D}}$. Then, as n_d is an isomorphism from d to $FG(d)$, we have that each object in $\mathcal{D}$ is isomorphic to an object in the image of F. We now show F is injective on morphisms, in particular, we suppose $F(f) = F(f')$ for $f, f' \in \mathrm{Mor}_{\mathcal{D}} : c \to c'$. Then, we have that $m_{c'}^{-1} \circ f \circ m_c = GF(f) = GF(f') = m_{c'}^{-1} \circ f' \circ m_c$. However, as m_c, m_c' are isomorphisms, this implies that $f = f'$. Hence, F is injective on morphisms. We note as well that by symmetry, we have that G is injective on morphisms.

To show surjectivity, we let $h : F(c) \to F(c')$ and wish to find $\overline{h} : c \to c'$ such that $F(\overline{h}) = h$. We claim that $\overline{h} = m_{c'} \circ G(h) \circ m_c^{-1}$ works as such. Applying GH, we have that $GF(\overline{h}) = m_{c'}^{-1} \circ \overline{h} \circ m_c = G(h)$. But G is faithful by the above, so $F(\overline{h}) = h$. □

2.)

Proposition. *For two rings A_1, A_2, there exists a bijection $\mathrm{Spec}\, A_1 \sqcup \mathrm{Spec}\, A_2 \to \mathrm{Spec}\, A_1 \times A_2$ by $\mathrm{Spec}\, A_i \ni x_{\mathfrak{p}} \mapsto x(\mathfrak{p} \times A_{i+1 \mod 2})$.*

Proof. We let $P = \{x(\mathfrak{p} \times A_{i+1 \mod 2}) : x_{\mathfrak{p}} \in \mathrm{Spec}\, A_i\} \subset \mathrm{Spec}\, A_1 \times A_2$. That P is indeed a subset of $\mathrm{Spec}\, A_1 \times A_2$ is given by construction: for $\mathfrak{p}$ prime in A_1, we have that $\mathfrak{p} \times A_1$ is an ideal, as it is a subgroup of $A_1 \times A_2$ by direct product construction, and as $(r_1 a, r_2 s) \in \mathfrak{p} \times A_2$ for all $(a, s) \in \mathfrak{p} \times A_2$ and $r_i \in A_i$. Finally, we note that $\mathfrak{p} \times A_2$ is indeed prime, as for any $(a, r)(b, s) \in \mathfrak{p} \times A_2$, we must have one of $a, b \in \mathfrak{p}$. Hence without loss of generality we may assume $a \in \mathfrak{p}$, and thus $(a, r) \in \mathfrak{p} \times A_2$, so $\mathfrak{p} \times A_2$ is prime; by symmetry this applies for those of the form $A_1 \times \mathfrak{q}$. We wish to show the reverse containment, that is, $\operatorname{Spec} A_1 \times A_2 \subset P$. We let I be an arbitrary prime ideal of the ring $A_1 \times A_2$. We let π_1, π_2 be the coordinate projection functions and have that $\pi_1(I), \pi_2(I)$ are necessarily prime ideals. We suppose for the sake of contradiction that neither π_1, π_2 have surjective image when applied to I. We then let $a \in A_1 \setminus \pi_1(I)$ and $b \in A_2 \setminus \pi_2(I)$. Then $(a, 0)(0, b) = (0, 0) \in I$, but $(a, 0), (0, b) \notin I$. Hence, all prime ideals of $A_1 \times A_2$ have surjective image under one of the projection maps, so $\operatorname{Spec} A_1 \times A_2 \subset P$. As P is in obvious bijection with $\operatorname{Spec} A_1 \sqcup \operatorname{Spec} A_2$, our proof is complete. □

3.)

Proposition. *Let $U \subset \operatorname{Spec} A$ be an open set containing all closed points of $\operatorname{Spec} A$. Then $U = \operatorname{Spec} A$.*

Proof. We begin with the lemma suggested by the wording of the problem.

Lemma 3.1. *Let V be a non-empty closed set in $\operatorname{Spec} A$. Then V contains a closed point.*

Proof. We let $E \subset A$ be any set generating the set $V = V(E)$. We let $\mathcal{P}$ denote the poset of prime ideals $\mathfrak{p} \supset E$ ordered by inclusion. Zorn's lemma then gives the existence of maximal elements; let one of these be $\mathfrak{m}$. Then, we claim $V(\mathfrak{m}) = \{x_\mathfrak{m}\}$, i.e. $x_\mathfrak{m}$ is a closed point. To see this, we have that because $x_\mathfrak{m} \in V$, $E \subset \mathfrak{m}$, so as V is inclusion reversing, $V(\mathfrak{m}) \subset V(E)$. On the other hand, by maximality of $\mathfrak{m}$ within $\mathcal{P}$, for any $\mathfrak{p} \supset E$, there exists some $a \in \mathfrak{m} \setminus \mathfrak{p}$, so $x_\mathfrak{p} \notin V(\mathfrak{m})$. Thus $V(\mathfrak{m})$ is a singleton and we have constructed a closed point within $V(E)$. ■

Remark 3.1. The ideal found in the proof of the lemma is indeed maximal (assuming $V(E)$ is nonempty)–any other ideal containing it must also contain E! Indeed this shows that all maximal ideals correspond to closed points in $\operatorname{Spec} A$ as we may take E to be 0.

Now, we suppose that U is an open subset of $\operatorname{Spec} A$ containing all closed points of $\operatorname{Spec} A$. Then, U^c is closed, but contains no closed points. Hence, $U^c = \emptyset$ and $U = \operatorname{Spec} A$.

□

4.)

Proposition. *Let k be a field with $\bar{k} = k$ and $A = k[t]$ the free algebra with one generator over k. Then the set of closed points in* $\operatorname{Spec} A$ (i) *can be identified with k and* (ii) *include all points of* $\operatorname{Spec} A$ *save the generic point* $[\langle 0 \rangle]$.

Proof. (i) By remark 3.1, we have that all maximal ideals correspond to closed points. We show the reverse containment: if x is a closed point, there exists some set $E \subset A$ such that $V(E) = \{x\}$–in other words, $\mathfrak{p}_x$ is the only ideal in A containing E. Thus, $\mathfrak{p}_x$ is maximal. Hence, closed points in $\operatorname{Spec} A$ may be identified in one-to-one correspondence with maximal ideals. As $k[t]$ is principal, maximal ideals correspond to irreducible elements modulo multiplication by a unit–we take the (unique!) monic generator of each maximal ideal to be a representative for its set of generators. As $k = \bar{k}$, we have that these irreducibles are necessarily degree 1, that is a complete set of representatives of generators of maximal ideals would be $\{(x - a) \ : \ a \in k\}$. Hence k is in bijection with $\{$closed points of $\operatorname{Spec} A\}$ by $a \mapsto x_{\langle x-a \rangle} \in \operatorname{Spec} A$. Furthermore *(ii)*, again as $k[x]$ is principal, any nonempty prime ideal is itself maximal and thus the only nonclosed point is indeed the generic point.

5.)

We let $K = k[x, y]$ where $k = \bar{k}$ and $X = \operatorname{Spec} k[x, y]$.

a.)

Proposition. *The closed points of X may be identified with k^2.*

Proof. We first show that any (proper) prime ideal $\mathfrak{p}$ which is not principal can be written $\mathfrak{p} = \langle (x - a), (y - b) \rangle$. we may find two elements $f(x, y), g(x, y)$ with no common factor. It is well-known that that $k[x, y]$ is Noetherian so we let $S = \{q_1(x, y), \ldots, q_n(x, y)\}$ be a generating set where n is minimal. We let $q_i(x, y), q_j(x, y)$ be arbitrary elements in S and let them be written $q_i(x, y) = f(x, y)p(x, y)$, $q_j(x, y) = g(x, y)p(x, y)$ where f and g share no common factors. We then have that both of $f(x, y)$ and $g(x, y)$ is in $\mathfrak{p}$, as $\mathfrak{p}$ is prime and if $p(x, y) \in \mathfrak{p}$, we may replace q_i and q_j with $p(x, y)$, hence contradicting the minimality of $\mathfrak{p}$. We then consider $K' \supset K$ where $K' = k(x)[y]$. Then, as f, g have no common factors and $k(x)[y]$ is Euclidean, we may find some linear combination $r(x, y)f(x, y) + s(x, y)g(x, y) = h(x)$ where $h(x)$ is a rational function in x (i.e. a unit in K'). By multiplying through by its denominator, WLOG we may assume $h(x) \in k[x]$. As $k[x]$ is a PID, we then have by the algebraic closure of k that some linear factor $(x - a) \in \mathfrak{p}$. By repeating this process in $K'' = k(y)[x]$, we can also find some linear term $(y - b) \in \mathfrak{p}$. As $K/\langle (x - a), (y - b) \rangle \approx k$, we have that $\langle (x-a),(y-b)\rangle$ is in fact maximal. Hence, all prime ideals in K which are not principal are maximal, and as a,b may be chosen to be arbitrary, we have identified k^2 with a subset of the maximal ideals of K. We now show that principal prime ideals in K are not maximal. We let $\langle f(x,y)\rangle$ be prime, of course implying $f(x,y)$ must be irreducible. As f is irreducible, we have that for $a \in k$ arbitrary, $f(a,y) = g(y) \neq 0$. As k is algebraically closed, we have that there exists some b such that $g(b) = 0$. Then, $f(x,y) \mapsto 0$ in the quotient map $K \to K/\langle (x-a),(x-b)\rangle$; hence $\langle f(x,y)\rangle \subset \langle (x-a),(x-b)\rangle$. □

b.)

Proposition. *The nonclosed points other than the generic point are given by the ideals of type $\langle f\rangle$ where $f \in k$ is irreducible*

Proof. A byproduct of our proof to part a) □

c.)

Proposition. *For $x \in X$, $\overline{\{x\}} = \{x\} \cup \{x \in k^2 \ : \ f(x) = 0\}$ where $x = \langle f\rangle$ in the case $\{x\}$ is not closed.*

Proof. This very nearly follows directly from parts a and b. We have that $\overline{\{x\}} = V(\mathfrak{p}_x) = \{x_\mathfrak{q} \ : \ \mathfrak{p} \subset \mathfrak{q}\}$. We have that for any principal prime ideals $\langle f\rangle, \langle g\rangle$ that $\langle g\rangle \not\subset \langle f\rangle$ as f is irreducible. Hence, the closure of $\{x\}$ contains $\{x\}$ and the maximal ideals containing $\{x\}$. As $\{x\}$ corresponds to $\langle f\rangle$, these are the maximal ideals $\langle (x-a),(y-b)\rangle$ such that $f \mapsto 0$ in the quotient $K \to K/\langle (x-a),(y-b)\rangle$, that is those such that $f(a,b) = 0$. This completes our proof. □

6.)

We take the following to be the definition of irreducible topological space:

Definition. A topological space X is said to be irreducible if there are no proper closed subsets X_1, X_2 such that $X = X_1 \cup X_2$. Equivalently,[1] X is irreducible if for any $\emptyset \neq U, V$ open in X, $U \cap V$ is nonempty.

Proposition. *For X an irreducible topological space and $U \subset X$ open, U is irreducible.*

Proof. We have that the open sets of U under the subspace topology are those written $V \cap U$ where $V \subset X$ is open. However, under the topology of X, we have that finite intersections of open sets are open. Thus, $V \cap U$ is open under the topology of X, and for any $W \subset U$ open under the topology of X, $W \cap U = W$. Hence, the topology of U can be written $\{W \subset U \ : \ W \text{ open in X}\}$. Then, for any $V, W \subset U$ open, we have that V, W are open in X, so $V \cap W \neq \emptyset$. However, as $V, W \subset U$ we have that $V \cap W \subset U$, thus proving the proposition. □

7.)

Let k be a finite field and A a k-algebra with finite dimension when considered as a k-module. We let the set of maximal ideals of A be denoted $\operatorname{Spm} A \subset \operatorname{Spec} A$ considered under the subspace topology.

Proposition. $\operatorname{Spm} A$ *is Hausdorff.*

Proof. We instead prove the following lemma, with a brief remark to tie together the loose threads at the end.

Lemma. $\operatorname{Spm} A$ *carries the discrete topology.*

Proof. We have that the closed sets of $\operatorname{Spm} A$ are those which can be written $V(E) \cap \operatorname{Spm} A$ where $V(E) \subset \operatorname{Spec} A$. For $x \in \operatorname{Spm} A$, we let $\mathfrak{p}_x$ be the associated (maximal) ideal in A. Then $V(\mathfrak{p}_x) = V(I(x)) = x$ by the Nullstellenstatz as $\mathfrak{p}_x = \operatorname{rad} \mathfrak{p}_x$. Hence, for any $x \in \operatorname{Spm} A$, we have that $\{x\}$ is closed. As $\mathcal{P}(\operatorname{Spm} A)$ is a finite set[2], we have that arbitrary unions of closed sets in $\operatorname{Spm} A$ are their selves closed. Thus for any arbitrary subset $S \subset \operatorname{Spm} A$, we have that S is closed, and equivalently, that S^c is open. Hence, all sets are clopen and our proof is complete. ■

This of course proves the main proposition of the problem; indeed, spaces carrying the discrete topology are trivially Hausdorff: for any two elements $x, y \in X$ where X is a topological space equipped with the discrete topology, $\{x\}, \{y\}$ both open and hence fulfill the separation requirement of the Hausdorff property. □

□

[1] A quick proof of this equivalence: suppose X is irreducible, U, V open, and $U \cap V = \emptyset$. Then, by de Morgan's laws, $\emptyset^c = X(U \cap V)^c = U^c \cup V^c$—a contradiction! The proof of the other direction is similar.

[2] Which follows from $\operatorname{Spm} A$ being a finite set, which in turn follows from $\mathcal{P}(A)$ being a finite set, which (finally) in turn follows from A being a finite set.

8.)

Proposition. *Let A be a k-algebra of finite type where k is a field. Then, for any closed subset $Y \subset X = \operatorname{Spec} A$, the closed points S of X are dense in Y.*

Proof. We shall take the following theorem as a lemma[3]

Theorem. *[Rot02, Prop. 11.67, 11.70] $K = k[x_1, \cdots, x_n]$ is a **Jacobson Ring,** i.e. for any prime ideal $\mathfrak{p} \triangleleft K$,*

$$\mathfrak{p} = \bigcap_{\substack{\mathfrak{m} \supset \mathfrak{p} \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}.$$

Now, onto our proof We let $I = \ker(K \to A)$ and note that $\operatorname{Spec} A \approx \operatorname{Spec} K/I \approx V(I)$ where $V(I)$ is considered under the induced topology as a closed subset of $\operatorname{Spec} A$ in canonical[4] homeomorphism. We have that as $Y = \overline{Y}$, $Y = V(E)$ for some $E \subset A$, and as $Y \subset V(I)$, we have by the inclusion-reversing nature of V, I that $E \supset I$. We seek to show that for any open set $D(E')$ where $E' \supset I$, there exists some point $x_\mathfrak{m}$ corresponding to maximal ideal $\mathfrak{m}$ such that $x_\mathfrak{m} \in D(E')$. We suppose the contrary, that there exists some $E' \supset I$ such that $D(E') \cap V(E) \neq \emptyset$ but $D(E') \cap V(E)$ contains no points corresponding to maximal ideals. We then have that $E' \subset J = \bigcap_{\substack{\mathfrak{m} \supset E \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}$, as all elements of E' vanish on $x_\mathfrak{m}$ for all maximal $\mathfrak{m}$. However, by the above theorem, we have that for any prime ideal $\mathfrak{p} \supset E$ (i.e. $x_\mathfrak{p} \in V(E)$), $\mathfrak{p} = \bigcap_{\substack{\mathfrak{m} \supset \mathfrak{p} \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m} \supset J$. Hence, we have that $E' \subset \mathfrak{p}$, so $D(E') \cap V(E) = \emptyset$, contradicting our assumption. This completes our proof. $\square$

References

[Rot02] Joseph J. Rotman. *Advanced modern algebra.* Prentice Hall, Inc., Upper Saddle River, NJ, 2002.

[3] A quick proof the hint implies the theorem: We have that $J(A) = \operatorname{rad} A$. Hence, for a prime ideal $\mathfrak{p}$, $J(A/\mathfrak{p}) = \operatorname{rad}(A/\mathfrak{p}) = 0$. Hence, by the bijection between ideals of $A/\mathfrak{p}$ and ideals containing $\mathfrak{p}$ of A, we let $q : A \to A/pfr$ and have that $\mathfrak{p} = q^{-1}(0) = q^{-1}(J(A/\mathfrak{p})) = q^{-1}\left(\bigcap_{\mathfrak{m} \subset A/\mathfrak{p}} \mathfrak{m}\right) = \bigcap_{\mathfrak{m} \supset \mathfrak{p}} \mathfrak{m}$.

[4] I'm not positive this is a completely rigorous usage of the word 'canonical,' but it is at the very least a correct colloquial usage.