MATH 8253 Homework I

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1.)

Proposition. *Let A be a ring, $S \subset A$ a multiplicative system, and $\tau : A \to A_S$ the localization homomorphism. Then, the induced map*

$$\tau^* : \operatorname{Spec} A_S \to \bigcap_{f \in S} D(f) \subset \operatorname{Spec} A$$

is a homeomorphism.

Proof.

- *Well-Defined* It is well-known that for any ring homomorphism, the pre-image of a prime ideal is a prime ideal. We let $\mathfrak{p} \lhd A_S$. We note that $\tau^{-1}(\mathfrak{p}) \neq \emptyset$ as $(\mathrm{Im}\tau)^c \subset A^\times$. Then, $\tau^{-1}(\mathfrak{p}) \cap S \neq \emptyset$, as else $\mathfrak{p}$ contains a unit. Hence, $\tau^*(x_\mathfrak{p}) = x_{\tau^{-1}(\mathfrak{p})} \in \bigcap_{f \in S} D(f)$.
- *Injectivity* From our previous observation that $(\mathrm{Im}\tau)^c \subset A_S^\times$, we have that for any proper ideal $I \lhd A_S$, $I \subset \mathrm{Im}\tau$. Thus, for any two prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$, we have that $\tau^{-1}(\mathfrak{p}_1 \Delta \mathfrak{p}_2) \neq \emptyset$ and hence $\tau^{-1}(\mathfrak{p}_1)$ and $\tau^{-1}(\mathfrak{p}_2)$ are distinct. It follows τ^* is injective.
- *Surjectivity* We let $x_\mathfrak{q} \in \bigcap_{f \in S} D(f)$ and let $\mathfrak{q} = \langle x_1, \ldots \rangle$. We claim that $\tau^{-1}\langle \tau\mathfrak{q} \rangle = \mathfrak{q}$. That $\mathfrak{q} \subset \tau^{-1}\langle \tau\mathfrak{q} \rangle$ is obvious. We suppose $r \in \tau^{-1}\tau\mathfrak{q}$. Then, $\frac{r}{1} \in \langle \tau\mathfrak{q} \rangle = \mathfrak{q}[S^{-1}]$, so $\frac{r}{1} = \frac{q}{s}$ for some $q \in \mathfrak{q}$, $s \in S$. Then, $s'(rs - q) = 0$ for some $s' \in S$. However, we must have that $(rs - q) \in \mathfrak{q}$ as $s \notin \mathfrak{q}$, which then implies $rs \in \mathfrak{q}$ which in turn implies by primality $r \in \mathfrak{q}$ as $s \notin \mathfrak{q}$. Thus, $\tau^{-1}\langle \tau\mathfrak{q} \rangle = \mathfrak{q}$, so $\tau^* x_{\langle \tau\mathfrak{q} \rangle} = x_\mathfrak{q}$ and τ^* is surjective.
- *Bicontinuity* We let $E \subset A$. Then, $(\tau^*)^{-1} D(E) = D_S(\tau E)$, which is open, so τ^* is continuous.

  To show continuity of $(\tau^*)^{-1}$, we let $E \subset A_S$ and claim $\tau^*(V_S(E)) = V(\tau^{-1}(E))$. We note that if $E \subset \mathfrak{p} \lhd A_S$ then certainly $\tau^{-1}(E) \subset \tau^{-1}(\mathfrak{p})$, so $\tau^* V_S(E) \subset V(\tau^{-1}(E))$. On the other hand, for $\tau^{-1}(E) \subset \mathfrak{q} \lhd A$, we have that $E \subset \tau\tau^{-1}(E) \subset \langle \tau(\mathfrak{q}) \rangle = (\tau^*)^{-1}(\mathfrak{q})$, so $V(\tau^{-1}(E)) \subset \tau^*(V_S(E))$ and our proof is complete.

□

2.)

Proposition. *Let X be an irreducible topological space such that there is some $x_0 \in X$ with $\overline{\{x_0\}} = X$. Then, for $\phi : X \to Y$ continuous,* (i) *$\overline{\mathrm{Im}(\phi)}$ is irreduicble, and* (ii) *$\overline{\{\phi(x)\}} = \overline{\mathrm{Im}(\phi)}$*

Proof. We begin with a quick lemma.

Lemma. *With the hypotheses of the proposition, for any $u \subset \overline{\mathrm{Im}(\phi)}$ open, $\phi^{-1}(u) = \emptyset$ if and only if $u = \emptyset$.*

Proof. One direction is obvious, we now show the other. We suppose $\phi^{-1}(u) = \emptyset$. Then, $u \cap \mathrm{Im}(\phi) = \emptyset$, so $\overline{\mathrm{Im}(\phi)} \setminus u$ is a closed subset of Y containing the image of ϕ and hence its closure, that is $\overline{\mathrm{Im}(\phi)} \subseteq \overline{\mathrm{Im}(\phi)} \setminus u$. Thus, $u = \emptyset$. ■

Now, onward to the proposition.

(i) We let $\emptyset \neq u, v \subset \overline{\mathrm{Im}(\phi)}$ be open. Then, $\phi^{-1}(u), \phi^{-1}(v)$ are nonempty open sets in X by the lemma, and hence $\emptyset \neq \phi^{-1}(u) \cap \phi^{-1}(v) = \phi^{-1}(u \cap v)$. Thus, for any two non-empty open sets in $\overline{\mathrm{Im}(\phi)}$, their intersection is non-empty and hence $\overline{\mathrm{Im}(\phi)}$ is irreducible.

(ii) We let $\phi(x) \notin v \subset \overline{\mathrm{Im}(\phi)}$ where v is open. Then, $x \notin \phi^{-1}(v)$, so $X \setminus \phi^{-1}(v)$ is a closed subset of X containing x and is hence the full set X. Thus, $\phi^{-1}(v)$ is empty, so by the lemma $v = \emptyset$. Hence, $\overline{\{\phi(x)\}} = \overline{\mathrm{Im}(\phi)}$

■

Corollary. *Let $\phi : A \to A'$ be a ring homomorphism and $\phi^* : \operatorname{Spec} A' \to \operatorname{Spec} A$ the corresponding map of spectra. If $\operatorname{Spec} A'$ is irreducible with ξ its generic point, then the closure $\overline{\phi^*(SpecA')}$ is irreducible and $\phi^*(\xi)$ is a generic point of it.*

Proof. Follows from the proposition, replacing $\operatorname{Spec} A'$ with X and ξ with x. $\square$

3.)

Proposition. *For R a principal ideal domain with natural injection $\iota : R \to R[t]$ and induced map $\pi : \operatorname{Spec} R[t] \to \operatorname{Spec} R$, $\pi^{-1}(s) \approx \mathbb{A}^1_{k(s)}$.*

Proof. We let $s \in \operatorname{Spec} R$ correspond to ideal $\mathfrak{p}_s \lhd R$. Then, $K(s) = (R/\mathfrak{p}_s)_{\mathfrak{p}_s} = R/\mathfrak{p}_s$ as in a PID, primality and maximality are equivalent. Thus, $\mathbb{A}^1_{k(s)} = \operatorname{Spec} k(s)[t] = \operatorname{Spec}(R/\mathfrak{p}_s)[t] = \operatorname{Spec}(R[t]/\mathfrak{p}_s[t]) \approx V(\mathfrak{p}_s[t]) \subset \mathbb{A}^1_R$. Thus, we need only show $V(\mathfrak{p}_s[t]) = \pi^{-1}(s)$. We note for $x \in \mathbb{A}^1_R$, $\pi(x) = s$ if and only if $\mathfrak{p}_s[t] \subset \mathfrak{q}_x \subset R[t]$, as $\mathfrak{p}_s$ is maximal in R. This is precisely the set $V(\mathfrak{p}_s[t])$. This completes our proof $\square$

4.)

We let A be a ring

a.)

Proposition. *For $f \in A$, we let $S(f) = \{g \in A \ : \ D(f) \subset D(g)\}$. Then $S(f)$ is a multiplicative system containing $\{f, f^2, f^3, \ldots\}$.*

Proof. We let $g, h \in S(f)$. Then, $D(gh) = D(g) \cap D(h) \supseteq D(f)$, so $gh \in S(f)$. We also note that as $A \to k(s)$ is a ring homomorphism, for $f \neq 0$ in $k(s)$, we have $f^n \neq 0$ as otherwise $k(s)$ would contain zero-devisors. Thus, for any $s \in D(f)$, $s \in D(f^n)$ for all n. $\square$

b.)

Proposition. *The localization map $\sigma : A_f \to A_{S(f)}$ is an isomorphism.*

Proof. We shall state and prove a lemma then show its equivalence to the proposition.

Lemma. *For $D(f) \subset D(g)$, and $\rho_f : A \to A_f$ the localization map, $\rho_f(g)$ is a unit in A_f.*

Proof. We rewrite our hypothesis as $V(f) \supset V(g)$ and have by the Nullstellensatz that $\operatorname{rad}(\langle f \rangle) = I(V(F)) \subset I(V(g)) = \operatorname{rad}(\langle g \rangle)$. In particular, we have that $f \in \operatorname{rad}(\langle g \rangle)$, so for some n, $f^n = gr$ with $r \in A$. Applying ρ_f to that equality yields $\frac{f^n}{1} = \frac{gr}{1}$, so dividing through by f^n gives the relation $\rho_f(g)\frac{r}{f^n} = 1$. $\blacksquare$

We have the below commutative diagram

$$\begin{tikzcd}
& A \arrow[dl, "\rho_f"'] \arrow[d, "\rho_{S(f)}"] \\
A_f \arrow[r, "\sigma"] & A_{S(f)}
\end{tikzcd}$$

We note that as f is mapped by $\rho_{S(f)}$ to the unit group of $A_{S(f)}$, the localization universal mapping property implies that σ is the unique map for which this diagram commutes. On the other hand, as $S(f)$ is mapped to the unit group of A_f by ρ_f, we have that there is a unique map ψ making the below diagram commutative.

$$\begin{tikzcd}
& A \arrow[dl, "\rho_f"'] \arrow[d, "\rho_{S(f)}"] \arrow[dr, "\rho_f"] & \\
A_f \arrow[r, "\sigma"] & A_{S(f)} \arrow[r, dashed, "\psi"] & A_f
\end{tikzcd}$$

However, as the identity is the unique map from A_f to A_f commuting with ρ_f, we must have that $\psi\sigma = \mathrm{id}_{A_f}$. This argument can be restated symmetrically, thus showing ψ and σ to be two-sided mutual inverses and hence isomorphisms. $\square$

c.)

Prompt. *Define a contravariant functor $\mathcal{O}_X : \mathbb{D}(X) \to \mathbf{Ring}$ by*

$$D(f) \mapsto A_{S(f)}$$

$$D(f) \subset D(g) \mapsto (A_{S(g)} \to A_{S(f)})$$

Response. We show functoriality. We first note that $D(f) \subset D(g)$ implies that for any $h \in S(g)$, $D(f) \subset D(g) \subset D(h)$, so $h \in S(f)$. Thus, $S(g) \subset S(f)$, so the morphism $A_{S(g)} \to A_{S(f)}$ is achieved simply by localizing $A_{S(g)}$ at $S(f)$. We note that $A_{S(f)} \to A_{S(f)}$ is indeed the identity morphism in **Ring** and thus $\mathcal{O}_X$ preserves identity. Finally, we suppose $D(f) \subset D(g) \subset D(h)$. Then, $A_{S(h)} \to A_{S(g)} \to A_{S(f)}$ is the composite of the localization of one set and a superset to it and hence the same map as $A_{S(h)} \to A_{S(f)}$ (this also follows immediately from a universal mapping property argument). Hence $\mathcal{O}_X$ is indeed a contravariant functor. □

d.)

Proposition. $\mathcal{O}_X^\# : \mathbb{D}^\#(X) \to \mathbf{Ring}$ *is naturally equivalent to the composition* $\mathcal{O}_X \circ F : \mathbb{D}^\#(X) \to \mathbb{D}(X) \to \mathbf{Ring}$

Proof. We define our natural transformation by $n_\bullet \in \mathrm{Mor}_{\mathbb{D}^\#(X)}(\bullet, -)$ and $m_\bullet \in \mathrm{Mor}_{\mathbf{Ring}}(\bullet, -)$.[1] We let $n_f = \mathrm{id}_f$ and $m_{A_f} = \sigma$ where σ is the isomorphism defined in part 4b. Then, we have the following diagram showing the paths of elements:

$$\begin{tikzcd} f \arrow[rr, mapsto, "\mathcal{O}^\#(X)"] \arrow[d, mapsto, "n_f"] & & A_f \arrow[d, mapsto, "m_f"] \\ f \arrow[r, mapsto, "F"] & D(f) \arrow[r, mapsto, "\mathcal{O}_X"] & A_{S(f)} \end{tikzcd}$$

It is immediately clear this diagram commutes. As all of $m_\bullet$ and $n_\bullet$ are isomorphisms, our natural transformation is a natural equivalence. □

5.)

a.)

Proposition. *Let A be a ring, X its spectrum, $f \in A$, X_f the spectrum of A_f, and $\tau^* : X_f \to D(f) \subset X$ the homeomorphism of problem 1. Then τ^* induces a bijection between basic open sets $D(g) \subset D(f)$ and those $D_f(g) \subset X_f$.*

Proof. We shall show that for $U \subset X_f$ open, $\tau^*(U)$ is basic if and only if U is basic. We first assume $\tau^*(U) = D(g)$ with $g \in A$. Then, as τ^* is a homeomorphism, $U = (\tau^*)^{-1}(D(g)) = D_f(\tau(g)) = D_f(\frac{g}{1})$. Conversely, consider $D_f(\frac{h}{f^n})$ where $n \geq 0$. We claim that $D_f(\frac{h}{f^n}) = D_f(\frac{h}{1})$. Indeed, as f^n is a unit in A_f, if $\frac{h}{f^n}$ maps to a nonzero element in $k(s)$ for any $s \in X_f$, then $f^n \frac{h}{f^n}$ is necessarily nonzero. Thus, $D_f(\frac{h}{f^n}) = D_f(\frac{h}{1}) = \tau(D(h))$. This completes our proof. □

b.)

Proposition 5.1. *The restriction of $\mathcal{O}_X$ to $\mathbb{D}(D(f))$ for some $f \in A$ is equivalent to* $\mathcal{O}_{X_f} : \mathbb{D}(X_f) \to \mathbf{Ring}$.

Proof. I'm ultimately a bit confused by this... We can only define a natural transformation between functors between the *same* two categories $\mathcal{C}$ and $\mathcal{D}$... It seems like a natural interpretation to require a natural equivalence $F : \mathbb{D}(D(f)) \to X_f$ which commutes with $\mathcal{O}_{X_f}$ and $\mathcal{O}_X$, that is a natural equivalence F such that the following commutes:

$$\begin{tikzcd} \mathbb{D}(D(f)) \arrow[r, "\mathcal{O}_X"] \arrow[d, "F"] & \mathbf{Ring} \\ \mathbb{D}(X_f) \arrow[ur, "\mathcal{O}_{X_f}"'] & \end{tikzcd}$$

We let $F(D(g)) = D_f(\frac{g}{1}) = (\tau^*)^{-1}(D(g))$, and $F(D(g) \hookrightarrow D(h)) = [(\tau^*)^{-1}(D(g)) \hookrightarrow (\tau^*)^{-1}(D(h))]$ (As τ is a homeomorphism, such inclusions are preserved). As F is in fact surjective (even better than essentially surjective!) in light of part a and is in its description explicitly bijective on morphisms, it is indeed a natural equivalence. Further, as $(\mathcal{O}_{X_f} \circ F)(D(g)) = (A_f)_g = A_g = \mathcal{O}_X(D(g))$ (4b,c), it is clear our diagram commutes on objects (that it commutes on morphisms is more or less trivially checked as each morphism set has cardinality at most one). This completes our "proof." □

[1] This feels like terrible and possibly not even well-defined notation, but I'm not sure how to express this otherwise?

6.)

Prompt. *Find two presheaves, one of which* (i) *fails locality but satisfies gluing and the other of which* (ii) *satisfies locality but fails gluing.*

Response. *(i)* We let X be some Euclidean[2] space, say $\mathbb{C}^n$ for $n \geq 1$ and define the "double-power" presheaf $P : \mathbf{Op}(X) \to \mathbf{Set}$ as follows: for an open set $U \subset X$, let $P(U) = \mathcal{P}(\mathcal{P}(U))$. For $V \subset U$, $S \in P(U)$, let $S|_V = \{s \in S \ : \ s \subset V\}$. Then, $S|_U = \{s \in S \ : \ s \subset U\} = S = \mathrm{id}_{\mathbf{Set}}(S)$, so the identity restriction axiom of presheaves is satisfied. In addition, for $S \in P(U)$, $W \subset V \subset U$, $(S|_V)|_W = \{s \in S \ : \ s \subset V\}|_W = \{s \in S \ : \ s \subset V \cap W\}$. As $V \cap W = W$ by assumption, this is the same set as $\{s \in S \ : \ s \subset W\} = S|_W$. Thus, P is indeed a presheaf. We claim it satisfies the gluing sheaf axiom: we let $\{U_i\}_{i\in I}$ be some open cover of X and $\{S_i\}_{i\in I}$ an element of $\prod_{i\in I} P(U_i)$ satisfying the intersection-compatibility condition of the gluing axiom. We claim that $S = \bigcup_{i\in I} S_i$ satisfies $S|_{U_i} = S_i$ for all i. Indeed, that $S_i \subset S|_{U_i}$ follows trivially from the way in which we defined restriction maps. To show the opposite containment, we suppose there exists some $\emptyset \neq W \in \left(S|_{U_i} \setminus S_i\right)$. Then, $W \in S_j$ for some j, and $W \subset U_i$ as otherwise it would not be an element of $S|_{U_i}$. Thus, $W \in \left(S_j|_{U_i \cap U_j} \setminus S_i|_{U_i \cap U_j}\right)$, contradicting our assumption of compatibility, and showing P to satisfy the gluing sheaf axiom. However, we claim it does not satisfy the locality axiom. To see this, we let $\{B_i\}_{i\in I}$ be an open covering of X by unit balls, V a ball of radius exceeding unity and S the singleton set $S = \{V\}$. Then, $V|_{B_i} = \emptyset$ for all i as clearly $V \not\subset B_i$, but $V \neq \emptyset$. This contradicts the locality sheaf axiom so indeed P is not a sheaf.

(ii) *(Note: I found reference to this one in Vakil, but proved its relevant properties myself)* We let $X = \mathbb{C}$ and let $\mathcal{B}_X$ be the "presheaf of bounded functions," that is, a functor from $\mathbb{D}(\mathbb{C})$ to **Ring** where $\mathcal{B}_X(u)$ is the ring of bounded functions on $u \subset \mathbb{C}$ open and $(u \subset v) \mapsto (f \mapsto f|_v)$, that is simple function restriction. Then, it is clear that $\mathcal{B}_X$ is a presheaf, and it quite clearly satisfies the locality axiom as any non-zero function on $\mathbb{C}$ certainly must restrict to a non-zero function on some set of an open cover. However, if we consider the function on $\mathbb{C}$ $f(x) = x$ and an open cover $\{u_i\}_{i\in I}$ consisting of bounded sets, we have that on each u_i, $f|_{u_i}$ is bounded, but $f \notin \mathcal{B}_X(\mathbb{C})$ as it is not bounded on the complex plane. Thus, $\mathcal{B}_X$ fails to be a sheaf because it fails the gluing axiom.

□

[2] well, I think any topological space with cardinality greater than 1 *should* work, but let's stick to Euclidean so I can make my counter-example super concrete so there's less to poke holes in