1.)

a.)

Proposition. *We let F be an Abelian group, with $\mathscr{F}$ the skyscraper sheaf supported at $x \in X$ associated to F and $\mathcal{F}$ the constant sheaf F on $\{x\}$. Then, $\mathscr{F} = \iota_*\mathcal{F}$ where ι_* is induced by $\iota : \{x\} \hookrightarrow X$.*

Proof. We recall that for $U \subset X$ open, $(\iota_*\mathcal{F})(U) = \mathcal{F}(\iota^{-1}(U))$. We note that

$$\iota^{-1}(U) = \begin{cases} \{x\} & x \in U \\ \emptyset & x \notin U \end{cases}$$

Thus,

$$(\iota_*\mathcal{F})(U) = \mathcal{F}(\iota^{-1}(U)) = \begin{cases} F & x \in U \\ 0 & x \notin U \end{cases} = \mathscr{F}(U)$$

□

b.)

Proposition. *We let X be a scheme, $x \in X$ a closed point, $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field, and F a $k(x)$-vector space. Then, there exists a skyscraper sheaf $\mathcal{F}$ with stalk $\mathcal{F}_x = F$ which is quasi-coherent.*

Proof. We note that $(\{x\}, \mathcal{O}_X)$ is a closed subscheme of X as $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_{\{x\}} = \mathcal{O}_{X,x}$ is surjective. Thus, as $\mathcal{F} = i_*F$, the direct image of the skyscraper sheaf on $\{x\}$, by part 1, $\mathcal{F}$ is quasi-coherent. □

2.)

We begin with a lemma; the proposition will follow as a corollary.

Lemma. *We let $A = k[t]$ for some field k and $E \subset k[t]$ a multiplicatively closed subset which is finitely generated as a semigroup. We let $A_E := A[E^{-1}]$. Then, $k(t)$ is not of finite rank as an A_E-module (via the natural inclusion $A_E \hookrightarrow k(t)$).*

Proof. We note that $k(t)$ is has irredundant generating set as a $k[t]$-algebra consisting of all elements of the form $1/p(t)$ where $p(t)$ is prime. We note that for any field k, the set of prime elements of $k[t]$ is an infinite set.[1] Thus, A_E is a finitely generated $k[t]$-subalgebra of $k(t)$, which is infinitely generated, so if $k(t)$ is a finitely-generated A_E-algebra it must be a finitely generated $k[t]$-algebra, contradicting the infinitude of prime elements. ■

Corollary. *The skyscraper sheaf $\mathscr{F}$ on $X = \mathbb{A}^1_k$ supported at the generic point with stalk $k(t)$ is not locally of finite rank and hence not coherent.*

Proof. We note that any open set in X is of the form $D(E)$ where $E \subset A$. A priori, E need not be a finite set, but we note that $D(E) = D(\langle E \rangle)$, and as $k[t]$ is Noetherian, we have that E has some finite generating set F. Hence, $D(E) = D(F)$, so $\mathcal{O}_X(D(E)) = A[F^{-1}]$ for some finite set $F \subset A$. Thus, by the lemma, we have that for any open set $U = D(E) \subset X$, $k(t) = \mathscr{F}(U)$ is an infinite-rank $\mathcal{O}_X(U)$-module. □

3.)

Proposition. *We let X be a scheme and $\mathcal{F}$, $\mathcal{G}$ quasicoherent $\mathcal{O}_X$-modules. Then, $\mathcal{F} \otimes \mathcal{G}$ is quasicoherent.*

Proof. By definition, we have that there exists an affine open cover $\{U_i\}_{i \in I}$ (resp. $\{U'_j\}$) of X such that on each U_i (resp. U'_j), $\mathcal{F}|_{U_i} \cong M_i^\sim$ (resp. $\mathcal{G}|_{U'_j} \cong M'_j \sim$) for some $\mathcal{O}_X(U_i)$-module M_i (resp. $\mathcal{O}_X(U'_j)$-module M'_j). We let W_{ij} be the open subscheme $U_i \cap U'_j$, and let $\{V_{ijk}\}_{k \in K_{ij}}$ be an affine open cover of W_{ij} for some indexing set K_{ij}. We note that as $V_{ijk} \subset U_i$ is an open subscheme of an affine open set on which $\mathcal{F}$ is associated, we have that $\mathcal{F}$ is associated on V_{ijk} with a similar statement holding for $\mathcal{G}$. Hence, $\mathcal{F}$ and $\mathcal{G}$ are both associated on each V_{ijk}, and thus their tensor product is as well. As $\bigcup_{(i,j) \in I \times J} \{V_{ijk}\}_{k \in K_{ij}}$ is an affine open cover of X, we have completed our proof. □

[1] As $kt]$ is a UFD, this follows from Euclid's argument for the infinitude of the primes.

4.)

a.)

Proposition. *A scheme X is quasi-compact if and only if it is a finite union of affine open subsets.*

Proof. ($\Longrightarrow$) We assume X is quasi-compact. By definition of scheme, there exists an affine open set $U_x \ni x$ for all $x \in X$. Then, $\{U_x\}_{x\in X}$ is an open cover of X so there exists some finite subcover $\{U_{x_i}\}_{i=1}^n$.

($\Longleftarrow$) It is a completely trivial exercise in point-set topology that any topological space which can be written as a finite union of quasi-compact sets is itself quasi-compact. As any affine scheme (and hence any affine open subset) is quasi-compact, the proof follows immediately. ☐

b.)

Definition 4.1. A morphism $\pi : X \to Y$ of schemes is quasi-compact if $\pi^{-1}(U)$ is quasi-compact for any quasi-compact $U \subset Y$.

Definition 4.2. A morphism $\pi : X \to Y$ of schemes is quasi-compact if $\pi^{-1}(U)$ is quasi-compact for any open affine $U \subset Y$.

Proposition. *Definitions 4.1 and 4.2 are equivalent.*

Proof. As all open affine sets are quasi-compact, it is immediately clear that definition 4.1 implies 4.2. On the other hand, we suppose π satisfies definition 4.2 and that $U \subset Y$ is quasi-compact. Then, we may cover U by affine open sets $\{W_i\}_{i\in I}$ and have that there exists a finite subcover $\{W_i\}_{i=1}^n$. Then, $\pi^{-1}(U) = \bigcup_{i=1}^n \pi^{-1}(W_i)$, and by assumption each $\pi^{-1}(W_i)$ is quasi-compact. Part a of this question implies the proposition. ☐

5.)

6.)

Proposition. *We let X be a topological space, $U \subset X$ with inclusion $j : U \hookrightarrow X$, and $\mathcal{F}$ a sheaf of sets on X. Then, $j^{-1}\mathcal{F}$ coincides with $\mathcal{F}|_U$.*

Proof. We recall that $(j^{-1}\mathcal{F})^{\text{pre}}(U) = \varinjlim_{V\supset U} \mathcal{F}(V)$. We recall as well that sheaves are recoverable from their stalks via sheafification—in particular if the stalks of presheaf $\mathcal{G}$ coincide with the stalks of sheaf $\mathcal{F}$, then $\mathcal{G}^+ = \mathcal{F}$. We let $p \in X$ and present the following diagram in which the black arrows commute by definition, with explanation of dashed arrows to follow.

$$
\begin{tikzcd}
\mathcal{F}(V_\alpha) \arrow[rr] \arrow[dr] \arrow[ddrrr] & & \mathcal{F}(V_\beta) \arrow[dl] \arrow[ddr] & & \mathcal{F}(V_\gamma) \arrow[rr] \arrow[ddl] \arrow[dr] & & \mathcal{F}(V_\delta) \arrow[dl] \arrow[ddlll] \\
& (j^{-1}\mathcal{F})^{\text{pre}}(U_a) \arrow[drr, dashed, "\exists! g_a"'] \arrow[ddrr] & & & & (j^{-1}\mathcal{F})^{\text{pre}}(U_b) \arrow[dll, dashed, "\exists! g_b"] \arrow[ddll, bend left] & \\
& & & \mathcal{F}_p \arrow[d, dashed, bend right, "\exists! f"'] & & & \\
& & & (j^{-1}\mathcal{F})_p \arrow[u, dashed, bend right, "\exists! h"'] & & &
\end{tikzcd}
$$

We note that each $\mathcal{F}(V_\alpha)$ has a map $\mathcal{F}(V_\alpha) \to (j^{-1}\mathcal{F})^{\text{pre}}(U_a) \to (j^{-1}\mathcal{F})_p$ commuting with $\mathcal{F}(V_\alpha \to \mathcal{F}(V_\beta)$ by definition of colimit. Thus, $(j^{-1}\mathcal{F})_p$ is a co-cone for $\mathcal{F}(V)$ with $V \ni p$, inducing the map f. We also note that $\{\mathcal{F}(V) \ : \ U \subset V\}$ is a subco-cone of $\{\mathcal{F}(V) \ : \ p \in V\}$, inducing the maps g_a, g_b in the diagram. Then, the maps g_a, g_b establish $\mathcal{F}_p$ as a co-cone for $(j^{-1}\mathcal{F})^{\text{pre}}(U)$ for all $U \ni p$, inducing the map h. By universality of the colimit construction, we have that f and h are mutual isomorphisms.

☐