MATH 8301 Homework V

David DeMark

9 October 2017

1.)

We let $\mathcal{C}$ be the category with a single object $*$ and morphisms $\mathrm{Hom}(*, *) = G$.

a.)

Proposition. *For functor $F : \mathcal{C} \to \mathrm{Vect}_k$, show that $V := F(*)$ is a representation of V.*

Proof. We define a representation of a group G as a pair (V, ϕ) where V is a vector space and $\phi : G \to GL(V)$ is a homomorphism. We note that the morphisms of Vect_k are linear maps, and further, as F takes morphisms $A \to B$ to morphsims $F(A) \to F(B)$, we have that the only morphisms in the image of F lie in $\mathrm{Hom}(F(*), F(*)$. Further, we note that for all $g \in \mathrm{Hom}_{\mathcal{C}}(*, *)$, there exists some g^{-1} with $g \circ g^{-1} = g^{-1} \circ g = \mathrm{id}_*$; as F preserves identity and respects composition, we have that $F(g) \circ F(g^{-1}) = F(g^{-1}) \circ F(g) = \mathrm{id}_V)$, that is, $F(g)$ is an invertible linear transformation. Thus, F induces a map $\mathrm{Hom}(*, *) \to GL(V)$, and as F respects composition and preserves identity, F may be viewed as a homomorphism. This completes our proof. □

b.)

Prompt. *Conversely, show a functor can be constructed from a representation (V, ϕ).*

Proof. We let $F : \mathcal{C} \to \mathrm{Vect}_k$ be our functor and let $F(*) = V$, with $F : \mathrm{Hom}_{\mathcal{C}}(*, *) \to \mathrm{Hom}_{\mathrm{Vect}_k}(V, V)$ being given by $F(g) = \phi(g)$. Then, as ϕ respects the group operation, it respects composition, and as ϕ respects the group identity, F respects the identity morphism and is hence functorial. □

2.)

a.)

Proposition. *We let $f : A \to B$ be a morphism in the category* Set. *Then, f is a [categorical] isomorphism if and only if f is a bijection.*

Proof. We suppose that $a, b \in A$ with $f(a) = f(b)$. Then, $\mathrm{id}_A(a) = g(f(a)) = g(f(b)) = \mathrm{id}_A(b)$, so $a = b$ and f is injective. On the other hand, we suppose there exists some $b \in B$ such that $b \notin \mathrm{im} f$. Then $f(g(b)) \neq b$, so we have established a contradiction and f is surjective and thus bijective.

On the other hand, we suppose f is a bijection. Then, f is invertible, with unique two-sided inverse f^{-1}, with $ff^{-1} = \mathrm{id}_A$ and $f^{-1}f = \mathrm{id}_B$. This completes our proof. □

b.)

Proposition. *We let $f : A \to B$ be a morphism in the category* Top. *f is an isomorphism if and only if it is a homeomorphism.*

Proof. We have that Top is a concrete category and hence a subcategory of Set. Thus, the isomorphisms of Set are bijections, and as the morphisms of Set are continuous maps, we have that f, g are continuous bijections and thus homeomorphisms.

On the other hand, we suppose f is a homeomorphism. Then, by definition, there exists a unique continuous two-sided inverse f^{-1} such that $ff^{-1} = \mathrm{id}_A$ and $f^{-1}f = \mathrm{id}_B$, so f is an isomorphism. □

c.)

Proposition. *We let $f : A \to B$ be a morphism in the category* hTop. *f is an isomorphism if and only if it is a homotopy equivalence.*

Proof. We suppose f is an isomorphism. Then, there exists some map $g : B \to A$ such that $fg \approxeq \mathrm{id}_B$ and $gf \approxeq \mathrm{id}_A$. Then, by definition, f is a homotopy equivalence. The converse follows just as obviously. □

d.)

Proposition. *All maps of the category defined in problem 1 are isomorphisms.*

Proof. By definition, for all $g \in G$, there exists some element g^{-1} such that $gg^{-1} = g^{-1}g = \mathrm{id}_G$. Then, viewing g, g^{-1} as morphisms in $\mathrm{Hom}_{\mathcal{C}}(*, *)$, g is an isomorphism. □

e.)

Proposition. *For $F : \mathcal{C} \to \mathcal{D}$ a functor, $f : X \to Y$ an isomorphism in $\mathcal{C}$, $F(f)$ is an isomorphism in $\mathcal{D}$.*

Proof. We have that $F(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{\mathcal{D}}$, and $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$. Thus, if $f : A \to B$ is an isomorphism in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, we have that $F(f) \circ F(g) = F(f \circ g) = F(\mathrm{id}_B) = \mathrm{id}_{F(B)}$, and $F(g) \circ F(f) = F(g \circ f) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. Thus, $F(f)$ is an isomorphism in $\mathcal{D}$ □

3.)

a.)

Let X and Y be topological spaces.

Proposition. *For $f : X \to Y$ continuous, the induced map $\phi_0(f) : [x] \mapsto [f(x)]$ is well-defined.*

Proof. We wish to show that for any x y, $f(x) = f(y)$. We let $\phi : [0, 1] \to X$ be a path from x to y. Then, $(f \circ \phi) : [0, 1] \to Y$ is continuous with $(f \circ \phi)(0) = f(x)$ and $(f \circ \phi)(1) = f(y)$. Hence, $f \circ \phi$ is a path connection from $f(x)$ to $f(y)$, so $f(x) \circ f(y)$ and $\pi_0(f)$ is well-defined. □

b.)

Proposition. π_0 *defines a functor* $\pi_0 : \mathrm{Top} \to \mathrm{Set}$.

Proof. There are two things to show: *(i)* for any $X \in \mathrm{Top}$, $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(x)}$, and *(ii)* for $f \in \mathrm{Hom}_{\mathrm{Top}}(A, B)$, $g \in \mathrm{Hom}_{\mathrm{Top}}(B \to C)$, $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$

(i) $\pi_0(\mathrm{id}_X) : [x] \mapsto [\mathrm{id}_X(x)] = [x]$ simply sends an equivalence class to itself, and is hence the identity morphism on $\pi_0(X)$.

(ii) $(\pi_0(g) \circ \pi_0(f)) : x \mapsto [f(x)] \mapsto \pi_0(g)[f(x)] = [g(f(x))] = \pi_0 * (f \circ g)[x]$

This completes our proof. □

c.)

Corollary. π_0 *defines a functor* $\pi_0 : \mathrm{hTop} \to \mathrm{Set}$.

Proof. We have already shown the functoriality of π_0 as a functor from Top to Set, so what is left to show is that π_0 respects equivalence classes of morphisms, that is, for $f, g \in \mathrm{Hom}_{\mathrm{Top}}(A, B)$ with $f \approxeq g$, $\pi_0(f) = \pi_0(g)$. We let $H : I \times A \to B$ be a homotopy from f to g, with $H(0, x) = f(x)$ and $H(1, x) = g(x)$. Then, for any $x_0 \in A$, $h : I \to B$ given by $t \mapsto H(t, x_0)$ gives a continuous map with $h(0) = f(x_0)$ and $h(1) = g(x_0)$ and is hence a path-connection from $f(x_0)$ to $g(x_0)$, so $f(x_0) \sim g(x_0)$ and $\pi_0(f) = \pi_0(g)$. Thus, π_0 is well-defined on morphisms of hTop, and is hence functorial as inherited from Top. □

d.)

Proposition. *If $X \approxeq Y$, then $\#\pi_0(X) = \#\pi_0(Y)$*

Proof. If $X \approxeq Y$, then there exists some isomorphism (homotopy equivalence) $f \in \mathrm{Hom}_{\mathrm{hTop}}(X, Y)$. Then, by question 2(e), $\pi_0(f) \in \mathrm{Hom}_{\mathrm{Set}}(\pi_0(X), \pi_0(Y))$ is an isomorphism in Set, and by problem 2(b), $\pi_0(f)$ is a bijection. This completes our proof. □