MATH 8301 Homework XI

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All references to theorems come from Allen Hatcher's *Algebraic Topology* unless otherwise stated.

1.)

We let $\overline{X}$ be path connected, G a group action on X, and $p : \overline{X} \to X := \overline{X}/G$ a covering map

a.)

Proposition. *For covering space maps $\overline{X} \to \tilde{X} \to X$, $\tilde{X} \cong \overline{X}/H$ for some $H < G$.*

Proof. We let $\hat{X} \to X$ be the universal cover of X and first restrict our attention to the case $\overline{X} = \hat{X}$. Then, proposition 1.39 implies that $F := \mathrm{Aut}(\hat{X}/X) \cong \pi_1(X, x_0)$. We note that for any $H < F$, we may construct a covering map $\tilde{X} := \hat{X}/H \to X$, which is unique up to based isomorphism by the observation that for $\tilde{X}_1 := \hat{X}/H_1$ and $\tilde{X}_2 := \hat{X}/H_2$, we then have that $\mathrm{Aut}(\hat{X}/\tilde{X}_1) = \mathrm{Aut}(\hat{X}/\tilde{X}_2)$ if and only if $H_1 = H_2$. Then, proposition 1.38 shows our special case, as we have shown that covering maps $\hat{X}/H \to X$ are in bijection with subgroups of F, and proposition 1.38 associates subgroups of F with covering maps $\overline{X} \to X$. Proposition 1.39 then states that for $\tilde{X} = \hat{X}/H$, $\pi_1(\tilde{X}, \tilde{x}_0) = H$, so indeed our association and that of proposition 1.38 one and the same.[1]

We now consider the case of a general covering map $\overline{X} \to X := \overline{X}/G$ with intermediate covering maps $\overline{X} \to \tilde{X} \to X$. As before, we let $\hat{X} \to X$ be a universal cover with $F := \mathrm{Aut}(\hat{X}/X) \cong \pi_1(X, x_0)$. Then, our previous work shows $\overline{X} = \hat{X}/N$ for some $N < F$ with $N \cong \pi_1(\overline{X}, \overline{x}_0)$. We have by proposition 1.40a that $\overline{X} \to X$ is normal and $G \cong F/N$. Then, as the subgroup poset of F is dual to the covering space poset of X (where $\tilde{X} \geq \overline{X}$ iff there exists a covering $\tilde{X} \to \overline{X}$)[2], we have by our special case that intermediate coverings $\overline{X} \to \tilde{X} \to X$ are in bijection with subgroups $\hat{H}$ such that $N < \hat{H} < F$, which are in bijection with subgroups $H < G = F/N$ by the map $\hat{H} \mapsto \hat{H}/N$. As each of the covering spaces $\overline{X}/H \to X$ exist uniquely by the same argument of above, our proof is now complete. $\square$

b.)

Proposition 1.1. *$\overline{X}/H_1 \cong \overline{X}/H_2$ if and only if H_1 and H_2 are conjugate*

Proof. Proposition 1.38 states that for covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$, $\tilde{X}_1 \cong \tilde{X}_2$ if and only if $\pi_1(\tilde{X}_1)$ and $\pi_1(\tilde{X}_2)$ are conjugate. We showed that $\tilde{X}_i = \hat{X}/\pi_1(\tilde{X}_i) := H_i$, and it is the case that all covering spaces $\hat{X}/H$ exist. Thus, the proposition follows trivially from part a. $\square$

c.)

Proposition 1.2. *$\tilde{X} := \overline{X}/H \to X$ is normal if and only if $H \trianglelefteq G$ in which case $\mathrm{Aut}(\tilde{X}/X) = G/H$.*

Proof. We have that $\tilde{X} = \overline{X}/H = \hat{X}/\hat{H}$ for some $\hat{H} < F$ with $N \trianglelefteq \hat{H} \cong \pi_1(\tilde{X}, \tilde{x}_0)$ and $H = \hat{H}/N$. We have by proposition 1.39 that $\tilde{X} \to X$ is normal if and only if $p_*(\pi_1(\tilde{X}, \tilde{x}_0) \cong \hat{H}$ is normal in $\pi_1(X, x_0)$. Finally, we note that $H \trianglelefteq G$ if and only if $\hat{H} \trianglelefteq F$. This completes our proof. $\square$

2.)

Proposition. *For any group G and $N \trianglelefteq G$, there exists a normal covering space $\overline{X} \to X$ with $\pi_1(X) \cong G$, $\pi_1(\overline{X}) \cong N$ and $\mathrm{Aut}(\overline{X}/X) \cong G/N$.*

[1] Well, technically to bridge that gap we also need the fact that for $p : \tilde{X} \to X$ a covering map, $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is an injection, which is given by proposition 1.31.

[2] We haven't quite justified this, but it is an immediate corollary of proposition 1.38 in conjunction with proposition 1.31.

Proof. We let $G = \langle g_\alpha \mid r_\beta \rangle$ with generators g_α indexed over A and relations r_β indexed over B. We then recall Hatcher's construction of a space X_G with $\pi_1(X_G, x_0) = G$: we let $Y = \left(\bigwedge_{\alpha\in A} S^1\right) \sqcup \left(\bigsqcup_{\beta\in B} D^2\right)$, and define an equivalence relation $\sim$ wich associates the boundary of the copy of D^2 indexed by β to the copies of S^1 associated to the letters in the word r_β. Then, $X := Y/\sim$ fulfills the desired property, and the universal cover $\hat{X}$ of X has $\mathrm{Aut}(\hat{X}/X) = G$. Our argument for question 1 then ensures the existence of $\overline{X}$ and gives its fundamental group and $\mathrm{Aut}(\overline{X}/X)$. □

3.)

We let $C_\bullet$ and $D_\bullet$ be chain complexes and $f, g, h : C_\bullet \to D_\bullet$ chain maps. If there exists a homotopy map $P : C_n \to D_{n+1}$ such that $P\partial + \partial P = f - g$, we say $f \sim g$.

Proposition. *Chain homotopy ($\sim$) is an equivalence relation.*

Proof. *Reflexivity:* We wish to show $f \sim f$. We let $P : C_n \to D_{n+1}$ be the 0 map. Then, $P\partial + \partial P = 0 = f - f$, so $f \sim f$.

Symmetry We suppose $f \sim g$ and let $P : C_n \to D_n$ be a chain homotopy between them. We let $P' = -P$ and then have that $P'\partial + \partial P' = -(P\partial + \partial P) = -(f - g) = g - f$ so $g \sim f$.

Transitivity We suppose $f \sim g$ and $g \sim h$. We let P_i be such that $P_1\partial + \partial P_1 = f - g$ and $P_2\partial + \partial P_2 = g - h$. We let $P = P_1 + P_2$ and then have that $P\partial + \partial P = (P_1 + P_2)\partial + \partial(P_1 + P_2) = (P_1\partial + \partial P_1) + (P_2\partial + \partial P_2) = (f - g) + (g - h) = f - h$ so $f \sim h$. □

4.)

Proposition. *We suppose $A \subset X$ with inclusion map $\iota : A \to X$ and that X retracts onto A. Then, $\iota_* : H_*(A) \to H_*(X)$ is an injection.*

Proof. We suppose X retracts onto A via the map τ, i.e. $\tau : X \to A$ is such that $\tau \circ \iota = \mathrm{id}_A$. We recall that $H_*(-)$ is *functorial,* that is, it preserves composition and the identity map. Thus, $\mathrm{id}_{H_*(A)} = H_*(\mathrm{id}_A) = H_*(\tau \circ \iota) = \tau_* \circ \iota_*$. We recall the following category-theoretic fact with obvious proof: if $\phi \in \mathrm{Hom}(A, B)$, $\psi \in \mathrm{Hom}(B, C)$ and $\psi \circ \phi$ is an isomorphism, then ϕ is a monomorphism and ψ is an epimorphism. Hence, ι_* is an injection, and though we have already completed our proof, we may as well note that τ_* is surjective. □

5.)

Prompt. *Let A be a finitely generated Abelian group. Construct a chain complex $C_\bullet$ such that $H_0(C_\bullet) = A$ but $H_i(C_\bullet) = 0$ for $i \neq 0$.*

Response. By the structure theorem for Abelian groups, $A = \bigoplus_{k=1}^n \mathbb{Z}/m_k$ for some $m_k \geq 0$. We let $N = \langle (0, \ldots, m_k, \ldots, 0) \mid k \in [n] \rangle$, so that $A = \mathbb{Z}^n/N$. Then, we claim that the following chain complex fulfils the desired properties: we let

$$C_i = \begin{cases} N & i = 1 \\ \mathbb{Z}^n & i = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Now, we have that ∂_1 is injective (the inclusion $N \to \mathbb{Z}^n$), so $H_1(C_*) = 0/0 = 0$. $\mathrm{im}\partial_1 = N$ and $\ker \partial_0 = \mathbb{Z}^n$ as $\mathrm{im}\partial_0 = 0$. Thus, $H_0(C_*) = \mathbb{Z}^n/N = A$. For all other i, we have that $C_i = 0$ so ∂_i is the 0 map and hence $H_i(C_*) = 0$. □