MATH 8301 Homework III

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Notation

- We let the topology of a topological space X be denoted $\mathcal{T}(X)$ unless this is insufficient in context to avoid confusion.
- We let the cardinality of a set or topological space (interpreted as a set) X be denoted $|X|$.
- We let S be a set and denote its power set as $\mathcal{P}(S)$
- When it is understood that our context is some metric space X, we let the open ball of radius δ around $x \in X$ be denoted $B_\delta(x)$ and its closed counterpart be denoted $\overline{B}_\delta(x)$

1.)

Let $(V, \mathcal{F})$ be a simplicial complex and $|(V, \mathcal{F})|$ its geometric realization.

Proposition. *$|(V,\mathcal{F})|$ is compact if and only if $|V| < \infty$.*

Proof. ($\Longleftarrow$) We let $|V| = m < \infty$. Then, as an individual n-simplex (where $n < m$) $\delta^n \subset \mathbb{R}^{n+1} \subseteq \mathbb{R}^m$ is closed under the standard topology of $\mathbb{R}^{n+1}$, which is itself closed as a subset of $\mathbb{R}^m$, we have that δ^n is closed under the standard topology of $\mathbb{R}^m$. Furthermore, we have that $|\mathcal{P}(V)| = 2^m < \infty$, so as $\mathcal{F} \subset \mathcal{P}$, we have that $(V, \mathcal{F})$ is composed of the union of finitely many simplices. Thus, $|(V,\mathcal{F})|$ is a finite union of closed sets and hence is itself closed. As $|(V,\mathcal{F})| \subset \overline{B}_1(\mathbf{0}) \subset \mathbb{R}^m$, we have that $|(V,\mathcal{F})|$ is a closed and bounded set and thus by Heine-Borel, is compact.

($\Longrightarrow$) We show the contrapositive, that is, if $|V| = \infty$ then $|(V,\mathcal{F})|$ is non-compact. We show this by explicitly constructing an open cover with no finite subcover. For $v \in V$, we let $U_v = B_{\frac{1}{4}}(\mathbf{v})$ where $\mathbf{v} \in |(V,\mathcal{F})|$ corresponds to $v \in V$. We then let $W = |(V,\mathcal{F})| \setminus \left(\bigcup_{v\in V} \overline{B}_{\frac{1}{8}}(\mathbf{v})\right)$. We claim that W is open under the topology of $|(V,\mathcal{F})|$. We consider the set $\tilde{W} = \mathbb{R}^{|V|} \setminus \left(\bigcup_{v\in V} \overline{B}_{\frac{1}{8}}(\mathbf{v})\right)$. Then, as any finite-dimensional subspace of $\mathbb{R}^{|V|}$ must contain only finitely many of the vertex-realizations $\mathbf{v}$, any projection of $\tilde{W}$ to $\mathbb{R}^d$ where d is finite is simply the space $\mathbb{R}^d$ with finitely many closed balls removed. Thus, $\tilde{W}$ is open, and as $W = |(V,\mathcal{F})| \cap \tilde{W}$, we have that W is indeed open. Then, $\mathcal{C} = \{U_v\}_{v\in V} \cup W$ is an open cover of $|(V,\mathcal{F})|$ and as $\mathbf{v}$ is an element of only one set in $\mathcal{C}$ for each $v \in V$, we have that no finite subcover could exist. Hence, $|(V,\mathcal{F})|$ is noncompact. □

2.)

Proposition. *$|(V,\mathcal{F})|$ is connected if and only if for any two vertices $v, w \in V$, there exists a sequences of vertices $v = v_0, v_1, v_2, \ldots, v_m$ such that each of $e_m = \{v_{m-1}, v_m\} \in \mathcal{F}$ for $m = 1, \ldots, n$.*

Proof. We first get one lemma out of the way, applying a theoretical sledgehammer to this hanging nail:

Lemma 2.1. *Let P be a smallest path in the graph theoretic sense (with no vertex repeats) between two vertices v, w embedded in a (not necessarily finite) graph G. Then P is finite.*

Proof. We construct the following ordering on $V(P)$: we let v be the minimal element of $(V(P), \leq)$, and let $a \leq b$ if any path $P' \subset P$ between v and b must necessarily pass through a. We note that as each vertex besides w has a unique successor, this is a well-ordering. We let $c(a)$ denote the cardinality of the minimal path $P_a \subset P$ from v to a. We claim that $c(a) \in \mathbb{N}$ for all $a \in V(P)$. We proceed by transfinite induction. Our claim holds trivially for v, as $c(v) = 0$. We then suppose that our claim holds for $a \in P$ and note that for b a successor to a, $c(b) \leq c(a) + 1$ as any path of length $c(a)$ from v to a can be extended by one edge to be a path from v to b. Finally, as only v does not have a predecessor in P, $(P, \leq)$ contains no limit ordinals, so our proof by transfinite induction is complete. and there exists a finite path between v and w. As P is minimal by assumption, we now have that P is finite. ■

Wow, that felt really dumb.

Anyway, moving on, we have another lemma...

Lemma 2.2. *Let Δ be the geometric realization of a single n-simplex for any n. Then Δ is path-connected*

Proof. We have that $\Delta = \{(t_1, \ldots, t_{n+1}) \ : \ t_i \geq 0 \,\forall i \text{ and } \sum_{i=1}^n t_i = 1\}$. We let $\mathbf{x} = (t_1, \ldots, t_{n+1}), \mathbf{y} = (s_1, \ldots, s_{n+1}) \in \Delta$ be arbitrary, and let $\Phi : [0,1] \to \Delta$ be given by $\Phi(\tau) = (1-\tau)\mathbf{x} + \tau\mathbf{y}$. We have that for any $0 \leq \tau_0 \leq 1$, $x_0 = (1-\tau_0)\mathbf{x} + \tau_0\mathbf{y}$ is, in each of its coordinates, a subtraction-free sum of non-negative numbers and is hence in the non-negative portion of $\mathbb{R}^{n+1}$. Furthermore, we have that for $x_0 = (j_1, \ldots, j_{n+1})$, $j_i = \tau_0 t_i + (1-\tau_0)s_i$. Thus, $\sum_{i=1}^n j_i = \tau_0 \left(\sum_{i=1}^n t_i\right) + (1-\tau_0)\left(\sum_{i=1}^n t_i\right) = \tau_0(1) + (1-\tau_0)(1) = 1$. Thus, $\Phi(\tau_0) \in \Delta$, is by construction a path-connection of x, y and as a polynomial is obviously continuous. ■

By the above lemma, we now may restrict our attention to $(V, \mathcal{F}')$ where $\mathcal{F}' = \{S \in \mathcal{F} \ : \ \text{card } S \leq 2\}$, that is, the 1-skeleton of $(V, \mathcal{F})$. We now attack the main theorem, this time assuming that $|(V, \mathcal{F})|$ is a complex of 0- and 1-simplices.

($\Longrightarrow$) We prove the contrapositive by contradiction: we assume that $|(V, \mathcal{F})|$ is disconnected by U, V but that for any two vertices, our path condition holds. We note that as U, V nonempty and as edges are path-connected by the lemma, we have that for edge e with $e \cap U \neq \emptyset$, we must have $\partial e = v_1, v_2 \in U$ and similarly for V. Thus, we may assume there is a vertex $v \in U$ and $w \in V$ with a sequences of vertices $v_0, \ldots, v_m$ and edges $e_1, \ldots, e_m$ between them. We let $k = \max\{v_k \in U\}$. Then, $e_{k+1} \cap U \supset v_k$ and is hence nonempty. However, by maximality of k, we have that $v_{k+1} \in V$. Hence, $U \cap e_{k+1}$, $V \cap e_{k+1}$ is a disconnection of e_{k+1}, contradicting the lemma.

($\Longleftarrow$) We again prove the contrapositive, this time directly. We suppose no path exists between vertices v, w. We let $V_1 = \{a \in V \ : \ \exists \text{ path of edges between } v \text{ and } a\}$, and $V_2 = V \setminus V_1$. We claim that all edges e with one vertex of V_1 as a boundary component contain two vertices of V_1 as boundary components. Indeed, suppose $v' \in e \cap V_1$ and $v'' = \partial e \setminus \{v'\}$, and let P be an edge-path from v to v'. Then, let $P' = P \cap e$. P' is an edge-path from v to v'', so $v'' \in V_1$. Hence, as $|(V, \mathcal{F})|$ is a complex of 1- and 0-simplices, we have that $|(V_1, \mathcal{F}_1)|$ is a maximally connected component of $|(V, \mathcal{F})|$. We let T be the union of the open tubular neighborhoods of radius 1/4 about all edges in $|(V_1, \mathcal{F}_1)|$ and let $U = \mathbb{R}^{|V|+1} \setminus \overline{T}$. Then $U \cap |(V, \mathcal{F})|, T \cap |(V, \mathcal{F})|$ is a disconnection of $|(V, \mathcal{F})|$, completing our proof. □

3.)

The figures accompanying this problem may be found on the next page.

Proposition 3.1. *The pairwise edge-identified hexagon with sides identified $abcab^{-1}c$ is homeomorphic to $N_2 = \mathbb{R}P^2 \# \mathbb{R}P^2 = a_1^2 a_2^2$, that is, the Klein Bottle.*

Proof. We first reorient $abcab^{-1}c = cabcab^{-1}$. We then make a cut d at the head of a and the tail of b^{-1} to transform our hexagon into the triangble cad and the pentagon $bcab^{-1}d^{-1} = ab^{-1}d^{-1}bc$ (Figure 1). We then reflect our triangle so that $cad = adc \mapsto c^{-1}d^{-1}a^{-1}$ and glue along a to return the hexagon $c^{-1}d^{-1}b^{-1}d^{-1}bc$ (Figure 2). We then reorient to $cc^{-1}d^{-1}b^{-1}d^{-1}b$ and "close up the hole" at c to return the square $d^{-1}b^{-1}d^{-1}b$ (Figure 3). This is well-known to be a representation of the Klein Bottle, but for completeness' sake, we reorient to $d^{-1}bd^{-1}b^{-1}$ and make another cut at the head of b and the head of d^{-1} (Figure 3) to yield the pair of triangles $d^{-1}e^{-1}b$ and $eb^{-1}d^{-1}$. We reflect the first of these such that $d^{-1}e^{-1}b \mapsto b^{-1}ed$ and reorient the second to $d^{-1}eb^{-1}$, then paste along d to yield $eb^{-1}b^{-1}e$ (Figure 4). We rename by $b \mapsto a_1^{-1}$ and $e \mapsto a_2$ to relabel our square $a_2 a_1^2 a_2$ then reorient to finally yield $a_1^2 a_2^2$. This completes our messy collage-making. □

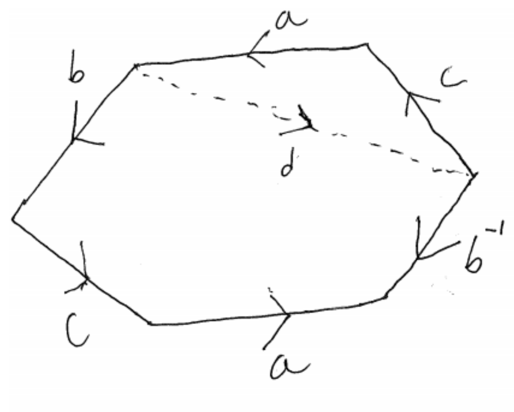

Figure 1: The hexagon $abcab^{-1}c$, with the cut d indicated

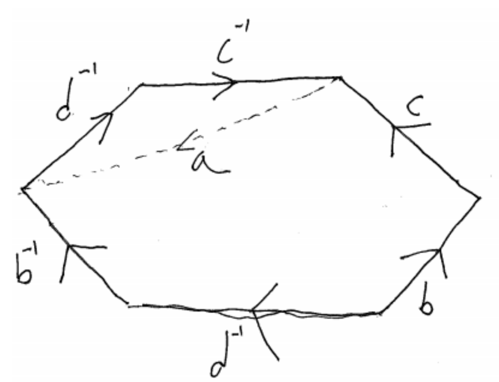

Figure 2: The hexagon $c^{-1}d^{-1}b^{-1}d^{-1}bc$, with the gluing along a indicated

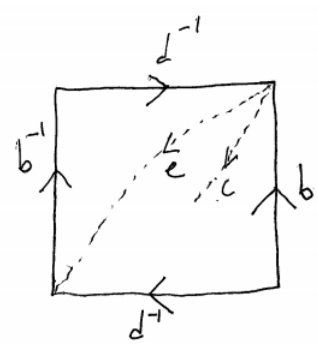

Figure 3: The square $d^{-1}bd^{-1}b^{-1}$, with the gluing along c and the to-be-made cut along e indicated

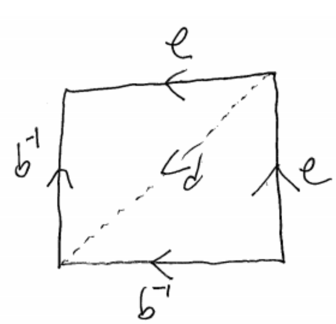

Figure 4: The square $eb^{-1}b^{-1}e$, with the gluing along d indicated