Prologue/Rapid-fire round: Midterm 1 Review

Do not take much time on this section—if this stuff isn't automatic for you, go back to chapters 2 and 3!!

-2.)

Compute the following limits or state they do not exist.

a.)

$$\lim_{x\to 3} \frac{x^2-9}{x^2+x-12}$$

Answer. $\frac{6}{7}$ □

b.)

$$\lim_{x\to\infty} \frac{6x^{-2}+2x^{-1}-1}{8x^{-5}+2}$$

Answer. $\frac{-1}{2}$ □

c.)

$$\lim_{x\to -2} \frac{|x^2-3x+10|}{x^2+5x+6}$$

Answer. DNE □

d.)

$$\lim_{x\to\infty} \frac{(x\cos(\sqrt{x}))^2}{x^3+7}$$

Answer. 0 □

-1.)

Given $f(x)$, find $f'(x)$ and use it to compute $T(x)$, the tangent line to $y=f(x)$ and $x=x_0$.

a.)

$f(x) = \dfrac{x^3 + 10x}{e^x}$
$x_0 = 1$

Answer.

$$f'(x) = \frac{-x^3 + 3x^2 - 10x + 10}{e^x}$$

$$T(x) = \frac{2(x-1) + 11}{e} = \frac{2x + 9}{e}$$

□

b.)

$f(x) = \sqrt{e^x + \tan x}$
$x_0 = 0$

Answer.

$$f'(x) = \frac{e^x + \sec^2(x)}{2\sqrt{e^x + \tan(x)}}$$

$$T(x) = x + 1$$

□

c.)

$f(x) = \ln(x^3 + 3^x)$
$x_0 = 2$

Answer.

$$f'(x) = \frac{3x^2 + 3^x \ln(3)}{x^3 + 3^x}$$

$$T(x) = \frac{3(4 + \ln 27)}{17}(x - 2) + \ln 17$$

□

0.)

Use the definition of derivative to find $f'(x)$. Check your work using rules of differentiation. Also, state the domain of f

$f(x) = \sqrt{2x^2 - 1}$

Answer.

$$f'(x) = \frac{x}{\sqrt{2x^2 - 1}}$$

(Hint: multiply by conjugate.) □

Implicit Differentiation and Applications (including rel8ed r8s)

1.)

Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y (Hint: try simplifying or otherwise manipulating the equation to make things less messy). Then, find the tangent line at the given point (x_0, y_0)

$$\frac{2^y}{x} = 2yx \qquad (x_0, y_0) = (1, 1)$$

Answer.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2^y + 2y}{x(\ln 2)2^y - 2x^3}$$

$$T(x) = \frac{2}{\ln 2 - 1}(x - 1) + 1$$

□

2.)

Use implicit differentiation to compute $f'(x)$ where $f(x) = \arccos x^2$ *without looking up the derivative of* $\arccos$. (Hint: you should get an answer in terms of x and $f(x)$ (or y if you so choose). Substitute in $y = \arccos x$, then draw a triangle to simplify it to the well-known version).

Solution. We start with the equation $y = \arccos(x^2)$ and apply cosine to both sides to yield $\cos y = x^2$. Differentiating, we have $-\frac{\mathrm{d}y}{\mathrm{d}x} \sin y = 2x$, or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{\sin y}$$

We have that $y = \arccos(x^2)$, so we substitute that in:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{\sin(\arccos(x^2))}$$

Recall that arccos takes as its input a ratio and spits out an angle. Let's draw a right triangle with one angle labeled as $\arccos(x^2)$ (Figure 1). Then, the ratio of the adjacent side to the hypotenuse is x^2, so why don't we let that adjacent side have length x^2 with the hypotenuse having length 1. The Pythagorean theorem then tells us that the opposite side must have length $\sqrt{1 - x^4}$.

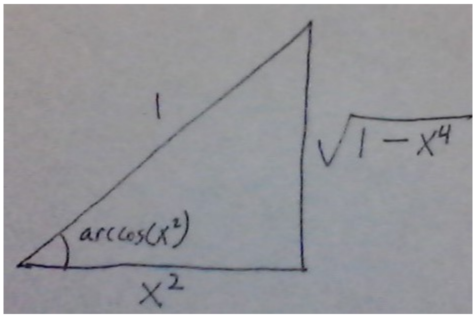

Figure 1: Our triangle, with side lenghts labeled.

As our hypotenuse has length 1, we now have that $\sin(\arccos(x^2)) = \sqrt{1 - x^4}$. We can finally conclude that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{\sqrt{1 - x^4}}$$

□

3.)

Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = x^{\sin x}$

Answer.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$$

□

4.)

A weather balloon is floating at a constant altitude of 50m above level ground. It is connected by a length of rope connecting it and a spool on the ground. The balloon is floating horizontally away from its tether at a constant rate of 3 m/s. Find the rate at which the angle (in radians!) its rope makes with the ground is changing when the rope is extended to a length of 626 m. You may assume that the rope is stretched fully taut.

Answer.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{-150}{624^2} = \frac{-25}{64896}$$

□

Approximations: Differentials & Newton's Method

5.)

Use Newton's method to estimate a root to the following functions to five decimal places from the given starting point (if one is given).

a.)

$f(x) = e^x - 3x$; $x_1 = 1.5$

Answer.

$$\begin{aligned} x_1 &= 1.5 \\ x_2 &= 1.51236 \\ x_3 &= 1.51213 \\ x_4 &= 1.51213 \end{aligned}$$

□

b.)

$f(x) = x^5 + x^4 - x^3 + x^2 - x + 1$

Answer.

$$\begin{aligned} x_1 &= -2 \\ x_2 &= -1.96774 \\ x_3 &= -1.96595 \\ x_4 &= -1.96595 \end{aligned}$$

□

6.)

Use Newton's method to estimate the quantity $\sqrt{3+2\sqrt[3]{4}}$

Answer. Use $f(x) = (x^2-3)^3 - 32 = x^6 - 9x^4 + 27x^2 - 59$

$$\begin{aligned} x_1 &= 3.00000 \\ x_2 &= 2.71605 \\ x_3 &= 2.54997 \\ x_4 &= 2.49156 \\ x_5 &= 2.48499 \\ x_6 &= 2.48491 \\ x_7 &= 2.48491 \end{aligned}$$

□

7.)

Find $\mathrm{d}y$ at x_0 given $\mathrm{d}x$.

a.)

$y = x^3 + 1$; $x_0 = 2$, $\mathrm{d}x = .01$ (Do this without a calculator.)

Answer. $\mathrm{d}y = .12 = \frac{3}{25}$ □

b.)

$y = \cos(2\pi x^2)$; $x = .4$, $\mathrm{d}x = .1$

Answer. $\mathrm{d}y \approx -0.424406$ □

Differential Behavior: Curve Sketching & The Mean Value Theorem

8.)

Show that $f(x) = \frac{e^x - e^{-x}}{2}$ does not have any root x_0 on the interval $(0, \infty)$. State which theorem you are using (if any) and justify that it may be applied.

Sketch. Proof by contradiction! Here's a step-by-step with many blanks left unfilled

- Show $f(0) = 0$.
- Assume another root $x_0 > 0$
- Note f is continuous & differentiable everywhere
- Use Rolle's theorem/MVT to show there must be some $x_* > 0$ such that $f'(x_*) = 0$.
- Show $f'(x)$ has no roots on $(0, \infty)$.
- Bask in the magnificent logical power of reductio ad absurdum.

□

9.)

Given f, sketch the curve and list any asymptotes of any type. State whether the curve has even or odd symmetry or neither. State as well the domains on which f is (i) continuous (ii) increasing/decreasing (iii) concave up/down. (Hint: the functions given should factor and/or simplify surprisingly nicely.)

a.)

$f(x) = x^5 - 2x^3 + x$

b.)

$$f(x) = \frac{x^2 + xe^{-x} + 2x + e^{-x} + 1}{x + 1}$$

Extrema and Optimization

10.)

Let $L(x) = 2x + 1$ and $P(x) = -15 + 8x - x^2$.

a.)

Find the x-coordinate for the point on the graph of $y = L(x)$ which has the shortest *vertical* distance to the curve $y = P(x)$.

Solution. The *vertical* distance between the two curves (i.e. the difference between the y-coordinates) can be given by $v(x) = |L(x) - P(x)| = |x^2 - 6x + 16|$. We notice that (by the quadratic formula giving a complex result and the fact that leading coefficient is positive) that $L(x) - P(x) > 0$ for all x, so we may safely drop the absolute value and instead minimize $v(x) = x^2 - 6x + 16$. Taking a derivative gives $v'(x) = 2x - 6$, and solving that for x gives $x = 3$. The second derivative $v''(x) = 2 > 0$, so this is indeed a local minimum, and as the only critical point, it must be our absolute minimum. Thus, the x-coordinate we're looking for is $x = 3$.

□

b.)

Find the x-coordinate for the point on the graph of $y = P(x)$ which has the shortest *total* distance to the origin $(0, 0)$.

Proof. The horizontal distance from $(x, P(x))$ to the origin is x, and the vertical distance is $P(x)$. The distance formula then gives $d(x) = \sqrt{x^2 + P(x)^2} = \sqrt{x^2 + (-15 + 8x - x^2)^2} = \sqrt{225 - 240x + 95x^2 - 16x^3 + x^4}$. Then, to minimize this, we take a derivative and set it equal to zero. The chain rule gives us:

$$d'(x) = \frac{2x^3 - 24x^2 + 95x - 120}{\sqrt{x^4 - 16x^3 + 95x^2 - 240x + 225}}$$

Oh yikes, this problem is actually too hard as well–we absolutely should NOT ask you to solve a gross cubic like that at any point. My bad!!

Anyway, the minimum is at

$$x = -\frac{\sqrt[3]{36 - \sqrt{1290}}}{6^{2/3}} - \frac{1}{\sqrt[3]{216 - 6\sqrt{1290}}} + 4 \approx 2.60823$$

but there is no expectation you're able to find that by hand. Sorry! □

11.)

(Go ahead and use a calculator on this one...)
Hans is planning his workout. He plans on using a stationary bike for some time, then going for a jog. He wants to burn 600 calories over the course of his workout. He burns $50 \log t_1$ calories in t_1 minutes on the stationary bicycle, and $75 \log t_2$ calories in t_2 minutes jogging. What is the least amount of time he could spend working out?[1]

[1] physiology/exercise science/nutrition majors: yes, I know this question is physically wildly unrealistic.

Solution. We want to minimize time, with the constraint that Hans must burn 600 calories. Let t_1 be time biking and t_2 time jogging. Then, our optimization equation is $T = t_1 + t_2$, and our constraint equation is $600 = 50\log t_1 + 75\log t_2$. Let's solve that for t_2. We have:

$$\begin{aligned} 600 - 50\log t_1 &= 75\log t_2 \\ \implies \log t_2 &= \frac{600 - 50\log t_1}{75} = \frac{24 - 2\log t_1}{3} \\ \implies t_2 &= e^{\frac{24-2\log t_1}{3}} \\ &= \frac{e^8}{e^{\frac{2}{3}\log t_1}} = e^8 t_1^{-\frac{2}{3}} \end{aligned}$$

Now, let's substitute that in: we get

$$T(t_1) = t_1 + e^8 t_1^{-\frac{2}{3}}$$

So, taking our derivative:

$$T'(t_1) = 1 - \frac{2}{3}e^8 t_1^{-\frac{5}{3}}$$

All that is left to do is solve for critical points, then double-check that we do indeed have the absolute maximum (incidentally, it occurs to me now that there are no endpoints to check because I accidentally wrote this problem such that biking/jogging for very very small amounts of time burns a number of calories approaching $-\infty$—in other words, if Hans ceases biking or jogging for a minute (in fact I think he has to do both at the same time), he will suddenly take in an infinite number of calories and become a singularity, inducing a black hole. I hope Hans doesn't stop biking or jogging any time soon. Oh well, let's find some critical points)

Setting T' equal to 0, we must solve

$$\begin{aligned} 0 &= 1 - \frac{2}{3}e^8 t_1^{-\frac{5}{3}} \\ \implies 1 &= \frac{2}{3}e^8 t_1^{-\frac{5}{3}} \\ \implies \frac{3}{2e^8} &= t_1^{-\frac{5}{3}} \\ \implies \frac{2e^8}{3} &= t_1^{\frac{5}{3}} \\ \implies \left(\frac{2e^8}{3}\right)^{\frac{3}{5}} &= t_1 \approx 95.2706 \end{aligned}$$

And that is indeed our only critical point, so by our observation above, it must be our minimizing t_1! Indeed, $T''(t_1) = \frac{10}{9}e^8 t_1^{-\frac{8}{3}}$, which is positive for any positive t_1, so we confirm that it is a local minimum by the second derivative test—as it is our only critical point and it is a local minimum, it must be a global minimum.

Finally, let's find t_2 and use that to find T. We have from above that $t_2 = e^8 t_1^{-\frac{2}{3}}$. Plugging in our minimizing value of t_1 gives us $t_2 \approx 142.906$, so Hans must slave away for $t_1 + t_2 \approx 238.176$ minutes. That will take forever! □

Odds & Ends: Indeterminates and Antiderivatives

12.)

Compute the following limits.

a.)

$\lim_{x\to 1} \frac{x^6-1}{\sin(2\pi x)}$

Answer.

$$\lim_{x\to 1} \frac{x^6-1}{\sin(2\pi x)} = \frac{3}{\pi}$$

□

b.)

$\lim_{x\to a} \frac{\cos x \ln(x-a)}{\ln(e^x-e^a)}$

Answer.

$$\lim_{x\to a} \frac{\cos x \ln(x-a)}{\ln(e^x-e^a)} = \cos(a)$$

□

c.)

$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{4x}$

Answer.

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{4x} = e^{12}$$

□

13.)

Compute the antiderivative of $\frac{x-x^3}{\sqrt[3]{x}}$

Answer. We should get $\frac{3}{5}x^{5/3} - \frac{3}{11}x^{11/3} + c$ □