MATH 8301 Homework VII

David DeMark

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1.)

We let $(X, *)$ be a based topological space such that for all $x \in X$, there exists a small contractible open neighborhood N_x such that $x \in N_x \subset X$. We let $f : S^n \to X$ be a based map with $n \geq 2$. We let $Y = X \sqcup D^{n+1}/\sim$ by $z \sim f(z)$.

a.)

Proposition. *The inclusion $X \to Y$ induces an isomorphism $\pi_1(X, *) \cong \pi_1(Y, *)$.*

Proof. We let $* = f(z)$ for some $z \in S^n$. We define the following subsets of Y: we let U be the union of X and the subset of Y corresponding to $\{\mathbf{x} \in D^{n+1} \ : \ .8 < \|\mathbf{x}\| \leq 1\}$, and we let V be the subset of Y corresponding to $\{\mathbf{x} \in D^{n+1} \ : \ \|\mathbf{x}\| < .9\}$. Then $Y = U \cup V$ with U and V open, so we may apply the Seifert–von Kampen theorem to yield $\pi_1(Y, *) = \pi_1(U, *) \star_{\pi_1(U \cap V, *)} \pi_1(V, *)$. U deformation retracts to X and hence has $\pi_1(U, *) \cong \pi_1(X, *)$. V is contractible and thus $\pi_1(V, *) = 1$. $U \cap V$ deformation-retracts to S^n and hence also has $\pi_1(U \cap V, *) = 1$. Thus, $\pi_1(Y, *) = \pi_1(X, *) \star_1 1 \cong \pi_1(X, *)$. $\square$

b.)

Proposition. *We let Y be a connected d-manifold and let $B \subseteq Y$ be an open neighborhood homeomorphic to $\mathbb{R}^d$. Then, $\pi_1(Y \setminus B, *) \cong \pi_1(Y, *)$ for any $* \in Y \setminus B$.*

Proof. We note that $\mathbb{R}^d \cong \text{int}D^d$ by the map $\mathbf{x} \mapsto \frac{2\tan^{-1}(\|\mathbf{x}\|)}{\pi\|\mathbf{x}\|}\mathbf{x}$. Hence, $\partial B \cong S^{n-1}$, so we may apply the previous proposition with $X = Y \setminus B$ and f any inclusion $S^{n-1} \to \partial B \subset Y \setminus B$. $\square$

c.)

Proposition. *Letting A and B be connected d-manifolds, $\pi_1(A\#B, *) = \pi_1(A, *) \star \pi_1(B, *)$.*

Proof. We fix some embedding D^d in both A and B By the previous part, we note $\pi_1(A \setminus \text{int}D^d, *) = \pi_1(A, *)$ and $\pi_1(B \setminus \text{int}D^d, *) = \pi_1(B, *)$. We note that for all $x \in \partial D^d$ in both A and B, there exists a neighborhood containing x homeomorphic to $\mathbb{R}^{d+}$ in $A \setminus \text{int}D^d$ or $B \setminus \text{int}D^d$. We may then let $\tilde{A} = A \setminus \text{int}D^d \cup T \subset A\#B$ where T is a small thickening of ∂D^d in $B \setminus \text{int}D^d \subset A\#B$ and let $\tilde{B}$ be defined analogously. Then, $A\#B = \tilde{A} \cup \tilde{B}$ with $\tilde{A} \cap \tilde{B} \simeq \partial D^d = S^d$, so application of Seifert–von Kampen completes the proof. $\square$

2.)

a.)

Proposition. *We let $\sim$ be a relation on D^n by $x \sim -x$ for all $x \in \partial D^n = S^{n-1}$. We let q be the associated quotient map. Then, $q(D^n) = D^n/\sim \cong \mathbb{R}P^n$.*

Proof. We identify D^n with the closed "upper" half-sphere[1] of S^n as follows: we let D^n be embedded in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ by

$$D^n = \{(\mathbf{x}, 0) \quad \|\mathbf{x}\| \leq 1\}$$

Then, we let $\phi : D^n \to S^n$ be defined by $(\mathbf{x}, 0) \mapsto (\mathbf{x}, \sqrt{1 - \|x\|^2})$, with image precisely the closed upper half-sphere of $\S^n$. As ϕ is polynomial, with inverse $\phi^{-1} : (\mathbf{x}, y) \mapsto (\mathbf{x}, 0)$ also polynomial, ϕ is bicontinuous and clearly bijective on its image and hence precisely the desired identification. We let $\sim'$ be an equivalence relation on S^n defined by $x \sim -x$, with associated quotient map q'. We first note that $q'|_{D^n} = q$ as q identifies the upper open half-sphere with the lower and $\text{int}D^n$ sits within the open upper half-sphere. Furthermore, q' and q have the same image, as for each equivalence class defined by $\sim'$, there is at least one representative in $D^n \subset S^n$. This completes our proof. $\square$

[1] Indeed, for the duration of this proof, we shall take the final coordinate of $\mathbb{R}^{n+1}$ to be "up and down," with positivity in that coordinate being "up."

b.)

Proposition. $\mathbb{R}P^n$ *can be constructed by attaching a copy of* D^n *to* $\mathbb{R}P^{n-1}$.

Proof. We let $q : S^{n-1} \to \mathbb{R}P^{n-1}$ be the standard quotient map and let $Y = \mathbb{R}P^{n-1} \sqcup D^n / \sim$ where $z \sim q(z)$ for all $z \in \partial D^n$. Then, $D^n / \sim' = Y$ where $\sim'|_{\partial D^n}$ is precisely that of q and $x \sim' x$ for all $x \in \text{int} D^n$. However, this is exactly the quotient map of problem 1! $\square$

c.)

Proposition. $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$.

Proof. We consider the polygonal model of $\mathbb{R}P^2$ aa. Then, $\pi_1(\mathbb{R}P^2) = F\{a\}/\langle aa \rangle \cong \mathbb{Z}/2$ $\square$

d.)

Proposition. $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$ *for all* $n > 1$

Proof. Follows immediately from the observation that problem 2b constructs $\mathbb{R}P^n$ in the fashion of Y in problem 1a with $\mathbb{R}P^{n-1}$ playing the role of X. This was a really neat problem set incidentally! $\square$