MATH 8301 Homework II

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Notation

- We let the topology of a topological space X be denoted $\mathcal{T}(X)$ unless this is insufficient in context to avoid confusion.
- We let the cardinality of a set or topological space (interpreted as a set) X be denoted $|X|$.

1.)

Proposition. *The image of a [(i) connected/ (ii) path-connected] space under a continuous map is [connected/ path-connected].*

Proof. (i) We show the contrapositive: if the image $Z = \psi(X)$ is disconnected where X is a topological space and $\psi : X \to Z$ is a continuous (de facto surjective) map, then X is disconnected. We let $Z = U \sqcup V$ where $\emptyset \neq U, V \in \mathcal{T}(Z)$. Then, as ψ is continuous, $\psi^{-1}(U), \psi^{-1}(V) \in \mathcal{T}(X)$, and as $\psi^{-1}(U) \cap \psi^{-1}(V) = \psi^{-1}(U \cap V) = \psi^{-1}(\emptyset) = \emptyset$, we have that $X = \psi^{-1}(U) \sqcup \psi^{-1}(V)$ and is hence disconnected.

(ii) We suppose topological space X is path-connected, with continuous map $\psi : X \to Z = \psi(X)$. We let $y, z \in Z$ be arbitrary and let $w, x \in X$ be chosen such that $\psi(w) = y$, $\psi(x) = z$. Then, there exists a continuous map $\alpha : [0,1] \to X$ with $\alpha(0) = w$, $\alpha(1) = x$ by assumption, and as a composition of continuous maps is itself continuous, we have that $\beta = \psi \circ \alpha : [0,1] \to Y$ is continuous with $\beta(0) = \psi(w) = y$ and $\beta(1) = \psi(x) = z$. As y, z were chosen arbitrarily, this completes our proof.

□

2.)

Proposition. *For M a d-manifold and $W \subset M$ open under $\mathcal{T}(M)$, W is a d-manifold.*

Proof. We consider W under the subspace topology.

Lemma. *$\mathcal{T}(W)$ coincides with $\mathcal{T}(M)$, i.e. $\mathcal{T}(W) = \{U \in \mathcal{T}(M) : U \subseteq W\}$.*

Proof. ($\supseteq$) We note that for any $U \subseteq W$ open in M, $U \cap W = U$. Hence, $U \in \mathcal{T}(W)$.
($\subseteq$) We let $U \in \mathcal{T}(W)$ be arbitrary. Then, there exists some $V \in \mathcal{T}(M)$ such that $U = V \cap W$. As finite intersections of open sets are open, we then have that $U \in \mathcal{T}(M)$. ■

Second-countability: As M is second-countable, there exists some basis $\{U_i\}_{i\in\mathbb{N}}$ for $\mathcal{T}(M)$. Then, as $\mathcal{T}(W) \subset \mathcal{T}(M)$, we have that for all $V \in \mathcal{T}(W)$, we have that for some indexing set J, $V = \bigcup_{j\in J} U_{i_j}$. Then, $V = W \cap V = W \cap \left(\bigcup_{j\in J} U_{i_j}\right) = \bigcup_{j\in J}(W \cap U_{i_j})$. Hence, $\{W \cap U_i\}_{i\in\mathbb{N}}$ is a countable basis for $\mathcal{T}(W)$.

Hausdorff: We let $x, y \in W$. Then, as M is Hausdorff, there exists some $(U \ni x), (V \ni y) \in \mathcal{T}(M)$ such that $U \cap V = \emptyset$. Then, $(U \cap W \ni x), (V \cap W \ni y) \in \mathcal{T}(W)$ and $(U \cap W) \cap (V \cap W) = \emptyset$.

Locally homeomorphic to $\mathbb{R}^d$: We let $x \in W$ and have that there exists $(U \ni x) \in \mathcal{T}(M)$ and homeomorphism $\phi : U \to V \subset \mathbb{R}^d$ where V is open. Then, as $U \cap W$ is an open set in $\mathcal{T}(M)$, by bicontinuity we have that $\phi^{-1}(U \cap W) \in \mathcal{T}(\mathbb{R}^d)$, and the bijectivity and bicontinuity of $\phi|_{U\cap W}$ are respectively inherited from bijectivity of ϕ and obvious from the lemma. □

3.)

Proposition. *For M_1, M_2 d-manifolds, $M = M_1 \sqcup M_2$ is a d-manifold.*

Proof. We let $\phi_i : M_i \to M$ be the canonical injection for $i = 1, 2$. Then, $\mathcal{T}(M)$ is the finest topology for which ϕ_i is continuous.

Second-countability. We let $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be bases for $\mathcal{T}(M_1)$ and $\mathcal{T}(M_2)$ respectively. We claim that $\{\phi_1(U_i)\}_{i\in I} \cup \{\phi_2(V_i)\}_{i\in I}$ is then a basis for $\mathcal{T}(M)$. Indeed, let $W \in \mathcal{T}(M)$ be arbitrary. Then, $W = \phi_1(\phi_1^{-1}(W)) \sqcup \phi_2(\phi_2^{-1}(W))$, and $\phi_1^{-1}(W) = \bigcup_{j\in J_1} U_{i_j}$ and $\phi_2^{-1}(W) = \bigcup_{k\in J_2} V_{i_k}$ for some indexing sets J_1, J_2. Applying ϕ_i to each equality, we have $\phi_1(\phi_1^{-1}(W)) = \bigcup_{j\in J_1} \phi_1(U_{i_j})$ and $\phi_2(\phi_2^{-1}(W)) = \bigcup_{k\in J_2} \phi_2(V_{i_k})$, so $W = \left(\bigcup_{j\in J_1} \phi_1(U_{i_j})\right) \sqcup \left(\bigcup_{k\in J_2} \phi_2(V_{i_k})\right)$. As the union of two countable sets is countable, M is then second-countable.

Hausdorff: We let $x, y \in M$ be arbitrary. If we have that $x, y \in M_i$ for one of $i = 1, 2$, then $\phi_i^{-1}(x), \phi_i^{-1}(y)$ are separated by disjoint open sets U, V, so x, y are separated by open sets $\phi_1(U), \phi_2(V)$ (which are open as $\mathcal{T}(M)$ is the *finest* topology on which ϕ_i is continuous.). If $x \in M_1$, $y \in M_2$ or vice-versa, we have that M_1, M_2 are open in $\mathcal{T}(M)$ and hence x, y are separated by disjoint open sets.

Locally homeomorphic to $\mathbb{R}^d$: We let $x \in M_i \subset M$. Then, there is some open set $(U \ni \phi^{-1}(x)) \subseteq M_i$ such that U is homeomorphic to some open $V \subset \mathbb{R}^d$ via homeomorphism $\alpha : U \to V$. As ϕ_i is a continuous injection and $\mathcal{T}(M)$ is the finest topology such that all ϕ_i are continuous, we then have that $\phi_i(U)$ is open in M, and ϕ_i is a homeomorphism from M_i to $M_i \subset M$. Hence, $\alpha \circ \tilde{\phi_i}^{-1} : \phi(U) \to V$ is a homeomorphism where $\tilde{\phi_i}^{-1}$ is taken to be the local inverse to ϕ_i on its image. ☐

4.)

a.)

Proposition. *All 0-manifolds carry the discrete topology.*

Proof. We let M be a 0-manifold. Then, for any $x \in M$, there exists a homeomorphism $\psi_x : U(x) \to V \subset \mathbb{R}^0 = \{0\}$ where $U(x)$ is an open neighborhood of x in X. However, as $|\mathbb{R}^0| = 1$ and V is a nonempty subset of $\mathbb{R}^0$, we have that $|V| = 1$, and as ψ is a bijection by definition, we have that $|U(x)| = 1$, i.e. $U(x) = \{x\}$. Hence, for any $x \in M$, $\{x\} \in \mathcal{T}(M)$, so $\mathcal{T}(M)$ is necessarily the discrete topology on M. ☐

b.)

Proposition. *Any nonempty, connected 0-manifold is a single point.*

Proof. We let M be a nonempty connected 0-manifold. As M is nonempty, we have that there is some element $x \in M$. We note that by part a), any set $U \subset M$ is open as all subsets are open under the discrete topology. Hence, $M = \{x\} \sqcup (M \setminus \{x\})$ is a presentation of M as the disjoint union of two open sets. As M is assumed to be connected and $\{x\}$ is assumed to be nonempty, we have that $M \setminus \{x\} = \emptyset$, so $|M| = 1$. ☐

c.)

Proposition. *Let M be a compact 0-manifold. Then, $|M| < \infty$.*

Proof. We let $M = \{x_\alpha \ : \ \alpha \in A\}$ be a 0-manifold where A is some indexing set and $x_\alpha = x_\beta \iff \alpha = \beta$. Then, $\{x_\alpha\}$ is open for any α as $\mathcal{T}(M)$ is necessarily the discrete topology. Hence, $C = \{\{x_\alpha\} \ : \ \alpha \in A\}$ is an open cover of M, and as x is contained in precisely one element of C for all $x \in M$, we have that C has no proper subcover. As M is assumed to be compact, this implies that $|C| < \infty$, and as C is in obvious bijection with A which is in turn in obvious bijection with M, this completes our proof. ☐

d.)

For this exercise, we take $m \in \mathbb{N}$ to be the set $m = \{\{\emptyset\}, 1, 2, \ldots, m-1\}$, and $0 \in \mathbb{N}$ to be the set $0 = \emptyset$.

Proposition. *Let* card : $\{$*homeomorphism classes of compact 0-manifolds*$\} \to \mathbb{N}$[1] *be defined by* $\mathrm{card}([M]) = m \iff$ *M is in bijection with m as sets where M is an equivalence class representative of $[M]$. Then* card *is well-defined and itself a bijection.*

[1]As $\mathbb{N}$ is the set which is often constructed as in the note before the proposition and is in canonical bijection with $\mathbb{Z}_{\geq 0}$, we let it stand in the place of $\mathbb{Z}_{\geq 0}$. Our argument does not change in any meaningful way by this substitution.

Proof. We first define a function $\bar{\mathrm{card}} : \{\text{compact 0-manifolds}\} \to \mathbb{Z}_{\geq 0}$ by the exact definition we gave before for card, with the exception that here for all M, M is in a singleton equivalence class. By part c), we have that for all such M, M is finite as a set, i.e. in bijection with some element of $\mathbb{N}$. Furthermore, we have for any $m \neq n \in \mathbb{N}$, we have that either $m \subset n$ or $n \subset m$ and hence m and n are not in bijection. As bijection induces an equivalence relation on sets, we have that $\bar{\mathrm{card}}$ is defined on all compact 0-manifolds, and well-defined. Furthermore, as all sets are open under the discrete topology, any map from a space under the discrete topology to any other topological space is continuous. Thus, if $M \approx m \approx M'$ where M, M' are compact 0-manifolds and $m \in \mathbb{N}$, the composition of bijections $M \to m \to M'$ is itself a bijection and bicontinuous as both M, M' are under the discrete topology. Hence, the homeomorphism classes of {compact 0-manifolds} are precisely $\bar{\mathrm{card}}^{-1}(m)$ where m ranges over $\mathbb{N}$. Finally, we show that each of these homeomorphism classes are non-empty: we note that the empty manifold $\emptyset$ is 1) vacuously a 0-manifold and 2) in bijection with $\emptyset = 0 \in \mathbb{N}$. For $0 < m \in \mathbb{N}$, we note that $\bigsqcup_{i=1}^{m} \mathbb{R}^0$, that is the disjoint union of m distinct copies of $\mathbb{R}^0$, is of cardinality m and hence in bijection with $m \in \mathbb{N}$. Thus, by construction card is itself a bijection and our proof is complete. $\square$