MATH 8301 Homework VIII

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1.)

We let $z_1, \dots, z_n \in \mathbb{C}$ be pairwise distinct and let $f(z) = (z - z_1)\dots(z - z_n)$. We let Y be the hyperelliptic curve associated to f. We let $p' : Y \to \mathbb{C}$ be defined by $(z, w) \mapsto z$. We let $X = \mathbb{C} \setminus \{z_1, \dots, z_n\}$ and let $\overline{X} := Y \setminus p^{-1}(\{z_1, \dots, z_n\})$. We let $p : \overline{X} \to X$ be the restriction of p' to $\overline{X}$.

a.)

Proposition 1.1. *p is a covering map of X relative to the open cover $R_\pm := \{z \in X \ : \ \pm\mathrm{re}(f(z)) > 0\}$ with $I_\pm$ defined analogously, replacing* re *with* im.

Proof. We first note that $\{R_+, R_-, I_+, I_-\}$ is indeed an open cover as removing $p^{-1}(z_i)$ removed all points with $\mathrm{re}(f(z)) = \mathrm{im}(f(z)) = 0$. Then, $p^{-1}(R_+) = \{(z, w) \in \overline{X} \ : \ \frac{-\pi}{4} < \arg(w) < \frac{\pi}{4}\} \sqcup \{(z, w) \in \overline{X} \ : \ \frac{3\pi}{4} < \arg(w) < \frac{5\pi}{4}\}$. Then, restricting to one of the two sets in disjoint union, it is clear that p is bijective as each of $(z, \sqrt{f(z)})$ $(z, -\sqrt{f(z)})$ is in a different component having fixed a branch of the square root function. Furthermore, as the square root function is continuous once a branch is chosen, $z \mapsto (z, \pm\sqrt{z})$ is a continuous inverse to p given compatible choices of component of $p^{-1}(R_+)$ and $\pm$. We give the components of the preimages of the other set of the open cover below; the arguments for p restricting to a homeomorphism are identical.[1]

$$p^{-1}(R_-) = \{(z, w) \in \overline{X} \ : \ \frac{\pi}{4} < \arg(w) < \frac{3\pi}{4}\} \sqcup \{(z, w) \in \overline{X} \ : \ \frac{5\pi}{4} < \arg(w) < \frac{7\pi}{4}\}$$
$$p^{-1}(I_+) = \{(z, w) \in \overline{X} \ : \ 0 < \arg(w) < \frac{\pi}{2}\} \sqcup \{(z, w) \in \overline{X} \ : \ \pi < \arg(w) < \frac{3\pi}{2}\}$$
$$p^{-1}(I_-) = \{(z, w) \in \overline{X} \ : \ \frac{\pi}{2} < \arg(w) < \pi\} \sqcup \{(z, w) \in \overline{X} \ : \ \frac{3\pi}{2} < \arg(w) < 2\pi\}$$

□

b.)

Proposition. *We let $\sigma : \overline{X} \to \overline{X}$ be given by $(z, w) \mapsto (z, -w)$. Then, σ is a homeomorphism with the property $p \circ \sigma = p$*

Proof. Trivial. □

Okay, fine. We note that σ is clearly bijective, as $p^{-1}(z)$ consists of two points (z, w) and $(z, -w)$ where $w = \sqrt{f(z)}$ given some choice of branch of square root function, and σ acts on $\overline{X}$ by permuting these. Furthermore, $\sigma^2(z, w) = (z, w)$, so σ is idempotent and hence a homeomorphism. The other statement of the proposition comes about as a byproduct of our observation that σ permutes elements of $p^{-1}(z)$ for all $z \in X$. □

c.)

Proposition. *σ and the identity are the only automorphisms of p.*

[1] For simplicity, my usage of arg is somewhat fast-and-loose—it should be totally clear what I mean and trivial to rephrase it totally correctly, but I just thought I should acknowledge that *technically* what I'm doing is not quite well-defined.

Proof. We suppose for the sake of contradiction that some third automorphsim τ exists which is distinct from σ and id. We recall from our previous argument that as $\#p^{-1}(z) = 2$ for all $z \in X$, we have that either $\tau|_{\{y\}} = \sigma|_{\{y\}}$ or $\tau|_{\{y\}} = \mathrm{id}|_{\{y\}}$ for each $y \in \overline{X}$. As τ is distinct from σ and id, we must have that both cases are realized for some y_1, y_2 in $\overline{X}$ respectively—in particular, we choose y_1 such that $\tau|_{\{y_1\}} = \mathrm{id}|_{\{y_1\}}$. We relabel $R_\pm$ $I_\pm$ to U_1, U_2, U_3, U_4 such that $p(y_1) \in U_1$ and $U_i \cap U_{i+1} \neq \emptyset$ where subscripts are taken mod 4. We now break into two cases:

Case 1 ($p(y_2) \in U_1$): We note that if $\tau(y_2) = \sigma(y_2)$, then $\tau(\sigma(y_2)) = y_2$ as otherwise τ is not a homeomorphism. As such, we may assume that y_2 is in the same connected component W of $p^{-1}(U_1)$ as y_1. We then let $\gamma : I \to W$ be a path between y_1 and y_2. Then, $\tau \circ \gamma$ is the composition of continuous functions and hence continuous. However, $\tau \circ \gamma(0)$ and $\tau \circ \gamma(1)$ are in different connected components of the target $p^{-1}(U_1)$ and hence $\tau \circ \gamma$ does not have connected image, contradicting our assumptions and establishing the case.

Case 2: $p(y_2) \notin U_1$ We label the maximal connected components of the sets $p^{-1}(U_i)$ $W_1, \ldots, W_8$, again such that $W_i \cap W_j \neq \emptyset$ and ($W_i = W_j \iff i = j$) where subscripts are taken mod 8. We let $y_1 \in W_1$ and let k be such that $y_2 \in W_k$. We then let $x_0 = y_1$, $x_k = y_2$ and $x_i \in W_i \cap W_{i+1}$ for $0 < i < k$. We let $\gamma_i : I \to W_i$ be a path between x_{i-1} and x_i. Then, at least one of the paths γ_i falls into the situation of case 1. □

d.)

Prompt. *For $z_1 = 0, z_2 = 2$ and $x_0 = 1$, compute $p^{-1}(x_0)$.*

Response. We note that $f(x_0) = -1$. Then, $p^{-1}(x_0) = \{(1, i), (1, -i)\}$. □

e.)

Prompt. *f may be seen as the action of an element of $S_{p^{-1}(x_0)}$. Compute that permutation.*

Proof. As f is a non-trivial permutation on a two-element set, it must be the unique possibility: that corresponding to (12) in the group isomorphism $S_2 \to S_{p^{-1}(x_0)}$. □

f.)

Prompt. *Let $\gamma : I \to X$ based at x_0 be given by $t \mapsto e^{2\pi i t}$. Find the two lifts of this path.*

Response. We fix the branch of the square root function which returns values in the upper half-plane and returns postive real values given positive real input. Then, we claim our two lifts are

$$L_\pm(t) = \begin{cases} \left(e^{2i\pi t}, \pm\sqrt{f(e^{2i\pi t})}\right) & t < \frac{1}{2} \\ \left(e^{2i\pi t}, \mp\sqrt{f(e^{2i\pi t})}\right) & t \geq \frac{1}{2} \end{cases}$$

Where L_+ is the path which lifts $\gamma(0)$ to $(1, i)$ and L_- is the path which lifts $\gamma(0)$ to $(1, -i)$. Indeed, to check our answer, we note that having chosen $L_\pm(0)$, we have chosen a pre-image of R_- which our path starts in. Then, when first $\mathrm{re}(f(\gamma(t))) = 0$ at $t_0 \approx .309$, we see that $f(\gamma(t_0)) \approx -2.542i$, so we must have that $L_\pm(t_0)$ is in the component of $p^{-1}(I_-)$ intersecting with the already-determined pre-image component of R_-—that is, the one residing to the same side of $\mathrm{im}(z) = 0$ as our lifted basepoint. Then, when next $\mathrm{im}(f(\gamma(t))) = 0$ at $t = .5$, we have that $f(\gamma(.5)) = 3$, so we must have that $L_\pm(.5)$ is in the component of $p^{-1}(R_+)$ intersecting with the previously-determined component of $p^{-1}(I_-)$. Then, for any $t \geq \frac{1}{2}$, we have that $\mathrm{im}(f(z)) > 1$, so our lifted path must reside in the component of $p^{-1}(I_+)$ which intersects with our previous component of $p^{-1}(R_+)$–indeed, at this point we pass from the upper to the lower half plane or vice versa in the w-coordinate, hence the piecewise function given. That we remain in our choice of $p^{-1}(I_+)$ for $t \in (.5, 1)$ shows that we remain in that half plane for $t \in (.5, 1]$ □

Remark 1.1. After having written that, I feel completely dirty and deeply ashamed.

g.)

Prompt. *Compute the permutation $\tilde{\sigma} \mapsto \gamma_x(1)$ on $p^{-1}(x_0)$*

Response. We see immediately from our formula that $\tilde{\sigma} : x \mapsto -x$. Hence, $\tilde{\sigma}$ coincides with σ from earlier! □

2.)

Proposition. *We let $L_1, \dots, L_n$ be $n > 0$ lines through the origin in $\mathbb{R}^3$ and let $X = \mathbb{R}^3 \setminus \bigcup_{i=1}^{n} L_i$. Then, for any $* \in X$, $\pi_1(X, *) = F_N$, the free group on $N := 2n - 1$ elements.*

Proof. We note that as $n > 0$, $\mathbf{0} \notin X \subset \mathbb{R}^3$. Hence, we may restrict the standard homotopy equivalence projection to the unit sphere $\mathbb{R}^3 \setminus \{\mathbf{0}\} \to \S^2$ to X. We then have that X is homotopic to $X' = \S^2 \setminus \left(S^2 \bigcup_{i=1}^{n} L_i \right)$. We note that any line through the origin has precisely two points of norm one, and hence we have that $\# S^2 \setminus X' = 2n$. We choose some point $z \in \# S^2 \setminus X'$ and recall the stereography homeomorphism $S^2 \setminus \{z\} \to D^2$. Restricting that homeomorphism to X', we now have that $X' \cong \mathrm{int} D^2 \setminus \{z_1, \dots, z_{2n-1}\}$ where each z_i, z_j are pairwise distinct and identify X' with that space. We note that if $n = 1$, X' then deformation-retracts to S^1, which has fundamental group $\mathbb{Z} = F_N$. We then proceed by strong induction for $n > 1$. We fix some sufficiently small epsilon so that we may use the finiteness of the z_i's to find some $-1 < y_0 < 1$ such that $d(y_0, \pi_y(z_i)\}) >$ where π_y is the projection map to the y-axis on D^2 and there exists some z_i, z_j such that $\pi_y(z_i) < y_0 < \pi_y(z_j)$. We then let $U_1 = \{(x, y) \in \mathrm{int} D^2 \ : \ y < y_0 + \epsilon\}$ and $U_2 = \{(x, y) \in \mathrm{int} D^2 \ : \ y > y_0 - \epsilon\}$. Then, $\{U_1, U_2\}$ is an open cover of X' with both components path-connected and with simply-connected intersection, so letting $* \in U_1 \cap U_2$, we have $\pi_1(X, *) \cong \pi_1(U_1, *) \star \pi_1(U_2, *)$. We note that our sets U_i are both homeomorphic to $\mathrm{int} D^2$ with some $0 < m_i < N$ points deleted, so by strong induction our claim is proven. □

3.)

Proposition. *We let $Z = X * Y := X \times Y \times I / \sim$ where $(x, y, 0) \sim (x', y, 0)$ and $(x, y, 1) \sim (x, y', 1)$ for any $x, x' \in X$ and $y, y' \in Y$. Then, Z is simply-connected.*

Proof. We begin by considering the case that Y is path-connected. We let $U = \{[(x, y, t)] \in Z \ : \ t < .6\}$ and $V = \{[(x, y, t)] \in Z \ : \ t > .4\}$. We note that as $U \cong \mathrm{cone}(X) \times Y$ and $V \cong X \times \mathrm{cone}(Y)$, both U and V are path-connected, and as the functor $\pi_1(-, *)$ respects products and cones are simply-connected, we have that $\pi_1(U, *) \cong \pi_1(Y, *)$ and $\pi_1(V, *) \cong \pi_1(X, *)$. We note that the intersection $U \cap V = \{[(x, y, t)] \in Z \ : \ .4 < t < .6\}$ is the product of three path connected spaces and deformation-retracts to $X \times Y$. Hence, $\pi_1(U \cap V, *) \cong \pi_1(X, *) \times \pi_1(Y, *)$ and we identify the two groups as such with the maps to U and V by inclusion inducing the projection maps $\rho_V : ([\gamma], [\zeta]) \mapsto [\gamma] \in \pi_1(V, *) \cong \pi_1(X, *))$ and $\rho_U : ([\gamma], [\zeta]) \mapsto [\zeta] \in \pi_1(U, *) \cong \pi_1(Y, *))$. Then, by Seifert-von Kampen, we have that $\pi_1(Z, *) = \pi_1(U, *) \star_{\pi_1(U \cap V, *)} \pi_1(V, *) \cong \pi_1(Y, *) \star_{\pi_1(X \times Y, *)} \pi_1(X, *)$. We claim that this is indeed the trivial group; indeed, we let $[\zeta] \in \pi_1(Y, *)$. Then, $[\zeta] = \rho_U([\zeta], [1])$, and as $\rho_V([\zeta], [1]) = [1]$, we have that in $\pi_1(Z, *)$, $[\zeta] = \rho_U([\zeta], [1]) \rho_V([\zeta], [1])^{-1} = [1]$, with a similar statement holding for any $[\gamma] \in \pi_1(X, *)$. Thus, as each of the generators of $\pi_1(Z, *)$ are the identity, we are left with the trivial group.

We now move on to the case that Y is not necessarily path-connected. We let $Y = \bigsqcup_{i \in I} Y_i$ over some indexing set I where the subspaces Y_i are the distinct maximally path-connected components of Y. We let $U_i \subset Y_i$ be some open contractible[2] subset of Y_i. We then let $V = (\bigcup_i U_i) \times X \times (.9, 1] /$ and let $Z_i = V \cup (Y_i \times X \times I) /$. We note that each of V, Z_i are open under the product topology, that they form an open cover, and they have intersection V which is homeomorphic to $\mathrm{cone}(\bigcup_i U_i) \times X$ and hence path-connected with fundamental group $\pi_1(V, *) \cong \pi_1(X, *)$. Furthermore, we note that each Z_i deformation retracts to $(Y_i \times X \times I) /$ as V is contractible by construction, and hence each Z_i falls into the case of the previous paragraph and has trivial fundamental group. Then, Seifert-von Kampen tells us that $\pi_1(Z, *) = \star_{\pi_1(V, *)} \pi_1(Z_i, *)$ with the product ranging over I. As this is isomorphic to a quotient group of the free group of a number of copies of the trivial group, it is trivial, proving our proposition. □

[2] I don't know how to make this proof work without the assumption some subset exists, but Y would have to be such a deeply pathological space for it not to hold that I don't feel all that guilty about it.