MATH 8253 Homework III

David DeMark

30 October 2017

1.)

We save this problem for after problem 3 so that we may use its result

2.)

Proposition. $\varinjlim_{G_i,f_{ij}} G_i = G := \star_{i\in I} G_i/N$ *where* N *is generated by elements of the form* $a_i a_j^{-1}$ *where there is some* k *such that* $f_{ik}(a_i) = f_{ij}(a_j)$ *for all* $i \le j$ *and* $\star$ *is taken to be the free product.*

Proof. We let $\rho_j : G_j \to G$ be the inclusion map $G_j \to \star_{i\in I} G_i$ composed with the quotient map $\star_{i\in I} G_i \to \star_{i\in I} G_i/N$. Then, (G, ρ_j) is a co-cone as $\rho_j(f_{ij}(a))\rho_i(f_{ii}(a))^{-1} \in N \implies \rho_j(f_{ij}(a)) = \rho_i(a)$. To see that it is universal, we let (C, σ_j) be another co-cone and first note that there is a unique map $\star_j G_j \to C$ commuting with σ_j by the universal property of the coproduct; hence, we may consider the σ_j maps fully determined by the map $\psi : \star_j G_j \to C$. We then note that $\sigma_j(a) = \sigma_i(b)$ for all $(a, b) \in G_j \times G_i$ such that $f_{jk}(a) = f_{ik}(b)$ for all i, j as C is a co-cone. Hence, $N \trianglelefteq \ker\psi$, so there is a unique map $G \to G/(\ker\psi/N)$ commuting with ψ by the universal property of the quotient map. This completes our proof. □

3.)

Proposition. *We let* A *be an integral domain. For any open set* $U \subset X$, $\mathcal{O}(U)$ *is canonically isomorphic to* $\bigcap_{x\in U} A_x$ *(viewing* A_x *as a subset of the ring* $\operatorname{frac}(A)$*).*

Proof. We break into two subclaims:

Claim. *For any* basic *open set* $U := D(f)$ *with* $f \in A$, $\mathcal{O}(U) \cong \bigcap_{x\in U} A_x$ *canonically.*

Proof. We have that $\mathcal{O}(U) = A_{S(f)}$ where $S(f) = \{g \in A \ : \ D(f) \subset D(g)\}$. We note that $\bigcap_{x\in U} A_x = \{\frac{a}{b} \in \operatorname{frac}(A) \ : \ b \notin \bigcup_{x\in U} \mathfrak{p}_x\} = A[S^{-1}]$ where $S := \left(\bigcup_{x\in U} \mathfrak{p}_x\right)^c$. We let $g \in S$. Then, for all $\mathfrak{p}$ such that $f \notin \mathfrak{p}$ (i.e. $x_\mathfrak{p} \in D(f) = U$),$g \notin \mathfrak{p} \implies D(g) \supset D(f) \implies S \subset S(f)$. Moreover, for any $g \in S(f)$, we have that $g \notin \mathfrak{p}_x$ for any $x \in U$, so $S(f) = S$. Thus, $\bigcap_{x\in U} A_x \cong A_{S(f)}$, and as $A_{S(f)}$ is the limit of all $\mathcal{O}(D(g)) \subset \mathcal{O}(D(f))$, we have that there is a canonical isomorphism $\bigcap_{x\in U} A_x \to A_{S(f)}$ commuting with restriction maps. ■

Claim. *For any* $U \subset X$ *open,* $\mathcal{O}(U) \cong \bigcap_{D(f)\subset U} \mathcal{O}(D(f))$

Proof. We have that $\mathcal{O}(U) = \varprojlim_{D(f)\subset U} \mathcal{O}(D(f)) = \varprojlim_{D(f)\subset U} A_f$. As A is a domain, each restriction map $A_f \to A_g$ where $D(g) \subset D(f)$ is an injection. Thus, for any $V, W \subset U$ basic open with $V \cap W \neq \emptyset$, we must have for any $s \in \mathcal{O}(U)$ that $s|_V = s|_W$ and hence $s \in \mathcal{O}(V) \cap \mathcal{O}(W)$ viewed as a subset of $\operatorname{frac}(A)$. We recall that $\operatorname{Spec} A$ is connected[1]. Considering the open cover of U by all basic open sets it contains, by taking a "walk" from $D(f)$ to $D(g)$ by a sequence $D(f) = D(f_0), D(f_1), \ldots, D(f_n)$ where $D(f_i) \cap D(f_{i+1}) \neq \emptyset$, we have for any $D(f), D(g) \subset U$, any $s \in \mathcal{O}(U)$ has $s \in D(f) \cap D(g)$. Thus, $\mathcal{O}(U) \subset \bigcap_{D(f)\subset U} \mathcal{O}(D(f))$. To show the reverse containment, we note that $\bigcap_{D(f)\subset U} \mathcal{O}(D(f))$ has obvious inclusion maps commuting with restriction to each $D(f) \subset U$, thus making it a cone. As the largest possible cone, we have that it must indeed be a universal cone and thus $\mathcal{O}(U) \cong \bigcap_{D(f)\subset U} \mathcal{O}(D(f))$ canonically. ■

These two claims together combine to prove the proposition. □

[1] Brief proof of this fact: suppose there exist $U := D(I_1), W := D(I_2)$ open such that $U \cap W = \emptyset$ and $U \cup W = X$. Then, $V(I_1) \cup V(I_2) = V(I_1 I_2) = X$, so $I_1 I_2 \subset \bigcap_{\mathfrak{p}\triangleleft A} \mathfrak{p}$. But as A is a domain, this last ideal is the zero ideal! Hence, one of I_1 or I_2 is the zero ideal, so either $U = \emptyset$ or $V = \emptyset$.

1.)

Proposition. *For $X = \operatorname{Spec}\mathbb{Z}$, $\mathcal{O}_X(U) = \mathbb{Z}[\{p_1, p_2, \ldots, p^n\}^{-1}]$ where $U = D(p_1p_2\ldots p_n)$*[2]*, with restriction maps all inclusions.*

Proof. We have that $\mathcal{O}_X(U) = \bigcap_{x_p \in U} \mathbb{Z}_p$[3] by the result of the previous problem. Viewed as a subset of $\mathbb{Q}$, this is $\{\frac{a}{b} \ : \ \gcd(a,b) = 1;\ b \notin p\mathbb{Z} \text{ for any} x_p \in U\}$, which can be rephrased $\mathbb{Z}[p_1p_2\ldots p_n^{-1}]$. Then, as each ring $\mathcal{O}_X(U)$ is an integral domain, all restriction maps are injective and hence inclusions. ☐

4.)

Corollary. *Any ring A is canonically isomorphic to the projective limit of all its localizations A_f for $f \in A$.*

Proof. Follows immediately from the fact that for *any*[4] open set $U \subset X = \operatorname{Spec} A$, $\mathcal{O}_X(U) = \varprojlim_{D(f)\subset U} \mathcal{O}_X(D(f)) = \varprojlim_{D(f)\subset U} A_f$ by letting $U = X$. ☐

5.)

Proposition. *$D(2,t) \subset \mathring{A}^1_{\mathbb{Z}}$ is not affine.*

Proof. We note that as $\langle 2,t\rangle$ is a maximum ideal, $U := D(2,t) = \mathring{A}^1_{\mathbb{Z}} \setminus \{x_{\langle 2,t\rangle}\}$. By the result of problem 3,

$$\mathcal{O}(U) = \bigcup_{x\in U} \mathbb{Z}[t]_{\mathfrak{p}_x} = \bigcup_{\langle 2,t\rangle \neq \mathfrak{p} \lhd \mathbb{Z}[t]} \mathbb{Z}[t]_{\mathfrak{p}}$$

. As $\mathbb{Z}[t]$ is a(n?)[5] UFD, we may rewrite this as

$$\mathcal{O}(U) = \left\{ \frac{f(t)}{g(t)} \in \mathbb{Z}(t) \quad g(t) = 1 \text{ or } \langle 2,t\rangle \text{ is the } \textit{only} \text{ ideal } \mathfrak{p} \text{ such that } g(t) \in \mathfrak{p} \right\}$$

. However, as $\mathbb{Z}[t]$ is a UFD, all elements are contained in a *principal* prime ideal, which $\langle 2,t\rangle$ is not. Thus, $g(t)$ may be assumed to be 1, so $\mathcal{O}(U) = \mathbb{Z}[t]$. We suppose for the sake of contradiction that U is affine. Then, $U \cong \operatorname{Spec}\mathcal{O}(U)$, so V induces a bijection between radical ideals of $\mathcal{O}(U)$ and varieties in $\operatorname{Spec}\mathcal{O}(U)$. However, $\langle 2,t\rangle$ is a radical ideal in $\mathcal{O}(U)$, but $V(2,t) = \emptyset$. This completes our proof. ☐

6.)

Proposition. *We let $\mathcal{O}(U)$ be the algebra of holomorphic functions on $U \subset \mathbb{C}$ open.*

i) *This presheaf is indeed a sheaf on $\mathbb{C}$*

ii) *The stalk $\mathcal{O}_0$ may be identified with $\{f(z) = \sum_{n=0}^{\infty} a_n z^n \ : \ f(z)$ converges for some $z \in \mathbb{C}\setminus\{0\}\}$*

Proof. *i)* Locality follows trivially; indeed if a holomorphic function is zero everywhere in any open set, it is a basic theorem of complex analysis that it is zero everywhere. Gluing follows nearly as trivially. We let $(u_i)_{i\in I}$ be an open cover of $\mathbb{C}$ and $(f_i)_{i\in I}$ a compatible I-touple of functions. Then,

$$F(x) = \left\{ f_i(x) \quad x \in u_i \right.$$

is well-defined as $f_i(x) = f_j(x)$ for any x in any $u_i \cap u_j$. We claim F is holomorphic. For any x in $\mathbb{C}$, we let N_x be a neighborhood of x such that $N_x \subset u_i$ for some u_i and have that as $F(x) = f_i(x)$ has all its complex derivatives in that neighborhood, it is holomorphic at x. As x was arbitrary, this shows that $F \in \mathcal{O}(\mathbb{C})$.

ii) We claim that $\mathcal{O}_0 = \{f(z) = \sum_{n=0}^{\infty} a_n z^n \ : \ f(z)$ convergent in some open $U \ni 0\}$. Indeed, as any holomorphic function on an open set is equal to its Taylor series centered at any point in that set, we may identify each element of each $\mathcal{O}_X(U)$ with its Maclaurin series. Then, as the construction from problem 1 generalizes immediately to rings or algebras, we identify $f \in \mathcal{O}_X(U)$ with $g \in \mathcal{O}_X(V)$ if $f = g$ on $U \cap V$, which occurs if and only if their Maclaurin series coincide. This proves our first claim. We then note that all functions of $\mathcal{O}_0$ converge *somewhere* and if a Maclaurin series converges at $z_0 \in \mathbb{C}$, it then converges in the open ball centered at zero with radius $\|z_0\|$. This identifies the two interpretations of $\mathcal{O}_0$ and proves the proposition. ☐

[2] that all open sets are of this form is a rephrasing of the hint
[3] That is, $\mathbb{Z}$ localized at the ideal p, not the p-adic integers–we shall not make reference to the completion of $\mathbb{Z}_p$ in this problem set.
[4] as opposed to just those which are not basic
[5] Is UFD pronounced "unique factorization domain" or "yoo-eff-dee?"

7.)

Proposition. *We let $\mathscr{F}$ be a presheaf on topological space X, and $\mathscr{F}^+$ its sheafification with natural sheafification map $\mathscr{F} \to \mathscr{F}^+$. Then, there is a canonical isomorphism between stalks $\mathscr{F}_p \to \mathscr{F}_p^+$.*

Proof. We recall that for $U \subset X$ open,

$$\mathscr{F}^+(U) = \{(s_x \in \mathscr{F}_x) \ : \ s_x \in \mathscr{F}_x;\ \forall\, x,\ \exists x \in V \subset U \text{ s.t. } \exists s \in \mathscr{F}(V) \text{ s.t. } s_y = s\big|_y \ \forall y \in V\}.$$

There is naturally a unique set[6] of morphisms $\rho_{U,x} : \mathscr{F}(U) \to \mathscr{F}_x^+$ for all $U \subset X$ open, $x \in U$ commuting with the restriction morphisms, that is $\mathscr{F}(U) \to \mathscr{F}^+(U) \to \mathscr{F}_p^+$, the composition of the sheafification map and the restriction to stalk map. Thus, $\mathscr{F}_x^+$ is a co-cone for $\mathscr{F}$, so there is a unique morphism $\phi_x : \mathscr{F}_x \to \mathscr{F}_x^+$ commuting with the restriction maps for $\mathscr{F}$ (and as a byproduct, the sheafification maps). On the other hand, there is an obvious set of maps $\sigma_{U,x:}\mathscr{F}^+(U) \to \mathscr{F}_x$ for any $x \in U \subset X$ open, that is $(s_y \in \mathscr{F}_y)_{y \in U} \mapsto s_x \in \mathscr{F}_x$. As for any $V \subset U$, the restriction $\mathscr{F}^+(U) \to \mathscr{F}^+(V)$ is the "tautological restriction map" $(s_y \in \mathscr{F}_y)_{y \in U} \mapsto (s_y \in \mathscr{F}_y)_{y \in V}$, it is clear that $\sigma_{U,x}$ forms a co-cone. Hence, there is a unique map $\psi_x : \mathscr{F}_x^+ \to \mathscr{F}_x$ commuting with $\sigma_{U,x}$. Then, $\phi_x \circ \psi_x : \mathscr{F}_x^+ \to \mathscr{F}_x^+$ gives a morphism commuting with the restriction maps of $\mathscr{F}^+$ and hence must be the identity by the universal mapping property of the colimit, with a mirrored statement holding for $\psi_x \circ \phi_x : \mathscr{F}_x \to \mathscr{F}_x$. Thus, as ϕ_x and ψ_x were unique and are inverse isomorphisms, our proof is complete. □

8.)

a.)

Proposition. *We consider a morphism of sheaves (with concrete target category[7]) $\phi : \mathscr{F} \to \mathscr{G}$ with maps $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ where both are sheaves over X. For any $p \in X$:*

i) $(\ker \phi)_p = \ker(\phi_p)$

ii) $(\mathrm{Im}\phi)_p = \mathrm{Im}(\phi_p)$

Proof. Proof by awful element-chasing:

i) We recall that in a concrete category, $\mathscr{F}_p = \bigsqcup_{U \ni p} \mathscr{F}(U)/\sim$, where $\sim$ is a relation (quotient by ideal, normal subgroup, etc.) allowing commutation. Then, the induced map $\phi_p : [x] \mapsto [\phi_U(x)]$ for some $U \ni x$ is well-defined. We suppose $[x] \in \ker(\phi_p)$. Then, $\phi_U(x) = 0 \in \mathscr{G}(U)$ for some $U \ni p$ and indeed any $p \in V \subset U$, so x has a representative in $(\ker \phi)(U)$ for any such V. Then, $[x] \in (\ker \phi)_p$, so $(\ker \phi)_p \supseteq \ker(\phi_p)$ On the other hand, we suppose $[x] \in (\ker \phi)_p$. Then, there exists some $U \ni p$ such that $\phi_U(x) = 0$, so $[\phi_U(x)] = \phi_p([x]) = [0]$. Thus, $(\ker \phi)_p = \ker(\phi_p)$.

ii) We suppose $[x] \in (\mathrm{Im}\phi)_p$. Then, there is some representative $x \in \mathscr{G}(U)$ where $p \in U$ such that $x = \phi_U(y)$ for some $y \in \mathscr{F}(U)$, so $\phi_p([y]) = [x]$, and $[x] \in \mathrm{Im}\phi_p$, so $(\mathrm{Im}\phi)_p \subseteq \mathrm{Im}\phi_p$. On the other hand, we let $[x] \in \mathrm{Im}\phi_p$ Then, there is some $[y] \in \mathscr{F}_p$ such that $\phi_p([y]) = [x]$, so there is some $U \ni p$ such that $[y]$ has a representative $y \in \mathscr{F}(U)$ where $\phi_U(y) = x \in [x]$. Thus, $x \in (\mathrm{Im}\phi)(U)$ implying $[x] \in (\mathrm{Im}\phi)_p$ so $\mathrm{Im}\phi_p = \mathrm{Im}\phi_p$.

□

b.)

Corollary. *ϕ is injective (resp. surjective) if and only if ϕ_p is injective (resp. surjective)*

Proof. We claim that for a sheaf $\mathscr{A}$, $A = 0$ if and only if $A_p = 0$ for all $p \in X$. Indeed, the locality axiom ensures this is true. Then, ϕ is injective if and only if $\ker \phi$ is the 0 sheaf, if and only if $(\ker \phi)_p = 0$, if and only if $\ker \phi_p = 0$ for all p by the previous result. The same follows for surjectivity by simply replacing ker with coker. □

[6] I'm ignoring some set-theoretic issues here; replacement with 'class' does not effect my argument.
[7] Following Heartshorne's lead. . .

c.)

Corollary. *Let $\mathcal{F} := \ldots \overset{\phi^{\mathrm{i-1}}}{\leftarrow} \mathscr{F}_{i-1} \overset{\phi^{\mathrm{i}}}{\leftarrow} \mathscr{F}_i \overset{\phi^{\mathrm{i+1}}}{\leftarrow} \ldots$ be a sequence of morphisms and sheaves. Then, $\mathcal{F}$ is exact if and only if the induced sequence $\mathcal{F}_p$ is exact for all $p \in X$.*

Proof. We suppose $\mathcal{F}$ is exact. Then, $\ker \phi^i = \mathrm{Im}\phi^{i+1}$ for all i, so by part a, we have $\ker(\phi^i_p) = \mathrm{Im}(\phi^{i+1}_p)$ for all p. Thus, for any $p \in X$, $\mathcal{F}_p$ is exact. On the other hand, as sheaves may be recovered uniquely from stalks, if the sequence $\mathcal{F}_p$ is exact at every point p, we may reconstruct the sheaves $\ker(\phi^i)$ and $\mathrm{Im}(\phi^{i+1})$ from their stalks and have by exactness at stalks that $\mathcal{F}$ is exact once again from part a. $\square$