Worksheet for 10/3—David's Solutions

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3 October 2017

1.)

Let $f(x)$ and $g(z)$ be continuous and differentiable functions.

a.)

Find formulas (in terms of f, f', g, g', x, z, x_0, and z_0) **for $T_f(x)$, the tangent line to the graph $y = f(x)$ taken at x_0, and $T_g(z)$, the tangent line to the graph $y = g(z)$ taken at z_0.** (these should be more or less the same formula, of course...)

Response. This is basically an application of point-slope form! Let's write that out for $T_f(x)$ with slope m and point (w_0, y_0)...

$$T_f(x) - y_0 = m(x - w_0)$$

Now, we need to figure out (a) what our slope is and (b) what our point is. We want $T_f(x)$ to give the formula for the *tangent line* of the curve $y = f(x)$ at $x = x_0$—i.e. we want $T_f(x)$ to "hit" $y = f(x)$ when $x = x_0$. The derivative tells us the slope of the tangent line, so our slope m is $f'(x_0)$. We also only know of one point on $T_f(x)$: it "hits" $y = f(x)$ when $x = x_0$. That means that we know the point $(x_0, f(x_0))$ must be on the graph $y = T_f(x)$, so $w_0 = x_0$ and $y_0 = f(x_0)$. We now get the formula:

$$T_f(x) - f(x_0) = f'(x_0)(x - x_0)$$

or, moving everything to one side to give a formula for T_f:

$$T_f(x) = f'(x_0)(x - x_0) + f(x_0).$$

The formula for T_g is found completely analogously:

$$T_g(z) = g'(z_0)(z - z_0) + g(z_0).$$

□

b.)

Let $z_0 = f(x_0)$. Adjust your formula for $T_g(z)$ appropriately.

Response. Okay! Literally all this part is asking is to replace every z_0 with a $f(x_0)$–we are simply declaring z_0 to be that.

$$T_g(z) = g'(z_0)(z - z_0) + g(z_0) = g'(f(x_0))\,(z - f(x_0)) + g(f(x_0))$$

□

Write out the formula for the composition $F(x) = T_g(T_f(x))$ (of course, $T_f(x)$ is still the tangent at x_0)

Response. Now, all we are being asked to do is substitute $z = T_f(x)$ in the formula for T_g to find $F(x) = T_g(T_f(x))$.

$$\begin{aligned} F(x) = T_g(T_f(x)) &= g'(f(x_0))\,(T_f(x) - f(x_0)) + g(f(x_0)) \\ &= g'(f(x_0))\,((f'(x_0)(x - x_0) + f(x_0)) - f(x_0)) + g(f(x_0)) \end{aligned}$$

That is correct, but very ugly. The $f(x_0)$'s cancel, which makes things considerably better.

$$F(x) = f'(x_0)g'(f(x_0))(x - x_0) + g(f(x_0))$$

□

c.)

Find the slope of $F(x)$. This should look familiar—what is it?? What do you think I'm trying to illustrate by putting this question on here?

Response. So! $F(x)$ is a ***linear*** function. If I give you a function of the form $y = a(x - x_0) + b$, what is its slope? That's right, it's a. Here, we can do the same thing to find that the slope of $F(x)$ is $f'(x_0)g'(f(x_0))$. This should look familiar... that's right it's the chain rule!

Remember how on tuesday, we looked at an animation of zooming into the function until it looks identical to the tangent line? That is precisely what's going on here! The point is, near x_0 and z_0, the tangent lines T_f and T_g are similar enough to our original function that they are *basically* interchangeable with the real thing. So, if we want the slope of the tangent line of the composition (that is, the tangent line for $(g \circ f)(x)$ at $x = x_0$, it is enough to compose the tangent lines at x_0 and $f(x_0)$! This is almost always harder to do than just actually use the chain rule, but I think it gives some neat intuition for what the chain rule is actually getting at–and a way to derive the formula without writing out the limit-definition of a derivative once!! □

I'll leave the rest of the worksheet for y'all to work out.

2.)

Compute the following derivatives. Write your answer in a form that *feels* simplified to you. (that is, if you find yourself expanding a 149-term polynomial, something has gone horribly *horribly* wrong).

a.)

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\cos(\sin(\theta)) =$$

b.)

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3 + x + 1)^{50} =$$

c.)

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{\frac{x+3}{2\sin(x)}}$$

3.)

Use the *definition of the derivative* to find $f'(x)$ **where** $f(x) = \sin x$**.** You may use the identity $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and the limit identities found on pages 191-192.