6 November 2017
Inverse limits cont'd...

Universal Property for Inverse Limits

- For all $i \leq j$: $\beta_i = \varphi_{j,i} \circ \beta_j$
- Further: given any homomorphism
  $g_i : N \to M_i$
  such that $g_i := \varphi_{j,i} \circ g_j \quad \forall i \leq j$,
  there exists a unique homomorphism

$$h : N \to \varprojlim_{I} M_i$$

such that for all i, the following diagram commutes

$$\begin{tikzcd}
N \arrow[r, "\exists!\, h"] \arrow[d, "g_i"'] & \varprojlim M_i \arrow[dl, "\beta_i"] \\
M_i &
\end{tikzcd}$$

i.e. $\beta_i \circ h = g_i$

This characterizes $\varprojlim M_i$ up to isomorphism.

__Ex.__ Let k be a field and let $I = \mathbb{N}$.
For $i \leq j$, let $\varphi_{ij} : k[x]/\langle x^j \rangle \longrightarrow k[x]/\langle x^i \rangle$

This surjection maps

$\overline{p(x)} \pmod{\langle x^j \rangle}$ to $\overline{p(x)} \pmod{\langle x^i \rangle}$

~~Then~~ Then there exists an isom of rings

$$\varprojlim_{i \in \mathbb{N}} k[x]/\langle x^i \rangle \xrightarrow{\sim} \underbrace{k[[x]]}_{\text{formal power series in } x.}$$

This sends a sequence of polynomials in x:

~~$(a_0, a_0 + a_1 x, a_0 + a_1 x + a_2 x^2, \ldots)_{i \geq 0}$~~

$$(a_0,\ a_0 + a_1 x,\ a_0 + a_1 x + a_2 x^2, \ldots)$$

to a formal power series

$$\sum_{j \geq 0} a_j x^j$$

<u>Note</u>: The ring is the completion of $k[x]$ w.r.t. the ideal $\langle x \rangle$. (coming soon!)

Coro 5: Let $\{M_i\}_{i\in I}$ be a direct or inverse system of R-modules. Let N be an R-module. Then

(a) $\mathrm{Hom}_R(\varinjlim M_i, N) \cong \varprojlim \mathrm{Hom}_R(M_i, N)$

(b) $\mathrm{Hom}_R(N, \varprojlim M_i) \cong \varprojlim \mathrm{Hom}_R(N, M_i)$.

Pf. Use definitions and the fact that Hom is left exact! $\square$

Rmk. $\{M_i\}$: direct (or inverse) system of R-modules (or rings) over $I = \mathbb{N}$.
Let (n_i) ~~strict~~ be a strictly increasing subsequence of $\mathbb{N}$. Then

$$\varinjlim M_i \cong \varinjlim M_{n_i}$$

$$\varprojlim M_i \cong \varprojlim M_{n_i}$$

As before, we can define $\varprojlim_I u_i$, given $u_i : M_i \to M_i'$ where $\{M_i\}_{i\in I}$ and $\{M_i'\}_{i\in I}$ are inverse systems over the same I.

Defn $\{M_i, \varphi_{j,i}\}_{I=\mathbb{N}}$: inverse system. This inverse system satisfies the Mittag-Leffler condition (ML) if for all i, the decreasing chain of submodules (of M_i),

$$\{ \varphi_{j,i}(M_j) \mid j \geq i \}$$

stabilizes.

i.e. for all i, there exists $n_i > i$ such that

$$\varphi_{j,i}(M_j) = \varphi_{n_i, i}(M_{n_i})$$

for all $j \geq n$.

<u>Note</u>. If the M_i's are artinian, or the $\varphi_{j,i}$ are all surjective, then ML is automatically satisfied.

<u>Thm</u> 6: Suppose we have three inverse systems of R-modules over $I = \mathbb{N}$,

$$\{ M_i', \varphi_{j,i}' \}, \{ M_i, \varphi_{j,i} \}, \{ M_i'', \varphi_{j,i}'' \}$$

along with compatible maps

$$u_i : M_i' \longrightarrow M_i$$

$$v_i : M_i \longrightarrow M_i''.$$

Then ...

(a) If $0 \to M_i' \xrightarrow{u_i} M_i \xrightarrow{v_i} M_i''$ is eventually exact, then

$$0 \to \varprojlim M_i' \xrightarrow{\varprojlim u_i} \varprojlim M_i \xrightarrow{\varprojlim v_i} \varprojlim M''$$

is exact.

(b) If $\{M_i', \varphi_{ji}''\}$ satisfies ML and

$$0 \to M_i' \to M_i \to M_i'' \to 0$$

is eventually exact, then so is

$$0 \to \varprojlim M_i' \leftarrow \varprojlim M_i \leftarrow \varprojlim M'' \to 0$$

Pf. Showing composition of two maps is 0 is easy. Use the definition of an inverse limit to get ker = im. □

Ex. Let $R = \mathbb{Z}$.

$$M_i = \mathbb{Z}, \quad \varphi_{i+1,i} : \mathbb{Z} \xrightarrow{3\cdot} \mathbb{Z}$$

$v_i \downarrow$

$$M_i'' = \mathbb{Z}/(2), \quad \varphi_{i+1,i}'' : \mathbb{Z}/(2) \xrightarrow{2\cdot} \mathbb{Z}/(2)$$

Note that $1 \equiv 3 \bmod 2$, so the v_i are compatible with $\{M_i\}$ and $\{M_i''\}$.

Now: $\varprojlim M_i = \varprojlim (\mathbb{Z} \xrightarrow{3\cdot} \mathbb{Z} \xrightarrow{3\cdot} \dots)$

$= \mathbb{Z}[\frac{1}{3}] = \mathbb{Z}[3^{-1}]$

& $\varprojlim M_i'' = \varprojlim (\mathbb{Z}/(2) \xrightarrow{1\cdot} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1\cdot} \dots)$

$= \mathbb{Z}/2\mathbb{Z}.$

Q: what is $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{3}], \mathbb{Z}/2\mathbb{Z})$?

A: $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{3}], \mathbb{Z}/2\mathbb{Z}) = 0$

since the $\frac{1}{3}$ is problematic.

$\frac{1}{3} \mapsto 1$

$1 \to (3 \text{ mod } 2) \equiv 1$

Why doesn't this work?

$\Rightarrow \varprojlim V_i = 0.$

So: ML is necc.

§ (7.1, 7.2): Completions

__Ex__ Let $R = k[x_1, \dots, x_n]/I$

$\mathfrak{m} = \langle x_1, \dots, x_n \rangle$

The completion of R w.r.t. $\mathfrak{m}$ is $\hat{R}^{\mathfrak{m}} = \hat{R}$ ($\hat{R}$ with a subscript $\mathfrak{m}$ on the hat... or we'll drop the $\mathfrak{m}$ when possible).

$$\hat{R} = k[[x_1, \dots, x_n]] \Big/ I\, k[[x_1, \dots, x_n]]$$

Geometric Motivation.

If $R \longleftrightarrow X$, then

$R_{\mathfrak{m}} \longleftrightarrow$ Zariski open nbd $X \setminus Z(\mathfrak{m})$

Here:

$\hat{R}^{\mathfrak{m}} \longleftrightarrow$ small (analytic) nbd of $Z(\mathfrak{m})$

__Ex.__ As above, let $R = k[x_1, \dots, x_n]/I$
Then

$$\hat{R}^{\mathfrak{m}} \longleftrightarrow B_\varepsilon(\vec{0}).$$

Defn: Let R be an abelian group (or ring). Let's consider a descending filtration of R:

$$R = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \cdots$$

where each $\mathfrak{m}_i$ is a subgroup. Define

$$\hat{R} := \varprojlim R/\mathfrak{m}_i$$

$$= \left\{ g = (g_1, g_2, \ldots) \in \prod_i R/\mathfrak{m}_i \;\middle|\; \begin{array}{l} g_i \equiv g_j \bmod \mathfrak{m}_i \\ \text{for all } j > i \end{array} \right\}$$

We call $\hat{R}$ the completion of R w.r.t. $\{\mathfrak{m}_i\}_{i \in \mathbb{N}}$.

Note. If R is a ring and the $\mathfrak{m}_i$ are ideals, then $\hat{R}$ is also a ring, since $R/\mathfrak{m}_i$ is a ring.

Special case. Suppose $(R, \mathfrak{m})$ (max'l ideal $\mathfrak{m}$) is local and $\mathfrak{m}_i = \mathfrak{m}^i$. Then $\{\mathfrak{m}_i\} = \{\mathfrak{m}^i\}$ is called the $\mathfrak{m}$-adic filtration of R. In this case, we say that $\hat{R}$ is the $\mathfrak{m}$-adic completion of R.

Note: "$(R, \mathfrak{m}, k)$ local" means $R/\mathfrak{m}$ is a field and R is local w. max ideal $\mathfrak{m}$.

Prop: Sppse R is a local ring w. maximal ideal $\mathfrak{m}$ (sometimes denoted $(R, \mathfrak{m})$). ~~and~~ Consider the $\mathfrak{m}$-adic completion $\hat{R}$ of R. Not only is $\hat{R}$ a local ring, but its unique maximal ideal is

$$\mathfrak{m}\hat{R} =: \hat{\mathfrak{m}}.$$

Pf. $$\hat{R}/\mathfrak{m}\hat{R} = \frac{\varprojlim (R \twoheadrightarrow R/\mathfrak{m} \twoheadrightarrow R/\mathfrak{m}^2 \twoheadrightarrow \cdots)}{\varprojlim (\mathfrak{m}R \twoheadrightarrow \mathfrak{m}R/\mathfrak{m}R \twoheadrightarrow \mathfrak{m}R/\mathfrak{m}^2 \twoheadrightarrow \cdots)}$$

$\simeq R/\mathfrak{m}$ is a field.

$\Rightarrow \mathfrak{m}\hat{R}$ is maximal in $\hat{R}$.

Consider $g = (g_1, g_2, g_3, \ldots) \in \hat{R}$

$$\left(\hat{R} \subseteq \prod_i R/\mathfrak{m}^i\right)$$

Then $g_1 = 0$ implies that $g \in \mathfrak{m}\hat{R}$. So:

$$g \notin \mathfrak{m}\hat{R} \Rightarrow g_1 \neq 0$$

$$\Rightarrow g_i \notin \mathfrak{m}(R/\mathfrak{m}^i)$$

$$\Rightarrow g_i \text{ is a unit for all } i. \text{ (b/c } R \text{ is local)}$$

Since $g_i \equiv g_j \pmod{\mathfrak{m}^i}$ for all $i \leq j$, we have:

$$g_j^{-1} \equiv g_i^{-1} \mod \mathfrak{m}^i \quad \forall i \leq j$$

$$\Rightarrow h := (g_1^{-1}, g_2^{-1}, \ldots) \in \hat{R}$$

(Check: $g \cdot h = 1$)

Further: $(R/\mathfrak{m}^i)_{\mathfrak{m}} \cong$ $(R_{\mathfrak{m}})/\mathfrak{m}^i_{\mathfrak{m}}$.

Thus $(R_{\mathfrak{m}})^{\wedge} \simeq (\hat{R})_{\mathfrak{m}\hat{R}}$ $\square$

Long Example: Let $p \in \mathbb{Z}$ be prime.

Let $\hat{\mathbb{Z}}^{\langle p \rangle} =: \mathbb{Z}_p$ (NOT: $\mathbb{Z}/p\mathbb{Z}$)

We call this the ring of p-adic integers.

i.e. For $p = 5$, $q = \frac{2}{3}$, the p-adic expansion of q is

$$q = 4 + 1(5) + (3)(5^2) + (1)(5^3) + \ldots$$

201 [redacted]

How to get that:

$$a_0 = \frac{2}{3} \pmod 5 \iff 3a_0 \equiv 2 \pmod 5$$
$$\iff a_0 = 4$$

$$(2/3 - 4) \equiv 5a_1 \pmod{5^2}$$

$$\iff -\frac{10}{3} \equiv 5a_1 \pmod{25}$$

$$\iff -\frac{2}{3} \equiv a_1 \pmod{\cancel{2}5}$$

$$\implies a_1 = 1.$$

$$\left(\frac{2}{3} - 4 - 1(5)\right) \equiv 25 \cdot a_2 \mod 5^3$$

$$\iff -\frac{1}{3} \equiv a_2 \pmod 5$$
$$\implies a_2 = 3$$

$$\vdots$$

Sanity check: multiply both sides by three to get:

$$2 = 12\cancel{4} + 3(5) + 9(5^2) + 3(5^3) + \cdots$$

Reducing both sides mod 5^{blah} gives 2. ✓

Source: Keith Conrad "p~~a~~-adic expansions of ratl #s"

~~[illegible]~~ 8 November 2017
Completions Cont'd.

Consider the natural map

$$R \longrightarrow \hat{R}^{m} \quad (= \varprojlim R/m^i)$$

If this map is an ~~[illegible]~~ isomorphism, then we say that R is complete w.r.t. m. When m is max'l, we say that R is a <u>complete local</u> ring. Further, if $\bigcap_j m^j = 0$, then we say that R is <u>separated</u> w.r.t. m.

<u>Thm</u> (7.1) Suppose that R is Noetherian and m is an R-ideal. Let $\hat{R} := \hat{R}^{m}$. Then
(a) $\hat{R}$ is Noetherian.
(b) $\hat{R}/m^j\hat{R} \cong R/m^j$

$\Rightarrow \hat{R}$ is complete w.r.t. $m\hat{R}$ and ~~[illegible]~~

$$gr_{m\hat{R}}\hat{R} = gr_m R.$$

<u>pf</u> (a) Will follow from thm 7.7 when R is local (see Eisenbud p. 196).

(b) If $\mathrm{in}(a_1), \dots, \mathrm{in}(a_r)$ generate $\mathrm{in}(I)$, then $a_1, \dots, a_r$ generate I. Further, $(\mathfrak{m}\hat{R})^n$ and $\mathfrak{m}^n\hat{R}$ generate the same initial ideals. So

$$\mathrm{gr}_{\mathfrak{m}} \hat{R} = \mathrm{gr}_{\mathfrak{m}} R. \qquad \square$$

<u>Thm</u> (7.2) Suppose R is Noetherian and $\mathfrak{m}$ is an R-ideal. $\hat{R} := \hat{R}^{\mathfrak{m}}$

Then

(a) If M is a f.g. R-module, then the natural map

$$\hat{R} \otimes_R M \longrightarrow \varprojlim M/\mathfrak{m}^i M =: \hat{M}$$

is an isomorphism. In particular if S is a f.g. ring, then

$$\hat{R} \otimes_R S = \hat{S}^{\mathfrak{m}S}$$

(b) Suppose $\hat{R}$ is flat as an R-module.

i.e. $\hat{R} \otimes_R (\text{blah})$ is exact.

Pf. We have ML (p. 195) in this case, so completions are exact.

(a) Second statement follows from the isomorphism. So begin with the case that $M = R$. It follows from the definitions that $\varprojlim$ and finite $\oplus$'s commute.

$\Rightarrow$ The result also holds for f.g. free modules.

Let M be a f.g. R-module and let $F \to G \to M \to 0$ be a free presentation of M.

Because inverse limits preserve exactness, we have the following comm. diagram w. exact rows:

$$\begin{tikzcd}
\hat{F} \arrow[r] & \hat{G} \arrow[r] & \hat{M} \arrow[r] & 0 \\
\hat{R} \otimes_R F \arrow[r] \arrow[u, "\cong"] & \hat{R} \otimes_R G \arrow[r] \arrow[u, "\cong"] & \hat{R} \otimes_R M \arrow[r] \arrow[u, "\cong (?)"'] & 0 \arrow[u, "\cong"]
\end{tikzcd}$$

Five lemma $\Rightarrow$ $\hat{R} \otimes_R M \to \hat{M}$ is an isomorphism.

(b) by prop 6.1 (p. 179), it is enough to show for all f.g. R-ideals I. The following map is injective:

$$I \otimes_R \hat{R} \longrightarrow I\hat{R} \subset \hat{R}$$

By part (a), $I \otimes_R \hat{R} \simeq \hat{I}$ and the ML condition implies that $\hat{I} = I\hat{R}$. So we have an injection

$$\hat{I} \hookrightarrow \hat{R} \qquad \square$$

__Thm__ (7.3, Hensel's lemma): Let R be a complete ring w.r.t. an ideal m. Let $f(x) \in R[x]$. If $a \in R$ is an approximate root* of $f(x)$, then there is a root b of f near* a. ~~in the sense that f(a)~~ Further, if $f'(a)$ is a nonzero divisor in R, then b is __unique__.

__upshot__: Use Thm 7.3 to prove something like the inverse function theorem.

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* In the sense that $f(x) \equiv 0 \pmod{f'(x)^2 m}$

* In the sense that $f(b) = 0$ and $b \equiv a \pmod{f'(a)\, m}$

Coro(7.4): Let k be a field and $f(t,x) \in k[t,x]$. Let $x=a$ be a simple root of $f(0,x)$, then there exists a unique power series $x(t) \in k[[t]]$ s.t. $x(0)=a$ and $f(t, x(t)) \equiv 0$ identically.

Recall. $f(0,x)$ w. simple root at $x=a$ means that $f_x(0,a) \neq 0$.

§7.4: Cohen's Structure Theorem

Idea: Complete, local, Noetherian rings have "nice" presentations via the p.s. ring

Thm (Cohen's Structure Thm, 7.7): Let R be a complete local Noetherian ring. Let m be a maximal ideal of R. Let K be the residue class field (i.e. $K = R/m$).
If R contains a field, then

$$R \cong K[[x_1, \dots, x_n]]/I$$

for some n and ideal I.

Deepest part: Show that R contains a coefficient field.

i.e. There exists a field inside R that maps isomorphically to $K = R/m$.

§7.6: Maps for power series rings

<u>Thm</u> (7.16): Let R be any ring and S be an R-algebra that is complete w.r.t. some ideal $\mathfrak{m}$. Given $f_1, f_2, \dots, f_n \in \mathfrak{m}$.

(a) $\exists!$ R-algebra homomorphism $\varphi: R[[x_1, \dots, x_n]] \to S$ via $x_i \longmapsto f_i$ for all i. and taking convergent sequences to convergent sequences*. (The map φ takes a power series

$$g(x_1, \dots, x_n) \mapsto g(f_1, \dots, f_n) \in S)$$

(b) If the induced map $R \to S/\mathfrak{m}$ is an epimorphism and $f_1, \dots, f_n$ generate $\mathfrak{m}$, then φ is an epimorphism

(c) If the induced map

$$\operatorname{gr}\varphi: \underbrace{R[x_1, \dots, x_n]}_{\simeq \operatorname{gr}_{\langle x \rangle} R[[x_1, \dots, x_n]]} \to \operatorname{gr}_{\mathfrak{m}} S$$

is a monomorphism, then φ is a monomorphism

* Zariski convergent

+ monomorphism $\Rightarrow$ map is injective.

pf (a) The unique k-algebra map

$$k[x_1,\dots,x_n] \longrightarrow S/\mathfrak{m}^t$$

via $\quad x_i \longmapsto \overline{f_i}$

factors through

$$\begin{tikzcd} k[x_1,\dots,x_n] \arrow[r] \arrow[d] & S/\mathfrak{m}^t \\ k[x_1,\dots,x_n]/\langle x_1,\dots,x_n\rangle^t \arrow[ur] & \end{tikzcd}$$

because $f_i \in \mathfrak{m}$.

$\Rightarrow \exists$ an induced map from $k[[x_1,\dots,x_n]]$ to $S = \varprojlim S/\mathfrak{m}^t$ since S is $\mathfrak{m}$-adically complete. Let's call this map φ. We have

$$\varphi : k[[x_1,\dots,x_n]] \to S$$

via $\quad x_i \longmapsto f_i$

Note: If we start with

$$g(\vec{x}) + \langle x_1,\dots,x_n\rangle^t \in k[\vec{x}]/\langle\vec{x}\rangle^t$$

then it maps to $g(f_1,\dots,f_n) + \mathfrak{m}^t \in S/\mathfrak{m}^t$.

~~ab~~ (b) Suppose $R \to S/\mathfrak{m}$ is surjective. Then

$$\langle x_1, \dots, x_n \rangle / \langle x_1, \dots, x_n \rangle^2$$

also surjects onto $\mathfrak{m}/\mathfrak{m}^2$.

Then $gr\varphi : R[\![x_1, \dots, x_n]\!] \twoheadrightarrow gr_{\mathfrak{m}} S$ is surjective.

Now, given $0 \neq g \in S$, let i be the largest # s.t. $g \in \mathfrak{m}^i$.*

Since $gr\,\varphi$ is surj, there exists $g_1 \in \langle x_1, \dots, x_n \rangle^i$ with

$$in(g_1) \xmapsto{gr(\varphi)} in(g)$$

$$\Rightarrow g - \varphi(g_1) \in \mathfrak{m}^{i+1}$$

Iterating this gives a sequence of elements

$$g_j \in \langle x_1, \dots, x_n \rangle^{i+j-1}$$

~~[illegible]~~

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* Such an i exists b/c S is complete, so $\bigcap \mathfrak{m}^i = 0$.

such that $g = \sum_{j=1}^{\infty} \varphi(g_j)$.

Because φ preserves infinite sums (by part (a)), we have

$$g = \varphi\left(\sum_{j=1}^{\infty} g_j\right)$$

where $\sum_{j=1}^{\infty} g_j \in R[\![x_1, \ldots, x_n]\!]$.

$\Rightarrow \varphi$ is surjective! ☺

(c) Suppose $0 \neq g \in R[\![x_1, \ldots, x_n]\!]$. Then $\mathrm{in}(g)$ is nonzero in $R[x_1, \ldots, x_n]$. Suppose that $\mathrm{in}(g)$ is of degree d.

By hypothesis, $\underbrace{(\mathrm{gr}\,\varphi)(\mathrm{in}(g))}_{\text{in degree } d \text{ part of } \mathrm{gr}_{\mathfrak{m}} S} \neq 0$

But $g \equiv \mathrm{in}\, g \mod \langle x_1, \ldots, x_n \rangle^{d+1}$

Then

$$\varphi(g) \equiv (\mathrm{gr}\,\varphi)(\mathrm{in}(g)) \mod \mathfrak{m}^{d+1}.$$

$\Rightarrow \varphi(g) \neq 0$

$\Rightarrow \varphi$ is injective. $\square$

Coro 7.17: Given $f \in R[[x]]$, if φ is the map

$$\varphi: R[[x]] \to R[[x]]$$

$$\text{via } p(x) \longmapsto p(f)$$

Then φ is an isomorphism $\iff f'(0) \in R^{\times}$.

Pf (of Hensel's lemma, p. 205). We'll do a pseudo-Taylor expansion. Let $e = f'(a)$. Choose $h(x)$ such that

$$f(a+ex) = f(a) + f'(a)ex + h(x)(ex)^2$$

$$= f(a) + e^2(x + x^2 h(x)) \quad (*)$$

Since $f'(a) = e$. By Theorem 7.16 (p. 208), there exists a hom

$$\varphi: R[[x]] \to R[[x]]$$
$$\text{via } ~~\text{[illegible]}~~ \; p(x) \mapsto p(x + x^2 h(x))$$

By Coro 7.17, φ is an isom. Apply φ^{-1} to $(*)$ and get

$$f(a + e\varphi^{-1}(x)) = f(a) + e^2 x$$
$$= f(a) + (f'(a))^2 x.$$