# MATH 8301 Homework IX

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## 1.)

We let $X$ be a path-connected topological space with $x_0 \in N \subset X$ where $N$ is contractible, $f : X \to X$ a continuous map fixing $x_0$, and $M_f$ its mapping torus with basepoint $m_0$ the image of $(x_0, 1/2)$ in $M_f$.

### a.)

**Proposition.** *$M_f$ is path-connected.*

*Proof.* We note that as $X$ and $I$ are path-connected, $X \times I$ is path-connected as the product of paths is itself a path. Thus, $M_f$ is path-connected, as the quotient of a path is also itself a path. $\square$

### b.)

**Proposition.** *We let $U$ be the image of $X \times (0, 1)$. Then, $\pi_1(U, m_0) = \pi_1(X, x_0)$.*

*Proof.* We note that each point in $U$ is in a singleton equivalence class in the quotient defining $M_f$. Thus, $U \cong X \times (0, 1)$, so $\pi_1(U, m_0) = \pi_1(X, x_0) \times \pi_1((0, 1), 1/2) = \pi_1(X, x_0)$. $\square$

### c.)

**Proposition.** *We let $V = X \times ([0, 1/3) \cup (2/3, 1]) \cup N \times I$. Then, $\pi_1(V, m_0) = \pi_1(X, x_0) \star \mathbb{Z}$.*

*Proof.* We let $W$ be a contractible neighborhood around $X_0$ such that $f(W) \subset N$ and $f(W)$ is contractible. We let $A = X \times ([0, 1/3) \cup (2/3, 1])$, and let $B = (f(W) \times [0, 1/3)) \cup (W \times (2/3, 1]) \cup N \times (1/5, 4/5)$. We let $m_0'$ be the image of $(x_0, 1)$. Then, $B$ deformation-retracts to $S_1$ by applying the appropriate contraction-retraction to each cross-section, so $\pi_1(B, m_0') \cong \mathbb{Z}$, and $A$ deformation retracts to $X \times 0$, which we show by constructing a map $H : A \times I \to A$ defined by

$$H((x, t), s) := \begin{cases} (x, (1 - s)t) & 0 < t < 1/3 \\ (x, (1 - s)t + s) & 2/3 < t < 1 \\ (x, t) & t = 0 \end{cases}$$

Thus, $\pi_1(A, m_0') \cong \pi_1(X, x_0)$. Finally, as $A \cap B$ deformation-retracts to $S_1 \setminus \{e^{i\theta} \ : \ 1/3 \leq \theta \leq 2/3\}$, we have that $\pi_1(A \cap B, m_0')$ is trivial. Thus Siefert-von Kampen yields the statement of the proposition. $\square$

### d.)

**Proposition.** *$\pi_1(U \cap V, m_0) \cong G \star \tilde{G}$ where $G \cong \pi_1(X, m_0) \cong \tilde{G}$.*

*Proof.* We let $A' = (X \times (2/3, 1)) \cup N \times (1/3, 4/5)$ and $B' = (X \times (0, 1/3)) \cup N \times (1/5, 2/3)$. Then, by contracting $(1/3, 1)$ to $\{4/5\}$ in the second coordinate, we see that $A'$ deformation-retracts to a space homeomorphic to $X$, as does $B'$ by a symmetric argument. Finally, $A' \cup B'$ is simply $N \times (1/3, 2/3)$, which is contractible, proving the proposition by an application of SvK. $\square$

### e.)

**Proposition.** *$H := \pi_1(M_f, m_0) = (G \star \mathbb{Z})/\langle t f_*(\gamma) t^{-1} f_*(\gamma^{-1}) \ : \ \gamma \in G\rangle$.*

*Proof.* We have from Siefert-von Kampen that $H = (G \star \mathbb{Z} \star \hat{G})/M$ where $\hat{G}$ is again a distinct isomorphic copy of $G$ and $M$ is some normal subgroup. We let $\iota_U$ and $\iota_V$ be the inclusions of $U \cap V$ to their indices. We claim that $(\iota_U)_*$ is simply the "forgetful" map, which evaluates elements of $G \star \tilde{G}$ by identifying $G$ and $\tilde{G}$. This can be seen by composing the inclusion $\iota_U$ with the deformation retract we used to compute $\pi_1(U, m_0)$. We claim as well that

$(\iota_V)_*$ acts by mapping $\gamma \mapsto \gamma$ for $\gamma \in G$ and $\tilde{\gamma} \mapsto t^{-1} f_*(\gamma) t$. To see this, we note that if $\tilde{\gamma} \in \tilde{G}$ (corresponding to $X \times (2/3, 1])$, we may "push $\gamma$ though" $X \times [0]$ so that after the homotopy, $\gamma$ is a path $\alpha$ from $m_0$ "the long way" to $(x_0, 1/4)$ concatenated with $f_*(\gamma)$ lying in $X \times \{1/4\}$, then concatenated with $\alpha^{-1}$. We let $\beta$ be a path from $(x_0, 1/4)$ to $m_0$ "the short way" and "stretch" the image of the path we've constructed to $\alpha^{-1}\beta^{-1} f_*(\gamma)\beta\alpha$. Noting $\beta\alpha = t \in \mathbb{Z}$ shows our claim. Then, quotienting by $(\iota_U)_*(\iota_V)_*^{-1}$ identifies $G$ and $\hat{G}$ and induces the relation of the statement of the proposition. □

### f.)

**Proposition 1.1.** $\mathbb{Z} \ltimes G \cong \mathbb{Z} \star G / \langle tgt^{-1}\phi(g^{-1})\rangle$

*Proof.* We note that $\mathbb{Z} \ltimes G$ is clearly a quotient of the free product $\mathbb{Z} \star G$ as it is generated by the same set $G \sqcup \mathbb{Z}$, identifying $\mathbb{Z}$ with $(\mathbb{Z}, [1])$ and identifying $G$ with $(0, G)$. As all commutation relations between elements of $G$ or between elements of $\mathbb{Z}$ are predetermined respectively by the group structure, what is left is to determine commutation relations between elements of $G$ and $\mathbb{Z}$. From the definition of $\mathbb{Z} \ltimes G$, we have that $gt = (0, g)(t, [1]) = (t, \phi(g)) = (t, [1])(0, \phi(g)) = t\phi(g)$. Thus, $t^{-1}gt = \phi(g)$. □

### g.)

**Proposition.** *We suppose $f$ is a homotopy equivalence. Then, $\pi_1(M_f, m_0) \cong \mathbb{Z} \ltimes G$*

*Proof.* Follows immediately from the presentation of $\pi_1(M_f, m_0)$ given above as well as the observation that if $f$ is a homotopy equivalence, then $f_*$ is an automorphism. □

## 2.)

**Proposition.** *Any map $f : \mathbb{R}P^2 \to S^1$ is nullhomotopic.*

*Proof.* We note that $\pi_1(\mathbb{R}P^2, x_0) \cong \mathbb{Z}/2$, which is a torsion group and hence has no non-zero quotient isomorphic to a subgroup of $\mathbb{Z} \cong \pi_1(S_1, 0)$. Thus, $f_*$ is the zero map for any such map. Hence, by the lifting criterion, there exists a lift $\tilde{f} : \mathbb{R}P^2 \to \mathbb{R}$. However, moreso than simply connected, $\mathbb{R}$ is contractible. We let $\tilde{H}$ be that contraction and let $H : \mathbb{R}P^2 \times I \to S^1$ be given by $H := p \circ \tilde{H} \circ \tilde{f}$. Then, $H$ is a nullhomotopy of $f = p \circ \tilde{f}$. □