Solutions for Quiz 7

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31 October 2017

Problem 1. *Let* $f(x) = \tan x$ *Show there is a value* c *in* $(0, \frac{\pi}{4})$ *such that* $f'(c) = \frac{4}{\pi}$. *State which theorem you are using and explain why the hypotheses are satisfied.*

Response. We note that $f(x) = \tan x = \frac{\sin x}{\cos x}$ is continuous on $[0, \frac{\pi}{4}]$ and differentiable on $(0, \frac{\pi}{4})$ because $\sin x$ and $\cos x$ are continuous and differentiable everywhere, so their quotient is everywhere it is defined —i.e. everywhere where $\cos x$ is not zero, and $\cos x = 0$ only for $x = \pm(2k+1)\frac{\pi}{2}$ where k is an integer and hence nowhere in $(0, \frac{\pi}{2})$. Hence, the **hypotheses** of the **Mean Value Theorem** are fulfilled, so we may apply it to conclude that there exists a $c \in (0, \frac{\pi}{4})$ such that

$$f'(c) = \frac{f(\frac{\pi}{4}) - f(0)}{\frac{\pi}{4} - 0} = \frac{1-0}{\frac{\pi}{4} - 0} = \frac{4}{\pi}$$

□

Problem 2. *Let* $f(x) = x^3 + 6x^2 + 9x - 2$ *Find the local maximum and minimum values of* f. *State which tests you are using.*

Response. We use the first derivative test to determine critical points of f. $f'(x) = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3) = 3(x+1)(x+3)$. By this factorization, we see f has critical points at $x = -1$ and $x = -3$. We then apply the second-derivative test to determine whether these are respectively maxima or minima. $f''(x) = 6x + 12 = 6(x+2)$. We see $f''(-3) = -6 < 0$, so there is a local maximum at $x = -3$. On the other hand, $f''(-1) = 6 > 0$, so we see there is a local minimum at $x = -1$.[1] Thus, we have a local maximum at $(-3, f(-3)) = (-3, -2)$ and a local minimum at $(-1, f(-1)) = (-1, -6)$. □

Problem 3. *Evaluate* $\lim_{x\to 1} \frac{x^4 - x^3 - x + 1}{x^3 - x^2 - x + 1}$

Response. We let $f(x) = x^4 - x^3 - x + 1$ and $g(x) = x^3 - x^2 - x + 1$, and let $L = \lim_{x\to 1} \frac{x^4 - x^3 - x + 1}{x^3 - x^2 - x + 1}$. We note $f(1) = 0 = g(1)$, so we apply L'Hopital's rule to yield

$$L = \lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{4x^3 - 3x^2 - 1}{3x^2 - 2x - 1}$$

We note $f'(1) = 0 = g'(1)$, so we apply L'Hopital once more and yield

$$L = \lim_{x\to 1} \frac{f''(x)}{g''(x)} = \lim_{x\to 1} \frac{12x^2 - 6x}{6x - 2} = \frac{12-6}{6-2} = \frac{3}{2}.$$

□

[1] This is not the only way to do this—one may also observe that $f'(x) > 0$ on $(-\infty, -3)$ and $(-1, \infty)$ while $f'(x) < 0$ on $(-3, -1)$ and reach the same conclusion that way.