MATH 8253 Homework IV

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To the Grader: I've been sick and overwhelmed this week and this is the best I can do. I'm sorry you have to wade through it.

1.)

Prompt. *Describe all open sets of $X = \operatorname{Spec} \mathbb{C}[t]/\langle t^2 - t\rangle$ and the restriction morphims of its structure sheaf $\mathcal{O}_X$.*

Response. We recall that for ring R and ideal $I \leq R$, there is a bijection between prime ideals of R containing I and prime ideals of R/I. Thus, the prime ideals of $\mathbb{C}[t]/\langle t^2 - t\rangle$ may be identified with those of $\mathbb{C}[t]$ containing $t^2 - t$ as $\mathbb{C}[t]$ is a PID. Moreover, again using that $\mathbb{C}[t]$ is a PID, we have that the only such ideals are those generated by divisors of $t^2 - t$, that is $\langle t - 1\rangle$ and $\langle t\rangle$. As both of these are closed points, X carries the discrete topology, so the only proper open sets are the singleton sets containing each. We consider $\mathcal{O}_X(\langle t\rangle) = D(t-1) = \left(\mathbb{C}[t]/\langle t^2 - t\rangle\right)_{t-1}$. By basic computations with the localization equivalence relation, we see that the kernel of $\mathbb{C}[t]/\langle t^2 - t\rangle \to \left(\mathbb{C}[t]/\langle t^2 - t\rangle\right)_{t-1}$ is the ideal $\langle t\rangle$, and note that this implies that the image of the map is isomorphic to $\mathbb{C}[t]/\langle t\rangle$. By the universal mapping property of localization, we may conclude that this is the whole of $\left(\mathbb{C}[t]/\langle t^2 - t\rangle\right)_{t-1}$. A similar argument (or application of the isomorphism $\mathbb{C}[t]/\langle t^2 - t\rangle \to \mathbb{C}[t]/\langle t^2 - t\rangle$ by $t \mapsto t - 1$) shows that $\mathcal{O}_X(D(t))$ $\square$

2.)

Proposition. $\operatorname{Spec} \mathbb{Z}$ *is the terminal object of* **AffSch***.*

Proof. We recall that locally ringed space morphisms between affine schemes are determined by their ring morphism on global sections. The proposition is therefore equivalent to the claim that $\mathbb{Z}$ is the initial object of **Ring**. This is indeed the case; as the free group on one generator, any group morphisim $\mathbb{Z} \to G$ is determined by the image of its generator $1 \in \mathbb{Z}$, and any ring morphism $\mathbb{Z} \to R$ must preserve multiplicative identity, uniquely determining the image of 1. Thus, for any ring R, there is a unique morphism $\mathbb{Z} \to R$, proving the equivalent claim to the proposition. $\square$

Corollary 2.1. **AffSch** *is in natural equivalence with the category of Affine Schemes over* $\operatorname{Spec} \mathbb{Z}$

Proof. Indeed, even better there is a categorical isomorphism between the two! This follows immediately from the fact that uniqueness of morphism to $\operatorname{Spec} \mathbb{Z}$ implies that any morphism between two Affine schemes X and Y commutes with their respective morphisms to $\operatorname{Spec} \mathbb{Z}$. $\square$

3.)

4.)

Prompt. *Suppose $\mathscr{F}$ and $\mathscr{G}$ are sheaves of Abelian groups on a topological space X. For any open set $U \subset X$, set $\underline{\operatorname{Hom}}(\mathscr{F}, \mathscr{G})(U) := \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$, where $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is the set of sheaf morphisms on U. Define the structure of a presheaf of Abelian groups of $\underline{\operatorname{Hom}}(\mathscr{F}, \mathscr{G})$ on X.*

Response. We have our global sections defined for us; what is left is to show that $\underline{\operatorname{Hom}}(\mathscr{F}, \mathscr{G})(U)$ is an Abelian group and define the restriction maps. Indeed, for arbitrary morphisms of Abelian groups $\phi : G \to H$ and $\psi : G \to H$, we may define $(\phi \cdot \psi) : G \to H$ by $(\phi \cdot \psi)(g) = \phi(g)\psi(g)$ for $g \in G$ and see that $(\psi \cdot \psi)(gh) = \phi(gh)\psi(gh) = \phi(g)\phi(h)\psi(g)\psi(h) = \phi(g)\psi(g)\phi(h)\psi(h) = (\phi \cdot \psi)(g)(\phi \cdot \psi)(h)$, showing $(\phi \cdot \psi)$ is indeed a morphism of Abelian groups. For $\phi, \psi \in \underline{\operatorname{Hom}}(\mathscr{F}, \mathscr{G})(U)$, we may define the same analogously, simply defining $(\phi \cdot \psi)(V)$ as $(\phi(V) \cdot \psi(V))$ for $V \subseteq U$ open. The restriction maps come about in a similarly straightforward manner; we recall that $\phi \in \underline{\operatorname{Hom}}(\mathscr{F}, \mathscr{G})(U)$ is the data of a set of maps $\phi(V) : \mathscr{F}(V) \to \mathscr{G}(V)$ for all open $V \subset U$. We may then define $\phi|_W$ to be the set of $\phi(V)$ where $V \subset W \subset U$ are open, and see that the sheaf morphism structure of ϕ ensures in a quite natural manner that our restriction maps commute with the Abelian group structure of $\underline{\operatorname{Hom}}(\mathscr{F}, \mathscr{G})(U)$. $\square$

Proposition. *With the presheaf structure of above, $\underline{\mathrm{Hom}}(\mathscr{F},\mathscr{G})$ is indeed a sheaf.*

Proof. **Locality:** We wish to show that for $U \subseteq X$ open, $0 \neq \phi \in \underline{\mathrm{Hom}}(\mathscr{F},\mathscr{G})(U)$, it is not the case that $\phi|_V = 0$ for all open $V \subset U$. We suppose the contrary: that ϕ is such a morphism and let $f \in \mathscr{F}(U)$ be such that $\phi(U)(f) = g \neq 0 \in \mathscr{G}(U)$. Then, as $\phi|_V = 0$ for all $V \subset U$, we have that $g|_V = \phi(V)(f) = \phi|_V(V)(f) = 0$, contradicting locality of $\mathscr{G}$.

Gluing: We let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of U and let $\{\phi_{U_\alpha}\}_{\alpha\in A}$ be a compatible set of sheaf morphisms. We wish to show that there exists some $\phi \in \underline{\mathrm{Hom}}(\mathscr{F},\mathscr{G})(U)$ such that $\phi|_{U_\alpha} = \phi_{U_\alpha}$. We construct ϕ as such: for $V \subset U_\alpha$, we let $\phi(V) = \phi_{U_\alpha}|_V$; our assumption of compatibility ensures this is well-defined. Otherwise, we let $\{V_\beta\}_{\beta\in B}$ be an open cover of V such that each V_β is contained within some U_α and let $r : B \to A$ be a (possibly not uniquely determined) set map[1] such that $V_\beta \subset U_{r(\beta)}$. We let $f \in \mathscr{F}(V)$ and consider $\overline{\phi(V)(f)} := \{\phi_{U_{r(\beta)}}(V_\beta)(f|_{V_\beta})\}_{\beta\in B}$. Then, by the sheaf morphism structure of $\phi_{U_{r(\beta)}}$ and our assumption of compatibility on $\{\phi_{U_\alpha}\}$, we have that $\overline{\phi(V)(f)}$ is a compatible set of sections on $\mathscr{G}|_V$. Thus, there exists a unique element $g \in \mathscr{G}|_V(V)$ such that $g|_{V_\beta} = \phi_{U_{r(\beta)}}(V_\beta)(f|_{V_\beta})$ by the gluing property of $\mathscr{G}|_V$; we let $\phi(V)(f) = g$. Then, it is clear that ϕ fits the desired properties. $\square$

5.)

a.)

Proposition. *We let F be an abelian group and x a closed point of the topological space X. We define the presheaf on X $\mathscr{F}$ by:*

$$\mathscr{F}(U) := \begin{cases} F & x \in U \\ 0 & x \notin U \end{cases} \tag{1}$$

Then, $\mathscr{F}$ is a sheaf.

Proof. We first show locality: we let $0 \neq f \in \mathscr{F}(X) = F$ where $x \in U$ and suppose $f|_V = 0$ for all $V \subset X$ open. Then, as all restriction maps are isomorphisms or the zero map, we have that $x \notin V$ for all $V \subset X$ open... wait why is that a problem? Why can we not have that?

To show gluing, we let $\{U_\alpha\}_\alpha$ be an open cover of X and $\{f_\alpha\}_\alpha$ be compatible. Then, as all restriction maps are either isomorphisms or the zero map, we have that for $x \in U_\alpha \cap U_\beta$, we must have $f_\alpha = f_\beta$, and for $x \notin U_\alpha \cap U_\beta$, we have that both restriction maps are the zero map. Hence, we now have for some fixed $f \in F$,

$$f_\alpha = \begin{cases} 0 & x \notin U_\alpha \\ f & x \in U_\alpha \end{cases}$$

Then, $f \in \mathscr{F}(X) = F$ satisfies the requirement for the gluing axiom. $\square$

b.)

Proposition. *The skyscraper sheaf is uniquely characterized by its stalks $\mathscr{F}_x = F$ and $\mathscr{F}_y = 0$ for $y \neq x$.*

6.)

a.)

Proposition. *For $X = \mathring{A}^1_k$ where k is a field, let $\mathscr{F}$ be the skyscraper sheaf supported at $\mathbf{0} := [(t)]$ with group $k(t)$ with the usual $k[t]$-module structure. Then, $\mathscr{F}$ is an $\mathcal{O}_X$-module, but not quasicoherent.*

Proof. We note for $\mathbf{0} \notin U \subset X$, $\mathscr{F}(U) = 0$ is trivially an $\mathcal{O}_X(U)$-module. For $\mathbf{0} \in U \subset X$, we claim that $k(t)$ has a natural $\mathcal{O}_X(U)$-module structure. By problem 3 of homework 3, $\mathcal{O}_X(U)$ may be identified with a subset of the field of fractions of $k[t]$, $k(t)$, and hence the $\mathcal{O}_X(U)$-module structure on $k(t)$ is given by standard multiplication in $k(t)$. However, $\mathscr{F}$ is not quasicoherent, as for $U = D(t)$, (again by the same homework problem coupled with flatness of $A[u^{-1}]$ for any ring A and multiplicative system u) $\mathscr{F}(U) = 0 \neq \mathcal{O}_X(U) \otimes \mathscr{F}(X) = k(t)$. $\square$

b.)

Proposition. *We let $X = \mathring{A}^1_k$ and $\mathscr{F}$ the skyscraper sheaf at $[\langle 0\rangle]$ with $k[t]$-module $k(t)$. Then $\mathscr{F}$ is quasicoherent.*

Proof. For any nonempty open set U, we have that $[\langle 0\rangle] \in U$, so $\mathscr{F}(U) = k(t) = \mathcal{O}_X(U) \otimes k(t) = \widetilde{k(t)}$. $\square$

[1] The actual details of r are not important here; it is pretty much a notational tool only.

7.)

Proposition (Heartshorne II.5.2(c))**.** *For an A-module M, we denote the sheaf associated to M on* $\operatorname{Spec} A$ *by $\tilde{M}$ or alternatively $(M)^\sim$ depending on clarity. Then, for $\{M_i\}$ a family of A-modules, $\bigoplus \tilde{M}_i \cong (\bigoplus M_i)^\sim$.*

Remark. Rather than use Heartshorne's definition, we take the definition of "sheaf associated to M" to be the one given in Vakil, defined over the basic open sets $D(f)$ as $\tilde{M}(D(F)) := M_f := A_f \otimes_A M$.

We use the following essential lemma:

Lemma. *We let R be a commutative unital ring.*[2] *For N, $\{M_i\}_i$ R-modules, $N \otimes_R (\oplus_i M_i) = \oplus_i (N \otimes_R M_i)$.*

Proof (of proposition). We recall that sheaves are uniquely recoverable from their data on distinguished open sets $D(f)$, $f \in A$. As such, we shall show equivalence only on basic opens $D(f)$; equivalence on basic opens then implies equivalence on arbitrary open sets U. We then have that $(\oplus M_i)^\sim(D(f)) = \mathcal{O}_X(D(f)) \otimes (\oplus M_i) = \oplus_i (\mathcal{O}_X(D(f)) \otimes M_i) = \oplus_i (\tilde{M}_i)(D(f))$

□

[2]as all rings are, of course