

You may *not* use your books, notes, graphing calculator, phones or any other internet devices on this exam. Please show all work clearly and legibly.

Problem	Points	Score
1	10	
2	10	
Total:	20	

1. (10 points) Find an equation for a tangent line to the curve defined by  $y^2 - x^2 - 3x + 6 = 0$  at the point  $(x_0, y_0) = (2, -2)$ .

*Response.* The tangent line to the curve is given by

$$T(x) = m(x - x_0) + y_0 \text{ where } m = \left( \frac{dy}{dx} \right) \Big|_{(x_0, y_0)}$$

First we differentiate with respect to both sides:

$$\begin{aligned} \frac{d}{dx}(y^2 - x^2 - 3x + 6) &= \frac{d}{dx}(0) \\ 2y \frac{dy}{dx} - 2x - 3 &= 0 \end{aligned}$$

Now we solve for  $\frac{dy}{dx}$ :

$$\begin{aligned} 2y \frac{dy}{dx} - 2x - 3 &= 0 \\ \implies 2y \frac{dy}{dx} &= 2x + 3 \\ \implies \frac{dy}{dx} &= \frac{2x + 3}{2y} \end{aligned}$$

So to find  $m = \left( \frac{dy}{dx} \right) \Big|_{(x_0, y_0)}$ , we just need to plug in  $(x_0, y_0) = (2, -2)$ :

$$\left( \frac{dy}{dx} \right) \Big|_{(x_0, y_0)} = \left( \frac{dy}{dx} \right) \Big|_{(2, -2)} = \frac{2(2) + 3}{2(-2)} = \frac{-7}{4}$$

From this we get that

$$T(x) = \frac{-7}{4}(x - 2) - 2 = \frac{-7}{4}x + \frac{3}{2}$$

□

2. Find  $y'$  (in terms of  $x$  and  $y$ ) for:

(a) (5 points)  $y = x^{\sqrt{x}}$

*Response.* We first rewrite (note that this step can be skipped) then apply natural log to both sides.

$$\begin{aligned} y &= \left(e^{\ln(x)}\right)^{\sqrt{x}} \\ &= e^{\ln(x)\sqrt{x}} \\ \implies \ln(y) &= \ln(x)\sqrt{x} \end{aligned}$$

We now differentiate each side w/r/t  $x$ :

$$\frac{d}{dx} \ln(y) = \frac{d}{dx} (\ln(x)\sqrt{x})$$

To compute  $\frac{d}{dx} \ln(y)$ , we use the chain rule. We let  $f(x) = \ln(x)$  and  $g(x) = y$ . Then,  $f'(x) = \frac{1}{x}$  and  $g'(x) = y'$ , so

$$\frac{d}{dx} \ln(y) = \frac{d}{dx} f(g(x)) = g'(x)f'(g(x)) = \frac{y'}{y}.$$

We compute the derivative of the other side using the product rule to yield

$$\begin{aligned} \frac{y'}{y} &= \frac{\sqrt{x}}{x} + \frac{\ln(x)}{2\sqrt{x}} \\ &= \frac{2 + \ln(x)}{2\sqrt{x}} \\ \implies y' &= \left(\frac{2 + \ln(x)}{2\sqrt{x}}\right) y \end{aligned}$$

Finally, to put it in terms of  $x$ , we substitute  $y = x^{\sqrt{x}}$  and yield

$$y' = \left(\frac{2 + \ln(x)}{2\sqrt{x}}\right) x^{\sqrt{x}}$$

□

(b) (5 points)  $x = \cos(y^2)$

*Response.* Don't overthink it! We can put this one in terms of  $x$  and  $y$ , so let's just differentiate each side:

$$\frac{d}{dx} x = \frac{d}{dx} \cos(y^2)$$

We use the chain rule on the right: we let  $f(x) = \cos(x)$  and  $g(x) = y^2$ . Then,  $f'(x) = -\sin(x)$  and  $g'(x) = 2yy'$ . Thus,

$$\frac{d}{dx} \cos(y^2) = \frac{d}{dx} f(g(x)) = g'(x)f'(g(x)) = 2yy'(-\sin(y^2)).$$

Now we have from differentiating both sides above:

$$\begin{aligned} 1 &= -2yy' \sin(y^2) \\ \implies \frac{-1}{2y \sin(y^2)} &= y' \end{aligned}$$

(Depending on what you did, there are several possible correct answers to this in mixed terms of  $x$  and  $y$  which are not obviously the same.)  $\square$