1.)

For each of the functions listed, **(i)** identify any points of discontinuity and **(ii)** say whether it is continuous from the left, right, or neither.

a.)

$$f(x) = \frac{x+4}{2x^2+7x-4}$$

Response. In Stewart, §2.5, theorem 4, it is stated that if $g(x)$ and $h(x)$ are continuous at a, then $\frac{g(x)}{h(x)}$ is continuous at a *unless* $h(a) = 0$. Polynomials are continuous everywhere, so we need only find the zeros of $2x^2 + 7x - 4$. By factoring $2x^2 + 7x - 4 = (2x - 1)(x + 4)$, we see that f is continuous everywhere except $x = \frac{1}{2}, -4$. As $f(\frac{1}{2})$ and $f(4)$ are undefined, f cannot possible be continuous (from the right or from the left, for that matter) at either point as the definition of continuity ($f(x)$ is continuous at $a \iff \lim_{x\to a} f(x) = f(a)$) requires $f(a)$ to be defined. □

b.)

$$f(x) = \frac{x^2-1}{x^2+1}$$

Response. $x^2 + 1$ is never 0 for any real x. Thus, $f(x)$ is continuous everywhere! □

c.)

$$f(x) = \begin{cases} \sqrt{1-x} & x \leq 1 \\ \frac{2x^3-2}{x-2} & 1 < x \leq 3 \\ 52\sin(\pi x) & 3 < x \end{cases}$$

Response. There are two steps to checking for discontinuities here: check for any discontinuities within each "leg" of the piecewise, then check for discontinuities at the "seams" (the points at which we change from one function to another). $\sqrt{1-x}$ is continuous everywhere it is defined except at $x = 1$ where it is continuous only from the left—however, we shouldn't take that to mean that f has a discontinuity at 1, as 1 is one of the "seams" of the piecewise, and we'll handle it later. $\frac{2x^3-2}{x-2}$ is undefined at $x = 2$, which is in the domain of the second leg of the piecewise, so we have a discontinuity (not continuous from right *or* left) at $x = 2$. Finally, $52\sin(\pi x)$ is continuous everywhere. What is left to check is the two "seams" at $x = 1$ and $x = 3$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{1-x} = 0 = f(1),$$

so f is at least continuous from the left there.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{2x^3-2}{x-2} = 0 = f(1)$$

so it is continuous from the right as well. Thus, f is continuous at 1. On the other hand,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{2x^3-2}{x-2} = 52 = f(3)$$

so f is continuous from the left at 3, but

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 52\sin(\pi x) = 52\sin(3\pi) = 0 \neq f(3),$$

so f is *not* continuous from the right at 3.

To summarize, we found two discontinuities: f is not continuous from either direction at $x = 2$, and f is continuous only from the left at $x = 3$. □

3.)

Evaluate the following limits.

a.)

$$\lim_{x\to 2}\frac{1}{x-2}$$

Response. $\displaystyle\lim_{x\to 2^-}\frac{1}{x-2} = -\infty$, $\displaystyle\lim_{x\to 2^+}\frac{1}{x-2} = \infty$, so $\displaystyle\lim_{x\to 2}\frac{1}{x-2}$ does not exist. □

b.)

$$\lim_{x\to 2}\frac{1}{(x-2)^2}$$

Response. $\displaystyle\lim_{x\to 2^-}\frac{1}{(x-2)^2} = \infty$, $\displaystyle\lim_{x\to 2^+}\frac{1}{(x-2)^2} = \infty$, so $\displaystyle\lim_{x\to 2}\frac{1}{(x-2)^2} = \infty$ □

c.)

$$\lim_{x\to\infty}\frac{2+5x^2}{1+x-x^2}$$

Response. Multiply through by $\dfrac{1/x^2}{1/x^2} = 1$ to get $\displaystyle\lim_{x\to\infty}\frac{\frac{2}{x^2}+5}{\frac{1}{x^2}+\frac{1}{x}-1} = -5$ as each c/x^r term goes to 0. □

d.)

$$\lim_{x\to-\infty}\frac{1-x^6}{1+x^5}$$

Response. Multiply through by $\dfrac{1/x^5}{1/x^5} = 1$ to get $\displaystyle\lim_{x\to-\infty}\frac{\frac{1}{x^5}-x}{\frac{1}{x^5}+1}$. Note $\displaystyle\lim_{x\to-\infty}(\frac{1}{x^5}-x) = \infty$, while $\displaystyle\lim_{x\to-\infty}\frac{1}{x^5}+1 = 1$. Thus, $\displaystyle\lim_{x\to-\infty}\frac{1-x^6}{1+x^5} = \infty$. □

e.)

$$\lim_{x\to\infty}\frac{\sqrt{1+4x^6}}{2-x^3}$$

Response. Multiply through by $1 = \frac{1/\sqrt{x^6}}{1/x^3}$ to get $\displaystyle\lim_{x\to\infty}\frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1}$. Note that $\displaystyle\lim_{x\to\infty}\sqrt{\frac{1}{x^6}+4} = \sqrt{\lim_{x\to\infty}\frac{1}{x^6}+4} = \sqrt{4} = 2$, while $\displaystyle\lim_{x\to\infty}\frac{2}{x^3}-1 = -1$. Thus, $\displaystyle\lim_{x\to\infty}\frac{\sqrt{1+4x^6}}{2-x^3} = -2$ □

f.)

$$\lim_{x\to\infty}e^{-x}\sin^2(x^2)$$

Response. Squeeze theorem! For any x, it is the case that $0 \le \sin^2(x^2) \le 1$, and $e^{-x} > 0$ for any x, so we can multiply through and now now have that $0 = 0 * e^{-x} \le e^{-x}\sin^2(x^2) \le e^{-x}$. The squeeze theorem now says $\displaystyle\lim_{x\to\infty} 0 \le \lim_{x\to\infty} e^{-x}\sin^2(x^2) \le \lim_{x\to\infty} e^{-x}$. Those two outer limits are 0, so we must have $\displaystyle\lim_{x\to\infty} e^{-x}\sin^2(x^2) = 0$. □