Practice Test for Midterm I

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Limits, finite and infinite

1.)

compute the following limits:

a.)

$\lim_{x\to 2} \frac{x^2-4}{x^2+3x-10} =$

Response. First, let's factor:

$$\lim_{x\to 2} \frac{x^2-4}{x^2+3x-10} = \lim_{x\to 2} \frac{(x+2)(x-2)}{(x+5)(x-2)} = \lim_{x\to 2} \frac{x+2}{x+5}$$

Now, as our denominator does not equal 0 when we plug our value in, we can do just that to see that the limit is $\frac{2+2}{2+5} = \frac{4}{7}$ $\square$

b.)

$\lim_{x\to 0} \frac{x^2-2x+1}{x^3-6} =$

Response. This one is easy–if we plug in $x = 0$, our denominator is nonzero to start with! Thus,

$$\lim_{x\to 0} \frac{x^2-2x+1}{x^3-6} = \frac{0^2-2*0+1}{0^3-6} = \frac{-1}{6}$$

$\square$

c.)

$\lim_{x\to -4} \frac{|x^2+8x+12|}{x+2} =$

Response. Here, we notice that $f(x) = x^2+8x+12 = (x+2)(x+6)$ is positive when $x < -6$, negative when $-6 < x < -2$ and positive again when $x > -2$. As we're taking a limit at -4, we can focus our attention at $-6 < x < -2$ (because we only care about the area immediately surrounding $x = -4$), so in the region we care about, $|f(x)| = -f(x)$. The denominator is non-zero when we plug in $x = -4$, so once we know what we're looking at up top, we are golden. Putting it together:

$$\lim_{x\to -4} \frac{|x^2+8x+12|}{x+2} = \lim_{x\to -4} \frac{-(x^2+8x+12)}{x+2} = \frac{-(16-32+12)}{(-4+2)} = \frac{4}{-2} = -2$$

$\square$

2.)

Compute more limits

a.)

$\lim_{x\to 1^-} \frac{x^2+2}{x^2-1} =$

Response. Here, we cannot make relevant calculations, and $1^2+2 = 3 \neq 0$ while $1^2-1=0$, so we are looking at an infinite limit of some sort. We're approaching from the left, so we may assume $x<1$. Then, $x^2<1$, so the denominator is negative. However, x^2+2 is positive for any real number x, so our only possible answer is $\lim_{x\to 1^-} \frac{x^2+2}{x^2-1} = -\infty$. In particular, because we have a 0 denominator with a finite nonzero numerator, we must have a vertical asymptote—so all that was left for us to do was figure out whether it was positive or negative. □

b.)

$\lim_{x\to\infty} \frac{cos^2(x)}{x+3} =$

Response. Wellp, you know what time it is: squeeze theorem time!:

Recall:

$$0 \le \cos^2(x) \le 1$$

Now, we can divide through by $x+3$ (note this does NOT reverse the inequality because $x+3$ is positive as $x\to+\infty$), and we get:

$$\frac{0}{x+3} = 0 \le \frac{\cos^2(x)}{x+3} \le \frac{1}{x+3}$$

Now, we add in our limits!

$$0 = \lim_{x\to\infty} 0 \le \lim_{x\to\infty} \frac{\cos^2(x)}{x+3} \le \lim_{x\to\infty} \frac{1}{x+3} = 0$$

Thus, the limit is zero! □

c.)

$\lim_{x\to-\infty} \frac{x^3-2x+2}{4x^3-6} =$

Response. This is a pretty standard horizontal asymptote question. Let's multiply through numerator and denominator by $1/x^3$

$$\lim_{x\to-\infty} \frac{x^3-2x+2}{4x^3-6} = \lim_{x\to-\infty} \frac{1-2x^{-2}+2x^{-3}}{4-6x^{-3}}$$

Now, as $x\to\pm\infty$, $x^{-r}\to 0$, so we can clear out our negative powers.

$$\begin{aligned}\lim_{x\to-\infty} \frac{1-2x^{-2}+2x^{-3}}{4-6x^{-3}} &= \frac{1-2(0)+2(0)}{4-6(0)} \\ &= \frac{1}{4}\end{aligned}$$

□

Continuity

3.)

Identify the points x at which $f(x)$ is *not* continuous.

$$f(x) = \begin{cases} x^2 & x < -5 \\ \frac{1}{x^2-9} & -5 \le x < 0 \\ \frac{x^2-1}{9e^x} & 0 \le x \end{cases}$$

Response. SO: First, we need to check each leg of the piecewise for discontinuities. x^2 is continuous everywhere by virtue of being a polynomial. Same story for $x^2 - 1$, and as $9e^x \neq 0$ for all x, we have that $\frac{x^2-1}{9e^x}$ is continuous everywhere as well. However, $\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)}$ is NOT defined at $x = -3$ (as well as $x = 3$, but at $x = 3$, we've moved on to a different part of the piecewise). This gives us our first discontinuity. What is left is to check at the "seams", that is $x = -5$ and $x = 0$. Now,

$$\lim_{x\to -5^-} f(x) = (-5)^2 = 25$$

but,

$$\begin{aligned} f(-5) = \lim_{x\to -5^+} f(x) &= \frac{1}{(-5)^2 - 9} \\ &= \frac{1}{16} \neq 25 \end{aligned}$$

As the left- and right-side limits do not agree at $x = -5$, we can conclude there is a discontinuity there as well.

Finally, at $x = 0$, we have:

$$\lim_{x\to 0^-} f(x) = \frac{1}{0^2 - 9} = \frac{-1}{9}$$

and

$$\lim_{x\to 0^+} f(x) = \frac{0^2 - 1}{9e^0} = \frac{-1}{9} = f(0)$$

As $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$, we do not have a discontinuity there. We can conclude that the only points of discontinuity are $x = -5$ and $x = -3$. □

4.)

Find a and b such that $f(x)$ is continuous everywhere.

$$f(x) = \begin{cases} ax + b & x < 0 \\ x^2 - a & 0 \le x < 2 \\ x^3 & 2 \le x \end{cases}$$

Response. First, notice that each of $ax+b$, $x^2 - a$, and x^3 are continuous everywhere so we again only need to focus on the "seams," now at $x = 0$ and $x = 2$. Let's start with the latter. We need that $f(2) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, so we write

$$\begin{aligned} f(x) &= 2^3 = 8 \\ \lim_{x\to 2^+} f(x) &= 2^3 = 8 \\ \lim_{x\to 2^-} f(x) &= 2^2 - a = 4 - a \end{aligned}$$

Thus, we now have that $8 = 4 - a$, or $a = -4$. That takes care of one of them! Next, let's look at $x = 0$. We need that $f(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$, so we write

$$\begin{aligned} f(0) &= 0^2 - a = 0^2 + 4 = 4 \\ \lim_{x\to 0^+} f(x) &= 0^2 - a = 4 \\ \lim_{x\to 0^-} f(x) &= a(0) + b = b \end{aligned}$$

Thus, as the limits need to match up, we have that $b = 4$. This gives our final answer of $a = -4$, $b = 4$.

□

5.)

Show that $f(x)$ achieves the value $f(c) = 1/2$ for some $0 \leq c \leq 5$. State which theorem you are using.

$$f(x) = \frac{1}{x-2}$$

Response. Well, we could just *find* c, but that's not the point of the question—let's just prove it exists instead.

The intermediate value theorem tells us that if $a < b$ and $f(a) < y^* < f(b)$ (or $f(a) > y^* > f(b)$) AND f is continuous on $[a, b]$, then there must be some c such that $a < c < b$ and $f(c) = y^*$. This is an OBVIOUS STATEMENT, once you parse what it means—but that's the hard part! Draw a picture or look at the pictures on page 122 (section 2.5) if you're confused.

ANYWAY: we want to find a, b such that f is continuous on $[a, b]$ and $f(a)$ is on the opposite side of $1/2$ from $f(b)$. Note that f has a discontinuity at $x = 2$, so we need that a and b are on the same side of $x = 2$. By trial and error, we come up with $a = 2.5$ and $b = 5$. Then $f(a) = 2$ and $f(b) = \frac{1}{3}$. As $\frac{1}{3} < \frac{1}{2} < 2$, the intermediate value theorem tells us that our desired c does indeed exist, and we are done!

Of course we could have just solved $1/(x-2) = 1/2$ to get $c = 4$, but that's no fun...

□

Definition of Derivative

6.)

Use the definition of the derivative to compute $f'(x)$. **Answers given only for parts a,b—you should be able to use the rules of differentiation to check your result for part c**

a.)

$$f(x) = 5x - 3$$

Response. This one's real easy.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(5(x+h)-3)-(5x-3)}{h} \\ &= \lim_{h\to 0} \frac{5x+5h-3-5x+3}{h} \\ &= \lim_{h\to 0} \frac{5h}{h} = 5. \end{aligned}$$

□

b.)

$$f(x) = \sqrt{1-2x}$$

Response. Well, I guess we should write out the definition of the derivative...

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{1-2(x+h)} - \sqrt{1-2x}}{h}$$

Now, we multiply by conjugate to simplify things up on the top.

$$\begin{aligned} \lim_{h\to 0} \frac{\sqrt{1-2(x+h)} - \sqrt{1-2x}}{h} &= \lim_{h\to 0} \frac{\sqrt{1-2(x+h)} - \sqrt{1-2x}}{h} \left(\frac{\sqrt{1-2(x+h)} + \sqrt{1-2x}}{\sqrt{1-2(x+h)} + \sqrt{1-2x}} \right) \\ &= \lim_{h\to 0} \frac{(1-2(x+h)) - (1-2x)}{h(\sqrt{1-2(x+h)} + \sqrt{1-2x})} \\ &= \lim_{h\to 0} \frac{1-2x-2h-1+2x}{h(\sqrt{1-2(x+h)} + \sqrt{1-2x})} \\ &= \lim_{h\to 0} \frac{-2h}{h(\sqrt{1-2(x+h)} + \sqrt{1-2x})} \\ &= \lim_{h\to 0} \frac{-2}{\sqrt{1-2(x+h)} + \sqrt{1-2x}} \end{aligned}$$

Now, since plugging in $h = 0$ does not make our denominator 0, we can do exactly that.

$$\begin{aligned} f'(x) &= \frac{-2}{\sqrt{1-2(x+0)}+\sqrt{1-2x}} \\ &= \frac{-2}{2\sqrt{1-2x}} \\ &= \frac{-1}{\sqrt{1-2x}} \end{aligned}$$

□

Rules of Differentiation

7.)

Compute the derivative f' of $f(x)$. Use that to find the equation for $T(x)$, the tangent line to $y = f(x)$ at $x = x_0$

a.)

$$f(x) = x^3 + 10x \qquad\qquad x = 3$$

Response. Ok, finding $f'(x)$ is a pretty simple application of the power rule: $f'(x) = 3x^2 + 10$. Now, we need to find the equation for the tangent line $T(x)$. In general, for T being the tangent at $x = x_0$, $T(x) = f'(x_0)(x - x_0) + f(x_0)$. Now, as $x_0 = 3$, $f(x_0) = 3^3 + 10 * 3 = 57$, and $f'(3) = 3(3^2) + 10 = 37$. Thus, we get the formula

$$T(x) = 37(x - 3) + 57.$$

□

b.)

$$f(x) = (x^2 + 3x)e^x \qquad\qquad x_0 = 2$$

Response. I believe it is time to catch up with our friend the product rule. We write $f(x) = g(x)h(x)$ with $g(x) = x^2+3x$ and $h(x) = e^x$. Then, $g'(x) = 2x + 3$ and $h'(x) = e^x$. The product rule gives $f'(x) = g'(x)h(x) + g(x)h'(x)$, so substituting some stuff in gives us

$$\begin{aligned} f'(x) &= (2x + 3)e^x + (x^2 + 3x)e^x \\ &= e^x(x^2 + 5x + 3) \end{aligned}$$

Now, for $x_0 = 2$, we use the same general formula $T(x) = f'(x_0)(x - x_0) + f(x_0)$. We find $f(2) = (2^2 + 3(2))e^2 = 10e^2$ and $f'(2) = e^2(2^2 + 5(2) + 3) = 17e^2$. Thus, our formula is $T(x) = 17e^2(x - 2) + 10e^2$. □

c.)

$$f(x) = \frac{x^2 \cos(x)}{x+1} \qquad\qquad x_0 = \pi$$

I'm not going to write up the whole procedure, but here are solutions if you'd like to check your answer. This problem is perhaps a couple shades more algebraically-intensive than the midterm will likely be.

$$f'(x) = \frac{(x^2 + 2x)\cos(x) - (x^3 + x^2)\sin(x)}{(x+1)^2} \tag{1}$$

$$T(x) = \left(\frac{1}{(1+\pi)^2} - 1\right)(x - \pi) - \frac{\pi^2}{1+\pi} \tag{2}$$

$$= -\frac{\pi^2 + 2\pi}{(\pi+1)^2}(x - \pi) - \frac{\pi^2}{1+\pi} \tag{3}$$

8.)

Compute the derivative f' of $f(x)$.

a.)

$$f(x) = \sin(x^2)$$

Response. Let $g(x) = \sin(x)$ and $h(x) = x^2$. Then, $g'(x) = \cos(x)$ and $h'(x) = 2x$. Chain rule: $f(x) = g(h(x))$, so $f'(x) = h'(x)g'(h(x)) = 2x\cos(x^2)$. □

b.)

$$f(x) = e^{\cos(x^3 - x)}$$

Response. Let $g(x) = e^x$ and $h(x) = \cos(x^3 - x)$. $g'(x) = e^x$, but we'll have to use the chain rule to find the derivative of $h(x)$. Let $j(x) = \cos(x)$ and $k(x) = x^3 - x$. Then, $j'(x) = -\sin(x)$ and $k'(x) = 3x^2 - 1$. Since $h(x) = j(k(x))$, $h'(x) = k'(x)j'(k(x)) = -(3x^2 - 1)\sin(x^3 - x)$. Now, we can compute that since $f(x) = g(h(x))$, $f'(x) = h'(x)g'(h(x)) = -(3x^2 - 1)\sin(x^3 - x)e^{\cos(x^3 - x)}$. □

c.)

$$f(x) = \frac{1}{\sqrt{x^3 - 8}}$$

Response. Let $g(x) = x^{-1/2}$ and $h(x) = x^3 - 8$. $g'(x) = \frac{-1}{2x^{3/2}}$ and $h'(x) = 3x^2$. Since $f(x) = g(h(x))$, $f'(x) = h'(x)g'(h(x)) = \frac{-3x^2}{2(x^3-8)^{3/2}}$ □

9.)

Let $f(x) = g(h(x))$. Compute the following using the below table of values or state that insufficient information is given with justification

x	g(x)	h(x)	g'(x)	h'(x)
1	7	3	8	2
2	6	1	-2	4
3	-1	4	-8	-2

a.)

$f'(2) =$

Response. Note $f'(x) = h'(x)g'(h(x))$, so $f'(2) = h'(2)g'(h(2))$. $h(2) = 1$, so this is $f'(2) = h'(2)g'(1) = 4 * 8 = 32$. □

b.)

$f(2) =$

Response. $f(2) = g(h(2))$. $h(2) = 1$, so $f(2) = g(1) = 7$.
(honestly this question was here to make sure you're reading carefully!) □

c.)

$f'(3) =$

Response. $f'(3) = h'(3)g'(h(3))$. $h(3) = 4$, so in order to know this, we need to know what $g'(4)$ is. But how on earth would we know that????? (insufficient information is given). □

d.)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)\Bigg|_{x=1} =$$

Response. The quotient rule says...

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

so evaluating it at $x = 1$ gives us

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)\Bigg|_{x=1} = \frac{g(1)f'(1) - f(1)g'(1)}{g(1)^2}$$

We know $g(1) = 7$, $f(1) = g(h(1)) = g(3) = -1$, and $g'(1) = 8$. We need to compute $f'(1) = g'(h(1))h'(1) = g'(3)h'(1) = -16$. Thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)\Bigg|_{x=1} &= \frac{g(1)f'(1) - f(1)g'(1)}{g(1)^2}\\ &= \frac{7 * (-16) - (-1)(8)}{(7)^2}\\ &= \frac{-104}{49} \approx -2.122\end{aligned}$$

$\square$