Math 1271 - Lectures 010 and 030
Fall 2017
Midterm 2 - A
11/09/17
Time Limit: 50 Minutes

Name (Print): David DeMarc (KEY)

Section 030 & 050 (518)

You may *not* use your books, notes, graphing calculator, phones or any other internet devices on this exam. Please show all work clearly and legibly.

Problem	Points	Score
1	15	
2	15	
3	20	
4	15	
5	20	
6	15	
Total:	100	

1. (15 points) Consider the equation $y = \sqrt{x-2}$

   (a) Compute Δy and dy at $x_0 = 3$ and $dx = \Delta x = 0.02$.

   (b) Use the previous part to compute an approximation to the number $\sqrt{1.02}$.

a) let $y = f(x) = \sqrt{x-2}$. Then, $f(x) = g(h(x))$ where

$g(x) = \sqrt{x}$ $\quad g'(x) = \frac{1}{2\sqrt{x}}$

$h(x) = x-2$ $\quad h'(x) = 1$

so $f'(x) = h'(x)\, g'(h(x)) = \frac{1}{2\sqrt{x-2}}$

$\Delta y = f(x_0 + \Delta x) - f(x_0) = f(3.02) - f(3)$

$= \sqrt{1.02} - \sqrt{1} \approx .00995$

$dy = f'(x_0)\, dx = \frac{1}{2\sqrt{3-2}}(.02) = \frac{1}{2}(.02) = .01$

b) $\sqrt{1.02} = f(3.02) \approx f(3) + dy = \sqrt{3-2} + .01$

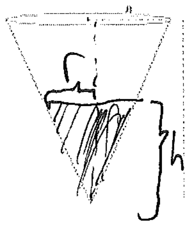
$= 1.01$

2. (15 points) A water tank has the shape of an inverted circular cone with base-radius $R = 3$m and height $H = 9$m. If water is being pumped into the tank at a rate of 3 m^3/min find:

   (a) the rate at which the water level is rising when the water is 6m deep.

   (b) the rate at which the radius of the water level is increasing when the water is 6m deep.

   Hint: The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.



note by similar triangles,

$$\frac{r}{3} = \frac{h}{9} \Rightarrow 3r = h$$

HAVE	WANT	WHEN:
(1) $V = \frac{1}{3}\pi r^2 h$	(a) $\frac{dh}{dt}$	$h = 6m$
(2) $\frac{dV}{dt} = 3\left(m^3/min\right)$		$\left(\Downarrow \text{ by } (3) \; r = 2m\right)$
(3) $h = 3r$	(b) $\frac{dr}{dt}$	

$$(1) \Rightarrow V = \frac{1}{3}\pi r^2 h \overset{(3)\downarrow}{=} \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{\pi h^3}{27}$$

$$\Rightarrow \frac{dV}{dt} = \frac{3\pi h^2}{27}\frac{dh}{dt} = \frac{\pi h^2}{9}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{dV}{dt}\cdot\frac{9}{\pi h^2} = (3m^3/min)\cdot\frac{9}{\pi h^2}$$

$$= \frac{27 \; m^3/min}{\pi h^2}$$

(a) $$\boxed{\Rightarrow \left.\frac{dh}{dt}\right|_{h=6m} = \frac{27\, m^3/min}{\pi (6m)^2} = \frac{3}{4\pi}\; m/min}$$

(b) (3) $h = 3r$

$$\Downarrow$$

$$\frac{dh}{dt} = 3\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{3}\frac{dh}{dt}$$

$$\Downarrow$$

$$\left.\frac{dr}{dt}\right|_{h=6m} = \frac{1}{3}\left.\frac{dh}{dt}\right|_{h=6m} = \frac{1}{4\pi}\; m/min.$$

3. (20 points) (a) State the definitions of critical and inflection numbers (or points).

A critical number of a function $f(x)$ is a value c in the domain of f s.t. $f'(c)=0$ or DNE.

An inflection point is a point in the domain of f at which f changes from concave up to concave down or vice-versa

(b) State the first and second derivative tests

FD test: $\boxed{\text{Suppose } c \text{ is a critical \# of } f.}$
if there is some $a<c$ & $b>c$ s.t. $f'(x)<0$ for $a<x<c$ & $f'(x)>0$ for $c<x<b$, then c is a local minimum.
if the inequalities are reversed, c is then a local max.

SD Test: if $f''(c)>0$, then c is a local minimum.
if $f''(c)<0$, then c is a local max.

(c) Consider the function $f(x)=\ln(x^2+4)$.

(i) Find the intervals of increase and decrease for $f(x)$.

(ii) Find the critical points of $f(x)$ and identify any local minima or maxima.

(iii) Find the intervals of concavity.

(iv) Find the inflection points of f.

$f(x)=h(g(x)) \Rightarrow f'(x)=g'(x)h'(g(x))$

$g(x)=x^2+4 \quad g'(x)=2x$

$h(x)=\ln(x) \quad h'(x)=1/x$

$=\dfrac{2x}{x^2+4}$

(i) note $x^2+4>0$ for all x. Thus $f'(x)>0 \Leftrightarrow 2x>0 \Leftrightarrow x>0$,
thus, f is increasing on $(0,\infty)$ and dec. on $(-\infty,0)$.

(ii) As $x^2+4>0$ for all x, f & f' are def'd for all x.
Thus all C.P. are of type $f'(x)=0$. We solve:

$0=\dfrac{2x}{x^2+4} \Rightarrow 0=2x \Rightarrow x=0.$ by FD-test, $x=0$ is a local min.

(iii). $\boxed{\begin{matrix} j(x)=2x & j'(x)=2 \\ k(x)=x^2+4 & k'(x)=2x \end{matrix}}$ $f'(x)=\dfrac{j(x)}{k(x)} \Rightarrow f''(x)=\dfrac{j'(x)k(x)-j(x)k'(x)}{k(x)^2}$

$=\dfrac{2(x^2+4)-(2x)^2}{(x^2+4)^2}=\dfrac{8-4x^2}{(x^2+4)^2}$

$=\dfrac{4(2-x^2)}{(x^2+4)^2}$

solve: $f''(x)=\dfrac{4(2-x^2)}{(x^2+4)^2}>0 \Rightarrow 4(2-x^2)>0$

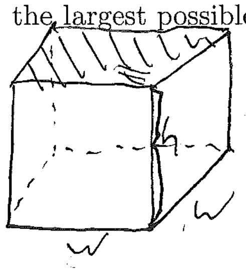
$\Rightarrow 2-x^2>0 \Rightarrow 2>x^2$

$\Rightarrow -\sqrt{2}<x<\sqrt{2}$

$\Rightarrow$ concave up on $(-\sqrt{2},\sqrt{2})$; concave down on $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$

(iv) POI: $x=\pm\sqrt{2}$

4. (15 points) If 200 cm^2 of material is available to make a closed box with a square base, find the largest possible volume.



WANT TO MAXIMIZE: $V = w^2 h$ (1)

constraint: $w \geq 0$, $h \geq 0$

SA: $w^2 + 4wh = 200\,\text{cm}^2$ (2)

$$(2) \Rightarrow w^2 + 4wh = 200\,\text{cm}^2 \Rightarrow 200 - w^2 = 4wh \Rightarrow h = \frac{200 - w^2}{4w}$$

$$h = 50w^{-1} - \frac{w}{4} \quad (3)$$

$$(1) \Rightarrow V = w^2 h \overset{(3)}{\Rightarrow} V = w^2\left(50w^{-1} - \frac{w}{4}\right)$$

$$\Rightarrow V(w) = 50w - \frac{w^3}{4}$$

find C.P.'s: On interval $w \in [0, \infty)$

$$V'(w) = 50 - \frac{3}{4}w^2$$

set $:= 0$:

$$0 = 50 - \frac{3}{4}w^2$$

$$\Rightarrow \frac{3}{4}w^2 = 50 \Rightarrow w^2 = \frac{200}{3}$$

$$\Rightarrow w = \sqrt{\frac{200}{3}} = 10\sqrt{2/3}$$

Only C.P. & $V(0) = 0 \Rightarrow$ maximizing $w_* = 10\sqrt{2/3} = \frac{10\sqrt{6}}{3}\,\text{cm}$

$$(3)\quad h_* = \frac{50}{w_*} - \frac{w_*}{4} = \frac{50}{10\sqrt{2/3}} - \frac{10\sqrt{2/3}}{4}$$

$$= 5\sqrt{3/2} - \frac{5}{\sqrt{6}} = \frac{15}{\sqrt{6}} - \frac{5}{\sqrt{6}} = \frac{10}{\sqrt{6}} = \frac{10\sqrt{6}}{6}\,\text{cm} = \frac{5\sqrt{6}}{3}$$

$$\Rightarrow V_* = h_* w_*^2 = \left(\frac{5\sqrt{6}}{3}\,\text{cm}\right)\left(\frac{10\sqrt{6}}{3}\,\text{cm}\right)^2 = \frac{5\sqrt{6}}{3} \cdot \frac{600}{9}\,\text{cm}^2 \quad \boxed{\frac{1000\sqrt{6}}{9}\,\text{cm}^3}$$

5. (20 points) Compute the value of the following limits:

(a)

$$\lim_{x\to\infty} x^2e^{-x^2} = \lim_{x\to\infty} \frac{x^2}{e^{x^2}} \quad \left(\frac{\infty}{\infty}\right)$$

L'hopital

$$= \lim_{x\to\infty} \frac{2x}{2x\,e^{x^2}} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = \lim_{x\to\infty} e^{-x^2} = 0$$

(b)

$$\lim_{x\to\infty} x^{1/x}$$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} \left(e^{\ln x}\right)^{1/x} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^{\left(\lim_{x\to\infty}\frac{\ln x}{x}\right)}$$

$$\lim_{x\to\infty} \left(\frac{\ln x}{x}\right) \quad \text{type } \frac{\infty}{\infty}$$

L'H

$$= \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} x^{-1} = 0.$$

6. (15 points) Find $f(t)$ where

$$f''(t) = \sqrt{t} - \sin(t),\ f(0) = 1,\ f'(0) = -1$$

$$= t^{1/2} - \sin(t)$$

Then, $f'(t) = \frac{2}{3}t^{3/2} + \cos(t) + c$

$$f'(0) = \frac{2}{3}0^{3/2} + \cos(0) + c = 1$$

$$= 1 + c \Rightarrow c = 0.$$

$$\Rightarrow f'(t) = \frac{2}{3}t^{3/2} + \cos(t)$$

$$\Rightarrow f(t) = \frac{4}{15}t^{5/2} + \sin(t) + d$$

$$f(0) = \frac{4}{15}(0)^{5/2} + \sin(0) + d = -1$$

$$= d \Rightarrow d = -1$$

$$\boxed{\Rightarrow f(t) = \frac{4}{15}t^{5/2} + \sin(t) - 1.}$$