1 Declarative system

$e,\ v,\ w,\ f,\ g,\ h$	$::=$			Ground value terms
	$\mid$	x		
	$\mid$	$\langle\rangle$		
	$\mid$	$v\ w$		
	$\mid$	$\mathbf{refl}$		
	$\mid$	$\lambda x.\, v$	bind x in v	
	$\mid$	$\mathbf{rec}_{eq}^{x.p.A}(e, v)$	bind x in A	
			bind p in A	
	$\mid$	$v : A$		
	$\mid$	$[w/x]v$	M	
	$\mid$	(v)	S	

$X,\ A,\ B,\ C,\ D$	$::=$		
	$\mid$	$\mathbf{eq}\, A\ v\ w$	
	$\mid$	1	
	$\mid$	$\Pi x : A.\, B$	bind x in B
	$\mid$	$\forall x : A.\, B$	bind x in B
	$\mid$	$[v/x]A$	M
	$\mid$	$[\Theta]\, A$	M
	$\mid$	(A)	S

$\boxed{\Gamma \vdash A}$ Declarative type well-formedness

$$\frac{}{\Gamma \vdash 1}\ \textsc{De\_wfT\_Unit}$$

$$\frac{\Gamma \vdash A \quad \Gamma, x : A \vdash B}{\Gamma \vdash \Pi x : A.\, B}\ \textsc{De\_wfT\_Pi}$$

$$\frac{\Gamma \vdash A \quad \Gamma, x : A \vdash B \quad x \in^{!} B\ ok}{\Gamma \vdash \forall x : A.\, B}\ \textsc{De\_wfT\_Forall}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash v \Leftarrow A \quad \Gamma \vdash w \Leftarrow A}{\Gamma \vdash \mathbf{eq}\, A\ v\ w}\ \textsc{De\_wfT\_Eq}$$

$\boxed{\Gamma \vdash v \Leftarrow A}$ Declarative checking

$$\frac{\Gamma \vdash A \quad \Gamma, x : A \vdash v \Leftarrow B}{\Gamma \vdash v \Leftarrow \forall x : A.\, B}\ \textsc{De\_check\_Gen}$$

$$\frac{\Gamma \vdash A \quad \Gamma, x : A \vdash v \Leftarrow B}{\Gamma \vdash \lambda x.\, v \Leftarrow \Pi x : A.\, B}\ \textsc{De\_check\_Pi}$$

$$\frac{\Gamma \vdash v \equiv w \Leftarrow A}{\Gamma \vdash \mathbf{refl} \Leftarrow \mathbf{eq}\, A\ v\ w}\ \textsc{De\_check\_Refl}$$

$$\frac{\Gamma \vdash v \Rightarrow A \quad \Gamma \vdash A \leqslant B}{\Gamma \vdash v \Leftarrow B}\ \textsc{De\_check\_Sub}$$

$\boxed{\Gamma \vdash v \Rightarrow A}$ Declarative inference

$$\frac{}{\Gamma \vdash \lozenge \Rightarrow 1}\ \textsc{De\_inf\_Unit}$$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x \Rightarrow A}\ \textsc{De\_inf\_Var}$$

$$\frac{\Gamma \vdash v \Rightarrow A \quad \Gamma \vdash A \trianglelefteq \Pi x : B.\, C}{\Gamma \vdash v\ w \Rightarrow [w/x]B}\ \textsc{De\_inf\_App}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash v \Leftarrow A}{\Gamma \vdash v : A \Rightarrow A}\ \textsc{De\_inf\_Ann}$$

$$\frac{\Gamma \vdash e \Rightarrow \mathbf{eq}\, A\, v\, w \quad \Gamma, x : A, p : \mathbf{eq}\, A\, v\, x \vdash X \quad \Gamma \vdash f \Leftarrow [\mathbf{refl}/p][v/x]X}{\Gamma \vdash \mathbf{rec}_{eq}^{x.p.X}(e, f) \Rightarrow [e/p][w/x]X}\ \textsc{De\_inf\_RecEq}$$

$\boxed{\Gamma \vdash A \trianglelefteq B}$ Top-level polymorphic instantiation

$$\frac{\Gamma \vdash w \Leftarrow A \quad \Gamma \vdash [w/x]B \trianglelefteq C}{\Gamma \vdash \forall x : A.\, B \trianglelefteq C}\ \textsc{De\_inst\_Forall}$$

$$\frac{A \neq \forall x : B.\, C}{\Gamma \vdash A \trianglelefteq A}\ \textsc{De\_inst\_Base}$$

$\boxed{\Gamma \vdash A \leqslant B}$ Declarative subtyping

$$\frac{\Gamma \vdash A_1 \equiv A_2 \quad \Gamma \vdash v_1 \equiv v_2 \Leftarrow A_1 \quad \Gamma \vdash w_1 \equiv w_2 \Leftarrow A_1}{\Gamma \vdash \mathbf{eq}\, A_1\, v_1\, w_1 \leqslant \mathbf{eq}\, A_2\, v_2\, w_2}\ \textsc{De\_subt\_Eq}$$

$$\frac{}{\Gamma \vdash 1 \leqslant 1}\ \textsc{De\_subt\_Unit}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \quad \Gamma, x : A_1 \vdash B_1 \leqslant B_2}{\Gamma \vdash \Pi x : A_1.\, B_1 \leqslant \Pi x : A_2.\, B_2}\ \textsc{De\_subt\_Pi}$$

$$\frac{\Gamma \vdash w \Leftarrow A \quad \Gamma \vdash [w/x]B \leqslant X}{\Gamma \vdash \forall x : A.\, B \leqslant X}\ \textsc{De\_subt\_ForallL}$$

$$\frac{\Gamma, x : A \vdash X \leqslant B}{\Gamma \vdash X \leqslant \forall x : A.\, B}\ \textsc{De\_subt\_ForallR}$$

$\boxed{\Gamma \vdash A \equiv B}$ Declarative type equivalence

$$\frac{}{\Gamma \vdash 1 \equiv 1}\ \textsc{De\_equivT\_Unit}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \quad \Gamma, x : A_1 \vdash B_1 \equiv B_2}{\Gamma \vdash \Pi x : A_1.\, B_1 \equiv \Pi x : A_2.\, B_2}\ \textsc{De\_equivT\_Pi}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \quad \Gamma, x : A_1 \vdash B_1 \equiv B_2}{\Gamma \vdash \forall x : A_1.\, B_1 \equiv \forall x : A_2.\, B_2}\ \textsc{De\_equivT\_Forall}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \quad \Gamma \vdash v_1 \equiv v_2 \Leftarrow A_1 \quad \Gamma \vdash w_1 \equiv w_2 \Leftarrow A_1}{\Gamma \vdash \mathbf{eq}\, A_1\, v_1\, w_1 \equiv \mathbf{eq}\, A_2\, v_2\, w_2}\ \textsc{De\_equivT\_Eq}$$

$\boxed{\Gamma \vdash v \equiv w \Leftarrow A}$ Declarative checking equivalence

$$\frac{\Gamma \vdash v \searrow v' \Leftarrow A \quad \Gamma \vdash w \searrow w' \Leftarrow A \quad \Gamma \vdash v' \equiv_n w' \Leftarrow A}{\Gamma \vdash v \equiv w \Leftarrow A} \quad \textsc{De\_equivChe\_Red}$$

$\boxed{\Gamma \vdash v \equiv w \Rightarrow A}$ Declarative inferring equivalence

$$\frac{\Gamma \vdash v \searrow v' \Rightarrow A \quad \Gamma \vdash w \searrow w' \Rightarrow A \quad \Gamma \vdash v' \equiv_n w' \Rightarrow A}{\Gamma \vdash v \equiv w \Rightarrow A} \quad \textsc{De\_equivInf\_Red}$$

$\boxed{\Gamma \vdash v \equiv_n w \Leftarrow A}$ Declarative checking equivalence for terms in WHNF

$$\frac{\Gamma, x : A \vdash v_1 \equiv v_2 \Leftarrow B}{\Gamma \vdash \lambda x.\, v_1 \equiv_n \lambda x.\, v_2 \Leftarrow \Pi x : A.\, B} \quad \textsc{De\_equivnChe\_Lam}$$

$$\frac{}{\Gamma \vdash \mathbf{refl} \equiv_n \mathbf{refl} \Leftarrow A} \quad \textsc{De\_equivnChe\_Refl}$$

$$\frac{\Gamma, x : A \vdash v \equiv_n w \Leftarrow B}{\Gamma \vdash v \equiv_n w \Leftarrow \forall x : A.\, B} \quad \textsc{De\_equivnChe\_Forall}$$

$$\frac{\Gamma \vdash v \equiv_n w \Rightarrow B \quad \Gamma \vdash B \leqslant A}{\Gamma \vdash v \equiv_n w \Leftarrow A} \quad \textsc{De\_equivnChe\_Subt}$$

$\boxed{\Gamma \vdash v \equiv_n w \Rightarrow A}$ Declarative inferring equivalence for terms in WHNF

$$\frac{x : A \in \Gamma}{\Gamma \vdash x \equiv_n x \Rightarrow A} \quad \textsc{De\_equivnInf\_Var}$$

$$\frac{}{\Gamma \vdash \langle\rangle \equiv_n \langle\rangle \Rightarrow 1} \quad \textsc{De\_equivnInf\_Unit}$$

$$\frac{\Gamma \vdash A \equiv A' \quad \Gamma \vdash v \equiv_n v' \Leftarrow A}{\Gamma \vdash (v : A) \equiv_n (v' : A') \Rightarrow A} \quad \textsc{De\_equivnInf\_Ann}$$

$$\frac{\Gamma \vdash f_1 \equiv_n f_2 \Rightarrow A \quad \Gamma \vdash A \trianglelefteq \Pi x : B.\, C \quad \Gamma \vdash v_1 \equiv v_2 \Leftarrow B}{\Gamma \vdash f_1\, v_1 \equiv_n f_2\, v_2 \Rightarrow [v_1/x]C} \quad \textsc{De\_equivnInf\_App}$$

$\boxed{\Gamma \vdash v \searrow w \Leftarrow A}$ Declarative checking reduction to WHNF

$$\frac{}{\Gamma \vdash \lambda x.\, e \searrow \lambda x.\, e \Leftarrow A} \quad \textsc{De\_redChe\_Lam}$$

$$\frac{}{\Gamma \vdash \mathbf{refl} \searrow \mathbf{refl} \Leftarrow A} \quad \textsc{De\_redChe\_Refl}$$

$$\frac{\Gamma, x : A \vdash v \searrow v' \Leftarrow B}{\Gamma \vdash v \searrow v' \Leftarrow \forall x : A.\, B} \quad \textsc{De\_redChe\_Forall}$$

$$\frac{\Gamma \vdash v \searrow v' \Rightarrow B \quad \Gamma \vdash B \leqslant A}{\Gamma \vdash v \searrow v' \Leftarrow A} \quad \textsc{De\_redChe\_Subt}$$

$\boxed{\Gamma \vdash v \searrow w \Rightarrow A}$ Declarative inferring CBN reduction to WHNF

$$\frac{x : A \in \Gamma}{\Gamma \vdash x \searrow x \Rightarrow A} \quad \textsc{De\_redInf\_Var}$$

$$\frac{}{\Gamma \vdash \langle\rangle \searrow \langle\rangle \Rightarrow 1} \quad \textsc{De\_redInf\_Refl}$$

$$\frac{\Gamma \vdash v \searrow v' \Leftarrow A}{\Gamma \vdash (v : A) \searrow (v' : A) \Rightarrow A} \quad \textsc{De\_redInf\_Ann}$$

$$\frac{\Gamma \vdash f \searrow ((\lambda y.\, w) : A) \Rightarrow A \quad \Gamma \vdash A \trianglelefteq \Pi x : B.\, C \quad \Gamma \vdash v \Leftarrow B}{\Gamma \vdash f\ v \searrow [v/y]w \Rightarrow [v/x]C} \quad \textsc{De\_redInf\_AppAbs}$$

$$\frac{\Gamma \vdash f \searrow f' \Rightarrow A \quad f' \neq (\lambda y.\, w) : D \quad \Gamma \vdash A \trianglelefteq \Pi x : B.\, C \quad \Gamma \vdash v \Leftarrow B}{\Gamma \vdash f\ v \searrow f'\ v \Rightarrow [v/x]C} \quad \textsc{De\_redInf\_App}$$

$$\frac{\Gamma \vdash e \searrow (\mathbf{refl} : A) \Rightarrow A \quad \Gamma \vdash A \trianglelefteq \mathbf{eq}\, B\ v\ w}{\Gamma \vdash \mathbf{rec}_{eq}^{x.p.X}(e, f) \searrow f \Rightarrow [e/p][w/x]X} \quad \textsc{De\_redInf\_RecRefl}$$

$$\frac{\Gamma \vdash e \searrow e' \Rightarrow \mathbf{eq}\, A\ v\ w \quad e' \neq \mathbf{refl} : \mathbf{eq}\, A\ v\ w}{\Gamma \vdash \mathbf{rec}_{eq}^{x.p.X}(e, f) \searrow \mathbf{rec}_{eq}^{x.p.X}(e', f) \Rightarrow [e/p][w/x]X} \quad \textsc{De\_redInf\_Rec}$$

2 Algorithmic system

$e,\ v,\ w$	$::=$		Potentially non-ground terms
	$\mid$	x	
	$\mid$	$\widehat{x}$	
	$\mid$	$\mathbf{refl}$	
	$\mid$	$\langle\rangle$	
	$\mid$	$v\ w$	
	$\mid$	$\lambda x.\ v$	bind x in v
	$\mid$	$\mathbf{rec}_{eq}^{x.p.\ A}(e, v)$	bind x in A bind p in A
	$\mid$	$v : A$	
	$\mid$	$[w/x]v$	M
	$\mid$	$[w/\widehat{x}]v$	M
	$\mid$	$[\Theta]e$	M
	$\mid$	(v)	S

$A,\ B$	$::=$		
	$\mid$	$\mathbf{eq}\ A\ v\ w$	
	$\mid$	1	
	$\mid$	$\Pi x : A.\ B$	bind x in B
	$\mid$	$\forall x : A.\ B$	bind x in B
	$\mid$	$[v/x]A$	M
	$\mid$	$[\Theta]A$	M
	$\mid$	(A)	S

$\boxed{\Theta_1 \models A \dashv \Theta_2}$ Algorithmic type well-formedness

$$\frac{}{\Theta \models 1 \dashv \Theta}\ \textsc{Al\_wfT\_Unit}$$

$$\frac{\Theta_1 \models A \dashv \Theta_2 \quad \Theta_2, x : [\Theta_2]A \models [\Theta_2]B \dashv \Theta_3}{\Theta_1 \models \Pi x : A.\ B \dashv \Theta_3}\ \textsc{Al\_wfT\_Pi}$$

$$\frac{\Theta_1 \models A \dashv \Theta_2 \quad \Theta_2, x : [\Theta_2]A \models [\Theta_2]B \dashv \Theta_3 \quad x \in^! B\ ok}{\Theta_1 \models \forall x : A.\ B \dashv \Theta_3}\ \textsc{Al\_wfT\_Forall}$$

$$\frac{\Theta_1 \models A \dashv \Theta_2 \quad \Theta_2 \models v \Leftarrow [\Theta_2]A \dashv \Theta_3 \quad \Theta_3 \models [\Theta_3]w \Leftarrow [\Theta_3]A \dashv \Theta_4}{\Theta_1 \models \mathbf{eq}\ A\ v\ w \dashv \Theta_4}\ \textsc{Al\_wfT\_Eq}$$

$\boxed{\Theta_1 \models v \Leftarrow A \dashv \Theta_2}$ Algorithmic checking