

$\alpha, \beta, \alpha, \beta, \gamma, \delta$	type variables
$n, m, i, j$	index variables
$x, y, z$	term variables



		$\hat{\alpha}^- : \approx N$	
		$(e)$	S
		$e_1 \ \& \ e_2$	M
$UC$	$::=$		unification constraint
		$\cdot$	
		$e$	
		$UC \setminus vars$	
		$UC vars$	
		$UC_1 \cup UC_2$	
		$\overline{UC_i}^i$	concatenate
		$(UC)$	S
		$UC' vars$	M
		$UC_1 \ \& \ UC_2$	M
		$UC_1 \cup UC_2$	M
		$ SC $	M
$SC$	$::=$		subtyping constraint
		$\cdot$	
		$e$	
		$SC \setminus vars$	
		$SC vars$	
		$SC_1 \cup SC_2$	
		$UC$	
		$\overline{SC_i}^i$	concatenate
		$(SC)$	S
		$SC' vars$	M
		$SC_1 \ \& \ SC_2$	M
$\hat{\sigma}$	$::=$		unification substitution
		$\cdot$	
		$P/\hat{\alpha}^+$	
		$N/\hat{\alpha}^-$	
		$\vec{P}/\vec{\alpha}^+$	
		$\vec{N}/\vec{\alpha}^-$	
		$(\hat{\sigma})$	S
		$\hat{\sigma}_1 \circ \hat{\sigma}_2$	
		$\overline{\hat{\sigma}_i}^i$	concatenate
		$\mathbf{nf}(\hat{\sigma}')$	M
		$\hat{\sigma}' vars$	M
$\hat{\tau}, \hat{\rho}$	$::=$		anti-unification substitution
		$\cdot$	
		$\hat{\alpha}^- : \approx N$	
		$\hat{\alpha}^- : \approx N$	
		$\vec{\alpha}^-/\vec{\alpha}^-$	
		$\vec{N}/\vec{\alpha}^-$	
		$\hat{\tau}_1 \cup \hat{\tau}_2$	
		$\overline{\hat{\tau}_i}^i$	concatenate
		$(\hat{\tau})$	S

	$\begin{array}{ l} \hat{\tau}' _{vars} \\ \hat{\tau}_1 \ \& \ \hat{\tau}_2 \end{array}$	$\begin{array}{l} \text{M} \\ \text{M} \end{array}$
$P, Q, R$	$\begin{array}{ l} \alpha^+ \\ \downarrow N \\ \exists \alpha^-. P \\ [\sigma]P \end{array}$	$\begin{array}{l} \text{positive types} \\ \\ \\ \text{M} \end{array}$
$N, M, K$	$\begin{array}{ l} \alpha^- \\ \uparrow P \\ \forall \alpha^+. N \\ P \rightarrow N \\ [\sigma]N \end{array}$	$\begin{array}{l} \text{negative types} \\ \\ \\ \\ \text{M} \end{array}$
$\overrightarrow{\alpha^+}, \overrightarrow{\beta^+}, \overrightarrow{\gamma^+}, \overrightarrow{\delta^+}$	$\begin{array}{ l} \cdot \\ \alpha^+ \\ \overrightarrow{\alpha^+} \\ \overrightarrow{\overrightarrow{\alpha^+}}^i \end{array}$	$\begin{array}{l} \text{positive variable list} \\ \text{empty list} \\ \text{a variable} \\ \text{a variable} \\ \text{concatenate lists} \end{array}$
$\overrightarrow{\alpha^-}, \overrightarrow{\beta^-}, \overrightarrow{\gamma^-}, \overrightarrow{\delta^-}$	$\begin{array}{ l} \cdot \\ \alpha^- \\ \overrightarrow{\alpha^-} \\ \overrightarrow{\overrightarrow{\alpha^-}}^i \end{array}$	$\begin{array}{l} \text{negative variables} \\ \text{empty list} \\ \text{a variable} \\ \text{variables} \\ \text{concatenate lists} \end{array}$
$\overrightarrow{\alpha^\pm}, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm}, \overrightarrow{\delta^\pm}$	$\begin{array}{ l} \cdot \\ \alpha^\pm \\ \overrightarrow{\mathbf{pa}} \\ \overrightarrow{\overrightarrow{\alpha^\pm}}^i \end{array}$	$\begin{array}{l} \text{positive or negative variable list} \\ \text{empty list} \\ \text{a variable} \\ \text{variables} \\ \text{concatenate lists} \end{array}$
$P, Q, R$	$\begin{array}{ l} \alpha^+ \\ \downarrow N \\ \exists \overrightarrow{\alpha^-}. P \\ [\sigma]P \\ [\hat{\tau}]P \\ [\hat{\sigma}]P \\ [\mu]P \\ (P) \\ P_1 \vee P_2 \\ \mathbf{nf}(P') \end{array}$	$\begin{array}{l} \text{multi-quantified positive types} \\ \\ \\ \text{M} \\ \text{M} \\ \text{M} \\ \text{M} \\ \text{S} \\ \text{M} \\ \text{M} \end{array}$
$N, M, K$	$\begin{array}{ l} \alpha^- \\ \uparrow P \end{array}$	$\begin{array}{l} \text{multi-quantified negative types} \end{array}$

		$P \rightarrow N$		
		$\forall \vec{\alpha}^+. N$		
		$[\sigma]N$	M	
		$[\hat{\tau}]N$	M	
		$[\mu]N$	M	
		$[\hat{\sigma}]N$	M	
		$(N)$	S	
		$\mathbf{nf}(N')$	M	
$\vec{P}, \vec{Q}$	::=			list of positive types
		$\cdot$		empty list
		$P$		a singel type
		$[\sigma]\vec{P}$	M	
		$\vec{P}_i$		concatenate lists
		$(\vec{P})$	S	
		$\mathbf{nf}(\vec{P}')$	M	
$\vec{N}, \vec{M}$	::=			list of negative types
		$\cdot$		empty list
		$N$		a singel type
		$[\sigma]\vec{N}$	M	
		$\vec{N}_i$		concatenate lists
		$(\vec{N})$	S	
		$\mathbf{nf}(\vec{N}')$	M	
$\Delta, \Gamma$	::=			declarative type context
		$\cdot$		empty context
		$\vec{\alpha}^+$		list of variables
		$\vec{\alpha}^-$		list of variables
		$\vec{\alpha}^\pm$		list of variables
		$vars$		
		$\vec{\Gamma}_i$		concatenate contexts
		$(\Gamma)$	S	
		$\Theta(\hat{\alpha}^+)$	M	
		$\Theta(\hat{\alpha}^-)$	M	
		$\Gamma_1 \cup \Gamma_2$	M	
$\Theta$	::=			algorithmic variable context
		$\cdot$		empty context
		$\vec{\alpha}\{\Delta\}$		from an ordered list of variables
		$\hat{\alpha}^+\{\Delta\}$		from a variable to a list
		$\vec{\Theta}_i$		concatenate contexts
		$(\Theta)$	S	
		$\Theta _{vars}$		leave only those variables that are in the set
		$\Theta_1 \cup \Theta_2$		
$\Xi$	::=			anti-unification type variable context
		$\cdot$		empty context
		$\vec{\alpha}^+$		list of positive variables
		$\vec{\alpha}^-$		list of negative variables

	$\mathbf{uv} \ N$		unification variables
	$\mathbf{uv} \ P$		unification variables
	$\overline{\Xi}_i^i$		concatenate contexts
	$(\Xi)$	S	
	$\Xi_1 \cup \Xi_2$		
	$\Xi_1 \cap \Xi_2$		
	$\Xi' _{vars}$	M	
	$\mathbf{dom}(UC)$	M	
	$\mathbf{dom}(SC)$	M	
	$\mathbf{dom}(\hat{\sigma})$	M	
	$\mathbf{dom}(\hat{\tau})$	M	
	$\mathbf{dom}(\Theta)$	M	
$\vec{\alpha}, \vec{\beta}$	$::=$		ordered positive or negative variables
	$\cdot$		empty list
	$\vec{\alpha}^+$		list of variables
	$\vec{\alpha}^-$		list of variables
	$\vec{\alpha}^\pm$		list of variables
	$\vec{\alpha}^{\sim}$		list of variables
	$\vec{\alpha}^+$		list of variables
	$\vec{\alpha}^-$		list of variables
	$\vec{\alpha}_1 \setminus vars$		setminus
	$\Gamma$		context
	$vars$		
	$\overline{\vec{\alpha}}_i^i$		concatenate contexts
	$(\vec{\alpha})$	S	parenthesis
	$[\mu]\vec{\alpha}$	M	apply moving to list
	$\mathbf{ord} \ vars \ \mathbf{in} \ P$	M	
	$\mathbf{ord} \ vars \ \mathbf{in} \ N$	M	
	$\mathbf{ord} \ vars \ \mathbf{in} \ P$	M	
	$\mathbf{ord} \ vars \ \mathbf{in} \ N$	M	
$vars$	$::=$		set of variables
	$\emptyset$		empty set
	$\mathbf{fv} \ P$		free variables
	$\mathbf{fv} \ N$		free variables
	$\mathbf{fv} \ \mathbf{im} P$		free variables
	$\mathbf{fv} \ \mathbf{im} N$		free variables
	$vars_1 \cap vars_2$		set intersection
	$vars_1 \cup vars_2$		set union
	$vars_1 \setminus vars_2$		set complement
	$\mathbf{mv} \ \mathbf{im} P$		movable variables
	$\mathbf{mv} \ \mathbf{im} N$		movable variables
	$\mathbf{fv} \ N$		free variables
	$\mathbf{fv} \ P$		free variables
	$(vars)$	S	parenthesis
	$\vec{\alpha}$		ordered list of variables
	$[\mu]vars$	M	apply moving to varset
	$\Xi$		anti-unification context
$\mu$	$::=$		

		$\cdot$	empty moving
		$pma1 \mapsto pma2$	Positive unit substitution
		$nma1 \mapsto nma2$	Positive unit substitution
		$\mu_1 \cup \mu_2$	M Set-like union of movings
		$\mu_1 \circ \mu_2$	M Composition
		$\overline{\mu_i}^i$	concatenate movings
		$\mu _{vars}$	M restriction on a set
		$\mu^{-1}$	M inversion
		$\mathbf{nf}(\mu')$	M
$\hat{\alpha}^\pm$	$::=$		positive/negative unification variable
		$\hat{\alpha}^\pm$	
$\hat{\alpha}^+$	$::=$		positive unification variable
		$\hat{\alpha}^+$	
		$\hat{\alpha}^+\{\Delta\}$	
		$\hat{\alpha}^\pm$	
$\hat{\alpha}^-, \hat{\beta}^-$	$::=$		negative unification variable
		$\hat{\alpha}^-$	
		$\hat{\alpha}^-_{\{N,M\}}$	
		$\hat{\alpha}^-\{\Delta\}$	
		$\hat{\alpha}^\pm$	
$\overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$	$::=$		positive unification variable list
		$\cdot$	empty list
		$\hat{\alpha}^+$	a variable
		$\hat{\alpha}^+$	from a normal variable, context unspecified
		$\overrightarrow{\alpha^+}^i$	
		$\alpha^+_i$	concatenate lists
$\overleftarrow{\alpha^-}, \overleftarrow{\beta^-}$	$::=$		negative unification variable list
		$\cdot$	empty list
		$\hat{\alpha}^-$	a variable
		$\Xi$	from an antiunification context
		$\overleftarrow{\hat{\alpha}^-}\{\Delta\}$	from a normal variable
		$\overleftarrow{\hat{\alpha}^-}$	from a normal variable, context unspecified
		$\overleftarrow{\alpha^-}^i$	
		$\alpha^-_i$	concatenate lists
$P, Q$	$::=$		a positive algorithmic type (potentially with metavariables)
		$\hat{\alpha}^+$	
		$\alpha^+$	
		$\downarrow N$	
		$\exists \alpha^+. P$	
		$[\sigma] P$	M
		$[\hat{\tau}] P$	M
		$[\mu] P$	M
		$[\hat{\sigma}] P$	M
		$[\overleftarrow{\alpha^-}/\alpha^-] P$	M
		$(P)$	S

		$\mathbf{nf}(P')$	M
$N, M$	$::=$	a negative algorithmic type (potentially with metavariables)	
		$\hat{\alpha}^-$	
		$\alpha^-$	
		$\uparrow P$	
		$P \rightarrow N$	
		$\forall \alpha^+. N$	
		$[\sigma]N$	M
		$[\hat{\tau}]N$	M
		$[\mu]N$	M
		$[\hat{\sigma}]N$	M
		$[\widetilde{\alpha^-}/\alpha^-]N$	M
		$(N)$	S
		$\mathbf{nf}(N')$	M
$auSol$	$::=$		
		$(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$	
		$(\Xi, N, \hat{\tau}_1, \hat{\tau}_2)$	
$terminals$	$::=$		
		$\exists$	
		$\forall$	
		$\uparrow$	
		$\downarrow$	
		$\rightarrow$	
		$\leftrightarrow$	
		$\in$	
		$\notin$	
		$\cdot$	
		$\perp$	
		$\leq$	
		$\geq$	
		$\mathcal{R}$	
		$\subset$	
		$\supset$	
		$\setminus$	
		$\cup$	
		$\mapsto$	
		$\approx$	
		$\approx^a$	
		$\emptyset$	
		$\circ$	
		$\Rightarrow$	
		$\models$	
		$\perp\!\!\!\perp$	
		$\neq$	
		$\equiv_n$	
		$\vee$	
		$\Leftrightarrow$	



	$ \begin{array}{ l} :\geq \\ :\simeq \\ \Lambda \\ \lambda \\ \mathbf{let}^{\exists} \\ \bullet \\ \Rightarrow\Rightarrow \\ \Leftarrow\Leftarrow \end{array} $	
$v, w$	$ \begin{array}{ l} ::= \\ x \\ \{c\} \\ (v : P) \\ (v) \end{array} $	<p>value terms</p> <p>M</p>
$\vec{v}$	$ \begin{array}{ l} ::= \\ \cdot \\ v \\ \overrightarrow{v}_i^i \end{array} $	<p>list of arguments</p> <p>concatenate</p>
$c, d$	$ \begin{array}{ l} ::= \\ (c : N) \\ \lambda x : P. c \\ \Lambda \alpha^+. c \\ \mathbf{return} \ v \\ \mathbf{let} \ x = v; c \\ \mathbf{let} \ x : P = v(\vec{v}); c \\ \mathbf{let} \ x = v(\vec{v}); c \\ \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c \end{array} $	<p>computation terms</p>
$vctx, \Phi$	$ \begin{array}{ l} ::= \\ \cdot \\ x : P \\ \overrightarrow{\Phi}_i^i \end{array} $	<p>variable context</p> <p>concatenate contexts</p>
<i>formula</i>	$ \begin{array}{ l} ::= \\ judgement \\ judgement \text{ unique} \\ formula_1 \ .. \ formula_n \\ \mu : vars_1 \leftrightarrow vars_2 \\ \mu \text{ is bijective} \\ x : P \in \Phi \\ UC_1 \subseteq UC_2 \\ UC_1 = UC_2 \\ SC_1 \subseteq SC_2 \\ e \in SC \\ e \in UC \\ vars_1 \subseteq vars_2 \\ vars_1 \subseteq vars_2 \subseteq vars_3 \\ vars_1 = vars_2 \end{array} $	

	$\text{vars is fresh}$ $\alpha^- \notin \text{vars}$ $\alpha^+ \notin \text{vars}$ $\alpha^- \in \text{vars}$ $\alpha^+ \in \text{vars}$ $\hat{\alpha}^+ \in \text{vars}$ $\hat{\alpha}^- \in \text{vars}$ $\hat{\alpha}^- \in \Theta$ $\hat{\alpha}^+ \in \Theta$ $\hat{\alpha}^- \notin \text{vars}$ $\hat{\alpha}^+ \notin \text{vars}$ $\hat{\alpha}^- \notin \Theta$ $\hat{\alpha}^+ \notin \Theta$ $\hat{\alpha}^- \in \Xi$ $\hat{\alpha}^- \notin \Xi$ $\hat{\alpha}^+ \in \Xi$ $\hat{\alpha}^+ \notin \Xi$ if any other rule is not applicable $\vec{\alpha}_1 = \vec{\alpha}_2$ $e_1 = e_2$ $e_1 = e_2$ $\hat{\sigma}_1 = \hat{\sigma}_2$ $N = M$ $\Theta \subseteq \Theta'$ $\vec{v}_1 = \vec{v}_2$ $N \neq M$ $P \neq Q$ $N \neq M$ $P \neq Q$ $P \neq Q$ $N \neq M$ $\vec{v}_1 \neq \vec{v}_2$ $\vec{\alpha}_1^+ \neq \vec{\alpha}_2^+$ $ \vec{\alpha}^-  +  \vec{\beta}^-  > 0$ $ \vec{\alpha}^+  +  \vec{\beta}^+  > 0$	
$A$	$::=$ $\Gamma; \Theta \models N \leq M = SC$ $\Gamma; \Theta \models P \geq Q = SC$	Negative subtyping Positive supertyping
$AT$	$::=$ $\Gamma; \Phi \models v : P$ $\Gamma; \Phi \models c : N$ $\Gamma; \Phi; \Theta_1 \models N \bullet \vec{v} \Rightarrow M = \Theta_2; SC$	Positive type inference Negative type inference Application type inference
$AU$	$::=$ $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 = (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$ $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 = (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$	
$SCM$	$::=$	

	$\begin{array}{ l} \Gamma \vdash e_1 \ \& \ e_2 = e_3 \\ \Theta \vdash SC_1 \& SC_2 = SC_3 \end{array}$	Subtyping Constraint Entry Merge Merge of subtyping constraints
<i>UCM</i>	$\begin{array}{ l} \Gamma \vdash e_1 \& e_2 = e_3 \\ \Theta \vdash UC_1 \& UC_2 = UC_3 \end{array}$	
<i>SATSCE</i>	$\begin{array}{ l} \Gamma \vdash P : e \\ \Gamma \vdash N : e \end{array}$	Positive type satisfies with the subtyping constraint entry Negative type satisfies with the subtyping constraint entry
<i>SING</i>	$\begin{array}{ l} e_1 \text{ singular with } P \\ e_1 \text{ singular with } N \\ SC \text{ singular with } \hat{\sigma} \end{array}$	Positive Subtyping Constraint Entry Is Singular Negative Subtyping Constraint Entry Is Singular Subtyping Constraint Is Singular
<i>E1</i>	$\begin{array}{ l} N \simeq_1^D M \\ P \simeq_1^D Q \\ \textcolor{gray}{P} \simeq_1^D \textcolor{gray}{Q} \\ \textcolor{gray}{N} \simeq_1^D \textcolor{gray}{M} \end{array}$	Negative multi-quantified type equivalence Positive multi-quantified type equivalence Positive unification type equivalence Positive unification type equivalence
<i>D1</i>	$\begin{array}{ l} \Gamma \vdash N \simeq_1^{\leq} M \\ \Gamma \vdash P \simeq_1^{\leq} Q \\ \Gamma \vdash N \leq_1 M \\ \Gamma \vdash P \geq_1 Q \\ \Gamma_2 \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : \Gamma_1 \\ \Gamma \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : vars \\ \Theta \vdash \hat{\sigma}_1 \simeq_1^{\leq} \hat{\sigma}_2 : vars \\ \Gamma \vdash \hat{\sigma}_1 \simeq_1^{\leq} \hat{\sigma}_2 : vars \\ \Gamma \vdash \Phi_1 \simeq_1^{\leq} \Phi_2 \end{array}$	Negative equivalence on MQ types Positive equivalence on MQ types Negative subtyping Positive supertyping Equivalence of substitutions Equivalence of substitutions Equivalence of unification substitutions Equivalence of unification substitutions Equivalence of contexts
<i>D0</i>	$\begin{array}{ l} \Gamma \vdash N \simeq_0^{\leq} M \\ \Gamma \vdash P \simeq_0^{\leq} Q \\ \Gamma \vdash N \leq_0 M \\ \Gamma \vdash P \geq_0 Q \end{array}$	Negative equivalence Positive equivalence Negative subtyping Positive supertyping
<i>DT</i>	$\begin{array}{ l} \Gamma; \Phi \vdash v : P \\ \Gamma; \Phi \vdash c : N \\ \Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M \end{array}$	Positive type inference Negative type inference Application type inference
<i>EQ</i>	$\begin{array}{ l} N = M \\ P = Q \\ \textcolor{gray}{P} = \textcolor{gray}{Q} \end{array}$	Negative type equality (alpha-equivalence) Positive type equality (alpha-equivalence)

$LUBF ::=$

- $P_1 \vee P_2 === Q$
- $\mathbf{ord\ vars\ in\ } P === \vec{\alpha}$
- $\mathbf{ord\ vars\ in\ } N === \vec{\alpha}$
- $\mathbf{ord\ vars\ in\ } P === \vec{\alpha}$
- $\mathbf{ord\ vars\ in\ } N === \vec{\alpha}$
- $\mathbf{nf}\ (N') === N$
- $\mathbf{nf}\ (P') === P$
- $\mathbf{nf}\ (N') === N$
- $\mathbf{nf}\ (P') === P$
- $\mathbf{nf}\ (\vec{N}') === \vec{N}$
- $\mathbf{nf}\ (\vec{P}') === \vec{P}$
- $\mathbf{nf}\ (\sigma') === \sigma$
- $\mathbf{nf}\ (\hat{\sigma}') === \hat{\sigma}$
- $\mathbf{nf}\ (\mu') === \mu$
- $\sigma'|_{vars}$
- $\hat{\sigma}'|_{vars}$
- $\hat{\tau}'|_{vars}$
- $\Xi'|_{vars}$
- $SC'|_{vars}$
- $UC'|_{vars}$
- $e_1 \ \& \ e_2$
- $e_1 \ \& \ e_2$
- $UC_1 \ \& \ UC_2$
- $UC_1 \cup UC_2$
- $\Gamma_1 \cup \Gamma_2$
- $SC_1 \ \& \ SC_2$
- $\hat{\tau}_1 \ \& \ \hat{\tau}_2$
- $\mathbf{dom}\ (UC) === \Xi$
- $\mathbf{dom}\ (SC) === \Xi$
- $\mathbf{dom}\ (\hat{\sigma}) === \Xi$
- $\mathbf{dom}\ (\hat{\tau}) === \Xi$
- $\mathbf{dom}\ (\Theta) === \Xi$
- $|SC| === UC$

$LUB ::=$

- $\Gamma \models P_1 \vee P_2 = Q$
- $\mathbf{upgrade}\ \Gamma \vdash P \text{ to } \Delta = Q$

Least Upper Bound (Least Common Supertype)

$Nrm ::=$

- $\mathbf{nf}\ (N) = M$
- $\mathbf{nf}\ (P) = Q$
- $\mathbf{nf}\ (N) = M$
- $\mathbf{nf}\ (P) = Q$

$Order ::=$

- $\mathbf{ord\ vars\ in\ } N = \vec{\alpha}$
- $\mathbf{ord\ vars\ in\ } P = \vec{\alpha}$
- $\mathbf{ord\ vars\ in\ } N = \vec{\alpha}$
- $\mathbf{ord\ vars\ in\ } P = \vec{\alpha}$

$U$	$::=$ $\mid \Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC$ Negative unification $\mid \Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC$ Positive unification
$WFT$	$::=$ $\mid \Gamma \vdash N$ Negative type well-formedness $\mid \Gamma \vdash P$ Positive type well-formedness $\mid \Gamma \vdash N$ Negative type well-formedness $\mid \Gamma \vdash P$ Positive type well-formedness $\mid \Gamma \vdash \vec{N}$ Negative type list well-formedness $\mid \Gamma \vdash \vec{P}$ Positive type list well-formedness
$WFAT$	$::=$ $\mid \Gamma; \Xi \vdash N$ Negative algorithmic type well-formedness $\mid \Gamma; \Xi \vdash P$ Positive algorithmic type well-formedness $\mid \Gamma; \Xi_2 \vdash \hat{\tau} : \Xi_1$ Antiunification substitution well-formedness $\mid \Gamma \vdash^{\supset} \Theta$ Unification context well-formedness $\mid \Gamma_1 \vdash \sigma : \Gamma_2$ Substitution signature $\mid \Theta \vdash \hat{\sigma} : \Xi$ Unification substitution signature $\mid \Gamma \vdash \hat{\sigma} : \Xi$ Unification substitution general signature $\mid \Theta \vdash \hat{\sigma} : UC$ Unification substitution satisfies unification constraint $\mid \Theta \vdash \hat{\sigma} : SC$ Unification substitution satisfies subtyping constraint $\mid \Gamma \vdash e$ Unification constraint entry well-formedness $\mid \Gamma \vdash e$ Subtyping constraint entry well-formedness $\mid \Gamma \vdash P : e$ Positive type satisfies unification constraint $\mid \Gamma \vdash N : e$ Negative type satisfies unification constraint $\mid \Gamma \vdash P : e$ Positive type satisfies subtyping constraint $\mid \Gamma \vdash N : e$ Negative type satisfies subtyping constraint $\mid \Theta \vdash UC : \Xi$ Unification constraint well-formedness with specified domain $\mid \Theta \vdash SC : \Xi$ Subtyping constraint well-formedness with specified domain $\mid \Theta \vdash UC$ Unification constraint well-formedness $\mid \Theta \vdash SC$ Subtyping constraint well-formedness $\mid \Gamma \vdash \vec{v}$ Argument List well-formedness $\mid \Gamma \vdash \Phi$ Context well-formedness $\mid \Gamma \vdash v$ Value well-formedness $\mid \Gamma \vdash c$ Computation well-formedness
$judgement$	$::=$ $\mid A$ $\mid AT$ $\mid AU$ $\mid SCM$ $\mid UCM$ $\mid SATSCE$ $\mid SING$ $\mid E1$ $\mid D1$ $\mid D0$ $\mid DT$ $\mid EQ$

	$LUB$
	$Nrm$
	$Order$
	$U$
	$WFT$
	$WFAT$
$user\_syntax$	$::=$
	$\alpha$
	$n$
	$x$
	$n$
	$\alpha^+$
	$\alpha^-$
	$\alpha^\pm$
	$\sigma$
	$e$
	$e$
	$UC$
	$SC$
	$\hat{\sigma}$
	$\hat{\tau}$
	$P$
	$N$
	$\overrightarrow{\alpha^+}$
	$\overrightarrow{\alpha^-}$
	$\overrightarrow{\alpha^\pm}$
	$P$
	$N$
	$\vec{P}$
	$\vec{N}$
	$\Gamma$
	$\Theta$
	$\Xi$
	$\vec{\alpha}$
	$vars$
	$\mu$
	$\hat{\alpha}^\pm$
	$\hat{\alpha}^+$
	$\hat{\alpha}^-$
	$\widetilde{\overrightarrow{\alpha^+}}$
	$\widetilde{\overrightarrow{\alpha^-}}$
	$\overline{P}$
	$\overline{N}$
	$auSol$
	$terminals$
	$v$
	$\vec{v}$
	$c$
	$vctx$

| *formula*

$\boxed{\Gamma; \Theta \models N \leq M \Rightarrow SC}$  Negative subtyping

$$\begin{array}{c}
\overline{\Gamma; \Theta \models \alpha^- \leq \alpha^- \Rightarrow} \quad \text{ANVAR} \\
\\
\frac{\Gamma; \Theta \models \mathbf{nf}(P) \stackrel{u}{\simeq} \mathbf{nf}(Q) \Rightarrow UC}{\Gamma; \Theta \models \uparrow P \leq \uparrow Q \Rightarrow UC} \quad \text{ASHIFTU} \\
\\
\frac{\Gamma; \Theta \models P \geq Q \Rightarrow SC_1 \quad \Gamma; \Theta \models N \leq M \Rightarrow SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC}{\Gamma; \Theta \models P \rightarrow N \leq Q \rightarrow M \Rightarrow SC} \quad \text{AARROW} \\
\\
\frac{\text{<<multiple parses>>}}{\Gamma; \Theta \models \forall \alpha^+. N \leq \forall \beta^+. M \Rightarrow SC \setminus \hat{\alpha}^+} \quad \text{AFORALL}
\end{array}$$

$\boxed{\Gamma; \Theta \models P \geq Q \Rightarrow SC}$  Positive supertyping

$$\begin{array}{c}
\overline{\Gamma; \Theta \models \alpha^+ \geq \alpha^+ \Rightarrow} \quad \text{APVAR} \\
\\
\frac{\Gamma; \Theta \models \mathbf{nf}(N) \stackrel{u}{\simeq} \mathbf{nf}(M) \Rightarrow UC}{\Gamma; \Theta \models \downarrow N \geq \downarrow M \Rightarrow UC} \quad \text{ASHIFTD} \\
\\
\frac{\Gamma, \vec{\beta}^-; \Theta, \vec{\hat{\alpha}}^- \{ \Gamma, \vec{\beta}^- \} \models [\vec{\hat{\alpha}}^- / \alpha^-] P \geq Q \Rightarrow SC}{\Gamma; \Theta \models \exists \alpha^+. P \geq \exists \beta^+. Q \Rightarrow SC \setminus \hat{\alpha}^-} \quad \text{AEXISTS} \\
\\
\frac{\hat{\alpha}^+ \{ \Delta \} \in \Theta \quad \mathbf{upgrade} \Gamma \vdash P \text{ to } \Delta = Q}{\Gamma; \Theta \models \hat{\alpha}^+ \geq P \Rightarrow (\hat{\alpha}^+ : \geq Q)} \quad \text{APUVAR}
\end{array}$$

$\boxed{\Gamma; \Phi \models v : P}$  Positive type inference

$$\begin{array}{c}
\frac{x : P \in \Phi}{\Gamma; \Phi \models x : \mathbf{nf}(P)} \quad \text{ATVAR} \\
\\
\frac{\Gamma; \Phi \models c : N}{\Gamma; \Phi \models \{c\} : \downarrow N} \quad \text{ATT HUNK} \\
\\
\frac{\Gamma \vdash Q \quad \Gamma; \Phi \models v : P \quad \Gamma; \cdot \models Q \geq P \Rightarrow \cdot}{\Gamma; \Phi \models (v : Q) : \mathbf{nf}(Q)} \quad \text{ATPANNOT}
\end{array}$$

$\boxed{\Gamma; \Phi \models c : N}$  Negative type inference

$$\begin{array}{c}
\frac{\Gamma \vdash M \quad \Gamma; \Phi \models c : N \quad \Gamma; \cdot \models N \leq M \Rightarrow \cdot}{\Gamma; \Phi \models (c : M) : \mathbf{nf}(M)} \quad \text{ATNANNOT} \\
\\
\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \lambda x : P. c : \mathbf{nf}(P \rightarrow N)} \quad \text{ATTLAM} \\
\\
\frac{\Gamma, \alpha^+; \Phi \models c : N}{\Gamma; \Phi \models \Lambda \alpha^+. c : \mathbf{nf}(\forall \alpha^+. N)} \quad \text{ATTLAM} \\
\\
\frac{\Gamma; \Phi \models v : P}{\Gamma; \Phi \models \mathbf{return} v : \uparrow P} \quad \text{ATRETURN} \\
\\
\frac{\Gamma; \Phi \models v : P \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \mathbf{let} x = v; c : N} \quad \text{ATVARLET}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash P \quad \Gamma; \Phi \models v : \downarrow M \quad \Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow \uparrow Q \models \Theta; SC_1 \quad \Gamma; \Theta \models \uparrow Q \leq \uparrow P \models SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \text{let } x : P = v(\vec{v}); c : N} \text{ ATAPPLETANN} \\
\\
\frac{\Gamma; \Phi \models v : \downarrow M \quad \Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow \uparrow Q \models \Theta; SC \quad \mathbf{uv} \ Q = \mathbf{dom}(SC) \quad SC \text{ singular with } \hat{\sigma} \quad \Gamma; \Phi, x : [\hat{\sigma}] Q \models c : N}{\Gamma; \Phi \models \text{let } x = v(\vec{v}); c : N} \text{ ATAPPLET} \\
\\
\frac{\Gamma; \Phi \models v : \exists \alpha^-. P \quad \Gamma, \alpha^-; \Phi, x : P \models c : N \quad \Gamma \vdash N}{\Gamma; \Phi \models \text{let}^3(\alpha^-, x) = v; c : N} \text{ ATUNPACK} \\
\\
\boxed{\Gamma; \Phi; \Theta_1 \models N \bullet \vec{v} \Rightarrow M \models \Theta_2; SC} \quad \text{Application type inference} \\
\\
\frac{}{\Gamma; \Phi; \Theta \models N \bullet \cdot \Rightarrow \mathbf{nf}(N) \models \Theta; \cdot} \text{ AEMPTYAPP} \\
\\
\frac{\Gamma; \Phi \models v : P \quad \Gamma; \Theta \models Q \geq P \models SC_1 \quad \Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rightarrow M \models \Theta'; SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC}{\Gamma; \Phi; \Theta \models Q \rightarrow N \bullet v, \vec{v} \Rightarrow M \models \Theta'; SC} \text{ ATARROWAPP} \\
\\
\frac{\begin{array}{c} \text{<<multiple parses>>} \\ \vec{v} \neq \cdot \quad \alpha^+ \neq \cdot \end{array}}{\text{<<multiple parses>>}} \text{ ATFORALLAPP} \\
\\
\boxed{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)} \\
\\
\frac{}{\Gamma \models \alpha^+ \stackrel{a}{\simeq} \alpha^+ \models (\cdot, \alpha^+, \cdot, \cdot)} \text{ AUPVAR} \\
\\
\frac{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \models (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N_1 \stackrel{a}{\simeq} \downarrow N_2 \models (\Xi, \downarrow M, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUSHIFTD} \\
\\
\frac{\alpha^- \cap \Gamma = \emptyset \quad \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \exists \alpha^-. P_1 \stackrel{a}{\simeq} \exists \alpha^-. P_2 \models (\Xi, \exists \alpha^-. Q, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUEXISTS} \\
\\
\boxed{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \models (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)} \\
\\
\frac{}{\Gamma \models \alpha^- \stackrel{a}{\simeq} \alpha^- \models (\cdot, \alpha^-, \cdot, \cdot)} \text{ AUNVAR} \\
\\
\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \uparrow P_1 \stackrel{a}{\simeq} \uparrow P_2 \models (\Xi, \uparrow Q, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUSHIFTU} \\
\\
\frac{\alpha^+ \cap \Gamma = \emptyset \quad \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \models (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \forall \alpha^+. N_1 \stackrel{a}{\simeq} \forall \alpha^+. N_2 \models (\Xi, \forall \alpha^+. M, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUFORALL} \\
\\
\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (\Xi_1, Q, \hat{\tau}_1, \hat{\tau}_2) \quad \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \models (\Xi_2, M, \hat{\tau}'_1, \hat{\tau}'_2)}{\Gamma \models P_1 \rightarrow N_1 \stackrel{a}{\simeq} P_2 \rightarrow N_2 \models (\Xi_1 \cup \Xi_2, Q \rightarrow M, \hat{\tau}_1 \cup \hat{\tau}'_1, \hat{\tau}_2 \cup \hat{\tau}'_2)} \text{ AUARROW} \\
\\
\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \stackrel{a}{\simeq} M \models (\hat{\alpha}_{\{N, M\}}^-, \hat{\alpha}_{\{N, M\}}^-, (\hat{\alpha}_{\{N, M\}}^- : \approx N), (\hat{\alpha}_{\{N, M\}}^- : \approx M))} \text{ AUAU} \\
\\
\boxed{\Gamma \vdash e_1 \& e_2 = e_3} \quad \text{Subtyping Constraint Entry Merge}
\end{array}$$



$$\frac{\Gamma \vdash P_1 \vee P_2 = Q}{\Gamma \vdash (\hat{\alpha}^+ : \geq P_1) \& (\hat{\alpha}^+ : \geq P_2) = (\hat{\alpha}^+ : \geq Q)} \quad \text{SCMESupSup}$$

$$\frac{\Gamma; \cdot \vdash P \geq Q = \cdot}{\Gamma \vdash (\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \geq Q) = (\hat{\alpha}^+ : \approx P)} \quad \text{SCMEEqSup}$$

$$\frac{\Gamma; \cdot \vdash Q \geq P = \cdot}{\Gamma \vdash (\hat{\alpha}^+ : \geq P) \& (\hat{\alpha}^+ : \approx Q) = (\hat{\alpha}^+ : \approx Q)} \quad \text{SCMESupEq}$$

$$\frac{\llbracket \text{multiple parses} \rrbracket}{\Gamma \vdash (\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \approx P') = (\hat{\alpha}^+ : \approx P)} \quad \text{SCMEPEqEq}$$

$$\frac{\llbracket \text{multiple parses} \rrbracket}{\Gamma \vdash (\hat{\alpha}^- : \approx N_1) \& (\hat{\alpha}^- : \approx N') = (\hat{\alpha}^- : \approx N)} \quad \text{SCMENEqEq}$$

$$\frac{\Theta \vdash SC_1 \& SC_2 = SC_3}{\Gamma \vdash e_1 \& e_2 = e_3} \quad \text{Merge of subtyping constraints}$$

$$\frac{\llbracket \text{multiple parses} \rrbracket}{\Gamma \vdash (\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \approx P') = (\hat{\alpha}^+ : \approx P)} \quad \text{UCMEPEqEq}$$

$$\frac{\llbracket \text{multiple parses} \rrbracket}{\Gamma \vdash (\hat{\alpha}^- : \approx N_1) \& (\hat{\alpha}^- : \approx N') = (\hat{\alpha}^- : \approx N)} \quad \text{UCMENEqEq}$$

$$\frac{\Theta \vdash UC_1 \& UC_2 = UC_3}{\Gamma \vdash P : e} \quad \text{Positive type satisfies with the subtyping constraint entry}$$

$$\frac{\Gamma \vdash P \geq_1 Q}{\Gamma \vdash P : (\hat{\alpha}^+ : \geq Q)} \quad \text{SATSCESup}$$

$$\frac{\llbracket \text{multiple parses} \rrbracket}{\Gamma \vdash P : (\hat{\alpha}^+ : \approx Q)} \quad \text{SATSCPEq}$$

$$\Gamma \vdash N : e \quad \text{Negative type satisfies with the subtyping constraint entry}$$

$$\frac{\llbracket \text{multiple parses} \rrbracket}{\Gamma \vdash N : (\hat{\alpha}^- : \approx M)} \quad \text{SATSCENEq}$$

$$e_1 \text{ singular with } P \quad \text{Positive Subtyping Constraint Entry Is Singular}$$

$$\frac{}{\hat{\alpha}^+ : \approx P \text{ singular with nf } (P)} \quad \text{SINGPEq}$$

$$\frac{}{\hat{\alpha}^+ : \geq \exists \alpha^- . \alpha^+ \text{ singular with } \alpha^+} \quad \text{SINGSUPVAR}$$

$$\frac{N \simeq_1^D \alpha_i^-}{\hat{\alpha}^+ : \geq \exists \alpha^- . \downarrow N \text{ singular with } \exists \alpha^- . \downarrow \alpha^-} \quad \text{SINGSUPSHIFT}$$

$$e_1 \text{ singular with } N \quad \text{Negative Subtyping Constraint Entry Is Singular}$$

$$\frac{}{\hat{\alpha}^- : \approx N \text{ singular with nf } (N)} \quad \text{SINGNEq}$$

$$SC \text{ singular with } \hat{\sigma} \quad \text{Subtyping Constraint Is Singular}$$

$$N \simeq_1^D M \quad \text{Negative multi-quantified type equivalence}$$

$$\frac{}{\alpha^- \simeq_1^D \alpha^-} \quad \text{E1NVAR}$$

$$\begin{array}{c}
\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad \text{E1SHIFTU} \\
\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \rightarrow N \simeq_1^D Q \rightarrow M} \quad \text{E1ARROW} \\
\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv} M = \emptyset \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv} M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv} N) \quad N \simeq_1^D [\mu]M}{\overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M} \quad \text{E1FORALL}
\end{array}$$

$\boxed{P \simeq_1^D Q}$  Positive multi-quantified type equivalence

$$\begin{array}{c}
\overline{\alpha^+ \simeq_1^D \alpha^+} \quad \text{E1PVAR} \\
\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad \text{E1SHIFTD} \\
\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv} Q = \emptyset \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv} Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv} P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \quad \text{E1EXISTS}
\end{array}$$

$\boxed{P \simeq_1^D Q}$  Positive unification type equivalence

$\boxed{N \simeq_1^D M}$  Positive unification type equivalence

$\boxed{\Gamma \vdash N \simeq_1^{\leq} M}$  Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leq_1 M \quad \Gamma \vdash M \leq_1 N}{\Gamma \vdash N \simeq_1^{\leq} M} \quad \text{D1NDEF}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leq} Q}$  Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash Q \geq_1 P}{\Gamma \vdash P \simeq_1^{\geq} Q} \quad \text{D1PDEF}$$

$\boxed{\Gamma \vdash N \leq_1 M}$  Negative subtyping

$$\begin{array}{c}
\overline{\Gamma \vdash \alpha^- \leq_1 \alpha^-} \quad \text{D1NVAR} \\
\frac{\text{<<multiple parses>>}}{\Gamma \vdash \uparrow P \leq_1 \uparrow Q} \quad \text{D1SHIFTU} \\
\frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash N \leq_1 M}{\Gamma \vdash P \rightarrow N \leq_1 Q \rightarrow M} \quad \text{D1ARROW} \\
\frac{\Gamma, \overrightarrow{\beta^+} \vdash \sigma : \overrightarrow{\alpha^+} \quad \Gamma, \overrightarrow{\beta^+} \vdash [\sigma]N \leq_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leq_1 \forall \overrightarrow{\beta^+}.M} \quad \text{D1FORALL}
\end{array}$$

$\boxed{\Gamma \vdash P \geq_1 Q}$  Positive supertyping

$$\begin{array}{c}
\overline{\Gamma \vdash \alpha^+ \geq_1 \alpha^+} \quad \text{D1PVAR} \\
\frac{\text{<<multiple parses>>}}{\Gamma \vdash \downarrow N \geq_1 \downarrow M} \quad \text{D1SHIFTD} \\
\frac{\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-} \quad \Gamma, \overrightarrow{\beta^-} \vdash [\sigma]P \geq_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geq_1 \exists \overrightarrow{\beta^-}.Q} \quad \text{D1EXISTS}
\end{array}$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : \Gamma_1}$  Equivalence of substitutions

$\Gamma \vdash \sigma_1 \simeq_1^\leq \sigma_2 : vars$	Equivalence of substitutions
$\Theta \vdash \hat{\sigma}_1 \simeq_1^\leq \hat{\sigma}_2 : vars$	Equivalence of unification substitutions
$\Gamma \vdash \hat{\sigma}_1 \simeq_1^\leq \hat{\sigma}_2 : vars$	Equivalence of unification substitutions
$\Gamma \vdash \Phi_1 \simeq_1^\leq \Phi_2$	Equivalence of contexts
$\Gamma \vdash N \simeq_0^\leq M$	Negative equivalence

$$\frac{\Gamma \vdash N \leq_0 M \quad \Gamma \vdash M \leq_0 N}{\Gamma \vdash N \simeq_0^\leq M} \quad \text{D0NDEF}$$

$$\boxed{\Gamma \vdash P \simeq_0^\leq Q} \quad \text{Positive equivalence}$$

$$\frac{\Gamma \vdash P \geq_0 Q \quad \Gamma \vdash Q \geq_0 P}{\Gamma \vdash P \simeq_0^\geq Q} \quad \text{D0PDEF}$$

$$\boxed{\Gamma \vdash N \leq_0 M} \quad \text{Negative subtyping}$$

$$\frac{}{\Gamma \vdash \alpha^- \leq_0 \alpha^-} \quad \text{D0NVAR}$$

$$\frac{\Gamma \vdash P \simeq_0^\leq Q}{\Gamma \vdash \uparrow P \leq_0 \uparrow Q} \quad \text{D0SHIFTU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leq_0 M \quad M \neq \forall \beta^+. M'}{\Gamma \vdash \forall \alpha^+. N \leq_0 M} \quad \text{D0FORALLL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leq_0 M}{\Gamma \vdash N \leq_0 \forall \alpha^+. M} \quad \text{D0FORALLR}$$

$$\frac{\Gamma \vdash P \geq_0 Q \quad \Gamma \vdash N \leq_0 M}{\Gamma \vdash P \rightarrow N \leq_0 Q \rightarrow M} \quad \text{D0ARROW}$$

$$\boxed{\Gamma \vdash P \geq_0 Q} \quad \text{Positive supertyping}$$

$$\frac{}{\Gamma \vdash \alpha^+ \geq_0 \alpha^+} \quad \text{D0PVAR}$$

$$\frac{\Gamma \vdash N \simeq_0^\leq M}{\Gamma \vdash \downarrow N \geq_0 \downarrow M} \quad \text{D0SHIFTD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geq_0 Q \quad Q \neq \exists \alpha^-. Q'}{\Gamma \vdash \exists \alpha^-. P \geq_0 Q} \quad \text{D0EXISTSL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geq_0 Q}{\Gamma \vdash P \geq_0 \exists \alpha^-. Q} \quad \text{D0EXISTSR}$$

$$\boxed{\Gamma; \Phi \vdash v : P} \quad \text{Positive type inference}$$

$$\frac{x : P \in \Phi}{\Gamma; \Phi \vdash x : P} \quad \text{DTVAR}$$

$$\frac{\Gamma; \Phi \vdash c : N}{\Gamma; \Phi \vdash \{c\} : \downarrow N} \quad \text{DTTHUNK}$$

$$\frac{\Gamma \vdash Q \quad \Gamma; \Phi \vdash v : P \quad \Gamma \vdash Q \geq_1 P}{\Gamma; \Phi \vdash (v : Q) : Q} \quad \text{DTPANNOT}$$

$$\frac{\text{<<multiple parses>>}}{\Gamma; \Phi \vdash v : P'} \quad \text{DTPEQUIV}$$

$$\boxed{\Gamma; \Phi \vdash c : N} \quad \text{Negative type inference}$$

$$\begin{array}{c}
\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \lambda x : P. c : P \rightarrow N} \quad \text{DTTLAM} \\
\\
\frac{\Gamma, \alpha^+; \Phi \vdash c : N}{\Gamma; \Phi \vdash \Lambda \alpha^+. c : \forall \alpha^+. N} \quad \text{DTTLAM} \\
\\
\frac{\Gamma; \Phi \vdash v : P}{\Gamma; \Phi \vdash \mathbf{return} v : \uparrow P} \quad \text{DTRETURN} \\
\\
\frac{\Gamma; \Phi \vdash v : P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x = v; c : N} \quad \text{DTVARIABLE} \\
\\
\frac{\Gamma; \Phi \vdash v : \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q \text{ unique} \quad \Gamma; \Phi, x : Q \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x = v(\vec{v}); c : N} \quad \text{DTAPPLET} \\
\\
\frac{\Gamma \vdash P \quad \Gamma; \Phi \vdash v : \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q \quad \Gamma \vdash \uparrow Q \leq_1 \uparrow P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x : P = v(\vec{v}); c : N} \quad \text{DTAPPLETANN} \\
\\
\frac{\begin{array}{c} \text{<<multiple parses>>} \\ \Gamma, \vec{\alpha}^-; \Phi, x : P \vdash c : N \quad \Gamma \vdash N \end{array}}{\Gamma; \Phi \vdash \mathbf{let}^3(\vec{\alpha}^-, x) = v; c : N} \quad \text{DTUNPACK} \\
\\
\frac{\begin{array}{c} \text{<<multiple parses>>} \\ \Gamma; \Phi \vdash (c : M) : M \end{array}}{\Gamma; \Phi \vdash (c : M) : M} \quad \text{DTNANNOT} \\
\\
\frac{\begin{array}{c} \text{<<multiple parses>>} \\ \Gamma; \Phi \vdash c : N' \end{array}}{\Gamma; \Phi \vdash c : N'} \quad \text{DTNEQUIV} \\
\\
\boxed{\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M} \quad \text{Application type inference} \\
\\
\frac{\begin{array}{c} \text{<<multiple parses>>} \\ \Gamma; \Phi \vdash N \bullet \cdot \Rightarrow N' \end{array}}{\Gamma; \Phi \vdash N \bullet \cdot \Rightarrow N'} \quad \text{DTEEMPTYAPP} \\
\\
\frac{\Gamma; \Phi \vdash v : P \quad \Gamma \vdash Q \geq_1 P \quad \Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M}{\Gamma; \Phi \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M} \quad \text{DTARROWAPP} \\
\\
\frac{\Gamma \vdash \sigma : \vec{\alpha}^+ \quad \Gamma; \Phi \vdash [\sigma]N \bullet \vec{v} \Rightarrow M \quad \vec{v} \neq \cdot \quad \vec{\alpha}^+ \neq \cdot}{\Gamma; \Phi \vdash \forall \alpha^+. N \bullet \vec{v} \Rightarrow M} \quad \text{DTFORALLAPP} \\
\\
\boxed{N = M} \quad \text{Negative type equality (alpha-equivalence)} \\
\boxed{P = Q} \quad \text{Positive type equality (alpha-equivalence)} \\
\boxed{P = Q} \\
\boxed{P_1 \vee P_2}
\end{array}$$

$\mathbf{ord} \text{ vars in } P$

$\mathbf{ord} \text{ vars in } N$

$\mathbf{ord} \text{ vars in } P$

$$\mathbf{ord} \, vars \, \mathbf{in} \, N$$

$$\mathbf{nf} \, (N')$$

$$\mathbf{nf} \, (P')$$

$$\mathbf{nf} \, (N')$$

$$\mathbf{nf} \, (P')$$

$$\mathbf{nf} \, (\vec{N}')$$

$$\mathbf{nf} \, (\vec{P}')$$

$$\mathbf{nf} \, (\sigma')$$

$$\mathbf{nf} \, (\hat{\sigma}')$$

$$\mathbf{nf} \, (\mu')$$

$$\sigma'|_{vars}$$

$$\hat{\sigma}'|_{vars}$$

$$\hat{\tau}'|_{vars}$$

$$\Xi'|_{vars}$$

$$SC'|_{vars}$$

$$UC' |_{vars}$$

$$e_1 \ \& \ e_2$$

$$e_1 \ \& \ e_2$$

$$UC_1 \ \& \ UC_2$$

$$UC_1 \cup UC_2$$

$$\Gamma_1 \cup \Gamma_2$$

$$SC_1 \ \& \ SC_2$$

$$\hat{\tau}_1 \ \& \ \hat{\tau}_2$$

$$\mathbf{dom} \left( UC \right)$$

$$\mathbf{dom} \left( SC \right)$$

$$\mathbf{dom} \left( \hat{\sigma} \right)$$

$$\mathbf{dom} \left( \hat{\tau} \right)$$

$$\mathbf{dom} \left( \Theta \right)$$

$$|SC|$$

$$\Gamma \models P_1 \vee P_2 = Q \quad \text{Least Upper Bound (Least Common Supertype)}$$

$$\begin{array}{c}
\overline{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVAR} \\
\\
\frac{\Gamma, \cdot \models \mathbf{nf}(\downarrow N) \stackrel{a}{\simeq} \mathbf{nf}(\downarrow M) \models (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N \vee \downarrow M = \exists \alpha^-. [\alpha^- / \Xi] P} \quad \text{LUBSHIFT} \\
\\
\frac{\Gamma, \alpha^-, \vec{\beta}^- \models P_1 \vee P_2 = Q}{\Gamma \models \exists \alpha^-. P_1 \vee \exists \beta^-. P_2 = Q} \quad \text{LUBEXISTS} \\
\\
\boxed{\text{upgrade} \Gamma \vdash P \text{ to } \Delta = Q} \\
\\
\frac{\Gamma = \Delta, \alpha^\pm \quad \vec{\beta}^\pm \text{ is fresh} \quad \vec{\gamma}^\pm \text{ is fresh} \quad \Delta, \vec{\beta}^\pm, \vec{\gamma}^\pm \models [\vec{\beta}^\pm / \alpha^\pm] P \vee [\vec{\gamma}^\pm / \alpha^\pm] P = Q}{\text{upgrade} \Gamma \vdash P \text{ to } \Delta = Q} \quad \text{LUBUPGRADE}
\end{array}$$

$$\boxed{\mathbf{nf}(N) = M}$$

$$\begin{array}{c}
\overline{\mathbf{nf}(\alpha^-) = \alpha^-} \quad \text{NRMNVAR} \\
\\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(\uparrow P) = \uparrow Q} \quad \text{NRMSHIFTU} \\
\\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(P \rightarrow N) = Q \rightarrow M} \quad \text{NRMARROW} \\
\\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(\forall \alpha^+. N) = \forall \alpha^{+'}. N'} \quad \text{NRMFORALL}
\end{array}$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\begin{array}{c}
\overline{\mathbf{nf}(\alpha^+) = \alpha^+} \quad \text{NRMPVAR} \\
\\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(\downarrow N) = \downarrow M} \quad \text{NRMSHIFTD} \\
\\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(\exists \alpha^-. P) = \exists \alpha^{-'}. P'} \quad \text{NRME EXISTS}
\end{array}$$

$$\boxed{\mathbf{nf}(N) = M}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^-) = \hat{\alpha}^-} \quad \text{NRMNUVAR}$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^+) = \hat{\alpha}^+} \quad \text{NRMPUVAR}$$

$$\boxed{\text{ord vars in } N = \vec{\alpha}}$$

$$\begin{array}{c}
\frac{\alpha^- \in \text{vars}}{\text{ord vars in } \alpha^- = \alpha^-} \quad \text{ONVARIN} \\
\\
\frac{\alpha^- \notin \text{vars}}{\text{ord vars in } \alpha^- = \cdot} \quad \text{ONVARININ} \\
\\
\frac{\text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \uparrow P = \vec{\alpha}} \quad \text{OSHIFTU}
\end{array}$$

$$\frac{\mathbf{ord\,vars\,in}\,P = \vec{\alpha}_1 \quad \mathbf{ord\,vars\,in}\,N = \vec{\alpha}_2}{\mathbf{ord\,vars\,in}\,P \rightarrow N = \vec{\alpha}_1, (\vec{\alpha}_2 \setminus \vec{\alpha}_1)} \quad \text{OARROW}$$

$$\frac{\mathbf{vars} \cap \vec{\alpha}^+ = \emptyset \quad \mathbf{ord\,vars\,in}\,N = \vec{\alpha}}{\mathbf{ord\,vars\,in}\,\forall \vec{\alpha}^+. N = \vec{\alpha}} \quad \text{Oforall}$$

$$\boxed{\mathbf{ord\,vars\,in}\,P = \vec{\alpha}}$$

$$\frac{\alpha^+ \in \mathbf{vars}}{\mathbf{ord\,vars\,in}\,\alpha^+ = \alpha^+} \quad \text{OPVARIN}$$

$$\frac{\alpha^+ \notin \mathbf{vars}}{\mathbf{ord\,vars\,in}\,\alpha^+ = \cdot} \quad \text{OPVARNIN}$$

$$\frac{\mathbf{ord\,vars\,in}\,N = \vec{\alpha}}{\mathbf{ord\,vars\,in}\,\downarrow N = \vec{\alpha}} \quad \text{OSHIFTD}$$

$$\frac{\mathbf{vars} \cap \vec{\alpha}^- = \emptyset \quad \mathbf{ord\,vars\,in}\,P = \vec{\alpha}}{\mathbf{ord\,vars\,in}\,\exists \vec{\alpha}^-. P = \vec{\alpha}} \quad \text{OEXISTS}$$

$$\boxed{\mathbf{ord\,vars\,in}\,N = \vec{\alpha}}$$

$$\frac{}{\mathbf{ord\,vars\,in}\,\hat{\alpha}^- = \cdot} \quad \text{ONUVAR}$$

$$\boxed{\mathbf{ord\,vars\,in}\,P = \vec{\alpha}}$$

$$\frac{}{\mathbf{ord\,vars\,in}\,\hat{\alpha}^+ = \cdot} \quad \text{OPUVAR}$$

$$\boxed{\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC} \quad \text{Negative unification}$$

$$\frac{}{\Gamma; \Theta \models \alpha^- \stackrel{u}{\simeq} \alpha^- \Rightarrow \cdot} \quad \text{UNVAR}$$

$$\frac{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC}{\Gamma; \Theta \models \uparrow P \stackrel{u}{\simeq} \uparrow Q \Rightarrow UC} \quad \text{USHIFTU}$$

$$\frac{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC_1 \quad \Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC_2}{\Gamma; \Theta \models P \rightarrow N \stackrel{u}{\simeq} Q \rightarrow M \Rightarrow UC_1 \ \& \ UC_2} \quad \text{UARROW}$$

$$\frac{\Gamma, \vec{\alpha}^+; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC}{\Gamma; \Theta \models \forall \vec{\alpha}^+. N \stackrel{u}{\simeq} \forall \vec{\alpha}^+. M \Rightarrow UC} \quad \text{Uforall}$$

$$\frac{\hat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma; \Theta \models \hat{\alpha}^- \stackrel{u}{\simeq} N \Rightarrow (\hat{\alpha}^- : \approx N)} \quad \text{UNUVAR}$$

$$\boxed{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC} \quad \text{Positive unification}$$

$$\frac{}{\Gamma; \Theta \models \alpha^+ \stackrel{u}{\simeq} \alpha^+ \Rightarrow \cdot} \quad \text{UPVAR}$$

$$\frac{\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC}{\Gamma; \Theta \models \downarrow N \stackrel{u}{\simeq} \downarrow M \Rightarrow UC} \quad \text{USHIFTD}$$

$$\frac{\Gamma, \vec{\alpha}^-; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC}{\Gamma; \Theta \models \exists \vec{\alpha}^-. P \stackrel{u}{\simeq} \exists \vec{\alpha}^-. Q \Rightarrow UC} \quad \text{UEXISTS}$$



$$\frac{\hat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma; \Theta \models \hat{\alpha}^+ \stackrel{u}{\approx} P \Rightarrow (\hat{\alpha}^+ : \approx P)} \text{UPUVAR}$$

$\boxed{\Gamma \vdash N}$  Negative type well-formedness

$$\begin{aligned} & \frac{\alpha^- \in \Gamma}{\Gamma \vdash \alpha^-} \text{WFTNVAR} \\ & \frac{\Gamma \vdash P}{\Gamma \vdash \uparrow P} \text{WFTSHIFTU} \\ & \frac{\Gamma \vdash P \quad \Gamma \vdash N}{\Gamma \vdash P \rightarrow N} \text{WFTARROW} \\ & \frac{\Gamma, \vec{\alpha}^+ \vdash N}{\Gamma \vdash \forall \alpha^+. N} \text{WFTFORALL} \end{aligned}$$

$\boxed{\Gamma \vdash P}$  Positive type well-formedness

$$\begin{aligned} & \frac{\alpha^+ \in \Gamma}{\Gamma \vdash \alpha^+} \text{WFTPVAR} \\ & \frac{\Gamma \vdash N}{\Gamma \vdash \downarrow N} \text{WFTSHIFTD} \\ & \frac{\Gamma, \vec{\alpha}^- \vdash P}{\Gamma \vdash \exists \alpha^-. P} \text{WFTEXISTS} \end{aligned}$$

$\boxed{\Gamma \vdash N}$  Negative type well-formedness

$\boxed{\Gamma \vdash P}$  Positive type well-formedness

$\boxed{\Gamma \vdash \vec{N}}$  Negative type list well-formedness

$\boxed{\Gamma \vdash \vec{P}}$  Positive type list well-formedness

$\boxed{\Gamma; \Xi \vdash N}$  Negative algorithmic type well-formedness

$$\begin{aligned} & \frac{\alpha^- \in \Gamma}{\Gamma; \Xi \vdash \alpha^-} \text{WFATNVAR} \\ & \frac{\hat{\alpha}^- \in \Xi}{\Gamma; \Xi \vdash \hat{\alpha}^-} \text{WFATNUVAR} \\ & \frac{\Gamma; \Xi \vdash P}{\Gamma; \Xi \vdash \uparrow P} \text{WFATSHIFTU} \\ & \frac{\Gamma; \Xi \vdash P \quad \Gamma; \Xi \vdash N}{\Gamma; \Xi \vdash P \rightarrow N} \text{WFATARROW} \\ & \frac{\Gamma, \vec{\alpha}^+; \Xi \vdash N}{\Gamma; \Xi \vdash \forall \alpha^+. N} \text{WFATFORALL} \end{aligned}$$

$\boxed{\Gamma; \Xi \vdash P}$  Positive algorithmic type well-formedness

$$\begin{aligned} & \frac{\alpha^+ \in \Gamma}{\Gamma; \Xi \vdash \alpha^+} \text{WFATPVAR} \\ & \frac{\hat{\alpha}^+ \in \Xi}{\Gamma; \Xi \vdash \hat{\alpha}^+} \text{WFATPUVAR} \\ & \frac{\Gamma; \Xi \vdash N}{\Gamma; \Xi \vdash \downarrow N} \text{WFATSHIFTD} \end{aligned}$$

$$\frac{\Gamma, \alpha^-; \Xi \vdash P}{\Gamma; \Xi \vdash \exists \alpha^-. P} \quad \text{WFATEXISTS}$$

$\boxed{\Gamma; \Xi_2 \vdash \hat{\tau} : \Xi_1}$	Antiunification substitution well-formedness
$\boxed{\Gamma \vdash^\Xi \Theta}$	Unification context well-formedness
$\boxed{\Gamma_1 \vdash \sigma : \Gamma_2}$	Substitution signature
$\boxed{\Theta \vdash \hat{\sigma} : \Xi}$	Unification substitution signature
$\boxed{\Gamma \vdash \hat{\sigma} : \Xi}$	Unification substitution general signature
$\boxed{\Theta \vdash \hat{\sigma} : UC}$	Unification substitution satisfies unification constraint
$\boxed{\Theta \vdash \hat{\sigma} : SC}$	Unification substitution satisfies subtyping constraint
$\boxed{\Gamma \vdash e}$	Unification constraint entry well-formedness
$\boxed{\Gamma \vdash e}$	Subtyping constraint entry well-formedness
$\boxed{\Gamma \vdash P : e}$	Positive type satisfies unification constraint
$\boxed{\Gamma \vdash N : e}$	Negative type satisfies unification constraint
$\boxed{\Gamma \vdash P : e}$	Positive type satisfies subtyping constraint
$\boxed{\Gamma \vdash N : e}$	Negative type satisfies subtyping constraint
$\boxed{\Theta \vdash UC : \Xi}$	Unification constraint well-formedness with specified domain
$\boxed{\Theta \vdash SC : \Xi}$	Subtyping constraint well-formedness with specified domain
$\boxed{\Theta \vdash UC}$	Unification constraint well-formedness
$\boxed{\Theta \vdash SC}$	Subtyping constraint well-formedness
$\boxed{\Gamma \vdash \vec{v}}$	Argument List well-formedness
$\boxed{\Gamma \vdash \Phi}$	Context well-formedness
$\boxed{\Gamma \vdash v}$	Value well-formedness

$$\overline{\Gamma \vdash x} \quad \text{WFATVAR}$$

$\boxed{\Gamma \vdash c}$	Computation well-formedness
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$$\frac{\Gamma \vdash v \quad \Gamma \vdash c \quad \Gamma \vdash \vec{v}}{\Gamma \vdash \mathbf{let} \ x = v(\vec{v}); c} \quad \text{WFATAPPLET}$$

Definition rules:            118 good      20 bad  
 Definition rule clauses: 241 good      21 bad