α, β	type variables
n, m, i, j	index variables

$\alpha^+,\ \beta^+$	$::=$			positive variable
	$\mid$	α^+		
$\alpha^-,\ \beta^-$	$::=$			negative variable
	$\mid$	α^-		
σ	$::=$			substitution
	$\mid$	$\cdot$		
	$\mid$	$P/a+$		
	$\mid$	$N/a-$		
	$\mid$	$\vec{P}/\overrightarrow{\alpha^+}$		
	$\mid$	$\vec{N}/\overrightarrow{\alpha^-}$		
	$\mid$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	$\mid$	$\overrightarrow{\widetilde{\alpha^-}}/\overrightarrow{\alpha^-}$		
	$\mid$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	$\mid$	$\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}$		
	$\mid$	$\vec{\alpha}_1/\vec{\alpha}_2$		
	$\mid$	$\overline{\sigma_i}^{\,i}$		concatenate
e	$::=$			entry of a unification solution
	$\mid$	$\widehat{\alpha}^+ :\approx P$		
	$\mid$	$\widehat{\alpha}^- :\approx N$		
	$\mid$	$\widehat{\alpha}^+ :\geqslant P$		
	$\mid$	(e)	S	
	$\mid$	$e_1\ \&\ e_2$	M	
$\widehat{\sigma}$	$::=$			unification solution (substitution)
	$\mid$	$\cdot$		
	$\mid$	e		
	$\mid$	$\widehat{\sigma}\backslash\overrightarrow{\widetilde{\alpha^+}}$		
	$\mid$	$\widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha^-}}$		
	$\mid$	$\widehat{\sigma}\backslash\widehat{\alpha}^+$		
	$\mid$	$\widehat{\sigma}\backslash\widehat{\alpha}^-$		
	$\mid$	$\widehat{\sigma}_1 \cup \widehat{\sigma}_2$		
	$\mid$	$\overline{\widehat{\sigma}_i}^{\,i}$		concatenate
	$\mid$	$(\widehat{\sigma})$	S	
	$\mid$	$\widehat{\sigma}_1\ \&\ \widehat{\sigma}_2$	M	
$P,\ Q$	$::=$			positive types
	$\mid$	$a+$		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists\alpha^-.P$		
	$\mid$	$[\sigma]P$	M	
$N,\ M$	$::=$			negative types
	$\mid$	$a-$		
	$\mid$	$\uparrow P$		
	$\mid$	$\forall\alpha^+.N$		
	$\mid$	$P \to N$		

	$\vert$	$[\sigma]N$	M	
$\overrightarrow{\alpha^+},\ \overrightarrow{\beta^+}$	$::=$			positive variable list
	$\vert$	$\cdot$		empty list
	$\vert$	α^+		a variable
	$\vert$	$\overline{\overrightarrow{\alpha^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^-},\ \overrightarrow{\beta^-}$	$::=$			negative variables
	$\vert$	$\cdot$		empty list
	$\vert$	α^-		a variable
	$\vert$	$\overline{\overrightarrow{\alpha^-}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$			multi-quantified positive types
	$\vert$	α^+		
	$\vert$	$\downarrow N$		
	$\vert$	$\exists\overrightarrow{\alpha^-}.P$		$P \neq \exists\ldots$
	$\vert$	$[\sigma]P$	M	
	$\vert$	$[\mu]P$	M	
	$\vert$	(P)	S	
	$\vert$	$P_1 \vee P_2$	M	
	$\vert$	$\mathbf{nf}\,(P')$	M	
$N,\ M$	$::=$			multi-quantified negative types
	$\vert$	α^-		
	$\vert$	$\uparrow P$		
	$\vert$	$P \to N$		
	$\vert$	$\forall\overrightarrow{\alpha^+}.N$		$N \neq \forall\ldots$
	$\vert$	$[\sigma]N$	M	
	$\vert$	$[\mu]N$	M	
	$\vert$	(N)	S	
	$\vert$	$\mathbf{nf}\,(N')$	M	
$\overrightarrow{P}$	$::=$			list of positive types
	$\vert$	$\cdot$		empty list
	$\vert$	P		a singel type
	$\vert$	$\overline{\overrightarrow{P}_i}^{\,i}$		concatenate lists
$\overrightarrow{N}$	$::=$			list of negative types
	$\vert$	$\cdot$		empty list
	$\vert$	N		a singel type
	$\vert$	$\overline{\overrightarrow{N}_i}^{\,i}$		concatenate lists
Γ	$::=$			declarative type context
	$\vert$	$\cdot$		empty context
	$\vert$	$\overrightarrow{\alpha^+}$		list of variables
	$\vert$	$\overrightarrow{\alpha^-}$		list of variables
	$\vert$	vars		
	$\vert$	$\overline{\Gamma_i}^{\,i}$		concatenate contexts
	$\vert$	(Γ)	S	

$\vec{\alpha},\ \vec{\beta}$	$::=$			ordered positive or negative variables
	$\vert$	$\cdot$		empty list
	$\vert$	$\overrightarrow{\alpha^+}$		list of variables
	$\vert$	$\overrightarrow{\alpha^-}$		list of variables
	$\vert$	$\vec{\alpha}_1 \backslash \mathit{vars}$		setminus
	$\vert$	Γ		context
	$\vert$	vars		
	$\vert$	$\overline{\vec{\alpha}_i}^{\,i}$		concatenate contexts
	$\vert$	$(\vec{\alpha})$	S	parenthesis
	$\vert$	$[\mu]\vec{\alpha}$	M	apply moving to list
	$\vert$	$\textbf{ord}\, \mathit{vars}\, \textbf{in}\, P$	M	
	$\vert$	$\textbf{ord}\, \mathit{vars}\, \textbf{in}\, N$	M	
	$\vert$	$\textbf{ord}\, \mathit{vars}\, \textbf{in}\, P$	M	
	$\vert$	$\textbf{ord}\, \mathit{vars}\, \textbf{in}\, N$	M	
vars	$::=$			set of variables
	$\vert$	$\varnothing$		empty set
	$\vert$	$\textbf{fv}\, P$		free variables
	$\vert$	$\textbf{fv}\, N$		free variables
	$\vert$	$\textbf{fv}\, P$		free variables
	$\vert$	$\textbf{fv}\, N$		free variables
	$\vert$	$\mathit{vars}_1 \cap \mathit{vars}_2$		set intersection
	$\vert$	$\mathit{vars}_1 \cup \mathit{vars}_2$		set union
	$\vert$	$\mathit{vars}_1 \backslash \mathit{vars}_2$		set complement
	$\vert$	$\textbf{mv}\, P$		movable variables
	$\vert$	$\textbf{mv}\, N$		movable variables
	$\vert$	$\textbf{uv}\, N$		unification variables
	$\vert$	$\textbf{uv}\, P$		unification variables
	$\vert$	$\textbf{fv}\, N$		free variables
	$\vert$	$\textbf{fv}\, P$		free variables
	$\vert$	(vars)	S	parenthesis
	$\vert$	$\{\vec{\alpha}\}$		ordered list of variables
	$\vert$	$[\mu]\mathit{vars}$	M	apply moving to varset
μ	$::=$			
	$\vert$	$\cdot$		empty moving
	$\vert$	$\widetilde{\alpha}^+_1 \mapsto \widetilde{\alpha}^+_2$		Positive unit substitution
	$\vert$	$\widetilde{\alpha}^-_1 \mapsto \widetilde{\alpha}^-_2$		Positive unit substitution
	$\vert$	$\mu_1 \cup \mu_2$	M	Set-like union of movings
	$\vert$	$\overline{\mu_i}^{\,i}$		concatenate movings
	$\vert$	$\mu\vert_{\mathit{vars}}$	M	restriction on a set
	$\vert$	μ^{-1}	M	inversion
n	$::=$			cohort index
	$\vert$	0		
	$\vert$	$n+1$		
$\widetilde{\alpha}^+$	$::=$			positive movable variable
	$\vert$	$\widetilde{\alpha}^{+n}$		

$\widetilde{\alpha}^{-}$	$::=$			negative movable variable
	$\|$	$\widetilde{\alpha}^{-n}$		
$\overrightarrow{\widetilde{\alpha}^{+}}$, $\overrightarrow{\widetilde{\beta}^{+}}$	$::=$			positive movable variable list
	$\|$	$\cdot$		empty list
	$\|$	$\widetilde{\alpha}^{+}$		a variable
	$\|$	$\overrightarrow{\widetilde{\alpha^{+n}}}$		from a non-movable variable
	$\|$	$\overline{\overrightarrow{\widetilde{\alpha}^{+}}_{i}}^{\,i}$		concatenate lists
$\overrightarrow{\widetilde{\alpha}^{-}}$, $\overrightarrow{\widetilde{\beta}^{-}}$	$::=$			negatiive movable variable list
	$\|$	$\cdot$		empty list
	$\|$	$\widetilde{\alpha}^{-}$		a variable
	$\|$	$\overrightarrow{\widetilde{\alpha^{-n}}}$		from a non-movable variable
	$\|$	$\overline{\overrightarrow{\widetilde{\alpha}^{-}}_{i}}^{\,i}$		concatenate lists
P, Q	$::=$			multi-quantified positive types with movable variables
	$\|$	α^{+}		
	$\|$	$\widetilde{\alpha}^{+}$		
	$\|$	$\downarrow N$		
	$\|$	$\exists\overrightarrow{\alpha^{-}}.P$		
	$\|$	$[\sigma]P$	M	
	$\|$	$[\mu]P$	M	
N, M	$::=$			multi-quantified negative types with movable variables
	$\|$	α^{-}		
	$\|$	$\widetilde{\alpha}^{-}$		
	$\|$	$\uparrow P$		
	$\|$	$P \rightarrow N$		
	$\|$	$\forall\overrightarrow{\alpha^{+}}.N$		
	$\|$	$[\sigma]N$	M	
	$\|$	$[\mu]N$	M	
$\widehat{\alpha}^{+}$	$::=$			positive unification variable
	$\|$	$\widehat{\alpha}^{+}$		
$\widehat{\alpha}^{-}$	$::=$			negative unification variable
	$\|$	$\widehat{\alpha}^{-}$		
	$\|$	$\widehat{\alpha}^{-}_{\{N,M\}}$		
$\overrightarrow{\widehat{\alpha}^{+}}$, $\overrightarrow{\widehat{\beta}^{+}}$	$::=$			positive unification variable list
	$\|$	$\cdot$		empty list
	$\|$	$\widehat{\alpha}^{+}$		a variable
	$\|$	$\overrightarrow{\widehat{\alpha}^{+}\,vars}$		from a normal variable
	$\|$	$\overrightarrow{\widehat{\alpha}^{+}}$		from a normal variable, context unspecified
	$\|$	$\overline{\overrightarrow{\widehat{\alpha}^{+}}_{i}}^{\,i}$		concatenate lists
$\overrightarrow{\widehat{\alpha}^{-}}$, $\overrightarrow{\widehat{\beta}^{-}}$	$::=$			negative unification variable list
	$\|$	$\cdot$		empty list
	$\|$	$\widehat{\alpha}^{-}$		a variable

	$\vert$	$\overrightarrow{\widehat{\alpha}^- \, vars}$		from a normal variable
	$\vert$	$\overrightarrow{\widehat{\alpha}^-}$		from a normal variable, context unspecified
	$\vert$	$\overline{\overrightarrow{\widehat{\alpha}^-}_i}^{\,i}$		concatenate lists
P, Q	$::=$			a positive algorithmic type (potentially with metavariables)
	$\vert$	α^+		
	$\vert$	$\widetilde{\alpha}^+$		
	$\vert$	$\widehat{\alpha}^+\{vars\}$		
	$\vert$	$\downarrow N$		
	$\vert$	$\exists\overrightarrow{\alpha^-}.P$		
	$\vert$	$[\sigma]P$	M	
	$\vert$	$[\mu]P$	M	
	$\vert$	$\mathbf{nf}\,(P')$	M	
N, M	$::=$			a negative algorithmic type (potentially with metavariables)
	$\vert$	α^-		
	$\vert$	$\widetilde{\alpha}^-$		
	$\vert$	$\widehat{\alpha}^-\{vars\}$		
	$\vert$	$\widehat{\alpha}^-$		
	$\vert$	$\uparrow P$		
	$\vert$	$P \rightarrow N$		
	$\vert$	$\forall\overrightarrow{\alpha^+}.N$		
	$\vert$	$[\sigma]N$	M	
	$\vert$	$[\mu]N$	M	
	$\vert$	$\mathbf{nf}\,(N')$	M	
terminals	$::=$			
	$\vert$	$\exists$		
	$\vert$	$\forall$		
	$\vert$	$\uparrow$		
	$\vert$	$\downarrow$		
	$\vert$	$\rightarrow$		
	$\vert$	$\leftrightarrow$		
	$\vert$	$\in$		
	$\vert$	$\notin$		
	$\vert$	$\cdot$		
	$\vert$	$\vdash$		
	$\vert$	$\leqslant$		
	$\vert$	$\geqslant$		
	$\vert$	$\simeq$		
	$\vert$	$\cup$		
	$\vert$	$\cap$		
	$\vert$	$\setminus$		
	$\vert$	$\subseteq$		
	$\vert$	$\mapsto$		
	$\vert$	$\stackrel{u}{\simeq}$		
	$\vert$	$\stackrel{a}{\simeq}$		
	$\vert$	$\varnothing$		
	$\vert$	$\models$		
	$\vert$	$\Dashv$		

	$\mid$	$\neq$
	$\mid$	$\equiv_n$
	$\mid$	$\vee$
	$\mid$	$\Downarrow$
	$\mid$	$:\geqslant$
	$\mid$	$:\simeq$

formula	$::=$		
	$\mid$	*judgement*	
	$\mid$	$formula_1 \quad .. \quad formula_n$	
	$\mid$	$\mu : vars_1 \leftrightarrow vars_2$	
	$\mid$	μ **is bijective**	
	$\mid$	$\widehat{\sigma}$ **is functional**	
	$\mid$	$\widehat{\sigma}_1 \in \widehat{\sigma}_2$	
	$\mid$	$vars_1 \subseteq vars_2$	
	$\mid$	$vars_1 = vars_2$	
	$\mid$	$vars$ **is fresh**	
	$\mid$	$\alpha^- \notin vars$	
	$\mid$	$\alpha^+ \notin vars$	
	$\mid$	$\alpha^- \in vars$	
	$\mid$	$\alpha^+ \in vars$	
	$\mid$	if any other rule is not applicable	
	$\mid$	$N \neq M$	
	$\mid$	$P \neq Q$	
E1A	$::=$		
	$\mid$	$n \models N \simeq_1^A M \dashv \mu$	Negative multi-quantified type equivalence (algorit
	$\mid$	$n \models P \simeq_1^A Q \dashv \mu$	Positive multi-quantified type equivalence (algorith
A	$::=$		
	$\mid$	$\Gamma \models \boxed{N} \leqslant M \dashv \widehat{\sigma}$	Negative subtyping
	$\mid$	$\Gamma \models \boxed{P} \geqslant Q \dashv \widehat{\sigma}$	Positive supertyping
E1	$::=$		
	$\mid$	$N \simeq_1^D M$	Negative multi-quantified type equivalence
	$\mid$	$P \simeq_1^D Q$	Positive multi-quantified type equivalence
D1	$::=$		
	$\mid$	$\Gamma \vdash N \simeq_1^{\leqslant} M$	Negative equivalence on MQ types
	$\mid$	$\Gamma \vdash P \simeq_1^{\leqslant} Q$	Positive equivalence on MQ types
	$\mid$	$\Gamma \vdash N \leqslant_1 M$	Negative subtyping
	$\mid$	$\Gamma \vdash P \geqslant_1 Q$	Positive supertyping
D0	$::=$		
	$\mid$	$\Gamma \vdash N \simeq_0^{\leqslant} M$	Negative equivalence
	$\mid$	$\Gamma \vdash P \simeq_0^{\leqslant} Q$	Positive equivalence
	$\mid$	$\Gamma \vdash N \leqslant_0 M$	Negative subtyping
	$\mid$	$\Gamma \vdash P \geqslant_0 Q$	Positive supertyping

LUBF	::=		
	|	$P_1 \vee P_2 === Q$	
	|	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P\ === \vec{\alpha}$	
	|	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N\ === \vec{\alpha}$	
	|	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P === \vec{\alpha}$	
	|	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N === \vec{\alpha}$	
	|	$\mathbf{nf}\,(N') === N$	
	|	$\mathbf{nf}\,(P') === P$	
	|	$\mathbf{nf}\,(N') === N$	
	|	$\mathbf{nf}\,(P') === P$	
	|	$e_1\ \&\ e_2$	
	|	$\widehat{\sigma}_1\ \&\ \widehat{\sigma}_2$	
LUB	::=		
	|	$P_1 \vee P_2 = Q$	Least Upper Bound (Least Common Supertype)
AU	::=		
	|	$\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)$	
	|	$\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)$	
Nrm	::=		
	|	$\mathbf{nf}\,(N) = M$	
	|	$\mathbf{nf}\,(P) = Q$	
	|	$\mathbf{nf}\,(N) = M$	
	|	$\mathbf{nf}\,(P) = Q$	
Order	::=		
	|	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \vec{\alpha}$	
	|	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}$	
	|	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N\ = \vec{\alpha}$	
	|	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P\ = \vec{\alpha}$	
SM	::=		
	|	$e_1\ \&\ e_2 = e_3$	Unification Solution Entry Merge
	|	$\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3$	Merge unification solutions
U	::=		
	|	$N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}$	Negative unification
	|	$P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}$	Positive unification
WF	::=		
	|	$\Gamma \vdash N$	Negative type well-formedness
	|	$\Gamma \vdash P$	Positive type well-formedness
	|	$\Gamma \vdash N$	Negative type well-formedness
	|	$\Gamma \vdash P$	Positive type well-formedness
judgement	::=		
	|	*E1A*	
	|	*A*	

| $E1$
| $D1$
| $D0$
| LUB
| AU
| Nrm
| $Order$
| SM
| U
| WF

$user\_syntax$ ::=
| α
| n
| α^+
| α^-
| σ
| e
| $\widehat{\sigma}$
| P
| N
| $\overrightarrow{\alpha^+}$
| $\overrightarrow{\alpha^-}$
| P
| N
| $\overrightarrow{P}$
| $\overrightarrow{N}$
| Γ
| $\overrightarrow{\alpha}$
| $vars$
| μ
| n
| $\widetilde{\alpha}^+$
| $\widetilde{\alpha}^-$
| $\overrightarrow{\widetilde{\alpha}^+}$
| $\overrightarrow{\widetilde{\alpha}^-}$
| P
| N
| $\widehat{\alpha}^+$
| $\widehat{\alpha}^-$
| $\overrightarrow{\widetilde{\alpha}^+}$
| $\overrightarrow{\widehat{\alpha}^-}$
| P
| N
| $terminals$
| $formula$

$\boxed{n \models N \simeq_1^A M \dashv \mu}$ Negative multi-quantified type equivalence (algorithmic)

$$\frac{}{n \models \alpha^- \simeq_1^A \alpha^- \dashv \cdot} \quad \text{E1ANVar}$$

$$\frac{n \models P \simeq_1^A Q \dashv \mu}{n \models \uparrow P \simeq_1^A \uparrow Q \dashv \mu} \quad \text{E1AShiftU}$$

$$\frac{n \models P \simeq_1^A Q \dashv \mu_1 \quad n \models N \simeq_1^A M \dashv \mu_2 \quad \mu_1 \cup \mu_2 \textbf{ is bijective}}{n \models P \to N \simeq_1^A Q \to M \dashv \mu_1 \cup \mu_2} \quad \text{E1AArrow}$$

$$\frac{n+1 \models [\overrightarrow{\widetilde{\alpha}^{+n}}/\overrightarrow{\alpha^+}]N \simeq_1^A [\overrightarrow{\widetilde{\beta}^{+n}}/\overrightarrow{\beta^+}]M \dashv \mu}{n \models \forall \overrightarrow{\alpha^+}.N \simeq_1^A \forall \overrightarrow{\beta^+}.M \dashv \mu|_{\mathbf{mv}\, M}} \quad \text{E1AForall}$$

$$\frac{}{n \models \widetilde{\alpha}^{-n} \simeq_1^A \widetilde{\beta}^{-n} \dashv \widetilde{\beta}^{-n} \mapsto \widetilde{\alpha}^{-n}} \quad \text{E1ANMVar}$$

$\boxed{n \models P \simeq_1^A Q \dashv \mu}$ Positive multi-quantified type equivalence (algorithmic)

$$\frac{}{n \models \alpha^+ \simeq_1^A \alpha^+ \dashv \cdot} \quad \text{E1APVar}$$

$$\frac{n \models N \simeq_1^A M \dashv \mu}{n \models \downarrow N \simeq_1^A \downarrow M \dashv \mu} \quad \text{E1AShiftD}$$

$$\frac{n+1 \models [\overrightarrow{\widetilde{\alpha}^{-n}}/\overrightarrow{\alpha^-}]P \simeq_1^A [\overrightarrow{\widetilde{\beta}^{-n}}/\overrightarrow{\beta^-}]Q \dashv \mu}{n \models \exists \overrightarrow{\alpha^-}.P \simeq_1^A \exists \overrightarrow{\beta^-}.Q \dashv \mu|_{\mathbf{mv}\, Q}} \quad \text{E1AExists}$$

$$\frac{}{n \models \widetilde{\alpha}^{+n} \simeq_1^A \widetilde{\beta}^{+n} \dashv \widetilde{\beta}^{+n} \mapsto \widetilde{\alpha}^{+n}} \quad \text{E1APMVar}$$

$\boxed{\Gamma \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma \models \alpha^- \leqslant \alpha^- \dashv \cdot} \quad \text{ANVar}$$

$$\frac{\mathbf{nf}\,(P) \stackrel{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma \models \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad \text{AShiftU}$$

$$\frac{\Gamma \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2} \quad \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \models [\overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma \models \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^+}} \quad \text{AForall}$$

$\boxed{\Gamma \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma \models \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad \text{APVar}$$

$$\frac{\mathbf{nf}\,(N) \stackrel{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma \models \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \quad \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \models [\overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma \models \exists \overrightarrow{\alpha^-}.P \geqslant \exists \overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}} \quad \text{AExists}$$

$$\frac{\mathbf{nf}\,(P) = P' \quad vars_1 = \mathbf{fv}\, P' \backslash vars \quad vars_2 \textbf{ is fresh}}{\Gamma \models \widehat{\alpha}^+\{vars\} \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant P' \vee [vars_2/vars_1]P')} \quad \text{APUVar}$$

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-}\quad \text{E1NVar}$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q}\quad \text{E1ShiftU}$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M}\quad \text{E1Arrow}$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M}\quad \text{E1Forall}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+}\quad \text{E1PVar}$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M}\quad \text{E1ShiftD}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q}\quad \text{E1Exists}$$

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M}\quad \text{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q}\quad \text{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-}\quad \text{D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q}\quad \text{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M}\quad \text{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M}\quad \text{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+}\quad \text{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M}\quad \text{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q'}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q}\quad \text{D1ExistsL}$$

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \; \text{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \; \text{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash a- \leqslant_0 a-} \; \text{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \; \text{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/a+]N \leqslant_0 M \quad M \neq \forall \beta^+ . M'}{\Gamma \vdash \forall \alpha^+ . N \leqslant_0 M} \; \text{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+ . M} \; \text{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \rightarrow N \leqslant_0 Q \rightarrow M} \; \text{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash a+ \geqslant_0 a+} \; \text{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \; \text{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/a-]P \geqslant_0 Q \quad Q \neq \exists \alpha^- . Q'}{\Gamma \vdash \exists \alpha^- . P \geqslant_0 Q} \; \text{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^- . Q} \; \text{D0ExistsR}$$

$\boxed{P_1 \vee P_2}$

$\boxed{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ P}$

$\boxed{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ N}$

$\boxed{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ P}$

$\boxed{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ N}$

$\boxed{\mathbf{nf}\,(N')}$

$\boxed{\mathbf{nf}\,(P')}$

$\boxed{\mathbf{nf}\,(N')}$

$\boxed{\mathbf{nf}\,(P')}$

$\boxed{e_1 \,\&\, e_2}$

$\boxed{\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}$

$\boxed{P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\alpha^+ \vee \alpha^+ = \alpha^+}\ \text{LUBVar}$$

$$\frac{(\mathbf{fv}\,N \cup \mathbf{fv}\,M) \models {\downarrow}N \overset{a}{\simeq} {\downarrow}M \dashv (P, \widehat{\sigma}_1, \widehat{\sigma}_2)}{{\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/(\mathbf{uv}\,P)]P}\ \text{LUBShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\overrightarrow{\beta^-}\} = \varnothing}{\exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = P_1 \vee P_2}\ \text{LUBExists}$$

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\alpha^+, \cdot, \cdot)}\ \text{AUPVar}$$

$$\frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models {\downarrow}N_1 \overset{a}{\simeq} {\downarrow}N_2 \dashv ({\downarrow}M, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \text{AUPShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\exists\overrightarrow{\alpha^-}.\,Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \text{AUPExists}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\alpha^-, \cdot, \cdot)}\ \text{AUNVar}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models {\uparrow}P_1 \overset{a}{\simeq} {\uparrow}P_2 \dashv ({\uparrow}Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \text{AUNShift}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}'_1, \widehat{\sigma}'_2)}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (Q \to M, \widehat{\sigma}_1 \cup \widehat{\sigma}'_1, \widehat{\sigma}_2 \cup \widehat{\sigma}'_2)}\ \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad \text{AUNAU}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-} \quad \textsc{NrmNVar}$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q} \quad \textsc{NrmShiftU}$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M} \quad \textsc{NrmArrow}$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'} \quad \textsc{NrmForall}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+} \quad \textsc{NrmPVar}$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M} \quad \textsc{NrmShiftD}$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'} \quad \textsc{NrmExists}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-\{vars\}) = \widehat{\alpha}^-\{vars\}} \quad \textsc{NrmNUVar}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+\{vars\}) = \widehat{\alpha}^+\{vars\}} \quad \textsc{NrmPUVar}$$

$\boxed{\mathbf{ord}\,vars\,\mathbf{in}\,N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^- = \alpha^-} \quad \textsc{ONVarIn}$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^- = \cdot} \quad \textsc{ONVarNIn}$$

$$\frac{\mathbf{ord}\,vars\,\mathbf{in}\,P = \vec{\alpha}}{\mathbf{ord}\,vars\,\mathbf{in}\uparrow P = \vec{\alpha}} \quad \textsc{OShiftU}$$

$$\frac{\mathbf{ord}\,vars\,\mathbf{in}\,P = \vec{\alpha}_1 \quad \mathbf{ord}\,vars\,\mathbf{in}\,N = \vec{\alpha}_2}{\mathbf{ord}\,vars\,\mathbf{in}\,P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})} \quad \textsc{OArrow}$$

$$\frac{vars \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \mathbf{ord}\,vars\,\mathbf{in}\,N = \vec{\alpha}}{\mathbf{ord}\,vars\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad \textsc{OForall}$$

$\boxed{\mathbf{ord}\,vars\,\mathbf{in}\,P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^+ = \alpha^+} \quad \textsc{OPVarIn}$$

$$\frac{\alpha^+ \notin vars}{\textbf{ord}\ vars\ \textbf{in}\ \alpha^+ = \cdot}\ \textsc{OPVarNIn}$$

$$\frac{\textbf{ord}\ vars\ \textbf{in}\ N = \vec{\alpha}}{\textbf{ord}\ vars\ \textbf{in}\ {\downarrow}N = \vec{\alpha}}\ \textsc{OShiftD}$$

$$\frac{vars \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \textbf{ord}\ vars\ \textbf{in}\ P = \vec{\alpha}}{\textbf{ord}\ vars\ \textbf{in}\ \exists\overrightarrow{\alpha^-}.P = \vec{\alpha}}\ \textsc{OExists}$$

$\boxed{\textbf{ord}\ vars\ \textbf{in}\ N = \vec{\alpha}}$

$$\frac{}{\textbf{ord}\ vars\ \textbf{in}\ \widehat{\alpha}^-\{vars'\} = \cdot}\ \textsc{ONUVar}$$

$\boxed{\textbf{ord}\ vars\ \textbf{in}\ P = \vec{\alpha}}$

$$\frac{}{\textbf{ord}\ vars\ \textbf{in}\ \widehat{\alpha}^+\{vars'\} = \cdot}\ \textsc{OPUVar}$$

$\boxed{e_1\ \&\ e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{}{(\widehat{\alpha}^+ :\geqslant P)\ \&\ (\widehat{\alpha}^+ :\geqslant Q) = (\widehat{\alpha}^+ :\geqslant P \vee Q)}\ \textsc{SMESupSup}$$

$$\frac{\textbf{fv}\,P \cup \textbf{fv}\,Q \models P \geqslant Q \dashv \widehat{\sigma}'}{(\widehat{\alpha}^+ :\approx P)\ \&\ (\widehat{\alpha}^+ :\geqslant Q) = (\widehat{\alpha}^+ :\approx P)}\ \textsc{SMEEqSup}$$

$$\frac{\textbf{fv}\,P \cup \textbf{fv}\,Q \models Q \geqslant P \dashv \widehat{\sigma}'}{(\widehat{\alpha}^+ :\geqslant P)\ \&\ (\widehat{\alpha}^+ :\approx Q) = (\widehat{\alpha}^+ :\approx Q)}\ \textsc{SMESupEq}$$

$$\frac{}{(\widehat{\alpha}^+ :\approx P)\ \&\ (\widehat{\alpha}^+ :\approx P) = (\widehat{\alpha}^+ :\approx P)}\ \textsc{SMEPEqEq}$$

$$\frac{}{(\widehat{\alpha}^- :\approx N)\ \&\ (\widehat{\alpha}^- :\approx N) = (\widehat{\alpha}^- :\approx N)}\ \textsc{SMENEqEq}$$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$\boxed{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\ \textsc{UNVar}$$

$$\frac{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{{\uparrow}P \overset{u}{\simeq} {\uparrow}Q \dashv \widehat{\sigma}}\ \textsc{UShiftU}$$

$$\frac{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1\ \&\ \widehat{\sigma}_2}\ \textsc{UArrow}$$

$$\frac{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\ \textsc{UForall}$$

$$\frac{\textbf{fv}\,N \subseteq vars}{\widehat{\alpha}^-\{vars\} \overset{u}{\simeq} N \dashv \widehat{\alpha}^- :\approx N}\ \textsc{UNUVar}$$

$\boxed{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\ \textsc{UPVar}$$

$$\frac{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{{\downarrow}N \overset{u}{\simeq} {\downarrow}M \dashv \widehat{\sigma}}\ \textsc{UShiftD}$$

$$\frac{\boxed{P} \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\exists \overrightarrow{\alpha^-}.\,\boxed{P} \overset{u}{\simeq} \exists \overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}} \quad \text{UEXISTS}$$

$$\frac{\mathbf{fv}\, P \subseteq vars}{\widehat{\alpha}^+\{vars\} \overset{u}{\simeq} P \dashv \widehat{\alpha}^+ :\approx P} \quad \text{UPUVAR}$$

$\boxed{\Gamma \vdash N}$ Negative type well-formedness

$\boxed{\Gamma \vdash P}$ Positive type well-formedness

$\boxed{\Gamma \vdash N}$ Negative type well-formedness

$\boxed{\Gamma \vdash P}$ Positive type well-formedness

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Definition rules:        88 good    0 bad
Definition rule clauses: 150 good    0 bad
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