1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
 & | & \alpha^+ & \\
 & | & {\downarrow}N & \\
 & | & \exists\alpha^-.P & \\
\end{array}
$$

$$
\begin{array}{llll}
N,\ M & ::= & & \text{negative types} \\
 & | & \alpha^- & \\
 & | & {\uparrow}P & \\
 & | & \forall\alpha^+.N & \\
 & | & P \to N & \\
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \ \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \ \textsc{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \ \textsc{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash {\uparrow}P \leqslant_0 {\uparrow}Q} \ \textsc{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall\beta^+.M'}{\Gamma \vdash \forall\alpha^+.N \leqslant_0 M} \ \textsc{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall\alpha^+.M} \ \textsc{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \ \textsc{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \ \textsc{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash {\downarrow}N \geqslant_0 {\downarrow}M} \ \textsc{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists\alpha^-.Q'}{\Gamma \vdash \exists\alpha^-.P \geqslant_0 Q} \ \textsc{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists\alpha^-.Q} \ \textsc{D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$P,\ Q$	$::=$			multi-quantified positive types
	$\mid$	α^+		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists\overrightarrow{\alpha^-}.P$		$P \neq \exists\ldots$
	$\mid$	(P)	S	
$N,\ M$	$::=$			multi-quantified negative types
	$\mid$	α^-		
	$\mid$	$\uparrow P$		
	$\mid$	$P \to N$		
	$\mid$	$\forall\overrightarrow{\alpha^+}.N$		$N \neq \forall\ldots$
	$\mid$	(N)	S	

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M}\ \textsc{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q}\ \textsc{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-}\ \textsc{D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q}\ \textsc{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M}\ \textsc{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M}\ \textsc{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+}\ \textsc{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M}\ \textsc{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q}\ \textsc{D1ExistsL}$$

2.3 Declarative Equivalence

$\boxed{N \simeq_1^{D} M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^{D} \alpha^-}\ (\textsc{Var}^{-\simeq_1^D})$$

$$\frac{P \simeq_1^{D} Q}{\uparrow P \simeq_1^{D} \uparrow Q}\ (\uparrow^{\simeq_1^D})$$

$$\frac{P \simeq_1^{D} Q \quad N \simeq_1^{D} M}{P \to N \simeq_1^{D} Q \to M}\ (\to^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\text{VAR}^+)$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, N = \vec{\alpha}}$

$$\frac{\alpha^- \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, \alpha^- = \alpha^-} \quad (\text{VAR}^-_{\in})$$

$$\frac{\alpha^- \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, \alpha^- = \cdot} \quad (\text{VAR}^-_{\notin})$$

$$\frac{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in} \uparrow P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, P = \vec{\alpha}_1 \quad \mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, N = \vec{\alpha}_2}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})} \quad (\to)$$

$$\frac{\mathit{vars} \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, P = \vec{\alpha}}$

$$\frac{\alpha^+ \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, \alpha^+ = \alpha^+} \quad (\text{VAR}^+_{\in})$$

$$\frac{\alpha^+ \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, \alpha^+ = \cdot} \quad (\text{VAR}^+_{\notin})$$

$$\frac{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in} \downarrow N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{\mathit{vars} \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, \exists \overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad (\exists)$$

$\boxed{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, \widehat{\alpha}^- = \cdot} \quad (\text{UVAR}^-)$$

$\boxed{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\, \mathbf{in}\, \widehat{\alpha}^+ = \cdot} \quad (\text{UVAR}^+)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\quad(\textsc{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\quad(\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\quad(\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'}\quad(\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\quad(\textsc{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\quad(\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'}\quad(\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\quad(\textsc{UVar}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\quad(\textsc{UVar}^+)$$

3.2 Unification

$\boxed{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\quad\textsc{UNVar}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\quad\textsc{UShiftU}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2}\quad\textsc{UArrow}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\quad\textsc{UForall}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)}\quad\textsc{UNUVar}$$

$\boxed{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\quad\textsc{UPVar}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\quad\textsc{UShiftD}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\quad\textsc{UExists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)}\quad\textsc{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma; \Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot} \quad \text{ANVar}$$

$$\frac{\Theta \models \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma; \Theta \models \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad \text{AShiftU}$$

$$\frac{\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2} \quad \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma; \Theta \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}} \quad \text{AForall}$$

$\boxed{\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma; \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad \text{APVar}$$

$$\frac{\Theta \models \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma; \Theta \models \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \quad \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma; \Theta \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}} \quad \text{AExists}$$

$$\frac{\mathbf{upgrade}\,\Gamma \vdash \mathbf{nf}\,(P)\,\mathbf{to}\,\Delta = Q}{\Gamma; \Theta \models \widehat{\alpha}^+\{\Delta\} \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)} \quad \text{APUVar}$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restrict an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\Delta \vdash \widehat{\alpha}^+ :\approx P$ or $\Delta \vdash \widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\Delta \vdash \widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries)**.** *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.
$\boxed{e_1 \;\&\; e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\Gamma \models P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)} \quad (\geqslant \& \geqslant)$$

$$\frac{\Gamma; \cdot \models P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)} \quad (\simeq \& \geqslant)$$

$$\frac{\Gamma; \cdot \models Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)} \quad (\geqslant \& \simeq)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)} \quad (\simeq \& \simeq^+)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N) \;\&\; (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)} \quad (\simeq \& \simeq^-)$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3. $\widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2 = \{e_1 \;\&\; e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, s.t.\ e_1 \text{ matches with } e_2\}$

3.5 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\dfrac{}{\{\mathbf{ord}\ vars\ \mathbf{in}\ N\} \equiv vars \cap \mathbf{fv}\ N}$	$\dfrac{N \simeq_1^D M}{\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ M}$	—
Normalization	$\dfrac{}{N \simeq_1^D \mathbf{nf}\,(N)}$	$\dfrac{N \simeq_1^D M}{\mathbf{nf}\,(N) = \mathbf{nf}\,(M)}$	—
Uppgrade	$\dfrac{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{Q \text{ is sound} \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$	$\dfrac{\begin{array}{l} Q' \text{ is sound} \\ \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \vDash P_1 \vee P_2 = Q}{Q \text{ is sound} \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \vDash P_1 \vee P_2 = Q}$	$\dfrac{\begin{array}{l} Q' \text{ is sound} \\ \Gamma \vDash P_1 \vee P_2 = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\begin{array}{l}(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \\ \text{is sound}\end{array} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat{\tau}_i : \Xi \\ [\hat{\tau}_i]\,Q = P_i \end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \hat{\tau}_1', \hat{\tau}_2')}{\begin{array}{l} \exists(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \text{ s.t.} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \end{array}}$	$\dfrac{\begin{array}{l} (\Xi', Q', \hat{\tau}_1', \hat{\tau}_2') \text{ is sound} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \end{array}}{\exists \Gamma; \Xi \vdash \hat{\tau} : \Xi' \text{ s.t. } [\hat{\tau}]\,Q' = Q}$
Unification (matching)	$\dfrac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}}{\hat{\sigma} \text{ is sound} \begin{cases} \Theta \vdash \hat{\sigma} \\ [\hat{\sigma}]\,P = Q \end{cases}}$	$\dfrac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}}$	—
Subtyping	$\dfrac{\Gamma;\, \Theta \vDash N \leqslant M \dashv \hat{\sigma}}{\hat{\sigma} \text{ is sound} \begin{cases} \Theta \vdash \hat{\sigma} \\ \Gamma \vdash [\hat{\sigma}]\,N \leqslant_1 M \end{cases}}$	$\dfrac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Gamma;\, \Theta \vDash N \leqslant M \dashv \hat{\sigma}}$	—

3.6 Least Upper Bound

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\dfrac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+}\ \text{LUBVar}$$

$$\dfrac{\Gamma, \cdot \vDash {\downarrow}N \overset{a}{\simeq} {\downarrow}M \dashv (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vDash {\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P}\ \text{LUBShift}$$

$$\dfrac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q}\ \text{LUBExists}$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

3.7 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}$

$$\dfrac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)}\ \text{AUPVar}$$

$$\dfrac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vDash {\downarrow}N_1 \overset{a}{\simeq} {\downarrow}N_2 \dashv (\Xi, {\downarrow}M, \hat{\tau}_1, \hat{\tau}_2)}\ \text{AUPShift}$$

$$\dfrac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vDash \exists\overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists\overrightarrow{\alpha^-}.\,Q, \hat{\tau}_1, \hat{\tau}_2)}\ \text{AUPExists}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)}\ \text{AUNVar}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)}\ \text{AUNShift}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)}\ \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))}\ \text{AUNAU}$$

4 Proofs

4.1 Variable Ordering

Definition 4 (Collision free bijection). *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is* ***collision free on sets*** *P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 1 (Soundness of variable ordering). *Variable ordering extracts precisely used free variables.*

− $\{\mathbf{ord}\, vars\, \mathbf{in}\, N\} \equiv vars \cap \mathbf{fv}\, N$ *(as sets)*

+ $\{\mathbf{ord}\, vars\, \mathbf{in}\, P\} \equiv vars \cap \mathbf{fv}\, P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\, vars\, \mathbf{in}\, N = \vec{\alpha}$ and $\mathbf{ord}\, vars\, \mathbf{in}\, P = \vec{\alpha}$ □

Corollary 1 (Additivity of ordering). *Variable ordering is additive (in terms of set union) with respect to its first argument.*

− $\{\mathbf{ord}\, (vars_1 \cup vars_2)\, \mathbf{in}\, N\} \equiv \{\mathbf{ord}\, vars_1\, \mathbf{in}\, N\} \cup \{\mathbf{ord}\, vars_2\, \mathbf{in}\, N\}$ *(as sets)*

+ $\{\mathbf{ord}\, (vars_1 \cup vars_2)\, \mathbf{in}\, P\} \equiv \{\mathbf{ord}\, vars_1\, \mathbf{in}\, P\} \cup \{\mathbf{ord}\, vars_2\, \mathbf{in}\, P\}$ *(as sets)*

Corollary 2 (Weakening of ordering). *Extending the first argument of the ordering with unused variables does not change the result.*

− $\mathbf{ord}\, (vars \cap \mathbf{fv}\, N)\, \mathbf{in}\, N = \mathbf{ord}\, vars\, \mathbf{in}\, N$

+ $\mathbf{ord}\, (vars \cap \mathbf{fv}\, P)\, \mathbf{in}\, P = \mathbf{ord}\, vars\, \mathbf{in}\, P$

Lemma 2 (Distributivity of renaming over variable ordering). *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

− *If μ is collision free on vars and $\mathbf{fv}\, N$ then $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = \mathbf{ord}\, ([\mu]vars)\, \mathbf{in}\, [\mu]N$*

+ *If μ is collision free on vars and $\mathbf{fv}\, P$ then $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, P) = \mathbf{ord}\, ([\mu]vars)\, \mathbf{in}\, [\mu]P$*

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$
Then
$$\begin{aligned}[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) \\ &= [\mu]\alpha^- && \text{by Rule } (\text{Var}^+_\in) \\ &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars) \\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \beta^- && \text{by Rule } (\text{Var}^+_\in)\text{, because } \beta^- \in [\mu]vars \\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^-\end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$
Notice that $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = \cdot$ by Rule $(\text{Var}^+_\notin)$. On the other hand, $\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- = \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \alpha^- = \cdot$ The latter equality is from Rule $(\text{Var}^+_\notin)$, because μ is collision free on *vars* and $\mathbf{fv}\, N$, so $\mathbf{fv}\, N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$
Then $[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) = [\mu](\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^-) = \cdot$ by Rule ($\mathrm{Var}^+_\notin$). To prove that $\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]\alpha^- = \cdot$, we apply Rule ($\mathrm{Var}^+_\notin$). Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

(i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.

(ii) Since μ is collision free on $vars$ and $\mathbf{fv}\ N$, $\mu(A \cap \mathbf{fv}\ N) \ni \mu(\alpha^-) \notin vars$.

d. $\alpha^- \notin A$ but $\alpha^- \in vars$
$\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]\alpha^- = \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ \alpha^- = \alpha^-$. The latter is by Rule ($\mathrm{Var}^+_\notin$), because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) = [\mu](\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = {\uparrow}P$

$$
\begin{aligned}
[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) &= [\mu](\mathbf{ord}\ vars\ \mathbf{in}\ {\uparrow}P) \\
&= [\mu](\mathbf{ord}\ vars\ \mathbf{in}\ P) && \text{by Rule } ({\uparrow}) \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]P && \text{by the induction hypothesis} \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ {\uparrow}[\mu]P && \text{by Rule } ({\uparrow}) \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]{\uparrow}P && \text{by the definition of substitution} \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]N
\end{aligned}
$$

Case 3. $N = P \to M$

$$
\begin{aligned}
[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) &= [\mu](\mathbf{ord}\ vars\ \mathbf{in}\ P \to M) \\
&= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})) && \text{where } \mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\ vars\ \mathbf{in}\ M = \vec{\alpha}_2 \\
&= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\}) \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\{\vec{\alpha}_1\}) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to case 1. Notice that } \mu \text{ is} \\
& && \text{collision free on } \{\vec{\alpha}_1\} \text{ and } \{\vec{\alpha}_2\} \text{ since } \{\vec{\alpha}_1\} \subseteq vars \text{ and } \{\vec{\alpha}_2\} \subseteq \mathbf{fv}\ N \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]N) &= (\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]P \to [\mu]M) \\
&= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \{\vec{\beta}_1\})) && \text{where } \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]M = \vec{\beta}_2 \\
& && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})
\end{aligned}
$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$
\begin{aligned}
[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) &= [\mu]\mathbf{ord}\ vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.M \\
&= [\mu]\mathbf{ord}\ vars\ \mathbf{in}\ M \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]M && \text{by the induction hypothesis}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]N) &= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]\forall\overrightarrow{\alpha^+}.M \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.[\mu]M \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]M
\end{aligned}
$$

□

Lemma 3 (Completeness of variable ordering)**.** *Variable ordering is invariant under equivalence.*

– *For* $N \simeq^D_1 M$ *and any vars,* $\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ M$ *(as lists)*

+ *For* $P \simeq^D_1 Q$ *and any vars,* $\mathbf{ord}\ vars\ \mathbf{in}\ P = \mathbf{ord}\ vars\ \mathbf{in}\ Q$ *(as lists)*

Proof. Mutual induction on $N \simeq^D_1 M$ and $P \simeq^D_1 Q$. □

4.2 Normaliztaion

Lemma 4. *Set of free variables is invariant under equivalence.*

– *If* $N \simeq^D_1 M$ *then* $\mathbf{fv}\ N \equiv \mathbf{fv}\ M$ *(as sets)*

+ *If* $P \simeq^D_1 Q$ *then* $\mathbf{fv}\ P \equiv \mathbf{fv}\ Q$ *(as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

Lemma 5. *Free variables are not changed by the normalization*

- $-$ $\mathbf{fv}\, N \equiv \mathbf{fv}\,\mathbf{nf}\,(N)$
- $+$ $\mathbf{fv}\, P \equiv \mathbf{fv}\,\mathbf{nf}\,(P)$

Proof. By straightforward induction on $\mathbf{nf}\,(N) = M$. □

Lemma 6 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

- $-$ $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$
- $+$ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Here equality means alpha-equivalence.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\,([\mu]N) = \mathbf{nf}\,([\mu]\alpha^-) = [\mu]\alpha^-$. The latter follows from the fact that $[\mu]\alpha^-$ is a variable, and thus, Rule (Var$^-$) is applicable.
$[\mu]\mathbf{nf}\,(N) = [\mu]\mathbf{nf}\,(\alpha^-) = [\mu]\alpha^-$.

Case 2. If the type is formed by $\to$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \to M$ then

$$\begin{aligned}
\mathbf{nf}\,([\mu]N) &= \mathbf{nf}\,([\mu](P \to M)) \\
&= \mathbf{nf}\,([\mu]P \to [\mu]M) && \text{By the congruence of substitution} \\
&= \mathbf{nf}\,([\mu]P) \to \mathbf{nf}\,([\mu]M) && \text{By the congruence of normalization, i.e. Rule } (\to) \\
&= [\mu]\mathbf{nf}\,(P) \to [\mu]\mathbf{nf}\,(M) && \text{By the induction hypothesis} \\
&= [\mu](\mathbf{nf}\,(P) \to \mathbf{nf}\,(M)) && \text{By the congruence of substitution} \\
&= [\mu]\mathbf{nf}\,(P \to M) && \text{By the congruence of normalization} \\
&= [\mu]\mathbf{nf}\,(N)
\end{aligned}$$

Case 3. $N = \forall\overrightarrow{\alpha^+}.M$

$$\begin{aligned}
[\mu]\mathbf{nf}\,(N) &= [\mu]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M) \\
&= [\mu]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M) && \text{Where } \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) = \overrightarrow{\alpha^+}'
\end{aligned}$$

$$\begin{aligned}
\mathbf{nf}\,([\mu]N) &= \mathbf{nf}\,([\mu]\forall\overrightarrow{\alpha^+}.M) \\
&= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\mu]M) && \text{Assuming } \{\overrightarrow{\alpha^+}\} \cap A = \varnothing \text{ and } \{\overrightarrow{\alpha^+}\} \cap B = \varnothing \\
&= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\mu]M) && \text{Where } \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,([\mu]M) = \overrightarrow{\beta^+} \\
&= \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\mu]M) && \text{As } \overrightarrow{\beta^+} = \overrightarrow{\alpha^+}' \text{ (see below)}
\end{aligned}$$

Notice that μ is free of collisions on $\{\overrightarrow{\alpha^+}\}$ and $\mathbf{fv}\,\mathbf{nf}\,(M)$ because

(i) $\mu(A \cap \{\overrightarrow{\alpha^+}\}) \cap \mathbf{fv}\,\mathbf{nf}\,(M) = \varnothing \cap \mathbf{fv}\,\mathbf{nf}\,(M) = \varnothing$ and
(ii) $\mu(A \cap \mathbf{fv}\,\mathbf{nf}\,(M)) \cap \{\overrightarrow{\alpha^+}\} \subseteq B \cap \{\overrightarrow{\alpha^+}\} = \varnothing$

$$\begin{aligned}
\overrightarrow{\beta^+} &= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,([\mu]M) \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{By the induction hypothesis} \\
&= \mathbf{ord}\,\{[\mu]\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{Since } \{\overrightarrow{\alpha^+}\} \cap A = \varnothing \\
&= [\mu]\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by lemma 2} \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{Since } \{\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M)\} \cap A \subseteq \{\overrightarrow{\alpha^+}\} \cap A = \varnothing \\
&= \overrightarrow{\alpha^+}'
\end{aligned}$$

To show alpha-equivalence of $[\mu]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\mu]M)$, we can assume that $\{\overrightarrow{\alpha^+}'\} \cap A = \varnothing$, and $\{\overrightarrow{\alpha^+}'\} \cap B = \varnothing$. Then $[\mu]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M) = \forall\overrightarrow{\alpha^+}'.[\mu]\mathbf{nf}\,(M) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\mu]M)$, the latter follows from the induction hypothesis.

Case 4. $P = \exists\overrightarrow{\alpha^-}.Q$
Same as for case 3.

□

Lemma 7 (Soundness of quantifier normalization)**.**

– $N \simeq_1^D \mathbf{nf}\,(N)$

+ $P \simeq_1^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var$^-$) and (Var$^+$)
By the corresponding equivalence rules.

Case 2. ($\uparrow$), ($\downarrow$), and ($\rightarrow$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall$), i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 4, $\mathbf{fv}\,N \equiv \mathbf{fv}\,N'$. Then by lemma 1, $\{\overrightarrow{\alpha^+}'\} \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$, and thus, $\{\overrightarrow{\alpha^+}'\} \cap \mathbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$.

To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^+}'.N'$, it suffices to provide a bijection $\mu : \{\overrightarrow{\alpha^+}'\} \cap \mathbf{fv}\,N' \leftrightarrow \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. ($\exists$) Same as for case 3.

□

Lemma 8 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

– *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

+ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. ($\forall^{\simeq_1^D}$)

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 1 and 5, the domain and the codomain of μ can be written as $\mu : \{\overrightarrow{\beta^+}'\} \leftrightarrow \{\overrightarrow{\alpha^+}'\}$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by lemma 6, the second—by the induction hypothesis.

(ii)
$$
\begin{aligned}
[\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}' \\
&= [\mu]\mathbf{ord}\,(\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 5 and corollary 2} \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 2, because } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing \\
& && \text{and } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved} \\
&= \mathbf{ord}\,(\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \text{ and } \{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 5 and corollary 2} \\
&= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'
\end{aligned}
$$

Case 2. ($\exists^{\simeq_1^D}$) Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

$\square$

4.3 Upper Bounds

Lemma 9 (Shape of the Supertypes). *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \text{for some } \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle\vert\; \begin{array}{c} [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \\ \text{for some } \Gamma \vdash N_i \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

Lemma 10 (Normalized Shape of the Supertypes). *For a normalized positive type* $P = \mathbf{nf}(P)$*, let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle\vert\; \begin{array}{c} [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \\ \text{for some } \Gamma \vdash N_i \end{array} \right\}$

hen $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 P\}$.

Lemma 11 (Soundness of the Least Upper Bound). *For types* $\Gamma \vdash P_1$*, and* $\Gamma \vdash P_2$*, if* $\Gamma \models P_1 \vee P_2 = Q$ *then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Lemma 12 (Completeness of the Least Upper Bound). *For types* $\Gamma \vdash P_1$*,* $\Gamma \vdash P_2$*, and* $\Gamma \vdash Q'$ *such that* $\Gamma \vdash Q' \geqslant_1 P_1$ *and* $\Gamma \vdash Q' \geqslant_1 P_2$*, there exists* Q *s.t.* $\Gamma \models P_1 \vee P_2 = Q$*, and* $\Gamma \vdash Q' \geqslant_1 Q$

Lemma 13 (Soundness of Upgrade). *For* $\Delta \subseteq \Gamma$*, suppose that* $\mathbf{upgrade}\, \Gamma \vdash P\, \mathbf{to}\, \Delta = Q$*. Then*

(i) $\Delta \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P$

Lemma 14 (Completeness of Upgrade). *For* $\Delta \subseteq \Gamma$*,* $\Gamma \vdash P$ *and* $\Delta \vdash Q'$*, such that* $\Gamma \vdash Q' \geqslant_1 P$*, there exists* Q *s.t.* $\mathbf{upgrade}\, \Gamma \vdash P\, \mathbf{to}\, \Delta = Q$*, and* $\Delta \vdash Q' \geqslant_1 Q$.