1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
& | & a+ & \\
& | & \downarrow N & \\
& | & \exists \alpha^-.P & \\
\\
N,\ M & ::= & & \text{negative types} \\
& | & a- & \\
& | & \uparrow P & \\
& | & \forall \alpha^+.N & \\
& | & P \to N &
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M}\ \text{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q}\ \text{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash a- \leqslant_0 a-}\ \text{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q}\ \text{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/a+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M}\ \text{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M}\ \text{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M}\ \text{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash a+ \geqslant_0 a+}\ \text{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M}\ \text{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/a-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q}\ \text{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q}\ \text{D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$$
\begin{array}{lcll}
P,\ Q & ::= & & \text{multi-quantified positive types} \\
& | & \alpha^+ & \\
& | & \downarrow N & \\
& | & \exists\overrightarrow{\alpha^-}.P & \quad P \neq \exists\ldots \\
\\
N,\ M & ::= & & \text{multi-quantified negative types} \\
& | & \alpha^- & \\
& | & \uparrow P & \\
& | & P \to N & \\
& | & \forall\overrightarrow{\alpha^+}.N & \quad N \neq \forall\ldots
\end{array}
$$

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M}\ \textsc{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q}\ \textsc{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-}\ \textsc{D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q}\ \textsc{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M}\ \textsc{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M}\ \textsc{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+}\ \textsc{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M}\ \textsc{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q'}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q}\ \textsc{D1ExistsL}$$

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-}\ \textsc{E1NVar}$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q}\ \textsc{E1ShiftU}$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M}\ \textsc{E1Arrow}$$

$$\frac{\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M}\ \textsc{E1Forall}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad \text{E1PVar}$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad \text{E1ShiftD}$$

$$\frac{\mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \quad \text{E1Exists}$$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = vars'}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \alpha^-} \quad \text{ONVarIn}$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \cdot} \quad \text{ONVarNIn}$$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^-\{vars'\} = \cdot} \quad \text{ONUVar}$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in} \uparrow P = \vec{\alpha}} \quad \text{OShiftU}$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ vars\ \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)} \quad \text{OArrow}$$

$$\frac{\mathbf{ord}\ (vars \backslash \overrightarrow{\alpha^+})\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad \text{OForall}$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = vars'}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \alpha^+} \quad \text{OPVarIn}$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \cdot} \quad \text{OPVarNIn}$$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^+\{vars'\} = \cdot} \quad \text{OPUVar}$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in} \downarrow N = \vec{\alpha}} \quad \text{OShiftD}$$

$$\frac{\mathbf{ord}\ (vars \backslash \overrightarrow{\alpha^-})\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \exists \overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad \text{OExists}$$

3.1.2 Quantifier Normalization

$\boxed{N \Downarrow M}$
$\boxed{P \Downarrow Q}$
$\boxed{N \Downarrow M}$

$$\frac{}{\alpha^- \Downarrow \alpha^-} \quad \text{NrmNVar}$$

$$\frac{}{\widehat{\alpha}^-\{vars\} \Downarrow \widehat{\alpha}^-\{vars\}} \quad \text{NrmNUVar}$$

$$\frac{P \Downarrow Q}{\uparrow P \Downarrow \uparrow Q} \quad \text{NrmShiftU}$$

$$\frac{P \Downarrow Q \quad N \Downarrow M}{P \to N \Downarrow Q \to M} \quad \textsc{NrmArrow}$$

$$\frac{N \Downarrow N' \quad \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\forall\overrightarrow{\alpha^+}.N \Downarrow \forall\overrightarrow{\alpha^+}'.N'} \quad \textsc{NrmForall}$$

$\boxed{P \Downarrow Q}$

$$\frac{}{\alpha^+ \Downarrow \alpha^+} \quad \textsc{NrmPVar}$$

$$\frac{}{\widehat{\alpha}^+\{vars\} \Downarrow \widehat{\alpha}^+\{vars\}} \quad \textsc{NrmPUVar}$$

$$\frac{N \Downarrow M}{\downarrow N \Downarrow \downarrow M} \quad \textsc{NrmShiftD}$$

$$\frac{P \Downarrow P' \quad \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\exists\overrightarrow{\alpha^-}.P \Downarrow \exists\overrightarrow{\alpha^-}'.P'} \quad \textsc{NrmExists}$$

3.2 Algorithmic Equivalence

$\boxed{n \vDash N \simeq_1^A M \dashv \mu}$ Negative multi-quantified type equivalence (algorithmic)

$$\frac{}{n \vDash \alpha^- \simeq_1^A \alpha^- \dashv \cdot} \quad \textsc{E1ANVar}$$

$$\frac{n \vDash P \simeq_1^A Q \dashv \mu}{n \vDash \uparrow P \simeq_1^A \uparrow Q \dashv \mu} \quad \textsc{E1AShiftU}$$

$$\frac{n \vDash P \simeq_1^A Q \dashv \mu_1 \quad n \vDash N \simeq_1^A M \dashv \mu_2 \quad \mu_1 \cup \mu_2\ \mathbf{is\ bijective}}{n \vDash P \to N \simeq_1^A Q \to M \dashv \mu_1 \cup \mu_2} \quad \textsc{E1AArrow}$$

$$\frac{n+1 \vDash [\overrightarrow{\widetilde{\alpha}^{+n}}/\overrightarrow{\alpha^+}]N \simeq_1^A [\overrightarrow{\widetilde{\beta}^{+n}}/\overrightarrow{\beta^+}]M \dashv \mu}{n \vDash \forall\overrightarrow{\alpha^+}.N \simeq_1^A \forall\overrightarrow{\beta^+}.M \dashv \mu|_{\mathbf{mv}\,M}} \quad \textsc{E1AForall}$$

$$\frac{}{n \vDash \widetilde{\alpha}^{-n} \simeq_1^A \widetilde{\beta}^{-n} \dashv \widetilde{\beta}^{-n} \mapsto \widetilde{\alpha}^{-n}} \quad \textsc{E1ANMVar}$$

$\boxed{n \vDash P \simeq_1^A Q \dashv \mu}$ Positive multi-quantified type equivalence (algorithmic)

$$\frac{}{n \vDash \alpha^+ \simeq_1^A \alpha^+ \dashv \cdot} \quad \textsc{E1APVar}$$

$$\frac{n \vDash N \simeq_1^A M \dashv \mu}{n \vDash \downarrow N \simeq_1^A \downarrow M \dashv \mu} \quad \textsc{E1AShiftD}$$

$$\frac{n+1 \vDash [\overrightarrow{\widetilde{\alpha}^{-n}}/\overrightarrow{\alpha^-}]P \simeq_1^A [\overrightarrow{\widetilde{\beta}^{-n}}/\overrightarrow{\beta^-}]Q \dashv \mu}{n \vDash \exists\overrightarrow{\alpha^-}.P \simeq_1^A \exists\overrightarrow{\beta^-}.Q \dashv \mu|_{\mathbf{mv}\,Q}} \quad \textsc{E1AExists}$$

$$\frac{}{n \vDash \widetilde{\alpha}^{+n} \simeq_1^A \widetilde{\beta}^{+n} \dashv \widetilde{\beta}^{+n} \mapsto \widetilde{\alpha}^{+n}} \quad \textsc{E1APMVar}$$

3.3 Unification

$\boxed{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot} \quad \textsc{UNVar}$$

$$\frac{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}} \quad \textsc{UShiftU}$$

$$\frac{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2} \quad \textsc{UArrow}$$

$$\frac{[\overrightarrow{\widetilde{\alpha^{+n}}/\alpha^{+}}]N \overset{u}{\simeq} [\overrightarrow{\widetilde{\beta^{+n}}/\beta^{+}}]M \dashv \widehat{\sigma}}{\forall\overrightarrow{\alpha^{+}}.N \overset{u}{\simeq} \forall\overrightarrow{\beta^{+}}.M \dashv \widehat{\sigma}} \quad \text{UForall}$$

$$\frac{\mathbf{fv}\, N \subseteq \mathit{vars}}{\widehat{\alpha}^{-}\{\mathit{vars}\} \overset{u}{\simeq} N \dashv \widehat{\alpha}^{-} :\approx N} \quad \text{UNUVar}$$

$\boxed{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\alpha^{+} \overset{u}{\simeq} \alpha^{+} \dashv \cdot} \quad \text{UPVar}$$

$$\frac{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}} \quad \text{UShiftD}$$

$$\frac{[\overrightarrow{\widetilde{\alpha^{-n}}/\alpha^{-}}]P \overset{u}{\simeq} [\overrightarrow{\widetilde{\beta^{-n}}/\beta^{-}}]Q \dashv \widehat{\sigma}}{\exists\overrightarrow{\alpha^{-}}.P \overset{u}{\simeq} \exists\overrightarrow{\beta^{-}}.Q \dashv \widehat{\sigma}} \quad \text{UExists}$$

$$\frac{\mathbf{fv}\, P \subseteq \mathit{vars}}{\widehat{\alpha}^{+}\{\mathit{vars}\} \overset{u}{\simeq} P \dashv \widehat{\alpha}^{+} :\approx P} \quad \text{UPUVar}$$

3.4 Algorithmic Subtyping

$\boxed{\Gamma \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma \models \alpha^{-} \leqslant \alpha^{-} \dashv \cdot} \quad \text{ANVar}$$

$$\frac{P \Downarrow P' \quad Q \Downarrow Q' \quad P' \overset{u}{\simeq} Q' \dashv \widehat{\sigma}}{\Gamma \models \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad \text{AShiftU}$$

$$\frac{\Gamma \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2} \quad \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^{+}} \models [\overrightarrow{\widehat{\alpha}^{+}\{\Gamma, \overrightarrow{\beta^{+}}\}/\alpha^{+}}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma \models \forall\overrightarrow{\alpha^{+}}.N \leqslant \forall\overrightarrow{\beta^{+}}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^{+}}} \quad \text{AForall}$$

$\boxed{\Gamma \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma \models \alpha^{+} \geqslant \alpha^{+} \dashv \cdot} \quad \text{APVar}$$

$$\frac{N \Downarrow N' \quad M \Downarrow M' \quad N' \overset{u}{\simeq} M' \dashv \widehat{\sigma}}{\Gamma \models \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \quad \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^{-}} \models [\overrightarrow{\widehat{\alpha}^{-}\{\Gamma, \overrightarrow{\beta^{-}}\}/\alpha^{-}}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma \models \exists\overrightarrow{\alpha^{-}}.P \geqslant \exists\overrightarrow{\beta^{-}}.Q \dashv \widehat{\sigma}} \quad \text{AExists}$$

$$\frac{\mathit{vars}_1 = \mathbf{fv}\, P\backslash\mathit{vars} \quad \mathit{vars}_2\, \mathbf{is\ fresh}}{\Gamma \models \widehat{\alpha}^{+}\{\mathit{vars}\} \geqslant P \dashv (\widehat{\alpha}^{+} :\geqslant P \vee [\mathit{vars}_2/\mathit{vars}_1]P)} \quad \text{APUVar}$$

3.5 Unification Solution Merge

$\boxed{e_1 \& e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{}{\widehat{\alpha}^{+} :\geqslant P \& \widehat{\alpha}^{+} :\geqslant Q = \widehat{\alpha}^{+} :\geqslant P \vee Q} \quad \text{SMEPSupSup}$$

$$\frac{\mathbf{fv}\, P \cup \mathbf{fv}\, Q \models P \geqslant Q \dashv \widehat{\sigma}'}{\widehat{\alpha}^{+} :\approx P \& \widehat{\alpha}^{+} :\geqslant Q = \widehat{\alpha}^{+} :\approx P} \quad \text{SMEPEqSup}$$

$$\frac{\mathbf{fv}\, P \cup \mathbf{fv}\, Q \models Q \geqslant P \dashv \widehat{\sigma}'}{\widehat{\alpha}^{+} :\geqslant P \& \widehat{\alpha}^{+} :\approx Q = \widehat{\alpha}^{+} :\approx Q} \quad \text{SMEPSupEq}$$

$$\frac{0 \models P \simeq_1^{A} Q \dashv \mu}{\widehat{\alpha}^{+} :\approx P \& \widehat{\alpha}^{+} :\approx Q = \widehat{\alpha}^{+} :\approx Q} \quad \text{SMEPEqEq}$$

$$\frac{0 \models N \simeq_1^A M \dashv \mu}{\widehat{\alpha}^- :\approx N \& \widehat{\alpha}^- :\approx M = \widehat{\alpha}^+ :\approx Q} \quad \text{SMENEqEq}$$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$$\frac{}{\cdot \& \widehat{\sigma} = \widehat{\sigma}} \quad \text{SMEmpty}$$

$$\frac{(\widehat{\alpha}^+ :\approx Q) \in \widehat{\sigma}_2 \quad 0 \models P \simeq_1^A Q \dashv \mu \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)} \quad \text{SMPEqEq}$$

$$\frac{(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\geqslant P \vee Q, \widehat{\sigma}_3)} \quad \text{SMPSupSup}$$

$$\frac{(\widehat{\alpha}^+ :\approx Q) \in \widehat{\sigma}_2 \quad \mathbf{fv}\, Q \cup \mathbf{fv}\, P \models Q \geqslant P \dashv \widehat{\sigma}' \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx Q, \widehat{\sigma}_3)} \quad \text{SMPSupEq}$$

$$\frac{(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \quad \mathbf{fv}\, Q \cup \mathbf{fv}\, P \models P \geqslant Q \dashv \widehat{\sigma}' \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)} \quad \text{SMPEqSup}$$

$$\frac{(\widehat{\alpha}^- :\approx M) \in \widehat{\sigma}_2 \quad 0 \models N \simeq_1^A M \dashv \mu \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^-) = \widehat{\sigma}_3}{(\widehat{\alpha}^- :\approx N, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^- :\approx N, \widehat{\sigma}_3)} \quad \text{SMNEqEq}$$

3.6 Least Upper Bound

$\boxed{P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVar}$$

$$\frac{N \Downarrow N' \quad M \Downarrow M' \quad (\mathbf{fv}\, N' \cup \mathbf{fv}\, M') \models \downarrow N' \overset{a}{\simeq} \downarrow M' \dashv (P, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\mathbf{uv}\, P]P} \quad \text{LUBShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing}{\exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = P_1 \vee P_2} \quad \text{LUBExists}$$

3.7 Antiunification

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\alpha^+, \cdot, \cdot)} \quad \text{AUPVar}$$

$$\frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \dashv (\downarrow M, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUPShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\exists \overrightarrow{\alpha^-}.Q, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUPExists}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\alpha^-, \cdot, \cdot)} \quad \text{AUNVar}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\uparrow Q, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUNShift}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1', \widehat{\sigma}_2')}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (Q \to M, \widehat{\sigma}_1 \cup \widehat{\sigma}_1', \widehat{\sigma}_2 \cup \widehat{\sigma}_2')} \quad \text{AUNArrow}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1', \widehat{\sigma}_2')}{\Gamma \models N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad \text{AUNAU}$$