1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
 & | & a+ & \\
 & | & \downarrow N & \\
 & | & \exists \alpha^-.P & \\
 & & & \\
N,\ M & ::= & & \text{negative types} \\
 & | & a- & \\
 & | & \uparrow P & \\
 & | & \forall \alpha^+.N & \\
 & | & P \to N &
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \text{ D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \text{ D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash a- \leqslant_0 a-} \text{ D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \text{ D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/a+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \text{ D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \text{ D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \text{ D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash a+ \geqslant_0 a+} \text{ D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \text{ D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/a-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \text{ D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \text{ D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{multi-quantified positive types} \\
 & | & \alpha^+ & \\
 & | & \downarrow N & \\
 & | & \exists\overrightarrow{\alpha^-}.P & P \neq \exists \ldots \\
 & & & \\
N,\ M & ::= & & \text{multi-quantified negative types} \\
 & | & \alpha^- & \\
 & | & \uparrow P & \\
 & | & P \to N & \\
 & | & \forall\overrightarrow{\alpha^+}.N & N \neq \forall \ldots
\end{array}
$$

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M}\ \text{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q}\ \text{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-}\ \text{D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q}\ \text{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M}\ \text{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M}\ \text{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+}\ \text{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M}\ \text{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q'}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q}\ \text{D1ExistsL}$$

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-}\ (\text{Var}^{-\simeq_1^D})$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q}\ (\uparrow^{\simeq_1^D})$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M}\ (\to^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M}\ (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\text{VAR}^+)$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists\overrightarrow{\alpha^-}.P \simeq_1^D \exists\overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\ vars\, \mathbf{in}\ N = vars'}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\ vars\, \mathbf{in}\ \alpha^- = \alpha^-} \quad \text{ONVARIN}$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\ vars\, \mathbf{in}\ \alpha^- = \cdot} \quad \text{ONVARNIN}$$

$$\frac{}{\mathbf{ord}\ vars\, \mathbf{in}\ \widehat{\alpha}^-\{vars'\} = \cdot} \quad \text{ONUVAR}$$

$$\frac{\mathbf{ord}\ vars\, \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\, \mathbf{in}\uparrow P = \vec{\alpha}} \quad \text{OSHIFTU}$$

$$\frac{\mathbf{ord}\ vars\, \mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ vars\, \mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ vars\, \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)} \quad \text{OARROW}$$

$$\frac{vars \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\ vars\, \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\, \mathbf{in}\ \forall\overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad \text{OFORALL}$$

$\boxed{\mathbf{ord}\ vars\, \mathbf{in}\ P = vars'}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\ vars\, \mathbf{in}\ \alpha^+ = \alpha^+} \quad \text{OPVARIN}$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\ vars\, \mathbf{in}\ \alpha^+ = \cdot} \quad \text{OPVARNIN}$$

$$\frac{}{\mathbf{ord}\ vars\, \mathbf{in}\ \widehat{\alpha}^+\{vars'\} = \cdot} \quad \text{OPUVAR}$$

$$\frac{\mathbf{ord}\ vars\, \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\, \mathbf{in}\downarrow N = \vec{\alpha}} \quad \text{OSHIFTD}$$

$$\frac{vars \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\ vars\, \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\, \mathbf{in}\ \exists\overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad \text{OEXISTS}$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$\boxed{\mathbf{nf}\,(P) = Q}$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-} \quad (\text{VAR}^-)$$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-\{vars\}) = \widehat{\alpha}^-\{vars\}} \quad (\text{UVAR}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q} \quad (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\ (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'}\ (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\ (\textsc{Var}^+)$$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+\{\mathit{vars}\}) = \widehat{\alpha}^+\{\mathit{vars}\}}\ (\textsc{UVar}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\ (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'}\ (\exists)$$

3.2 Unification

$\boxed{N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\alpha^- \stackrel{u}{\simeq} \alpha^- \dashv \cdot}\ \textsc{UNVar}$$

$$\frac{P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}{\uparrow P \stackrel{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\ \textsc{UShiftU}$$

$$\frac{P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}_2}{P \to N \stackrel{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2}\ \textsc{UArrow}$$

$$\frac{N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}{\forall\overrightarrow{\alpha^+}.N \stackrel{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\ \textsc{UForall}$$

$$\frac{\mathbf{fv}\,N \subseteq \mathit{vars}}{\widehat{\alpha}^-\{\mathit{vars}\} \stackrel{u}{\simeq} N \dashv \widehat{\alpha}^- :\approx N}\ \textsc{UNUVar}$$

$\boxed{P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\alpha^+ \stackrel{u}{\simeq} \alpha^+ \dashv \cdot}\ \textsc{UPVar}$$

$$\frac{N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}{\downarrow N \stackrel{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\ \textsc{UShiftD}$$

$$\frac{P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}{\exists\overrightarrow{\alpha^-}.P \stackrel{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\ \textsc{UExists}$$

$$\frac{\mathbf{fv}\,P \subseteq \mathit{vars}}{\widehat{\alpha}^+\{\mathit{vars}\} \stackrel{u}{\simeq} P \dashv \widehat{\alpha}^+ :\approx P}\ \textsc{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma \vDash N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma \vDash \alpha^- \leqslant \alpha^- \dashv \cdot}\ \textsc{ANVar}$$

$$\frac{\mathbf{nf}\,(P) \stackrel{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma \vDash \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}}\ \textsc{AShiftU}$$

$$\frac{\Gamma \vDash P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma \vDash N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma \vDash P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2}\ \textsc{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vDash [\overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}/\alpha^+}]\, N \leqslant M \dashv \widehat{\sigma}}{\Gamma \vDash \forall \overrightarrow{\alpha^+}.\, N \leqslant \forall \overrightarrow{\beta^+}.M \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^+}} \quad \text{AForall}$$

$\boxed{\Gamma \vDash P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad \text{APVar}$$

$$\frac{\mathbf{nf}\,(N) \stackrel{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma \vDash \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \quad \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vDash [\overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}/\alpha^-}]\, P \geqslant Q \dashv \widehat{\sigma}}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.\, P \geqslant \exists \overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}} \quad \text{AExists}$$

$$\frac{\mathbf{nf}\,(P) = P' \quad vars_1 = \mathbf{fv}\, P' \backslash vars \quad vars_2\ \textbf{is fresh}}{\Gamma \vDash \widehat{\alpha}^+\{vars\} \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant P' \vee [vars_2/vars_1]P')} \quad \text{APUVar}$$

3.4 Unification Solution Merge

$\boxed{e_1 \& e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{}{\widehat{\alpha}^+ :\geqslant P \& \widehat{\alpha}^+ :\geqslant Q = \widehat{\alpha}^+ :\geqslant P \vee Q} \quad \text{SMEPSupSup}$$

$$\frac{\mathbf{fv}\, P \cup \mathbf{fv}\, Q \vDash P \geqslant Q \dashv \widehat{\sigma}'}{\widehat{\alpha}^+ :\approx P \& \widehat{\alpha}^+ :\geqslant Q = \widehat{\alpha}^+ :\approx P} \quad \text{SMEPEqSup}$$

$$\frac{\mathbf{fv}\, P \cup \mathbf{fv}\, Q \vDash Q \geqslant P \dashv \widehat{\sigma}'}{\widehat{\alpha}^+ :\geqslant P \& \widehat{\alpha}^+ :\approx Q = \widehat{\alpha}^+ :\approx Q} \quad \text{SMEPSupEq}$$

$$\frac{}{\widehat{\alpha}^+ :\approx P \& \widehat{\alpha}^+ :\approx P = \widehat{\alpha}^+ :\approx P} \quad \text{SMEPEqEq}$$

$$\frac{}{\widehat{\alpha}^- :\approx N \& \widehat{\alpha}^- :\approx N = \widehat{\alpha}^- :\approx N} \quad \text{SMENEqEq}$$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$$\frac{}{\cdot \& \widehat{\sigma} = \widehat{\sigma}} \quad \text{SMEmpty}$$

$$\frac{\begin{array}{c}(\widehat{\alpha}^+ :\approx P) \in \widehat{\sigma}_2 \\ \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3\end{array}}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)} \quad \text{SMPEqEq}$$

$$\frac{(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\geqslant P \vee Q, \widehat{\sigma}_3)} \quad \text{SMPSupSup}$$

$$\frac{\begin{array}{l}(\widehat{\alpha}^+ :\approx Q) \in \widehat{\sigma}_2 \quad \mathbf{fv}\, Q \cup \mathbf{fv}\, P \vDash Q \geqslant P \dashv \widehat{\sigma}' \\ \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3\end{array}}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx Q, \widehat{\sigma}_3)} \quad \text{SMPSupEq}$$

$$\frac{\begin{array}{l}(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \quad \mathbf{fv}\, Q \cup \mathbf{fv}\, P \vDash P \geqslant Q \dashv \widehat{\sigma}' \\ \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3\end{array}}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)} \quad \text{SMPEqSup}$$

$$\frac{\begin{array}{c}(\widehat{\alpha}^- :\approx N) \in \widehat{\sigma}_2 \\ \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^-) = \widehat{\sigma}_3\end{array}}{(\widehat{\alpha}^- :\approx N, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^- :\approx N, \widehat{\sigma}_3)} \quad \text{SMNEqEq}$$

3.5 Least Upper Bound

$\boxed{P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVar}$$

$$\frac{(\mathbf{fv}\, N \cup \mathbf{fv}\, M) \vDash \downarrow N \stackrel{a}{\simeq} \downarrow M \dashv (P, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\mathbf{uv}\, P]P} \quad \text{LUBShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing}{\exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = P_1 \vee P_2} \quad \text{LUBExists}$$

3.6 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\alpha^+, \cdot, \cdot)} \quad \text{AUPVar}$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \vDash {\downarrow} N_1 \overset{a}{\simeq} {\downarrow} N_2 \dashv ({\downarrow} M, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUPShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\exists \overrightarrow{\alpha^-}.Q, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUPExists}$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\alpha^-, \cdot, \cdot)} \quad \text{AUNVar}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \vDash {\uparrow} P_1 \overset{a}{\simeq} {\uparrow} P_2 \dashv ({\uparrow} Q, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUNShift}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}'_1, \widehat{\sigma}'_2)}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (Q \to M, \widehat{\sigma}_1 \cup \widehat{\sigma}'_1, \widehat{\sigma}_2 \cup \widehat{\sigma}'_2)} \quad \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad \text{AUNAU}$$

4 Proofs

4.1 Normaliztaion

4.1.1 Auxiliary properties

Lemma 1. *Set of free variables is invariant under equivalence.*

- *If* $N \simeq_1^D M$ *then* $\mathbf{fv}\, N \equiv \mathbf{fv}\, M$ *(as sets)*

+ *If* $P \simeq_1^D Q$ *then* $\mathbf{fv}\, P \equiv \mathbf{fv}\, Q$ *(as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

Lemma 2 (Distributivity of renaming over variable ordering)**.** *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$, and B is disjoint with vars.*

- *If B is disjoint with* $\mathbf{fv}\, N$ *then* $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = \mathbf{ord}\, ([\mu] vars)\, \mathbf{in}\, [\mu]N$

+ *If B is disjoint with* $\mathbf{fv}\, P$ *then* $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, P) = \mathbf{ord}\, ([\mu] vars)\, \mathbf{in}\, [\mu]P$

Proof. Mutual induction on N and P.

Proof (should be easy)

□

Lemma 3 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming.*
Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then

- $\mathbf{nf}\, ([\mu]N) = [\mu]\mathbf{nf}\, (N)$

+ $\mathbf{nf}\, ([\mu]P) = [\mu]\mathbf{nf}\, (P)$

Here equality means alpha-equivalence.

Proof. Mutual induction on iN and iP.

Write a little bit about the exists/forall case

□

4.1.2 Soundness

Lemma 4 (Soundness of variable ordering)**.** *Variable ordering extracts precisely used free variables.*

$-$ $\mathbf{ord}\, vars\, \mathbf{in}\ N \equiv vars \cap \mathbf{fv}\ N$ *(as sets)*

$+$ $\mathbf{ord}\, vars\, \mathbf{in}\ P \equiv vars \cap \mathbf{fv}\ P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\, vars\, \mathbf{in}\ N = \vec{\alpha}$ and $\mathbf{ord}\, vars\, \mathbf{in}\ P = \vec{\alpha}$ □

Corollary 1 (Additivity of ordering)**.** $-$ $\mathbf{ord}\,(vars_1 \cup vars_2)\, \mathbf{in}\, N \equiv \mathbf{ord}\, vars_1\, \mathbf{in}\, N \cup \mathbf{ord}\, vars_2\, \mathbf{in}\, N$ *(as sets)*

$+$ $\mathbf{ord}\,(vars_1 \cup vars_2)\, \mathbf{in}\, P \equiv \mathbf{ord}\, vars_1\, \mathbf{in}\, P \cup \mathbf{ord}\, vars_2\, \mathbf{in}\, P$ *(as sets)*

Corollary 2 (Weakening of ordering)**.** $-$ $\mathbf{ord}\,(vars \cap \mathbf{fv}\, N)\, \mathbf{in}\, N \equiv \mathbf{ord}\, vars\, \mathbf{in}\, N$ *(as sets)*

$+$ $\mathbf{ord}\,(vars \cap \mathbf{fv}\, P)\, \mathbf{in}\, P \equiv \mathbf{ord}\, vars\, \mathbf{in}\, P$ *(as sets)*

Lemma 5 (Soundness of quantifier normalization)**.** *Normalization respects equivalence.*

$-$ $N \simeq_1^D \mathbf{nf}\,(N)$

$+$ $P \simeq_1^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

- (Var^-) and (Var^+) by the corresponding equivalence rules.
- ($\uparrow$), ($\downarrow$), and ($\to$) by the induction hypothesis and the corresponding congruent equivalence rules.
- ($\forall$) From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 1, $\mathbf{fv}\, N \equiv \mathbf{fv}\, N'$. Then by lemma 4, $\overrightarrow{\alpha^{+}}' \equiv \overrightarrow{\alpha^{+}} \cap \mathbf{fv}\, N' \equiv \overrightarrow{\alpha^{+}} \cap \mathbf{fv}\, N$, and thus, $\overrightarrow{\alpha^{+}}' \cap \mathbf{fv}\, N' \equiv \overrightarrow{\alpha^{+}} \cap \mathbf{fv}\, N$.
  To prove $\forall\overrightarrow{\alpha^{+}}.N \simeq_1^D \forall\overrightarrow{\alpha^{+}}'.N'$, it suffices to provide a bijection $\mu : \overrightarrow{\alpha^{+}}' \cap \mathbf{fv}\, N' \leftrightarrow \overrightarrow{\alpha^{+}} \cap \mathbf{fv}\, N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.
- ($\exists$) Same as for ($\forall$).

□

Corollary 3. *Free variables are not changed by the normalization*

$-$ $\mathbf{fv}\, N \equiv \mathbf{fv}\, \mathbf{nf}\,(N)$

$+$ $\mathbf{fv}\, P \equiv \mathbf{fv}\, \mathbf{nf}\,(P)$

Proof. Immediately from lemmas 1 and 5. □

4.1.3 Completeness

Lemma 6 (Completeness of variable ordering)**.** *Variable ordering is invariant under equivalence.*

$-$ *For* $N \simeq_1^D M$ *and any vars, if* $\mathbf{ord}\, vars\, \mathbf{in}\ N = \vec{\alpha}_1$ *and* $\mathbf{ord}\, vars\, \mathbf{in}\ M = \vec{\alpha}_2$*, then* $\vec{\alpha}_1 = \vec{\alpha}_2$ *(as lists)*

$+$ *For* $P \simeq_1^D Q$ *and any vars, if* $\mathbf{ord}\, vars\, \mathbf{in}\ P = \vec{\alpha}_1$ *and* $\mathbf{ord}\, vars\, \mathbf{in}\ Q = \vec{\alpha}_2$*, then* $\vec{\alpha}_1 = \vec{\alpha}_2$ *(as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. □

Lemma 7 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

$-$ *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

$+$ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

- ($\forall^{\simeq_1^D}$)
  From the definition of the normalization,

– $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N)$

– $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that by lemma 4 and corollary 3, the domain and codomain of μ can be written as $\mu : \overrightarrow{\beta^+}' \leftrightarrow \overrightarrow{\alpha^+}'$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by lemma 3, the second—by the induction hypothesis.

(ii)
$$
\begin{aligned}
[\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}' \\
&= [\mu]\mathbf{ord}\,(\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by corollaries 2 and 3} \\
&= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 2, because } \overrightarrow{\alpha^+}' \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \\
& && \text{and } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \\
&= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved} \\
&= \mathbf{ord}\,(\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \text{ and } \overrightarrow{\beta^+} \cap \mathbf{fv}\,M \\
&= \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{by corollaries 2 and 3} \\
&= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'
\end{aligned}
$$

- ($\exists^{\simeq^D_1}$) Same as for ($\forall^{\simeq^D_1}$).
- Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□