1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
& | & \alpha^+ & \\
& | & \downarrow N & \\
& | & \exists \alpha^-.P & \\
\\
N,\ M & ::= & & \text{negative types} \\
& | & \alpha^- & \\
& | & \uparrow P & \\
& | & \forall \alpha^+.N & \\
& | & P \to N &
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \text{ D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \text{ D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \text{ D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \text{ D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \text{ D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \text{ D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \text{ D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \text{ D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \text{ D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \text{ D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \text{ D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$P,\ Q$	$::=$			multi-quantified positive types
	$\mid$	α^+		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists\overrightarrow{\alpha^-}.P$		$P \neq \exists\ldots$
	$\mid$	(P)	S	
$N,\ M$	$::=$			multi-quantified negative types
	$\mid$	α^-		
	$\mid$	$\uparrow P$		
	$\mid$	$P \to N$		
	$\mid$	$\forall\overrightarrow{\alpha^+}.N$		$N \neq \forall\ldots$
	$\mid$	(N)	S	

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M}\ \text{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q}\ \text{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-}\ \text{D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q}\ \text{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M}\ \text{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M}\ \text{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+}\ \text{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M}\ \text{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q}\ \text{D1ExistsL}$$

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-}\ (\text{Var}^{-\simeq_1^D})$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q}\ (\uparrow^{\simeq_1^D})$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M}\ (\to^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\textsc{Var}^+)$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \alpha^-} \quad (\textsc{Var}^-_{\in})$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \cdot} \quad (\textsc{Var}^-_{\notin})$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \uparrow P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ vars\ \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})} \quad (\to)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \alpha^+} \quad (\textsc{Var}^+_{\in})$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \cdot} \quad (\textsc{Var}^+_{\notin})$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \downarrow N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \exists \overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad (\exists)$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^- = \cdot} \quad (\textsc{UVar}^-)$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^+ = \cdot} \quad (\textsc{UVar}^+)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\ (\textsc{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\ (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\ (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'}\ (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\ (\textsc{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\ (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'}\ (\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\ (\textsc{UVar}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\ (\textsc{UVar}^+)$$

3.2 Unification

$\boxed{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\ \textsc{UNVar}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\ \textsc{UShiftU}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\ \textsc{UArrow}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\ \textsc{UForall}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)}\ \textsc{UNUVar}$$

$\boxed{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\ \textsc{UPVar}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\ \textsc{UShiftD}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\ \textsc{UExists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)}\ \textsc{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma; \Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot} \quad \text{ANVar}$$

$$\frac{\Theta \models \mathbf{nf}(P) \stackrel{u}{\simeq} \mathbf{nf}(Q) \dashv \widehat{\sigma}}{\Gamma; \Theta \models \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad \text{AShiftU}$$

$$\frac{\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2} \quad \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma; \Theta \models \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^+}} \quad \text{AForall}$$

$\boxed{\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma; \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad \text{APVar}$$

$$\frac{\Theta \models \mathbf{nf}(N) \stackrel{u}{\simeq} \mathbf{nf}(M) \dashv \widehat{\sigma}}{\Gamma; \Theta \models \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \quad \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma; \Theta \models \exists \overrightarrow{\alpha^-}.P \geqslant \exists \overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}} \quad \text{AExists}$$

$$\frac{\mathbf{upgrade}\, \Gamma \vdash \mathbf{nf}(P)\, \mathbf{to}\, \Delta = Q}{\Gamma; \Theta \models \widehat{\alpha}^+\{\Delta\} \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)} \quad \text{APUVar}$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restrict an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\Delta \vdash \widehat{\alpha}^+ :\approx P$ or $\Delta \vdash \widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\Delta \vdash \widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries)**.** *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.
$\boxed{e_1 \mathbin{\&} e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\Gamma \models P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1) \mathbin{\&} (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)} \quad (\geqslant \mathbin{\&} \geqslant)$$

$$\frac{\Gamma; \cdot \models P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \mathbin{\&} (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)} \quad (\simeq \mathbin{\&} \geqslant)$$

$$\frac{\Gamma; \cdot \models Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P) \mathbin{\&} (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)} \quad (\geqslant \mathbin{\&} \simeq)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \mathbin{\&} (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)} \quad (\simeq \mathbin{\&} \simeq^+)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N) \mathbin{\&} (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)} \quad (\simeq \mathbin{\&} \simeq^-)$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3. $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2 = \{e_1 \mathbin{\&} e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, \textit{s.t. } e_1 \textit{ matches with } e_2\}$

3.5 Least Upper Bound

$\boxed{\Gamma \models P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \text{ LUBVar}$$

$$\frac{\Gamma, \cdot \models \downarrow N \overset{a}{\simeq} \downarrow M \dashv (\Xi, \boxed{P}, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\boxed{P}} \text{ LUBShift}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q}{\Gamma \models \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q} \text{ LUBExists}$$

$\boxed{\textbf{upgrade}\,\Gamma \vdash P\,\textbf{to}\,\Delta = Q}$

3.6 Antiunification

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \text{ AUPVar}$$

$$\frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \dashv (\Xi, \downarrow M, \widehat{\tau}_1, \widehat{\tau}_2)} \text{ AUPShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists \overrightarrow{\alpha^-}.\, Q, \widehat{\tau}_1, \widehat{\tau}_2)} \text{ AUPExists}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)} \text{ AUNVar}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)} \text{ AUNShift}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)} \text{ AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \text{ AUNAU}$$

4 Proofs

4.1 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\dfrac{}{\{\mathbf{ord}\ vars\ \mathbf{in}\ N\} \equiv vars \cap \mathbf{fv}\ N}$	$\dfrac{N \simeq_1^D M}{\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ M}$	—
Normalization	$\dfrac{}{N \simeq_1^D \mathbf{nf}\,(N)}$	$\dfrac{N \simeq_1^D M}{\mathbf{nf}\,(N) = \mathbf{nf}\,(M)}$	—
Uppgrade	$\dfrac{\mathbf{upgrade}\ \Gamma \vdash P\ \mathbf{to}\ \Delta = Q}{Q \text{ is sound} \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\ \Gamma \vdash P\ \mathbf{to}\ \Delta = Q}$	$\dfrac{\begin{array}{c} Q' \text{ is sound} \\ \mathbf{upgrade}\ \Gamma \vdash P\ \mathbf{to}\ \Delta = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \vDash P_1 \vee P_2 = Q}{Q \text{ is sound} \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \vDash P_1 \vee P_2 = Q}$	$\dfrac{\begin{array}{c} Q' \text{ is sound} \\ \Gamma \vDash P_1 \vee P_2 = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\begin{array}{l}(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \\ \text{is sound}\end{array} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat{\tau}_i : \Xi \\ [\hat{\tau}_i]\, Q = P_i \end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \hat{\tau}_1', \hat{\tau}_2')}{\begin{array}{l} \exists (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \text{ s.t.} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \end{array}}$	$\dfrac{\begin{array}{c} (\Xi', Q', \hat{\tau}_1', \hat{\tau}_2') \text{ is sound} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \end{array}}{\exists \Gamma; \Xi \vdash \hat{\tau} : \Xi' \text{ s.t. } [\hat{\tau}]\, Q' = Q}$
Unification (matching)	$\dfrac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}}{\hat{\sigma} \text{ is sound} \begin{cases} \Theta \vdash \hat{\sigma} \\ [\hat{\sigma}]\, P = Q \end{cases}}$	$\dfrac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}}$	—
Subtyping	$\dfrac{\Gamma;\, \Theta \vDash N \leqslant M \dashv \hat{\sigma}}{\hat{\sigma} \text{ is sound} \begin{cases} \Theta \vdash \hat{\sigma} \\ \Gamma \vdash [\hat{\sigma}]\, N \leqslant_1 M \end{cases}}$	$\dfrac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Gamma;\, \Theta \vDash N \leqslant M \dashv \hat{\sigma}}$	—

4.2 Variable Ordering

Definition 4 (Collision free bijection). *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is **collision free on sets** P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 1 (Soundness of variable ordering). *Variable ordering extracts precisely used free variables.*

− $\{\mathbf{ord}\ vars\ \mathbf{in}\ N\} \equiv vars \cap \mathbf{fv}\ N$ *(as sets)*

+ $\{\mathbf{ord}\ vars\ \mathbf{in}\ P\} \equiv vars \cap \mathbf{fv}\ P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}$ and $\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}$ □

Corollary 1 (Additivity of ordering). *Variable ordering is additive (in terms of set union) with respect to its first argument.*

− $\{\mathbf{ord}\,(vars_1 \cup vars_2)\ \mathbf{in}\ N\} \equiv \{\mathbf{ord}\ vars_1\ \mathbf{in}\ N\} \cup \{\mathbf{ord}\ vars_2\ \mathbf{in}\ N\}$ *(as sets)*

+ $\{\mathbf{ord}\,(vars_1 \cup vars_2)\ \mathbf{in}\ P\} \equiv \{\mathbf{ord}\ vars_1\ \mathbf{in}\ P\} \cup \{\mathbf{ord}\ vars_2\ \mathbf{in}\ P\}$ *(as sets)*

Corollary 2 (Weakening of ordering). *Extending the first argument of the ordering with unused variables does not change the result.*

− $\mathbf{ord}\,(vars \cap \mathbf{fv}\ N)\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ N$

+ $\mathbf{ord}\,(vars \cap \mathbf{fv}\ P)\ \mathbf{in}\ P = \mathbf{ord}\ vars\ \mathbf{in}\ P$

Lemma 2 (Distributivity of renaming over variable ordering). *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

- *If μ is collision free on vars and* $\mathbf{fv}\,N$ *then* $[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) = \mathbf{ord}\,([\mu]\mathit{vars})\,\mathbf{in}\,[\mu]N$

+ *If μ is collision free on vars and* $\mathbf{fv}\,P$ *then* $[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P) = \mathbf{ord}\,([\mu]\mathit{vars})\,\mathbf{in}\,[\mu]P$

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in \mathit{vars}$
Then
$$\begin{aligned}[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^-) \\ &= [\mu]\alpha^- && \text{by Rule } (\mathrm{Var}^+_\in) \\ &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]\mathit{vars}) \\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,\beta^- && \text{by Rule } (\mathrm{Var}^+_\in)\text{, because } \beta^- \in [\mu]\mathit{vars} \\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\alpha^-\end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin \mathit{vars}$
Notice that $[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. On the other hand, $\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,\alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_\notin)$, because μ is collision free on *vars* and $\mathbf{fv}\,N$, so $\mathbf{fv}\,N \ni \alpha^- \notin \mu(A \cap \mathit{vars}) \cup \mathit{vars} \supseteq [\mu]\mathit{vars}$.

c. $\alpha^- \in A$ but $\alpha^- \notin \mathit{vars}$
Then $[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. To prove that $\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_\notin)$. Let us show that $[\mu]\alpha^- \notin [\mu]\mathit{vars}$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]\mathit{vars} \subseteq \mu(A \cap \mathit{vars}) \cup \mathit{vars}$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap \mathit{vars}) \cup \mathit{vars}$.

(i) If there is an element $x \in A \cap \mathit{vars}$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin \mathit{vars}$. This way, $\mu(\alpha^-) \notin \mu(A \cap \mathit{vars})$.

(ii) Since μ is collision free on *vars* and $\mathbf{fv}\,N$, $\mu(A \cap \mathbf{fv}\,N) \ni \mu(\alpha^-) \notin \mathit{vars}$.

d. $\alpha^- \notin A$ but $\alpha^- \in \mathit{vars}$
$\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_\notin)$, because $\alpha^- = [\mu]\alpha^- \in [\mu]\mathit{vars}$ since $\alpha^- \in \mathit{vars}$. On the other hand, $[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$
$$\begin{aligned}[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\uparrow P) \\ &= [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P) && \text{by Rule } (\uparrow) \\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]P && \text{by the induction hypothesis} \\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,\uparrow[\mu]P && \text{by Rule } (\uparrow) \\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\uparrow P && \text{by the definition of substitution} \\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]N\end{aligned}$$

Case 3. $N = P \to M$
$$\begin{aligned}[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P \to M) \\ &= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})) && \text{where } \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,M = \vec{\alpha}_2 \\ &= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\}) \\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\{\vec{\alpha}_1\}) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to case 1. Notice that } \mu \text{ is} \\ & && \text{collision free on } \{\vec{\alpha}_1\} \text{ and } \{\vec{\alpha}_2\} \text{ since } \{\vec{\alpha}_1\} \subseteq \mathit{vars} \text{ and } \{\vec{\alpha}_2\} \subseteq \mathbf{fv}\,N \\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})\end{aligned}$$
$$\begin{aligned}(\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]N) &= (\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]P \to [\mu]M) \\ &= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \{\vec{\beta}_1\})) && \text{where } \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]M = \vec{\beta}_2 \\ & && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})\end{aligned}$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$
$$\begin{aligned}[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) &= [\mu]\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.M \\ &= [\mu]\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,M \\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]M && \text{by the induction hypothesis}\end{aligned}$$

$$
\begin{aligned}
(\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]N) &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\forall\overrightarrow{\alpha^+}.M \\
&= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.[\mu]M \\
&= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]M
\end{aligned}
$$

□

Lemma 3 (Completeness of variable ordering)**.** *Variable ordering is invariant under equivalence.*

– *For* $N \simeq_1^D M$ *and any vars,* $\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,M$ *(as lists)*

+ *For* $P \simeq_1^D Q$ *and any vars,* $\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,Q$ *(as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. □

4.3 Normaliztaion

Lemma 4. *Set of free variables is invariant under equivalence.*

– *If* $N \simeq_1^D M$ *then* $\mathbf{fv}\,N \equiv \mathbf{fv}\,M$ *(as sets)*

+ *If* $P \simeq_1^D Q$ *then* $\mathbf{fv}\,P \equiv \mathbf{fv}\,Q$ *(as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

Lemma 5. *Free variables are not changed by the normalization*

– $\mathbf{fv}\,N \equiv \mathbf{fv}\,\mathbf{nf}\,(N)$

+ $\mathbf{fv}\,P \equiv \mathbf{fv}\,\mathbf{nf}\,(P)$

Proof. By straightforward induction on $\mathbf{nf}\,(N) = M$. □

Lemma 6 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that* μ *is a bijection between two sets of variables* $\mu : A \leftrightarrow B$*. Then*

– $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$

+ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Here equality means alpha-equivalence.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\,([\mu]N) = \mathbf{nf}\,([\mu]\alpha^-) = [\mu]\alpha^-$. The latter follows from the fact that $[\mu]\alpha^-$ is a variable, and thus, Rule (Var$^-$) is applicable.
$[\mu]\mathbf{nf}\,(N) = [\mu]\mathbf{nf}\,(\alpha^-) = [\mu]\alpha^-$.

Case 2. If the type is formed by $\rightarrow$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \rightarrow M$ then

$$
\begin{aligned}
\mathbf{nf}\,([\mu]N) &= \mathbf{nf}\,([\mu](P \rightarrow M)) \\
&= \mathbf{nf}\,([\mu]P \rightarrow [\mu]M) && \text{By the congruence of substitution} \\
&= \mathbf{nf}\,([\mu]P) \rightarrow \mathbf{nf}\,([\mu]M) && \text{By the congruence of normalization, i.e. Rule } (\rightarrow) \\
&= [\mu]\mathbf{nf}\,(P) \rightarrow [\mu]\mathbf{nf}\,(M) && \text{By the induction hypothesis} \\
&= [\mu](\mathbf{nf}\,(P) \rightarrow \mathbf{nf}\,(M)) && \text{By the congruence of substitution} \\
&= [\mu]\mathbf{nf}\,(P \rightarrow M) && \text{By the congruence of normalization} \\
&= [\mu]\mathbf{nf}\,(N)
\end{aligned}
$$

Case 3. $N = \forall\overrightarrow{\alpha^+}.M$

$$
\begin{aligned}
[\mu]\mathbf{nf}\,(N) &= [\mu]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M) \\
&= [\mu]\forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,(M) && \text{Where } \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) = \overrightarrow{\alpha^{+\prime}}
\end{aligned}
$$

$$\begin{aligned}
\mathbf{nf}\,([\mu]N) &= \mathbf{nf}\,([\mu]\forall\overrightarrow{\alpha^+}.M) \\
&= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\mu]M) && \text{Assuming } \{\overrightarrow{\alpha^+}\} \cap A = \varnothing \text{ and } \{\overrightarrow{\alpha^+}\} \cap B = \varnothing \\
&= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\mu]M) && \text{Where } \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,([\mu]M) = \overrightarrow{\beta^+} \\
&= \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\mu]M) && \text{As } \overrightarrow{\beta^+} = \overrightarrow{\alpha^+}' \text{ (see below)}
\end{aligned}$$

Notice that μ is free of collisions on $\{\overrightarrow{\alpha^+}\}$ and $\mathbf{fv}\,\mathbf{nf}\,(M)$ because

(i) $\mu(A \cap \{\overrightarrow{\alpha^+}\}) \cap \mathbf{fv}\,\mathbf{nf}\,(M) = \varnothing \cap \mathbf{fv}\,\mathbf{nf}\,(M) = \varnothing$ and

(ii) $\mu(A \cap \mathbf{fv}\,\mathbf{nf}\,(M)) \cap \{\overrightarrow{\alpha^+}\} \subseteq B \cap \{\overrightarrow{\alpha^+}\} = \varnothing$

$$\begin{aligned}
\overrightarrow{\beta^+} &= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,([\mu]M) \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{By the induction hypothesis} \\
&= \mathbf{ord}\,\{[\mu]\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{Since } \{\overrightarrow{\alpha^+}\} \cap A = \varnothing \\
&= [\mu]\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by lemma 2} \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{Since } \{\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M)\} \cap A \subseteq \{\overrightarrow{\alpha^+}\} \cap A = \varnothing \\
&= \overrightarrow{\alpha^+}'
\end{aligned}$$

To show alpha-equivalence of $[\mu]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\mu]M)$, we can assume that $\{\overrightarrow{\alpha^+}'\} \cap A = \varnothing$, and $\{\overrightarrow{\alpha^+}'\} \cap B = \varnothing$. Then $[\mu]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M) = \forall\overrightarrow{\alpha^+}'.[\mu]\mathbf{nf}\,(M) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\mu]M)$, the latter follows from the induction hypothesis.

Case 4. $P = \exists\overrightarrow{\alpha^-}.Q$
Same as for case 3.

□

Lemma 7 (Soundness of quantifier normalization)**.**

− $N \simeq_1^D \mathbf{nf}\,(N)$

+ $P \simeq_1^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var^-) and (Var^+)
By the corresponding equivalence rules.

Case 2. ($\uparrow$), ($\downarrow$), and ($\rightarrow$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall$), i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 4, $\mathbf{fv}\,N \equiv \mathbf{fv}\,N'$. Then by lemma 1, $\{\overrightarrow{\alpha^+}'\} \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$, and thus, $\{\overrightarrow{\alpha^+}'\} \cap \mathbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$.

To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^+}'.N'$, it suffices to provide a bijection $\mu : \{\overrightarrow{\alpha^+}'\} \cap \mathbf{fv}\,N' \leftrightarrow \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. ($\exists$) Same as for case 3.

□

Lemma 8 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

− *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

+ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. ($\forall^{\simeq_1^D}$)

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 1 and 5, the domain and the codomain of μ can be written as $\mu : \{\overrightarrow{\beta^+}'\} \leftrightarrow \{\overrightarrow{\alpha^+}'\}$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by lemma 6, the second—by the induction hypothesis.

(ii)
$$
\begin{aligned}
[\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}' \\
&= [\mu]\mathbf{ord}\,(\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 5 and corollary 2} \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 2, because } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing \\
& && \text{and } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved} \\
&= \mathbf{ord}\,(\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \text{ and } \{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 5 and corollary 2} \\
&= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'
\end{aligned}
$$

Case 2. ($\exists^{\simeq_1^D}$) Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

$\square$

4.4 Upper Bounds

Lemma 9 (Shape of the Supertypes)**.** *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \textit{for some } \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \\ \textit{for some } \Gamma \vdash N_i \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

Lemma 10 (Normalized Shape of the Supertypes)**.** *For a normalized positive type* $P = \mathbf{nf}\,(P)$*, let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \\ \textit{for some } \Gamma \vdash N_i \end{array} \right\}$

hen $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 P\}$.

Lemma 11 (Soundness of the Least Upper Bound)**.** *For types* $\Gamma \vdash P_1$*, and* $\Gamma \vdash P_2$*, if* $\Gamma \vDash P_1 \vee P_2 = Q$ *then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Lemma 12 (Completeness of the Least Upper Bound)**.** *For types* $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, *and* $\Gamma \vdash Q'$ *such that* $\Gamma \vdash Q' \geqslant_1 P_1$ *and* $\Gamma \vdash Q' \geqslant_1 P_2$, *there exists* Q *s.t.* $\Gamma \models P_1 \vee P_2 = Q$, *and* $\Gamma \vdash Q' \geqslant_1 Q$

Lemma 13 (Soundness of Upgrade)**.** *For* $\Delta \subseteq \Gamma$, *suppose that* $\mathbf{upgrade}\, \Gamma \vdash P \,\mathbf{to}\, \Delta = Q$. *Then*

(i) $\Delta \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P$

Lemma 14 (Completeness of Upgrade)**.** *For* $\Delta \subseteq \Gamma$, $\Gamma \vdash P$ *and* $\Delta \vdash Q'$, *such that* $\Gamma \vdash Q' \geqslant_1 P$, *there exists* Q *s.t.* $\mathbf{upgrade}\, \Gamma \vdash P \,\mathbf{to}\, \Delta = Q$, *and* $\Delta \vdash Q' \geqslant_1 Q$.