α, β, α, β	type variables
n, m, i, j	index variables

$\alpha^+,\ \beta^+$	$::=$			positive variable
	$\vert$	α^+		
$\alpha^-,\ \beta^-$	$::=$			negative variable
	$\vert$	α^-		
σ	$::=$			substitution
	$\vert$	$\cdot$		
	$\vert$	P/α^+		
	$\vert$	N/α^-		
	$\vert$	$\vec{P}/\overrightarrow{\alpha^+}$		
	$\vert$	$\vec{N}/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^-}}/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	$\vert$	$\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha^-}}$		
	$\vert$	$\vec{\alpha}_1/\vec{\alpha}_2$		
	$\vert$	$\overline{\sigma_i}^{\,i}$		concatenate
e	$::=$			entry of a unification solution
	$\vert$	$\Gamma \vdash \widehat{\alpha}^+ :\approx P$		
	$\vert$	$\Gamma \vdash \widehat{\alpha}^- :\approx N$		
	$\vert$	$\Gamma \vdash \widehat{\alpha}^+ :\geqslant P$		
	$\vert$	(e)	S	
	$\vert$	$e_1\ \&\ e_2$	M	
$\widehat{\sigma}$	$::=$			unification solution (substitution)
	$\vert$	$\cdot$		
	$\vert$	e		
	$\vert$	$\widehat{\sigma}\backslash\overrightarrow{\widetilde{\alpha^+}}$		
	$\vert$	$\widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha^-}}$		
	$\vert$	$\widehat{\sigma}\backslash\widehat{\alpha}^+$		
	$\vert$	$\widehat{\sigma}\backslash\widehat{\alpha}^-$		
	$\vert$	$\widehat{\sigma}_1 \cup \widehat{\sigma}_2$		
	$\vert$	$\overline{\widehat{\sigma}_i}^{\,i}$		concatenate
	$\vert$	$(\widehat{\sigma})$	S	
	$\vert$	$\widehat{\sigma}_1\ \&\ \widehat{\sigma}_2$	M	
$\widehat{\tau}$	$::=$			anti-unification substitution
	$\vert$	$\cdot$		
	$\vert$	$\widehat{\alpha}^- :\approx N$		
	$\vert$	$\widehat{\alpha}^- :\approx N$		
	$\vert$	$\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha^-}}$		
	$\vert$	$\widehat{\tau}_1 \cup \widehat{\tau}_2$		
	$\vert$	$\overline{\widehat{\tau}_i}^{\,i}$		concatenate
	$\vert$	$(\widehat{\tau})$	S	
$P,\ Q$	$::=$			positive types

	$\|$	α^{+}		
	$\|$	$\downarrow N$		
	$\|$	$\exists \alpha^{-}.P$		
	$\|$	$[\sigma] P$	M	
$N,\ M$	$::=$			negative types
	$\|$	α^{-}		
	$\|$	$\uparrow P$		
	$\|$	$\forall \alpha^{+}.N$		
	$\|$	$P \rightarrow N$		
	$\|$	$[\sigma] N$	M	
$\overrightarrow{\alpha^{+}},\ \overrightarrow{\beta^{+}}$	$::=$			positive variable list
	$\|$	$\cdot$		empty list
	$\|$	α^{+}		a variable
	$\|$	$\overrightarrow{\overrightarrow{\alpha^{+}}_{i}}^{i}$		concatenate lists
$\overrightarrow{\alpha^{-}},\ \overrightarrow{\beta^{-}}$	$::=$			negative variables
	$\|$	$\cdot$		empty list
	$\|$	α^{-}		a variable
	$\|$	$\overrightarrow{\overrightarrow{\alpha^{-}}_{i}}^{i}$		concatenate lists
$P,\ Q$	$::=$			multi-quantified positive types
	$\|$	α^{+}		
	$\|$	$\downarrow N$		
	$\|$	$\exists \overrightarrow{\alpha^{-}}.P$		$P \neq \exists \ldots$
	$\|$	$[\sigma] P$	M	
	$\|$	$[\widehat{\tau}] P$	M	
	$\|$	$[\widehat{\sigma}] P$	M	
	$\|$	$[\mu] P$	M	
	$\|$	(P)	S	
	$\|$	$\mathbf{nf}\,(P')$	M	
$N,\ M$	$::=$			multi-quantified negative types
	$\|$	α^{-}		
	$\|$	$\uparrow P$		
	$\|$	$P \rightarrow N$		
	$\|$	$\forall \overrightarrow{\alpha^{+}}.N$		$N \neq \forall \ldots$
	$\|$	$[\sigma] N$	M	
	$\|$	$[\mu] N$	M	
	$\|$	$[\widehat{\sigma}] N$	M	
	$\|$	(N)	S	
	$\|$	$\mathbf{nf}\,(N')$	M	
$\overrightarrow{P}$	$::=$			list of positive types
	$\|$	$\cdot$		empty list
	$\|$	P		a singel type
	$\|$	$\overrightarrow{\overrightarrow{P}_{i}}^{i}$		concatenate lists

$\overrightarrow{N}$	$::=$			list of negative types
	$\mid$	$\cdot$		empty list
	$\mid$	N		a singel type
	$\mid$	$\overline{\overrightarrow{N}_i}^{\,i}$		concatenate lists
$\Delta,\ \Gamma$	$::=$			declarative type context
	$\mid$	$\cdot$		empty context
	$\mid$	$\overrightarrow{\alpha^+}$		list of variables
	$\mid$	$\overrightarrow{\alpha^-}$		list of variables
	$\mid$	$vars$		
	$\mid$	$\overline{\Gamma_i}^{\,i}$		concatenate contexts
	$\mid$	(Γ)	S	
Θ	$::=$			unification type variable context
	$\mid$	$\cdot$		empty context
	$\mid$	$\overrightarrow{\widetilde{\alpha}^+}$		list of variables
	$\mid$	$\overrightarrow{\widehat{\alpha}^-}$		list of variables
	$\mid$	$\overline{\Theta_i}^{\,i}$		concatenate contexts
	$\mid$	(Θ)	S	
Ξ	$::=$			anti-unification type variable context
	$\mid$	$\cdot$		empty context
	$\mid$	$\overrightarrow{\widetilde{\alpha}^+}$		list of variables
	$\mid$	$\overrightarrow{\widehat{\alpha}^-}$		list of variables
	$\mid$	$\overline{\Xi_i}^{\,i}$		concatenate contexts
	$\mid$	(Ξ)	S	
	$\mid$	$\Xi_1 \cup \Xi_2$		
$\vec{\alpha},\ \vec{\beta}$	$::=$			ordered positive or negative variables
	$\mid$	$\cdot$		empty list
	$\mid$	$\overrightarrow{\alpha^+}$		list of variables
	$\mid$	$\overrightarrow{\alpha^-}$		list of variables
	$\mid$	$\vec{\alpha}_1 \backslash vars$		setminus
	$\mid$	Γ		context
	$\mid$	$vars$		
	$\mid$	$\overline{\vec{\alpha}_i}^{\,i}$		concatenate contexts
	$\mid$	$(\vec{\alpha})$	S	parenthesis
	$\mid$	$[\mu]\vec{\alpha}$	M	apply moving to list
	$\mid$	$\mathbf{ord}\, vars\, \mathbf{in}\, P$	M	
	$\mid$	$\mathbf{ord}\, vars\, \mathbf{in}\, N$	M	
	$\mid$	$\mathbf{ord}\, vars\, \mathbf{in}\, P$	M	
	$\mid$	$\mathbf{ord}\, vars\, \mathbf{in}\, N$	M	
$vars$	$::=$			set of variables
	$\mid$	$\varnothing$		empty set
	$\mid$	$\mathbf{fv}\, P$		free variables
	$\mid$	$\mathbf{fv}\, N$		free variables
	$\mid$	$\mathbf{fv}\, P$		free variables
	$\mid$	$\mathbf{fv}\, N$		free variables
	$\mid$	$vars_1 \cap vars_2$		set intersection

	|	$vars_1 \cup vars_2$		set union
	|	$vars_1 \backslash vars_2$		set complement
	|	$\mathbf{mv}\, P$		movable variables
	|	$\mathbf{mv}\, N$		movable variables
	|	$\mathbf{uv}\, N$		unification variables
	|	$\mathbf{uv}\, P$		unification variables
	|	$\mathbf{fv}\, N$		free variables
	|	$\mathbf{fv}\, P$		free variables
	|	$(vars)$	S	parenthesis
	|	$\{\vec{\alpha}\}$		ordered list of variables
	|	$[\mu]vars$	M	apply moving to varset
μ	::=			
	|	$\cdot$		empty moving
	|	$\widetilde{\alpha}^+_1 \mapsto \widetilde{\alpha}^+_2$		Positive unit substitution
	|	$\widetilde{\alpha}^-_1 \mapsto \widetilde{\alpha}^-_2$		Positive unit substitution
	|	$\mu_1 \cup \mu_2$	M	Set-like union of movings
	|	$\overline{\mu_i}^{\,i}$		concatenate movings
	|	$\mu\vert_{vars}$	M	restriction on a set
	|	μ^{-1}	M	inversion
n	::=			cohort index
	|	0		
	|	$n+1$		
$\widetilde{\alpha}^+$	::=			positive movable variable
	|	$\widetilde{\alpha}^{+n}$		
$\widetilde{\alpha}^-$	::=			negative movable variable
	|	$\widetilde{\alpha}^{-n}$		
$\overrightarrow{\widetilde{\alpha}^+}$, $\overrightarrow{\widetilde{\beta}^+}$	::=			positive movable variable list
	|	$\cdot$		empty list
	|	$\widetilde{\alpha}^+$		a variable
	|	$\overrightarrow{\widetilde{\alpha}^{+n}}$		from a non-movable variable
	|	$\overline{\overrightarrow{\widetilde{\alpha}^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\widetilde{\alpha}^-}$, $\overrightarrow{\widetilde{\beta}^-}$	::=			negatiive movable variable list
	|	$\cdot$		empty list
	|	$\widetilde{\alpha}^-$		a variable
	|	$\overrightarrow{\widetilde{\alpha}^{-n}}$		from a non-movable variable
	|	$\overline{\overrightarrow{\widetilde{\alpha}^-}_i}^{\,i}$		concatenate lists
P, Q	::=			multi-quantified positive types with movable variables
	|	α^+		
	|	$\widetilde{\alpha}^+$		
	|	$\downarrow N$		
	|	$\exists \overrightarrow{\alpha^-}.P$		
	|	$[\sigma]P$	M	

	$\vert$	$[\mu]P$	M	
$N,\ M$	$::=$			multi-quantified negative types with movable variables
	$\vert$	α^-		
	$\vert$	$\widetilde{\alpha}^-$		
	$\vert$	$\uparrow P$		
	$\vert$	$P \to N$		
	$\vert$	$\forall \overrightarrow{\alpha^+}.N$		
	$\vert$	$[\sigma]N$	M	
	$\vert$	$[\mu]N$	M	
$\widehat{\alpha}^+$	$::=$			positive unification variable
	$\vert$	$\widehat{\alpha}^+$		
	$\vert$	$\widehat{\alpha}^+\{\Delta\}$		
$\widehat{\alpha}^-$	$::=$			negative unification variable
	$\vert$	$\widehat{\alpha}^-$		
	$\vert$	$\widehat{\alpha}^-_{\{N,M\}}$		
	$\vert$	$\widehat{\alpha}^-\{\Delta\}$		
$\overrightarrow{\widetilde{\alpha^+}},\ \overrightarrow{\widetilde{\beta^+}}$	$::=$			positive unification variable list
	$\vert$	$\cdot$		empty list
	$\vert$	$\widehat{\alpha}^+$		a variable
	$\vert$	$\overrightarrow{\widehat{\alpha}^+\{\Delta\}}$		from a normal variable
	$\vert$	$\overrightarrow{\widehat{\alpha}^+}$		from a normal variable, context unspecified
	$\vert$	$\overline{\overrightarrow{\widetilde{\alpha^+}}_i}^{\,i}$		concatenate lists
$\overrightarrow{\widehat{\alpha^-}},\ \overrightarrow{\widehat{\beta^-}}$	$::=$			negative unification variable list
	$\vert$	$\cdot$		empty list
	$\vert$	$\widehat{\alpha}^-$		a variable
	$\vert$	Ξ		from an antiunification context
	$\vert$	$\overrightarrow{\widehat{\alpha}^-\{\Delta\}}$		from a normal variable
	$\vert$	$\overrightarrow{\widehat{\alpha}^-}$		from a normal variable, context unspecified
	$\vert$	$\overline{\overrightarrow{\widehat{\alpha^-}}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$			a positive algorithmic type (potentially with metavariables)
	$\vert$	α^+		
	$\vert$	$\widetilde{\alpha}^+$		
	$\vert$	$\widehat{\alpha}^+$		
	$\vert$	$\downarrow N$		
	$\vert$	$\exists \overrightarrow{\alpha^-}.P$		
	$\vert$	$[\sigma]P$	M	
	$\vert$	$[\widehat{\tau}]P$	M	
	$\vert$	$[\mu]P$	M	
	$\vert$	$\mathbf{nf}\,(P')$	M	
$N,\ M$	$::=$			a negative algorithmic type (potentially with metavariables)
	$\vert$	α^-		
	$\vert$	$\widehat{\alpha}^-$		

	$\vert$	$\uparrow P$	
	$\vert$	$P \to N$	
	$\vert$	$\forall \overrightarrow{\alpha^+}. N$	
	$\vert$	$[\sigma] N$	M
	$\vert$	$[\mu] N$	M
	$\vert$	$\mathbf{nf}\,(N')$	M
auSol	$::=$		
	$\vert$	$(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$	
terminals	$::=$		
	$\vert$	$\exists$	
	$\vert$	$\forall$	
	$\vert$	$\uparrow$	
	$\vert$	$\downarrow$	
	$\vert$	$\rightarrow$	
	$\vert$	$\leftrightarrow$	
	$\vert$	$\in$	
	$\vert$	$\notin$	
	$\vert$	$\cdot$	
	$\vert$	$\vdash$	
	$\vert$	$\leqslant$	
	$\vert$	$\geqslant$	
	$\vert$	$\simeq$	
	$\vert$	$\cup$	
	$\vert$	$\cap$	
	$\vert$	$\setminus$	
	$\vert$	$\subseteq$	
	$\vert$	$\mapsto$	
	$\vert$	$\stackrel{u}{\simeq}$	
	$\vert$	$\stackrel{a}{\simeq}$	
	$\vert$	$\varnothing$	
	$\vert$	$\vDash$	
	$\vert$	$\dashv$	
	$\vert$	$\neq$	
	$\vert$	$\equiv_n$	
	$\vert$	$\vee$	
	$\vert$	$\Downarrow$	
	$\vert$	$:\geqslant$	
	$\vert$	$:\simeq$	
formula	$::=$		
	$\vert$	*judgement*	
	$\vert$	$formula_1 \quad .. \quad formula_n$	
	$\vert$	$\mu : vars_1 \leftrightarrow vars_2$	
	$\vert$	μ **is bijective**	
	$\vert$	$\widehat{\sigma}$ **is functional**	
	$\vert$	$\widehat{\sigma}_1 \in \widehat{\sigma}_2$	
	$\vert$	$vars_1 \subseteq vars_2$	
	$\vert$	$vars_1 = vars_2$	

	$\|$	$vars$ **is fresh**	
	$\|$	$\alpha^- \notin vars$	
	$\|$	$\alpha^+ \notin vars$	
	$\|$	$\alpha^- \in vars$	
	$\|$	$\alpha^+ \in vars$	
	$\|$	$\widehat{\alpha}^- \in \Theta$	
	$\|$	$\widehat{\alpha}^+ \in \Theta$	
	$\|$	if any other rule is not applicable	
	$\|$	$N \neq M$	
	$\|$	$P \neq Q$	
A	$::=$		
	$\|$	$\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}$	Negative subtyping
	$\|$	$\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}$	Positive supertyping
AU	$::=$		
	$\|$	$\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$	
	$\|$	$\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$	
$E1$	$::=$		
	$\|$	$N \simeq_1^D M$	Negative multi-quantified type equivalence
	$\|$	$P \simeq_1^D Q$	Positive multi-quantified type equivalence
	$\|$	$P \simeq Q$	
$D1$	$::=$		
	$\|$	$\Gamma \vdash N \simeq_1^{\leqslant} M$	Negative equivalence on MQ types
	$\|$	$\Gamma \vdash P \simeq_1^{\leqslant} Q$	Positive equivalence on MQ types
	$\|$	$\Gamma \vdash N \leqslant_1 M$	Negative subtyping
	$\|$	$\Gamma \vdash P \geqslant_1 Q$	Positive supertyping
$D0$	$::=$		
	$\|$	$\Gamma \vdash N \simeq_0^{\leqslant} M$	Negative equivalence
	$\|$	$\Gamma \vdash P \simeq_0^{\leqslant} Q$	Positive equivalence
	$\|$	$\Gamma \vdash N \leqslant_0 M$	Negative subtyping
	$\|$	$\Gamma \vdash P \geqslant_0 Q$	Positive supertyping
EQ	$::=$		
	$\|$	$N = M$	Negative type equality (alpha-equivalence)
	$\|$	$P = Q$	Positive type equuality (alphha-equivalence)
	$\|$	$P = Q$	
$LUBF$	$::=$		
	$\|$	**ord** $vars$ **in** $P === \vec{\alpha}$	
	$\|$	**ord** $vars$ **in** $N === \vec{\alpha}$	
	$\|$	**ord** $vars$ **in** $P === \vec{\alpha}$	
	$\|$	**ord** $vars$ **in** $N === \vec{\alpha}$	
	$\|$	**nf** $(N') === N$	
	$\|$	**nf** $(P') === P$	
	$\|$	**nf** $(N') === N$	

	|	$\mathbf{nf}\,(P') === P$	
	|	$e_1 \;\&\; e_2$	
	|	$\widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2$	
LUB	::=		
	|	$\Gamma \vDash P_1 \vee P_2 = Q$	Least Upper Bound (Least Common Supertype)
	|	$\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$	
Nrm	::=		
	|	$\mathbf{nf}\,(N) = M$	
	|	$\mathbf{nf}\,(P) = Q$	
	|	$\mathbf{nf}\,(N) = M$	
	|	$\mathbf{nf}\,(P) = Q$	
Order	::=		
	|	$\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}$	
	|	$\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}$	
	|	$\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}$	
	|	$\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}$	
SM	::=		
	|	$e_1 \;\&\; e_2 = e_3$	Unification Solution Entry Merge
	|	$\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3$	Merge unification solutions
U	::=		
	|	$\Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}$	Negative unification
	|	$\Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}$	Positive unification
WF	::=		
	|	$\Gamma \vdash N$	Negative type well-formedness
	|	$\Gamma \vdash P$	Positive type well-formedness
	|	$\Gamma \vdash N$	Negative type well-formedness
	|	$\Gamma \vdash P$	Positive type well-formedness
	|	$\Gamma; \Xi \vdash P$	Positive anti-unification type well-formedness
	|	$\Gamma; \Xi_2 \vdash \widehat{\tau} : \Xi_1$	Antiunification substitution well-formedness
	|	$\Theta \vdash \widehat{\sigma}$	Unification substitution well-formedness
	|	$\Gamma \vdash \Theta$	Unification context well-formedness
judgement	::=		
	|	*A*	
	|	*AU*	
	|	*E1*	
	|	*D1*	
	|	*D0*	
	|	*EQ*	
	|	*LUB*	
	|	*Nrm*	
	|	*Order*	
	|	*SM*	

$$
\begin{array}{lll}
 & | & U \\
 & | & \mathit{WF} \\
\\
\mathit{user\_syntax} & ::= & \\
 & | & \alpha \\
 & | & n \\
 & | & \alpha^{+} \\
 & | & \alpha^{-} \\
 & | & \sigma \\
 & | & e \\
 & | & \widehat{\sigma} \\
 & | & \widehat{\tau} \\
 & | & P \\
 & | & N \\
 & | & \overrightarrow{\alpha^{+}} \\
 & | & \overrightarrow{\alpha^{-}} \\
 & | & P \\
 & | & N \\
 & | & \overrightarrow{P} \\
 & | & \overrightarrow{N} \\
 & | & \Gamma \\
 & | & \Theta \\
 & | & \Xi \\
 & | & \overrightarrow{\alpha} \\
 & | & \mathit{vars} \\
 & | & \mu \\
 & | & n \\
 & | & \widetilde{\alpha}^{+} \\
 & | & \widetilde{\alpha}^{-} \\
 & | & \overrightarrow{\widetilde{\alpha}^{+}} \\
 & | & \overrightarrow{\widetilde{\alpha}^{-}} \\
 & | & P \\
 & | & N \\
 & | & \widehat{\alpha}^{+} \\
 & | & \widehat{\alpha}^{-} \\
 & | & \overrightarrow{\widetilde{\alpha}^{+}} \\
 & | & \overrightarrow{\widehat{\alpha}^{-}} \\
 & | & P \\
 & | & N \\
 & | & \mathit{auSol} \\
 & | & \mathit{terminals} \\
 & | & \mathit{formula}
\end{array}
$$

$\boxed{\Gamma;\, \Theta \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma;\, \Theta \models \alpha^{-} \leqslant \alpha^{-} \dashv \cdot} \quad \text{ANVar}$$

$$\frac{\Theta \models \mathbf{nf}\,(P) \stackrel{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma;\, \Theta \models {\uparrow}P \leqslant {\uparrow}Q \dashv \widehat{\sigma}} \quad \text{AShiftU}$$

$$\frac{\Gamma;\,\Theta \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma;\,\Theta \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma;\,\Theta \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\ \textsc{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+};\,\Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma;\,\Theta \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}}\ \textsc{AForall}$$

$\boxed{\Gamma;\,\Theta \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma;\,\Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \textsc{APVar}$$

$$\frac{\Theta \models \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma;\,\Theta \models {\downarrow}N \geqslant {\downarrow}M \dashv \widehat{\sigma}}\ \textsc{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-};\,\Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma;\,\Theta \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}}\ \textsc{AExists}$$

$$\frac{\mathbf{upgrade}\,\Gamma \vdash \mathbf{nf}\,(P)\,\mathbf{to}\,\Delta = Q}{\Gamma;\,\Theta \models \widehat{\alpha}^+\{\Delta\} \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)}\ \textsc{APUVar}$$

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)}\ \textsc{AUPVar}$$

$$\frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models {\downarrow}N_1 \overset{a}{\simeq} {\downarrow}N_2 \dashv (\Xi, {\downarrow}M, \widehat{\tau}_1, \widehat{\tau}_2)}\ \textsc{AUPShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists\overrightarrow{\alpha^-}.Q, \widehat{\tau}_1, \widehat{\tau}_2)}\ \textsc{AUPExists}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)}\ \textsc{AUNVar}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models {\uparrow}P_1 \overset{a}{\simeq} {\uparrow}P_2 \dashv (\Xi, {\uparrow}Q, \widehat{\tau}_1, \widehat{\tau}_2)}\ \textsc{AUNShift}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)}\ \textsc{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))}\ \textsc{AUNAU}$$

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-}\ \textsc{E1NVar}$$

$$\frac{P \simeq_1^D Q}{{\uparrow}P \simeq_1^D {\uparrow}Q}\ \textsc{E1ShiftU}$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M}\ \textsc{E1Arrow}$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M} \textsc{E1Forall}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \textsc{E1PVar}$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \textsc{E1ShiftD}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists\overrightarrow{\alpha^-}.P \simeq_1^D \exists\overrightarrow{\beta^-}.Q} \textsc{E1Exists}$$

$\boxed{P \simeq Q}$

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \textsc{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \textsc{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \textsc{D1NVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \textsc{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \textsc{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M} \textsc{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \textsc{D1PVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \textsc{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q} \textsc{D1ExistsL}$$

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \textsc{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \quad \text{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \quad \text{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \quad \text{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \quad \text{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \quad \text{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \quad \text{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \quad \text{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \quad \text{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \quad \text{D0ExistsR}$$

$\boxed{N = M}$ Negative type equality (alpha-equivalence)

$\boxed{P = Q}$ Positive type equuality (alphha-equivalence)

$\boxed{P = Q}$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P}$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N}$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P}$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N}$

$\boxed{\mathbf{nf}\ (N')}$

$\boxed{\mathbf{nf}\ (P')}$

$\boxed{\mathbf{nf}\ (N')}$

$\boxed{\mathbf{nf}\,(P')}$

$\boxed{e_1 \,\&\, e_2}$

$\boxed{\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}$

$\boxed{\Gamma \models P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVAR}$$

$$\frac{\Gamma, \cdot \models {\downarrow}N \overset{a}{\simeq} {\downarrow}M \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models {\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P} \quad \text{LUBSHIFT}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q}{\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q} \quad \text{LUBEXISTS}$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-} \quad \text{NRMNVAR}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,({\uparrow}P) = {\uparrow}Q} \quad \text{NRMSHIFTU}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(P \to N) = Q \to M} \quad \text{NRMARROW}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+{}'}.N'} \quad \text{NRMFORALL}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+} \quad \text{NRMPVAR}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,({\downarrow}N) = {\downarrow}M} \quad \text{NRMSHIFTD}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-{}'}.P'} \quad \text{NRMEXISTS}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-} \quad \text{NRMNUVAR}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+} \quad \text{NRMPUVAR}$$

$\boxed{\textbf{ord}\ vars\ \textbf{in}\ N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\textbf{ord}\ vars\ \textbf{in}\ \alpha^- = \alpha^-}\ \textsc{ONVarIn}$$

$$\frac{\alpha^- \notin vars}{\textbf{ord}\ vars\ \textbf{in}\ \alpha^- = \cdot}\ \textsc{ONVarNIn}$$

$$\frac{\textbf{ord}\ vars\ \textbf{in}\ P = \vec{\alpha}}{\textbf{ord}\ vars\ \textbf{in}\ \uparrow P = \vec{\alpha}}\ \textsc{OShiftU}$$

$$\frac{\textbf{ord}\ vars\ \textbf{in}\ P = \vec{\alpha}_1 \quad \textbf{ord}\ vars\ \textbf{in}\ N = \vec{\alpha}_2}{\textbf{ord}\ vars\ \textbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})}\ \textsc{OArrow}$$

$$\frac{vars \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \textbf{ord}\ vars\ \textbf{in}\ N = \vec{\alpha}}{\textbf{ord}\ vars\ \textbf{in}\ \forall\overrightarrow{\alpha^+}.N = \vec{\alpha}}\ \textsc{OForall}$$

$\boxed{\textbf{ord}\ vars\ \textbf{in}\ P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\textbf{ord}\ vars\ \textbf{in}\ \alpha^+ = \alpha^+}\ \textsc{OPVarIn}$$

$$\frac{\alpha^+ \notin vars}{\textbf{ord}\ vars\ \textbf{in}\ \alpha^+ = \cdot}\ \textsc{OPVarNIn}$$

$$\frac{\textbf{ord}\ vars\ \textbf{in}\ N = \vec{\alpha}}{\textbf{ord}\ vars\ \textbf{in}\ \downarrow N = \vec{\alpha}}\ \textsc{OShiftD}$$

$$\frac{vars \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \textbf{ord}\ vars\ \textbf{in}\ P = \vec{\alpha}}{\textbf{ord}\ vars\ \textbf{in}\ \exists\overrightarrow{\alpha^-}.P = \vec{\alpha}}\ \textsc{OExists}$$

$\boxed{\textbf{ord}\ vars\ \textbf{in}\ N = \vec{\alpha}}$

$$\frac{}{\textbf{ord}\ vars\ \textbf{in}\ \widehat{\alpha}^- = \cdot}\ \textsc{ONUVar}$$

$\boxed{\textbf{ord}\ vars\ \textbf{in}\ P = \vec{\alpha}}$

$$\frac{}{\textbf{ord}\ vars\ \textbf{in}\ \widehat{\alpha}^+ = \cdot}\ \textsc{OPUVar}$$

$\boxed{e_1\ \&\ e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{\Gamma \models P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1)\ \&\ (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)}\ \textsc{SMESupSup}$$

$$\frac{\Gamma; \cdot \models P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P)\ \&\ (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\ \textsc{SMEEqSup}$$

$$\frac{\Gamma; \cdot \models Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P)\ \&\ (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)}\ \textsc{SMESupEq}$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P)\ \&\ (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\ \textsc{SMEPEqEq}$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N)\ \&\ (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)}\ \textsc{SMENEqEq}$$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$\boxed{\Theta \models N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Theta \models \alpha^- \stackrel{u}{\simeq} \alpha^- \dashv \cdot}\ \textsc{UNVar}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}} \quad \text{USHIFTU}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2} \quad \text{UARROW}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \forall \overrightarrow{\alpha^+}. N \overset{u}{\simeq} \forall \overrightarrow{\alpha^+}. M \dashv \widehat{\sigma}} \quad \text{UFORALL}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)} \quad \text{UNUVAR}$$

$\boxed{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot} \quad \text{UPVAR}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}} \quad \text{USHIFTD}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \exists \overrightarrow{\alpha^-}. P \overset{u}{\simeq} \exists \overrightarrow{\alpha^-}. Q \dashv \widehat{\sigma}} \quad \text{UEXISTS}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)} \quad \text{UPUVAR}$$

$\boxed{\Gamma \vdash N}$ Negative type well-formedness
$\boxed{\Gamma \vdash P}$ Positive type well-formedness
$\boxed{\Gamma \vdash N}$ Negative type well-formedness
$\boxed{\Gamma \vdash P}$ Positive type well-formedness
$\boxed{\Gamma ; \Xi \vdash P}$ Positive anti-unification type well-formedness
$\boxed{\Gamma ; \Xi_2 \vdash \widehat{\tau} : \Xi_1}$ Antiunification substitution well-formedness
$\boxed{\Theta \vdash \widehat{\sigma}}$ Unification substitution well-formedness
$\boxed{\Gamma \vdash \Theta}$ Unification context well-formedness

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Definition rules:         72 good    7 bad
Definition rule clauses: 130 good    7 bad
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