1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
 & | & \alpha^+ & \\
 & | & {\downarrow}N & \\
 & | & \exists\alpha^-.P & \\
 & & & \\
N,\ M & ::= & & \text{negative types} \\
 & | & \alpha^- & \\
 & | & {\uparrow}P & \\
 & | & \forall\alpha^+.N & \\
 & | & P \to N &
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M}\ \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q}\ \textsc{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-}\ \textsc{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash {\uparrow}P \leqslant_0 {\uparrow}Q}\ \textsc{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall\beta^+.M'}{\Gamma \vdash \forall\alpha^+.N \leqslant_0 M}\ \textsc{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall\alpha^+.M}\ \textsc{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M}\ \textsc{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+}\ \textsc{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash {\downarrow}N \geqslant_0 {\downarrow}M}\ \textsc{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists\alpha^-.Q'}{\Gamma \vdash \exists\alpha^-.P \geqslant_0 Q}\ \textsc{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists\alpha^-.Q}\ \textsc{D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$P,\ Q$	$::=$			multi-quantified positive types
	$\mid$	α^+		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists\overrightarrow{\alpha^-}.P$		$P \neq \exists\ldots$
	$\mid$	(P)	S	
$N,\ M$	$::=$			multi-quantified negative types
	$\mid$	α^-		
	$\mid$	$\uparrow P$		
	$\mid$	$P \to N$		
	$\mid$	$\forall\overrightarrow{\alpha^+}.N$		$N \neq \forall\ldots$
	$\mid$	(N)	S	

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \quad (\simeq_1^{\leqslant}{}^-)$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \quad (\simeq_1^{\leqslant}{}^+)$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad (\text{VAR}^{-\leqslant_1})$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \quad (\uparrow^{\leqslant_1})$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \quad (\to^{\leqslant_1})$$

$$\frac{\mathbf{fv}\, N \cap \{\overrightarrow{\beta^+}\} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M} \quad (\forall^{\leqslant_1})$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad (\text{VAR}^{+\geqslant_1})$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \quad (\downarrow^{\geqslant_1})$$

$$\frac{\mathbf{fv}\, P \cap \{\overrightarrow{\beta^-}\} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q} \quad (\exists^{\geqslant_1})$$

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-} \quad (\text{VAR}^{-\simeq_1^D})$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad (\uparrow^{\simeq_1^D})$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M} \quad (\to^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M} \; (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \; (\text{VAR}^+)$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \; (\downarrow^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \; (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^- = \alpha^-} \; (\text{VAR}^-_{\in})$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^- = \cdot} \; (\text{VAR}^-_{\notin})$$

$$\frac{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in}\, \uparrow P = \vec{\alpha}} \; (\uparrow)$$

$$\frac{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}_1 \quad \mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}_2}{\mathbf{ord}\; vars\, \mathbf{in}\, P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})} \; (\to)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}} \; (\forall)$$

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^+ = \alpha^+} \; (\text{VAR}^+_{\in})$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^+ = \cdot} \; (\text{VAR}^+_{\notin})$$

$$\frac{\mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in}\, \downarrow N = \vec{\alpha}} \; (\downarrow)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in}\, \exists \overrightarrow{\alpha^-}.P = \vec{\alpha}} \; (\exists)$$

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\; vars\, \mathbf{in}\, \widehat{\alpha}^- = \cdot} \; (\text{UVAR}^-)$$

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\; vars\, \mathbf{in}\, \widehat{\alpha}^+ = \cdot} \; (\text{UVAR}^+)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-} \quad (\text{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q} \quad (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M} \quad (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'} \quad (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+} \quad (\text{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M} \quad (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'} \quad (\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-} \quad (\text{UVar}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+} \quad (\text{UVar}^+)$$

3.2 Unification

$\boxed{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot} \quad \text{UNVar}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}} \quad \text{UShiftU}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2} \quad \text{UArrow}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}} \quad \text{UForall}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)} \quad \text{UNUVar}$$

$\boxed{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot} \quad \text{UPVar}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}} \quad \text{UShiftD}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}} \quad \text{UExists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)} \quad \text{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma; \Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot} \quad \text{ANVar}$$

$$\frac{\Theta \models \mathbf{nf}\,(P) \stackrel{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma; \Theta \models {\uparrow}P \leqslant {\uparrow}Q \dashv \widehat{\sigma}} \quad \text{AShiftU}$$

$$\frac{\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2} \quad \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma; \Theta \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}} \quad \text{AForall}$$

$\boxed{\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma; \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad \text{APVar}$$

$$\frac{\Theta \models \mathbf{nf}\,(N) \stackrel{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma; \Theta \models {\downarrow}N \geqslant {\downarrow}M \dashv \widehat{\sigma}} \quad \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma; \Theta \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}} \quad \text{AExists}$$

$$\frac{\mathbf{upgrade}\,\Gamma \vdash \mathbf{nf}\,(P)\,\mathbf{to}\,\Delta = Q}{\Gamma; \Theta \models \widehat{\alpha}^+\{\Delta\} \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)} \quad \text{APUVar}$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restrict an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\Delta \vdash \widehat{\alpha}^+ :\approx P$ or $\Delta \vdash \widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\Delta \vdash \widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries)**.** *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.
$\boxed{e_1 \;\&\; e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\Gamma \models P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)} \quad (\geqslant \& \geqslant)$$

$$\frac{\Gamma; \cdot \models P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)} \quad (\simeq \& \geqslant)$$

$$\frac{\Gamma; \cdot \models Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)} \quad (\geqslant \& \simeq)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)} \quad (\simeq \& \simeq^+)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N) \;\&\; (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)} \quad (\simeq \& \simeq^-)$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3.
$$\begin{aligned}\widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2 = &\{e_1 \;\&\; e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, \textit{s.t. } e_1 \textit{ matches with } e_2\}\\ &\cup \{e_1 \mid e_1 \in \widehat{\sigma}_1,\; \textit{s.t. } \forall e_2 \in \widehat{\sigma}_2, e_1 \textit{ does not match with } e_2\}\\ &\cup \{e_2 \mid e_2 \in \widehat{\sigma}_2,\; \textit{s.t. } \forall e_1 \in \widehat{\sigma}_1, e_1 \textit{ does not match with } e_2\}\end{aligned}$$

3.5 Least Upper Bound

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+} \ (\textsc{Var}^{\vee})$$

$$\frac{\Gamma, \cdot \vDash {\downarrow} N \overset{a}{\simeq} {\downarrow} M \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash {\downarrow} N \vee {\downarrow} M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi] P} \ ({\downarrow}^{\vee})$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q} \ (\exists^{\vee})$$

$\boxed{\mathbf{upgrade}\, \Gamma \vdash P\, \mathbf{to}\, \Delta = Q}$

3.6 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \ \text{AUPVar}$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash {\downarrow} N_1 \overset{a}{\simeq} {\downarrow} N_2 \dashv (\Xi, {\downarrow} M, \widehat{\tau}_1, \widehat{\tau}_2)} \ \text{AUPShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists \overrightarrow{\alpha^-}.\, Q, \widehat{\tau}_1, \widehat{\tau}_2)} \ \text{AUPExists}$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)} \ \text{AUNVar}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash {\uparrow} P_1 \overset{a}{\simeq} {\uparrow} P_2 \dashv (\Xi, {\uparrow} Q, \widehat{\tau}_1, \widehat{\tau}_2)} \ \text{AUNShift}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)} \ \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \ \text{AUNAU}$$

4 Proofs

4.1 Declarative Subtyping

Lemma 1 (Free Variable Propagation)**.** *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context* Γ*,*

- $-$ *if* $\Gamma \vdash N \leqslant_1 M$ *then* $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$
- $+$ *if* $\Gamma \vdash P \geqslant_1 Q$ *then* $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$

Proof. Mutual induction on $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash P \geqslant_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
It is self-evident that $\{\alpha^-\} \subseteq \{\alpha^-\}$.

Case 2. $\Gamma \vdash {\uparrow} P \leqslant_1 {\uparrow} Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq_1^{\leqslant} Q$), we have $\Gamma \vdash P \geqslant_1 Q$. Then by the induction hypothesis, $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$. The desired inclusion inclusion holds, since $\mathbf{fv}\,({\uparrow} P) = \mathbf{fv}\,(P)$ and $\mathbf{fv}\,({\uparrow} Q) = \mathbf{fv}\,(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant_1 Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$ and $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$. Then $\mathbf{fv}\,(P \to N) = \mathbf{fv}\,(P) \cup \mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(Q) \cup \mathbf{fv}\,(M) = \mathbf{fv}\,(Q \to M)$.

Case 4. $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$

$$\begin{aligned} \mathbf{fv}\,\forall\overrightarrow{\alpha^+}.N &\subseteq \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \setminus \{\overrightarrow{\beta^+}\} && \text{here } \{\overrightarrow{\beta^+}\} \text{ is excluded by the premise } \mathbf{fv}\,N \cap \{\overrightarrow{\beta^+}\} = \varnothing \\ &\subseteq \mathbf{fv}\,M \setminus \{\overrightarrow{\beta^+}\} && \text{by the induction hypothesis, } \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \subseteq \mathbf{fv}\,M \\ &\subseteq \mathbf{fv}\,\forall\overrightarrow{\beta^+}.M \end{aligned}$$

Case 5. The positive cases are symmetric.

□

4.2 Substitution

Lemma 2 (Substitution strengthening)**.** *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

+ *if* $\Gamma_1 \vdash P$ *then* $[\sigma]P = [\sigma|_{\mathbf{fv}\,P}]P$*,*

− *if* $\Gamma_1 \vdash N$ *then* $[\sigma]N = [\sigma|_{\mathbf{fv}\,N}]N$

Proof. Ilya: todo □

Lemma 3 (Substitution preserves subtyping)**.** *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

+ *if* $\Gamma, \Gamma_1 \vdash P$*,* $\Gamma, \Gamma_1 \vdash Q$*, and* $\Gamma, \Gamma_1 \vdash P \geqslant_1 Q$ *then* $\Gamma, \Gamma_2 \vdash [\sigma]P \geqslant_1 [\sigma]Q$

− *if* $\Gamma, \Gamma_1 \vdash N$*,* $\Gamma, \Gamma_1 \vdash M$*, and* $\Gamma, \Gamma_1 \vdash N \leqslant_1 M$ *then* $\Gamma, \Gamma_2 \vdash [\sigma]N \leqslant_1 [\sigma]M$

Proof. Ilya: todo □

Corollary 1 (Substitution preserves subtyping)**.** *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

+ *if* $\Gamma, \Gamma_1 \vdash P$*,* $\Gamma, \Gamma_1 \vdash Q$*, and* $\Gamma, \Gamma_1 \vdash P \simeq_1^{\leqslant} Q$ *then* $\Gamma, \Gamma_2 \vdash [\sigma]P \simeq_1^{\leqslant} [\sigma]Q$

− *if* $\Gamma, \Gamma_1 \vdash N$*,* $\Gamma, \Gamma_1 \vdash M$*, and* $\Gamma, \Gamma_1 \vdash N \simeq_1^{\leqslant} M$ *then* $\Gamma, \Gamma_2 \vdash [\sigma]N \simeq_1^{\leqslant} [\sigma]M$

4.3 Type well-formedness

Lemma 4 (Well-formedness agrees with substitution)**.** *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

+ $\Gamma, \Gamma_1 \vdash P \Leftrightarrow \Gamma, \Gamma_2 \vdash [\sigma]P$

− $\Gamma, \Gamma_1 \vdash N \Leftrightarrow \Gamma, \Gamma_2 \vdash [\sigma]N$

Proof. Ilya: todo □

Corollary 2. *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

+ $\Gamma_1, \Gamma_2 \vdash P \Leftrightarrow \Gamma_2 \vdash [\sigma]P$

− $\Gamma_1, \Gamma_2 \vdash N \Leftrightarrow \Gamma_2 \vdash [\sigma]N$

Lemma 5 (Equivalent Contexts)**.** *In the well-formedness judgment, only used variables matter:*

+ *if* $\{\Gamma_1\} \cap \mathbf{fv}\,P = \{\Gamma_2\} \cap \mathbf{fv}\,P$ *then* $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$*,*

− *if* $\{\Gamma_1\} \cap \mathbf{fv}\,N = \{\Gamma_2\} \cap \mathbf{fv}\,N$ *then* $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$*.*

Proof. By simple mutual induction on P and Q. □

4.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\dfrac{}{\{\mathbf{ord}\ vars\,\mathbf{in}\ N\} \equiv vars \cap \mathbf{fv}\ N}$	$\dfrac{N \simeq_1^D M}{\mathbf{ord}\ vars\,\mathbf{in}\ N = \mathbf{ord}\ vars\,\mathbf{in}\ M}$	—
Normalization	$\dfrac{}{N \simeq_1^D \mathbf{nf}\,(N)}$	$\dfrac{N \simeq_1^D M}{\mathbf{nf}\,(N) = \mathbf{nf}\,(M)}$	—
Equivalence	$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq_1^D Q}{\Gamma \vdash P \simeq_1^{\leqslant} Q}$	$\dfrac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{P \simeq_1^D Q}$	—
Uppgrade	$\dfrac{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{Q \text{ is sound} \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$	$\dfrac{\begin{array}{c} Q' \text{ is sound} \\ \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \vDash P_1 \vee P_2 = Q}{Q \text{ is sound} \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \vDash P_1 \vee P_2 = Q}$	$\dfrac{\begin{array}{c} Q' \text{ is sound} \\ \Gamma \vDash P_1 \vee P_2 = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\begin{array}{l}(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \\ \text{is sound}\end{array} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \widehat{\tau}_i : \Xi \\ [\widehat{\tau}_i]\, Q = P_i \end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)}{\begin{array}{l} \exists (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \text{ s.t.} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \end{array}}$	$\dfrac{\begin{array}{c} (\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2) \text{ is sound} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \end{array}}{\exists \Gamma; \Xi \vdash \widehat{\tau} : \Xi' \text{ s.t. } [\widehat{\tau}]\, Q' = Q}$
Unification (matching)	$\dfrac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\widehat{\sigma} \text{ is sound} \begin{cases} \Theta \vdash \widehat{\sigma} \\ [\widehat{\sigma}]\, P = Q \end{cases}}$	$\dfrac{\exists \text{ sound } \widehat{\sigma}'}{\exists \widehat{\sigma} \text{ s.t. } \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$	—
Subtyping	$\dfrac{\Gamma;\, \Theta \vDash N \leqslant M \dashv \widehat{\sigma}}{\widehat{\sigma} \text{ is sound} \begin{cases} \Theta \vdash \widehat{\sigma} \\ \Gamma \vdash [\widehat{\sigma}]\, N \leqslant_1 M \end{cases}}$	$\dfrac{\exists \text{ sound } \widehat{\sigma}'}{\exists \widehat{\sigma} \text{ s.t. } \Gamma;\, \Theta \vDash N \leqslant M \dashv \widehat{\sigma}}$	—

4.5 Anti-unification

Lemma 6 (Soundness of the anti-unification algorithm).

Lemma 7 (Completeness of the anti-unification algorithm).

Lemma 8 (Initiality of the anti-unification algorithm).

4.6 Variable Ordering

Definition 4 (Collision free bijection). *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is* ***collision free on sets*** *P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 9 (Soundness of variable ordering). *Variable ordering extracts precisely used free variables.*

$-$ $\{\mathbf{ord}\ vars\,\mathbf{in}\ N\} \equiv vars \cap \mathbf{fv}\ N$ *(as sets)*

$+$ $\{\mathbf{ord}\ vars\,\mathbf{in}\ P\} \equiv vars \cap \mathbf{fv}\ P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\ vars\,\mathbf{in}\ N = \vec{\alpha}$ and $\mathbf{ord}\ vars\,\mathbf{in}\ P = \vec{\alpha}$ □

Corollary 3 (Additivity of ordering). *Variable ordering is additive (in terms of set union) with respect to its first argument.*

– $\{\textbf{ord}\,(vars_1 \cup vars_2)\,\textbf{in}\,N\} \equiv \{\textbf{ord}\,vars_1\,\textbf{in}\,N\} \cup \{\textbf{ord}\,vars_2\,\textbf{in}\,N\}$ *(as sets)*

\+ $\{\textbf{ord}\,(vars_1 \cup vars_2)\,\textbf{in}\,P\} \equiv \{\textbf{ord}\,vars_1\,\textbf{in}\,P\} \cup \{\textbf{ord}\,vars_2\,\textbf{in}\,P\}$ *(as sets)*

Corollary 4 (Weakening of ordering)**.** *Extending the first argument of the ordering with unused variables does not change the result.*

– $\textbf{ord}\,(vars \cap \textbf{fv}\,N)\,\textbf{in}\,N = \textbf{ord}\,vars\,\textbf{in}\,N$

\+ $\textbf{ord}\,(vars \cap \textbf{fv}\,P)\,\textbf{in}\,P = \textbf{ord}\,vars\,\textbf{in}\,P$

Lemma 10 (Distributivity of renaming over variable ordering)**.** *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

– *If μ is collision free on vars and $\textbf{fv}\,N$ then $[\mu](\textbf{ord}\,vars\,\textbf{in}\,N) = \textbf{ord}\,([\mu]vars)\,\textbf{in}\,[\mu]N$*

\+ *If μ is collision free on vars and $\textbf{fv}\,P$ then $[\mu](\textbf{ord}\,vars\,\textbf{in}\,P) = \textbf{ord}\,([\mu]vars)\,\textbf{in}\,[\mu]P$*

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$
Then
$$\begin{aligned}[\mu](\textbf{ord}\,vars\,\textbf{in}\,N) &= [\mu](\textbf{ord}\,vars\,\textbf{in}\,\alpha^-)\\ &= [\mu]\alpha^- && \text{by Rule } (\mathrm{Var}^+_{\in})\\ &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars)\\ &= \textbf{ord}\,[\mu]vars\,\textbf{in}\,\beta^- && \text{by Rule } (\mathrm{Var}^+_{\in}) \text{, because } \beta^- \in [\mu]vars\\ &= \textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]\alpha^-\end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$
Notice that $[\mu](\textbf{ord}\,vars\,\textbf{in}\,N) = [\mu](\textbf{ord}\,vars\,\textbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_{\notin})$. On the other hand, $\textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]\alpha^- = \textbf{ord}\,[\mu]vars\,\textbf{in}\,\alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_{\notin})$, because μ is collision free on *vars* and $\textbf{fv}\,N$, so $\textbf{fv}\,N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$
Then $[\mu](\textbf{ord}\,vars\,\textbf{in}\,N) = [\mu](\textbf{ord}\,vars\,\textbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_{\notin})$. To prove that $\textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_{\notin})$. Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

(i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.

(ii) Since μ is collision free on *vars* and $\textbf{fv}\,N$, $\mu(A \cap \textbf{fv}\,N) \ni \mu(\alpha^-) \notin vars$.

d. $\alpha^- \notin A$ but $\alpha^- \in vars$
$\textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]\alpha^- = \textbf{ord}\,[\mu]vars\,\textbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_{\notin})$, because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\textbf{ord}\,vars\,\textbf{in}\,N) = [\mu](\textbf{ord}\,vars\,\textbf{in}\,\alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$
$$\begin{aligned}[\mu](\textbf{ord}\,vars\,\textbf{in}\,N) &= [\mu](\textbf{ord}\,vars\,\textbf{in}\,\uparrow P)\\ &= [\mu](\textbf{ord}\,vars\,\textbf{in}\,P) && \text{by Rule } (\uparrow)\\ &= \textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]P && \text{by the induction hypothesis}\\ &= \textbf{ord}\,[\mu]vars\,\textbf{in}\,\uparrow[\mu]P && \text{by Rule } (\uparrow)\\ &= \textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]\uparrow P && \text{by the definition of substitution}\\ &= \textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]N\end{aligned}$$

Case 3. $N = P \to M$
$$\begin{aligned}[\mu](\textbf{ord}\,vars\,\textbf{in}\,N) &= [\mu](\textbf{ord}\,vars\,\textbf{in}\,P \to M)\\ &= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})) && \text{where } \textbf{ord}\,vars\,\textbf{in}\,P = \vec{\alpha}_1 \text{ and } \textbf{ord}\,vars\,\textbf{in}\,M = \vec{\alpha}_2\\ &= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})\\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\{\vec{\alpha}_1\}) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to case 1. Notice that } \mu \text{ is}\\ & && \text{collision free on } \{\vec{\alpha}_1\} \text{ and } \{\vec{\alpha}_2\} \text{ since } \{\vec{\alpha}_1\} \subseteq vars \text{ and } \{\vec{\alpha}_2\} \subseteq \textbf{fv}\,N\\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})\end{aligned}$$
$$\begin{aligned}(\textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]N) &= (\textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]P \to [\mu]M)\\ &= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \{\vec{\beta}_1\})) && \text{where } \textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]P = \vec{\beta}_1 \text{ and } \textbf{ord}\,[\mu]vars\,\textbf{in}\,[\mu]M = \vec{\beta}_2\\ & && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2,\\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})\end{aligned}$$

Case 4. $N = \forall \overrightarrow{\alpha^+}.M$

$$\begin{aligned}[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu]\mathbf{ord}\, vars\, \mathbf{in}\, \forall\overrightarrow{\alpha^+}.M\\ &= [\mu]\mathbf{ord}\, vars\, \mathbf{in}\, M\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]M \quad \text{by the induction hypothesis}\end{aligned}$$

$$\begin{aligned}(\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]N) &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\forall\overrightarrow{\alpha^+}.M\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \forall\overrightarrow{\alpha^+}.[\mu]M\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]M\end{aligned}$$

□

Lemma 11 (Ordering is not affected by independent substitutions)**.** *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2, and vars is a set of variables disjoint with both Γ_1 and Γ_2. Then*

− $\mathbf{ord}\, vars\, \mathbf{in}\, [\sigma]N = \mathbf{ord}\, vars\, \mathbf{in}\, N$

+ $\mathbf{ord}\, vars\, \mathbf{in}\, [\sigma]P = \mathbf{ord}\, vars\, \mathbf{in}\, P$

Proof. `Ilya: Should be easy` □

Lemma 12 (Completeness of variable ordering)**.** *Variable ordering is invariant under equivalence. For arbitrary vars,*

− *If $N \simeq_1^D M$ then $\mathbf{ord}\, vars\, \mathbf{in}\, N = \mathbf{ord}\, vars\, \mathbf{in}\, M$ (as lists)*

+ *If $P \simeq_1^D Q$ then $\mathbf{ord}\, vars\, \mathbf{in}\, P = \mathbf{ord}\, vars\, \mathbf{in}\, Q$ (as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. □

4.7 Normaliztaion

Lemma 13. *Set of free variables is invariant under equivalence.*

− *If $N \simeq_1^D M$ then $\mathbf{fv}\, N \equiv \mathbf{fv}\, M$ (as sets)*

+ *If $P \simeq_1^D Q$ then $\mathbf{fv}\, P \equiv \mathbf{fv}\, Q$ (as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

Lemma 14. *Free variables are not changed by the normalization*

− $\mathbf{fv}\, N \equiv \mathbf{fv}\, \mathbf{nf}\,(N)$

+ $\mathbf{fv}\, P \equiv \mathbf{fv}\, \mathbf{nf}\,(P)$

Proof. By straightforward induction on $\mathbf{nf}\,(N) = M$. □

Lemma 15 (Soundness of quantifier normalization)**.**

− $N \simeq_1^D \mathbf{nf}\,(N)$

+ $P \simeq_1^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var$^-$) and (Var$^+$)
By the corresponding equivalence rules.

Case 2. ($\uparrow$), ($\downarrow$), and ($\rightarrow$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall$), i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 13, $\mathbf{fv}\, N \equiv \mathbf{fv}\, N'$. Then by lemma 9, $\{\overrightarrow{\alpha^+}'\} \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N$, and thus, $\{\overrightarrow{\alpha^+}'\} \cap \mathbf{fv}\, N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N$.

To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^+}'.N'$, it suffices to provide a bijection $\mu : \{\overrightarrow{\alpha^+}'\} \cap \mathbf{fv}\, N' \leftrightarrow \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. ($\exists$) Same as for case 3.

$\square$

Corollary 5 (Normalization preserves ordering)**.** *For any vars,*

- $-$ $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \mathbf{nf}\,(N) = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ M$
- $+$ $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \mathbf{nf}\,(P) = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ Q$

Proof. Immediately from lemmas 12 and 15. $\square$

Lemma 16 (Distributivity of normalization over substitution)**.** *Normalization of a term distributes over substitution. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2. Then*

- $-$ $\mathbf{nf}\,([\sigma]N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$
- $+$ $\mathbf{nf}\,([\sigma]P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P)$

where $\mathbf{nf}\,(\sigma)$ *means pointwise normalization:* $[\mathbf{nf}\,(\sigma)]\alpha^- = \mathbf{nf}\,([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.

Case 2. $P = \alpha^+$
Similar to case 1.

Case 3. If the type is formed by $\rightarrow$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \rightarrow M$ then

$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma](P \rightarrow M))$	
$= \mathbf{nf}\,([\sigma]P \rightarrow [\sigma]M)$	By the congruence of substitution
$= \mathbf{nf}\,([\sigma]P) \rightarrow \mathbf{nf}\,([\sigma]M)$	By the congruence of normalization, i.e. Rule ($\rightarrow$)
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) \rightarrow [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$	By the induction hypothesis
$= [\mathbf{nf}\,(\sigma)](\mathbf{nf}\,(P) \rightarrow \mathbf{nf}\,(M))$	By the congruence of substitution
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P \rightarrow M)$	By the congruence of normalization
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$	

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M)$	
$= [\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$	Where $\overrightarrow{\alpha^+}' = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M$ (the latter is by corollary 5)

$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\forall\overrightarrow{\alpha^+}.M)$	
$= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\sigma]M)$	Assuming $\{\overrightarrow{\alpha^+}\} \cap \{\Gamma_1\} = \varnothing$ and $\{\overrightarrow{\alpha^+}\} \cap \{\Gamma_2\} = \varnothing$
$= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\sigma]M)$	Where $\overrightarrow{\beta^+} = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,([\sigma]M) = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\sigma]M$ (the latter is by corollary 5)
$= \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\sigma]M)$	By lemma 11, $\overrightarrow{\beta^+} = \overrightarrow{\alpha^+}'$ since $\{\overrightarrow{\alpha^+}\}$ is disjoint with Γ_1 and Γ_2
$= \forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$	By the induction hypothesis

To show alpha-equivalence of $[\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$, we can assume that $\{\overrightarrow{\alpha^+}'\} \cap \{\Gamma_1\} = \varnothing$, and $\{\overrightarrow{\alpha^+}'\} \cap \{\Gamma_2\} = \varnothing$.

Case 5. $P = \exists\overrightarrow{\alpha^-}.Q$
Same as for case 4.

$\square$

Corollary 6 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

- $-$ $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$
- $+$ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Proof. Immediately from lemma 16, after noticing that $\mathbf{nf}\,(\mu) = \mu$. $\square$

Lemma 17 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

$-$ *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

$+$ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. $(\forall^{\simeq_1^D})$

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^{+\prime}}$ is $\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^{+\prime}}.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^{+\prime}}$ is $\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 9 and 14, the domain and the codomain of μ can be written as $\mu : \{\overrightarrow{\beta^{+\prime}}\} \leftrightarrow \{\overrightarrow{\alpha^{+\prime}}\}$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^{+\prime}}.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^{+\prime}} = \overrightarrow{\alpha^{+\prime}}$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by corollary 6, the second—by the induction hypothesis.

(ii)
$$
\begin{aligned}
[\mu]\overrightarrow{\beta^{+\prime}} &= [\mu]\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^{+\prime}}\\
&= [\mu]\mathbf{ord}\,(\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 14 and corollary 4}\\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 10, because } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing\\
& && \text{and } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing\\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved}\\
&= \mathbf{ord}\,(\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \text{ and } \{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M\\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 14 and corollary 4}\\
&= \overrightarrow{\alpha^{+\prime}} && \text{by the definition of } \overrightarrow{\alpha^{+\prime}}
\end{aligned}
$$

Case 2. $(\exists^{\simeq_1^D})$ Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

$\square$

Lemma 18 (Idempotence of normalization)**.** *Normalization is idempotent*

$-$ $\mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$

$+$ $\mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

Proof. By applying lemma 17 to lemma 15. $\square$

Lemma 19. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized. Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then*

$+$ $[\sigma]P$ *is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \textit{is normal}\\ P & \textit{is normal}\end{cases}$

$-$ $[\sigma]N$*is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \textit{is normal}\\ N & \textit{is normal}\end{cases}$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$
Then N is always normal, and the normality of $\sigma|_{\{\alpha^-\}}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$\begin{aligned}
[\sigma](P \to M) \text{ is normal} &\iff [\sigma]P \to [\sigma]M \text{ is normal} && \text{by the substitution congruence}\\
&\iff \begin{cases}[\sigma]P & \text{is normal}\\ [\sigma]M & \text{is normal}\end{cases} && \text{by congruence of normality \texttt{Ilya: lemma?}}\\
&\iff \begin{cases}P & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(P)} & \text{is normal}\\ M & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(M)} & \text{is normal}\end{cases} && \text{by the induction hypothesis}\\
&\iff \begin{cases}P \to M & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(P)\cup\mathbf{fv}\,(M)} & \text{is normal}\end{cases} \iff \begin{cases}P \to M & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(P\to M)} & \text{is normal}\end{cases}
\end{aligned}$$

Case 3. $N = \uparrow P$
By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$\begin{aligned}
[\sigma](\forall\overrightarrow{\alpha^+}.M) \text{ is normal} &\iff (\forall\overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} && \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing\\
&\iff \begin{cases}[\sigma]M \text{ is normal}\\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+}\end{cases} && \text{by the definition of normalization}\\
&\iff \begin{cases}[\sigma]M \text{ is normal}\\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M = \overrightarrow{\alpha^+}\end{cases} && \text{by lemma 11}\\
&\iff \begin{cases}\sigma|_{\mathbf{fv}\,(M)} \text{ is normal}\\ M \text{ is normal}\\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M = \overrightarrow{\alpha^+}\end{cases} && \text{by the induction hypothesis}\\
&\iff \begin{cases}\sigma|_{\mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M)} \text{ is normal}\\ \forall\overrightarrow{\alpha^+}.M \text{ is normal}\end{cases} && \text{since } \mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M);\ \text{by the definition of normalization}
\end{aligned}$$

Case 5. $P = \ldots$
The positive cases are done in the same way as the negative ones.

$\square$

4.8 Equivalence

Lemma 20 (Type well-formedness is invariant under equivalence). *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,
− *if* $N \simeq_1^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. `Ilya: todo` $\square$

Corollary 7 (Normalization preserves well-formedness).

+ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}\,(P)$,
− $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}\,(N)$

Proof. Immediately from lemmas 15 and 20. $\square$

Corollary 8 (Normalization preserves well-formedness of substitution).
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}\,(\sigma) : \Gamma_1$

Lemma 21 (Soundness of equivalence). *Declarative equivalence implies mutual subtyping.*

+ *if* $\Gamma \vdash P$, $\Gamma \vdash Q$, *and* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \simeq_1^{\leqslant} Q$,
− *if* $\Gamma \vdash N$, $\Gamma \vdash M$, *and* $N \simeq_1^D M$ *then* $\Gamma \vdash N \simeq_1^{\leqslant} M$.

Proof. We prove it by mutual induction on $P \simeq_1^D Q$ and $N \simeq_1^D M$.

Case 1. $\alpha^- \simeq_1^D \alpha^-$
Then $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$ by Rule ($\text{Var}^{-\leqslant_1}$), which immediately implies $\Gamma \vdash \alpha^- \simeq_1^{\leqslant} \alpha^-$ by Rule ($\simeq_1^{\leqslant} -$).

Case 2. $\uparrow P \simeq_1^D \uparrow Q$
Then by inversion of Rule ($\uparrow^{\leqslant_1}$), $P \simeq_1^D Q$, and from the induction hypothesis, $\Gamma \vdash P \simeq_1^{\leqslant} Q$, and (by symmetry) $\Gamma \vdash Q \simeq_1^{\leqslant} P$.

When Rule ($\uparrow^{\leqslant_1}$) is applied to $\Gamma \vdash P \simeq_1^{\leqslant} Q$, it gives us $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$; when it is applied to $\Gamma \vdash Q \simeq_1^{\leqslant} P$, we obtain $\Gamma \vdash \uparrow Q \leqslant_1 \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \simeq_1^{\leqslant} \uparrow Q$.

Case 3. $P \to N \simeq_1^D Q \to M$
Then by inversion of Rule ($\to^{\leqslant_1}$), $P \simeq_1^D Q$ and $N \simeq_1^D M$. By the induction hypothesis, $\Gamma \vdash P \simeq_1^{\leqslant} Q$ and $\Gamma \vdash N \simeq_1^{\leqslant} M$, which means by inversion: (i) $\Gamma \vdash P \geqslant_1 Q$, (ii) $\Gamma \vdash Q \geqslant_1 P$, (iii) $\Gamma \vdash N \leqslant_1 M$, (iv) $\Gamma \vdash M \leqslant_1 N$. Applying Rule ($\to^{\leqslant_1}$) to (i) and (iii), we obtain $\Gamma \vdash P \to N \leqslant_1 Q \to M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \to M \leqslant_1 P \to N$. Together, it implies $\Gamma \vdash P \to N \simeq_1^{\leqslant} Q \to M$.

Case 4. $\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M$
Then by inversion, there exists bijection $\mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N)$, such that $N \simeq_1^D [\mu]M$. By the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu]M$. From corollary 1 and the fact that μ is bijective, we also have $\Gamma, \overrightarrow{\beta^+} \vdash [\mu^{-1}]N \simeq_1^{\leqslant} M$.

Let us construct a substitution $\overrightarrow{\alpha^+} \vdash \overrightarrow{P}/\overrightarrow{\beta^+} : \overrightarrow{\beta^+}$ by extending μ with arbitrary positive types on $\{\overrightarrow{\beta^+}\} \backslash \mathbf{fv}\, M$.

Notice that $[\mu]M = [\overrightarrow{P}/\overrightarrow{\beta^+}]M$, and therefore, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu]M$ implies $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{P}/\overrightarrow{\beta^+}]M \leqslant_1 N$. Then by Rule ($\forall^{\leqslant_1}$), $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \leqslant_1 \forall \overrightarrow{\alpha^+}.N$.

Analogously, we construct the substitution from μ^{-1}, and use it to instantiate $\overrightarrow{\alpha^+}$ in the application of Rule ($\forall^{\leqslant_1}$) to infer $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M$.

This way, $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \leqslant_1 \forall \overrightarrow{\alpha^+}.N$ and $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M$ gives us $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \simeq_1^{\leqslant} \forall \overrightarrow{\alpha^+}.N$.

□

Lemma 22 (Completeness of equivalence)**.** *Mutual subtyping implies declarative equivalence.*

+ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $P \simeq_1^D Q$,

– *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $N \simeq_1^D M$.

Proof. `Ilya: todo` □

4.9 Upper Bounds

Lemma 23 (Decomposition of the quantifier rule)**.** *Ilya: move somewhere* *Whenever the quantifier rule (Rule ($\exists^{\geqslant_1}$) or Rule ($\forall^{\leqslant_1}$)) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.*

– *If* $\Gamma \vdash N \leqslant_1 \forall \overrightarrow{\beta^+}.M$ *then* $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant_1 M$.

+ *If* $\Gamma \vdash P \geqslant_1 \exists \overrightarrow{\beta^-}.Q$ *then* $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant_1 Q$.

Lemma 24 (Characterization of the Supertypes)**.** *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists \overrightarrow{\alpha^-}.\beta^+ \mid \textit{for } \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists \overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle\vert\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-}, M', \textit{and } \overrightarrow{N} \textit{ s.t.} \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Then the last rule that is applied to infer $\Gamma \vdash Q \geqslant_1 \beta^+$ must be either Rule ($\text{Var}^{+\geqslant_1}$) or Rule ($\exists^{\geqslant_1}$). The former case means that $Q = \beta^+$. In the latter case, $Q = \exists\overrightarrow{\alpha^-}.Q'$, where Q' has no outer existential quantifiers. Then by inversion of Rule ($\exists^{\geqslant_1}$), $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]Q' \geqslant_1 \beta^+$ for some $\overrightarrow{N}$. This time, to infer this judgment, only Rule ($\text{Var}^{+\geqslant_1}$) is applicable, which means that $Q' = \beta^+$, and then $Q = \exists\overrightarrow{\alpha^-}.\beta^+$.

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

Then if $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'$, then by lemma 23, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$, and $\mathbf{fv}\, Q \cap \{\overrightarrow{\beta^-}\} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule ($\exists^{\geqslant_1}$). This way, $\{Q \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,(Q) \cap \{\overrightarrow{\beta^-}\} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = \downarrow M$

Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\nexists}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists\overrightarrow{\beta^-}.Q'$

Then the last applied rule to infer $\Gamma \vdash Q \geqslant_1 \downarrow M$ must be Rule ($\downarrow^{\geqslant_1}$), which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq_1^{\leqslant} M$, then by lemma 22 and Rule ($\downarrow^{\simeq_1^D}$), $\downarrow M' \simeq_1^D \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq_1^D \downarrow M\} = \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\nexists}$.

In the other direction,
$$\begin{aligned} \downarrow M' \simeq_1^D \downarrow M &\Rightarrow \Gamma \vdash \downarrow M' \simeq_1^{\leqslant} \downarrow M && \text{by lemma 21, since } \Gamma \vdash \downarrow M' \text{ by lemma 20} \\ &\Rightarrow \Gamma \vdash \downarrow M' \geqslant_1 \downarrow M && \text{by inversion} \end{aligned}$$

b. $Q = \exists\overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)

Then the last rule applied to infer $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q' \geqslant_1 \downarrow M$ must be Rule ($\exists^{\geqslant_1}$). Inversion of this rule gives us $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \geqslant_1 \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.*a*, $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \simeq_1^D \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists\overrightarrow{\beta^-}.\downarrow M' \in \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).

In the other direction,
$$\begin{aligned} [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^D \downarrow M &\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^{\leqslant} \downarrow M && \text{by lemma 21, since } \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \text{ by lemma 20} \\ &\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \geqslant_1 \downarrow M && \text{by inversion} \\ &\Rightarrow \Gamma \vdash \exists\overrightarrow{\beta^-}.\downarrow M' \geqslant_1 \downarrow M && \text{by Rule } (\exists^{\geqslant_1}) \end{aligned}$$

□

Lemma 25 (Characterization of the Normalized Supertypes)**.** *For a normalized positive type $P = \mathbf{nf}\,(P)$, let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-}\textit{, } M'\textit{, and } \overrightarrow{N} \textit{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M'\textit{, and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \end{array} \right\}$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Then from lemma 24, $\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\beta^+) \mid \text{for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

$$\begin{aligned} \mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') &= \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ not using } \overrightarrow{\beta^-} \\ &= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothesis} \\ &= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}\,(Q) = \mathbf{fv}\,Q \text{ by lemma 14} \\ &= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{by lemma 24} \\ &= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} && \text{by the definition of UB} \\ &= \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} && \text{by lemma 24} \end{aligned}$$

Case 3. $P = \downarrow M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 1. *Suppose that $\nu : A \to A$ is an idempotent function, P is a predicate on A, $F : A \to B$ is a function. Then*

$$\begin{aligned}&\{F(\nu x) \mid x \in A \ s.t.\ P(\nu x)\} = \\ =&\{F(x) \mid x \in A \ s.t.\ \nu x = x \ and\ P(x)\}.\end{aligned}$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 2. *For functions F and ν, and predicates P and Q,*

$$\begin{aligned}&\{F(\nu x) \mid x \in A \ s.t.\ Q(\nu x) \ and\ P(x)\} = \\ =&\{F(\nu x) \mid x \in A \ s.t.\ Q(\nu x) \ and\ (\exists x' \in A \ s.t.\ P(x') \ and\ \nu x' = \nu x)\}.\end{aligned}$$

$$
\begin{aligned}
&\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \downarrow M\} = \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \downarrow M)\} && \text{by lemma 24} \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash N_i, \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} && \text{by the definition of UB} \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} && \text{we reassigned the substitution } \overrightarrow{N}/\overrightarrow{\alpha^-} \text{ as } \sigma \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} && \text{by lemma 2} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} && \text{by the definition of normalization} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } \mathbf{nf}\,([\sigma|_{\mathbf{fv}\,M'}]\downarrow M') = \mathbf{nf}\,(\downarrow M) \end{array} \right\} && \begin{array}{l}\text{from lemmas 15 and 17, equivalence of types can be} \\ \text{replaced with the equality of their normal forms}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})]\downarrow\mathbf{nf}\,(M') = \downarrow\mathbf{nf}\,(M) \end{array} \right\} && \text{by congruence of normalization and lemma 16} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} && \begin{array}{l}\text{by lemma 19, } \downarrow M' \text{ and } \sigma|_{\mathbf{fv}\,M'} \text{ are already normal,} \\ \text{since the result of the substitution is normal;} \\ M \text{ is normal by assumption}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ (\exists \sigma' \text{ s.t. } \Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,(\downarrow M')} = \sigma'|_{\mathbf{fv}\,(\downarrow M')}) \\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \text{ and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} && \begin{array}{l}\text{We apply observation 2 (with } \nu\sigma = \sigma|_{\mathbf{fv}\,M'}\text{, and} \\ P(\sigma) = \Gamma \vdash \sigma : \overrightarrow{\alpha^-})\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv}\,M'} : \overrightarrow{\alpha^-}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} && \begin{array}{l}\text{Notice that} \\ \text{“}\exists \sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,(\downarrow M')} = \sigma'|_{\mathbf{fv}\,(\downarrow M')})\text{”} \\ \text{is equivalent to } \Gamma \vdash \sigma|_{\mathbf{fv}\,(\downarrow M')} : \overrightarrow{\alpha^-}'\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} && \begin{array}{l}\text{We apply observation 1 to the restriction of } \sigma\text{, and} \\ \text{remove } \sigma|_{\mathbf{fv}\,M'} = \sigma \text{ as it follows from } \Gamma \vdash \sigma : \overrightarrow{\alpha^-}'\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, \overrightarrow{\alpha^-}',\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} && \text{by lemma 5, since } \{\Gamma, \overrightarrow{\alpha^-}\} \cap \mathbf{fv}\,M' = \{\Gamma, \overrightarrow{\alpha^-}'\} \cap \mathbf{fv}\,M' \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}'\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} && \text{We apply observation 1 to the ordering of } \overrightarrow{\alpha^-} \\
&= \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}'\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-}' \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}']\downarrow M' = \downarrow M \end{array} \right\} && \text{By reassigning } \sigma \text{ explicitly as } \overrightarrow{N}/\overrightarrow{\alpha^-}' \\
&= \mathsf{NFUB}(\downarrow M) && \text{by definition}
\end{aligned}
$$

□

Observation 3. *Upper bounds of a type do not depend on the context as soon as the type are well-formed in it.*
If $\Gamma_1 \vdash M$ *and* $\Gamma_2 \vdash M$ *then* $\mathsf{UB}(\Gamma_1 \vdash M) = \mathsf{UB}(\Gamma \vdash M)$ *and* $\mathsf{NFUB}(\Gamma_1 \vdash M) = \mathsf{NFUB}(\Gamma \vdash M)$

Proof. We prove both inclusions by induction on $\Gamma_1 \vdash M$. Notice that if $[\sigma]M' \simeq_1^D M$ and $\Gamma_2 \vdash M$ then the types from the range of $\sigma|_{\mathbf{fv}\,M'}$ are well-formed in 2 Ilya: lemma. □

Lemma 26 (Soundness of the Least Upper Bound)**.** *For types* $\Gamma \vdash P_1$*, and* $\Gamma \vdash P_2$*, if* $\Gamma \vDash P_1 \vee P_2 = Q$ *then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Proof. Induction on $\Gamma \models P_1 \vee P_2 = Q$.

Case 1. $\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+$
Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule ($\mathrm{Var}^{+\geqslant_1}$).

Case 2. $\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q$
Then by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applied to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q$. Then

(i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,

(ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$,

(iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}\,(Q) \cap \{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}\} = \mathbf{fv}\,(Q) \cap \{\Gamma\}$ (and then apply lemma 5). The inclusion right-to-left is self-evident. To show $\mathbf{fv}\,(Q) \cap \{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}\} \subseteq \mathbf{fv}\,(Q) \cap \{\Gamma\}$, we prove that $\mathbf{fv}\,(Q) \subseteq \{\Gamma\}$

$\mathbf{fv}\,(Q) \subseteq \mathbf{fv}\,P_1 \cap \mathbf{fv}\,P_2$ by lemma 1

$\subseteq (\{\Gamma, \overrightarrow{\alpha^-}\} \backslash \{\overrightarrow{\beta^-}\}) \cap (\{\Gamma, \overrightarrow{\beta^-}\} \backslash \{\overrightarrow{\alpha^-}\})$ since $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_1$, $\mathbf{fv}\,(P_1) \subseteq \{\Gamma, \overrightarrow{\alpha^-}\} = \{\Gamma, \overrightarrow{\alpha^-}\} \backslash \{\overrightarrow{\beta^-}\}$ (the latter is because by the Barendregt's convention, $\{\Gamma, \overrightarrow{\alpha^-}\} \cap \{\overrightarrow{\beta^-}\} = \varnothing$); similarly, $\mathbf{fv}\,(P_2) \subseteq \{\Gamma, \overrightarrow{\beta^-}\} \backslash \{\overrightarrow{\alpha^-}\}$

$\subseteq \{\Gamma\}$

To show $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\alpha^-}.P_1$, we apply Rule ($\exists^{\geqslant_1}$). Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant_1 P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \models \downarrow N \vee \downarrow M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$ By the inversion, $\Gamma, \cdot \models \downarrow N \overset{a}{\simeq} \downarrow M \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$. Then by lemma 6,

(i) $\Gamma; \Xi \vdash P$, then by **??**,

$$\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]P \tag{1}$$

(ii) $\Gamma; \cdot \vdash \widehat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_2 : \Xi$. Assuming that $\Xi = \widehat{\beta}_1^-, .., \widehat{\beta}_n^-$, the antiunification solutions $\widehat{\tau}_1$ and $\widehat{\tau}_2$ can be put explicitly as $\widehat{\tau}_1 = (\widehat{\beta}_1^- :\approx N_1, .., \widehat{\beta}_n^- :\approx N_n)$, and $\widehat{\tau}_2 = (\widehat{\beta}_1^- :\approx M_1, .., \widehat{\beta}_n^- :\approx M_n)$. Then

$$\widehat{\tau}_1 = (\overrightarrow{N}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \text{ (as substitutions)} \tag{2}$$

$$\widehat{\tau}_2 = (\overrightarrow{M}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \text{ (as substitutions)} \tag{3}$$

(iii) $[\widehat{\tau}_1]Q = P_1$ and $[\widehat{\tau}_2]Q = P_1$, which, by 2 and 3, means

$$[\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P = \downarrow N \tag{4}$$

$$[\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P = \downarrow M \tag{5}$$

Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$ follows directly from 1.

To show $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 \downarrow N$, we apply Rule ($\exists^{\geqslant_1}$), instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{N}$. Then $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 \downarrow N$ follows from 4 and reflexivity of subtyping (**??**).

Analogously, instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{M}$, gives us $\Gamma \vdash [\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 \downarrow M$ (from 5), and hence, $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 \downarrow M$.

□

Lemma 27 (Completeness of the Least Upper Bound)**.** *For normalized types* $\Gamma \vdash P_1$*,* $\Gamma \vdash P_2$*, and* $\Gamma \vdash Q$ *such that* $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$*, there exists* Q' *s.t.* $\Gamma \models P_1 \vee P_2 = Q'$*.*

Proof. Induction on the pair (P_1, P_2). From lemma 25, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$. Let us consider the cases what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_1$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$ where $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

$$
\begin{aligned}
\text{Then } Q &\in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_2.Q_2) \\
&\subseteq \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_2 \vdash Q_2) && \text{from the definition of UB} \\
&= \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_2) && \text{by observation 3, weakening and exchange} \\
&= \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1\} \cap \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2\} && \text{by lemma 24,}
\end{aligned}
$$

meaning that $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$. Then after one step, the algorithm terminates by the induction hypothesis. In other words, $\exists Q'$ s.t. $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vDash Q_1 \vee Q_2 = Q'$, and thus, Rule ($\exists^\vee$) is applicable.

Case 2. $P_1 = \alpha^+$ and $P_2 = {\downarrow}N$

Then the set of common upper bounds of ${\downarrow}N$ and α^+ is empty, and thus, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$ gives a contradiction:

$$
\begin{aligned}
Q &\in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash {\downarrow}N) \\
&= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.{\downarrow}M' \mid \cdots\} && \text{by the definition of UB} \\
&= \varnothing && \text{since } \alpha^+ \neq {\downarrow}M' \text{ for any } M'
\end{aligned}
$$

Case 3. $P_1 = {\downarrow}N$ and $P_2 = \alpha^+$

Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)

Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$$
\begin{aligned}
Q &\in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \beta^+) \\
&= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\beta^+ \mid \cdots\} && \text{by the definition of UB} \\
&= \varnothing && \text{since } \alpha^+ \neq \beta^+
\end{aligned}
$$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Then the algorithm terminates in one step (Rule (Var$^\vee$)): $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$.

Case 6. $P_1 = {\downarrow}M_1$ and $P_2 = {\downarrow}M_2$

Then on the next step, the algorithm tries to anti-unify ${\downarrow}M_1$ and ${\downarrow}M_2$. By lemma 7, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$$
\begin{aligned}
&\mathbf{nf}\,(Q) \in \mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_2.Q_2) \\
&= \begin{array}{l}
\left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_1 \end{array} \right\} \\
\cap \\
\left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \overrightarrow{N}_1,\ \Gamma \vdash \overrightarrow{N}_2,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_2 \end{array} \right\}
\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M',\ \overrightarrow{N}_1 \text{ and } \overrightarrow{N}_2 \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \overrightarrow{N}_1,\ \Gamma \vdash \overrightarrow{N}_2,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_1, \text{ and } [\overrightarrow{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_2 \end{array} \right\}
\end{aligned}
$$

The fact that the latter set is non-empty means that there exist $\overrightarrow{\alpha^-}$, M', $\overrightarrow{N}_1$ and $\overrightarrow{N}_2$ such that

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$,

(ii) $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma \vdash \overrightarrow{N}_1$,

(iii) $[\overrightarrow{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_1$ and $[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_2$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative antiunification variable $\widehat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\widehat{\alpha}^-}$. Let us show that $(\overrightarrow{\widehat{\alpha}^-},\ [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M',\ \overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-},\ \overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-})$ is a sound anti-unifier of ${\downarrow}M_1$ and ${\downarrow}M_2$ in context Γ:

- $\overrightarrow{\widehat{\alpha}^-}$ is negative by construction,
- $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M'$ because $\Gamma, \overrightarrow{\alpha^-} \vdash {\downarrow}M'$ `Ilya: lemma!`,
- $\Gamma; \cdot \vdash (\overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma; \cdot \vdash (\overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N}_2$,

- $[\overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_1$; analogously, $[\overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = i[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_2$.

Then by the completeness of the anti-unification (lemma 7), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it.

$\square$

Lemma 28 (Initiality of the Least Upper Bound). *For normalized types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q$ such that $\Gamma \vdash Q \geqslant_1 P_1$ and $\Gamma \vdash Q \geqslant_1 P_2$, If $\Gamma \models P_1 \vee P_2 = Q'$ then $\Gamma \vdash Q \geqslant_1 Q'$.*

Proof. By induction on a pair (P_1, P_2), similarly to the proof of lemma 27.

Let us consider the cases what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_1$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$ where $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then by the same reasoning as in case 1 of the proof of lemma 27, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$.

On the other hand, the inversion of $\Gamma \models \exists\overrightarrow{\beta^-}_1.Q_1 \vee \exists\overrightarrow{\beta^-}_2.Q_2 = Q'$ gives us $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \models Q_1 \vee Q_2 = Q'$. Hence, by the induction hypothesis, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$.

Since both Q and Q' are sound, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$ can be strengthened to $\Gamma \vdash Q \geqslant_1 Q'$. `Ilya: lemma!`

Case 2. ($P_1 = \alpha^+$ and $P_2 = {\downarrow}N$) or ($P_1 = {\downarrow}N$ and $P_2 = \alpha^+$) or ($P_1 = \alpha^+$ and $P_2 = \beta^+$)

By the same argument as in case 2 of the proof of lemma 27, the set of common supertypes of P_1 and P_2 is empty, hence contradiction.

Case 3. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Since $Q \in \mathsf{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists\overrightarrow{\alpha^-}.\alpha^+$. Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant_1 \alpha^+$ by Rule ($\exists^{\geqslant_1}$): $\overrightarrow{\alpha^-}$ can be instantiated with arbitrary negative types (for example $\forall\beta^+.{\uparrow}\beta^+$), since the substitution for unused variables does not change the term $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule ($\mathrm{Var}^{+\geqslant_1}$).

Case 4. $P_1 = {\downarrow}M_1$ and $P_2 = {\downarrow}M_2$

By the same reasoning as in case 6 of the proof of lemma 27, $\mathbf{nf}\,(Q) = \exists\overrightarrow{\alpha^-}.{\downarrow}M'$ for some $\overrightarrow{\alpha^-}$ and ${\downarrow}M'$ such that there exist $\overrightarrow{N}_1$ and $\overrightarrow{N}_2$ such that:

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$,

(ii) $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma \vdash \overrightarrow{N}_1$,

(iii) $[\overrightarrow{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_1$ and $[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_2$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative antiunification variable $\widehat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\widehat{\alpha}^-}$. As shown in case 6 of the proof of lemma 27, $(\overrightarrow{\widehat{\alpha}^-},\ [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M',\ \overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-},\ \overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-})$ is a sound anti-unifier of ${\downarrow}M_1$ and ${\downarrow}M_2$ in context Γ.

By the inversion of $\Gamma \models {\downarrow}M_1 \vee {\downarrow}M_2 = Q'$, we conclude that $Q' = \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, where $(\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$ is the result of the antiunification of ${\downarrow}M_1$ and ${\downarrow}M_2$ in context Γ.

Then by the initiality of the anti-unification (lemma 8), there exisits $\widehat{\tau}$ such that $\Gamma; \Xi \vdash \widehat{\tau} : \overrightarrow{\widehat{\alpha}^-}$ and $[\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}$, (ii) $\widehat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi][\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi]P$. In particular, from the reflexivity of subtyping: $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]{\downarrow}M' \geqslant_1 [\overrightarrow{\beta^-}/\Xi]P$.

It allows us to show $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant_1 Q'$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.{\downarrow}M' \geqslant_1 \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, by applying Rule ($\exists^{\geqslant_1}$), instantiating $\overrightarrow{\alpha^-}$ with respect to σ. Finally, $\Gamma \vdash Q \geqslant_1 Q'$ since $\Gamma \vdash \mathbf{nf}\,(Q) \simeq_1^{\leqslant} Q$, and equivalence implies subtyping by `Ilya: lemma`.

$\square$

Lemma 29 (Soundness of Upgrade). *For $\Delta \subseteq \Gamma$, suppose that* **upgrade** $\Gamma \vdash P$ **to** $\Delta = Q$*.*

(i) $\Delta \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P$

Lemma 30 (Completeness of Upgrade). *For $\Delta \subseteq \Gamma$, $\Gamma \vdash P$ and $\Delta \vdash Q'$, such that $\Gamma \vdash Q' \geqslant_1 P$, there exists Q s.t.* **upgrade** $\Gamma \vdash P$ **to** $\Delta = Q$*, and $\Delta \vdash Q' \geqslant_1 Q$.*