1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$\begin{array}{lcll} P,\ Q & ::= & & \text{positive types} \\ & | & \alpha^+ & \\ & | & \downarrow N & \\ & | & \exists \alpha^-.P & \end{array}$$

$$\begin{array}{lcll} N,\ M & ::= & & \text{negative types} \\ & | & \alpha^- & \\ & | & \uparrow P & \\ & | & \forall \alpha^+.N & \\ & | & P \to N & \end{array}$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \ \text{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \ \text{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \ \text{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \ \text{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \ \text{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \ \text{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \ \text{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \ \text{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \ \text{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \ \text{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \ \text{D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{multi-quantified positive types} \\
& \mid & \alpha^+ & \\
& \mid & \downarrow N & \\
& \mid & \exists \overrightarrow{\alpha^-}.P & \quad P \neq \exists \ldots \\
& \mid & (P) & \mathsf{S}
\end{array}
$$

$$
\begin{array}{llll}
N,\ M & ::= & & \text{multi-quantified negative types} \\
& \mid & \alpha^- & \\
& \mid & \uparrow P & \\
& \mid & P \to N & \\
& \mid & \forall \overrightarrow{\alpha^+}.N & \quad N \neq \forall \ldots \\
& \mid & (N) & \mathsf{S}
\end{array}
$$

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \quad \text{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \quad \text{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad (\text{Var}^{-\leqslant_1})$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \quad (\uparrow^{\leqslant_1})$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \quad (\to^{\leqslant_1})$$

$$\frac{\mathbf{fv}\, N \cap \{\overrightarrow{\beta^+}\} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\leqslant_1})$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad (\text{Var}^{+\geqslant_1})$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \quad (\downarrow^{\geqslant_1})$$

$$\frac{\mathbf{fv}\, P \cap \{\overrightarrow{\beta^-}\} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\geqslant_1})$$

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-} \quad (\text{Var}^{-\simeq_1^D})$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad (\uparrow^{\simeq_1^D})$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M} \quad (\to^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M} \; (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \; (\text{VAR}^+)$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \; (\downarrow^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \; (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^- = \alpha^-} \; (\text{VAR}^-_\in)$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^- = \cdot} \; (\text{VAR}^-_\notin)$$

$$\frac{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in}\, \uparrow P = \vec{\alpha}} \; (\uparrow)$$

$$\frac{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}_1 \quad \mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}_2}{\mathbf{ord}\; vars\, \mathbf{in}\, P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})} \; (\to)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}} \; (\forall)$$

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^+ = \alpha^+} \; (\text{VAR}^+_\in)$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^+ = \cdot} \; (\text{VAR}^+_\notin)$$

$$\frac{\mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in}\, \downarrow N = \vec{\alpha}} \; (\downarrow)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in}\, \exists \overrightarrow{\alpha^-}.P = \vec{\alpha}} \; (\exists)$$

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\; vars\, \mathbf{in}\, \widehat{\alpha}^- = \cdot} \; (\text{UVAR}^-)$$

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\; vars\, \mathbf{in}\, \widehat{\alpha}^+ = \cdot} \; (\text{UVAR}^+)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\quad(\textsc{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\quad(\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\quad(\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'}\quad(\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\quad(\textsc{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\quad(\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'}\quad(\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\quad(\text{U}\textsc{Var}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\quad(\text{U}\textsc{Var}^+)$$

3.2 Unification

$\boxed{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\quad\text{UNV}\textsc{ar}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\quad\text{US}\textsc{hift}\text{U}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2}\quad\text{UA}\textsc{rrow}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\quad\text{UF}\textsc{orall}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)}\quad\text{UNUV}\textsc{ar}$$

$\boxed{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\quad\text{UPV}\textsc{ar}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\quad\text{US}\textsc{hift}\text{D}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\quad\text{UE}\textsc{xists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)}\quad\text{UPUV}\textsc{ar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma;\, \Theta \vDash N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma;\, \Theta \vDash \alpha^- \leqslant \alpha^- \dashv \cdot}\ \textsc{ANVar}$$

$$\frac{\Theta \vDash \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma;\, \Theta \vDash \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}}\ \textsc{AShiftU}$$

$$\frac{\Gamma;\, \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma;\, \Theta \vDash N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma;\, \Theta \vDash P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\ \textsc{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+};\, \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma;\, \Theta \vDash \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}}\ \textsc{AForall}$$

$\boxed{\Gamma;\, \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma;\, \Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \textsc{APVar}$$

$$\frac{\Theta \vDash \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma;\, \Theta \vDash \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}}\ \textsc{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-};\, \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vDash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma;\, \Theta \vDash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}}\ \textsc{AExists}$$

$$\frac{\mathbf{upgrade}\,\Gamma \vdash \mathbf{nf}\,(P)\,\mathbf{to}\,\Delta = Q}{\Gamma;\, \Theta \vDash \widehat{\alpha}^+\{\Delta\} \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)}\ \textsc{APUVar}$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restrict an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\Delta \vdash \widehat{\alpha}^+ :\approx P$ or $\Delta \vdash \widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\Delta \vdash \widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries)**.** *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.
$\boxed{e_1 \,\&\, e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\Gamma \vDash P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)}\ (\geqslant \& \geqslant)$$

$$\frac{\Gamma;\, \cdot \vDash P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\ (\simeq \& \geqslant)$$

$$\frac{\Gamma;\, \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)}\ (\geqslant \& \simeq)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\ (\simeq \& \simeq^+)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N) \,\&\, (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)}\ (\simeq \& \simeq^-)$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3. $\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 = \{e_1 \,\&\, e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, \textit{s.t. } e_1 \textit{ matches with } e_2\}$
$\cup \{e_1 \mid e_1 \in \widehat{\sigma}_1,\ \textit{s.t. } \forall e_2 \in \widehat{\sigma}_2, e_1 \textit{ does not match with } e_2\}$
$\cup \{e_2 \mid e_2 \in \widehat{\sigma}_2,\ \textit{s.t. } \forall e_1 \in \widehat{\sigma}_1, e_1 \textit{ does not match with } e_2\}$

3.5 Least Upper Bound

$\boxed{\Gamma \models P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+}\ \textsc{LUBVar}$$

$$\frac{\Gamma, \cdot \models {\downarrow}N \overset{a}{\simeq} {\downarrow}M \dashv (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models {\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P}\ \textsc{LUBShift}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q}{\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q}\ \textsc{LUBExists}$$

$\boxed{\textbf{upgrade}\,\Gamma \vdash P\,\textbf{to}\,\Delta = Q}$

3.6 Antiunification

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)}\ \textsc{AUPVar}$$

$$\frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models {\downarrow}N_1 \overset{a}{\simeq} {\downarrow}N_2 \dashv (\Xi, {\downarrow}M, \hat{\tau}_1, \hat{\tau}_2)}\ \textsc{AUPShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists\overrightarrow{\alpha^-}.\,Q, \hat{\tau}_1, \hat{\tau}_2)}\ \textsc{AUPExists}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)}\ \textsc{AUNVar}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models {\uparrow}P_1 \overset{a}{\simeq} {\uparrow}P_2 \dashv (\Xi, {\uparrow}Q, \hat{\tau}_1, \hat{\tau}_2)}\ \textsc{AUNShift}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \hat{\tau}_1, \hat{\tau}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \hat{\tau}_1', \hat{\tau}_2')}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \hat{\tau}_1 \cup \hat{\tau}_1', \hat{\tau}_2 \cup \hat{\tau}_2')}\ \textsc{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \overset{a}{\simeq} M \dashv (\hat{\alpha}^-_{\{N,M\}}, \hat{\alpha}^-_{\{N,M\}}, (\hat{\alpha}^-_{\{N,M\}} :\approx N), (\hat{\alpha}^-_{\{N,M\}} :\approx M))}\ \textsc{AUNAU}$$

4 Proofs

4.1 Declarative Subtyping

Lemma 1 (Free Variable Propagation)**.** *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context* Γ*,*

- $-$ *if* $\Gamma \vdash N \leqslant_1 M$ *then* $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$
- $+$ *if* $\Gamma \vdash P \geqslant_1 Q$ *then* $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$

Proof. Mutual induction on $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash P \geqslant_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
It is self-evident that aa.

Case 2. $\Gamma \vdash {\uparrow}P \leqslant_1 {\uparrow}Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq_1^{\leqslant} Q$), we have $\Gamma \vdash P \geqslant_1 Q$. Then by the induction hypothesis, $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$. The desired inclusion inclusion holds, since $\mathbf{fv}\,({\uparrow}P) = \mathbf{fv}\,(P)$ and $\mathbf{fv}\,({\uparrow}Q) = \mathbf{fv}\,(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant_1 Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$ and $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$. Then $\mathbf{fv}\,(P \to N) = \mathbf{fv}\,(P) \cup \mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(Q) \cup \mathbf{fv}\,(M) = \mathbf{fv}\,(Q \to M)$.

Case 4. $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$

$$\begin{aligned}
\mathbf{fv}\,\forall\overrightarrow{\alpha^+}.N &\subseteq \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \setminus \{\overrightarrow{\beta^+}\} && \text{here } \{\overrightarrow{\beta^+}\} \text{ is excluded by the premise } \mathbf{fv}\,N \cap \{\overrightarrow{\beta^+}\} = \varnothing \\
&\subseteq \mathbf{fv}\,M \setminus \{\overrightarrow{\beta^+}\} && \text{by the induction hypothesis, } \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \subseteq \mathbf{fv}\,M \\
&\subseteq \mathbf{fv}\,\forall\overrightarrow{\beta^+}.M
\end{aligned}$$

Case 5. The positive cases are symmetric.

□

4.2 Substitution

Lemma 2 (Substitution strengthening)**.** *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

+ *if* $\Gamma_1 \vdash P$ *then* $[\sigma]P = [\sigma|_{\mathbf{fv}\,P}]P$*,*
− *if* $\Gamma_1 \vdash N$ *then* $[\sigma]N = [\sigma|_{\mathbf{fv}\,N}]N$

Proof. `Ilya: todo` □

4.3 Type well-formedness

Lemma 3 (Well-formedness agrees with substitution)**.** *Suppose that* $\Gamma_1 \vdash \sigma : \Gamma_2$*. Then*

+ $\Gamma_1, \Gamma_2 \vdash P \Leftrightarrow \Gamma_1 \vdash [\sigma]P$
− $\Gamma_1, \Gamma_2 \vdash N \Leftrightarrow \Gamma_1 \vdash [\sigma]N$

Lemma 4 (Equivalent Contexts)**.** *In the well-formedness judgment, only used variables matter:*

+ *if* $\{\Gamma_1\} \cap \mathbf{fv}\,P = \{\Gamma_2\} \cap \mathbf{fv}\,P$ *then* $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$*,*
− *if* $\{\Gamma_1\} \cap \mathbf{fv}\,N = \{\Gamma_2\} \cap \mathbf{fv}\,N$ *then* $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$*.*

Proof. By simple mutual induction on P and Q. □

4.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\dfrac{}{\{\mathbf{ord}\ vars\ \mathbf{in}\ N\} \equiv vars \cap \mathbf{fv}\ N}$	$\dfrac{N \simeq_1^D M}{\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ M}$	—
Normalization	$\dfrac{}{N \simeq_1^D \mathbf{nf}\,(N)}$	$\dfrac{N \simeq_1^D M}{\mathbf{nf}\,(N) = \mathbf{nf}\,(M)}$	—
Equivalence	$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq_1^D Q}{\Gamma \vdash P \simeq_1^{\leqslant} Q}$	$\dfrac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{P \simeq_1^D Q}$	—
Uppgrade	$\dfrac{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{Q \text{ is sound} \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$	$\dfrac{\begin{array}{l} Q' \text{ is sound} \\ \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \vDash P_1 \vee P_2 = Q}{Q \text{ is sound} \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \vDash P_1 \vee P_2 = Q}$	$\dfrac{\begin{array}{l} Q' \text{ is sound} \\ \Gamma \vDash P_1 \vee P_2 = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2)}{\begin{array}{l} (\Xi, Q, \hat\tau_1, \hat\tau_2) \\ \text{is sound} \end{array} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat\tau_i : \Xi \\ [\hat\tau_i]\, Q = P_i \end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \hat\tau_1', \hat\tau_2')}{\begin{array}{l} \exists (\Xi, Q, \hat\tau_1, \hat\tau_2) \text{ s.t.} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2) \end{array}}$	$\dfrac{\begin{array}{l} (\Xi', Q', \hat\tau_1', \hat\tau_2') \text{ is sound} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2) \end{array}}{\exists \Gamma; \Xi \vdash \hat\tau : \Xi' \text{ s.t. } [\hat\tau]\, Q' = Q}$
Unification (matching)	$\dfrac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \hat\sigma}{\hat\sigma \text{ is sound} \begin{cases} \Theta \vdash \hat\sigma \\ [\hat\sigma]\, P = Q \end{cases}}$	$\dfrac{\exists \text{ sound } \hat\sigma'}{\exists \hat\sigma \text{ s.t. } \Theta \vDash P \overset{u}{\simeq} Q \dashv \hat\sigma}$	—
Subtyping	$\dfrac{\Gamma;\, \Theta \vDash N \leqslant M \dashv \hat\sigma}{\hat\sigma \text{ is sound} \begin{cases} \Theta \vdash \hat\sigma \\ \Gamma \vdash [\hat\sigma]\, N \leqslant_1 M \end{cases}}$	$\dfrac{\exists \text{ sound } \hat\sigma'}{\exists \hat\sigma \text{ s.t. } \Gamma;\, \Theta \vDash N \leqslant M \dashv \hat\sigma}$	—

4.5 Variable Ordering

Definition 4 (Collision free bijection). *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is **collision free on sets** P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 5 (Soundness of variable ordering). *Variable ordering extracts precisely used free variables.*

− $\{\mathbf{ord}\ vars\ \mathbf{in}\ N\} \equiv vars \cap \mathbf{fv}\ N$ *(as sets)*

+ $\{\mathbf{ord}\ vars\ \mathbf{in}\ P\} \equiv vars \cap \mathbf{fv}\ P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}$ and $\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}$ □

Corollary 1 (Additivity of ordering). *Variable ordering is additive (in terms of set union) with respect to its first argument.*

− $\{\mathbf{ord}\,(vars_1 \cup vars_2)\ \mathbf{in}\ N\} \equiv \{\mathbf{ord}\ vars_1\ \mathbf{in}\ N\} \cup \{\mathbf{ord}\ vars_2\ \mathbf{in}\ N\}$ *(as sets)*

+ $\{\mathbf{ord}\,(vars_1 \cup vars_2)\ \mathbf{in}\ P\} \equiv \{\mathbf{ord}\ vars_1\ \mathbf{in}\ P\} \cup \{\mathbf{ord}\ vars_2\ \mathbf{in}\ P\}$ *(as sets)*

Corollary 2 (Weakening of ordering). *Extending the first argument of the ordering with unused variables does not change the result.*

− $\mathbf{ord}\,(vars \cap \mathbf{fv}\ N)\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ N$

$+$ $\mathbf{ord}\,(vars \cap \mathbf{fv}\,P)\,\mathbf{in}\,P = \mathbf{ord}\,vars\,\mathbf{in}\,P$

Lemma 6 (Distributivity of renaming over variable ordering)**.** *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

- *If μ is collision free on vars and $\mathbf{fv}\,N$ then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]N$*

+ *If μ is collision free on vars and $\mathbf{fv}\,P$ then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]P$*

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$
Then
$$\begin{aligned}[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-)\\ &= [\mu]\alpha^- && \text{by Rule }(\mathrm{Var}^+_\in)\\ &= \beta^- && \text{for some }\beta^- \in B\text{ (notice that }\beta^- \in [\mu]vars)\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\beta^- && \text{by Rule }(\mathrm{Var}^+_\in)\text{, because }\beta^- \in [\mu]vars\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^-\end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$
Notice that $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. On the other hand, $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_\notin)$, because μ is collision free on $vars$ and $\mathbf{fv}\,N$, so $\mathbf{fv}\,N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$
Then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. To prove that $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_\notin)$. Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

(i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.

(ii) Since μ is collision free on $vars$ and $\mathbf{fv}\,N$, $\mu(A \cap \mathbf{fv}\,N) \ni \mu(\alpha^-) \notin vars$.

d. $\alpha^- \notin A$ but $\alpha^- \in vars$
$\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_\notin)$, because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$
$$\begin{aligned}[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\uparrow P)\\ &= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) && \text{by Rule }(\uparrow)\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P && \text{by the induction hypothesis}\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\uparrow[\mu]P && \text{by Rule }(\uparrow)\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\uparrow P && \text{by the definition of substitution}\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N\end{aligned}$$

Case 3. $N = P \to M$
$$\begin{aligned}[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P \to M)\\ &= [\mu](\vec\alpha_1, (\vec\alpha_2\backslash\{\vec\alpha_1\})) && \text{where }\mathbf{ord}\,vars\,\mathbf{in}\,P = \vec\alpha_1\text{ and }\mathbf{ord}\,vars\,\mathbf{in}\,M = \vec\alpha_2\\ &= [\mu]\vec\alpha_1, [\mu](\vec\alpha_2\backslash\{\vec\alpha_1\})\\ &= [\mu]\vec\alpha_1, ([\mu]\vec\alpha_2\backslash[\mu]\{\vec\alpha_1\}) && \text{by induction on }\vec\alpha_2\text{; the inductive step is similar to case 1. Notice that }\mu\text{ is}\\ & && \text{collision free on }\{\vec\alpha_1\}\text{ and }\{\vec\alpha_2\}\text{ since }\{\vec\alpha_1\} \subseteq vars\text{ and }\{\vec\alpha_2\} \subseteq \mathbf{fv}\,N\\ &= [\mu]\vec\alpha_1, ([\mu]\vec\alpha_2\backslash\{[\mu]\vec\alpha_1\})\end{aligned}$$
$$\begin{aligned}(\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N) &= (\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P \to [\mu]M)\\ &= (\vec\beta_1, (\vec\beta_2\backslash\{\vec\beta_1\})) && \text{where }\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P = \vec\beta_1\text{ and }\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M = \vec\beta_2\\ & && \text{then by the induction hypothesis, }\vec\beta_1 = [\mu]\vec\alpha_1,\ \vec\beta_2 = [\mu]\vec\alpha_2,\\ &= [\mu]\vec\alpha_1, ([\mu]\vec\alpha_2\backslash\{[\mu]\vec\alpha_1\})\end{aligned}$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$\begin{aligned}[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) &= [\mu]\mathbf{ord}\ vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.M\\ &= [\mu]\mathbf{ord}\ vars\ \mathbf{in}\ M\\ &= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]M \qquad \text{by the induction hypothesis}\end{aligned}$$

$$\begin{aligned}(\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]N) &= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]\forall\overrightarrow{\alpha^+}.M\\ &= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.[\mu]M\\ &= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]M\end{aligned}$$

□

Lemma 7 (Ordering is not affected by independent substitutions). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2, and vars is a set of variables disjoint with both Γ_1 and Γ_2. Then*

− $\mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]N = \mathbf{ord}\ vars\ \mathbf{in}\ N$

+ $\mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]P = \mathbf{ord}\ vars\ \mathbf{in}\ P$

Proof. `Ilya: Should be easy` □

Lemma 8 (Completeness of variable ordering). *Variable ordering is invariant under equivalence. For arbitrary vars,*

− *If $N \simeq_1^D M$ then $\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ M$ (as lists)*

+ *If $P \simeq_1^D Q$ then $\mathbf{ord}\ vars\ \mathbf{in}\ P = \mathbf{ord}\ vars\ \mathbf{in}\ Q$ (as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. □

4.6 Normaliztaion

Lemma 9. *Set of free variables is invariant under equivalence.*

− *If $N \simeq_1^D M$ then $\mathbf{fv}\ N \equiv \mathbf{fv}\ M$ (as sets)*

+ *If $P \simeq_1^D Q$ then $\mathbf{fv}\ P \equiv \mathbf{fv}\ Q$ (as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

Lemma 10. *Free variables are not changed by the normalization*

− $\mathbf{fv}\ N \equiv \mathbf{fv}\ \mathbf{nf}\,(N)$

+ $\mathbf{fv}\ P \equiv \mathbf{fv}\ \mathbf{nf}\,(P)$

Proof. By straightforward induction on $\mathbf{nf}\,(N) = M$. □

Lemma 11 (Soundness of quantifier normalization).

− $N \simeq_1^D \mathbf{nf}\,(N)$

+ $P \simeq_1^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var^-) and (Var^+)
By the corresponding equivalence rules.

Case 2. ($\uparrow$), ($\downarrow$), and ($\rightarrow$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall$), i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 9, $\mathbf{fv}\ N \equiv \mathbf{fv}\ N'$. Then by lemma 5, $\{\overrightarrow{\alpha^{+\prime}}\} \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\ N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\ N$, and thus, $\{\overrightarrow{\alpha^{+\prime}}\} \cap \mathbf{fv}\ N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\ N$.

To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^{+\prime}}.N'$, it suffices to provide a bijection $\mu : \{\overrightarrow{\alpha^{+\prime}}\} \cap \mathbf{fv}\ N' \leftrightarrow \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\ N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. ($\exists$) Same as for case 3.

□

Corollary 3 (Normalization preserves ordering). *For any vars,*

$-$ $\mathbf{ord}\ vars\ \mathbf{in}\ \mathbf{nf}\,(N) = \mathbf{ord}\ vars\ \mathbf{in}\ M$

$+$ $\mathbf{ord}\ vars\ \mathbf{in}\ \mathbf{nf}\,(P) = \mathbf{ord}\ vars\ \mathbf{in}\ Q$

Proof. Immediately from lemmas 8 and 11. □

Lemma 12 (Distributivity of normalization over substitution). *Normalization of a term distributes over substitution. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2. Then*

$-$ $\mathbf{nf}\,([\sigma]N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$

$+$ $\mathbf{nf}\,([\sigma]P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P)$

where $\mathbf{nf}\,(\sigma)$ *means pointwise normalization:* $[\mathbf{nf}\,(\sigma)]\alpha^- = \mathbf{nf}\,([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.

Case 2. $P = \alpha^+$
Similar to case 1.

Case 3. If the type is formed by $\rightarrow$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \rightarrow M$ then

$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma](P \rightarrow M))$	
$= \mathbf{nf}\,([\sigma]P \rightarrow [\sigma]M)$	By the congruence of substitution
$= \mathbf{nf}\,([\sigma]P) \rightarrow \mathbf{nf}\,([\sigma]M)$	By the congruence of normalization, i.e. Rule ($\rightarrow$)
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) \rightarrow [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$	By the induction hypothesis
$= [\mathbf{nf}\,(\sigma)](\mathbf{nf}\,(P) \rightarrow \mathbf{nf}\,(M))$	By the congruence of substitution
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P \rightarrow M)$	By the congruence of normalization
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$	

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M)$
$= [\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$ Where $\overrightarrow{\alpha^+}' = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M$ (the latter is by corollary 3)

$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\forall\overrightarrow{\alpha^+}.M)$	
$= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\sigma]M)$	Assuming $\{\overrightarrow{\alpha^+}\} \cap \{\Gamma_1\} = \varnothing$ and $\{\overrightarrow{\alpha^+}\} \cap \{\Gamma_2\} = \varnothing$
$= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\sigma]M)$	Where $\overrightarrow{\beta^+} = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,([\sigma]M) = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\sigma]M$ (the latter is by corollary 3)
$= \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\sigma]M)$	By lemma 7, $\overrightarrow{\beta^+} = \overrightarrow{\alpha^+}'$ since $\{\overrightarrow{\alpha^+}\}$ is disjoint with Γ_1 and Γ_2
$= \forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$	By the induction hypothesis

To show alpha-equivalence of $[\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$, we can assume that $\{\overrightarrow{\alpha^+}'\} \cap \{\Gamma_1\} = \varnothing$, and $\{\overrightarrow{\alpha^+}'\} \cap \{\Gamma_2\} = \varnothing$.

Case 5. $P = \exists\overrightarrow{\alpha^-}.Q$
Same as for case 4.

□

Corollary 4 (Commutativity of normalization and renaming). *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

$-$ $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$

$+$ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Proof. Immediately from lemma 12, after noticing that $\mathbf{nf}\,(\mu) = \mu$. $\square$

Lemma 13 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

- $-$ *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$
- $+$ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. $(\forall^{\simeq_1^D})$

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 5 and 10, the domain and the codomain of μ can be written as $\mu : \{\overrightarrow{\beta^+}'\} \leftrightarrow \{\overrightarrow{\alpha^+}'\}$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by corollary 4, the second—by the induction hypothesis.

(ii)
$$
\begin{aligned}
[\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}' \\
&= [\mu]\mathbf{ord}\,(\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 10 and corollary 2} \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 6, because } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing \\
& && \text{and } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved} \\
&= \mathbf{ord}\,(\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \text{ and } \{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 10 and corollary 2} \\
&= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'
\end{aligned}
$$

Case 2. $(\exists^{\simeq_1^D})$ Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

$\square$

Lemma 14 (Idempotence of normalization)**.** *Normalization is idempotent*

- $-$ $\mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$
- $+$ $\mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

Proof. By applying lemma 13 to lemma 11. $\square$

Lemma 15. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized. Suppose that* σ *is a substitution* $\Gamma_2 \vdash \sigma : \Gamma_1$, P *is a positive type* ($\Gamma_1 \vdash P$), N *is a negative type* ($\Gamma_1 \vdash N$). *Then*

$$+\ [\sigma]P \textit{ is normal} \iff \begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \textit{is normal} \\ P & \textit{is normal} \end{cases}$$

$$-\ [\sigma]N \textit{is normal} \iff \begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \textit{is normal} \\ N & \textit{is normal} \end{cases}$$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$
Then N is always normal, and the normality of $\sigma|_{\{\alpha^-\}}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$
\begin{aligned}
[\sigma](P \to M) \text{ is normal} &\iff [\sigma]P \to [\sigma]M \text{ is normal} && \text{by the substitution congruence} \\
&\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases} && \text{by congruence of normality } \texttt{Ilya: lemma?} \\
&\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(M)} & \text{is normal} \end{cases} && \text{by the induction hypothesis} \\
&\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P) \cup \mathbf{fv}\,(M)} & \text{is normal} \end{cases} \iff && \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P \to M)} & \text{is normal} \end{cases}
\end{aligned}
$$

Case 3. $N = \uparrow P$
By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall \overrightarrow{\alpha^+}.M$

$$
\begin{aligned}
[\sigma](\forall \overrightarrow{\alpha^+}.M) \text{ is normal} &\iff (\forall \overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} && \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing \\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+} \end{cases} && \text{by the definition of normalization} \\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} && \text{by lemma 7} \\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} && \text{by the induction hypothesis} \\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(\forall \overrightarrow{\alpha^+}.M)} \text{ is normal} \\ \forall \overrightarrow{\alpha^+}.M \text{ is normal} \end{cases} && \begin{array}{l} \text{since } \mathbf{fv}\,(\forall \overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M); \\ \text{by the definition of normalization} \end{array}
\end{aligned}
$$

Case 5. $P = \dots$
The positive cases are done in the same way as the negative ones.

□

4.7 Equivalence

Lemma 16 (Type well-formedness is invariant under equivalence). *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,

− *if* $N \simeq_1^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. `Ilya: todo` □

Corollary 5 (Normalization preserves well-formedness).

+ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}\,(P)$,

− $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}\,(N)$

Proof. Immediately from lemmas 11 and 16. □

Corollary 6 (Normalization preserves well-formedness of substitution).
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}\,(\sigma) : \Gamma_1$

Lemma 17 (Soundness of equivalence). *Declarative equivalence implies mutual subtyping.*

+ *if* $\Gamma \vdash P$, $\Gamma \vdash Q$, *and* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \simeq_1^{\leqslant} Q$,

$-$ *if* $\Gamma \vdash N$, $\Gamma \vdash M$, *and* $N \simeq_1^{D} M$ *then* $\Gamma \vdash N \simeq_1^{\leqslant} M$.

Proof. `Ilya: todo` $\square$

Lemma 18 (Completeness of equivalence). *Mutual subtyping implies declarative equivalence.*

$+$ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $P \simeq_1^{D} Q$,

$-$ *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $N \simeq_1^{D} M$.

Proof. `Ilya: todo` $\square$

4.8 Upper Bounds

Lemma 19 (Decomposition of the quantifier rule). ***Ilya: move somewhere*** *Whenever the quantifier rule (Rule* $(\exists^{\geqslant_1})$ *or Rule* $(\forall^{\leqslant_1})$*) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.*

$-$ *If* $\Gamma \vdash N \leqslant_1 \forall\overrightarrow{\beta^+}.M$ *then* $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant_1 M$.

$+$ *If* $\Gamma \vdash P \geqslant_1 \exists\overrightarrow{\beta^-}.Q$ *then* $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant_1 Q$.

Lemma 20 (Shape of the Supertypes). *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \textit{for } \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash {\downarrow}M$	$\left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle\vert\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-}, M', \textit{and } \overrightarrow{N} \textit{ s.t.} \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' \simeq_1^{D} {\downarrow}M \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Then the last rule that is applied to infer $\Gamma \vdash Q \geqslant_1 \beta^+$ must be either Rule $(\mathrm{Var}^{+\geqslant_1})$ or Rule $(\exists^{\geqslant_1})$. The former case means that $Q = \beta^+$. In the latter case, $Q = \exists\overrightarrow{\alpha^-}.Q'$, where Q' has no outer existential quantifiers. Then by inversion of Rule $(\exists^{\geqslant_1})$, $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]Q' \geqslant_1 \beta^+$ for some $\overrightarrow{N}$. This time, to infer this judgment, only Rule $(\mathrm{Var}^{+\geqslant_1})$ is applicable, which means that $Q' = \beta^+$, and then $Q = \exists\overrightarrow{\alpha^-}.\beta^+$.

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$
Then if $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'$, then by lemma 19, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$, and $\mathbf{fv}\,Q \cap \{\overrightarrow{\beta^-}\} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule $(\exists^{\geqslant_1})$. This way, $\{Q \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,(Q) \cap \{\overrightarrow{\beta^-}\} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = {\downarrow}M$
Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\nexists}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists\overrightarrow{\beta^-}.Q'$
Then the last applied rule to infer $\Gamma \vdash Q \geqslant_1 {\downarrow}M$ must be Rule $({\downarrow}^{\geqslant_1})$, which means $Q = {\downarrow}M'$, and by inversion, $\Gamma \vdash M' \simeq_1^{\leqslant} M$, then by lemma 18 and Rule $({\downarrow}^{\simeq_1^D})$, ${\downarrow}M' \simeq_1^{D} {\downarrow}M$. This way, $Q = {\downarrow}M' \in \{{\downarrow}M' \mid {\downarrow}M' \simeq_1^{D} {\downarrow}M\} = \mathsf{UB}(\Gamma \vdash {\downarrow}M)|_{\nexists}$.
In the other direction,
$$\begin{aligned} {\downarrow}M' \simeq_1^{D} {\downarrow}M &\Rightarrow \Gamma \vdash {\downarrow}M' \simeq_1^{\leqslant} {\downarrow}M && \text{by lemma 17, since } \Gamma \vdash {\downarrow}M' \text{ by lemma 16} \\ &\Rightarrow \Gamma \vdash {\downarrow}M' \geqslant_1 {\downarrow}M && \text{by inversion} \end{aligned}$$

b. $Q = \exists\overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)
Then the last rule applied to infer $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q' \geqslant_1 {\downarrow}M$ must be Rule $(\exists^{\geqslant_1})$. Inversion of this rule gives us $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \geqslant_1 {\downarrow}M$ for some $\Gamma \vdash N_i$. Notice that $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.*a*, $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \simeq_1^{D} {\downarrow}M$, which is only possible if $Q' = {\downarrow}M'$. This way, $Q = \exists\overrightarrow{\beta^-}.{\downarrow}M' \in \mathsf{UB}(\Gamma \vdash {\downarrow}M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).
In the other direction,
$$\begin{aligned} [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \simeq_1^{D} {\downarrow}M &\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \simeq_1^{\leqslant} {\downarrow}M && \text{by lemma 17, since } \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \text{ by lemma 16} \\ &\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \geqslant_1 {\downarrow}M && \text{by inversion} \\ &\Rightarrow \Gamma \vdash \exists\overrightarrow{\beta^-}.{\downarrow}M' \geqslant_1 {\downarrow}M && \text{by Rule } (\exists^{\geqslant_1}) \end{aligned}$$

$\square$

Lemma 21 (Normalized Shape of the Supertypes)**.** *For a normalized positive type $P = \mathbf{nf}\,(P)$, let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle\vert\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \text{and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\, \Gamma, \overrightarrow{\alpha^-} \vdash M',\, \text{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \end{array} \right\}$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \;\mid\; \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Then from lemma 20, $\{\mathbf{nf}\,(Q) \;\mid\; \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\beta^+) \;\mid\; \text{for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

$$\begin{aligned}
\mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') &= \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ not using } \overrightarrow{\beta^-} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothesis} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}\,(Q) = \mathbf{fv}\,Q \text{ by lemma 10} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{by lemma 20} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} && \text{by the definition of } \mathsf{UB} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} && \text{by lemma 20}
\end{aligned}$$

Case 3. $P = \downarrow M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 1. *Suppose that $\nu : A \to A$ is an idempotent function, P is a predicate on A, $F : A \to B$ is a function. Then*

$$\begin{aligned}
&\{F(\nu x) \mid x \in A \text{ s.t. } P(\nu x)\} = \\
=&\{F(x) \mid x \in A \text{ s.t. } \nu x = x \text{ and } P(x)\}.
\end{aligned}$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 2. *For functions F and ν, and predicates P and Q,*

$$\begin{aligned}
&\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } P(x)\} = \\
=&\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } (\exists x' \in A \text{ s.t. } P(x') \text{ and } \nu x' = \nu x)\}.
\end{aligned}$$

$$\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 {\downarrow}M\} =$$

$$= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash {\downarrow}M)\}$$
by lemma 20

$$= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \overrightarrow{N} \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash N_i, \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\}$$
by the definition of UB

$$= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\}$$
we reassigned the substitution $\overrightarrow{N}/\overrightarrow{\alpha^-}$ as σ

$$= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\}$$
by lemma 2

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\}$$
by the definition of normalization

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } \mathbf{nf}\,([\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M') = \mathbf{nf}\,({\downarrow}M) \end{array} \right\}$$
from lemmas 11 and 13, equivalence of types can be replaced with the equality of their normal forms

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})]{\downarrow}\mathbf{nf}\,(M') = {\downarrow}\mathbf{nf}\,(M) \end{array} \right\}$$
by congruence of normalization and lemma 12

$$= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\}$$
by lemma 15, ${\downarrow}M'$ and $\sigma|_{\mathbf{fv}\,M'}$ are already normal, since the result of the substitution is normal; M is normal by assumption

$$= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ (\exists\sigma' \text{ s.t. } \Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')}) \\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\}$$
We apply observation 2 (with $\nu\sigma = \sigma|_{\mathbf{fv}\,M'}$, and $P(\sigma) = \Gamma \vdash \sigma : \overrightarrow{\alpha^-}$)

$$= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv}\,M'} : \overrightarrow{\alpha^-}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\}$$
Notice that "$\exists\sigma'$ s.t. $(\Gamma \vdash \sigma' : \overrightarrow{\alpha^-}$ and $\sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')})$" is equivalent to $\Gamma \vdash \sigma|_{\mathbf{fv}\,({\downarrow}M')} : \overrightarrow{\alpha^-}'$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\}$$
We apply observation 1 to the restriction of σ, and remove $\sigma|_{\mathbf{fv}\,M'} = \sigma$ as it follows from $\Gamma \vdash \sigma : \overrightarrow{\alpha^-}'$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ \overrightarrow{\alpha^-}',\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\}$$
by lemma 4, since $\{\Gamma, \overrightarrow{\alpha^-}\} \cap \mathbf{fv}\,M' = \{\Gamma, \overrightarrow{\alpha^-}'\} \cap \mathbf{fv}\,M'$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}'\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\}$$
We apply observation 1 to the ordering of $\overrightarrow{\alpha^-}$

$$= \left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\ M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}'\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-}' \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}']{\downarrow}M' = {\downarrow}M \end{array} \right\}$$
By reassigning σ explicitly as $\overrightarrow{N}/\overrightarrow{\alpha^-}'$

$$= \mathsf{NFUB}({\downarrow}M)$$
by definition

□

Lemma 22 (Soundness of the Least Upper Bound). *For types $\Gamma \vdash P_1$, and $\Gamma \vdash P_2$, if $\Gamma \vDash P_1 \vee P_2 = Q$ then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Proof. Induction on $\Gamma \vDash P_1 \vee P_2 = Q$.

Case 1. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$
Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule ($\mathrm{Var}^{+\geqslant_1}$).

Case 2. $\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q$

Then by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applied to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q$. Then

(i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,

(ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$,

(iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}\,(Q) \cap \{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}\} = \mathbf{fv}\,(Q) \cap \{\Gamma\}$ (and then apply lemma 4). The inclusion right-to-left is self-evident. To show $\mathbf{fv}\,(Q) \cap \{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}\} \subseteq \mathbf{fv}\,(Q) \cap \{\Gamma\}$, we prove that $\mathbf{fv}\,(Q) \subseteq \{\Gamma\}$

$\mathbf{fv}\,(Q) \subseteq \mathbf{fv}\,P_1 \cap \mathbf{fv}\,P_2$ by lemma 1

$\subseteq (\{\Gamma, \overrightarrow{\alpha^-}\} \backslash \{\overrightarrow{\beta^-}\}) \cap (\{\Gamma, \overrightarrow{\beta^-}\} \backslash \{\overrightarrow{\alpha^-}\})$ since $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_1$, $\mathbf{fv}\,(P_1) \subseteq \{\Gamma, \overrightarrow{\alpha^-}\} = \{\Gamma, \overrightarrow{\alpha^-}\} \backslash \{\overrightarrow{\beta^-}\}$ (the latter is because by the Barendregt's convention, $\{\Gamma, \overrightarrow{\alpha^-}\} \cap \{\overrightarrow{\beta^-}\} = \varnothing$); similarly, $\mathbf{fv}\,(P_2) \subseteq \{\Gamma, \overrightarrow{\beta^-}\} \backslash \{\overrightarrow{\alpha^-}\}$

$\subseteq \{\Gamma\}$

To show $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\alpha^-}.P_1$, we apply Rule $(\exists^{\geqslant_1})$. Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant_1 P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \models {\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$ By the inversion, $\Gamma, \cdot \models {\downarrow}N \overset{a}{\simeq} {\downarrow}M \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$. Then by **??**,

(i) $\Gamma; \Xi \vdash P$

To show $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$, notice that $\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]P$, which follows from $\Gamma; \Xi \vdash P$ by **??**.

To show $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}N$, we apply Rule **??**. Notice that $\widehat{\tau}_i$ corresponds to $\overrightarrow{N}/\overrightarrow{\alpha^-}$

□

Lemma 23 (Completeness of the Least Upper Bound). *For types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q'$ such that $\Gamma \vdash Q' \geqslant_1 P_1$ and $\Gamma \vdash Q' \geqslant_1 P_2$, there exists Q s.t. $\Gamma \models P_1 \vee P_2 = Q$, and $\Gamma \vdash Q' \geqslant_1 Q$*

Lemma 24 (Soundness of Upgrade). *For $\Delta \subseteq \Gamma$, suppose that* $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$.

(i) $\Delta \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P$

Lemma 25 (Completeness of Upgrade). *For $\Delta \subseteq \Gamma$, $\Gamma \vdash P$ and $\Delta \vdash Q'$, such that $\Gamma \vdash Q' \geqslant_1 P$, there exists Q s.t.* $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$, *and $\Delta \vdash Q' \geqslant_1 Q$.*