1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{lcll}
P,\ Q & ::= & & \text{positive types} \\
& \mid & \alpha^{+} & \\
& \mid & {\downarrow}N & \\
& \mid & \exists\alpha^{-}.P & \\
\\
N,\ M & ::= & & \text{negative types} \\
& \mid & \alpha^{-} & \\
& \mid & {\uparrow}P & \\
& \mid & \forall\alpha^{+}.N & \\
& \mid & P \to N &
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq^{\leqslant}_0 M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq^{\leqslant}_0 M} \quad \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq^{\leqslant}_0 Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq^{\leqslant}_0 Q} \quad \textsc{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^{-} \leqslant_0 \alpha^{-}} \quad \textsc{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq^{\leqslant}_0 Q}{\Gamma \vdash {\uparrow}P \leqslant_0 {\uparrow}Q} \quad \textsc{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^{+}]N \leqslant_0 M \quad M \neq \forall\beta^{+}.M'}{\Gamma \vdash \forall\alpha^{+}.N \leqslant_0 M} \quad \textsc{D0ForallL}$$

$$\frac{\Gamma, \alpha^{+} \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall\alpha^{+}.M} \quad \textsc{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \quad \textsc{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^{+} \geqslant_0 \alpha^{+}} \quad \textsc{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq^{\leqslant}_0 M}{\Gamma \vdash {\downarrow}N \geqslant_0 {\downarrow}M} \quad \textsc{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^{-}]P \geqslant_0 Q \quad Q \neq \exists\alpha^{-}.Q'}{\Gamma \vdash \exists\alpha^{-}.P \geqslant_0 Q} \quad \textsc{D0ExistsL}$$

$$\frac{\Gamma, \alpha^{-} \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists\alpha^{-}.Q} \quad \textsc{D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$P,\ Q$ ::= multi-quantified positive types
| α^+
| $\downarrow N$
| $\exists \overrightarrow{\alpha^-}.P$ $\quad P \neq \exists \ldots$
| (P) S

$N,\ M$ ::= multi-quantified negative types
| α^-
| $\uparrow P$
| $P \to N$
| $\forall \overrightarrow{\alpha^+}.N$ $\quad N \neq \forall \ldots$
| (N) S

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \quad (\simeq_1^{\leqslant}{}^-)$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \quad (\simeq_1^{\leqslant}{}^+)$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad (\text{VAR}^{-\leqslant_1})$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \quad (\uparrow^{\leqslant_1})$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \quad (\to^{\leqslant_1})$$

$$\frac{\mathbf{fv}\, N \cap \{\overrightarrow{\beta^+}\} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\leqslant_1})$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad (\text{VAR}^{+\geqslant_1})$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \quad (\downarrow^{\geqslant_1})$$

$$\frac{\mathbf{fv}\, P \cap \{\overrightarrow{\beta^-}\} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\geqslant_1})$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1}$ Equivalence of substitutions

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-} \quad (\text{VAR}^{-\simeq_1^D})$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad (\uparrow^{\simeq_1^D})$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M} \quad (\to^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M} \ (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \ (\text{VAR}^+)$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \ (\downarrow^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \ (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^- = \alpha^-} \ (\text{VAR}^-_{\in})$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^- = \cdot} \ (\text{VAR}^-_{\notin})$$

$$\frac{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in} \uparrow P = \vec{\alpha}} \ (\uparrow)$$

$$\frac{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}_1 \quad \mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}_2}{\mathbf{ord}\; vars\, \mathbf{in}\, P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})} \ (\to)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}} \ (\forall)$$

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^+ = \alpha^+} \ (\text{VAR}^+_{\in})$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\; vars\, \mathbf{in}\, \alpha^+ = \cdot} \ (\text{VAR}^+_{\notin})$$

$$\frac{\mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in} \downarrow N = \vec{\alpha}} \ (\downarrow)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\; vars\, \mathbf{in}\, \exists \overrightarrow{\alpha^-}.P = \vec{\alpha}} \ (\exists)$$

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\; vars\, \mathbf{in}\, \widehat{\alpha}^- = \cdot} \ (\text{UVAR}^-)$$

$\boxed{\mathbf{ord}\; vars\, \mathbf{in}\, P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\; vars\, \mathbf{in}\, \widehat{\alpha}^+ = \cdot} \ (\text{UVAR}^+)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\quad(\textsc{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\quad(\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\quad(\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'}\quad(\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\quad(\textsc{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\quad(\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'}\quad(\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\quad(\textsc{UVar}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\quad(\textsc{UVar}^+)$$

3.2 Unification

$\boxed{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\quad\textsc{UNVar}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\quad\textsc{UShiftU}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\quad\textsc{UArrow}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\quad\textsc{UForall}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)}\quad\textsc{UNUVar}$$

$\boxed{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\quad\textsc{UPVar}$$

$$\frac{\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\quad\textsc{UShiftD}$$

$$\frac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\quad\textsc{UExists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)}\quad\textsc{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma;\, \Theta \vDash N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma;\, \Theta \vDash \alpha^- \leqslant \alpha^- \dashv \cdot}\ \textsc{ANVar}$$

$$\frac{\Theta \vDash \mathbf{nf}\,(P) \stackrel{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma;\, \Theta \vDash \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}}\ \textsc{AShiftU}$$

$$\frac{\Gamma;\, \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma;\, \Theta \vDash N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma;\, \Theta \vDash P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\ \textsc{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+};\, \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma;\, \Theta \vDash \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^+}}\ \textsc{AForall}$$

$\boxed{\Gamma;\, \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma;\, \Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \textsc{APVar}$$

$$\frac{\Theta \vDash \mathbf{nf}\,(N) \stackrel{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma;\, \Theta \vDash \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}}\ \textsc{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-};\, \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vDash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma;\, \Theta \vDash \exists \overrightarrow{\alpha^-}.P \geqslant \exists \overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}}\ \textsc{AExists}$$

$$\frac{\mathbf{upgrade}\, \Gamma \vdash \mathbf{nf}\,(P)\, \mathbf{to}\, \Delta = Q}{\Gamma;\, \Theta \vDash \widehat{\alpha}^+\{\Delta\} \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)}\ \textsc{APUVar}$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restrict an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\Delta \vdash \widehat{\alpha}^+ :\approx P$ or $\Delta \vdash \widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\Delta \vdash \widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries)**.** *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.
$\boxed{e_1 \,\&\, e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\Gamma \vDash P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)}\ (\geqslant \& \geqslant)$$

$$\frac{\Gamma;\, \cdot \vDash P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\ (\simeq \& \geqslant)$$

$$\frac{\Gamma;\, \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)}\ (\geqslant \& \simeq)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\ (\simeq \& \simeq^+)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N) \,\&\, (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)}\ (\simeq \& \simeq^-)$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3.
$$\begin{aligned}\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 = &\{e_1 \,\&\, e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, s.t.\ e_1 \textit{ matches with } e_2\} \\ &\cup \{e_1 \mid e_1 \in \widehat{\sigma}_1,\ s.t.\ \forall e_2 \in \widehat{\sigma}_2, e_1 \textit{ does not match with } e_2\} \\ &\cup \{e_2 \mid e_2 \in \widehat{\sigma}_2,\ s.t.\ \forall e_1 \in \widehat{\sigma}_1, e_1 \textit{ does not match with } e_2\}\end{aligned}$$

3.5 Least Upper Bound

$\boxed{\Gamma \models P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \quad (\text{VAR}^{\vee})$$

$$\frac{\Gamma, \cdot \models {\downarrow}N \overset{a}{\simeq} {\downarrow}M \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models {\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P} \quad ({\downarrow}^{\vee})$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q}{\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q} \quad (\exists^{\vee})$$

$\boxed{\textbf{upgrade}\, \Gamma \vdash P\, \textbf{to}\, \Delta = Q}$

3.6 Antiunification

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \quad \text{AUPVAR}$$

$$\frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models {\downarrow}N_1 \overset{a}{\simeq} {\downarrow}N_2 \dashv (\Xi, {\downarrow}M, \widehat{\tau}_1, \widehat{\tau}_2)} \quad \text{AUPSHIFT}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists\overrightarrow{\alpha^-}.Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad \text{AUPEXISTS}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)} \quad \text{AUNVAR}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models {\uparrow}P_1 \overset{a}{\simeq} {\uparrow}P_2 \dashv (\Xi, {\uparrow}Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad \text{AUNSHIFT}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)} \quad \text{AUNARROW}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad \text{AUNAU}$$

4 Proofs

4.1 Declarative Subtyping

Lemma 1 (Free Variable Propagation)**.** *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context* Γ*,*

- $-$ *if* $\Gamma \vdash N \leqslant_1 M$ *then* $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$
- $+$ *if* $\Gamma \vdash P \geqslant_1 Q$ *then* $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$

Proof. Mutual induction on $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash P \geqslant_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
It is self-evident that $\{\alpha^-\} \subseteq \{\alpha^-\}$.

Case 2. $\Gamma \vdash {\uparrow}P \leqslant_1 {\uparrow}Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq^{\leqslant}_1 Q$), we have $\Gamma \vdash P \geqslant_1 Q$. Then by the induction hypothesis, $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$. The desired inclusion inclusion holds, since $\mathbf{fv}({\uparrow}P) = \mathbf{fv}(P)$ and $\mathbf{fv}({\uparrow}Q) = \mathbf{fv}(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant_1 Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$ and $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$. Then $\mathbf{fv}(P \to N) = \mathbf{fv}(P) \cup \mathbf{fv}(N) \subseteq \mathbf{fv}(Q) \cup \mathbf{fv}(M) = \mathbf{fv}(Q \to M)$.

Case 4. $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$

$$\begin{aligned}\mathbf{fv}\, \forall\overrightarrow{\alpha^+}.N &\subseteq \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \setminus \{\overrightarrow{\beta^+}\} && \text{here } \{\overrightarrow{\beta^+}\} \text{ is excluded by the premise } \mathbf{fv}\, N \cap \{\overrightarrow{\beta^+}\} = \varnothing \\ &\subseteq \mathbf{fv}\, M \setminus \{\overrightarrow{\beta^+}\} && \text{by the induction hypothesis, } \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \subseteq \mathbf{fv}\, M \\ &\subseteq \mathbf{fv}\, \forall\overrightarrow{\beta^+}.M\end{aligned}$$

Case 5. The positive cases are symmetric.

□

4.2 Substitution

Lemma 2 (Substitution strengthening)**.** *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then*

+ *if* $\Gamma_1 \vdash P$ *then* $[\sigma]P = [\sigma|_{\mathbf{fv}\, P}]P$,

− *if* $\Gamma_1 \vdash N$ *then* $[\sigma]N = [\sigma|_{\mathbf{fv}\, N}]N$

Proof. Ilya: todo □

Lemma 3 (Substitution preserves subtyping)**.** *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then*

+ *if* $\Gamma, \Gamma_1 \vdash P$, $\Gamma, \Gamma_1 \vdash Q$, *and* $\Gamma, \Gamma_1 \vdash P \geqslant_1 Q$ *then* $\Gamma, \Gamma_2 \vdash [\sigma]P \geqslant_1 [\sigma]Q$

− *if* $\Gamma, \Gamma_1 \vdash N$, $\Gamma, \Gamma_1 \vdash M$, *and* $\Gamma, \Gamma_1 \vdash N \leqslant_1 M$ *then* $\Gamma, \Gamma_2 \vdash [\sigma]N \leqslant_1 [\sigma]M$

Proof. Ilya: todo □

Corollary 1 (Substitution preserves subtyping)**.** *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then*

+ *if* $\Gamma, \Gamma_1 \vdash P$, $\Gamma, \Gamma_1 \vdash Q$, *and* $\Gamma, \Gamma_1 \vdash P \simeq_1^{\leqslant} Q$ *then* $\Gamma, \Gamma_2 \vdash [\sigma]P \simeq_1^{\leqslant} [\sigma]Q$

− *if* $\Gamma, \Gamma_1 \vdash N$, $\Gamma, \Gamma_1 \vdash M$, *and* $\Gamma, \Gamma_1 \vdash N \simeq_1^{\leqslant} M$ *then* $\Gamma, \Gamma_2 \vdash [\sigma]N \simeq_1^{\leqslant} [\sigma]M$

4.3 Type well-formedness

Lemma 4 (Well-formedness agrees with substitution)**.** *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then*

+ $\Gamma, \Gamma_1 \vdash P \;\Leftrightarrow\; \Gamma, \Gamma_2 \vdash [\sigma]P$

− $\Gamma, \Gamma_1 \vdash N \;\Leftrightarrow\; \Gamma, \Gamma_2 \vdash [\sigma]N$

Proof. Ilya: todo □

Corollary 2. *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then*

+ $\Gamma_1, \Gamma_2 \vdash P \;\Leftrightarrow\; \Gamma_2 \vdash [\sigma]P$

− $\Gamma_1, \Gamma_2 \vdash N \;\Leftrightarrow\; \Gamma_2 \vdash [\sigma]N$

Lemma 5 (Equivalent Contexts)**.** *In the well-formedness judgment, only used variables matter:*

+ *if* $\{\Gamma_1\} \cap \mathbf{fv}\, P = \{\Gamma_2\} \cap \mathbf{fv}\, P$ *then* $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$,

− *if* $\{\Gamma_1\} \cap \mathbf{fv}\, N = \{\Gamma_2\} \cap \mathbf{fv}\, N$ *then* $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$.

Proof. By simple mutual induction on P and Q. □

4.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\dfrac{}{\{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N\} \equiv \mathit{vars} \cap \mathbf{fv}\ N}$	$\dfrac{N \simeq^{D}_{1} M}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ M}$	—
Normalization	$\dfrac{}{N \simeq^{D}_{1} \mathbf{nf}\,(N)}$	$\dfrac{N \simeq^{D}_{1} M}{\mathbf{nf}\,(N) = \mathbf{nf}\,(M)}$	—
Equivalence	$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq^{D}_{1} Q}{\Gamma \vdash P \simeq^{\leqslant}_{1} Q}$	$\dfrac{\Gamma \vdash P \simeq^{\leqslant}_{1} Q}{P \simeq^{D}_{1} Q}$	—
Uppgrade	$\dfrac{\mathbf{upgrade}\ \Gamma \vdash P\ \mathbf{to}\ \Delta = Q}{Q \text{ is sound} \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\ \Gamma \vdash P\ \mathbf{to}\ \Delta = Q}$	$\dfrac{\begin{array}{l} Q' \text{ is sound} \\ \mathbf{upgrade}\ \Gamma \vdash P\ \mathbf{to}\ \Delta = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \vDash P_1 \vee P_2 = Q}{Q \text{ is sound} \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \vDash P_1 \vee P_2 = Q}$	$\dfrac{\begin{array}{l} Q' \text{ is sound} \\ \Gamma \vDash P_1 \vee P_2 = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\begin{array}{l} (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \\ \text{is sound} \end{array} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat{\tau}_i : \Xi \\ [\hat{\tau}_i] Q = P_i \end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \hat{\tau}_1', \hat{\tau}_2')}{\begin{array}{l} \exists (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \text{ s.t.} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \end{array}}$	$\dfrac{\begin{array}{l} (\Xi', Q', \hat{\tau}_1', \hat{\tau}_2') \text{ is sound} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \end{array}}{\exists \Gamma; \Xi \vdash \hat{\tau} : \Xi' \text{ s.t. } [\hat{\tau}] Q' = Q}$
Unification (matching)	$\dfrac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}}{\hat{\sigma} \text{ is sound} \begin{cases} \Theta \vdash \hat{\sigma} \\ [\hat{\sigma}] P = Q \end{cases}}$	$\dfrac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}}$	—
Subtyping	$\dfrac{\Gamma; \Theta \vDash N \leqslant M \dashv \hat{\sigma}}{\hat{\sigma} \text{ is sound} \begin{cases} \Theta \vdash \hat{\sigma} \\ \Gamma \vdash [\hat{\sigma}] N \leqslant_1 M \end{cases}}$	$\dfrac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Gamma; \Theta \vDash N \leqslant M \dashv \hat{\sigma}}$	—

4.5 Anti-unification

Lemma 6 (Soundness of the anti-unification algorithm)**.**

Lemma 7 (Completeness of the anti-unification algorithm)**.**

Lemma 8 (Initiality of the anti-unification algorithm)**.**

4.6 Variable Ordering

Definition 4 (Collision free bijection)**.** *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is* ***collision free on sets*** *P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 9 (Soundness of variable ordering)**.** *Variable ordering extracts precisely used free variables.*

$-$ $\{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N\} \equiv \mathit{vars} \cap \mathbf{fv}\ N$ *(as sets)*

$+$ $\{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P\} \equiv \mathit{vars} \cap \mathbf{fv}\ P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \vec{\alpha}$ and $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}$ ∎

Corollary 3 (Additivity of ordering)**.** *Variable ordering is additive (in terms of set union) with respect to its first argument.*

– $\{\textbf{ord}\,(\mathit{vars}_1 \cup \mathit{vars}_2)\,\textbf{in}\,N\} \equiv \{\textbf{ord}\,\mathit{vars}_1\,\textbf{in}\,N\} \cup \{\textbf{ord}\,\mathit{vars}_2\,\textbf{in}\,N\}$ *(as sets)*

+ $\{\textbf{ord}\,(\mathit{vars}_1 \cup \mathit{vars}_2)\,\textbf{in}\,P\} \equiv \{\textbf{ord}\,\mathit{vars}_1\,\textbf{in}\,P\} \cup \{\textbf{ord}\,\mathit{vars}_2\,\textbf{in}\,P\}$ *(as sets)*

Corollary 4 (Weakening of ordering)**.** *Extending the first argument of the ordering with unused variables does not change the result.*

– $\textbf{ord}\,(\mathit{vars} \cap \textbf{fv}\,N)\,\textbf{in}\,N = \textbf{ord}\,\mathit{vars}\,\textbf{in}\,N$

+ $\textbf{ord}\,(\mathit{vars} \cap \textbf{fv}\,P)\,\textbf{in}\,P = \textbf{ord}\,\mathit{vars}\,\textbf{in}\,P$

Lemma 10 (Distributivity of renaming over variable ordering)**.** *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

– *If μ is collision free on vars and $\textbf{fv}\,N$ then $[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) = \textbf{ord}\,([\mu]\mathit{vars})\,\textbf{in}\,[\mu]N$*

+ *If μ is collision free on vars and $\textbf{fv}\,P$ then $[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,P) = \textbf{ord}\,([\mu]\mathit{vars})\,\textbf{in}\,[\mu]P$*

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in \mathit{vars}$
Then
$$\begin{aligned}[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) &= [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\alpha^-) \\ &= [\mu]\alpha^- && \text{by Rule } (\mathrm{Var}^+_\in) \\ &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]\mathit{vars}) \\ &= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,\beta^- && \text{by Rule } (\mathrm{Var}^+_\in)\text{, because } \beta^- \in [\mu]\mathit{vars} \\ &= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\alpha^-\end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin \mathit{vars}$
Notice that $[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) = [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. On the other hand, $\textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\alpha^- = \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,\alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_\notin)$, because μ is collision free on *vars* and $\textbf{fv}\,N$, so $\textbf{fv}\,N \ni \alpha^- \notin \mu(A \cap \mathit{vars}) \cup \mathit{vars} \supseteq [\mu]\mathit{vars}$.

c. $\alpha^- \in A$ but $\alpha^- \notin \mathit{vars}$
Then $[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) = [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. To prove that $\textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_\notin)$. Let us show that $[\mu]\alpha^- \notin [\mu]\mathit{vars}$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]\mathit{vars} \subseteq \mu(A \cap \mathit{vars}) \cup \mathit{vars}$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap \mathit{vars}) \cup \mathit{vars}$.

(i) If there is an element $x \in A \cap \mathit{vars}$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin \mathit{vars}$. This way, $\mu(\alpha^-) \notin \mu(A \cap \mathit{vars})$.

(ii) Since μ is collision free on *vars* and $\textbf{fv}\,N$, $\mu(A \cap \textbf{fv}\,N) \ni \mu(\alpha^-) \notin \mathit{vars}$.

d. $\alpha^- \notin A$ but $\alpha^- \in \mathit{vars}$
$\textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\alpha^- = \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_\notin)$, because $\alpha^- = [\mu]\alpha^- \in [\mu]\mathit{vars}$ since $\alpha^- \in \mathit{vars}$. On the other hand, $[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) = [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$
$$\begin{aligned}[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) &= [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\uparrow P) \\ &= [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,P) && \text{by Rule } (\uparrow) \\ &= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]P && \text{by the induction hypothesis} \\ &= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,\uparrow[\mu]P && \text{by Rule } (\uparrow) \\ &= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\uparrow P && \text{by the definition of substitution} \\ &= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]N\end{aligned}$$

Case 3. $N = P \to M$
$$\begin{aligned}[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) &= [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,P \to M) \\ &= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})) && \text{where } \textbf{ord}\,\mathit{vars}\,\textbf{in}\,P = \vec{\alpha}_1 \text{ and } \textbf{ord}\,\mathit{vars}\,\textbf{in}\,M = \vec{\alpha}_2 \\ &= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\}) \\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\{\vec{\alpha}_1\}) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to case 1. Notice that } \mu \text{ is} \\ & && \text{collision free on } \{\vec{\alpha}_1\} \text{ and } \{\vec{\alpha}_2\} \text{ since } \{\vec{\alpha}_1\} \subseteq \mathit{vars} \text{ and } \{\vec{\alpha}_2\} \subseteq \textbf{fv}\,N \\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})\end{aligned}$$

$$\begin{aligned}(\textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]N) &= (\textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]P \to [\mu]M) \\ &= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \{\vec{\beta}_1\})) && \text{where } \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]P = \vec{\beta}_1 \text{ and } \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]M = \vec{\beta}_2 \\ & && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})\end{aligned}$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$\begin{aligned}[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) &= [\mu]\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\forall\overrightarrow{\alpha^+}.M\\ &= [\mu]\textbf{ord}\,\mathit{vars}\,\textbf{in}\,M\\ &= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]M \qquad \text{by the induction hypothesis}\end{aligned}$$

$$\begin{aligned}(\textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]N) &= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\forall\overrightarrow{\alpha^+}.M\\ &= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,\forall\overrightarrow{\alpha^+}.[\mu]M\\ &= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]M\end{aligned}$$

□

Lemma 11 (Ordering is not affected by independent substitutions). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2, and vars is a set of variables disjoint with both Γ_1 and Γ_2. Then*

– $\textbf{ord}\,\mathit{vars}\,\textbf{in}\,[\sigma]N = \textbf{ord}\,\mathit{vars}\,\textbf{in}\,N$

+ $\textbf{ord}\,\mathit{vars}\,\textbf{in}\,[\sigma]P = \textbf{ord}\,\mathit{vars}\,\textbf{in}\,P$

Proof. `Ilya: Should be easy` □

Lemma 12 (Completeness of variable ordering). *Variable ordering is invariant under equivalence. For arbitrary vars,*

– *If $N \simeq_1^D M$ then $\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N = \textbf{ord}\,\mathit{vars}\,\textbf{in}\,M$ (as lists)*

+ *If $P \simeq_1^D Q$ then $\textbf{ord}\,\mathit{vars}\,\textbf{in}\,P = \textbf{ord}\,\mathit{vars}\,\textbf{in}\,Q$ (as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. □

4.7 Normaliztaion

Lemma 13. *Set of free variables is invariant under equivalence.*

– *If $N \simeq_1^D M$ then $\textbf{fv}\,N \equiv \textbf{fv}\,M$ (as sets)*

+ *If $P \simeq_1^D Q$ then $\textbf{fv}\,P \equiv \textbf{fv}\,Q$ (as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

Lemma 14. *Free variables are not changed by the normalization*

– $\textbf{fv}\,N \equiv \textbf{fv}\,\textbf{nf}\,(N)$

+ $\textbf{fv}\,P \equiv \textbf{fv}\,\textbf{nf}\,(P)$

Proof. By straightforward induction on $\textbf{nf}\,(N) = M$. □

Lemma 15 (Soundness of quantifier normalization).

– $N \simeq_1^D \textbf{nf}\,(N)$

+ $P \simeq_1^D \textbf{nf}\,(P)$

Proof. Mutual induction on $\textbf{nf}\,(N) = M$ and $\textbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var^-) and (Var^+)
By the corresponding equivalence rules.

Case 2. ($\uparrow$), ($\downarrow$), and ($\rightarrow$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall$), i.e. $\textbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}{}'.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 13, $\textbf{fv}\,N \equiv \textbf{fv}\,N'$. Then by lemma 9, $\{\overrightarrow{\alpha^+}{}'\} \equiv \{\overrightarrow{\alpha^+}\} \cap \textbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \textbf{fv}\,N$, and thus, $\{\overrightarrow{\alpha^+}{}'\} \cap \textbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \textbf{fv}\,N$.

To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^+}{}'.N'$, it suffices to provide a bijection $\mu : \{\overrightarrow{\alpha^+}{}'\} \cap \textbf{fv}\,N' \leftrightarrow \{\overrightarrow{\alpha^+}\} \cap \textbf{fv}\,N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = \mathit{id}$.

Case 4. ($\exists$) Same as for case 3.

□

Corollary 5 (Normalization preserves ordering). *For any vars,*

- $-$ $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \mathbf{nf}\,(N) = \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, M$
- $+$ $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \mathbf{nf}\,(P) = \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, Q$

Proof. Immediately from lemmas 12 and 15. □

Lemma 16 (Distributivity of normalization over substitution). *Normalization of a term distributes over substitution. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2. Then*

- $-$ $\mathbf{nf}\,([\sigma]N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$
- $+$ $\mathbf{nf}\,([\sigma]P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P)$

where $\mathbf{nf}\,(\sigma)$ means pointwise normalization: $[\mathbf{nf}\,(\sigma)]\alpha^- = \mathbf{nf}\,([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.

Case 2. $P = \alpha^+$
Similar to case 1.

Case 3. If the type is formed by $\to$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \to M$ then

$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma](P \to M))$	
$= \mathbf{nf}\,([\sigma]P \to [\sigma]M)$	By the congruence of substitution
$= \mathbf{nf}\,([\sigma]P) \to \mathbf{nf}\,([\sigma]M)$	By the congruence of normalization, i.e. Rule ($\to$)
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) \to [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$	By the induction hypothesis
$= [\mathbf{nf}\,(\sigma)](\mathbf{nf}\,(P) \to \mathbf{nf}\,(M))$	By the congruence of substitution
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P \to M)$	By the congruence of normalization
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$	

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M)$
$= [\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}{}'.\mathbf{nf}\,(M)$ Where $\overrightarrow{\alpha^+}{}' = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M$ (the latter is by corollary 5)

$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\forall\overrightarrow{\alpha^+}.M)$	
$= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\sigma]M)$	Assuming $\{\overrightarrow{\alpha^+}\} \cap \{\Gamma_1\} = \varnothing$ and $\{\overrightarrow{\alpha^+}\} \cap \{\Gamma_2\} = \varnothing$
$= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\sigma]M)$	Where $\overrightarrow{\beta^+} = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,([\sigma]M) = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\sigma]M$ (the latter is by corollary 5)
$= \forall\overrightarrow{\alpha^+}{}'.\mathbf{nf}\,([\sigma]M)$	By lemma 11, $\overrightarrow{\beta^+} = \overrightarrow{\alpha^+}{}'$ since $\{\overrightarrow{\alpha^+}\}$ is disjoint with Γ_1 and Γ_2
$= \forall\overrightarrow{\alpha^+}{}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$	By the induction hypothesis

To show alpha-equivalence of $[\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}{}'.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^+}{}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$, we can assume that $\{\overrightarrow{\alpha^+}{}'\} \cap \{\Gamma_1\} = \varnothing$, and $\{\overrightarrow{\alpha^+}{}'\} \cap \{\Gamma_2\} = \varnothing$.

Case 5. $P = \exists\overrightarrow{\alpha^-}.Q$
Same as for case 4.

□

Corollary 6 (Commutativity of normalization and renaming). *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

- $-$ $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$
- $+$ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Proof. Immediately from lemma 16, after noticing that $\mathbf{nf}\,(\mu) = \mu$. $\square$

Lemma 17 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

$-$ *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

$+$ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. $(\forall^{\simeq_1^D})$

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 9 and 14, the domain and the codomain of μ can be written as $\mu : \{\overrightarrow{\beta^+}'\} \leftrightarrow \{\overrightarrow{\alpha^+}'\}$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by corollary 6, the second—by the induction hypothesis.

(ii)
$$
\begin{aligned}
[\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}' \\
&= [\mu]\mathbf{ord}\,(\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 14 and corollary 4} \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 10, because } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing \\
& && \text{and } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved} \\
&= \mathbf{ord}\,(\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \text{ and } \{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 14 and corollary 4} \\
&= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'
\end{aligned}
$$

Case 2. $(\exists^{\simeq_1^D})$ Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

$\square$

Lemma 18 (Idempotence of normalization)**.** *Normalization is idempotent*

$-$ $\mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$

$+$ $\mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

Proof. By applying lemma 17 to lemma 15. $\square$

Lemma 19. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized. Suppose that* σ *is a substitution* $\Gamma_2 \vdash \sigma : \Gamma_1$*,* P *is a positive type* $(\Gamma_1 \vdash P)$*,* N *is a negative type* $(\Gamma_1 \vdash N)$*. Then*

$$+\ [\sigma]P \textit{ is normal} \iff \begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \textit{is normal} \\ P & \textit{is normal} \end{cases}$$

$$-\ [\sigma]N \textit{is normal} \iff \begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \textit{is normal} \\ N & \textit{is normal} \end{cases}$$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$
Then N is always normal, and the normality of $\sigma|_{\{\alpha^-\}}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$
\begin{aligned}
[\sigma](P \to M) \text{ is normal} &\iff [\sigma]P \to [\sigma]M \text{ is normal} && \text{by the substitution congruence}\\
&\iff \begin{cases}[\sigma]P & \text{is normal}\\ [\sigma]M & \text{is normal}\end{cases} && \text{by congruence of normality } \texttt{Ilya: lemma?}\\
&\iff \begin{cases}P & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(P)} & \text{is normal}\\ M & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(M)} & \text{is normal}\end{cases} && \text{by the induction hypothesis}\\
&\iff \begin{cases}P \to M & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(P)\cup\mathbf{fv}\,(M)} & \text{is normal}\end{cases} \iff \begin{cases}P \to M & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(P\to M)} & \text{is normal}\end{cases}
\end{aligned}
$$

Case 3. $N = \uparrow P$
By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$
\begin{aligned}
[\sigma](\forall\overrightarrow{\alpha^+}.M) \text{ is normal} &\iff (\forall\overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} && \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing\\
&\iff \begin{cases}[\sigma]M \text{ is normal}\\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+}\end{cases} && \text{by the definition of normalization}\\
&\iff \begin{cases}[\sigma]M \text{ is normal}\\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M = \overrightarrow{\alpha^+}\end{cases} && \text{by lemma 11}\\
&\iff \begin{cases}\sigma|_{\mathbf{fv}\,(M)} \text{ is normal}\\ M \text{ is normal}\\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M = \overrightarrow{\alpha^+}\end{cases} && \text{by the induction hypothesis}\\
&\iff \begin{cases}\sigma|_{\mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M)} \text{ is normal}\\ \forall\overrightarrow{\alpha^+}.M \text{ is normal}\end{cases} && \text{since } \mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M);\ \text{by the definition of normalization}
\end{aligned}
$$

Case 5. $P = \ldots$
The positive cases are done in the same way as the negative ones.

□

4.8 Equivalence

Lemma 20 (Type well-formedness is invariant under equivalence)**.** *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,

− *if* $N \simeq_1^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. `Ilya: todo` □

Corollary 7 (Normalization preserves well-formedness)**.**

+ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}\,(P)$,

− $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}\,(N)$

Proof. Immediately from lemmas 15 and 20. □

Corollary 8 (Normalization preserves well-formedness of substitution)**.**
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}\,(\sigma) : \Gamma_1$

Lemma 21 (Soundness of equivalence)**.** *Declarative equivalence implies mutual subtyping.*

+ *if* $\Gamma \vdash P$, $\Gamma \vdash Q$, *and* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \simeq_1^{\leqslant} Q$,

− *if* $\Gamma \vdash N$, $\Gamma \vdash M$, *and* $N \simeq_1^D M$ *then* $\Gamma \vdash N \simeq_1^{\leqslant} M$.

Proof. We prove it by mutual induction on $P \simeq_1^D Q$ and $N \simeq_1^D M$.

Case 1. $\alpha^- \simeq_1^D \alpha^-$
Then $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$ by Rule ($\mathrm{Var}^{-\leqslant_1}$), which immediately implies $\Gamma \vdash \alpha^- \simeq_1^{\leqslant} \alpha^-$ by Rule ($\simeq_1^{\leqslant} -$).

Case 2. $\uparrow P \simeq_1^D \uparrow Q$
Then by inversion of Rule ($\uparrow^{\leqslant_1}$), $P \simeq_1^D Q$, and from the induction hypothesis, $\Gamma \vdash P \simeq_1^{\leqslant} Q$, and (by symmetry) $\Gamma \vdash Q \simeq_1^{\leqslant} P$.

When Rule ($\uparrow^{\leqslant_1}$) is applied to $\Gamma \vdash P \simeq_1^{\leqslant} Q$, it gives us $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$; when it is applied to $\Gamma \vdash Q \simeq_1^{\leqslant} P$, we obtain $\Gamma \vdash \uparrow Q \leqslant_1 \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \simeq_1^{\leqslant} \uparrow Q$.

Case 3. $P \to N \simeq_1^D Q \to M$
Then by inversion of Rule ($\to^{\leqslant_1}$), $P \simeq_1^D Q$ and $N \simeq_1^D M$. By the induction hypothesis, $\Gamma \vdash P \simeq_1^{\leqslant} Q$ and $\Gamma \vdash N \simeq_1^{\leqslant} M$, which means by inversion: (i) $\Gamma \vdash P \geqslant_1 Q$, (ii) $\Gamma \vdash Q \geqslant_1 P$, (iii) $\Gamma \vdash N \leqslant_1 M$, (iv) $\Gamma \vdash M \leqslant_1 N$. Applying Rule ($\to^{\leqslant_1}$) to (i) and (iii), we obtain $\Gamma \vdash P \to N \leqslant_1 Q \to M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \to M \leqslant_1 P \to N$. Together, it implies $\Gamma \vdash P \to N \simeq_1^{\leqslant} Q \to M$.

Case 4. $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M$
Then by inversion, there exists bijection $\mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N)$, such that $N \simeq_1^D [\mu]M$. By the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu]M$. From corollary 1 and the fact that μ is bijective, we also have $\Gamma, \overrightarrow{\beta^+} \vdash [\mu^{-1}]N \simeq_1^{\leqslant} M$.

Let us construct a subsitution $\overrightarrow{\alpha^+} \vdash \overrightarrow{P}/\overrightarrow{\beta^+} : \overrightarrow{\beta^+}$ by extending μ with arbitrary positive types on $\{\overrightarrow{\beta^+}\}\backslash\mathbf{fv}\, M$.

Notice that $[\mu]M = [\overrightarrow{P}/\overrightarrow{\beta^+}]M$, and therefore, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu]M$ implies $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{P}/\overrightarrow{\beta^+}]M \leqslant_1 N$. Then by Rule ($\forall^{\leqslant_1}$), $\Gamma \vdash \forall\overrightarrow{\beta^+}.M \leqslant_1 \forall\overrightarrow{\alpha^+}.N$.

Analogously, we construct the substitution from μ^{-1}, and use it to instantiate $\overrightarrow{\alpha^+}$ in the application of Rule ($\forall^{\leqslant_1}$) to infer $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$.

This way, $\Gamma \vdash \forall\overrightarrow{\beta^+}.M \leqslant_1 \forall\overrightarrow{\alpha^+}.N$ and $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$ gives us $\Gamma \vdash \forall\overrightarrow{\beta^+}.M \simeq_1^{\leqslant} \forall\overrightarrow{\alpha^+}.N$.

Case 5. For the cases of the positive types, the proofs are symmetric.

$\square$

Lemma 22 (Completeness of equivalence)**.** *Mutual subtyping implies declarative equivalence.*

+ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $P \simeq_1^D Q$,

− *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $N \simeq_1^D M$.

Proof. Ilya: todo $\square$

Lemma 23. *Informally, this lemma says that if* $\Gamma, vars_1 \vdash [\sigma_{21}]P \geqslant_1 Q$ *and* $\Gamma, vars_2 \vdash [\sigma_{12}]Q \geqslant_1 P$ *holds for substitutions* σ_{12} *and* σ_{21} *then these substitutions are in fact mutually inverse bijections between variables* $vars_1$ *and* $vars_1$*.*

+ *For* $\Gamma, vars_2 \vdash P$, $\Gamma, vars_1 \vdash Q$, $\Gamma, vars_2 \vdash \sigma_{12} : vars_1$, $\Gamma, vars_1 \vdash \sigma_{21} : vars_2$, *suppose that:*

  1. $\{n \mid \alpha^{\pm n} \in vars_1 \cap \mathbf{fv}\, Q\} = \{n \mid \alpha^{\pm n} \in vars_2 \cap \mathbf{fv}\, P\}$,
  2. $\Gamma, vars_1 \vdash [\sigma_{21}]P \geqslant_1 Q$,
  3. $\Gamma, vars_2 \vdash [\sigma_{12}]Q \geqslant_1 P$.

  Then there exists a bijection $\mu : vars_1 \cap \mathbf{fv}\, Q \leftrightarrow vars_2 \cap \mathbf{fv}\, P$ *such that:*

  1. μ *preserves cohorts:* $\mu(\alpha^{\pm n}) = \beta^{\pm n}$ *(*$\alpha^{\pm n}$ *and* $\beta^{\pm n}$ *have the same cohort label* $\pm n$*).*
  2. $\Gamma, vars_2 \vdash \sigma_{12} \simeq_1^{\leqslant} \mu : (vars_1 \cap \mathbf{fv}\, Q)$ *(the equivalence is pointwise)*
  3. $\Gamma, vars_1 \vdash \sigma_{21} \simeq_1^{\leqslant} \mu^{-1} : (vars_2 \cap \mathbf{fv}\, P)$ *(the equivalence is pointwise)*

− *For* $\Gamma, vars_2 \vdash N$, $\Gamma, vars_1 \vdash M$, $\Gamma, vars_2 \vdash \sigma_{12} : vars_1$, $\Gamma, vars_1 \vdash \sigma_{21} : vars_2$, *suppose that:*

  1. $\{n \mid \alpha^{\pm n} \in vars_1 \cap \mathbf{fv}\, M\} = \{n \mid \beta^{\pm n} \in vars_2 \cap \mathbf{fv}\, N\}$,
  2. $\Gamma, vars_1 \vdash [\sigma_{21}]N \leqslant_1 M$,
  3. $\Gamma, vars_2 \vdash [\sigma_{12}]M \leqslant_1 N$.

  Then there exists a bijection $\mu : vars_1 \cap \mathbf{fv}\, M \leftrightarrow vars_2 \cap \mathbf{fv}\, N$ *such that:*

  1. μ *preserves cohorts:* $\mu(\alpha^{\pm n}) = \beta^{\pm n}$ *(*$\alpha^{\pm n}$ *and* $\beta^{\pm n}$ *have the same cohort label* $\pm n$*).*

2. $\Gamma, vars_2 \vdash \sigma_{12} \simeq_1^{\leqslant} \mu : (vars_1 \cap \mathbf{fv}\, M)$ *(the equivalence is pointwise)*
3. $\Gamma, vars_1 \vdash \sigma_{21} \simeq_1^{\leqslant} \mu^{-1} : (vars_2 \cap \mathbf{fv}\, N)$ *(the equivalence is pointwise)*

Proof. Mutual induction on the pair of sizes of inference trees: $\Gamma, vars_1 \vdash [\sigma_{21}]N \leqslant_1 M$ and $\Gamma, vars_2 \vdash [\sigma_{12}]M \leqslant_1 N$ (or the corresponding trees in the positive case).

Case 1. $N = \forall\overrightarrow{\delta^+}.\beta^{-n}$, for $\beta^{-n} \in vars_2$ and possibly empty $\overrightarrow{\delta^+}$

Then by **??** Ilya: lemma $\Gamma, vars_2 \vdash [\sigma_{12}]M \leqslant_1 \forall\overrightarrow{\delta^+}.\beta^{-n}$ means that $[\sigma_{12}]M = \forall\overrightarrow{\gamma^+}.\beta^{-n}$, which (since $\beta^{-n} \notin \mathbf{fv}\, M$) is only possible when $M = \forall\overrightarrow{\gamma^{+\prime}}.\alpha^{-n'}$ (where $\alpha^{-n'} \in vars_1$ and $\overrightarrow{\gamma^{+\prime}}$ is possibly empty), and $\sigma_{12}(\alpha^{-n'}) = \forall\overrightarrow{\gamma^{+\prime\prime}}.\beta^{-n}$. Notice that by the first condition, the sets of cohort labels of $vars_1 \cap \mathbf{fv}\, M$ and $vars_2 \cap \mathbf{fv}\, N$ must be equal, which means $n = n'$. Also notice that $\Gamma, vars_2 \vdash \forall\overrightarrow{\gamma^{+\prime\prime}}.\beta^{-n} \simeq_1^{\leqslant} \beta^{-n}$.

Then $\Gamma, vars_1 \vdash [\sigma_{21}]N \leqslant_1 M$ becomes $\Gamma, vars_1 \vdash [\sigma_{21}]\forall\overrightarrow{\delta^+}.\beta^{-n} \leqslant_1 \forall\overrightarrow{\gamma^{+\prime}}.\alpha^{-n}$, which by **??** Ilya: lemma implies that $[\sigma_{21}]\forall\overrightarrow{\delta^+}.\beta^{-n} = \forall\overrightarrow{\delta^{+\prime}}.\alpha^{-n}$, and thus, $\sigma_{21}(\beta^{-n}) = \forall\overrightarrow{\delta^{+\prime\prime}}.\alpha^{-n}$. Notice that $\Gamma, vars_1 \vdash \forall\overrightarrow{\delta^{+\prime\prime}}.\alpha^{-n} \simeq_1^{\leqslant} \alpha^{-n}$.

This way, we can take $\mu = \alpha^{-n} \mapsto \beta^{-n}$, which by construction is a bijection preserving cohorts. Moreover, $\Gamma, vars_2 \vdash \forall\overrightarrow{\gamma^{+\prime\prime}}.\beta^{-n} \simeq_1^{\leqslant} \beta^{-n}$ means $\Gamma, vars_2 \vdash [\sigma_{12}]\alpha^{-n} \simeq_1^{\leqslant} [\mu]\alpha^{-n}$ implying $\Gamma, vars_2 \vdash \sigma_{12} \simeq_1^{\leqslant} \mu : (vars_1 \cap \mathbf{fv}\, M)$; and $\Gamma, vars_1 \vdash \forall\overrightarrow{\delta^{+\prime\prime}}.\alpha^{-n} \simeq_1^{\leqslant} \alpha^{-n}$ means $\Gamma, vars_1 \vdash [\sigma_{21}]\beta^{-n} \simeq_1^{\leqslant} [\mu^{-1}]\beta^{-n}$ implying $\Gamma, vars_1 \vdash \sigma_{21} \simeq_1^{\leqslant} \mu^{-1} : (vars_2 \cap \mathbf{fv}\, N)$.

Case 2. The last rule to infer $\Gamma, vars_1 \vdash [\sigma_{21}]N \leqslant_1 M$ was Rule $(\mathrm{Var}^{-\leqslant_1})$, i.e. $[\sigma_{21}]N = M = \gamma^-$.
Then the case when $\gamma^- \in vars_2$, has been covered by case 1, so we assume that $\gamma^- \notin vars_2$, and thus, $N = \gamma^-$.

Notice that $\gamma^- \notin vars_1$ because otherwise, $vars_1 \cap \mathbf{fv}\, N \neq \varnothing$, which would contradict with $\Gamma, vars_2 \vdash N$.

Then $\mathbf{fv}\, N \cap vars_2 = \{\gamma^-\} \cap vars_2 = \varnothing$ and $\mathbf{fv}\, M \cap vars_1 = \{\gamma^-\} \cap vars_1 = \varnothing$. Hence, we take the empty $\mu : \varnothing \leftrightarrow \varnothing$, which vacuously satisfies the required properties.

Case 3. The last rule to infer $\Gamma, vars_1 \vdash [\sigma_{21}]N \leqslant_1 M$ was Rule $(\uparrow^{\leqslant_1})$, i.e. $[\sigma_{21}]N = \uparrow P$, $M = \uparrow Q$, and $\Gamma, vars_1 \vdash P \simeq_1^{\leqslant} Q$
Since N is not a variable from the domain of σ_{21} (which has been covered by case 1), the substitution applied to N must preserve its outer shape. Specifically, $[\sigma_{21}]N = \uparrow P$ means $[\sigma_{21}]N = [\sigma_{21}]\uparrow P' = \uparrow[\sigma_{21}]P' = \uparrow P$, i.e. $N = \uparrow P'$ and $[\sigma_{21}]P' = P$. In particular, $\Gamma, vars_1 \vdash P \simeq_1^{\leqslant} Q$ implies $\Gamma, vars_1 \vdash P \geqslant_1 Q$, i.e. $\Gamma, vars_1 \vdash [\sigma_{21}]P' \geqslant_1 Q$.

In addition, $\Gamma, vars_2 \vdash [\sigma_{12}]M \leqslant_1 N$ becomes $\Gamma, vars_2 \vdash \uparrow[\sigma_{12}]Q \leqslant_1 \uparrow P'$, which is only inferable by Rule $(\uparrow^{\leqslant_1})$, meaning that $\Gamma, vars_2 \vdash [\sigma_{12}]Q \simeq_1^{\leqslant} P'$, and in particular, $\Gamma, vars_2 \vdash [\sigma_{12}]Q \geqslant_1 P'$.

Notice that the tree inferring $\Gamma, vars_2 \vdash [\sigma_{12}]Q \geqslant_1 P'$ is a proper subtree of $\Gamma, vars_2 \vdash [\sigma_{12}]M \leqslant_1 N$. Analogously, $\Gamma, vars_1 \vdash [\sigma_{21}]P' \geqslant_1 Q$ is a proper subtree of $\Gamma, vars_1 \vdash [\sigma_{21}]N \leqslant_1 M$. This way, we apply the induction hypothesis to $\Gamma, vars_1 \vdash [\sigma_{21}]P' \geqslant_1 Q$ and $\Gamma, vars_2 \vdash [\sigma_{12}]Q \geqslant_1 P'$ (notice that $vars_1$, $vars_2$, and the sets of free variables of the types did not change) and obtain exactly what we aimed.

Case 4. The last rule to infer $\Gamma, vars_1 \vdash [\sigma_{21}]N \leqslant_1 M$ was Rule $(\forall^{\leqslant_1})$, i.e. $[\sigma_{21}]N = \forall\overrightarrow{\gamma^+}.N'$, $M = \forall\overrightarrow{\delta^+}.M'$, and $\Gamma, vars_1, \overrightarrow{\delta^+} \vdash [\overrightarrow{P}/\overrightarrow{\gamma^+}]N' \leqslant_1 M'$ for $\Gamma, vars_1, \overrightarrow{\delta^+} \vdash P_i$

Since N does not have the shape of $\forall\overrightarrow{\delta^+}.\beta^{-n}$, for $\beta^{-n} \in vars_2$ (which has been covered by case 1), the substitution applied to N must preserve its outer shape. Specifically, $[\sigma_{21}]N = \forall\overrightarrow{\gamma^+}.N'$ means that N "starts with" $\forall\overrightarrow{\gamma^+}$, i.e. $[\sigma_{21}]N = [\sigma_{21}]\forall\overrightarrow{\gamma^+}.N'' = \forall\overrightarrow{\gamma^+}.[\sigma_{21}]N'' = \forall\overrightarrow{\gamma^+}.N'$, where $N = \forall\overrightarrow{\gamma^+}.N''$ and $[\sigma_{21}]N'' = N'$.

This way, $\Gamma, vars_1, \overrightarrow{\delta^+} \vdash [\overrightarrow{P}/\overrightarrow{\gamma^+}]N' \leqslant_1 M'$ becomes $\Gamma, vars_1, \overrightarrow{\delta^+} \vdash [\overrightarrow{P}/\overrightarrow{\gamma^+}][\sigma_{21}]N'' \leqslant_1 M'$. Notice that the tree inferring this judgment is a proper subtree of $\Gamma, vars_1 \vdash [\sigma_{21}]N \leqslant_1 M$.

On the other hand, $\Gamma, vars_2 \vdash [\sigma_{12}]M \leqslant_1 N$ becomes $\Gamma, vars_2 \vdash \forall\overrightarrow{\delta^+}.[\sigma_{12}]M' \leqslant_1 \forall\overrightarrow{\gamma^+}.N''$ (where either $\overrightarrow{\delta^+}$ or $\overrightarrow{\gamma^+}$ is non-empty), which is only inferable by Rule $(\forall^{\leqslant_1})$, meaning that $\Gamma, vars_2, \overrightarrow{\gamma^+} \vdash [\overrightarrow{Q}/\overrightarrow{\delta^+}][\sigma_{12}]M' \leqslant_1 N''$ for some $\Gamma, vars_2, \overrightarrow{\gamma^+} \vdash Q_i$. Notice that the tree inferring this judgment is a proper subtree of $\Gamma, vars_2 \vdash [\sigma_{12}]M \leqslant_1 N$.

Let us label $\overrightarrow{\gamma^+}$ and $\overrightarrow{\delta^+}$ with a sufficiently large cohort label m such that $m > n$ for any $\alpha^{+n} \in vars_1 \cup vars_2$. Then we merge $\overrightarrow{Q}/\overrightarrow{\delta^{+m}}$ and σ_{12} denoting the resulting substitution as σ'_{12} $((\Gamma, vars_2, \overrightarrow{\gamma^{+m}}) \vdash \sigma'_{12} : (vars_1, \overrightarrow{\delta^{+m}}))$.

What do we mean by merging? The codomains of $\overrightarrow{Q}/\overrightarrow{\delta^{+m}}$ and σ_{12} are $\Gamma, vars_2, \overrightarrow{\gamma^{+m}}$ and $\Gamma, vars_2$, respectively, which are disjoint with both of their domains ($\overrightarrow{\delta^{+m}}$ and $vars_1$, respectively). In turn, the domains of $\overrightarrow{Q}/\overrightarrow{\delta^{+m}}$ and σ_{12} are themselves mutually disjoint. This way, by merging we mean $\overrightarrow{Q}/\overrightarrow{\delta^{+m}} \cup \sigma_{12}$ (relationally) or $\overrightarrow{Q}/\overrightarrow{\delta^{+m}} \circ \sigma_{12}$ or $\sigma_{12} \circ \overrightarrow{Q}/\overrightarrow{\delta^{+m}}$, since these three substitutions are equal.

Analogously, we merge $\overrightarrow{P}/\overrightarrow{\gamma^{+m}}$ and σ_{21} denoting the resulting substitution as σ'_{21} $((\Gamma, vars_1, \overrightarrow{\delta^{+m}}) \vdash \sigma'_{21} : (vars_2, \overrightarrow{\gamma^{+m}}))$.

We wish to apply the induction hypothesis to $\Gamma, vars_1, \overrightarrow{\delta^{+m}} \vdash [\sigma'_{21}]N'' \leqslant_1 M'$ and $\Gamma, vars_2, \overrightarrow{\gamma^{+m}} \vdash [\sigma'_{12}]M' \leqslant_1 N''$. To do so, we need to show that the cohort labels of $(vars_2 \cup \{\overrightarrow{\gamma^{+m}}\}) \cap \mathbf{fv}\, N''$ coincide with those of $(vars_1 \cup \{\overrightarrow{\delta^{+m}}\}) \cap \mathbf{fv}\, M'$ (as sets).

Assertion. *The set of cohorts of* $(\mathit{vars}_2 \cup \{\overrightarrow{\gamma^{+m}}\}) \cap \mathbf{fv}\, N''$ *is equal to the set of cohorts of* $(\mathit{vars}_1 \cup \{\overrightarrow{\delta^{+m}}\}) \cap \mathbf{fv}\, M'$.

Proof. $(\mathit{vars}_2 \cup \{\overrightarrow{\gamma^{+m}}\}) \cap \mathbf{fv}\, N'' = (\mathit{vars}_2 \cup \{\overrightarrow{\gamma^{+m}}\}) \cap (\mathbf{fv}\, N \cup (\{\overrightarrow{\gamma^{+m}}\} \cap \mathbf{fv}\, N''))$

$= \mathit{vars}_2 \cap \mathbf{fv}\, N \cup \{\overrightarrow{\gamma^{+m}}\} \cap \mathbf{fv}\, N''$ because vars_2 and $\mathbf{fv}\, N$ are disjoint with $\overrightarrow{\gamma^{+m}}$

$(\mathit{vars}_1 \cup \{\overrightarrow{\delta^{+m}}\}) \cap \mathbf{fv}\, M' = \mathit{vars}_1 \cap \mathbf{fv}\, M \cup \{\overrightarrow{\delta^{+m}}\} \cap \mathbf{fv}\, M'$ analogously

Since the cohort labels of $\mathit{vars}_2 \cap \mathbf{fv}\, N$ coincide with those of $\mathit{vars}_1 \cap \mathbf{fv}\, M$ by the assumption, it suffices to prove that the cohort labels of $\{\overrightarrow{\gamma^{+m}}\} \cap \mathbf{fv}\, N''$ and of $\{\overrightarrow{\delta^{+m}}\} \cap \mathbf{fv}\, M'$ coincide. Note that these sets have the same cohorts labels m, i.e. it is required to show that these sets are either both empty or both non-empty, i.e. $\{\overrightarrow{\gamma^{+m}}\} \cap \mathbf{fv}\, N'' = \varnothing \iff \{\overrightarrow{\delta^{+m}}\} \cap \mathbf{fv}\, M' = \varnothing$

($\Leftarrow$) Suppose that $\{\overrightarrow{\delta^{+m}}\} \cap \mathbf{fv}\, M' = \varnothing$. Then by **??**, since $\Gamma, \mathit{vars}_1, \overrightarrow{\delta^{+m}} \vdash [\sigma'_{21}]N'' \leqslant_1 M'$, $\mathbf{fv}\, [\sigma'_{21}]N'' \subseteq \mathbf{fv}\, M'$, implying that $\{\overrightarrow{\delta^{+m}}\} \cap \mathbf{fv}\, [\sigma'_{21}]N'' = \varnothing$, and by context strengthening **??**, $\Gamma, \mathit{vars}_1, \overrightarrow{\delta^{+m}} \vdash [\sigma'_{21}]N''$ reduces to $\Gamma, \mathit{vars}_1 \vdash [\sigma'_{21}]N''$.
Let us restrict σ'_{21} to the set of free variables of $\mathbf{fv}\, N''$. Then the domain and codomain of σ'_{21} are the following: $\Gamma, \mathit{vars}_1 \vdash \sigma'_{21}|_{\mathbf{fv}\, N''} : \{\mathit{vars}_2, \overrightarrow{\gamma^{+m}}\} \cap \mathbf{fv}\, N$. By **??**, $[\sigma'_{21}]N'' = [\sigma'_{21}|_{\mathbf{fv}\, N''}]N''$, and then $\Gamma, \mathit{vars}_1 \vdash [\sigma'_{21}]N'' \leqslant_1 M'$, can be rewritten as $\Gamma, \mathit{vars}_1 \vdash [\sigma'_{21}|_{\mathbf{fv}\, N''}]N'' \leqslant_1 M'$.
Let us apply substitution σ_{12} ($\Gamma, \mathit{vars}_2 \vdash \sigma_{12} : \mathit{vars}_1$) to both sides of this judgment to obtain: $\Gamma, \mathit{vars}_2 \vdash [\sigma_{12} \circ \sigma'_{21}|_{\mathbf{fv}\, N''}]N'' \leqslant_1 [\sigma_{12}]M'$. Using the transitivity **??**, let us compose this subtyping judgment with $\Gamma, \mathit{vars}_2, \overrightarrow{\gamma^{+m}} \vdash [\sigma_{12}]M' \leqslant_1 N''$ to form $\Gamma, \mathit{vars}_2, \overrightarrow{\gamma^{+m}} \vdash [\sigma_{12} \circ \sigma'_{21}|_{\mathbf{fv}\, N''}]N'' \leqslant_1 N''$.
What is the codomain of $\sigma_{12} \circ \sigma'_{21}|_{\mathbf{fv}\, N''}$? By composing $\Gamma, \mathit{vars}_1 \vdash \sigma'_{21}|_{\mathbf{fv}\, N''} : \{\mathit{vars}_2, \overrightarrow{\gamma^{+m}}\} \cap \mathbf{fv}\, N$ and $\Gamma, \mathit{vars}_2 \vdash \sigma_{12} : \mathit{vars}_1$, we have $\Gamma, \mathit{vars}_2 \vdash \sigma_{12} \circ \sigma'_{21}|_{\mathbf{fv}\, N''} : \{\mathit{vars}_2, \overrightarrow{\gamma^{+m}}\} \cap \mathbf{fv}\, N$, i.e. $\sigma_{12} \circ \sigma'_{21}|_{\mathbf{fv}\, N''}$ is a substitution contracting $\overrightarrow{\gamma^{+m}}$. Then by **??**, $\Gamma, \mathit{vars}_2, \overrightarrow{\gamma^{+m}} \vdash [\sigma_{12} \circ \sigma'_{21}|_{\mathbf{fv}\, N''}]N'' \leqslant_1 N''$ implies that $\{\overrightarrow{\gamma^{+m}}\} \cap \mathbf{fv}\, N'' = \varnothing$.

($\Rightarrow$) Analogous to the previous case.

□

This way, we can apply the induction hypothesis to $\Gamma, \mathit{vars}_1, \overrightarrow{\delta^{+m}} \vdash [\sigma'_{21}]N'' \leqslant_1 M'$ and $\Gamma, \mathit{vars}_2, \overrightarrow{\gamma^{+m}} \vdash [\sigma'_{12}]M' \leqslant_1 N''$, and obtain $\mu : (\mathit{vars}_1 \cup \{\overrightarrow{\delta^{+m}}\}) \cap \mathbf{fv}\, M' \leftrightarrow (\mathit{vars}_2 \cup \{\overrightarrow{\gamma^{+m}}\}) \cap \mathbf{fv}\, N''$ such that:

1. μ preserves cohorts,
2. $\Gamma, \mathit{vars}_2, \overrightarrow{\gamma^{+m}} \vdash \sigma'_{12} \simeq_1^{\leqslant} \mu : (\mathit{vars}_1 \cup \{\overrightarrow{\delta^{+m}}\}) \cap \mathbf{fv}\, M'$, and
3. $\Gamma, \mathit{vars}_1, \overrightarrow{\delta^{+m}} \vdash \sigma'_{21} \simeq_1^{\leqslant} \mu^{-1} : (\mathit{vars}_2 \cup \{\overrightarrow{\gamma^{+m}}\}) \cap \mathbf{fv}\, N''$

Let us decompose μ into the union of two bijections: $\mu_1 \sqcup \mu_2$, where μ_1 is defined on the variables whose cohort labels are less than m, and μ_2 is defined on the variables whose cohort labels are equal to m. Notice that since μ preserves cohorts, this union is disjoint: the *range* of μ_1 is the variables with cohorts $< m$ and the *range* of μ_2 is the variables with cohorts $= m$.

Recalling how m is chosen, notice that the signatures of μ_1 and μ_2 are the following:

1. $\mu_1 : \mathit{vars}_1 \cap \mathbf{fv}\, M' \leftrightarrow \mathit{vars}_2 \cap \mathbf{fv}\, N''$ and
2. $\mu_2 : \{\overrightarrow{\delta^{+m}}\} \cap \mathbf{fv}\, M' \leftrightarrow \{\overrightarrow{\gamma^{+m}}\} \cap \mathbf{fv}\, N''$.

This way, μ_1 is a cohort-preserving bijection with the required signature, and what is left to show is the equivalences $\mu_1 \simeq_1^{\leqslant} \sigma_{12}$, and $\mu_1^{-1} \simeq_1^{\leqslant} \sigma_{21}$:

1. $\Gamma, \mathit{vars}_2 \vdash \sigma_{12} \simeq_1^{\leqslant} \mu_1 : (\mathit{vars}_1 \cap \mathbf{fv}\, M)$
   Let us take an arbitrary $\alpha^{\pm} \in \mathit{vars}_1 \cap \mathbf{fv}\, M$.
   $[\sigma_{12}]\alpha^{\pm} = [\sigma_{12} \circ \vec{Q}/\overrightarrow{\delta^{+m}}]\alpha^{\pm}$ because $\alpha^{\pm} \neq \delta_i^{+m}$
   $= [\sigma'_{12}]\alpha^{\pm}$ by the definition of σ'_{12}
   $\simeq_1^{\leqslant} [\mu]\alpha^{\pm}$ since $\Gamma, \mathit{vars}_2, \overrightarrow{\gamma^{+m}} \vdash \sigma'_{12} \simeq_1^{\leqslant} \mu : ((\mathit{vars}_1 \cup \{\overrightarrow{\delta^{+m}}\}) \cap \mathbf{fv}\, N)$
   $= [\mu_1]\alpha^{\pm}$ since $\mu_1 = \mu|_{\mathit{vars}_1 \cap \mathbf{fv}\, M'}$ and $\alpha^{\pm} \in \mathit{vars}_1 \cap \mathbf{fv}\, M \subseteq \mathit{vars}_1 \cap \mathbf{fv}\, M'$
   This way, $\Gamma, \mathit{vars}_2, \overrightarrow{\gamma^{+m}} \vdash [\sigma_{12}]\alpha^{\pm} \simeq_1^{\leqslant} [\mu_1]\alpha^{\pm}$, which, considering that the codomains of σ_{12} and μ_1 are in Γ, vars_2, can be strengthen to $\Gamma, \mathit{vars}_2 \vdash [\sigma_{12}]\alpha^{\pm} \simeq_1^{\leqslant} [\mu_1]\alpha^{\pm}$.
2. $\Gamma, \mathit{vars}_1 \vdash \sigma_{21} \simeq_1^{\leqslant} \mu_1^{-1} : (\mathit{vars}_2 \cap \mathbf{fv}\, N)$ is proved analogously.

Case 5. The last rule to infer $\Gamma, \mathit{vars}_1 \vdash [\sigma_{21}]N \leqslant_1 M$ was Rule ($\rightarrow^{\leqslant_1}$), i.e. $[\sigma_{21}]N = P \rightarrow N'$, and $M = Q \rightarrow M'$; then by inverting this rule, $\Gamma, \mathit{vars}_1 \vdash P \geqslant_1 Q$ and $\Gamma, \mathit{vars}_1 \vdash N' \leqslant_1 M'$.

$[\sigma_{21}]N = P \rightarrow N'$ means that either N is a variable from the domain of σ_{21} (which has been covered by case 1) or $[\sigma_{21}]N = [\sigma_{21}](P' \rightarrow N'') = [\sigma_{21}]P' \rightarrow [\sigma_{21}]N'' = P \rightarrow N'$, where $N = P' \rightarrow N''$, $[\sigma_{21}]P' = P$, and $[\sigma_{21}]N'' = N'$.

This way, $\Gamma \vdash P \geqslant_1 Q$ and $\Gamma \vdash N' \leqslant_1 M'$ can be rewritten as $\Gamma \vdash [\sigma_{21}]P' \geqslant_1 Q$ and $\Gamma \vdash [\sigma_{21}]N'' \leqslant_1 M'$.

In addition, $\Gamma, \mathit{vars}_2 \vdash [\sigma_{12}]M \leqslant_1 N$ becomes $\Gamma, \mathit{vars}_2 \vdash [\sigma_{12}](Q \to M') \leqslant_1 P' \to N''$, implying by inversion $\Gamma, \mathit{vars}_2 \vdash [\sigma_{12}]Q \geqslant_1 P'$ and $\Gamma, \mathit{vars}_2 \vdash [\sigma_{12}]M' \leqslant_1 N''$.

□

4.9 Upper Bounds

Lemma 24 (Decomposition of the quantifier rule). *Ilya: move somewhere* *Whenever the quantifier rule (Rule $(\exists^{\geqslant_1})$ or Rule $(\forall^{\leqslant_1})$) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.*

– *If* $\Gamma \vdash N \leqslant_1 \forall\overrightarrow{\beta^+}.M$ *then* $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant_1 M$.

+ *If* $\Gamma \vdash P \geqslant_1 \exists\overrightarrow{\beta^-}.Q$ *then* $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant_1 Q$.

Lemma 25 (Characterization of the Supertypes). *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \text{for } \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle\vert\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M',\ \text{and } \overrightarrow{N} \text{ s.t.} \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ \text{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Then the last rule that is applied to infer $\Gamma \vdash Q \geqslant_1 \beta^+$ must be either Rule $(\mathrm{Var}^{+\geqslant_1})$ or Rule $(\exists^{\geqslant_1})$. The former case means that $Q = \beta^+$. In the latter case, $Q = \exists\overrightarrow{\alpha^-}.Q'$, where Q' has no outer existential quantifiers. Then by inversion of Rule $(\exists^{\geqslant_1})$, $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]Q' \geqslant_1 \beta^+$ for some $\overrightarrow{N}$. This time, to infer this judgment, only Rule $(\mathrm{Var}^{+\geqslant_1})$ is applicable, which means that $Q' = \beta^+$, and then $Q = \exists\overrightarrow{\alpha^-}.\beta^+$.

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$
Then if $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'$, then by lemma 24, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$, and $\mathbf{fv}\, Q \cap \{\overrightarrow{\beta^-}\} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule $(\exists^{\geqslant_1})$. This way, $\{Q \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,(Q) \cap \{\overrightarrow{\beta^-}\} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = \downarrow M$
Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\nexists}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists\overrightarrow{\beta^-}.Q'$
Then the last applied rule to infer $\Gamma \vdash Q \geqslant_1 \downarrow M$ must be Rule $(\downarrow^{\geqslant_1})$, which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq_1^{\leqslant} M$, then by lemma 22 and Rule $(\downarrow^{\simeq_1^D})$, $\downarrow M' \simeq_1^D \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq_1^D \downarrow M\} = \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\nexists}$.
In the other direction,

$$\begin{aligned} \downarrow M' \simeq_1^D \downarrow M &\Rightarrow \Gamma \vdash \downarrow M' \simeq_1^{\leqslant} \downarrow M && \text{by lemma 21, since } \Gamma \vdash \downarrow M' \text{ by lemma 20} \\ &\Rightarrow \Gamma \vdash \downarrow M' \geqslant_1 \downarrow M && \text{by inversion} \end{aligned}$$

b. $Q = \exists\overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)
Then the last rule applied to infer $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q' \geqslant_1 \downarrow M$ must be Rule $(\exists^{\geqslant_1})$. Inversion of this rule gives us $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \geqslant_1 \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.*a*, $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \simeq_1^D \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists\overrightarrow{\beta^-}.\downarrow M' \in \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).
In the other direction,

$$\begin{aligned} [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^D \downarrow M &\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^{\leqslant} \downarrow M && \text{by lemma 21, since } \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \text{ by lemma 20} \\ &\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \geqslant_1 \downarrow M && \text{by inversion} \\ &\Rightarrow \Gamma \vdash \exists\overrightarrow{\beta^-}.\downarrow M' \geqslant_1 \downarrow M && \text{by Rule } (\exists^{\geqslant_1}) \end{aligned}$$

$\square$

Lemma 26 (Characterization of the Normalized Supertypes)**.** *For a normalized positive type $P = \mathbf{nf}(P)$, let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle\vert\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \end{array} \right\}$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Then from lemma 25, $\{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}(\exists\overrightarrow{\alpha^-}.\beta^+) \mid \text{for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

$$\begin{aligned}
\mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') &= \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ not using } \overrightarrow{\beta^-} \\
&= \{\mathbf{nf}(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothesis} \\
&= \{\mathbf{nf}(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}(Q) = \mathbf{fv}\,Q \text{ by lemma 14} \\
&= \{\mathbf{nf}(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{by lemma 25} \\
&= \{\mathbf{nf}(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} && \text{by the definition of } \mathsf{UB} \\
&= \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} && \text{by lemma 25}
\end{aligned}$$

Case 3. $P = \downarrow M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 1. *Suppose that $\nu : A \to A$ is an idempotent function, P is a predicate on A, $F : A \to B$ is a function. Then*

$$\begin{aligned} &\{F(\nu x) \mid x \in A \text{ s.t. } P(\nu x)\} = \\ =&\{F(x) \mid x \in A \text{ s.t. } \nu x = x \text{ and } P(x)\}. \end{aligned}$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 2. *For functions F and ν, and predicates P and Q,*

$$\begin{aligned} &\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } P(x)\} = \\ =&\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } (\exists x' \in A \text{ s.t. } P(x') \text{ and } \nu x' = \nu x)\}. \end{aligned}$$

Observation 3. *There exist positive and negative types well-formed in empty context, hence, a type substitution can be extended to an arbitrary domain (if its values on the domain extension are irrelevant). Specifically, Suppose that $vars_1 \subseteq vars_2$. Then $\Gamma \vdash \sigma|_{vars_1} : vars_1$ implies $\exists\ \sigma'$ s.t. $\Gamma \vdash \sigma' : vars_2$ and $\sigma|_{vars_1} = \sigma'|_{vars_1}$.*

$$
\begin{aligned}
&\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 {\downarrow}M\} = \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash {\downarrow}M)\} && \text{by lemma 25} \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash N_i, \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by the definition of } \mathsf{UB} \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{we reassigned the substitution } \overrightarrow{N}/\overrightarrow{\alpha^-} \text{ as } \sigma \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by lemma 2} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by the definition of normalization} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } \mathbf{nf}\,([\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M') = \mathbf{nf}\,({\downarrow}M) \end{array} \right\} && \begin{array}{l}\text{from lemmas 15 and 17, equivalence of types can be} \\ \text{replaced with the equality of their normal forms}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})]{\downarrow}\mathbf{nf}\,(M') = {\downarrow}\mathbf{nf}\,(M) \end{array} \right\} && \text{by congruence of normalization and lemma 16} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{by lemma 19, } {\downarrow}M' \text{ and } \sigma|_{\mathbf{fv}\,M'} \text{ are already normal,} \\ \text{since the result of the substitution is normal;} \\ M \text{ is normal by assumption}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ (\exists\sigma' \text{ s.t. } \Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')}) \\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{We apply observation 2 (with } \nu\sigma = \sigma|_{\mathbf{fv}\,M'}\text{, and} \\ P(\sigma) = \Gamma \vdash \sigma : \overrightarrow{\alpha^-}\text{)}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv}\,M'} : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{Notice that} \\ \text{“}\exists\sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')})\text{”} \\ \text{is equivalent to } \Gamma \vdash \sigma|_{\mathbf{fv}\,({\downarrow}M')} : \overrightarrow{\alpha^-}{}' \text{ (observation 3)}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{We apply observation 1 to the restriction of } \sigma\text{, and} \\ \text{remove } \sigma|_{\mathbf{fv}\,M'} = \sigma \text{ as it follows from } \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}'\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}{}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{by lemma 5, since } \{\Gamma, \overrightarrow{\alpha^-}\} \cap \mathbf{fv}\,M' = \{\Gamma, \overrightarrow{\alpha^-}{}'\} \cap \mathbf{fv}\,M' \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}{}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}{}'\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{We apply observation 1 to the ordering of } \overrightarrow{\alpha^-} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}{}'\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}', \\ \Gamma \vdash N_i,\, \Gamma, \overrightarrow{\alpha^-}{}' \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}{}']{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{By reassigning } \sigma \text{ explicitly as } \overrightarrow{N}/\overrightarrow{\alpha^-}{}' \\
&= \mathsf{NFUB}({\downarrow}M) && \text{by definition}
\end{aligned}
$$

□

Observation 4. *Upper bounds of a type do not depend on the context as soon as the type are well-formed in it.*
If $\Gamma_1 \vdash M$ *and* $\Gamma_2 \vdash M$ *then* $\mathsf{UB}(\Gamma_1 \vdash M) = \mathsf{UB}(\Gamma \vdash M)$ *and* $\mathsf{NFUB}(\Gamma_1 \vdash M) = \mathsf{NFUB}(\Gamma \vdash M)$

Proof. We prove both inclusions by induction on $\Gamma_1 \vdash M$. Notice that if $[\sigma]M' \simeq_1^D M$ and $\Gamma_2 \vdash M$ then the types from the range of $\sigma|_{\mathbf{fv}\,M'}$ are well-formed in 2 Ilya: lemma. □

Lemma 27 (Soundness of the Least Upper Bound). *For types* $\Gamma \vdash P_1$*, and* $\Gamma \vdash P_2$*, if* $\Gamma \vDash P_1 \vee P_2 = Q$ *then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Proof. Induction on $\Gamma \models P_1 \vee P_2 = Q$.

Case 1. $\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+$
Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule $(\mathrm{Var}^{+\geqslant_1})$.

Case 2. $\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q$
Then by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applied to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q$. Then

(i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,

(ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$,

(iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}\,(Q) \cap \{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}\} = \mathbf{fv}\,(Q) \cap \{\Gamma\}$ (and then apply lemma 5). The inclusion right-to-left is self-evident. To show $\mathbf{fv}\,(Q) \cap \{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}\} \subseteq \mathbf{fv}\,(Q) \cap \{\Gamma\}$, we prove that $\mathbf{fv}\,(Q) \subseteq \{\Gamma\}$

$\mathbf{fv}\,(Q) \subseteq \mathbf{fv}\,P_1 \cap \mathbf{fv}\,P_2$ by lemma 1

$\subseteq (\{\Gamma, \overrightarrow{\alpha^-}\} \backslash \{\overrightarrow{\beta^-}\}) \cap (\{\Gamma, \overrightarrow{\beta^-}\} \backslash \{\overrightarrow{\alpha^-}\})$ since $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_1$, $\mathbf{fv}\,(P_1) \subseteq \{\Gamma, \overrightarrow{\alpha^-}\} = \{\Gamma, \overrightarrow{\alpha^-}\} \backslash \{\overrightarrow{\beta^-}\}$ (the latter is because by the Barendregt's convention, $\{\Gamma, \overrightarrow{\alpha^-}\} \cap \{\overrightarrow{\beta^-}\} = \varnothing$); similarly, $\mathbf{fv}\,(P_2) \subseteq \{\Gamma, \overrightarrow{\beta^-}\} \backslash \{\overrightarrow{\alpha^-}\}$

$\subseteq \{\Gamma\}$

To show $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\alpha^-}.P_1$, we apply Rule $(\exists^{\geqslant_1})$. Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant_1 P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \models {\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$ By the inversion, $\Gamma, \cdot \models {\downarrow}N \stackrel{a}{\simeq} {\downarrow}M \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$. Then by lemma 6,

(i) $\Gamma; \Xi \vdash P$, then by **??**,

$$\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]P \tag{1}$$

(ii) $\Gamma; \cdot \vdash \widehat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_2 : \Xi$. Assuming that $\Xi = \widehat{\beta}_1^-, .., \widehat{\beta}_n^-$, the antiunification solutions $\widehat{\tau}_1$ and $\widehat{\tau}_2$ can be put explicitly as $\widehat{\tau}_1 = (\widehat{\beta}_1^- :\approx N_1, .., \widehat{\beta}_n^- :\approx N_n)$, and $\widehat{\tau}_2 = (\widehat{\beta}_1^- :\approx M_1, .., \widehat{\beta}_n^- :\approx M_n)$. Then

$$\widehat{\tau}_1 = (\overrightarrow{N}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \text{ (as substitutions)} \tag{2}$$

$$\widehat{\tau}_2 = (\overrightarrow{M}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \text{ (as substitutions)} \tag{3}$$

(iii) $[\widehat{\tau}_1]Q = P_1$ and $[\widehat{\tau}_2]Q = P_1$, which, by 2 and 3, means

$$[\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P = {\downarrow}N \tag{4}$$

$$[\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P = {\downarrow}M \tag{5}$$

Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$ follows directly from 1.

To show $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}N$, we apply Rule $(\exists^{\geqslant_1})$, instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{N}$. Then $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}N$ follows from 4 and reflexivity of subtyping (**??**).

Analogously, instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{M}$, gives us $\Gamma \vdash [\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}M$ (from 5), and hence, $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}M$.

$\square$

Lemma 28 (Completeness of the Least Upper Bound)**.** *For normalized types* $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, *and* $\Gamma \vdash Q$ *such that* $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$, *there exists* Q' *s.t.* $\Gamma \models P_1 \vee P_2 = Q'$.

Proof. Induction on the pair (P_1, P_2). From lemma 26, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$. Let us consider the cases what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_1$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$ where $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then $Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_2.Q_2)$

$$\begin{aligned}
&\subseteq \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_2 \vdash Q_2) && \text{from the definition of UB}\\
&= \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_2) && \text{by observation 4, weakening and exchange}\\
&= \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1\} \cap \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2\} && \text{by lemma 25,}
\end{aligned}$$

meaning that $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$. Then after one step, the algorithm terminates by the induction hypothesis. In other words, $\exists Q'$ s.t. $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vDash Q_1 \vee Q_2 = Q'$, and thus, Rule ($\exists^\vee$) is applicable.

Case 2. $P_1 = \alpha^+$ and $P_2 = \downarrow N$

Then the set of common upper bounds of $\downarrow N$ and α^+ is empty, and thus, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$ gives a contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \downarrow N)$

$$\begin{aligned}
&= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\downarrow M' \mid \cdots\} && \text{by the definition of UB}\\
&= \varnothing && \text{since } \alpha^+ \neq \downarrow M' \text{ for any } M'
\end{aligned}$$

Case 3. $P_1 = \downarrow N$ and $P_2 = \alpha^+$

Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)

Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \beta^+)$

$$\begin{aligned}
&= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\beta^+ \mid \cdots\} && \text{by the definition of UB}\\
&= \varnothing && \text{since } \alpha^+ \neq \beta^+
\end{aligned}$$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Then the algorithm terminates in one step (Rule (Var$^\vee$)): $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$.

Case 6. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$

Then on the next step, the algorithm tries to anti-unify $\downarrow M_1$ and $\downarrow M_2$. By lemma 7, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$\mathbf{nf}\,(Q) \in \mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_2.Q_2)$

$$\begin{aligned}
&= \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\, M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_1 \end{array} \right\} \\
&\quad \cap \\
&\quad \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\, M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \overrightarrow{N}_1,\ \Gamma \vdash \overrightarrow{N}_2,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_2 \end{array} \right\} \\
&= \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M',\ \overrightarrow{N}_1 \text{ and } \overrightarrow{N}_2 \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\, M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \overrightarrow{N}_1,\ \Gamma \vdash \overrightarrow{N}_2,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_1, \text{ and } [\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_2 \end{array} \right\}
\end{aligned}$$

The fact that the latter set is non-empty means that there exist $\overrightarrow{\alpha^-}$, M', $\overrightarrow{N}_1$ and $\overrightarrow{N}_2$ such that

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$,

(ii) $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma \vdash \overrightarrow{N}_1$,

(iii) $[\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_1$ and $[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_2$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative antiunification variable $\widehat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\widehat{\alpha}^-}$. Let us show that $(\overrightarrow{\widehat{\alpha}^-},\ [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M',\ \overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-},\ \overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-})$ is a sound anti-unifier of $\downarrow M_1$ and $\downarrow M_2$ in context Γ:

- $\overrightarrow{\widehat{\alpha}^-}$ is negative by construction,
- $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M'$ because $\Gamma, \overrightarrow{\alpha^-} \vdash \downarrow M'$ Ilya: lemma!,
- $\Gamma; \cdot \vdash (\overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma; \cdot \vdash (\overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N}_2$,

- $[\overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_1$; analogously, $[\overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = i[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_2$.

Then by the completeness of the anti-unification (lemma 7), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it.

$\square$

Lemma 29 (Initiality of the Least Upper Bound). *For normalized types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q$ such that $\Gamma \vdash Q \geqslant_1 P_1$ and $\Gamma \vdash Q \geqslant_1 P_2$, If $\Gamma \models P_1 \vee P_2 = Q'$ then $\Gamma \vdash Q \geqslant_1 Q'$.*

Proof. By induction on a pair (P_1, P_2), similarly to the proof of lemma 28.

Let us consider the cases what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_1$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$ where $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then by the same reasoning as in case 1 of the proof of lemma 28, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$.

On the other hand, the inversion of $\Gamma \models \exists\overrightarrow{\beta^-}_1.Q_1 \vee \exists\overrightarrow{\beta^-}_2.Q_2 = Q'$ gives us $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \models Q_1 \vee Q_2 = Q'$. Hence, by the induction hypothesis, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$.

Since both Q and Q' are sound, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$ can be strengthened to $\Gamma \vdash Q \geqslant_1 Q'$. Ilya: lemma!

Case 2. ($P_1 = \alpha^+$ and $P_2 = \downarrow N$) or ($P_1 = \downarrow N$ and $P_2 = \alpha^+$) or ($P_1 = \alpha^+$ and $P_2 = \beta^+$)
By the same argument as in case 2 of the proof of lemma 28, the set of common supertypes of P_1 and P_2 is empty, hence contradiction.

Case 3. $P_1 = \alpha^+$ and $P_2 = \alpha^+$
Since $Q \in \mathsf{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists\overrightarrow{\alpha^-}.\alpha^+$. Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant_1 \alpha^+$ by Rule $(\exists^{\geqslant_1})$: $\overrightarrow{\alpha^-}$ can be instantiated with arbitrary negative types (for example $\forall\beta^+.\uparrow\beta^+$), since the substitution for unused variables does not change the term $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule $(\mathrm{Var}^{+\geqslant_1})$.

Case 4. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$
By the same reasoning as in case 6 of the proof of lemma 28, $\mathbf{nf}\,(Q) = \exists\overrightarrow{\alpha^-}.\downarrow M'$ for some $\overrightarrow{\alpha^-}$ and $\downarrow M'$ such that there exist $\overrightarrow{N}_1$ and $\overrightarrow{N}_2$ such that:

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$,

(ii) $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma \vdash \overrightarrow{N}_1$,

(iii) $[\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_1$ and $[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_2$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative antiunification variable $\widehat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\widehat{\alpha}^-}$. As shown in case 6 of the proof of lemma 28, $(\overrightarrow{\widehat{\alpha}^-},\ [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M',\ \overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-},\ \overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-})$ is a sound anti-unifier of $\downarrow M_1$ and $\downarrow M_2$ in context Γ.

By the inversion of $\Gamma \models \downarrow M_1 \vee \downarrow M_2 = Q'$, we conclude that $Q' = \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, where $(\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$ is the result of the antiunification of $\downarrow M_1$ and $\downarrow M_2$ in context Γ.

Then by the initiality of the anti-unification (lemma 8), there exisits $\widehat{\tau}$ such that $\Gamma; \Xi \vdash \widehat{\tau} : \overrightarrow{\widehat{\alpha}^-}$ and $[\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}$, (ii) $\widehat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]\downarrow M' = [\overrightarrow{\beta^-}/\Xi][\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{\beta^-}/\Xi]P$. In particular, from the reflexivity of subtyping: $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]\downarrow M' \geqslant_1 [\overrightarrow{\beta^-}/\Xi]P$.

It allows us to show $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant_1 Q'$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\downarrow M' \geqslant_1 \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, by applying Rule $(\exists^{\geqslant_1})$, instantiating $\overrightarrow{\alpha^-}$ with respect to σ. Finally, $\Gamma \vdash Q \geqslant_1 Q'$ since $\Gamma \vdash \mathbf{nf}\,(Q) \simeq_1^{\leqslant} Q$, and equivalence implies subtyping by Ilya: lemma.

$\square$

Lemma 30 (Soundness of Upgrade). *For $\Delta \subseteq \Gamma$, suppose that* **upgrade** $\Gamma \vdash P$ **to** $\Delta = Q$.

(i) $\Delta \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P$

Lemma 31 (Completeness of Upgrade). *For $\Delta \subseteq \Gamma$, $\Gamma \vdash P$ and $\Delta \vdash Q'$, such that $\Gamma \vdash Q' \geqslant_1 P$, there exists Q s.t.* **upgrade** $\Gamma \vdash P$ **to** $\Delta = Q$, *and $\Delta \vdash Q' \geqslant_1 Q$.*