1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
& | & a+ & \\
& | & \downarrow N & \\
& | & \exists \alpha^- . P & \\
\end{array}
$$

$$
\begin{array}{llll}
N,\ M & ::= & & \text{negative types} \\
& | & a- & \\
& | & \uparrow P & \\
& | & \forall \alpha^+ . N & \\
& | & P \to N & \\
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \ \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \ \textsc{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash a- \leqslant_0 a-} \ \textsc{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \ \textsc{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/a+]N \leqslant_0 M \quad M \neq \forall \beta^+ . M'}{\Gamma \vdash \forall \alpha^+ . N \leqslant_0 M} \ \textsc{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+ . M} \ \textsc{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \ \textsc{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash a+ \geqslant_0 a+} \ \textsc{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \ \textsc{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/a-]P \geqslant_0 Q \quad Q \neq \exists \alpha^- . Q'}{\Gamma \vdash \exists \alpha^- . P \geqslant_0 Q} \ \textsc{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^- . Q} \ \textsc{D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$P,\ Q$	$::=$		multi-quantified positive types
	$\mid$	α^+	
	$\mid$	$\downarrow N$	
	$\mid$	$\exists\overrightarrow{\alpha^-}.P$	$P \neq \exists \ldots$
	$\mid$	(P)	S

$N,\ M$	$::=$		multi-quantified negative types
	$\mid$	α^-	
	$\mid$	$\uparrow P$	
	$\mid$	$P \rightarrow N$	
	$\mid$	$\forall\overrightarrow{\alpha^+}.N$	$N \neq \forall \ldots$
	$\mid$	(N)	S

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \ \text{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \ \text{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \ \text{D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \ \text{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \rightarrow N \leqslant_1 Q \rightarrow M} \ \text{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M} \ \text{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \ \text{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \ \text{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q'}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q} \ \text{D1ExistsL}$$

2.3 Declarative Equivalence

$\boxed{N \simeq_1^{D} M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^{D} \alpha^-} \ (\text{Var}^{-\simeq_1^{D}})$$

$$\frac{P \simeq_1^{D} Q}{\uparrow P \simeq_1^{D} \uparrow Q} \ (\uparrow^{\simeq_1^{D}})$$

$$\frac{P \simeq_1^{D} Q \quad N \simeq_1^{D} M}{P \rightarrow N \simeq_1^{D} Q \rightarrow M} \ (\rightarrow^{\simeq_1^{D}})$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M} \quad (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\mathrm{VAR}^+)$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists\overrightarrow{\alpha^-}.P \simeq_1^D \exists\overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}$
$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}$
$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \alpha^-} \quad (\mathrm{VAR}^-_{\in})$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \cdot} \quad (\mathrm{VAR}^-_{\notin})$$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^-\{vars'\} = \cdot} \quad (\mathrm{UVAR}^-)$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\uparrow P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ vars\ \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})} \quad (\to)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \alpha^+} \quad (\mathrm{VAR}^+_{\in})$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \cdot} \quad (\mathrm{VAR}^+_{\notin})$$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^+\{vars'\} = \cdot} \quad (\mathrm{UVAR}^+)$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\downarrow N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \exists\overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad (\exists)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$\boxed{\mathbf{nf}\,(P) = Q}$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\quad(\textsc{Var}^-)$$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-\{vars\}) = \widehat{\alpha}^-\{vars\}}\quad(\textsc{UVar}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\quad(\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\quad(\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'}\quad(\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\quad(\textsc{Var}^+)$$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+\{vars\}) = \widehat{\alpha}^+\{vars\}}\quad(\textsc{UVar}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\quad(\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'}\quad(\exists)$$

3.2 Unification

$\boxed{N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\alpha^- \stackrel{u}{\simeq} \alpha^- \dashv \cdot}\quad\textsc{UNVar}$$

$$\frac{P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}{\uparrow P \stackrel{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\quad\textsc{UShiftU}$$

$$\frac{P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}_2}{P \to N \stackrel{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2}\quad\textsc{UArrow}$$

$$\frac{N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}{\forall\overrightarrow{\alpha^+}.N \stackrel{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\quad\textsc{UForall}$$

$$\frac{\mathbf{fv}\,N \subseteq vars}{\widehat{\alpha}^-\{vars\} \stackrel{u}{\simeq} N \dashv \widehat{\alpha}^- :\approx N}\quad\textsc{UNUVar}$$

$\boxed{P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\alpha^+ \stackrel{u}{\simeq} \alpha^+ \dashv \cdot}\quad\textsc{UPVar}$$

$$\frac{N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}{\downarrow N \stackrel{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\quad\textsc{UShiftD}$$

$$\frac{P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}{\exists\overrightarrow{\alpha^-}.P \stackrel{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\quad\textsc{UExists}$$

$$\frac{\mathbf{fv}\,P \subseteq vars}{\widehat{\alpha}^+\{vars\} \stackrel{u}{\simeq} P \dashv \widehat{\alpha}^+ :\approx P}\quad\textsc{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma \models \alpha^- \leqslant \alpha^- \dashv \cdot}\ \textsc{ANVar}$$

$$\frac{\mathbf{nf}\,(P) \stackrel{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma \models {\uparrow}P \leqslant {\uparrow}Q \dashv \widehat{\sigma}}\ \textsc{AShiftU}$$

$$\frac{\Gamma \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2}\ \textsc{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \models [\overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}}/\overrightarrow{\alpha^+}]\,N \leqslant M \dashv \widehat{\sigma}}{\Gamma \models \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^+}}\ \textsc{AForall}$$

$\boxed{\Gamma \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \textsc{APVar}$$

$$\frac{\mathbf{nf}\,(N) \stackrel{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma \models {\downarrow}N \geqslant {\downarrow}M \dashv \widehat{\sigma}}\ \textsc{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \models [\overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}}/\overrightarrow{\alpha^-}]\,P \geqslant Q \dashv \widehat{\sigma}}{\Gamma \models \exists \overrightarrow{\alpha^-}.P \geqslant \exists \overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}}\ \textsc{AExists}$$

$$\frac{\mathbf{nf}\,(P) = P' \quad vars_1 = \mathbf{fv}\,P' \backslash vars \quad vars_2\ \mathbf{is\ fresh}}{\Gamma \models \widehat{\alpha}^+\{vars\} \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant P' \vee [vars_2/vars_1]P')}\ \textsc{APUVar}$$

3.4 Unification Solution Merge

$\boxed{e_1 \& e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{}{\widehat{\alpha}^+ :\geqslant P \& \widehat{\alpha}^+ :\geqslant Q = \widehat{\alpha}^+ :\geqslant P \vee Q}\ \textsc{SMEPSupSup}$$

$$\frac{\mathbf{fv}\,P \cup \mathbf{fv}\,Q \models P \geqslant Q \dashv \widehat{\sigma}'}{\widehat{\alpha}^+ :\approx P \& \widehat{\alpha}^+ :\geqslant Q = \widehat{\alpha}^+ :\approx P}\ \textsc{SMEPEqSup}$$

$$\frac{\mathbf{fv}\,P \cup \mathbf{fv}\,Q \models Q \geqslant P \dashv \widehat{\sigma}'}{\widehat{\alpha}^+ :\geqslant P \& \widehat{\alpha}^+ :\approx Q = \widehat{\alpha}^+ :\approx Q}\ \textsc{SMEPSupEq}$$

$$\frac{}{\widehat{\alpha}^+ :\approx P \& \widehat{\alpha}^+ :\approx P = \widehat{\alpha}^+ :\approx P}\ \textsc{SMEPEqEq}$$

$$\frac{}{\widehat{\alpha}^- :\approx N \& \widehat{\alpha}^- :\approx N = \widehat{\alpha}^- :\approx N}\ \textsc{SMENEqEq}$$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$$\frac{}{\cdot \& \widehat{\sigma} = \widehat{\sigma}}\ \textsc{SMEmpty}$$

$$\frac{\begin{array}{c}(\widehat{\alpha}^+ :\approx P) \in \widehat{\sigma}_2 \\ \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3\end{array}}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)}\ \textsc{SMPEqEq}$$

$$\frac{(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\geqslant P \vee Q, \widehat{\sigma}_3)}\ \textsc{SMPSupSup}$$

$$\frac{\begin{array}{l}(\widehat{\alpha}^+ :\approx Q) \in \widehat{\sigma}_2 \quad \mathbf{fv}\,Q \cup \mathbf{fv}\,P \models Q \geqslant P \dashv \widehat{\sigma}' \\ \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3\end{array}}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx Q, \widehat{\sigma}_3)}\ \textsc{SMPSupEq}$$

$$\frac{\begin{array}{l}(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \quad \mathbf{fv}\,Q \cup \mathbf{fv}\,P \models P \geqslant Q \dashv \widehat{\sigma}' \\ \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3\end{array}}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)}\ \textsc{SMPEqSup}$$

$$\frac{\begin{array}{c}(\widehat{\alpha}^- :\approx N) \in \widehat{\sigma}_2 \\ \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^-) = \widehat{\sigma}_3\end{array}}{(\widehat{\alpha}^- :\approx N, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^- :\approx N, \widehat{\sigma}_3)}\ \textsc{SMNEqEq}$$

3.5 Least Upper Bound

$\boxed{P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVar}$$

$$\frac{(\mathbf{fv}\, N \cup \mathbf{fv}\, M) \vDash {\downarrow} N \overset{a}{\simeq} {\downarrow} M \dashv (P, \widehat{\sigma}_1, \widehat{\sigma}_2)}{{\downarrow} N \vee {\downarrow} M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/(\mathbf{uv}\, P)]P} \quad \text{LUBShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\overrightarrow{\beta^-}\} = \varnothing}{\exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = P_1 \vee P_2} \quad \text{LUBExists}$$

3.6 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\alpha^+, \cdot, \cdot)} \quad \text{AUPVar}$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \vDash {\downarrow} N_1 \overset{a}{\simeq} {\downarrow} N_2 \dashv ({\downarrow} M, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUPShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\exists \overrightarrow{\alpha^-}.Q, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUPExists}$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\alpha^-, \cdot, \cdot)} \quad \text{AUNVar}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \vDash {\uparrow} P_1 \overset{a}{\simeq} {\uparrow} P_2 \dashv ({\uparrow} Q, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUNShift}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}'_1, \widehat{\sigma}'_2)}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (Q \to M, \widehat{\sigma}_1 \cup \widehat{\sigma}'_1, \widehat{\sigma}_2 \cup \widehat{\sigma}'_2)} \quad \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad \text{AUNAU}$$

4 Proofs

4.1 Variable Ordering

Definition 1 (Collision free bijection)**.** *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is* ***collision free on sets*** *P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 1 (Soundness of variable ordering)**.** *Variable ordering extracts precisely used free variables.*

$-$ $\{\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, N\} \equiv \mathit{vars} \cap \mathbf{fv}\, N$ *(as sets)*

$+$ $\{\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, P\} \equiv \mathit{vars} \cap \mathbf{fv}\, P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, N = \vec{\alpha}$ and $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, P = \vec{\alpha}$ □

Corollary 1 (Additivity of ordering)**.** *Variable ordering is additive (in terms of set union) with respect to its first argument.*

$-$ $\{\mathbf{ord}\, (\mathit{vars}_1 \cup \mathit{vars}_2)\, \mathbf{in}\, N\} \equiv \{\mathbf{ord}\, \mathit{vars}_1\, \mathbf{in}\, N\} \cup \{\mathbf{ord}\, \mathit{vars}_2\, \mathbf{in}\, N\}$ *(as sets)*

$+$ $\{\mathbf{ord}\, (\mathit{vars}_1 \cup \mathit{vars}_2)\, \mathbf{in}\, P\} \equiv \{\mathbf{ord}\, \mathit{vars}_1\, \mathbf{in}\, P\} \cup \{\mathbf{ord}\, \mathit{vars}_2\, \mathbf{in}\, P\}$ *(as sets)*

Corollary 2 (Weakening of ordering)**.** *Extending the first argument of the ordering with unused variables does not change the result.*

- $-$ $\mathbf{ord}\,(vars \cap \mathbf{fv}\,N)\,\mathbf{in}\,N = \mathbf{ord}\,vars\,\mathbf{in}\,N$
- $+$ $\mathbf{ord}\,(vars \cap \mathbf{fv}\,P)\,\mathbf{in}\,P = \mathbf{ord}\,vars\,\mathbf{in}\,P$

Lemma 2 (Distributivity of renaming over variable ordering)**.** *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

- $-$ *If μ is collision free on vars and $\mathbf{fv}\,N$ then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]N$*
- $+$ *If μ is collision free on vars and $\mathbf{fv}\,P$ then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]P$*

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$
Then
$$\begin{aligned}[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-)\\ &= [\mu]\alpha^- && \text{by Rule } (\mathrm{Var}^+_\in)\\ &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars)\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\beta^- && \text{by Rule } (\mathrm{Var}^+_\in)\text{, because } \beta^- \in [\mu]vars\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^-\end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$
Notice that $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. On the other hand, $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_\notin)$, because μ is collision free on $vars$ and $\mathbf{fv}\,N$, so $\mathbf{fv}\,N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$
Then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. To prove that $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_\notin)$. Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

- (i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.
- (ii) Since μ is collision free on $vars$ and $\mathbf{fv}\,N$, $\mu(A \cap \mathbf{fv}\,N) \ni \mu(\alpha^-) \notin vars$.

d. $\alpha^- \notin A$ but $\alpha^- \in vars$
$\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_\notin)$, because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$
$$\begin{aligned}[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\uparrow P)\\ &= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) && \text{by Rule } (\uparrow)\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P && \text{by the induction hypothesis}\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\uparrow[\mu]P && \text{by Rule } (\uparrow)\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\uparrow P && \text{by the definition of substitution}\\ &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N\end{aligned}$$

Case 3. $N = P \to M$
$$\begin{aligned}[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P \to M)\\ &= [\mu](\vec\alpha_1, (\vec\alpha_2 \backslash \{\vec\alpha_1\})) && \text{where } \mathbf{ord}\,vars\,\mathbf{in}\,P = \vec\alpha_1 \text{ and } \mathbf{ord}\,vars\,\mathbf{in}\,M = \vec\alpha_2\\ &= [\mu]\vec\alpha_1, [\mu](\vec\alpha_2 \backslash \{\vec\alpha_1\})\\ &= [\mu]\vec\alpha_1, ([\mu]\vec\alpha_2 \backslash [\mu]\{\vec\alpha_1\}) && \text{by induction on } \vec\alpha_2\text{; the inductive step is similar to case 1. Notice that } \mu \text{ is}\\ & && \text{collision free on } \{\vec\alpha_1\} \text{ and } \{\vec\alpha_2\} \text{ since } \{\vec\alpha_1\} \subseteq vars \text{ and } \{\vec\alpha_2\} \subseteq \mathbf{fv}\,N\\ &= [\mu]\vec\alpha_1, ([\mu]\vec\alpha_2 \backslash \{[\mu]\vec\alpha_1\})\end{aligned}$$
$$\begin{aligned}(\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N) &= (\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P \to [\mu]M)\\ &= (\vec\beta_1, (\vec\beta_2 \backslash \{\vec\beta_1\})) && \text{where } \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P = \vec\beta_1 \text{ and } \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M = \vec\beta_2\\ & && \text{then by the induction hypothesis, } \vec\beta_1 = [\mu]\vec\alpha_1,\ \vec\beta_2 = [\mu]\vec\alpha_2,\\ &= [\mu]\vec\alpha_1, ([\mu]\vec\alpha_2 \backslash \{[\mu]\vec\alpha_1\})\end{aligned}$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$
\begin{aligned}
[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) &= [\mu]\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.M\\
&= [\mu]\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,M\\
&= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]M \qquad \text{by the induction hypothesis}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]N) &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\forall\overrightarrow{\alpha^+}.M\\
&= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.[\mu]M\\
&= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]M
\end{aligned}
$$

□

Lemma 3 (Completeness of variable ordering)**.** *Variable ordering is invariant under equivalence.*

– *For* $N \simeq_1^D M$ *and any vars, if* $\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}_1$ *and* $\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,M = \vec{\alpha}_2$*, then* $\vec{\alpha}_1 = \vec{\alpha}_2$ *(as lists)*

+ *For* $P \simeq_1^D Q$ *and any vars, if* $\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}_1$ *and* $\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,Q = \vec{\alpha}_2$*, then* $\vec{\alpha}_1 = \vec{\alpha}_2$ *(as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. □

4.2 Normaliztaion

Lemma 4. *Set of free variables is invariant under equivalence.*

– *If* $N \simeq_1^D M$ *then* $\mathbf{fv}\,N \equiv \mathbf{fv}\,M$ *(as sets)*

+ *If* $P \simeq_1^D Q$ *then* $\mathbf{fv}\,P \equiv \mathbf{fv}\,Q$ *(as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

Lemma 5. *Free variables are not changed by the normalization*

– $\mathbf{fv}\,N \equiv \mathbf{fv}\,\mathbf{nf}\,(N)$

+ $\mathbf{fv}\,P \equiv \mathbf{fv}\,\mathbf{nf}\,(P)$

Proof. By straightforward induction on $\mathbf{nf}\,(N) = M$. □

Lemma 6 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that* μ *is a bijection between two sets of variables* $\mu : A \leftrightarrow B$*. Then*

– $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$

+ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Here equality means alpha-equivalence.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$

$\mathbf{nf}\,([\mu]N) = \mathbf{nf}\,([\mu]\alpha^-) = [\mu]\alpha^-$. The latter follows from the fact that $[\mu]\alpha^-$ is a variable, and thus, Rule (Var^-) is applicable.

$[\mu]\mathbf{nf}\,(N) = [\mu]\mathbf{nf}\,(\alpha^-) = [\mu]\alpha^-$.

Case 2. If the type is formed by $\to$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \to M$ then

$$
\begin{aligned}
\mathbf{nf}\,([\mu]N) &= \mathbf{nf}\,([\mu](P \to M))\\
&= \mathbf{nf}\,([\mu]P \to [\mu]M) && \text{By the congruence of substitution}\\
&= \mathbf{nf}\,([\mu]P) \to \mathbf{nf}\,([\mu]M) && \text{By the congruence of normalization, i.e. Rule } (\to)\\
&= [\mu]\mathbf{nf}\,(P) \to [\mu]\mathbf{nf}\,(M) && \text{By the induction hypothesis}\\
&= [\mu](\mathbf{nf}\,(P) \to \mathbf{nf}\,(M)) && \text{By the congruence of substitution}\\
&= [\mu]\mathbf{nf}\,(P \to M) && \text{By the congruence of normalization}\\
&= [\mu]\mathbf{nf}\,(N)
\end{aligned}
$$

Case 3. $N = \forall \overrightarrow{\alpha^+}.M$

$$\begin{aligned}
[\mu]\mathbf{nf}(N) &= [\mu]\mathbf{nf}(\forall \overrightarrow{\alpha^+}.M) \\
&= [\mu]\forall \overrightarrow{\alpha^+}'.\mathbf{nf}(M) && \text{Where } \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}(M) = \overrightarrow{\alpha^+}'
\end{aligned}$$

$$\begin{aligned}
\mathbf{nf}([\mu]N) &= \mathbf{nf}([\mu]\forall \overrightarrow{\alpha^+}.M) \\
&= \mathbf{nf}(\forall \overrightarrow{\alpha^+}.[\mu]M) && \text{Assuming } \{\overrightarrow{\alpha^+}\} \cap A = \varnothing \text{ and } \{\overrightarrow{\alpha^+}\} \cap B = \varnothing \\
&= \forall \overrightarrow{\beta^+}.\mathbf{nf}([\mu]M) && \text{Where } \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}([\mu]M) = \overrightarrow{\beta^+} \\
&= \forall \overrightarrow{\alpha^+}'.\mathbf{nf}([\mu]M) && \text{As } \overrightarrow{\beta^+} = \overrightarrow{\alpha^+}' \text{ (see below)}
\end{aligned}$$

Notice that μ is free of collisions on $\{\overrightarrow{\alpha^+}\}$ and $\mathbf{fv}\,\mathbf{nf}(M)$ because

(i) $\mu(A \cap \{\overrightarrow{\alpha^+}\}) \cap \mathbf{fv}\,\mathbf{nf}(M) = \varnothing \cap \mathbf{fv}\,\mathbf{nf}(M) = \varnothing$ and

(ii) $\mu(A \cap \mathbf{fv}\,\mathbf{nf}(M)) \cap \{\overrightarrow{\alpha^+}\} \subseteq B \cap \{\overrightarrow{\alpha^+}\} = \varnothing$

$$\begin{aligned}
\overrightarrow{\beta^+} &= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}([\mu]M) \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\mu]\mathbf{nf}(M) && \text{By the induction hypothesis} \\
&= \mathbf{ord}\,\{[\mu]\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\mu]\mathbf{nf}(M) && \text{Since } \{\overrightarrow{\alpha^+}\} \cap A = \varnothing \\
&= [\mu]\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}(M) && \text{by lemma 2} \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}(M) && \text{Since } \{\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}(M)\} \cap A \subseteq \{\overrightarrow{\alpha^+}\} \cap A = \varnothing \\
&= \overrightarrow{\alpha^+}'
\end{aligned}$$

To show alpha-equivalence of $[\mu]\forall \overrightarrow{\alpha^+}'.\mathbf{nf}(M)$ and $\forall \overrightarrow{\alpha^+}'.\mathbf{nf}([\mu]M)$, we can assume that $\{\overrightarrow{\alpha^+}'\} \cap A = \varnothing$, and $\{\overrightarrow{\alpha^+}'\} \cap B = \varnothing$. Then $[\mu]\forall \overrightarrow{\alpha^+}'.\mathbf{nf}(M) = \forall \overrightarrow{\alpha^+}'.[\mu]\mathbf{nf}(M) = \forall \overrightarrow{\alpha^+}'.\mathbf{nf}([\mu]M)$, the latter follows from the induction hypothesis.

Case 4. $P = \exists \overrightarrow{\alpha^-}.Q$
Same as for case 3.

$\square$

Lemma 7 (Soundness of quantifier normalization)**.**

- $-$ $N \simeq_1^D \mathbf{nf}(N)$
- $+$ $P \simeq_1^D \mathbf{nf}(P)$

Proof. Mutual induction on $\mathbf{nf}(N) = M$ and $\mathbf{nf}(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var$^-$) and (Var$^+$)
By the corresponding equivalence rules.

Case 2. ($\uparrow$), ($\downarrow$), and ($\rightarrow$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall$), i.e. $\mathbf{nf}(\forall \overrightarrow{\alpha^+}.N) = \forall \overrightarrow{\alpha^+}'.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 4, $\mathbf{fv}\,N \equiv \mathbf{fv}\,N'$. Then by lemma 1, $\{\overrightarrow{\alpha^+}'\} \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$, and thus, $\{\overrightarrow{\alpha^+}'\} \cap \mathbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$.

To prove $\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\alpha^+}'.N'$, it suffices to provide a bijection $\mu : \{\overrightarrow{\alpha^+}'\} \cap \mathbf{fv}\,N' \leftrightarrow \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. ($\exists$) Same as for case 3.

$\square$

Lemma 8 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

- $-$ *If* $N \simeq_1^D M$ *then* $\mathbf{nf}(N) = \mathbf{nf}(M)$
- $+$ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}(P) = \mathbf{nf}(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. $(\forall^{\simeq_1^D})$

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 1 and 5, the domain and the codomain of μ can be written as $\mu : \{\overrightarrow{\beta^+}'\} \leftrightarrow \{\overrightarrow{\alpha^+}'\}$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by lemma 6, the second—by the induction hypothesis.

(ii)
$$
\begin{aligned}
[\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}' \\
&= [\mu]\mathbf{ord}\,(\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 5 and corollary 2} \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 2, because } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing \\
& && \text{and } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved} \\
&= \mathbf{ord}\,(\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \text{ and } \{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 5 and corollary 2} \\
&= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'
\end{aligned}
$$

Case 2. $(\exists^{\simeq_1^D})$ Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□