# 1 The Vanilla System

First, we present the top-level system, which is easy to understand.

## 1.1 Grammar

$$
\begin{array}{lcll}
P,\ Q & ::= & & \text{positive types} \\
 & | & \alpha^+ & \\
 & | & \downarrow N & \\
 & | & \exists \alpha^-.P & \\
\\
N,\ M & ::= & & \text{negative types} \\
 & | & \alpha^- & \\
 & | & \uparrow P & \\
 & | & \forall \alpha^+.N & \\
 & | & P \to N &
\end{array}
$$

## 1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \ \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \ \textsc{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \ \textsc{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \ \textsc{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \ \textsc{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \ \textsc{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \ \textsc{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \ \textsc{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \ \textsc{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \ \textsc{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \ \textsc{D0ExistsR}$$

# 2 Multi-Quantified System

## 2.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{multi-quantified positive types} \\
 & | & \alpha^+ & \\
 & | & \downarrow N & \\
 & | & \exists \overrightarrow{\alpha^-}.P & P \neq \exists \ldots \\
 & | & (P) & \mathsf{S}
\end{array}
$$

$$
\begin{array}{llll}
N,\ M & ::= & & \text{multi-quantified negative types} \\
 & | & \alpha^- & \\
 & | & \uparrow P & \\
 & | & P \rightarrow N & \\
 & | & \forall \overrightarrow{\alpha^+}.N & N \neq \forall \ldots \\
 & | & (N) & \mathsf{S}
\end{array}
$$

## 2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \quad \text{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \quad \text{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad (\text{Var}^{-\leqslant_1})$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \quad (\uparrow^{\leqslant_1})$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \rightarrow N \leqslant_1 Q \rightarrow M} \quad (\rightarrow^{\leqslant_1})$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\leqslant_1})$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad (\text{Var}^{+\geqslant_1})$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \quad (\downarrow^{\geqslant_1})$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\geqslant_1})$$

## 2.3 Declarative Equivalence

$\boxed{N \simeq_1^{D} M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^{D} \alpha^-} \quad (\text{Var}^{-\simeq_1^{D}})$$

$$\frac{P \simeq_1^{D} Q}{\uparrow P \simeq_1^{D} \uparrow Q} \quad (\uparrow^{\simeq_1^{D}})$$

$$\frac{P \simeq_1^{D} Q \quad N \simeq_1^{D} M}{P \rightarrow N \simeq_1^{D} Q \rightarrow M} \quad (\rightarrow^{\simeq_1^{D}})$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M} \quad (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\text{VAR}^+)$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists\overrightarrow{\alpha^-}.P \simeq_1^D \exists\overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

# 3 Algorithm

## 3.1 Normalization

### 3.1.1 Ordering

$\boxed{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, N = \vec{\alpha}}$

$$\frac{\alpha^- \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\alpha^- = \alpha^-} \quad (\text{VAR}^-_{\in})$$

$$\frac{\alpha^- \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\alpha^- = \cdot} \quad (\text{VAR}^-_{\notin})$$

$$\frac{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\uparrow P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, P = \vec{\alpha}_1 \quad \mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, N = \vec{\alpha}_2}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})} \quad (\to)$$

$$\frac{\mathit{vars} \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, P = \vec{\alpha}}$

$$\frac{\alpha^+ \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\alpha^+ = \alpha^+} \quad (\text{VAR}^+_{\in})$$

$$\frac{\alpha^+ \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\alpha^+ = \cdot} \quad (\text{VAR}^+_{\notin})$$

$$\frac{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\downarrow N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{\mathit{vars} \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\exists\overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad (\exists)$$

$\boxed{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\widehat{\alpha}^- = \cdot} \quad (\text{UVAR}^-)$$

$\boxed{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\, P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\widehat{\alpha}^+ = \cdot} \quad (\text{UVAR}^+)$$

### 3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\quad (\text{VAR}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\quad (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\quad (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'}\quad (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\quad (\text{VAR}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\quad (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'}\quad (\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\quad (\text{UVAR}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\quad (\text{UVAR}^+)$$

## 3.2 Unification

$\boxed{\Theta \models N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Theta \models \alpha^- \stackrel{u}{\simeq} \alpha^- \dashv \cdot}\quad \text{UNVAR}$$

$$\frac{\Theta \models P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \uparrow P \stackrel{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\quad \text{USHIFTU}$$

$$\frac{\Theta \models P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Theta \models N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Theta \models P \to N \stackrel{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2}\quad \text{UARROW}$$

$$\frac{\Theta \models N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \forall\overrightarrow{\alpha^+}.N \stackrel{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\quad \text{UFORALL}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Theta \models \widehat{\alpha}^- \stackrel{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)}\quad \text{UNUVAR}$$

$\boxed{\Theta \models P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Theta \models \alpha^+ \stackrel{u}{\simeq} \alpha^+ \dashv \cdot}\quad \text{UPVAR}$$

$$\frac{\Theta \models N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \models \downarrow N \stackrel{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\quad \text{USHIFTD}$$

$$\frac{\Theta \models P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \models \exists\overrightarrow{\alpha^-}.P \stackrel{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\quad \text{UEXISTS}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Theta \models \widehat{\alpha}^+ \stackrel{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)}\quad \text{UPUVAR}$$

### 3.3 Algorithmic Subtyping

$\boxed{\Gamma;\Theta \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma;\Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot}\ \text{ANVar}$$

$$\frac{\Theta \models \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma;\Theta \models \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}}\ \text{AShiftU}$$

$$\frac{\Gamma;\Theta \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma;\Theta \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma;\Theta \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\ \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+};\Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma,\overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma;\Theta \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}}\ \text{AForall}$$

$\boxed{\Gamma;\Theta \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma;\Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \text{APVar}$$

$$\frac{\Theta \models \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma;\Theta \models \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}}\ \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-};\Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma,\overrightarrow{\beta^-}\}} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma;\Theta \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}}\ \text{AExists}$$

$$\frac{\mathbf{upgrade}\,\Gamma \vdash \mathbf{nf}\,(P)\,\mathbf{to}\,\Delta = Q}{\Gamma;\Theta \models \widehat{\alpha}^+\{\Delta\} \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)}\ \text{APUVar}$$

### 3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restrict an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\Delta \vdash \widehat{\alpha}^+ :\approx P$ or $\Delta \vdash \widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\Delta \vdash \widehat{\alpha}^+ :\geqslant P$).

**Definition 1** (Matching Entries)**.** *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

**Definition 2.**
$\boxed{e_1 \,\&\, e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\Gamma \models P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)}\ (\geqslant \& \geqslant)$$

$$\frac{\Gamma;\cdot \models P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\ (\simeq \& \geqslant)$$

$$\frac{\Gamma;\cdot \models Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)}\ (\geqslant \& \simeq)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\ (\simeq \& \simeq^+)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N) \,\&\, (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)}\ (\simeq \& \simeq^-)$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

**Definition 3.** 
$$\begin{aligned}\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 = &\{e_1 \,\&\, e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, \textit{s.t. } e_1 \textit{ matches with } e_2\}\\ &\cup \{e_1 \mid e_1 \in \widehat{\sigma}_1,\ \textit{s.t. } \forall e_2 \in \widehat{\sigma}_2, e_1 \textit{ does not match with } e_2\}\\ &\cup \{e_2 \mid e_2 \in \widehat{\sigma}_2,\ \textit{s.t. } \forall e_1 \in \widehat{\sigma}_1, e_1 \textit{ does not match with } e_2\}\end{aligned}$$

### 3.5 Least Upper Bound

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+}\ \text{LUBVar}$$

$$\frac{\Gamma, \cdot \vDash {\downarrow} N \overset{a}{\simeq} {\downarrow} M \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash {\downarrow} N \vee {\downarrow} M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi] P}\ \text{LUBShift}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q}\ \text{LUBExists}$$

$\boxed{\textbf{upgrade}\, \Gamma \vdash P\, \textbf{to}\, \Delta = Q}$

### 3.6 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)}\ \text{AUPVar}$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash {\downarrow} N_1 \overset{a}{\simeq} {\downarrow} N_2 \dashv (\Xi, {\downarrow} M, \widehat{\tau}_1, \widehat{\tau}_2)}\ \text{AUPShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists \overrightarrow{\alpha^-}.\, Q, \widehat{\tau}_1, \widehat{\tau}_2)}\ \text{AUPExists}$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)}\ \text{AUNVar}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash {\uparrow} P_1 \overset{a}{\simeq} {\uparrow} P_2 \dashv (\Xi, {\uparrow} Q, \widehat{\tau}_1, \widehat{\tau}_2)}\ \text{AUNShift}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)}\ \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))}\ \text{AUNAU}$$

# 4 Proofs

## 4.1 Substitution

**Lemma 1** (Substitution strengthening)**.** *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then*

+ *if* $\Gamma_1 \vdash P$ *then* $[\sigma]P = [\sigma|_{\mathbf{fv}\, P}]P$,

− *if* $\Gamma_1 \vdash N$ *then* $[\sigma]N = [\sigma|_{\mathbf{fv}\, N}]N$

*Proof.* Ilya: todo ∎

## 4.2 Type well-formedness

**Lemma 2** (Well-formedness agrees with substitution)**.** *Suppose that* $\Gamma_1 \vdash \sigma : \Gamma_2$*. Then*

+ $\Gamma_1, \Gamma_2 \vdash P \Leftrightarrow \Gamma_1 \vdash [\sigma]P$

− $\Gamma_1, \Gamma_2 \vdash N \Leftrightarrow \Gamma_1 \vdash [\sigma]N$

**Lemma 3** (Equivalent Contexts)**.** *In the well-formedness judgment, only used variables matter:*

+ *if* $\{\Gamma_1\} \cap \mathbf{fv}\, P = \{\Gamma_2\} \cap \mathbf{fv}\, P$ *then* $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$,

− *if* $\{\Gamma_1\} \cap \mathbf{fv}\, N = \{\Gamma_2\} \cap \mathbf{fv}\, N$ *then* $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$.

*Proof.* By simple mutual induction on $P$ and $Q$. □

## 4.3 Overview

| Algorithm | Soundness | Completeness | Initiality |
|---|---|---|---|
| Ordering | $\dfrac{}{\{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\ N\} \equiv \mathit{vars} \cap \mathbf{fv}\, N}$ | $\dfrac{N \simeq_1^D M}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\ N = \mathbf{ord}\ \mathit{vars}\,\mathbf{in}\ M}$ | — |
| Normalization | $\dfrac{}{N \simeq_1^D \mathbf{nf}\,(N)}$ | $\dfrac{N \simeq_1^D M}{\mathbf{nf}\,(N) = \mathbf{nf}\,(M)}$ | — |
| Equivalence | $\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq_1^D Q}{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ | $\dfrac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{P \simeq_1^D Q}$ | — |
| Uppgrade | $\dfrac{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{Q \text{ is sound} \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$ | $\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$ | $\dfrac{Q' \text{ is sound} \quad \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{\Delta \vdash Q' \geqslant_1 Q}$ |
| LUB | $\dfrac{\Gamma \models P_1 \vee P_2 = Q}{Q \text{ is sound} \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$ | $\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \models P_1 \vee P_2 = Q}$ | $\dfrac{Q' \text{ is sound} \quad \Gamma \models P_1 \vee P_2 = Q}{\Delta \vdash Q' \geqslant_1 Q}$ |
| Anti-unification | $\dfrac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \text{ is sound} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \widehat{\tau}_i : \Xi \\ [\widehat{\tau}_i] Q = P_i \end{cases}}$ | $\dfrac{\exists \text{ sound } (\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)}{\exists (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \text{ s.t. } \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$ | $\dfrac{(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2) \text{ is sound} \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\exists \Gamma; \Xi \vdash \widehat{\tau} : \Xi' \text{ s.t. } [\widehat{\tau}] Q' = Q}$ |
| Unification (matching) | $\dfrac{\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\widehat{\sigma} \text{ is sound} \begin{cases} \Theta \vdash \widehat{\sigma} \\ [\widehat{\sigma}] P = Q \end{cases}}$ | $\dfrac{\exists \text{ sound } \widehat{\sigma}'}{\exists \widehat{\sigma} \text{ s.t. } \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ | — |
| Subtyping | $\dfrac{\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}}{\widehat{\sigma} \text{ is sound} \begin{cases} \Theta \vdash \widehat{\sigma} \\ \Gamma \vdash [\widehat{\sigma}] N \leqslant_1 M \end{cases}}$ | $\dfrac{\exists \text{ sound } \widehat{\sigma}'}{\exists \widehat{\sigma} \text{ s.t. } \Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}}$ | — |

## 4.4 Variable Ordering

**Definition 4** (Collision free bijection)**.** *We say that a bijection* $\mu : A \leftrightarrow B$ *between sets of variables is* ***collision free on sets*** $P$ *and* $Q$ *if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

**Lemma 4** (Soundness of variable ordering)**.** *Variable ordering extracts precisely used free variables.*

− $\{\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N\} \equiv \mathit{vars} \cap \textbf{fv}\,N$ *(as sets)*

+ $\{\textbf{ord}\,\mathit{vars}\,\textbf{in}\,P\} \equiv \mathit{vars} \cap \textbf{fv}\,P$ *(as sets)*

*Proof.* Straightforward mutual induction on $\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N = \vec{\alpha}$ and $\textbf{ord}\,\mathit{vars}\,\textbf{in}\,P = \vec{\alpha}$ □

**Corollary 1** (Additivity of ordering)**.** *Variable ordering is additive (in terms of set union) with respect to its first argument.*

− $\{\textbf{ord}\,(\mathit{vars}_1 \cup \mathit{vars}_2)\,\textbf{in}\,N\} \equiv \{\textbf{ord}\,\mathit{vars}_1\,\textbf{in}\,N\} \cup \{\textbf{ord}\,\mathit{vars}_2\,\textbf{in}\,N\}$ *(as sets)*

+ $\{\textbf{ord}\,(\mathit{vars}_1 \cup \mathit{vars}_2)\,\textbf{in}\,P\} \equiv \{\textbf{ord}\,\mathit{vars}_1\,\textbf{in}\,P\} \cup \{\textbf{ord}\,\mathit{vars}_2\,\textbf{in}\,P\}$ *(as sets)*

**Corollary 2** (Weakening of ordering)**.** *Extending the first argument of the ordering with unused variables does not change the result.*

− $\textbf{ord}\,(\mathit{vars} \cap \textbf{fv}\,N)\,\textbf{in}\,N = \textbf{ord}\,\mathit{vars}\,\textbf{in}\,N$

+ $\textbf{ord}\,(\mathit{vars} \cap \textbf{fv}\,P)\,\textbf{in}\,P = \textbf{ord}\,\mathit{vars}\,\textbf{in}\,P$

**Lemma 5** (Distributivity of renaming over variable ordering)**.** *Suppose that $\mu$ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

− *If $\mu$ is collision free on vars and $\textbf{fv}\,N$ then $[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) = \textbf{ord}\,([\mu]\mathit{vars})\,\textbf{in}\,[\mu]N$*

+ *If $\mu$ is collision free on vars and $\textbf{fv}\,P$ then $[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,P) = \textbf{ord}\,([\mu]\mathit{vars})\,\textbf{in}\,[\mu]P$*

*Proof.* Mutual induction on $N$ and $P$.

**Case 1**. $N = \alpha^-$
let us consider four cases:

*a.* $\alpha^- \in A$ and $\alpha^- \in \mathit{vars}$
Then
$$\begin{aligned}
[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) &= [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\alpha^-) \\
&= [\mu]\alpha^- && \text{by Rule } (\mathrm{Var}^+_\in) \\
&= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]\mathit{vars}) \\
&= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,\beta^- && \text{by Rule } (\mathrm{Var}^+_\in)\text{, because } \beta^- \in [\mu]\mathit{vars} \\
&= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\alpha^-
\end{aligned}$$

*b.* $\alpha^- \notin A$ and $\alpha^- \notin \mathit{vars}$
Notice that $[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) = [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. On the other hand, $\textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\alpha^- = \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,\alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_\notin)$, because $\mu$ is collision free on $\mathit{vars}$ and $\textbf{fv}\,N$, so $\textbf{fv}\,N \ni \alpha^- \notin \mu(A \cap \mathit{vars}) \cup \mathit{vars} \supseteq [\mu]\mathit{vars}$.

*c.* $\alpha^- \in A$ but $\alpha^- \notin \mathit{vars}$
Then $[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) = [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. To prove that $\textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_\notin)$. Let us show that $[\mu]\alpha^- \notin [\mu]\mathit{vars}$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]\mathit{vars} \subseteq \mu(A \cap \mathit{vars}) \cup \mathit{vars}$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap \mathit{vars}) \cup \mathit{vars}$.

(i) If there is an element $x \in A \cap \mathit{vars}$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of $\mu$, which contradicts with $\alpha^- \notin \mathit{vars}$. This way, $\mu(\alpha^-) \notin \mu(A \cap \mathit{vars})$.

(ii) Since $\mu$ is collision free on $\mathit{vars}$ and $\textbf{fv}\,N$, $\mu(A \cap \textbf{fv}\,N) \ni \mu(\alpha^-) \notin \mathit{vars}$.

*d.* $\alpha^- \notin A$ but $\alpha^- \in \mathit{vars}$
$\textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\alpha^- = \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_\notin)$, because $\alpha^- = [\mu]\alpha^- \in [\mu]\mathit{vars}$ since $\alpha^- \in \mathit{vars}$. On the other hand, $[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) = [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\alpha^-) = [\mu]\alpha^- = \alpha^-$.

**Case 2**. $N = \uparrow P$
$$\begin{aligned}
[\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,N) &= [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,\uparrow P) \\
&= [\mu](\textbf{ord}\,\mathit{vars}\,\textbf{in}\,P) && \text{by Rule } (\uparrow) \\
&= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]P && \text{by the induction hypothesis} \\
&= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,\uparrow[\mu]P && \text{by Rule } (\uparrow) \\
&= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]\uparrow P && \text{by the definition of substitution} \\
&= \textbf{ord}\,[\mu]\mathit{vars}\,\textbf{in}\,[\mu]N
\end{aligned}$$

**Case 3.** $N = P \to M$

$$
\begin{aligned}
[\mu](\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N) &= [\mu](\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P \to M) \\
&= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})) && \text{where } \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ M = \vec{\alpha}_2 \\
&= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\}) \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\{\vec{\alpha}_1\}) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to case 1. Notice that } \mu \text{ is} \\
& && \text{collision free on } \{\vec{\alpha}_1\} \text{ and } \{\vec{\alpha}_2\} \text{ since } \{\vec{\alpha}_1\} \subseteq \mathit{vars} \text{ and } \{\vec{\alpha}_2\} \subseteq \mathbf{fv}\ N \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]N) &= (\mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]P \to [\mu]M) \\
&= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \{\vec{\beta}_1\})) && \text{where } \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]M = \vec{\beta}_2 \\
& && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})
\end{aligned}
$$

**Case 4.** $N = \forall\overrightarrow{\alpha^+}.M$

$$
\begin{aligned}
[\mu](\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N) &= [\mu]\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.M \\
&= [\mu]\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ M \\
&= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]M && \text{by the induction hypothesis} \\
(\mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]N) &= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]\forall\overrightarrow{\alpha^+}.M \\
&= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.[\mu]M \\
&= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]M
\end{aligned}
$$

□

**Lemma 6** (Ordering is not affected by independent substitutions). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. $\sigma$ maps variables from $\Gamma_1$ into types taking free variables from $\Gamma_2$, and vars is a set of variables disjoint with both $\Gamma_1$ and $\Gamma_2$. Then*

- $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ [\sigma]N = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N$

+ $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ [\sigma]P = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P$

*Proof.* Ilya: Should be easy □

**Lemma 7** (Completeness of variable ordering). *Variable ordering is invariant under equivalence. For arbitrary vars,*

- *If $N \simeq_1^D M$ then $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ M$ (as lists)*

+ *If $P \simeq_1^D Q$ then $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ Q$ (as lists)*

*Proof.* Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. □

## 4.5 Normaliztaion

**Lemma 8.** *Set of free variables is invariant under equivalence.*

- *If $N \simeq_1^D M$ then $\mathbf{fv}\ N \equiv \mathbf{fv}\ M$ (as sets)*

+ *If $P \simeq_1^D Q$ then $\mathbf{fv}\ P \equiv \mathbf{fv}\ Q$ (as sets)*

*Proof.* Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

**Lemma 9.** *Free variables are not changed by the normalization*

- $\mathbf{fv}\ N \equiv \mathbf{fv}\ \mathbf{nf}\,(N)$

+ $\mathbf{fv}\ P \equiv \mathbf{fv}\ \mathbf{nf}\,(P)$

*Proof.* By straightforward induction on $\mathbf{nf}\,(N) = M$. □

**Lemma 10** (Soundness of quantifier normalization).

- $N \simeq_1^D \mathbf{nf}\,(N)$

+ $P \simeq_1^D \mathbf{nf}\,(P)$

*Proof.* Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

**Case 1.** (Var$^-$) and (Var$^+$)
By the corresponding equivalence rules.

**Case 2.** ($\uparrow$), ($\downarrow$), and ($\rightarrow$)
By the induction hypothesis and the corresponding congruent equivalence rules.

**Case 3.** ($\forall$), i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 8, $\mathbf{fv}\,N \equiv \mathbf{fv}\,N'$. Then by lemma 4, $\{\overrightarrow{\alpha^+}'\} \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$, and thus, $\{\overrightarrow{\alpha^+}'\} \cap \mathbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$.

To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^+}'.N'$, it suffices to provide a bijection $\mu : \{\overrightarrow{\alpha^+}'\} \cap \mathbf{fv}\,N' \leftrightarrow \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

**Case 4.** ($\exists$) Same as for case 3.

$\square$

**Corollary 3** (Normalization preserves ordering)**.** *For any vars,*

$-$ $\mathbf{ord}\ vars\ \mathbf{in}\ \mathbf{nf}\,(N) = \mathbf{ord}\ vars\ \mathbf{in}\ M$

$+$ $\mathbf{ord}\ vars\ \mathbf{in}\ \mathbf{nf}\,(P) = \mathbf{ord}\ vars\ \mathbf{in}\ Q$

*Proof.* Immediately from lemmas 7 and 10. $\square$

**Lemma 11** (Distributivity of normalization over substitution)**.** *Normalization of a term distributes over substitution. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. $\sigma$ maps variables from $\Gamma_1$ into types taking free variables from $\Gamma_2$. Then*

$-$ $\mathbf{nf}\,([\sigma]N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$

$+$ $\mathbf{nf}\,([\sigma]P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P)$

*where* $\mathbf{nf}\,(\sigma)$ *means pointwise normalization:* $[\mathbf{nf}\,(\sigma)]\alpha^- = \mathbf{nf}\,([\sigma]\alpha^-)$.

*Proof.* Mutual induction on $N$ and $P$.

**Case 1.** $N = \alpha^-$
$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.

**Case 2.** $P = \alpha^+$
Similar to case 1.

**Case 3.** If the type is formed by $\rightarrow$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \rightarrow M$ then

| | |
|---|---|
| $\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma](P \rightarrow M))$ | |
| $= \mathbf{nf}\,([\sigma]P \rightarrow [\sigma]M)$ | By the congruence of substitution |
| $= \mathbf{nf}\,([\sigma]P) \rightarrow \mathbf{nf}\,([\sigma]M)$ | By the congruence of normalization, i.e. Rule ($\rightarrow$) |
| $= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) \rightarrow [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$ | By the induction hypothesis |
| $= [\mathbf{nf}\,(\sigma)](\mathbf{nf}\,(P) \rightarrow \mathbf{nf}\,(M))$ | By the congruence of substitution |
| $= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P \rightarrow M)$ | By the congruence of normalization |
| $= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$ | |

**Case 4.** $N = \forall\overrightarrow{\alpha^+}.M$

| | |
|---|---|
| $[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M)$ | |
| $= [\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$ | Where $\overrightarrow{\alpha^+}' = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M$ (the latter is by corollary 3) |

| | |
|---|---|
| $\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\forall\overrightarrow{\alpha^+}.M)$ | |
| $= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\sigma]M)$ | Assuming $\{\overrightarrow{\alpha^+}\} \cap \{\Gamma_1\} = \varnothing$ and $\{\overrightarrow{\alpha^+}\} \cap \{\Gamma_2\} = \varnothing$ |
| $= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\sigma]M)$ | Where $\overrightarrow{\beta^+} = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,([\sigma]M) = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\sigma]M$ (the latter is by corollary 3) |
| $= \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\sigma]M)$ | By lemma 6, $\overrightarrow{\beta^+} = \overrightarrow{\alpha^+}'$ since $\{\overrightarrow{\alpha^+}\}$ is disjoint with $\Gamma_1$ and $\Gamma_2$ |
| $= \forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$ | By the induction hypothesis |

To show alpha-equivalence of $[\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^{+}}'.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^{+}}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$, we can assume that $\{\overrightarrow{\alpha^{+}}'\} \cap \{\Gamma_1\} = \varnothing$, and $\{\overrightarrow{\alpha^{+}}'\} \cap \{\Gamma_2\} = \varnothing$.

**Case 5.** $P = \exists\overrightarrow{\alpha^{-}}.Q$
Same as for case 4.

□

**Corollary 4** (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that $\mu$ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

– $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$

+ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

*Proof.* Immediately from lemma 11, after noticing that $\mathbf{nf}\,(\mu) = \mu$. □

**Lemma 12** (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

– *If $N \simeq_1^D M$ then $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$*

+ *If $P \simeq_1^D Q$ then $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$*

*(Here equality means alpha-equivalence)*

*Proof.* Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

**Case 1.** $(\forall^{\simeq_1^D})$

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^{+}}.N) = \forall\overrightarrow{\alpha^{+}}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^{+}}'$ is $\mathbf{ord}\,\{\overrightarrow{\alpha^{+}}\}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^{+}}.M) = \forall\overrightarrow{\beta^{+}}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^{+}}'$ is $\mathbf{ord}\,\{\overrightarrow{\beta^{+}}\}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\{\overrightarrow{\beta^{+}}\} \cap \mathbf{fv}\,M) \leftrightarrow (\{\overrightarrow{\alpha^{+}}\} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 4 and 9, the domain and the codomain of $\mu$ can be written as $\mu : \{\overrightarrow{\beta^{+}}'\} \leftrightarrow \{\overrightarrow{\alpha^{+}}'\}$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^{+}}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^{+}}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^{+}}' = \overrightarrow{\alpha^{+}}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by corollary 4, the second—by the induction hypothesis.

(ii)
$$
\begin{aligned}
[\mu]\overrightarrow{\beta^{+}}' &= [\mu]\mathbf{ord}\,\{\overrightarrow{\beta^{+}}\}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^{+}}' \\
&= [\mu]\mathbf{ord}\,(\{\overrightarrow{\beta^{+}}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 9 and corollary 2} \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^{+}}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 5, because } \{\overrightarrow{\alpha^{+}}\} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \{\overrightarrow{\alpha^{+}}\} \cap \mathbf{fv}\,M = \varnothing \\
& && \text{and } \{\overrightarrow{\alpha^{+}}\} \cap \mathbf{fv}\,N \cap (\{\overrightarrow{\beta^{+}}\} \cap \mathbf{fv}\,M) \subseteq \{\overrightarrow{\alpha^{+}}\} \cap \mathbf{fv}\,M = \varnothing \\
&= \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^{+}}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved} \\
&= \mathbf{ord}\,(\{\overrightarrow{\alpha^{+}}\} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \{\overrightarrow{\alpha^{+}}\} \cap \mathbf{fv}\,N \text{ and } \{\overrightarrow{\beta^{+}}\} \cap \mathbf{fv}\,M \\
&= \mathbf{ord}\,\{\overrightarrow{\alpha^{+}}\}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 9 and corollary 2} \\
&= \overrightarrow{\alpha^{+}}' && \text{by the definition of } \overrightarrow{\alpha^{+}}'
\end{aligned}
$$

**Case 2.** $(\exists^{\simeq_1^D})$ Same as for case 1.

**Case 3.** Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□

**Lemma 13** (Idempotence of normalization)**.** *Normalization is idempotent*

– $\mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$

+ $\mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

*Proof.* By applying lemma 12 to lemma 10. □

**Lemma 14.** *The result of a substitution is normalized if and only if the initial type and the substitution are normalized. Suppose that $\sigma$ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, $P$ is a positive type ($\Gamma_1 \vdash P$), $N$ is a negative type ($\Gamma_1 \vdash N$). Then*

+ $[\sigma]P$ *is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \textit{is normal} \\ P & \textit{is normal} \end{cases}$

− $[\sigma]N$*is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \textit{is normal} \\ N & \textit{is normal} \end{cases}$

*Proof.* Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

**Case 1.** $N = \alpha^-$
Then $N$ is always normal, and the normality of $\sigma|_{\{\alpha^-\}}$ by the definition means $[\sigma]\alpha^-$ is normal.

**Case 2.** $N = P \to M$

$$\begin{aligned}
[\sigma](P \to M) \text{ is normal} &\iff [\sigma]P \to [\sigma]M \text{ is normal} && \text{by the substitution congruence} \\
&\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases} && \text{by congruence of normality } \texttt{Ilya: lemma?} \\
&\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(M)} & \text{is normal} \end{cases} && \text{by the induction hypothesis} \\
&\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P) \cup \mathbf{fv}\,(M)} & \text{is normal} \end{cases} \iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P \to M)} & \text{is normal} \end{cases}
\end{aligned}$$

**Case 3.** $N = \uparrow P$
By congruence and the inductive hypothesis, similar to case 2

**Case 4.** $N = \forall \overrightarrow{\alpha^+}.M$

$$\begin{aligned}
[\sigma](\forall \overrightarrow{\alpha^+}.M) \text{ is normal} &\iff (\forall \overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} && \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing \\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+} \end{cases} && \text{by the definition of normalization} \\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} && \text{by lemma 6} \\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} && \text{by the induction hypothesis} \\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(\forall \overrightarrow{\alpha^+}.M)} \text{ is normal} \\ \forall \overrightarrow{\alpha^+}.M \text{ is normal} \end{cases} && \begin{array}{l} \text{since } \mathbf{fv}\,(\forall \overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M); \\ \text{by the definition of normalization} \end{array}
\end{aligned}$$

**Case 5.** $P = \dots$
The positive cases are done in the same way as the negative ones.

□

## 4.6 Equivalence

**Lemma 15** (Type well-formedness is invariant under equivalence)**.** *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,

− *if* $N \simeq_1^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

*Proof.* Ilya: todo □

**Corollary 5** (Normalization preserves well-formedness)**.**

$+$ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}\,(P)$,

$-$ $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}\,(N)$

*Proof.* Immediately from lemmas 10 and 15. □

**Corollary 6** (Normalization preserves well-formedness of substitution)**.**
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}\,(\sigma) : \Gamma_1$

**Lemma 16** (Soundness of equivalence)**.** *Declarative equivalence implies mutual subtyping.*

$+$ *if* $\Gamma \vdash P$, $\Gamma \vdash Q$, *and* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \simeq_1^{\leqslant} Q$,

$-$ *if* $\Gamma \vdash N$, $\Gamma \vdash M$, *and* $N \simeq_1^D M$ *then* $\Gamma \vdash N \simeq_1^{\leqslant} M$.

*Proof.* `Ilya: todo` □

**Lemma 17** (Completeness of equivalence)**.** *Mutual subtyping implies declarative equivalence.*

$+$ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $P \simeq_1^D Q$,

$-$ *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $N \simeq_1^D M$.

*Proof.* `Ilya: todo` □

## 4.7 Upper Bounds

**Lemma 18** (Decomposition of the quantifier rule)**.** *`Ilya: move somewhere` Whenever the quantifier rule (Rule* $(\exists^{\geqslant_1})$ *or Rule* $(\forall^{\leqslant_1})$*) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.*

$-$ *If* $\Gamma \vdash N \leqslant_1 \forall\overrightarrow{\beta^+}.M$ *then* $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant_1 M$.

$+$ *If* $\Gamma \vdash P \geqslant_1 \exists\overrightarrow{\beta^-}.Q$ *then* $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant_1 Q$.

**Lemma 19** (Shape of the Supertypes)**.** *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

| $\Gamma \vdash P$ | $\mathsf{UB}(\Gamma \vdash P)$ |
|---|---|
| $\Gamma \vdash \beta^+$ | $\{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \textit{for } \overrightarrow{\alpha^-}\}$ |
| $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$ | $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)$ *not using* $\overrightarrow{\beta^-}$ |
| $\Gamma \vdash \downarrow M$ | $\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-},\ M',\ \textit{and } \overrightarrow{N} \textit{ s.t.} \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\}$ |

*Then* $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

*Proof.* By induction on $\Gamma \vdash P$.

**Case 1.** $P = \beta^+$
Then the last rule that is applied to infer $\Gamma \vdash Q \geqslant_1 \beta^+$ must be either Rule $(\mathrm{Var}^{+\geqslant_1})$ or Rule $(\exists^{\geqslant_1})$. The former case means that $Q = \beta^+$. In the latter case, $Q = \exists\overrightarrow{\alpha^-}.Q'$, where $Q'$ has no outer existential quantifiers. Then by inversion of Rule $(\exists^{\geqslant_1})$, $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]Q' \geqslant_1 \beta^+$ for some $\overrightarrow{N}$. This time, to infer this judgment, only Rule $(\mathrm{Var}^{+\geqslant_1})$ is applicable, which means that $Q' = \beta^+$, and then $Q = \exists\overrightarrow{\alpha^-}.\beta^+$.

**Case 2.** $P = \exists\overrightarrow{\beta^-}.P'$
Then if $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'$, then by lemma 18, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$, and $\mathbf{fv}\,Q \cap \{\overrightarrow{\beta^-}\} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule $(\exists^{\geqslant_1})$. This way, $\{Q \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,(Q) \cap \{\overrightarrow{\beta^-}\} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')$.

**Case 3.** $P = \downarrow M$
Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\nexists}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

*a.* $Q \neq \exists\overrightarrow{\beta^-}.Q'$

Then the last applied rule to infer $\Gamma \vdash Q \geqslant_1 {\downarrow}M$ must be Rule $({\downarrow}^{\geqslant_1})$, which means $Q = {\downarrow}M'$, and by inversion, $\Gamma \vdash M' \simeq_1^{\leqslant} M$, then by lemma 17 and Rule $({\downarrow}^{\simeq_1^D})$, ${\downarrow}M' \simeq_1^D {\downarrow}M$. This way, $Q = {\downarrow}M' \in \{{\downarrow}M' \mid {\downarrow}M' \simeq_1^D {\downarrow}M\} = \mathsf{UB}(\Gamma \vdash {\downarrow}M)|_{\nexists}$.

In the other direction, ${\downarrow}M' \simeq_1^D {\downarrow}M \Rightarrow \Gamma \vdash {\downarrow}M' \simeq_1^{\leqslant} {\downarrow}M$ by lemma 16, since $\Gamma \vdash {\downarrow}M'$ by lemma 15

$\Rightarrow \Gamma \vdash {\downarrow}M' \geqslant_1 {\downarrow}M$ by inversion

*b.* $Q = \exists\overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)

Then the last rule applied to infer $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q' \geqslant_1 {\downarrow}M$ must be Rule $(\exists^{\geqslant_1})$. Inversion of this rule gives us $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \geqslant_1 {\downarrow}M$ for some $\Gamma \vdash N_i$. Notice that $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.*a*, $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \simeq_1^D {\downarrow}M$, which is only possible if $Q' = {\downarrow}M'$. This way, $Q = \exists\overrightarrow{\beta^-}.{\downarrow}M' \in \mathsf{UB}(\Gamma \vdash {\downarrow}M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).

In the other direction, $[\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \simeq_1^D {\downarrow}M \Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \simeq_1^{\leqslant} {\downarrow}M$ by lemma 16, since $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M'$ by lemma 15

$\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \geqslant_1 {\downarrow}M$ by inversion

$\Rightarrow \Gamma \vdash \exists\overrightarrow{\beta^-}.{\downarrow}M' \geqslant_1 {\downarrow}M$ by Rule $(\exists^{\geqslant_1})$

□

**Lemma 20** (Normalized Shape of the Supertypes). *For a normalized positive type $P = \mathbf{nf}(P)$, let us define the set of normalized upper bounds in the following way:*

| $\Gamma \vdash P$ | $\mathsf{NFUB}(\Gamma \vdash P)$ |
|---|---|
| $\Gamma \vdash \beta^+$ | $\{\beta^+\}$ |
| $\Gamma \vdash \exists\overrightarrow{\beta^-}.P$ | $\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)$ *not using* $\overrightarrow{\beta^-}$ |
| $\Gamma \vdash {\downarrow}M$ | $\left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M',\ \text{and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M \end{array} \right\}$ |

*Then* $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 P\}$.

*Proof.* By induction on $\Gamma \vdash P$.

**Case 1.** $P = \beta^+$

Then from lemma 19, $\{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}(\exists\overrightarrow{\alpha^-}.\beta^+) \mid \text{for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

**Case 2.** $P = \exists\overrightarrow{\beta^-}.P'$

$$\begin{aligned}
\mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') &= \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ not using } \overrightarrow{\beta^-} \\
&= \{\mathbf{nf}(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothesis} \\
&= \{\mathbf{nf}(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}(Q) = \mathbf{fv}\,Q \text{ by lemma 9} \\
&= \{\mathbf{nf}(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{by lemma 19} \\
&= \{\mathbf{nf}(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} && \text{by the definition of UB} \\
&= \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} && \text{by lemma 19}
\end{aligned}$$

**Case 3.** $P = {\downarrow}M$

In the following reasoning, we will use the following principle of variable replacement.

**Observation 1.** *Suppose that $\nu : A \to A$ is an idempotent function, $P$ is a predicate on $A$, $F : A \to B$ is a function. Then*

$$\begin{aligned} &\{F(\nu x) \mid x \in A \text{ s.t. } P(\nu x)\} = \\ =&\{F(x) \mid x \in A \text{ s.t. } \nu x = x \text{ and } P(x)\}. \end{aligned}$$

In our case, the idempotent $\nu$ will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

**Observation 2.** *For functions $F$ and $\nu$, and predicates $P$ and $Q$,*

$$\begin{aligned} &\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } P(x)\} = \\ =&\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } (\exists x' \in A \text{ s.t. } P(x') \text{ and } \nu x' = \nu x)\}. \end{aligned}$$

$$
\begin{aligned}
&\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 {\downarrow}M\} = \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash {\downarrow}M)\} && \text{by lemma 19} \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash N_i, \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by the definition of } \mathsf{UB} \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{we reassigned the substitution } \overrightarrow{N}/\overrightarrow{\alpha^-} \text{ as } \sigma \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by lemma 1} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by the definition of normalization} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } \mathbf{nf}\,([\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M') = \mathbf{nf}\,({\downarrow}M) \end{array} \right\} && \begin{array}{l}\text{from lemmas 10 and 12, equivalence of types can be} \\ \text{replaced with the equality of their normal forms}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})]{\downarrow}\mathbf{nf}\,(M') = {\downarrow}\mathbf{nf}\,(M) \end{array} \right\} && \text{by congruence of normalization and lemma 11} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{by lemma 14, } {\downarrow}M' \text{ and } \sigma|_{\mathbf{fv}\,M'} \text{ are already normal,} \\ \text{since the result of the substitution is normal;} \\ M \text{ is normal by assumption}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ (\exists\sigma' \text{ s.t. } \Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')}) \\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{We apply observation 2 (with } \nu\sigma = \sigma|_{\mathbf{fv}\,M'}, \text{ and} \\ P(\sigma) = \Gamma \vdash \sigma : \overrightarrow{\alpha^-})\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv}\,M'} : \overrightarrow{\alpha^-}',\; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{Notice that} \\ \text{“}\exists\sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,(\underrightarrow{{\downarrow}M'})} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')})\text{”} \\ \text{is equivalent to } \Gamma \vdash \sigma|_{\mathbf{fv}\,({\downarrow}M')} : \overrightarrow{\alpha^-}'\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{We apply observation 1 to the restriction of } \sigma\text{, and the} \\ \text{remove } \sigma|_{\mathbf{fv}\,M'} = \sigma \text{ as it follows from } \Gamma \vdash \sigma : \overrightarrow{\alpha^-}'\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, \overrightarrow{\alpha^-}',\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{From lemma 3, since } \{\Gamma, \overrightarrow{\alpha^-}\} \cap \mathbf{fv}\,M' = \{\Gamma, \overrightarrow{\alpha^-}'\} \cap \mathbf{fv} \\
&= \left\{ \exists\overrightarrow{\alpha^-}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\; \mathbf{ord}\,\{\overrightarrow{\alpha^-}'\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{We apply observation 1 to the ordering of } \overrightarrow{\alpha^-} \\
&= \left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}'\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\, \Gamma, \overrightarrow{\alpha^-}' \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}']{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{By reassigning } \sigma \text{ explicitly as } \overrightarrow{N}/\overrightarrow{\alpha^-}'
\end{aligned}
$$

□

**Lemma 21** (Soundness of the Least Upper Bound)**.** *For types $\Gamma \vdash P_1$, and $\Gamma \vdash P_2$, if $\Gamma \vDash P_1 \vee P_2 = Q$ then*

*(i)* $\Gamma \vdash Q$

*(ii)* $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

**Lemma 22** (Completeness of the Least Upper Bound)**.** *For types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q'$ such that $\Gamma \vdash Q' \geqslant_1 P_1$ and $\Gamma \vdash Q' \geqslant_1 P_2$, there exists $Q$ s.t. $\Gamma \vDash P_1 \vee P_2 = Q$, and $\Gamma \vdash Q' \geqslant_1 Q$*

**Lemma 23** (Soundness of Upgrade)**.** *For $\Delta \subseteq \Gamma$, suppose that* $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$*. Then*

*(i)* $\Delta \vdash Q$

*(ii)* $\Gamma \vdash Q \geqslant_1 P$

**Lemma 24** (Completeness of Upgrade)**.** *For* $\Delta \subseteq \Gamma$*,* $\Gamma \vdash P$ *and* $\Delta \vdash Q'$*, such that* $\Gamma \vdash Q' \geqslant_1 P$*, there exists* $Q$ *s.t.* $\mathbf{upgrade}\, \Gamma \vdash P\, \mathbf{to}\, \Delta = Q$*, and* $\Delta \vdash Q' \geqslant_1 Q$*.*