Contents

1 Declarative Type Systems

1.1 Grammar

We assume that there is an infinite set of positive and negative *type* variables. Positive type variables are denoted as α^+, β^+, γ^+, etc. Negative type variables are denoted as α^-, β^-, γ^-, etc. We assume there is an infinite set of *term* variables, which are denoted as x, y, z, etc. A list of objects (variables, types or terms) is denoted by an overline arrow. For instance, $\overrightarrow{\alpha^+}$ is a list of positive type variables, $\overrightarrow{\beta^-}$ is a list of negative type variables, $\vec{v}$ is a list of values, which are arguments of a function. $\mathbf{fv}\,(P)$ and $\mathbf{fv}\,(N)$ denote the set of free variables in a type P and N, respectively.

$P,\ Q,\ R$	$::=$		positive declarative types
	$\mid$	α^+	
	$\mid$	$\downarrow N$	
	$\mid$	$\exists\overrightarrow{\alpha^-}.P$	
$N,\ M,\ K$	$::=$		negative declarative types
	$\mid$	α^-	
	$\mid$	$\uparrow P$	
	$\mid$	$P \rightarrow N$	
	$\mid$	$\forall\overrightarrow{\alpha^+}.N$	

1.2 Equalities

For simplicity, assume alpha-equivalent terms equal. This way, we assume that substitutions do not capture bound variables. Besides, we equate $\forall\overrightarrow{\alpha^+}.\forall\overrightarrow{\beta^+}.N$ with $\forall\overrightarrow{\alpha^+},\overrightarrow{\beta^+}.N$, as well as $\exists\overrightarrow{\alpha^-}.\exists\overrightarrow{\beta^-}.P$ with $\exists\overrightarrow{\alpha^-},\overrightarrow{\beta^-}.P$, and lift these equations transitively and congruently to the whole system.

1.3 Contexts and Well-formedness

Definition 1 (Declarative Type Context)**.**
Declarative type context Γ is represented by a set of type variables. The concatenation Γ_1, Γ_2 means the union of two contexts $\Gamma_1 \cup \Gamma_2$.

$\Gamma \vdash P$ and $\Gamma \vdash N$ denote that the type is well-formed in the context Γ, which, in fact, means that each free type variable of the type is contained in Γ (it will be shown later in lemma 3 and section 5.3).

$\boxed{\Gamma \vdash N}$ Negative type well-formedness

$$\frac{\alpha^- \in \Gamma}{\Gamma \vdash \alpha^-}\ \textsc{WFTNVar}$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash \uparrow P}\ \textsc{WFTShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash N}{\Gamma \vdash P \rightarrow N}\ \textsc{WFTArrow}$$

$$\frac{\Gamma, \overrightarrow{\alpha^+} \vdash N}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N}\ \textsc{WFTForall}$$

$\boxed{\Gamma \vdash P}$ Positive type well-formedness

$$\frac{\alpha^+ \in \Gamma}{\Gamma \vdash \alpha^+}\ \textsc{WFTPVar}$$

$$\frac{\Gamma \vdash N}{\Gamma \vdash \downarrow N}\ \textsc{WFTShiftD}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-} \vdash P}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P}\ \textsc{WFTExists}$$

1.4 Substitutions

Definition 2 (Substitution). *Substitutions (denoted as σ) are represented by total functions form variables to types, preserving the polarity.*

Definition 3 (Substitution Application). *Substitution application (denoted as $[\sigma]P$ or $[\sigma]N$) is defined congruently as follows:*

- $[\sigma]\alpha^+ = \sigma(\alpha^+)$;
- $[\sigma]\alpha^- = \sigma(\alpha^-)$;
- $[\sigma]{\downarrow}N = {\downarrow}[\sigma]N$;
- $[\sigma]{\uparrow}P = {\uparrow}[\sigma]P$;
- $[\sigma]\exists\overrightarrow{\alpha^-}.Q = \exists\overrightarrow{\alpha^-}.[\sigma]Q$,
- $[\sigma]\forall\overrightarrow{\alpha^+}.N = \forall\overrightarrow{\alpha^+}.[\sigma]N$ *(here we assume that $\overrightarrow{\alpha^-}$ and $\overrightarrow{\alpha^+}$ are lists of fresh variables, that is the variable capture never happens);*
- $[\sigma](P \to N) = [\sigma]P \to [\sigma]N$.

Definition 4 (Substitution Signature). *The signature $\Gamma' \vdash \sigma : \Gamma$ means that*

1. *for any $\alpha^\pm \in \Gamma, \Gamma' \vdash [\sigma]\alpha^\pm$; and*
2. *for any $\alpha^\pm \notin \Gamma', [\sigma]\alpha^\pm = \alpha^\pm$.*

A substitution can be restricted to a set of variables. The restricted substitution is define as expected.

Definition 5 (Subsitution Restriction). *The specification $\sigma|_{vars}$ is defined as a function such that*

1. *$\sigma|_{vars}(\alpha^\pm) = \sigma(\alpha^\pm)$, if $\alpha^\pm \in vars$; and*
2. *$\sigma|_{vars}(\alpha^\pm) = \alpha^\pm$, if $\alpha^\pm \notin vars$.*

Two substitutions can be composed in two ways: $\sigma_2 \circ \sigma_1$ corresponds to a consecutive application of σ_1 and σ_2, while $\sigma_2 \lll \sigma_1$ depends on a signature of σ_1 and modifies σ_1 by applying σ_2 to its results on the domain.

Definition 6 (Substitution Composition). *$\sigma_2 \circ \sigma_1$ is defined as a function such that $\sigma_2 \circ \sigma_1(\alpha^\pm) = \sigma_2(\sigma_1(\alpha^\pm))$.*

Definition 7 (Monadic Substitution Composition). *Suppose that $\Gamma' \vdash \sigma_1 : \Gamma$. Then we define $\sigma_2 \lll \sigma_1$ as $(\sigma_2 \circ \sigma_1)|_\Gamma$.*

Notice that the result of $\sigma_2 \lll \sigma_1$ depends on the specification of σ_1, which is not unique. However, we assume that the used specification clear from the context of the proof.

Definition 8 (Equivalent Substitutions). *The substitution equivalence judgement $\Gamma' \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \Gamma$ indicates that on the domain Γ, the result of σ_1 and σ_2 are equivalent in context Γ'. Formally, for any $\alpha^\pm \in \Gamma, \Gamma' \vdash [\sigma_1]\alpha^\pm \simeq^{\leqslant} [\sigma_2]\alpha^\pm$.*

Sometimes it is convenient to construct substitution explicitly mapping each variable from a list (or a set) to a type. Such substitutions are denoted as $\vec{P}/\overrightarrow{\alpha^+}$ and $\vec{N}/\overrightarrow{\alpha^-}$, where $\vec{P}$ and $\vec{N}$ are lists of the corresponding types.

Definition 9 (Explicit Substitution).

– *Suppose that $\overrightarrow{\alpha^-}$ is a list of negative type variables, and $\vec{N}$ is a list of negative types of the same length Then $\vec{N}/\overrightarrow{\alpha^-}$ denotes a substitution such that*

  1. *for $\alpha_i^+ \in \overrightarrow{\alpha^-}$, $[\vec{N}/\overrightarrow{\alpha^-}]\alpha_i^+ = N_i$;*
  2. *for $\beta^+ \notin \overrightarrow{\alpha^-}$, $[\vec{N}/\overrightarrow{\alpha^-}]\beta^+ = \beta^+$.*

+ *Positive explicit substitution $\vec{P}/\overrightarrow{\alpha^+}$ is defined symmetrically.*

1.5 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq^{\leqslant} M}$ Negative subtyping-induced equivalence

$$\frac{\Gamma \vdash N \leqslant M \quad \Gamma \vdash M \leqslant N}{\Gamma \vdash N \simeq^{\leqslant} M} \quad (\simeq^{\leqslant}-)$$

$\boxed{\Gamma \vdash P \simeq^{\leqslant} Q}$ Positive subtyping-induced equivalence

$$\frac{\Gamma \vdash P \geqslant Q \quad \Gamma \vdash Q \geqslant P}{\Gamma \vdash P \simeq^{\leqslant} Q} \quad (\simeq^{\leqslant}+)$$

$\boxed{\Gamma \vdash N \leqslant M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant \alpha^-} \quad (\textsc{Var}^{-\leqslant})$$

$$\frac{\Gamma \vdash P \simeq^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant \uparrow Q} \quad (\uparrow^{\leqslant})$$

$$\frac{\Gamma \vdash P \geqslant Q \quad \Gamma \vdash N \leqslant M}{\Gamma \vdash P \to N \leqslant Q \to M} \quad (\to^{\leqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash \sigma : \overrightarrow{\alpha^+} \quad \Gamma, \overrightarrow{\beta^+} \vdash [\sigma]N \leqslant M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\leqslant})$$

$\boxed{\Gamma \vdash P \geqslant Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant \alpha^+} \quad (\textsc{Var}^{+\geqslant})$$

$$\frac{\Gamma \vdash N \simeq^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant \downarrow M} \quad (\downarrow^{\geqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-} \quad \Gamma, \overrightarrow{\beta^-} \vdash [\sigma]P \geqslant Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\geqslant})$$

1.6 Declarative Equivalence

$\boxed{N \simeq^{D} M}$ Negative type equivalence

$$\frac{}{\alpha^- \simeq^{D} \alpha^-} \quad (\textsc{Var}^{-\simeq^{D}})$$

$$\frac{P \simeq^{D} Q}{\uparrow P \simeq^{D} \uparrow Q} \quad (\uparrow^{\simeq^{D}})$$

$$\frac{P \simeq^{D} Q \quad N \simeq^{D} M}{P \to N \simeq^{D} Q \to M} \quad (\to^{\simeq^{D}})$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq^{D} [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq^{D} \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\simeq^{D}})$$

$\boxed{P \simeq^{D} Q}$ Positive type equivalence

$$\frac{}{\alpha^+ \simeq^{D} \alpha^+} \quad (\textsc{Var}^{+\simeq^{D}})$$

$$\frac{N \simeq^{D} M}{\downarrow N \simeq^{D} \downarrow M} \quad (\downarrow^{\simeq^{D}})$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq^{D} [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq^{D} \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq^{D}})$$

$\boxed{P \simeq^{D} Q}$ Positive unification type equivalence

$\boxed{N \simeq^{D} M}$ Positive unification type equivalence

2 Algorithmic Type System

2.1 Grammar

In the algorithmic system, we extend the grammar of types by adding positive and negative *algorithmic variables* ($\widehat{\alpha}^+$, $\widehat{\beta}^+$, $\widehat{\gamma}^+$, etc. and $\widehat{\alpha}^-$, $\widehat{\beta}^-$, $\widehat{\gamma}^-$, etc.). They represent the unknown types, which will be inferred by the algorithm. This way, we add two base cases to the grammar of positive and negative types, and use highlight to denote that the type can potentially contain algorithmic variables.

Definition 10 (Algorithmic Types)**.**

$$
\begin{array}{lcll}
P,\ Q & ::= & & \textit{a positive algorithmic type (potentially with algorithmic variables)} \\
& | & \widehat{\alpha}^+ & \\
& | & \alpha^+ & \\
& | & \downarrow N & \\
& | & \exists \overrightarrow{\alpha^-}.\, P &
\end{array}
$$

$$
\begin{array}{lcll}
N,\ M & ::= & & \textit{a negative algorithmic type (potentially with algorithmic variables)} \\
& | & \widehat{\alpha}^- & \\
& | & \alpha^- & \\
& | & \uparrow P & \\
& | & P \to N & \\
& | & \forall \overrightarrow{\alpha^+}.\, N &
\end{array}
$$

2.2 Variable Algorithmization

In several places of our algorithm, in particular, during anti-unification, we turn a declarative type into the algorithmic one via replacing certain type variables with fresh algorithmic variables.

Definition 11 (Variable Algorithmization)**.** *Suppose that $\overrightarrow{\alpha^-}$ is a list of negative type variables and $\overrightarrow{\widehat{\alpha}^-}$ is a list of negative algorithmic variables of the same length. Then $\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}$ is a substitution-like procedure turning each $\alpha_i^- \in \overrightarrow{\alpha^-}$ into $\widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-}$.*

Conversely, we can turn algorithmic type variables into declarative type variables via *dealgorithmization*.

Definition 12 (Variable Dealgorithmization)**.** *Suppose that $\overrightarrow{\widehat{\alpha}^-}$ is a list of negative algorithmic variables and $\overrightarrow{\alpha^-}$ is a list of negative type variables of the same length. Then $\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}$ is a substitution-like procedure turning each $\widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-}$ into $\alpha_i^- \in \overrightarrow{\alpha^-}$.*

2.3 Contexts and Well-formedness

Definition 13 (Algorithmic Type Context)**.**
Algorithmic type context Ξ is represented by a set of algorithmic *type variables ($\widehat{\alpha}^+$, $\widehat{\alpha}^-$, $\widehat{\beta}^+$, $\dots$). The concatenation Ξ_1, Ξ_2 means the union of two contexts $\Xi_1 \cup \Xi_2$.*

$\Gamma; \Xi \vdash P$ and $\Gamma; \Xi \vdash N$ are used to denote that the algorithmic type is well-formed in the contexts Γ and Ξ, which means that each algorithmic variable of the type is contained in Ξ, and each free declarative type variable of the type is contained in Γ.

$\boxed{\Gamma; \Xi \vdash N}$ Negative algorithmic type well-formedness

$$\frac{\alpha^- \in \Gamma}{\Gamma; \Xi \vdash \alpha^-}\ \text{WFATNV}\textsc{ar}$$

$$\frac{\widehat{\alpha}^- \in \Xi}{\Gamma; \Xi \vdash \widehat{\alpha}^-}\ \text{WFATNUV}\textsc{ar}$$

$$\frac{\Gamma; \Xi \vdash P}{\Gamma; \Xi \vdash \uparrow P}\ \text{WFATS}\textsc{hift}\text{U}$$

$$\frac{\Gamma; \Xi \vdash P \quad \Gamma; \Xi \vdash N}{\Gamma; \Xi \vdash P \to N}\ \text{WFATA}\textsc{rrow}$$

$$\frac{\Gamma, \overrightarrow{\alpha^+}; \Xi \vdash N}{\Gamma; \Xi \vdash \forall \overrightarrow{\alpha^+}.\, N}\ \text{WFATF}\textsc{orall}$$

$\boxed{\Gamma;\Xi \vdash P}$ Positive algorithmic type well-formedness

$$\frac{\alpha^+ \in \Gamma}{\Gamma;\Xi \vdash \alpha^+}\ \text{WFATPVar}$$

$$\frac{\widehat{\alpha}^+ \in \Xi}{\Gamma;\Xi \vdash \widehat{\alpha}^+}\ \text{WFATPUVar}$$

$$\frac{\Gamma;\Xi \vdash N}{\Gamma;\Xi \vdash \downarrow N}\ \text{WFATShiftD}$$

$$\frac{\Gamma,\overrightarrow{\alpha^-};\Xi \vdash P}{\Gamma;\Xi \vdash \exists\overrightarrow{\alpha^-}.P}\ \text{WFATExists}$$

Algorithmic Type Context are used in the unification algorithm. In the subtyping algorithm, the context needs to remember additional information. In the subtyping context, each algorithmic variable is associated with a context it must be instantiated in (i.e. the context in which the type replacing the variable must be well-formed). This association is represented by *algorithmic subtyping context* Θ.

Definition 14 (Algorithmic Subtyping Context)**.**
Algorithmic Subtyping Context Θ is represented by a set of entries of form $\widehat{\alpha}^+\{\Gamma\}$ and $\widehat{\alpha}^-\{\Gamma\}$, where $\widehat{\alpha}^+$ and $\widehat{\alpha}^-$ are algorithmic variables, and Γ is a context in which they must be instantiated. We assume that no two entries associating the same variable appear in Θ.

dom (Θ) *denotes the set of variables appearing in Θ:* **dom** $(\Theta) = \{\widehat{\alpha}^\pm \mid \widehat{\alpha}^\pm\{\Gamma\} \in \Theta\}$.

If $\widehat{\alpha}^\pm\{\Gamma\} \in \Theta$, we denote Γ as $\Theta(\widehat{\alpha}^\pm)$.

2.4 Subsitutions

Substitution that operates on algorithmic type variables is denoted as $\widehat{\sigma}$. It is defined as a total function from algorithmic type variables to non-algorithmic types, preserving the polarity.

The signature $\Theta \vdash \widehat{\sigma} : \Xi$ means that $\Xi \subseteq \mathbf{dom}(\Theta)$ and $\widehat{\sigma}$ maps each algorithmic variable from Ξ to a type well-formed in $\Theta(\widehat{\alpha}^\pm)$; and for each variable not appearing in $\mathbf{dom}(\Theta)$, it acts as identity.

Definition 15 (Signature of Algorithmic Substitution)**.**

- $\Theta \vdash \widehat{\sigma} : \Xi$ *means that*
  1. *for any $\widehat{\alpha}^\pm \in \Xi$, there exists Γ such that $\widehat{\alpha}^\pm\{\Gamma\} \in \Theta$ and $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^\pm$;*
  2. *for any $\widehat{\alpha}^\pm \notin \Xi$, $[\widehat{\sigma}]\widehat{\alpha}^\pm = \widehat{\alpha}^\pm$.*
- $\Gamma \vdash \widehat{\sigma} : \Xi$ *means that*
  1. *for any $\widehat{\alpha}^\pm \in \Xi$, $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^\pm$;*
  2. *for any $\widehat{\alpha}^\pm \notin \Xi$, $[\widehat{\sigma}]\widehat{\alpha}^\pm = \widehat{\alpha}^\pm$.*

Anti-unification substitution is denoted as $\widehat{\tau}$ and $\widehat{\rho}$. In contrast to algorithmic substitution $\widehat{\sigma}$, it allows mapping algorithmic variables to *algorithmic* terms.

The pair of contexts Γ and Ξ, in which the results of the anti-unification substitutions are formed, is fixed for the whole substitution. This way, $\Gamma;\Xi_2 \vdash \widehat{\tau} : \Xi_1$ means that $\widehat{\tau}$ maps each negative algorithmic variable appearing in Ξ_1 to a term well-formed in Γ and Ξ_2.

Definition 16 (Signature of Anti-unification substitution)**.** $\Gamma;\Xi_2 \vdash \widehat{\tau} : \Xi_1$ *means that*

1. *for any $\widehat{\alpha}^- \in \Xi_1$, $\Gamma;\Xi_2 \vdash [\widehat{\tau}]\widehat{\alpha}^-$ and*
2. *for any $\widehat{\alpha}^- \notin \Xi_1$, $[\widehat{\tau}]\widehat{\alpha}^- = \widehat{\alpha}^-$.*

2.5 Normalization

2.5.1 Ordering

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \alpha^-}\ (\text{Var}^-_\in)$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \cdot}\ (\text{Var}^-_\notin)$$

$$\frac{\mathbf{ord}\, vars\, \mathbf{in}\, P = \overrightarrow{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in} \uparrow P = \overrightarrow{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\, vars\, \mathbf{in}\, P = \overrightarrow{\alpha}_1 \quad \mathbf{ord}\, vars\, \mathbf{in}\, N = \overrightarrow{\alpha}_2}{\mathbf{ord}\, vars\, \mathbf{in}\, P \to N = \overrightarrow{\alpha}_1, (\overrightarrow{\alpha}_2 \backslash \overrightarrow{\alpha}_1)} \quad (\to)$$

$$\frac{vars \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\, vars\, \mathbf{in}\, N = \overrightarrow{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}. N = \overrightarrow{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\, vars\, \mathbf{in}\, P = \overrightarrow{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^+ = \alpha^+} \quad (\text{VAR}^+_{\in})$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^+ = \cdot} \quad (\text{VAR}^+_{\notin})$$

$$\frac{\mathbf{ord}\, vars\, \mathbf{in}\, N = \overrightarrow{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in} \downarrow N = \overrightarrow{\alpha}} \quad (\downarrow)$$

$$\frac{vars \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\, vars\, \mathbf{in}\, P = \overrightarrow{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in}\, \exists \overrightarrow{\alpha^-}. P = \overrightarrow{\alpha}} \quad (\exists)$$

$\boxed{\mathbf{ord}\, vars\, \mathbf{in}\, N = \overrightarrow{\alpha}}$

$$\frac{}{\mathbf{ord}\, vars\, \mathbf{in}\, \widehat{\alpha}^- = \cdot} \quad (\text{UVAR}^-)$$

$\boxed{\mathbf{ord}\, vars\, \mathbf{in}\, P = \overrightarrow{\alpha}}$

$$\frac{}{\mathbf{ord}\, vars\, \mathbf{in}\, \widehat{\alpha}^+ = \cdot} \quad (\text{UVAR}^+)$$

2.5.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-} \quad (\text{VAR}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q} \quad (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M} \quad (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\, \overrightarrow{\alpha^+}\, \mathbf{in}\, N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall \overrightarrow{\alpha^+}. N) = \forall \overrightarrow{\alpha^+}'. N'} \quad (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+} \quad (\text{VAR}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M} \quad (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\, \overrightarrow{\alpha^-}\, \mathbf{in}\, P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists \overrightarrow{\alpha^-}. P) = \exists \overrightarrow{\alpha^-}'. P'} \quad (\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-} \quad (\text{UVAR}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+} \quad (\text{UVAR}^+)$$

We also define normalization of a substitution pointwise:

Definition 17 (Substitution Normalization)**.** *For a substitution σ, we define $\mathbf{nf}\,(\sigma)$ as a substitution that maps $\alpha^\pm$ into $\mathbf{nf}\,([\sigma]\alpha^\pm)$.*

2.6 Unification

$\boxed{\Gamma;\Theta \models N \overset{u}{\simeq} M \dashv UC}$ Negative unification

$$\frac{}{\Gamma;\Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\ (\text{VAR}^{-\overset{u}{\simeq}})$$

$$\frac{\Gamma;\Theta \models P \overset{u}{\simeq} Q \dashv UC}{\Gamma;\Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv UC}\ (\uparrow^{\overset{u}{\simeq}})$$

$$\frac{\Gamma;\Theta \models P \overset{u}{\simeq} Q \dashv UC_1 \quad \Gamma;\Theta \models N \overset{u}{\simeq} M \dashv UC_2}{\Gamma;\Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv UC_1 \mathbin{\&} UC_2}\ (\to^{\overset{u}{\simeq}})$$

$$\frac{\Gamma, \overrightarrow{\alpha^+};\Theta \models N \overset{u}{\simeq} M \dashv UC}{\Gamma;\Theta \models \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv UC}\ (\forall^{\overset{u}{\simeq}})$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma;\Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\widehat{\alpha}^- :\approx N)}\ (\text{UVAR}^{-\overset{u}{\simeq}})$$

$\boxed{\Gamma;\Theta \models P \overset{u}{\simeq} Q \dashv UC}$ Positive unification

$$\frac{}{\Gamma;\Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\ (\text{VAR}^{+\overset{u}{\simeq}})$$

$$\frac{\Gamma;\Theta \models N \overset{u}{\simeq} M \dashv UC}{\Gamma;\Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv UC}\ (\downarrow^{\overset{u}{\simeq}})$$

$$\frac{\Gamma, \overrightarrow{\alpha^-};\Theta \models P \overset{u}{\simeq} Q \dashv UC}{\Gamma;\Theta \models \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv UC}\ (\exists^{\overset{u}{\simeq}})$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma;\Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\widehat{\alpha}^+ :\approx P)}\ (\text{UVAR}^{+\overset{u}{\simeq}})$$

2.7 Algorithmic Subtyping

$\boxed{\Gamma;\ \Theta \models N \leqslant M \dashv SC}$ Negative subtyping

$$\frac{}{\Gamma;\ \Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot}\ (\text{VAR}^{-\leqslant})$$

$$\frac{\Gamma;\Theta \models \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv UC}{\Gamma;\ \Theta \models \uparrow P \leqslant \uparrow Q \dashv UC}\ (\uparrow^{\leqslant})$$

$$\frac{\Gamma;\ \Theta \models P \geqslant Q \dashv SC_1 \quad \Gamma;\ \Theta \models N \leqslant M \dashv SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC}{\Gamma;\ \Theta \models P \to N \leqslant Q \to M \dashv SC}\ (\to^{\leqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^+};\ \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma, \overrightarrow{\beta^+}\} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv SC}{\Gamma;\ \Theta \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv SC\backslash\overrightarrow{\widehat{\alpha}^+}}\ (\forall^{\leqslant})$$

$\boxed{\Gamma;\ \Theta \models P \geqslant Q \dashv SC}$ Positive supertyping

$$\frac{}{\Gamma;\ \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ (\text{VAR}^{+\geqslant})$$

$$\frac{\Gamma;\Theta \models \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv UC}{\Gamma;\ \Theta \models \downarrow N \geqslant \downarrow M \dashv UC}\ (\downarrow^{\geqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^-};\ \Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv SC}{\Gamma;\ \Theta \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv SC\backslash\overrightarrow{\widehat{\alpha}^-}}\ (\exists^{\geqslant})$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \mathbf{upgrade}\,\Gamma \vdash P\ \mathbf{to}\,\Delta = Q}{\Gamma;\ \Theta \models \widehat{\alpha}^+ \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant Q)}\ (\text{UVAR}^{\geqslant})$$

2.8 Constraints

Unification and subtyping algorithms are based on the constraint generation. The constraints are represented by set of constraint entries.

Definition 18 (Unification Constraint)**.**

- *Unification entry (denoted as e) is an expression of shape* $\widehat{\alpha}^+ :\approx P$ *or* $\widehat{\alpha}^- :\approx N$*;*
- *unification constraint (denoted as UC) is a set of unification constraint entries.*

However, in subtyping we need to consider more general kind of constraints. Specifically, subtyping constraint entries can restrict a variable not only to be equivalent to a certain type, but also to be a supertype of a positive type.

Definition 19 (Subtyping Constraint)**.**

- *Subtyping entry (denoted as e) is an expression of shape* $\widehat{\alpha}^+ :\geqslant P$*,* $\widehat{\alpha}^- :\approx N$*, or* $\widehat{\alpha}^+ :\approx P$*;*
- *subtyping constraint (denoted as SC) is a set of subtyping constraint entries.*

Definition 20 (Well-formed Constraint Entry)**.** *We say that a constraint entry is well-formed in a context* Γ *if the type it restricts the unification variable to is well-formed in* Γ*.*

- $\Gamma \vdash \widehat{\alpha}^+ :\geqslant P$ *iff* $\Gamma \vdash P$*;*
- $\Gamma \vdash \widehat{\alpha}^+ :\approx P$ *iff* $\Gamma \vdash P$*;*
- $\Gamma \vdash \widehat{\alpha}^- :\approx N$ *iff* $\Gamma \vdash N$*.*

Definition 21 (Well-formed Constraint)**.** *We say that a constraint is well-formed in a subtyping context* Θ *if all its entries are well-formed in the corresponding elements of* Θ*. More formally,* $\Theta \vdash UC$ *iff for every* $e \in UC$*, such that e restricts* $\widehat{\alpha}^\pm$*, there exists* $\widehat{\alpha}^\pm\{\Gamma\} \in \Theta$ *and* $\Gamma \vdash e$*.*

Next, define the least upper bound for two subtyping constraints. First, we define the least upper bound for entries, and then extend it to the set of entries.

Definition 22 (Matching Entries)**.** *We call two unification constraint entries or two subtyping constraint entries matching if they are restricting the same unification variable.*

Two matching entries formed in the same context Γ can be merged in the following way:

Definition 23 (Merge of Matching Constraint Entries)**.**
$\boxed{\Gamma \vdash e_1 \ \&\ e_2 = e_3}$ *Subtyping Constraint Entry Merge*

$$\frac{\Gamma \vDash P_1 \vee P_2 = Q}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P_1) \ \&\ (\widehat{\alpha}^+ :\geqslant P_2) = (\widehat{\alpha}^+ :\geqslant Q)} \quad (\geqslant \&^+ \geqslant)$$

$$\frac{\Gamma; \cdot \vDash P \geqslant Q \dashv \cdot}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \ \&\ (\widehat{\alpha}^+ :\geqslant Q) = (\widehat{\alpha}^+ :\approx P)} \quad (\simeq \&^+ \geqslant)$$

$$\frac{\Gamma; \cdot \vDash Q \geqslant P \dashv \cdot}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P) \ \&\ (\widehat{\alpha}^+ :\approx Q) = (\widehat{\alpha}^+ :\approx Q)} \quad (\geqslant \&^+ \simeq)$$

$$\frac{\mathbf{nf}\,(P) = \mathbf{nf}\,(P')}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \ \&\ (\widehat{\alpha}^+ :\approx P') = (\widehat{\alpha}^+ :\approx P)} \quad (\simeq \&^+ \simeq)$$

$$\frac{\mathbf{nf}\,(N) = \mathbf{nf}\,(N')}{\Gamma \vdash (\widehat{\alpha}^- :\approx N_1) \ \&\ (\widehat{\alpha}^- :\approx N') = (\widehat{\alpha}^- :\approx N)} \quad (\simeq \&^- \simeq)$$

Unification constraint entries are a special case of subtyping constraint entries. They are merged using the same algorithm (definition 23). Notice that the merge of two matching unification constraint entries is a unification constraint entry.

Lemma 1 (Merge of Matching Unification Constraint Entries is well-defined)**.** *Suppose that* $\Gamma \vdash e_1$ *and* $\Gamma \vdash e_2$ *are unification constraint entries. Then the merge of* e_1 *and* e_2 $\Gamma \vdash e_1 \& e_2 = e$ *according to definition 23, is a unification constraint entry.*

Proof. Since e_1 and e_2 are matching unification constraint entries, they have the shape $(\widehat{\alpha}^+ :\approx P_1, \widehat{\alpha}^+ :\approx P_2)$ or $(\widehat{\alpha}^- :\approx N_1, \widehat{\alpha}^- :\approx N_2)$. Then the merge of e_1 and e_2 can only be defined by Rule $(\simeq \&^+ \simeq)$ or Rule $(\simeq \&^- \simeq)$. In both cases the result, if it exists, is a unification constraint entry: in the first case, the result has shape $\widehat{\alpha}^+ :\approx P_1$, in the second case, the result has shape $\widehat{\alpha}^- :\approx N_1$. $\square$

Definition 24 (Merge of Subtyping Constraints). *Suppose that $\Theta \vdash SC_1$ and $\Theta \vdash SC_2$. Then $\Theta \vdash SC_1 \& SC_2 = SC$ defines a set such that $e \in SC$ iff either*

- *$e \in SC_1$ and there is no matching $e' \in SC_2$; or*
- *$e \in SC_2$ and there is no matching $e' \in SC_1$; or*
- *$\Theta(\widehat{\alpha}^{\pm}) \vdash e_1 \& e_2 = e$ for some $e_1 \in SC_1$ and $e_2 \in SC_2$ such that e_1 matches with e_2 restricting variable $\widehat{\alpha}^{\pm}$.*

Unification constraints can be considered as a special case of subtyping constraints, and the merge of unification constraints is defined as the merge of subtyping constraints. Then it is easy to see that the merge of two unification constraints is a unification constraint.

Lemma 2 (Merge of Unification Constraints is well-defined). *Suppose that $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$ are unification constraints. Then the merge of UC_1 and UC_2 $\Theta \vdash UC_1 \& UC_2 = UC$ according to definition 24, is a unification constraint.*

Proof. UC consists of unmatched entries of UC_1 and UC_2, which are *unification* constraint entries by assumption, and merge of matching entries, which also are *unification* constraint entries by **??**. □

2.9 Constraint Satisfaction

$\boxed{\Gamma \vdash P : e}$ Positive type satisfies with the subtyping constraint entry

$$\frac{\Gamma \vdash P \geqslant Q}{\Gamma \vdash P : (\widehat{\alpha}^{+} :\geqslant Q)} \quad \text{SATSCESUP}$$

$$\frac{\Gamma \vdash P \simeq^{\leqslant} Q}{\Gamma \vdash P : (\widehat{\alpha}^{+} :\approx Q)} \quad \text{SATSCEPEQ}$$

$\boxed{\Gamma \vdash N : e}$ Negative type satisfies with the subtyping constraint entry

$$\frac{\Gamma \vdash N \simeq^{\leqslant} M}{\Gamma \vdash N : (\widehat{\alpha}^{-} :\approx M)} \quad \text{SATSCENEQ}$$

2.10 Least Upper Bound

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^{+} \vee \alpha^{+} = \alpha^{+}} \quad (\text{VAR}^{\vee})$$

$$\frac{\Gamma, \cdot \vDash \mathbf{nf}\,(\downarrow N) \overset{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^{-}}.[\overrightarrow{\alpha^{-}}/\Xi]\, P} \quad (\downarrow^{\vee})$$

$$\frac{\Gamma, \overrightarrow{\alpha^{-}}, \overrightarrow{\beta^{-}} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^{-}}.P_1 \vee \exists \overrightarrow{\beta^{-}}.P_2 = Q} \quad (\exists^{\vee})$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

$$\frac{\Gamma = \Delta, \overrightarrow{\alpha^{\pm}} \quad \overrightarrow{\beta^{\pm}}\ \textbf{is fresh} \quad \overrightarrow{\gamma^{\pm}}\ \textbf{is fresh} \quad \Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vDash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P = Q}{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q} \quad (\text{UPG})$$

2.11 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^{+} \overset{a}{\simeq} \alpha^{+} \dashv (\cdot, \alpha^{+}, \cdot, \cdot)} \quad (\text{VAR}^{+\overset{a}{\simeq}})$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \dashv (\Xi, \downarrow M, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\downarrow^{\overset{a}{\simeq}})$$

$$\frac{\overrightarrow{\alpha^{-}} \cap \Gamma = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^{-}}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^{-}}.P_2 \dashv (\Xi, \exists \overrightarrow{\alpha^{-}}.\, Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\exists^{\overset{a}{\simeq}})$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\cdot, \alpha^-, \cdot, \cdot)} \quad (\textsc{Var}^{-\overset{a}{\simeq}})$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models {\uparrow}P_1 \overset{a}{\simeq} {\uparrow}P_2 \dashv (\Xi, {\uparrow}Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad ({\uparrow}^{\overset{a}{\simeq}})$$

$$\frac{\overrightarrow{\alpha^+} \cap \Gamma = \varnothing \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \forall\overrightarrow{\alpha^+}.N_1 \overset{a}{\simeq} \forall\overrightarrow{\alpha^+}.N_2 \dashv (\Xi, \forall\overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\forall^{\overset{a}{\simeq}})$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}_1', \widehat{\tau}_2')}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}_1', \widehat{\tau}_2 \cup \widehat{\tau}_2')} \quad (\to^{\overset{a}{\simeq}})$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad (\mathrm{AU}^-)$$

3 Declarative Typing

3.1 Grammar

$$\begin{array}{llll}
v,\ w & ::= & & \text{value terms} \\
 & | & x & \\
 & | & \{c\} & \\
 & | & (v : P) & \\
 & & & \\
c,\ d & ::= & & \text{computation terms} \\
 & | & (c : N) & \\
 & | & \lambda x : P.c & \\
 & | & \Lambda\alpha^+.c & \\
 & | & \mathbf{return}\, v & \\
 & | & \mathbf{let}\, x = v; c & \\
 & | & \mathbf{let}\, x : P = v(\overrightarrow{v}); c & \\
 & | & \mathbf{let}\, x = v(\overrightarrow{v}); c & \\
 & | & \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c &
\end{array}$$

3.2 Declarative Type Inference

$\boxed{\Gamma; \Phi \vdash v \colon P}$ Positive type inference

$$\frac{x : P \in \Phi}{\Gamma; \Phi \vdash x \colon P} \quad \textsc{DTVar}$$

$$\frac{\Gamma; \Phi \vdash c \colon N}{\Gamma; \Phi \vdash \{c\} \colon {\downarrow}N} \quad \textsc{DTThunk}$$

$$\frac{\Gamma \vdash Q \quad \Gamma; \Phi \vdash v \colon P \quad \Gamma \vdash Q \geqslant P}{\Gamma; \Phi \vdash (v : Q) \colon Q} \quad \textsc{DTPAnnot}$$

$$\frac{\Gamma; \Phi \vdash v \colon P \quad \Gamma \vdash P \simeq^{\leqslant} P'}{\Gamma; \Phi \vdash v \colon P'} \quad \textsc{DTPEquiv}$$

$\boxed{\Gamma; \Phi \vdash c \colon N}$ Negative type inference

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \vdash c \colon N}{\Gamma; \Phi \vdash \lambda x : P.c \colon P \to N} \quad \textsc{DTtLam}$$

$$\frac{\Gamma, \alpha^+; \Phi \vdash c \colon N}{\Gamma; \Phi \vdash \Lambda\alpha^+.c \colon \forall\alpha^+.N} \quad \textsc{DTTLam}$$

$$\frac{\Gamma; \Phi \vdash v \colon P}{\Gamma; \Phi \vdash \textbf{return}\, v \colon \uparrow P}\ \textsc{DTReturn}$$

$$\frac{\Gamma; \Phi \vdash v \colon P \quad \Gamma; \Phi, x : P \vdash c \colon N}{\Gamma; \Phi \vdash \textbf{let}\, x = v; c \colon N}\ \textsc{DTVarLet}$$

$$\frac{\Gamma; \Phi \vdash v \colon \downarrow M \quad \Gamma; \Phi \vdash M \bullet \overrightarrow{v} \Rrightarrow \uparrow Q \text{ unique} \quad \Gamma; \Phi, x : Q \vdash c \colon N}{\Gamma; \Phi \vdash \textbf{let}\, x = v(\overrightarrow{v}); c \colon N}\ \textsc{DTAppLet}$$

$$\frac{\begin{array}{l}\Gamma \vdash P \quad \Gamma; \Phi \vdash v \colon \downarrow M \\ \Gamma; \Phi \vdash M \bullet \overrightarrow{v} \Rrightarrow \uparrow Q \quad \Gamma \vdash \uparrow Q \leqslant \uparrow P \quad \Gamma; \Phi, x : P \vdash c \colon N\end{array}}{\Gamma; \Phi \vdash \textbf{let}\, x : P = v(\overrightarrow{v}); c \colon N}\ \textsc{DTAppLetAnn}$$

$$\frac{\begin{array}{l}\Gamma; \Phi \vdash v \colon \exists \overrightarrow{\alpha^-}.P \quad \textbf{nf}\,(\exists \overrightarrow{\alpha^-}.P) = \exists \overrightarrow{\alpha^-}.P \\ \Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \vdash c \colon N \quad \Gamma \vdash N\end{array}}{\Gamma; \Phi \vdash \textbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c \colon N}\ \textsc{DTUnpack}$$

$$\frac{\Gamma \vdash M \quad \Gamma; \Phi \vdash c \colon N \quad \Gamma \vdash N \simeq^{\leqslant} M}{\Gamma; \Phi \vdash (c : M) \colon M}\ \textsc{DTNAnnot}$$

$$\frac{\Gamma; \Phi \vdash c \colon N \quad \Gamma \vdash N \simeq^{\leqslant} N'}{\Gamma; \Phi \vdash c \colon N'}\ \textsc{DTNEquiv}$$

$\boxed{\Gamma; \Phi \vdash N \bullet \overrightarrow{v} \Rrightarrow M}$ Application type inference

$$\frac{\Gamma \vdash N \simeq^{\leqslant} N'}{\Gamma; \Phi \vdash N \bullet \cdot \Rrightarrow N'}\ \textsc{DTEmptyApp}$$

$$\frac{\Gamma; \Phi \vdash v \colon P \quad \Gamma \vdash Q \geqslant P \quad \Gamma; \Phi \vdash N \bullet \overrightarrow{v} \Rrightarrow M}{\Gamma; \Phi \vdash Q \to N \bullet v, \overrightarrow{v} \Rrightarrow M}\ \textsc{DTArrowApp}$$

$$\frac{\begin{array}{l}\Gamma \vdash \sigma : \overrightarrow{\alpha^+} \quad \Gamma; \Phi \vdash [\sigma]N \bullet \overrightarrow{v} \Rrightarrow M \\ \overrightarrow{v} \neq \cdot \quad \overrightarrow{\alpha^+} \neq \cdot\end{array}}{\Gamma; \Phi \vdash \forall \overrightarrow{\alpha^+}.N \bullet \overrightarrow{v} \Rrightarrow M}\ \textsc{DTForallApp}$$

4 Algorithmic Typing

4.1 Algorithmic Type Inference

$\boxed{\Gamma; \Phi \vDash v \colon P}$ Positive type inference

$$\frac{x : P \in \Phi}{\Gamma; \Phi \vDash x \colon \textbf{nf}\,(P)}\ \textsc{ATVar}$$

$$\frac{\Gamma; \Phi \vDash c \colon N}{\Gamma; \Phi \vDash \{c\} \colon \downarrow N}\ \textsc{ATThunk}$$

$$\frac{\Gamma \vdash Q \quad \Gamma; \Phi \vDash v \colon P \quad \Gamma; \cdot \vDash Q \geqslant P \dashv \cdot}{\Gamma; \Phi \vDash (v : Q) \colon \textbf{nf}\,(Q)}\ \textsc{ATPAnnot}$$

$\boxed{\Gamma; \Phi \vDash c \colon N}$ Negative type inference

$$\frac{\Gamma \vdash M \quad \Gamma; \Phi \vDash c \colon N \quad \Gamma; \cdot \vDash N \leqslant M \dashv \cdot}{\Gamma; \Phi \vDash (c : M) \colon \textbf{nf}\,(M)}\ \textsc{ATNAnnot}$$

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \vDash c \colon N}{\Gamma; \Phi \vDash \lambda x : P.c \colon \textbf{nf}\,(P \to N)}\ \textsc{ATtLam}$$

$$\frac{\Gamma, \alpha^+; \Phi \vDash c \colon N}{\Gamma; \Phi \vDash \Lambda \alpha^+.c \colon \textbf{nf}\,(\forall \alpha^+.N)}\ \textsc{ATTLam}$$

$$\frac{\Gamma; \Phi \vDash v \colon P}{\Gamma; \Phi \vDash \textbf{return}\, v \colon \uparrow P}\ \textsc{ATReturn}$$

$$\frac{\Gamma; \Phi \vDash v \colon P \quad \Gamma; \Phi, x : P \vDash c \colon N}{\Gamma; \Phi \vDash \textbf{let}\, x = v; c \colon N}\ \textsc{ATVarLet}$$

$$\frac{\begin{array}{l}\Gamma \vdash P \quad \Gamma; \Phi \vDash v \colon {\downarrow}M \\ \Gamma; \Phi; \cdot \vDash M \bullet \vec{v} \Rrightarrow {\uparrow}Q \dashv \Theta; SC_1 \quad \Gamma;\ \Theta \vDash {\uparrow}Q \leqslant {\uparrow}P \dashv SC_2 \\ \Theta \vdash SC_1 \& SC_2 = SC \quad \Gamma; \Phi, x : P \vDash c \colon N\end{array}}{\Gamma; \Phi \vDash \mathbf{let}\, x : P = v(\vec{v}); c \colon N} \text{ ATAPPLETANN}$$

$$\frac{\begin{array}{l}\Gamma; \Phi \vDash v \colon {\downarrow}M \quad \Gamma; \Phi; \cdot \vDash M \bullet \vec{v} \Rrightarrow {\uparrow}Q \dashv \Theta; SC \\ \mathbf{uv}\ Q = \mathbf{dom}\,(SC) \quad SC\ \mathbf{singular\ with}\ \widehat{\sigma} \\ \Gamma; \Phi, x : [\widehat{\sigma}]\, Q \vDash c \colon N\end{array}}{\Gamma; \Phi \vDash \mathbf{let}\, x = v(\vec{v}); c \colon N} \text{ ATAPPLET}$$

$$\frac{\Gamma; \Phi \vDash v \colon \exists \overrightarrow{\alpha^-}.P \quad \Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \vDash c \colon N \quad \Gamma \vdash N}{\Gamma; \Phi \vDash \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c \colon N} \text{ ATUNPACK}$$

$\boxed{\Gamma; \Phi; \Theta_1 \vDash N \bullet \vec{v} \Rrightarrow M \dashv \Theta_2; SC}$ Application type inference

$$\frac{}{\Gamma; \Phi; \Theta \vDash N \bullet \cdot \Rrightarrow \mathbf{nf}\,(N) \dashv \Theta; \cdot} \text{ ATEMPTYAPP}$$

$$\frac{\begin{array}{l}\Gamma; \Phi \vDash v \colon P \quad \Gamma;\ \Theta \vDash Q \geqslant P \dashv SC_1 \quad \Gamma; \Phi; \Theta \vDash N \bullet \vec{v} \Rrightarrow M \dashv \Theta'; SC_2 \\ \Theta \vdash SC_1 \& SC_2 = SC\end{array}}{\Gamma; \Phi; \Theta \vDash Q \to N \bullet v, \vec{v} \Rrightarrow M \dashv \Theta'; SC} \text{ ATARROWAPP}$$

$$\frac{\begin{array}{l}\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \vec{v} \Rrightarrow M \dashv \Theta'; SC \\ \vec{v} \neq \cdot \quad \overrightarrow{\alpha^+} \neq \cdot\end{array}}{\Gamma; \Phi; \Theta \vDash \forall \overrightarrow{\alpha^+}.N \bullet \vec{v} \Rrightarrow M \dashv \Theta'; SC|_{\mathbf{uv}\,(N) \cup \mathbf{uv}\,(M)}} \text{ ATFORALLAPP}$$

4.2 Singularity

$\boxed{e_1\ \mathbf{singular\ with}\ P}$ Positive Subtyping Constraint Entry Is Singular

$$\frac{}{\widehat{\alpha}^+ :\approx P\ \mathbf{singular\ with\ nf}\,(P)} \text{ SINGPEQ}$$

$$\frac{}{\widehat{\alpha}^+ :\geqslant \exists \overrightarrow{\alpha^-}.\alpha^+\ \mathbf{singular\ with}\ \alpha^+} \text{ SINGSUPVAR}$$

$$\frac{\mathbf{nf}\,(N) = \alpha_i^-}{\widehat{\alpha}^+ :\geqslant \exists \overrightarrow{\alpha^-}.{\downarrow}N\ \mathbf{singular\ with}\ \exists \alpha^-.{\downarrow}\alpha^-} \text{ SINGSUPSHIFT}$$

$\boxed{e_1\ \mathbf{singular\ with}\ N}$ Negative Subtyping Constraint Entry Is Singular

$$\frac{}{\widehat{\alpha}^- :\approx N\ \mathbf{singular\ with\ nf}\,(N)} \text{ SINGNEQ}$$

$\boxed{SC\ \mathbf{singular\ with}\ \widehat{\sigma}}$ Subtyping Constraint Is Singular

5 Properties of the Declarative Type System

5.1 Type Well-formedness

Lemma 3 (Soundness of type well-formedness)**.**

+ *if* $\Gamma \vdash P$ *then* $\mathbf{fv}\,(P) \subseteq \Gamma$*;*

− *if* $\Gamma \vdash N$ *then* $\mathbf{fv}\,(N) \subseteq \Gamma$.

Proof. The proof is done by a simple structural induction on $\Gamma \vdash P$ and mutually, $\Gamma \vdash N$.

Case 1. $\Gamma \vdash \alpha^{\pm}$ means by inversion that $\alpha^{\pm} \in \Gamma$, that is, $\alpha^{\pm} = \mathbf{fv}\,(\alpha^{\pm}) \subseteq \Gamma$.

Case 2. $\Gamma \vdash Q \to M$ means by inversion that $\Gamma \vdash Q$ and $\Gamma \vdash M$. Then by the induction hypothesis, $\mathbf{fv}\,(Q) \subseteq \Gamma$ and $\mathbf{fv}\,(M) \subseteq \Gamma$, and hence, $\mathbf{fv}\,(Q \to M) = \mathbf{fv}\,(Q) \cup \mathbf{fv}\,(M) \subseteq \Gamma$.

Case 3. the cases when $P = {\downarrow}N'$ or $N = {\uparrow}P'$ are proven analogously.

Case 4. $\Gamma \vdash \forall\overrightarrow{\alpha^+}.M$ means by inversion that $\Gamma, \overrightarrow{\alpha^+} \vdash M$. Then by the induction hypothesis, $\mathbf{fv}\,(M) \subseteq \Gamma, \overrightarrow{\alpha^+}$, and hence, $\mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M)\backslash\overrightarrow{\alpha^+} \subseteq \Gamma, \overrightarrow{\alpha^+}\backslash\overrightarrow{\alpha^+} = \Gamma$.

Case 5. The case $P = \exists\overrightarrow{\alpha^-}.Q$ is proven analogously.

$\square$

Lemma 4 (Completeness of type well-formedness)**.** *In the well-formedness judgment, only used variables matter:*

+ *if* $\Gamma_1 \cap \mathbf{fv}\,P = \Gamma_2 \cap \mathbf{fv}\,P$ *then* $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$,

− *if* $\Gamma_1 \cap \mathbf{fv}\,N = \Gamma_2 \cap \mathbf{fv}\,N$ *then* $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$.

Proof. By simple mutual induction on P and N. $\square$

Corollary 1 (Context Strengthening)**.**

+ *If* $\Gamma \vdash P$ *then* $\mathbf{fv}\,(P) \vdash P$*;*

− *If* $\Gamma \vdash N$ *then* $\mathbf{fv}\,(N) \vdash N$*.*

Proof. It follows from section 5.3 and lemma 3.

+ By lemma 3, $\mathbf{fv}\,(P) \subseteq \Gamma$, and hence, $\Gamma \cap \mathbf{fv}\,P = \mathbf{fv}\,P$, which makes section 5.3 applicable fore contexts Γ and $\mathbf{fv}\,(P)$.

− The negative case is proven analogously.

Corollary 2 (Well-formedness Context Weakening)**.** *Suppose that* $\Gamma_1 \subseteq \Gamma_2$*, then*

+ *if* $\Gamma_1 \vdash P$ *then* $\Gamma_2 \vdash P$*,*

− *if* $\Gamma_1 \vdash N$ *then* $\Gamma_2 \vdash N$*.*

Proof. By lemma 3, $\Gamma_1 \vdash P$ implies $\mathbf{fv}\,(P) \subseteq \Gamma_1$, which means that $\mathbf{fv}\,(P) \subseteq \Gamma_2$, and thus, $\mathbf{fv}\,(P) = \mathbf{fv}\,(P) \cap \Gamma_1 = \mathbf{fv}\,(P) \cap \Gamma_2$. Then by section 5.3, $\Gamma_2 \vdash P$. The negative case is symmetric. $\square$

Corollary 3. *Suppose that all the types below are well-formed in* Γ *and* $\Gamma' \subseteq \Gamma$*. Then*

+ $\Gamma \vdash P \simeq^{\leqslant} Q$ *implies* $\Gamma' \vdash P \iff \Gamma' \vdash Q$

− $\Gamma \vdash N \simeq^{\leqslant} M$ *implies* $\Gamma' \vdash N \iff \Gamma' \vdash M$

Proof. From section 5.3 and corollary 6. $\square$

Lemma 5 (Well-formedness agrees with substitution)**.** *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

+ $\Gamma, \Gamma_1 \vdash P$ *implies* $\Gamma, \Gamma_2 \vdash [\sigma]P$*, and*

− $\Gamma, \Gamma_1 \vdash N$ *implies* $\Gamma, \Gamma_2 \vdash [\sigma]N$*.*

Proof. We prove it by induction on $\Gamma, \Gamma_1 \vdash P$ and mutually, on $\Gamma, \Gamma_1 \vdash N$. Let us consider the last rule used in the derivation.

Case 1. Rule WFTPVar, i.e. P is α^+.
By inversion, $\alpha^+ \in \Gamma, \Gamma_1$, then

- if $\alpha^+ \in \Gamma_1$ then $\Gamma_2 \vdash [\sigma]\alpha^+$, and by weakening (corollary 2), $\Gamma, \Gamma_2 \vdash [\sigma]\alpha^+$;
- if $\alpha^+ \in \Gamma\backslash\Gamma_1$ then $[\sigma]\alpha^+ = \alpha^+$, and by Rule WFTPVar, $\Gamma, \Gamma_2 \vdash \alpha^+$.

Case 2. Rule WFTShiftU, i.e. P is $\downarrow N$.
Then $\Gamma, \Gamma_1 \vdash \downarrow N$ means $\Gamma, \Gamma_1 \vdash N$ by inversion, and by the induction hypothesis, $\Gamma, \Gamma_2 \vdash [\sigma]N$. Then by Rule WFTShiftU, $\Gamma, \Gamma_2 \vdash \downarrow[\sigma]N$, which by definition of substitution is rewritten as $\Gamma, \Gamma_2 \vdash [\sigma]\downarrow N$.

Case 3. Rule WFTExists, i.e. P is $\exists\overrightarrow{\alpha^-}.Q$.
Then $\Gamma, \Gamma_1 \vdash \exists\overrightarrow{\alpha^-}.Q$ means $\Gamma, \overrightarrow{\alpha^-}, \Gamma_1 \vdash Q$ by inversion, and by the induction hypothesis, $\Gamma, \overrightarrow{\alpha^-}, \Gamma_2 \vdash [\sigma]Q$. Then by Rule WFTExists, $\Gamma, \overrightarrow{\alpha^-}, \Gamma_2 \vdash \exists\overrightarrow{\alpha^-}.[\sigma]Q$, which by definition of substitution is rewritten as $\Gamma, \Gamma_2 \vdash [\sigma]\exists\overrightarrow{\alpha^-}.Q$.

Case 4. The negative cases are proved symmetrically.

$\square$

5.2 Substitution

Lemma 6 (Substitution strengthening)**.** *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that σ is a substitution, P and N are types. Then*

$+$ $[\sigma]P = [\sigma|_{\mathbf{fv}\,P}]P$,

$-$ $[\sigma]N = [\sigma|_{\mathbf{fv}\,N}]N$

Proof. First, we strengthen the statement by saying that one can restrict the substitution to an arbitrary superset of the free variables of the substitution subject:

$+$ $[\sigma]P = [\sigma|_{vars}]P$, for any $vars \supseteq \mathbf{fv}\,P$, and

$-$ $[\sigma]N = [\sigma|_{vars}]N$, for any $vars \supseteq \mathbf{fv}\,N$.

Then the proof is a straightforward induction on P and mutually, on N. For the base cases:

Case 1. $N = \alpha^-$
Then $[\sigma]\alpha^- = \sigma|_{vars}(\alpha^-)$ by definition, since $\alpha^- \in \mathbf{fv}\,\alpha^- \subseteq vars$.

Case 2. $N = P \to M$
Then $[\sigma](P \to M) = [\sigma]P \to [\sigma]M$ by definition. Since $\mathbf{fv}\,P \subseteq \mathbf{fv}\,(P \to M) \subseteq vars$, the induction hypothesis is applicable to $[\sigma]P$: $[\sigma]P = [\sigma|_{vars}]P$. Analogously, and $[\sigma]M = [\sigma|_{vars}]M$. Then $[\sigma](P \to M) = [\sigma|_{vars}]P \to [\sigma|_{vars}]M = [\sigma|_{vars}](P \to M)$.

Case 3. $N = \uparrow P$ is proved analogously to the previous case.

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$ (where $\overrightarrow{\alpha^+}$ is not empty)
Then $[\sigma]\forall\overrightarrow{\alpha^+}.M = \forall\overrightarrow{\alpha^+}.[\sigma]M$ by definition. Let us assume $\overrightarrow{\alpha^+}$ are fresh variables, it means that $\sigma(\alpha^\pm) = \alpha^\pm$ for any $\alpha^\pm \in \overrightarrow{\alpha^+}$, and thus, $[\sigma|_{vars}] = [\sigma|_{(vars \cup \overrightarrow{\alpha^+})}]$ immediately from the definition.

Since $vars \subseteq \mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M) = \mathbf{fv}\,M \backslash \overrightarrow{\alpha^+}$, $vars \cup \overrightarrow{\alpha^+} \subseteq \mathbf{fv}\,(M)$. Then by the induction hypothesis, $[\sigma]M = [\sigma|_{(vars \cup \overrightarrow{\alpha^+})}]M$. Finally, $[\sigma]\forall\overrightarrow{\alpha^+}.M = \forall\overrightarrow{\alpha^+}.[\sigma|_{(vars \cup \overrightarrow{\alpha^+})}]M = \forall\overrightarrow{\alpha^+}.[\sigma|_{vars}]M = [\sigma|_{vars}]\forall\overrightarrow{\alpha^+}.M$.

Case 5. The positive cases are proves symmetrically.

$\square$

Lemma 7 (Signature of a restricted substitution)**.** *If $\Gamma_2 \vdash \sigma : \Gamma_1$ then $\Gamma_2 \vdash \sigma|_{vars} : \Gamma_1 \cap vars$.*

Proof. Let us take an arbitrary $\alpha^\pm \in \Gamma_1 \cap vars$. Since $\alpha^\pm \in \Gamma_1$, $\Gamma_2 \vdash [\sigma]\alpha^\pm$ by the signature of σ.

Let us take an arbitrary $\alpha^\pm \notin \Gamma_1 \cap vars$. If $\alpha^\pm \notin vars$ then $[\sigma|_{vars}]\alpha^\pm = \alpha^\pm$ by definition of restriction. If $\alpha^\pm \in vars \backslash \Gamma_1$ then $[\sigma|_{vars}]\alpha^\pm = [\sigma]\alpha^\pm$ by definition and $[\sigma]\alpha^\pm = \alpha^\pm$ by the signature of σ. $\square$

Lemma 8. *Suppose that σ is a substitution with signature $\Gamma_2 \vdash \sigma : \Gamma_1$. Then if vars is disjoint from Γ_1, then $\sigma|_{vars} = \mathsf{id}$.*

Proof. Let us take an arbitrary $\alpha^\pm$. If $\alpha^\pm \notin vars$ then $[\sigma|_{vars}]\alpha^\pm = \alpha^\pm$ by definition.

If $\alpha^\pm \in vars$ then $\alpha^\pm \notin \Gamma_1$ by assumption. Then $[\sigma|_{vars}]\alpha^\pm = [\sigma]\alpha^\pm$ by definition of restricted substitution, and since $\Gamma_2 \vdash \sigma : \Gamma_1$, we have $[\sigma]\alpha^\pm = \alpha^\pm$. $\square$

Corollary 4 (Application of a disjoint substitution)**.** *Suppose that σ is a substitution with signature $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

$+$ *if $\Gamma_1 \cap \mathbf{fv}\,(Q) = \varnothing$ then $[\sigma]Q = Q$;*

$-$ *if $\Gamma_1 \cap \mathbf{fv}\,(N) = \varnothing$ then $[\sigma]N = N$.*

Lemma 9 (Substitution range weakening)**.** *Suppose that $\Gamma_2 \subseteq \Gamma_2'$ are contexts and σ is a substitution. Then $\Gamma_2 \vdash \sigma : \Gamma_1$ implies $\Gamma_2' \vdash \sigma : \Gamma_1$.*

Proof. For any $\alpha^\pm \in \Gamma_1$, $\Gamma_2 \vdash \sigma : \Gamma_1$ gives us $\Gamma_2 \vdash [\sigma]\alpha^\pm$, which can be weakened to $\Gamma_2' \vdash [\sigma]\alpha^\pm$ by corollary 2. This way, $\Gamma_2' \vdash \sigma : \Gamma_1$. $\square$

Lemma 10. *Suppose that $\Gamma' \subseteq \Gamma$, σ_1 and σ_2 are substitutions of signature $\Gamma \vdash \sigma_i : \Gamma'$. Then*

$+$ *for a type $\Gamma \vdash P$, if $\Gamma \vdash [\sigma_1]P \simeq^{\leqslant} [\sigma_2]P$ then $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \mathbf{fv}\,P \cap \Gamma'$;*

$-$ *for a type $\Gamma \vdash N$, if $\Gamma \vdash [\sigma_1]N \simeq^{\leqslant} [\sigma_2]N$ then $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \mathbf{fv}\,N \cap \Gamma'$.*

Proof. Let us make an additional assumption that σ_1, σ_2, and the mentioned types are normalized. If they are not, we normalize them first.

Notice that the normalization preserves the set of free variables (lemma 30), well-formedness (corollary 16), and equivalence (lemma 46), and distributes over substitution (lemma 32). This way, the assumed and desired properties are equivalent to their normalized versions.

We prove it by induction on the structure of P and mutually, N. Let us consider the shape of this type.

Case 1. $P = \alpha^+ \in \Gamma'$. Then $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \mathbf{fv}\, P \cap \Gamma'$ means $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \alpha^+$, i.e. $\Gamma \vdash [\sigma_1]\alpha^+ \simeq^{\leqslant} [\sigma_2]\alpha^+$, which holds by assumption.

Case 2. $P = \alpha^+ \in \Gamma \backslash \Gamma'$. Then $\mathbf{fv}\, P \cap \Gamma' = \varnothing$, so $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \mathbf{fv}\, P \cap \Gamma'$ holds vacuously.

Case 3. $P = \downarrow N$. Then the induction hypothesis is applicable to type N:

1. N is normalized,
2. $\Gamma \vdash N$ by inversion of $\Gamma \vdash \downarrow N$,
3. $\Gamma \vdash [\sigma_1]N \simeq^{\leqslant} [\sigma_2]N$ holds by inversion of $\Gamma \vdash [\sigma_1]\downarrow N \simeq^{\leqslant} [\sigma_2]\downarrow N$, i.e. $\Gamma \vdash \downarrow[\sigma_1]N \simeq^{\leqslant} \downarrow[\sigma_2]N$.

This way, we obtain $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \mathbf{fv}\, N \cap \Gamma'$, which implies the required equivalence since $\mathbf{fv}\, P \cap \Gamma' = \mathbf{fv} \downarrow N \cap \Gamma' = \mathbf{fv}\, N \cap \Gamma'$.

Case 4. $P = \exists \overrightarrow{\alpha^-}.Q$ Then the induction hypothesis is applicable to type Q well-formed in context $\Gamma, \overrightarrow{\alpha^-}$:

1. $\Gamma' \subseteq \Gamma, \overrightarrow{\alpha^-}$ since $\Gamma' \subseteq \Gamma$,
2. $\Gamma, \overrightarrow{\alpha^-} \vdash \sigma_i : \Gamma'$ by weakening,
3. Q is normalized,
4. $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ by inversion of $\Gamma \vdash \exists \overrightarrow{\alpha^-}.Q$,
5. Notice that $[\sigma_i]\exists \overrightarrow{\alpha^-}.Q$ is normalized, and thus, $[\sigma_1]\exists \overrightarrow{\alpha^-}.Q \simeq^{D} [\sigma_2]\exists \overrightarrow{\alpha^-}.Q$ implies $[\sigma_1]\exists \overrightarrow{\alpha^-}.Q = [\sigma_2]\exists \overrightarrow{\alpha^-}.Q$ (by lemma 46).). This equality means $[\sigma_1]Q = [\sigma_2]Q$, which implies $\Gamma \vdash [\sigma_1]Q \simeq^{\leqslant} [\sigma_2]Q$.

Case 5. $N = P \to M$

□

Lemma 11 (Substitution composition well-formedness). *If $\Gamma_1' \vdash \sigma_1 : \Gamma_1$ and $\Gamma_2' \vdash \sigma_2 : \Gamma_2$, then $\Gamma_1', \Gamma_2' \vdash \sigma_2 \circ \sigma_1 : \Gamma_1, \Gamma_2$.*

Lemma 12 (Substitution monadic composition well-formedness). *If $\Gamma_1' \vdash \sigma_1 : \Gamma_1$ and $\Gamma_2' \vdash \sigma_2 : \Gamma_2$, then $\Gamma_2' \vdash \sigma_2 \lll \sigma_1 : \Gamma_1$.*

Lemma 13 (Substitution composition). *If $\Gamma_1' \vdash \sigma_1 : \Gamma_1$, $\Gamma_2' \vdash \sigma_2 : \Gamma_2$, $\Gamma_1 \cap \Gamma_2' = \varnothing$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$ then $\sigma_2 \circ \sigma_1 = (\sigma_2 \lll \sigma_1) \circ \sigma_2$.*

Corollary 5 (Substitution composition commutativity). *If $\Gamma_1' \vdash \sigma_1 : \Gamma_1$, $\Gamma_2' \vdash \sigma_2 : \Gamma_2$, and $\Gamma_1 \cap \Gamma_2 = \varnothing$, $\Gamma_1 \cap \Gamma_2' = \varnothing$, and $\Gamma_1' \cap \Gamma_2 = \varnothing$ then $\sigma_2 \circ \sigma_1 = \sigma_1 \circ \sigma_2$.*

Proof. by lemma 13, $\sigma_2 \circ \sigma_1 = (\sigma_2 \lll \sigma_1) \circ \sigma_2$. Since the codomain of σ_1 is Γ_1', and it is disjoint with the domain of σ_2, $\sigma_2 \lll \sigma_1 = \sigma_1$. □

Lemma 14 (Substitution domain weakening). *If $\Gamma_2 \vdash \sigma : \Gamma_1$ then $\Gamma_2, \Gamma' \vdash \sigma : \Gamma_1, \Gamma'$*

Proof. If the variable $\alpha^{\pm}$ is in Γ_1 then $\Gamma_2 \vdash [\sigma]\alpha^{\pm}$ by assumption, and then $\Gamma_2, \Gamma' \vdash [\sigma]\alpha^{\pm}$ by weakening. If the variable $\alpha^{\pm}$ is in $\Gamma' \backslash \Gamma_1$ then $[\sigma]\alpha^{\pm} = \alpha^{\pm} \in \Gamma'\Gamma_2, \Gamma'$, and thus, $\Gamma_2, \Gamma' \vdash \alpha^{\pm}$. □

Lemma 15 (Free variables after substitution). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, then*

+ *for a type P, the free variables of $[\sigma]P$ are bounded in the following way: $\mathbf{fv}\,(P)\backslash\Gamma_1 \subseteq \mathbf{fv}\,([\sigma]P) \subseteq (\mathbf{fv}\,(P)\backslash\Gamma_1) \cup \Gamma_2$;*
- *for a type N, the free variables of $[\sigma]P$ are bounded in the following way: $\mathbf{fv}\,(N)\backslash\Gamma_1 \subseteq \mathbf{fv}\,([\sigma]N) \subseteq (\mathbf{fv}\,(N)\backslash\Gamma_1) \cup \Gamma_2$.*

Proof. We prove it by structural induction on P and mutually, on N.

Case 1. $P = \alpha^+$

If $\alpha^+ \in \Gamma_1$ then $\Gamma_2 \vdash [\sigma]\alpha^+$, and by lemma 3, $\mathbf{fv}\,([\sigma]\alpha^+) \subseteq \Gamma_2$. $\mathbf{fv}\,(\alpha^+)\backslash\Gamma_1 = \varnothing$, so $\mathbf{fv}\,([\sigma]P)\backslash\Gamma_1 \subseteq \mathbf{fv}\,([\sigma]\alpha^+)$ vacuously.

If $\alpha^+ \notin \Gamma_1$ then $[\sigma]\alpha^+ = \alpha^+$, and $\mathbf{fv}\,([\sigma]\alpha^+) = \alpha^+ = \alpha^+\backslash\Gamma_1$.

Case 2. $P = \exists\overrightarrow{\alpha^-}.Q$
Then we need to show that $\mathbf{fv}([\sigma]P) = \mathbf{fv}([\sigma]Q)\backslash\overrightarrow{\alpha^-}$ is a subset of $(\mathbf{fv}(P)\backslash\Gamma_1) \cup \Gamma_2$ and a superset of $\mathbf{fv}(P)\backslash\Gamma_1$. Notice that $\mathbf{fv}(P) = \mathbf{fv}(Q)\backslash\overrightarrow{\alpha^-}$ by definition. This way, we need to show that $\mathbf{fv}(Q)\backslash\overrightarrow{\alpha^-}\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]Q)\backslash\overrightarrow{\alpha^-} \subseteq (\mathbf{fv}(Q)\backslash\overrightarrow{\alpha^-}\backslash\Gamma_1) \cup \Gamma_2$,

By the induction hypothesis, $\mathbf{fv}([\sigma]Q) \subseteq (\mathbf{fv}(Q)\backslash\Gamma_1) \cup \Gamma_2$. So for the second inclusion, it suffices to show that $((\mathbf{fv}(Q)\backslash\Gamma_1) \cup \Gamma_2)\backslash\overrightarrow{\alpha^-} \subseteq (\mathbf{fv}(Q)\backslash\overrightarrow{\alpha^-}\backslash\Gamma_1) \cup \Gamma_2$, which holds by set theoretical reasoning.

Also by the induction hypothesis, $\mathbf{fv}(Q)\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]Q)$, and thus, by subtracting $\overrightarrow{\alpha^-}$ from both sides, $\mathbf{fv}(Q)\backslash\overrightarrow{\alpha^-}\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]Q)\backslash\overrightarrow{\alpha^-}$.

Case 3. The case $N = \forall\overrightarrow{\alpha^+}.M$ is proved analogously.

Case 4. $N = P \to M$
Then $\mathbf{fv}([\sigma]N) = \mathbf{fv}([\sigma]P) \cup \mathbf{fv}([\sigma]M)$. By the induction hypothesis,

1. $\mathbf{fv}(P)\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]P) \subseteq (\mathbf{fv}(P)\backslash\Gamma_1) \cup \Gamma_2$ and
2. $\mathbf{fv}(M)\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]M) \subseteq (\mathbf{fv}(M)\backslash\Gamma_1) \cup \Gamma_2$.

We unite these inclusions vertically and obtain $\mathbf{fv}(P)\backslash\Gamma_1 \cup \mathbf{fv}(M)\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]N) \subseteq ((\mathbf{fv}(P)\backslash\Gamma_1) \cup \Gamma_2) \cup ((\mathbf{fv}(M)\backslash\Gamma_1) \cup \Gamma_2)$, which is equivalent to $(\mathbf{fv}(P) \cup \mathbf{fv}(M))\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]N) \subseteq (\mathbf{fv}(P) \cup \mathbf{fv}(M))\backslash\Gamma_1 \cup \Gamma_2$. Since $\mathbf{fv}(P) \cup \mathbf{fv}(M) = \mathbf{fv}(N)$, $\mathbf{fv}(N)\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]N) \subseteq (\mathbf{fv}(N)\backslash\Gamma_1) \cup \Gamma_2$.

Case 5. The cases when $P = \downarrow M$ and $N = \uparrow Q$ are proved analogously

□

Lemma 16 (Free variables of a variable image)**.** *Suppose that σ is an arbitrary substitution, Then*

$+$ *if $\alpha^\pm \in \mathbf{fv}(P)$ then $\mathbf{fv}([\sigma]\alpha^\pm) \subseteq \mathbf{fv}([\sigma]P)$,*

$-$ *if $\alpha^\pm \in \mathbf{fv}(N)$ then $\mathbf{fv}([\sigma]\alpha^\pm) \subseteq \mathbf{fv}([\sigma]N)$.*

Proof. By straightforward mutual induction on P and N. □

5.3 Declarative Subtyping

Lemma 17 (Free Variable Propagation)**.** *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context Γ,*

- $-$ *if $\Gamma \vdash N \leqslant M$ then $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$*
- $+$ *if $\Gamma \vdash P \geqslant Q$ then $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$*

Proof. Mutual induction on $\Gamma \vdash N \leqslant M$ and $\Gamma \vdash P \geqslant Q$.

Case 1. $\Gamma \vdash \alpha^- \leqslant \alpha^-$
It is self-evident that $\alpha^- \subseteq \alpha^-$.

Case 2. $\Gamma \vdash \uparrow P \leqslant \uparrow Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq^{\leqslant} Q$), we have $\Gamma \vdash P \geqslant Q$. Then by the induction hypothesis, $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$. The desired inclusion holds, since $\mathbf{fv}(\uparrow P) = \mathbf{fv}(P)$ and $\mathbf{fv}(\uparrow Q) = \mathbf{fv}(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$ and $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$. Then $\mathbf{fv}(P \to N) = \mathbf{fv}(P) \cup \mathbf{fv}(N) \subseteq \mathbf{fv}(Q) \cup \mathbf{fv}(M) = \mathbf{fv}(Q \to M)$.

Case 4. $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M$

$$\begin{aligned}
\mathbf{fv}\,\forall\overrightarrow{\alpha^+}.N &\subseteq \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \setminus \overrightarrow{\beta^+} && \text{here } \overrightarrow{\beta^+} \text{ is excluded by the premise } \mathbf{fv}\,N \cap \overrightarrow{\beta^+} = \varnothing \\
&\subseteq \mathbf{fv}\,M \setminus \overrightarrow{\beta^+} && \text{by the induction hypothesis, } \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \subseteq \mathbf{fv}\,M \\
&\subseteq \mathbf{fv}\,\forall\overrightarrow{\beta^+}.M
\end{aligned}$$

Case 5. The positive cases are symmetric.

□

Corollary 6 (Free Variables of mutual subtypes)**.**

$-$ *If $\Gamma \vdash N \simeq^{\leqslant} M$ then $\mathbf{fv}\,N = \mathbf{fv}\,M$,*

+ *If* $\Gamma \vdash P \simeq^{\leqslant} Q$ *then* $\mathbf{fv}\, P = \mathbf{fv}\, Q$

Lemma 18 (Decomposition of quantifier rules). *Assuming that* $\overrightarrow{\alpha^+}$, $\overrightarrow{\beta^+}$, $\overrightarrow{\alpha^-}$, *and* $\overrightarrow{\alpha^-}$ *are disjoint from* Γ,

$-_R$ $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M$ *holds if and only if* $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M$;

$+_R$ $\Gamma \vdash P \geqslant \exists\overrightarrow{\beta^-}.Q$ *holds if and only if* $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant Q$;

$-_L$ *suppose* $M \neq \forall \ldots$ *then* $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant M$ *holds if and only if* $\Gamma \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant M$ *for some* $\Gamma \vdash \overrightarrow{P}$;

$+_L$ *suppose* $Q \neq \exists \ldots$ *then* $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant Q$ *holds if and only if* $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant Q$ *for some* $\Gamma \vdash \overrightarrow{N}$.

Proof.

$-_R$ Let us prove both directions.

- $\Rightarrow$ Let us assume $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M$. $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M$. Let us decompose M as $\forall\overrightarrow{\beta^+}'.M'$ where M' does not start with $\forall$, and decompose N as $\forall\overrightarrow{\alpha^+}.N'$ where N' does not start with $\forall$. If $\overrightarrow{\beta^+}$ is empty, then $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M$ holds by assumption. Otherwise, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.\forall\overrightarrow{\beta^+}'.M$ is inferred by Rule ($\forall^{\leqslant}$), and by inversion: $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\beta^+}' \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$ for some $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\beta^+}' \vdash \overrightarrow{P}$. Then again by Rule ($\forall^{\leqslant}$) with the same $\overrightarrow{P}$, $\Gamma, \overrightarrow{\beta^+} \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}'.M'$, that is $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M$.
- $\Leftarrow$ let us assume $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M$, and let us decompose N as $\forall\overrightarrow{\alpha^+}.N'$ where N' does not start with $\forall$, and M as $\forall\overrightarrow{\beta^+}'.M'$ where M' does not start with $\forall$. if $\overrightarrow{\alpha^+}$ and $\overrightarrow{\beta^+}'$ are empty then $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M$ is turned into $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M$ by Rule ($\forall^{\leqslant}$). Otherwise, $\Gamma, \overrightarrow{\beta^+} \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}'.M'$ is inferred by Rule ($\forall^{\leqslant}$), that is $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\beta^+}' \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$ for some $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\beta^+}' \vdash \overrightarrow{P}$. Then by Rule ($\forall^{\leqslant}$) again, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}, \overrightarrow{\beta^+}'.M'$, in other words, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.\forall\overrightarrow{\beta^+}'.M'$, that is $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M$.

$-_L$ Suppose $M \neq \forall \ldots$. Let us prove both directions.

- $\Rightarrow$ Let us assume $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant M$. then if $\overrightarrow{\alpha^+} = \cdot$, $\Gamma \vdash N \leqslant M$ holds immediately. Otherwise, let us decompose iN as $\forall\overrightarrow{\alpha^+}'.N'$ where N' does not start with $\forall$. Then $\Gamma \vdash \forall\overrightarrow{\alpha^+}.\forall\overrightarrow{\alpha^+}'.N' \leqslant M'$ is inferred by Rule ($\forall^{\leqslant}$), and by inversion, there exist $\Gamma \vdash \overrightarrow{P}, \overrightarrow{P}'$ such that $\Gamma \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}][\overrightarrow{P}'/\overrightarrow{\alpha^+}']N' \leqslant M'$ (the decomposition of substitutions is possible since $\overrightarrow{\alpha^+} \cap \Gamma = \varnothing$). Then by Rule ($\forall^{\leqslant}$) again, $\Gamma \vdash \forall\overrightarrow{\alpha^+}'.[\overrightarrow{P}'/\overrightarrow{\alpha^+}']N' \leqslant M'$ (notice that $[\overrightarrow{P}'/\overrightarrow{\alpha^+}']N'$ cannot start with $\forall$).
- $\Leftarrow$ Let us assume $\Gamma \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant M$ for some $\Gamma \vdash \overrightarrow{P}$. let us decompose iN as $\forall\overrightarrow{\alpha^+}'.N'$ where N' does not start with $\forall$. Then $\Gamma \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]\forall\overrightarrow{\alpha^+}'.N' \leqslant M'$ or, equivalently, $\Gamma \vdash \forall\overrightarrow{\alpha^+}'.[\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$ is inferred by Rule ($\forall^{\leqslant}$) (notice that $[\overrightarrow{P}/\overrightarrow{\alpha^+}]N'$ cannot start with $\forall$). By inversion, there exist $\Gamma \vdash \overrightarrow{P}'$ such that $\Gamma \vdash [\overrightarrow{P}'/\overrightarrow{\alpha^+}'][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$. Since $\overrightarrow{\alpha^+}'$ is disjoint from the free variables of $\overrightarrow{P}$ and from $\overrightarrow{\alpha^+}$, the composition of $\overrightarrow{P}'/\overrightarrow{\alpha^+}'$ and $\overrightarrow{P}/\overrightarrow{\alpha^+}$ can be joined into a single substitution well-formed in Γ. Then by Rule ($\forall^{\leqslant}$) again, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant M$.

$+$ The positive cases are proved symmetrically.

□

Corollary 7 (Redundant quantifier elimination).

$-_L$ *Suppose that* $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,(N) = \varnothing$ *then* $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant M$ *holds if and only if* $\Gamma \vdash N \leqslant M$;

$-_R$ *Suppose that* $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,(M) = \varnothing$ *then* $\Gamma \vdash N \leqslant \forall\overrightarrow{\alpha^+}.M$ *holds if and only if* $\Gamma \vdash N \leqslant M$;

$+_L$ *Suppose that* $\overrightarrow{\alpha^-} \cap \mathbf{fv}\,(P) = \varnothing$ *then* $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant Q$ *holds if and only if* $\Gamma \vdash P \geqslant Q$.

$+_R$ *Suppose that* $\overrightarrow{\alpha^-} \cap \mathbf{fv}\,(Q) = \varnothing$ *then* $\Gamma \vdash P \geqslant \exists\overrightarrow{\alpha^-}.Q$ *holds if and only if* $\Gamma \vdash P \geqslant Q$.

Proof. $-_R$ Suppose that $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,(M) = \varnothing$ then by lemma 18, $\Gamma \vdash N \leqslant \forall\overrightarrow{\alpha^+}.M$ is equivalent to $\Gamma, \overrightarrow{\alpha^+} \vdash N \leqslant M$, By , since $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,(N) = \varnothing$ and $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,(M) = \varnothing$, $\Gamma, \overrightarrow{\alpha^+} \vdash N \leqslant M$ is equivalent to $\Gamma \vdash N \leqslant M$.

$-_L$ Suppose that $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,(N) = \varnothing$. Let us decompose M as $\forall\overrightarrow{\beta^+}.M'$ where M' does not start with $\forall$. By lemma 18, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M'$ is equivalent to $\Gamma, \overrightarrow{\beta^+} \vdash \forall\overrightarrow{\alpha^+}.N \leqslant M'$, which is equivalent to existence of $\Gamma, \overrightarrow{\beta^+} \vdash \overrightarrow{P}$ such that $\Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant M'$. Since $[\overrightarrow{P}/\overrightarrow{\alpha^+}]N = N$, the latter is equivalent to $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M'$, which is equivalent to $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M'$. $\Gamma, \overrightarrow{\beta^+} \vdash \overrightarrow{P}$ can be chosen arbitrary, for example, $\overrightarrow{P}_i = \exists\alpha^-.\downarrow\alpha^-$.

$+$ The positive cases are proved symmetrically.

□

Lemma 19 (Subtypes and supertypes of a variable). *Assuming $\Gamma \vdash \alpha^-$, $\Gamma \vdash \alpha^+$, $\Gamma \vdash N$, and $\Gamma \vdash P$,*

+ *if $\Gamma \vdash P \geqslant \exists\overrightarrow{\alpha^-}.\alpha^+$ or $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant P$ then $P = \exists\overrightarrow{\beta^-}.\alpha^+$ (for some potentially empty $\overrightarrow{\beta^-}$)*

− *if $\Gamma \vdash N \leqslant \forall\overrightarrow{\alpha^+}.\alpha^-$ or $\Gamma \vdash \forall\overrightarrow{\alpha^+}.\alpha^- \leqslant N$ then $N = \forall\overrightarrow{\beta^+}.\alpha^-$ (for some potentially empty $\overrightarrow{\beta^+}$)*

Proof. We prove by induction on the tree inferring $\Gamma \vdash P \geqslant \exists\overrightarrow{\alpha^-}.\alpha^+$ or $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant P$ or or $\Gamma \vdash N \leqslant \forall\overrightarrow{\alpha^+}.\alpha^-$ or $\Gamma \vdash \forall\overrightarrow{\alpha^+}.\alpha^- \leqslant N$. Let us consider which one of these judgments is inferred.

Case 1. $\Gamma \vdash P \geqslant \exists\overrightarrow{\alpha^-}.\alpha^+$

If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\mathrm{Var}^{+\geqslant}$), which implies that $\overrightarrow{\alpha^-}$ is empty and $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists^{\geqslant}$). By inverting this rule, $P = \exists\overrightarrow{\beta^-}.P'$ where P' does not start with $\exists$ and $\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]P' \geqslant \alpha^+$ for some $\Gamma, \overrightarrow{\alpha^-} \vdash N_i$.

By the induction hypothesis, $[\overrightarrow{N}/\overrightarrow{\beta^-}]P' = \exists\overrightarrow{\gamma^-}.\alpha^+$. What shape can P' have? As mentioned, it does not start with $\exists$, and it cannot start with $\uparrow$ (otherwise, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]P'$ would also start with $\uparrow$ and would not be equal to $\exists\overrightarrow{\beta^-}.\alpha^+$). This way, P' is a *positive* variable. As such, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]P' = P'$, and then $P' = \exists\overrightarrow{\gamma^-}.\alpha^+$ meaning that $\overrightarrow{\gamma^-}$ is empty and $P' = \alpha^+$. This way, $P = \exists\overrightarrow{\beta^-}.P' = \exists\overrightarrow{\beta^-}.\alpha^+$, as required.

Case 2. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant P$

If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\mathrm{Var}^{+\geqslant}$), which implies that $\overrightarrow{\alpha^-}$ is empty and $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists^{\geqslant}$). By inverting this rule, $P = \exists\overrightarrow{\beta^-}.Q$ where $\Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ \geqslant Q$ and Q does not start with $\exists$. Notice that since α^+ is positive, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash \alpha^+ \geqslant Q$.

By the induction hypothesis, $Q = \exists\overrightarrow{\beta^-}'.\alpha^+$, and since Q does not start with $\exists$, $\overrightarrow{\beta^-}'$ is empty This way, $P = \exists\overrightarrow{\beta^-}.Q = \exists\overrightarrow{\beta^-}.\alpha^+$, as required.

Case 3. The negative cases ($\Gamma \vdash N \leqslant \forall\overrightarrow{\alpha^+}.\alpha^-$ and $\Gamma \vdash \forall\overrightarrow{\alpha^+}.\alpha^- \leqslant N$) are proved analogously.

∎

Corollary 8 (Variables have no proper subtypes and supertypes). *Assuming that all mentioned types are well-formed in Γ,*

$$\begin{array}{ccccccc}
\Gamma \vdash P \geqslant \alpha^+ & \iff & P = \exists\overrightarrow{\beta^-}.\alpha^+ & \iff & \Gamma \vdash P \simeq^{\leqslant} \alpha^+ & \iff & P \simeq^{D} \alpha^+ \\
\Gamma \vdash \alpha^+ \geqslant P & \iff & P = \exists\overrightarrow{\beta^-}.\alpha^+ & \iff & \Gamma \vdash P \simeq^{\leqslant} \alpha^+ & \iff & P \simeq^{D} \alpha^+ \\
\Gamma \vdash N \leqslant \alpha^- & \iff & N = \forall\overrightarrow{\beta^+}.\alpha^- & \iff & \Gamma \vdash N \simeq^{\leqslant} \alpha^- & \iff & N \simeq^{D} \alpha^- \\
\Gamma \vdash \alpha^- \leqslant N & \iff & N = \forall\overrightarrow{\beta^+}.\alpha^- & \iff & \Gamma \vdash N \simeq^{\leqslant} \alpha^- & \iff & N \simeq^{D} \alpha^-
\end{array}$$

Proof. Notice that $\Gamma \vdash \exists\overrightarrow{\beta^-}.\alpha^+ \simeq^{\leqslant} \alpha^+$ and $\exists\overrightarrow{\beta^-}.\alpha^+ \simeq^{D} \alpha^+$ and apply lemma 19. ∎

Lemma 20 (Subtyping context irrelevance). *Suppose that all the mentioned types are well-formed in Γ_1 and Γ_2. Then*

+ *$\Gamma_1 \vdash P \geqslant Q$ is equivalent to $\Gamma_2 \vdash P \geqslant Q$;*

− *$\Gamma_1 \vdash N \leqslant M$ is equivalent to $\Gamma_2 \vdash N \leqslant M$.*

Proof. We prove it by induction on the size of $\Gamma_1 \vdash P \geqslant Q$ and mutually, the size of $\Gamma_1 \vdash N \leqslant M$.

All the cases except Rule ($\exists^{\geqslant}$) and Rule ($\forall^{\leqslant}$) are proven congruently: first, we apply the inversion to $\Gamma_1 \vdash P \geqslant Q$ to obtain the premises of the corresponding rule X, then we apply the induction hypothesis to each premise, and build the inference tree (with Γ_2) by the same rule X.

Suppose that the judgement is inferred by Rule ($\exists^{\geqslant}$). Then we are proving that $\Gamma_1 \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q$ implies $\Gamma_2 \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q$ (the other implication is proven symmetrically).

By inversion of $\Gamma_1 \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q$, we obtain σ such that $\Gamma_1, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $\Gamma_1, \overrightarrow{\beta^-} \vdash [\sigma]P \geqslant Q$. By lemma 17, $\mathbf{fv}([\sigma]P) \subseteq \mathbf{fv}(Q)$.

From the well-formedness statements $\Gamma_i \vdash \exists\overrightarrow{\alpha^-}.P$ and $\Gamma_i \vdash \exists\overrightarrow{\beta^-}.Q$ we have:

- $\Gamma_1, \overrightarrow{\alpha^-} \vdash P$, which also means $\Gamma_1, \overrightarrow{\beta^-} \vdash [\sigma]P$ by lemma 5;

- $\Gamma_2, \overrightarrow{\alpha^-} \vdash P$;
- $\Gamma_1, \overrightarrow{\beta^-} \vdash Q$; and
- $\Gamma_2, \overrightarrow{\beta^-} \vdash Q$, which means $\mathbf{fv}\,(Q) \subseteq \Gamma_2, \overrightarrow{\beta^-}$ by lemma 3, and combining it with $\mathbf{fv}\,([\sigma]P) \subseteq \mathbf{fv}\,(Q)$, we have $\mathbf{fv}\,([\sigma]P) \subseteq \Gamma_2, \overrightarrow{\beta^-}$.

Let us construct a substitution σ_0 in the following way:

$$\begin{cases} [\sigma_0]\alpha_i^- = [\sigma]\alpha_i^- & \text{for } \alpha_i^- \in \overrightarrow{\alpha^-} \cap \mathbf{fv}\,(P) \\ [\sigma_0]\alpha_i^- = \forall\gamma^+.\uparrow\gamma^+ & \text{for } \alpha_i^- \in \overrightarrow{\alpha^-} \backslash \mathbf{fv}\,(P) \\ [\sigma_0]\gamma^\pm = \gamma^\pm & \text{for any other } \gamma^\pm \end{cases}$$

Notice that

1. $[\sigma_0]P = [\sigma]P$. Since $\sigma_0|_{\mathbf{fv}\,(P)} = \sigma|_{\mathbf{fv}\,(P)}$ as functions (which follows from the construction of σ_0 and the signature of σ), $[\sigma_0]P = [\sigma_0|_{\mathbf{fv}\,(P)}]P = [\sigma|_{\mathbf{fv}\,(P)}]P = [\sigma]P$ (where the first and the last equalities are by lemma 6).
2. $\mathbf{fv}\,([\sigma]P) \vdash \sigma_0 : \overrightarrow{\alpha^-}$. To show that, let us consider α_i^-
   - if $\alpha_i^- \in \overrightarrow{\alpha^-} \backslash \mathbf{fv}\,(P)$ then $\cdot \vdash [\sigma_0]\alpha_i^-$, which can be weakened to $\mathbf{fv}\,([\sigma]P) \vdash [\sigma_0]\alpha_i^-$;
   - if $\alpha_i^- \in \overrightarrow{\alpha^-} \cap \mathbf{fv}\,(P)$, we have $[\sigma_0]\alpha_i^- = [\sigma]\alpha_i^-$, and thus, by specification of σ, $\Gamma_1, \overrightarrow{\beta^+} \vdash [\sigma_0]\alpha_i^-$. By corollary 1, it means $\mathbf{fv}\,([\sigma_0]\alpha_i^-) \vdash [\sigma_0]\alpha_i^-$, which we weaken (corollary 2) to $\mathbf{fv}\,([\sigma]P) \vdash [\sigma_0]\alpha_i^-$ (since $\mathbf{fv}\,([\sigma_0]\alpha_i^-) \subseteq \mathbf{fv}\,([\sigma_0]P)$ by lemma 16, and $[\sigma_0]P = [\sigma]P$, as noted above).

By corollary 1, $\Gamma_1, \overrightarrow{\beta^-} \vdash [\sigma]P$ implies $\mathbf{fv}\,([\sigma]P) \vdash [\sigma]P$, which, since $\mathbf{fv}\,([\sigma]P) \subseteq \Gamma_2, \overrightarrow{\beta^-}$, is weakened to $\Gamma_2, \overrightarrow{\beta^-} \vdash [\sigma]P$. and rewritten as $\Gamma_2, \overrightarrow{\beta^-} \vdash [\sigma_0]P$.

Notice that the premises of the induction hold:

1. $\Gamma_i, \overrightarrow{\beta^-} \vdash [\sigma_0]P$,
2. $\Gamma_i, \overrightarrow{\beta^-} \vdash Q$, and
3. $\Gamma_1, \overrightarrow{\beta^-} \vdash [\sigma_0]P \geqslant Q$, notice that the tree inferring this judgement is the same tree inferring $\Gamma_1, \overrightarrow{\beta^-} \vdash [\sigma]P \geqslant Q$ (since $[\sigma_0]P = [\sigma]P$), i.e., it is a subtree of $\Gamma_1 \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q$.

This way, by the induction hypothesis, $\Gamma_2, \overrightarrow{\beta^-} \vdash [\sigma_0]P \geqslant Q$. Combining it with $\Gamma_2, \overrightarrow{\beta^-} \vdash \sigma_0 : \overrightarrow{\alpha^-}$ by Rule ($\exists^{\geqslant}$), we obtain $\Gamma_2 \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q$.

The case of $\Gamma_1 \vdash \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M$ is symmetric. ☐

Lemma 21 (Weakening of subtyping context)**.** *Suppose Γ_1 and Γ_2 are contexts and $\Gamma_1 \subseteq \Gamma_2$. Then*

+ $\Gamma_1 \vdash P \geqslant Q$ *implies* $\Gamma_2 \vdash P \geqslant Q$;

− $\Gamma_1 \vdash N \leqslant M$ *implies* $\Gamma_2 \vdash N \leqslant M$.

Proof. ☐

Lemma 22 (Reflexivity of subtyping)**.** *Assuming all the types are well-formed in Γ,*

− $\Gamma \vdash N \leqslant N$

+ $\Gamma \vdash P \geqslant P$

Proof. Let us prove it by the size of N and mutually, P.

Case 1. $N = \alpha^-$
Then $\Gamma \vdash \alpha^- \leqslant \alpha^-$ is inferred immediately by Rule ($\mathrm{Var}^{-\leqslant}$).

Case 2. $N = \forall\overrightarrow{\alpha^+}.N'$ where $\overrightarrow{\alpha^+}$ is not empty
First, we rename $\overrightarrow{\alpha^+}$ to fresh $\overrightarrow{\beta^+}$ in $\forall\overrightarrow{\alpha^+}.N'$ to avoid name clashes: $\forall\overrightarrow{\alpha^+}.N' = \forall\overrightarrow{\beta^+}.[\overrightarrow{\alpha^+}/\overrightarrow{\beta^+}]N'$. Then to infer $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.[\overrightarrow{\alpha^+}/\overrightarrow{\beta^+}]N'$ we can apply Rule ($\forall^{\leqslant}$), instantiating $\overrightarrow{\alpha^+}$ with $\overrightarrow{\beta^+}$:

- $\mathbf{fv}\,N \cap \overrightarrow{\beta^+} = \varnothing$ by choice of $\overrightarrow{\beta^+}$,
- $\Gamma, \overrightarrow{\beta^+} \vdash \beta_i^+$,

- $\Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{\beta^+}/\overrightarrow{\alpha^+}]N' \leqslant [\overrightarrow{\beta^+}/\overrightarrow{\alpha^+}]N'$ by the induction hypothesis, since the size of $[\overrightarrow{\beta^+}/\overrightarrow{\alpha^+}]N'$ is equal to the size of N', which is smaller than the size of $N = \forall\overrightarrow{\alpha^+}.N'$.

Case 3. $N = P \to M$
Then $\Gamma \vdash P \to M \leqslant P \to M$ is inferred by Rule $(\to^{\leqslant})$, since $\Gamma \vdash P \geqslant P$ and $\Gamma \vdash M \leqslant M$ hold the induction hypothesis.

Case 4. $N = \uparrow P$
Then $\Gamma \vdash \uparrow P \leqslant \uparrow P$ is inferred by Rule $(\uparrow^{\leqslant})$, since $\Gamma \vdash P \geqslant P$ holds by the induction hypothesis.

Case 5. The positive cases are symmetric to the negative ones.

□

Lemma 23 (Substitution preserves subtyipng)**.** *Suppose that all mentioned types are well-formed in* Γ_1*, and* σ *is a substitution* $\Gamma_2 \vdash \sigma : \Gamma_1$*.*

− *If* $\Gamma_1 \vdash N \leqslant M$ *then* $\Gamma_2 \vdash [\sigma]N \leqslant [\sigma]M$*.*

+ *If* $\Gamma_1 \vdash P \geqslant Q$ *then* $\Gamma_2 \vdash [\sigma]P \geqslant [\sigma]Q$*.*

Proof. We prove it by induction on the size of the derivation of $\Gamma_1 \vdash N \leqslant M$ and mutually, $\Gamma_1 \vdash P \geqslant Q$. Let us consider the last rule used in the derivation:

Case 1. Rule $(\mathrm{Var}^{-\leqslant})$. Then by inversion, $N = \alpha^-$ and $M = \alpha^-$. By reflexivity of subtyping (lemma 22), we have $\Gamma_2 \vdash [\sigma]\alpha^- \leqslant [\sigma]\alpha^-$, i.e. $\Gamma_2 \vdash [\sigma]N \leqslant [\sigma]M$, as required.

Case 2. Rule $(\forall^{\leqslant})$. Then by inversion, $N = \forall\overrightarrow{\alpha^+}.N'$, $M = \forall\overrightarrow{\beta^+}.M'$, where $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is not empty. Moreover, $\Gamma_1, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$ for some $\Gamma_1, \overrightarrow{\beta^+} \vdash \overrightarrow{P}$, and $\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing$.

Notice that since the derivation of $\Gamma_1, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$ is a subderivation of the derivation of $\Gamma \vdash N \leqslant M$, its size is smaller, and hence, the induction hypothesis applies ($\Gamma_1, \overrightarrow{\beta^+} \vdash \sigma : \Gamma_1, \overrightarrow{\beta^+}$ by) : $\Gamma_2, \overrightarrow{\beta^+} \vdash [\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant [\sigma]M'$.

Notice that by convention, $\overrightarrow{\alpha^+}$ and $\overrightarrow{\beta^+}$ are fresh, and thus, $[\sigma]\forall\overrightarrow{\alpha^+}.N' = \forall\overrightarrow{\alpha^+}.[\sigma]N'$ and $[\sigma]\forall\overrightarrow{\beta^+}.M' = \forall\overrightarrow{\beta^+}.[\sigma]M'$, which means that the required $\Gamma_2, \Gamma \vdash [\sigma]\forall\overrightarrow{\alpha^+}.N' \leqslant [\sigma]\forall\overrightarrow{\beta^+}.M'$ is rewritten as $\Gamma_2, \Gamma \vdash \forall\overrightarrow{\alpha^+}.[\sigma]N' \leqslant \forall\overrightarrow{\beta^+}.[\sigma]M'$.

To infer it, we apply Rule $(\forall^{\leqslant})$, instantiating α_i^+ with $[\sigma]P_i$:

- $\mathbf{fv}\,[\sigma]N \cap \overrightarrow{\beta^+} = \varnothing$;
- $\Gamma_2, \Gamma, \overrightarrow{\beta^+} \vdash [\sigma]P_i$, by lemma 5 since from the inversion, $\Gamma_1, \Gamma, \overrightarrow{\beta^+} \vdash P_i$;
- $\Gamma, \overrightarrow{\beta^+} \vdash [[\sigma]\overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N' \leqslant [\sigma]M'$ holds by lemma 13: Since $\overrightarrow{\alpha^+}$ is fresh, it is disjoint with the domain and the codomain of σ (Γ_1 and Γ_2), and thus, $[\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' = [\sigma \ll \overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N' = [[\sigma]\overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N'$. Then $\Gamma_2, \Gamma, \overrightarrow{\beta^+} \vdash [\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant [\sigma]M'$ holds by the induction hypothesis.

Case 3. Rule $(\to^{\leqslant})$. Then by inversion, $N = P \to N_1$, $M = Q \to M_1$, $\Gamma \vdash P \geqslant Q$, and $\Gamma \vdash N_1 \leqslant M_1$. And by the induction hypothesis, $\Gamma' \vdash [\sigma]P \geqslant [\sigma]Q$ and $\Gamma' \vdash [\sigma]N_1 \leqslant [\sigma]M_1$. Then $\Gamma' \vdash [\sigma]N \leqslant [\sigma]M$, i.e. $\Gamma' \vdash [\sigma]P \to [\sigma]N_1 \leqslant [\sigma]Q \to [\sigma]M_1$, is inferred by Rule $(\to^{\leqslant})$.

Case 4. Rule $(\uparrow^{\leqslant})$. Then by inversion, $N = \uparrow P$, $M = \uparrow Q$, and $\Gamma \vdash P \simeq^{\leqslant} Q$, which by inversion means that $\Gamma \vdash P \geqslant Q$ and $\Gamma \vdash Q \geqslant P$. Then the induction hypothesis applies, and we have $\Gamma' \vdash [\sigma]P \geqslant [\sigma]Q$ and $\Gamma' \vdash [\sigma]Q \geqslant [\sigma]P$. Then by sequential application of Rule $(\simeq^{\leqslant}{}^{-})$ and Rule $(\uparrow^{\leqslant})$ to these judgments, we have $\Gamma' \vdash \uparrow[\sigma]P \leqslant \uparrow[\sigma]Q$, i.e. $\Gamma' \vdash [\sigma]N \leqslant [\sigma]M$, as required.

Case 5. The positive cases are proved symmetrically.

□

Corollary 9 (Substitution preserves subtyping induced equivalence)**.** *Suppose that* $\Gamma \vdash \sigma : \Gamma_1$*. Then*

+ *if* $\Gamma_1 \vdash P$*,* $\Gamma_1 \vdash Q$*, and* $\Gamma_1 \vdash P \simeq^{\leqslant} Q$ *then* $\Gamma \vdash [\sigma]P \simeq^{\leqslant} [\sigma]Q$

− *if* $\Gamma_1 \vdash N$*,* $\Gamma_1 \vdash M$*, and* $\Gamma_1 \vdash N \simeq^{\leqslant} M$ *then* $\Gamma \vdash [\sigma]N \simeq^{\leqslant} [\sigma]M$

Lemma 24 (Transitivity of subtyping)**.** *Assuming the types are well-formed in* Γ*,*

− *if* $\Gamma \vdash N_1 \leqslant N_2$ *and* $\Gamma \vdash N_2 \leqslant N_3$ *then* $\Gamma \vdash N_1 \leqslant N_3$*,*

+ *if* $\Gamma \vdash P_1 \geqslant P_2$ *and* $\Gamma \vdash P_2 \geqslant P_3$ *then* $\Gamma \vdash P_1 \geqslant P_3$*.*

Proof. To prove it, we formulate a stronger property, which will imply the required one, taking $\sigma = \Gamma \vdash \mathsf{id} : \Gamma$.
Assuming all the types are well-formed in Γ,

– if $\Gamma \vdash N \leqslant M_1$, $\Gamma \vdash M_2 \leqslant K$, and for $\Gamma' \vdash \sigma : \Gamma$, $[\sigma]M_1 = [\sigma]M_2$ then $\Gamma' \vdash [\sigma]N \leqslant [\sigma]K$

+ if $\Gamma \vdash P \geqslant Q_1$, $\Gamma \vdash Q_2 \geqslant R$, and for $\Gamma' \vdash \sigma : \Gamma$, $[\sigma]Q_1 = [\sigma]Q_2$ then $\Gamma' \vdash [\sigma]P \geqslant [\sigma]R$

We prove it by induction on $\mathsf{size}(\Gamma \vdash N \leqslant M_1) + \mathsf{size}(\Gamma \vdash M_2 \leqslant K)$ and mutually, on $\mathsf{size}(\Gamma \vdash P \geqslant Q_1) + \mathsf{size}(\Gamma \vdash Q_2 \geqslant R)$. First, let us consider the 3 important cases.

Case 1. Let us consider the case when $M_1 = \forall \overrightarrow{\beta^+}_1.\alpha^-$. Then by lemma 19, $\Gamma \vdash N \leqslant M_1$ means that $N = \forall \overrightarrow{\alpha^+}.\alpha^-$. $[\sigma]M_1 = [\sigma]M_2$ means that $\forall \overrightarrow{\beta^+}_1.[\sigma]\alpha^- = [\sigma]M_2$. Applying σ to both sides of $\Gamma \vdash M_2 \leqslant K$ (by lemma 23), we obtain $\Gamma' \vdash [\sigma]M_2 \leqslant [\sigma]K$, that is $\Gamma' \vdash \forall \overrightarrow{\beta^+}_1.[\sigma]\alpha^- \leqslant [\sigma]K$. Since $\mathbf{fv}([\sigma]\alpha^-) \subseteq \Gamma, \alpha^-$, it is disjoint from $\overrightarrow{\alpha^+}$ and $\overrightarrow{\beta^+}_1$, This way, by corollary 7, $\Gamma' \vdash \forall \overrightarrow{\beta^+}_1.[\sigma]\alpha^- \leqslant [\sigma]K$ is equivalent to $\Gamma' \vdash [\sigma]\alpha^- \leqslant [\sigma]K$, which is equivalent to $\Gamma' \vdash \forall \overrightarrow{\alpha^+}.[\sigma]\alpha^- \leqslant [\sigma]K$, that is $\Gamma' \vdash [\sigma]N \leqslant [\sigma]K$.

Case 2. Let us consider the case when $M_2 = \forall \overrightarrow{\beta^+}_2.\alpha^-$. This case is symmetric to the previous one. Notice that lemma 19 and corollary 7 are agnostic to the side on which the the quantifiers occur, and thus, the proof stays the same.

Case 3. Let us decompose the types, by extracting the outer quantifiers:

- $N = \forall \overrightarrow{\alpha^+}.N'$, where $N' \neq \forall \ldots$,
- $M_1 = \forall \overrightarrow{\beta^+}_1.M_1'$, where $M_1' \neq \forall \ldots$,
- $M_2 = \forall \overrightarrow{\beta^+}_2.M_2'$, where $M_2' \neq \forall \ldots$,
- $K = \forall \overrightarrow{\gamma^+}.K'$, where $K' \neq \forall \ldots$.

and assume that at least one of $\overrightarrow{\alpha^+}$, $\overrightarrow{\beta^+}_1$, $\overrightarrow{\beta^+}_2$, and $\overrightarrow{\gamma^+}$ is not empty. Since $[\sigma]M_1 = [\sigma]M_2$, we have $\forall \overrightarrow{\beta^+}_1.[\sigma]M_1' = \forall \overrightarrow{\beta^+}_2.[\sigma]M_2'$, and since M_i' are not variables (which was covered by the previous cases) and do not start with $\forall$, $[\sigma]M_i'$ do not start with $\forall$ either, which means $\overrightarrow{\beta^+}_1 = \overrightarrow{\beta^+}_2$ and $[\sigma]M_1' = [\sigma]M_2'$. Let us rename $\overrightarrow{\beta^+}_1$ and $\overrightarrow{\beta^+}_2$ to $\overrightarrow{\beta^+}$. Then $M_1 = \forall \overrightarrow{\beta^+}.M_1'$ and $M_2 = \forall \overrightarrow{\beta^+}.M_2'$.

By lemma 18 applied twice to $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N' \leqslant \forall \overrightarrow{\beta^+}.M_1'$ and to $\Gamma \vdash \forall \overrightarrow{\beta^+}.M_2' \leqslant \forall \overrightarrow{\gamma^+}.K'$, we have the following:

1. $\Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M_1'$ for some $\Gamma, \overrightarrow{\beta^+} \vdash \overrightarrow{P}$;
2. $\Gamma, \overrightarrow{\gamma^+} \vdash [\overrightarrow{Q}/\overrightarrow{\beta^+}]M_2' \leqslant K'$ for some $\Gamma, \overrightarrow{\gamma^+} \vdash \overrightarrow{Q}$.

And since at least one of $\overrightarrow{\alpha^+}$, $\overrightarrow{\beta^+}$, and $\overrightarrow{\gamma^+}$ is not empty, either $\Gamma \vdash N \leqslant M_1$ or $\Gamma \vdash M_2 \leqslant K$ is inferred by Rule ($\forall^{\leqslant}$), meaning that either $\Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M_1'$ is a proper subderivation of $\Gamma \vdash N \leqslant M_1$ or $\Gamma, \overrightarrow{\gamma^+} \vdash [\overrightarrow{Q}/\overrightarrow{\beta^+}]M_2' \leqslant K'$ is a proper subderivation of $\Gamma \vdash M_2 \leqslant K$.

Notice that we can weaken and rearrange the contexts without changing the sizes of the derivations: $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\gamma^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M_1'$ and $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\gamma^+} \vdash [\overrightarrow{Q}/\overrightarrow{\beta^+}]M_2' \leqslant K'$. This way, the sum of the sizes of these derivations is smaller than the sum of the sizes of $\Gamma \vdash N \leqslant M_1$ and $\Gamma \vdash M_2 \leqslant K$. Let us apply the induction hypothesis to these derivations, with the substitution $\Gamma', \overrightarrow{\gamma^+} \vdash \sigma \circ (\overrightarrow{Q}/\overrightarrow{\beta^+}) : \Gamma, \overrightarrow{\beta^+}, \overrightarrow{\gamma^+}$ (section 6.9). To apply the induction hypothesis, it is left to show that $\sigma \circ (\overrightarrow{Q}/\overrightarrow{\beta^+})$ unifies M_1' and $[\overrightarrow{Q}/\overrightarrow{\beta^+}]M_2'$:

$$\begin{aligned}
[\sigma \circ \overrightarrow{Q}/\overrightarrow{\beta^+}]M_1' &= [\sigma][\overrightarrow{Q}/\overrightarrow{\beta^+}]M_1' \\
&= [[\sigma]\overrightarrow{Q}/\overrightarrow{\beta^+}][\sigma]M_2' && \text{by lemma 13} \\
&= [[\sigma]\overrightarrow{Q}/\overrightarrow{\beta^+}][\sigma]M_2' && \text{Since } [\sigma]M_1' = [\sigma]M_2' \\
&= [\sigma][\overrightarrow{Q}/\overrightarrow{\beta^+}]M_2' && \text{by lemma 13} \\
&= [\sigma][\overrightarrow{Q}/\overrightarrow{\beta^+}][\overrightarrow{Q}/\overrightarrow{\beta^+}]M_2' && \text{Since } \Gamma, \overrightarrow{\gamma^+} \vdash \overrightarrow{Q}, \text{ and } (\Gamma, \overrightarrow{\gamma^+}) \cap \overrightarrow{\beta^+} = \varnothing \\
&= [\sigma \circ \overrightarrow{Q}/\overrightarrow{\beta^+}][\overrightarrow{Q}/\overrightarrow{\beta^+}]M_2'
\end{aligned}$$

This way the induction hypothesis gives us $\Gamma', \overrightarrow{\gamma^+} \vdash [\sigma][\overrightarrow{Q}/\overrightarrow{\beta^+}][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant [\sigma][\overrightarrow{Q}/\overrightarrow{\beta^+}]K'$, and since $\Gamma, \overrightarrow{\gamma^+} \vdash K'$, $[\overrightarrow{Q}/\overrightarrow{\beta^+}]K' =$

K', that is $\Gamma', \overrightarrow{\gamma^+} \vdash [\sigma][\overrightarrow{Q}/\overrightarrow{\beta^+}][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant [\sigma]K'$. Let us rewrite the substitution that we apply to N':

$$\begin{aligned}
[\sigma \circ \overrightarrow{Q}/\overrightarrow{\beta^+} \circ \overrightarrow{P}/\overrightarrow{\alpha^+}]N' &= [(\sigma \lll \overrightarrow{Q}/\overrightarrow{\beta^+}) \circ \sigma \circ \overrightarrow{P}/\overrightarrow{\alpha^+}]N' && \text{by lemma 13}\\
&= [(\sigma \lll \overrightarrow{Q}/\overrightarrow{\beta^+}) \circ (\sigma \lll \overrightarrow{P}/\overrightarrow{\alpha^+}) \circ \sigma]N' && \text{by lemma 13}\\
&= [(((\sigma \lll \overrightarrow{Q}/\overrightarrow{\beta^+}) \circ \sigma) \lll \overrightarrow{P}/\overrightarrow{\alpha^+}) \circ \sigma]N' && \text{Since } \mathbf{fv}\,([\sigma]N') \cap \overrightarrow{\beta^+} = \varnothing\\
&= [((\sigma \circ \overrightarrow{Q}/\overrightarrow{\beta^+}) \lll \overrightarrow{P}/\overrightarrow{\alpha^+}) \circ \sigma]N' && \text{by lemma 13}\\
&= [(\sigma \circ \overrightarrow{Q}/\overrightarrow{\beta^+}) \lll \overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N'
\end{aligned}$$

Notice that $(\sigma \circ \overrightarrow{Q}/\overrightarrow{\beta^+}) \lll \overrightarrow{P}/\overrightarrow{\alpha^+}$ is a substitution that turns α_i^+ into $[\sigma \circ \overrightarrow{Q}/\overrightarrow{\beta^+}]P_i$, where $\Gamma', \overrightarrow{\gamma^+} \vdash [\sigma \circ \overrightarrow{Q}/\overrightarrow{\beta^+}]P_i$. This way, $\Gamma', \overrightarrow{\gamma^+} \vdash [(\sigma \circ \overrightarrow{Q}/\overrightarrow{\beta^+}) \lll \overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N' \leqslant [\sigma]K'$ means $\Gamma \vdash \forall\overrightarrow{\alpha^+}.[\sigma]N' \leqslant \forall\overrightarrow{\gamma^+}.[\sigma]K'$ by lemma 18, that is $\Gamma \vdash [\sigma]N \leqslant [\sigma]K$, as required.

Now, we can assume that neither $\Gamma \vdash N \leqslant M_1$ nor $\Gamma \vdash M_2 \leqslant K$ is inferred by Rule $(\forall^{\leqslant})$, and that neither M_1 nor M_2 is equivalent to a variable. Because of that, $[\sigma]M_1 = [\sigma]M_2$ means that M_1 and M_2 have the same outer constructor. Let us consider the shape of M_1.

Case 1. $M_1 = \alpha^-$ this case has been considered;

Case 2. $M_1 = \forall\overrightarrow{\beta^+}.M_1'$ this case has been considered;

Case 3. $M_1 = \uparrow Q_1$. Then as noted above, $[\sigma]M_1 = [\sigma]M_2$ means that $M_2 = \uparrow Q_2$ and $[\sigma]Q_1 = [\sigma]Q_2$. Moreover, $\Gamma \vdash N \leqslant \uparrow Q_1$ can only be inferred by Rule $(\uparrow^{\leqslant})$, and thus, $N = \uparrow P$, and by inversion, $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash Q_1 \geqslant P$. Analogously, $\Gamma \vdash \uparrow Q_2 \leqslant K$ means that $K = \uparrow R$, $\Gamma \vdash Q_2 \geqslant R$, and $\Gamma \vdash R \geqslant Q_2$.

Notice that the derivations of $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash Q_1 \geqslant P$ are proper sub-derivations of $\Gamma \vdash N \leqslant M_1$, and the derivations of $\Gamma \vdash Q_2 \geqslant R$ and $\Gamma \vdash R \geqslant Q_2$ are proper sub-derivations of $\Gamma \vdash M_2 \leqslant K$. This way, the induction hypothesis is applicable:

- applying the induction hypothesis to $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash Q_2 \geqslant R$ with $\Gamma' \vdash \sigma : \Gamma$ unifying Q_1 and Q_2, we obtain $\Gamma' \vdash [\sigma]P \geqslant [\sigma]R$;
- applying the induction hypothesis to $\Gamma \vdash R \geqslant Q_2$ and $\Gamma \vdash Q_1 \geqslant P$ with $\Gamma' \vdash \sigma : \Gamma$ unifying Q_2 and Q_1, we obtain $\Gamma' \vdash [\sigma]R \geqslant [\sigma]P$.

This way, by Rule $(\uparrow^{\leqslant})$, $\Gamma' \vdash [\sigma]N \leqslant [\sigma]K$, as required.

Case 4. $M_1 = Q_1 \to M_1'$. Then as noted above, $[\sigma]M_1 = [\sigma]M_2$ means that $M_2 = Q_2 \to M_2'$, $[\sigma]Q_1 = [\sigma]Q_2$, and $[\sigma]M_1' = [\sigma]M_2'$. Moreover, $\Gamma \vdash N \leqslant Q_1 \to M_1'$ can only be inferred by Rule $(\to^{\leqslant})$, and thus, $N = P \to N'$, and by inversion, $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash N' \leqslant M_1'$. Analogously, $\Gamma \vdash Q_2 \to M_2' \leqslant K$ means that $K = R \to K'$, $\Gamma \vdash Q_2 \geqslant R$, and $\Gamma \vdash M_2' \leqslant K'$.

Notice that the derivations of $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash N' \leqslant M_1'$ are proper sub-derivations of $\Gamma \vdash P \to N' \leqslant Q_1 \to M_1'$, and the derivations of $\Gamma \vdash Q_2 \geqslant R$ and $\Gamma \vdash M_2' \leqslant K'$ are proper sub-derivations of $\Gamma \vdash Q_2 \to M_2' \leqslant R \to K'$. This way, the induction hypothesis is applicable:

- applying the induction hypothesis to $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash Q_2 \geqslant R$ with $\Gamma' \vdash \sigma : \Gamma$ unifying Q_1 and Q_2, we obtain $\Gamma' \vdash [\sigma]P \geqslant [\sigma]R$;
- applying the induction hypothesis to $\Gamma \vdash N' \leqslant M_1'$ and $\Gamma \vdash M_2' \leqslant K'$ with $\Gamma' \vdash \sigma : \Gamma$ unifying M_1' and M_2', we obtain $\Gamma' \vdash [\sigma]N' \leqslant [\sigma]K'$.

This way, by Rule $(\to^{\leqslant})$, $\Gamma' \vdash [\sigma]P \to [\sigma]N' \leqslant [\sigma]R \to [\sigma]K'$, that is $\Gamma' \vdash [\sigma]N \leqslant [\sigma]K$, as required.

After that we consider all the analogous positive cases, and prove them symmetrically. □

Corollary 10 (Transitivity of equivalence)**.** *Assuming the types are well-formed in* Γ*,*

$-$ *if* $\Gamma \vdash N_1 \simeq^{\leqslant} N_2$ *and* $\Gamma \vdash N_2 \simeq^{\leqslant} N_3$ *then* $\Gamma \vdash N_1 \simeq^{\leqslant} N_3$*,*

$+$ *if* $\Gamma \vdash P_1 \simeq^{\leqslant} P_2$ *and* $\Gamma \vdash P_2 \simeq^{\leqslant} P_3$ *then* $\Gamma \vdash P_1 \simeq^{\leqslant} P_3$*.*

5.4 Variable Ordering

Definition 25 (Collision free bijection)**.** *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is* ***collision free on sets*** *P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 25 (Soundness of variable ordering)**.** *Variable ordering extracts used free variables.*

$-$ $\mathbf{ord}\, vars\, \mathbf{in}\, N = vars \cap \mathbf{fv}\, N$ *(as sets)*

$+$ $\mathbf{ord}\, vars\, \mathbf{in}\, P = vars \cap \mathbf{fv}\, P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\, vars\, \mathbf{in}\, N = \vec{\alpha}$ and $\mathbf{ord}\, vars\, \mathbf{in}\, P = \vec{\alpha}$ ☐

Corollary 11 (Additivity of ordering)**.** *Variable ordering is additive (in terms of set union) with respect to its first argument.*

$-$ $\mathbf{ord}\, (vars_1 \cup vars_2)\, \mathbf{in}\, N \equiv \mathbf{ord}\, vars_1\, \mathbf{in}\, N \cup \mathbf{ord}\, vars_2\, \mathbf{in}\, N$ *(as sets)*

$+$ $\mathbf{ord}\, (vars_1 \cup vars_2)\, \mathbf{in}\, P \equiv \mathbf{ord}\, vars_1\, \mathbf{in}\, P \cup \mathbf{ord}\, vars_2\, \mathbf{in}\, P$ *(as sets)*

Corollary 12 (Weakening of ordering)**.** *Extending the first argument of the ordering with unused variables does not change the result.*

$-$ $\mathbf{ord}\, (vars \cap \mathbf{fv}\, N)\, \mathbf{in}\, N = \mathbf{ord}\, vars\, \mathbf{in}\, N$

$+$ $\mathbf{ord}\, (vars \cap \mathbf{fv}\, P)\, \mathbf{in}\, P = \mathbf{ord}\, vars\, \mathbf{in}\, P$

Corollary 13 (Idempotency of ordering)**.**

$-$ *If* $\mathbf{ord}\, vars\, \mathbf{in}\, N = \vec{\alpha}$ *then* $\mathbf{ord}\, \vec{\alpha}\, \mathbf{in}\, N = \vec{\alpha}$*,*

$+$ *If* $\mathbf{ord}\, vars\, \mathbf{in}\, P = \vec{\alpha}$ *then* $\mathbf{ord}\, \vec{\alpha}\, \mathbf{in}\, P = \vec{\alpha}$*;*

Proof. By lemma 25 and corollary 12. ☐

☐

Lemma 26 (Distributivity of renaming over variable ordering)**.** *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

$-$ *If μ is collision free on vars and* $\mathbf{fv}\, N$ *then* $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = \mathbf{ord}\, ([\mu]vars)\, \mathbf{in}\, [\mu]N$

$+$ *If μ is collision free on vars and* $\mathbf{fv}\, P$ *then* $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, P) = \mathbf{ord}\, ([\mu]vars)\, \mathbf{in}\, [\mu]P$

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$
Then
$$\begin{aligned} [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) \\ &= [\mu]\alpha^- && \text{by Rule } (\text{Var}^+_\in) \\ &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars) \\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \beta^- && \text{by Rule } (\text{Var}^+_\in)\text{, because } \beta^- \in [\mu]vars \\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- \end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$
Notice that $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = \cdot$ by Rule $(\text{Var}^+_\notin)$. On the other hand, $\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- = \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \alpha^- = \cdot$ The latter equality is from Rule $(\text{Var}^+_\notin)$, because μ is collision free on *vars* and $\mathbf{fv}\, N$, so $\mathbf{fv}\, N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$
Then $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = \cdot$ by Rule $(\text{Var}^+_\notin)$. To prove that $\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- = \cdot$, we apply Rule $(\text{Var}^+_\notin)$. Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

(i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.

(ii) Since μ is collision free on *vars* and $\mathbf{fv}\, N$, $\mu(A \cap \mathbf{fv}\, N) \ni \mu(\alpha^-) \notin \mathit{vars}$.

d. $\alpha^- \notin A$ but $\alpha^- \in \mathit{vars}$
$\mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]\alpha^- = \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, \alpha^- = \alpha^-$. The latter is by Rule ($\mathrm{Var}^+_{\notin}$), because $\alpha^- = [\mu]\alpha^- \in [\mu]\mathit{vars}$ since $\alpha^- \in \mathit{vars}$. On the other hand, $[\mu](\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$

$$
\begin{aligned}
[\mu](\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \uparrow P) \\
&= [\mu](\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, P) && \text{by Rule } (\uparrow) \\
&= \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]P && \text{by the induction hypothesis} \\
&= \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, \uparrow [\mu]P && \text{by Rule } (\uparrow) \\
&= \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]\uparrow P && \text{by the definition of substitution} \\
&= \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]N
\end{aligned}
$$

Case 3. $N = P \to M$

$$
\begin{aligned}
[\mu](\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, P \to M) \\
&= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)) && \text{where } \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, M = \vec{\alpha}_2 \\
&= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \vec{\alpha}_1) \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to case 1. Notice that } \mu \text{ is} \\
& && \text{collision free on } \vec{\alpha}_1 \text{ and } \vec{\alpha}_2 \text{ since } \vec{\alpha}_1 \subseteq \mathit{vars} \text{ and } \vec{\alpha}_2 \subseteq \mathbf{fv}\, N \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]N) &= (\mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]P \to [\mu]M) \\
&= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \vec{\beta}_1)) && \text{where } \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]M = \vec{\beta}_2 \\
& && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)
\end{aligned}
$$

Case 4. $N = \forall \overrightarrow{\alpha^+}. M$

$$
\begin{aligned}
[\mu](\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, N) &= [\mu]\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}. M \\
&= [\mu]\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, M \\
&= \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]M && \text{by the induction hypothesis}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]N) &= \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]\forall \overrightarrow{\alpha^+}. M \\
&= \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}. [\mu]M \\
&= \mathbf{ord}\, [\mu]\mathit{vars}\, \mathbf{in}\, [\mu]M
\end{aligned}
$$

□

Lemma 27 (Ordering is not affected by independent substitutions)**.** *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2, and vars is a set of variables disjoint with both Γ_1 and Γ_2, N and P are types. Then*

− $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, [\sigma]N = \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, N$

+ $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, [\sigma]P = \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, P$

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
If $\alpha^- \notin \Gamma_1$ then $[\sigma]\alpha^- = \alpha^-$ and $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, [\sigma]\alpha^- = \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \alpha^-$, as requried. If $\alpha^- \in \Gamma_1$ then $\alpha^- \notin \mathit{vars}$, so $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \alpha^- = \cdot$. Moreover, $\Gamma_2 \vdash \sigma : \Gamma_1$ means $\mathbf{fv}\, ([\sigma]\alpha^-) \subseteq \Gamma_2$, and thus, as a set, $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, [\sigma]\alpha^- = \mathit{vars} \cap \mathbf{fv}\, ([\sigma]\alpha^-) \subseteq \mathit{vars} \cap \Gamma_2 = \cdot$.

Case 2. $N = \forall \overrightarrow{\alpha^+}. M$
We can assume $\overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing$ and $\overrightarrow{\alpha^+} \cap \mathit{vars} = \varnothing$. Then

$$
\begin{aligned}
\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, [\sigma]N &= \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, [\sigma]\forall \overrightarrow{\alpha^+}. M \\
&= \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}. [\sigma]M \\
&= \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, [\sigma]M && \text{by the induction hypothesis} \\
&= \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, M \\
&= \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}. M \\
&= \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, N
\end{aligned}
$$

Case 3. $N = \uparrow P$

$$
\begin{aligned}
\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,[\sigma]N &= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,[\sigma]\uparrow P \\
&= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\uparrow[\sigma]P && \text{by the definition of substitution} \\
&= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,[\sigma]P && \text{by the induction hypothesis} \\
&= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P && \text{by the definition of substitution} \\
&= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\uparrow P && \text{by the definition of ordering} \\
&= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N
\end{aligned}
$$

Case 4. $N = P \to M$

$$
\begin{aligned}
\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,[\sigma]N &= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,[\sigma](P \to M) \\
&= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,([\sigma]P \to [\sigma]M) && \text{by the definition of substitution} \\
&= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,[\sigma]P, (\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,[\sigma]M \backslash \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,[\sigma]P) && \text{by the definition of ordering} \\
&= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P, (\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,M \backslash \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P) && \text{by the induction hypothesis} \\
&= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P \to M && \text{by the definition of ordering} \\
&= \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N
\end{aligned}
$$

Case 5. The proofs of the positive cases are symmetric.

□

Lemma 28 (Completeness of variable ordering)**.** *Variable ordering is invariant under equivalence. For arbitrary vars,*

− *If* $N \simeq^D M$ *then* $\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,M$ *(as lists)*

\+ *If* $P \simeq^D Q$ *then* $\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,Q$ *(as lists)*

Proof. Mutual induction on $N \simeq^D M$ and $P \simeq^D Q$. □

5.5 Normaliztaion

Lemma 29. *Set of free variables is invariant under equivalence.*

− *If* $N \simeq^D M$ *then* $\mathbf{fv}\,N \equiv \mathbf{fv}\,M$ *(as sets)*

\+ *If* $P \simeq^D Q$ *then* $\mathbf{fv}\,P \equiv \mathbf{fv}\,Q$ *(as sets)*

Proof. Straightforward mutual induction on $N \simeq^D M$ and $P \simeq^D Q$ □

Lemma 30. *Free variables are not changed by the normalization*

− $\mathbf{fv}\,N \equiv \mathbf{fv}\,\mathbf{nf}\,(N)$

\+ $\mathbf{fv}\,P \equiv \mathbf{fv}\,\mathbf{nf}\,(P)$

Proof. By straightforward induction on N and mutually on P. □

Lemma 31 (Soundness of normalization)**.**

− $N \simeq^D \mathbf{nf}\,(N)$

\+ $P \simeq^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var$^-$) and (Var$^+$)
By the corresponding equivalence rules.

Case 2. ($\uparrow$), ($\downarrow$), and ($\to$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall$), i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'$
From the induction hypothesis, we know that $N \simeq^D N'$. In particular, by lemma 29, $\mathbf{fv}\,N \equiv \mathbf{fv}\,N'$. Then by lemma 25, $\overrightarrow{\alpha^+}' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$, and thus, $\overrightarrow{\alpha^+}' \cap \mathbf{fv}\,N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$.

To prove $\forall\overrightarrow{\alpha^+}.N \simeq^D \forall\overrightarrow{\alpha^+}'.N'$, it suffices to provide a bijection $\mu : \overrightarrow{\alpha^+}' \cap \mathbf{fv}\,N' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$ such that $N \simeq^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. ($\exists$) Same as for case 3.

$\square$

Corollary 14 (Normalization preserves ordering)**.** *For any vars,*

$-$ $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \mathbf{nf}\,(N) = \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, M$

$+$ $\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, \mathbf{nf}\,(P) = \mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, Q$

Proof. Immediately from lemmas 28 and 31. $\square$

Lemma 32 (Distributivity of normalization over substitution)**.** *Normalization of a term distributes over substitution. Suppose that σ is a substitution, N and P are types. Then*

$-$ $\mathbf{nf}\,([\sigma]N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$

$+$ $\mathbf{nf}\,([\sigma]P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P)$

where $\mathbf{nf}\,(\sigma)$ *means pointwise normalization:* $[\mathbf{nf}\,(\sigma)]\alpha^- = \mathbf{nf}\,([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.

Case 2. $P = \alpha^+$
Similar to case 1.

Case 3. If the type is formed by $\to$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \to M$ then

$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma](P \to M))$	
$= \mathbf{nf}\,([\sigma]P \to [\sigma]M)$	By the congruence of substitution
$= \mathbf{nf}\,([\sigma]P) \to \mathbf{nf}\,([\sigma]M)$	By the congruence of normalization, i.e. Rule ($\to$)
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) \to [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$	By the induction hypothesis
$= [\mathbf{nf}\,(\sigma)](\mathbf{nf}\,(P) \to \mathbf{nf}\,(M))$	By the congruence of substitution
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P \to M)$	By the congruence of normalization
$= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$	

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M)$	
$= [\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$	Where $\overrightarrow{\alpha^+}' = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(M) = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M$ (the latter is by corollary 14)

$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\forall\overrightarrow{\alpha^+}.M)$	
$= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\sigma]M)$	Assuming $\overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing$ and $\overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$
$= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\sigma]M)$	Where $\overrightarrow{\beta^+} = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,([\sigma]M) = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M$ (the latter is by corollary 14)
$= \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\sigma]M)$	By lemma 27, $\overrightarrow{\beta^+} = \overrightarrow{\alpha^+}'$ since $\overrightarrow{\alpha^+}$ is disjoint with Γ_1 and Γ_2
$= \forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$	By the induction hypothesis

To show alpha-equivalence of $[\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$, we can assume that $\overrightarrow{\alpha^+}' \cap \Gamma_1 = \varnothing$, and $\overrightarrow{\alpha^+}' \cap \Gamma_2 = \varnothing$.

Case 5. $P = \exists\overrightarrow{\alpha^-}.Q$
Same as for case 4.

$\square$

Corollary 15 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

$-$ $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$

+ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Proof. Immediately from lemma 32, after noticing that $\mathbf{nf}\,(\mu) = \mu$. □

Lemma 33 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

– *If* $N \simeq^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

+ *If* $P \simeq^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

Proof. Mutual induction on $N \simeq^D M$ and $P \simeq^D Q$.

Case 1. $(\forall^{\simeq^D})$

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 25 and 30, the domain and the codomain of μ can be written as $\mu : \overrightarrow{\beta^+}' \leftrightarrow \overrightarrow{\alpha^+}'$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by corollary 15, the second—by the induction hypothesis.

(ii)
$$
\begin{aligned}
[\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}' \\
&= [\mu]\mathbf{ord}\,(\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 30 and corollary 12} \\
&= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 26, because } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \\
& && \text{and } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \\
&= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved} \\
&= \mathbf{ord}\,(\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \text{ and } \overrightarrow{\beta^+} \cap \mathbf{fv}\,M \\
&= \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 30 and corollary 12} \\
&= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'
\end{aligned}
$$

Case 2. $(\exists^{\simeq^D})$ Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□

Lemma 34 (Idempotence of normalization)**.** *Normalization is idempotent*

– $\mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$

+ $\mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

Proof. By applying lemma 33 to lemma 31. □

Lemma 35. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized. Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then*

+ $[\sigma]P$ *is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \textit{is normal} \\ P & \textit{is normal} \end{cases}$

– $[\sigma]N$ *is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \textit{is normal} \\ N & \textit{is normal} \end{cases}$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$

Then N is always normal, and the normality of $\sigma|_{\alpha^-}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$
\begin{aligned}
[\sigma](P \to M) \text{ is normal} &\iff [\sigma]P \to [\sigma]M \text{ is normal} && \text{by the substitution congruence}\\
&\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases}\\
&\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(M)} & \text{is normal} \end{cases} && \text{by the induction hypothesis}\\
&\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P) \cup \mathbf{fv}\,(M)} & \text{is normal} \end{cases}\\
&\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P \to M)} & \text{is normal} \end{cases}
\end{aligned}
$$

Case 3. $N = \uparrow P$

By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$
\begin{aligned}
[\sigma](\forall\overrightarrow{\alpha^+}.M) \text{ is normal} &\iff (\forall\overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} && \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing\\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+} \end{cases} && \text{by the definition of normalization}\\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} && \text{by lemma 27}\\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} && \text{by the induction hypothesis}\\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M)} \text{ is normal} \\ \forall\overrightarrow{\alpha^+}.M \text{ is normal} \end{cases} && \text{since } \mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M);\ \text{by the definition of normalization}
\end{aligned}
$$

Case 5. $P = \ldots$

The positive cases are done in the same way as the negative ones.

□

5.6 Equivalence

Lemma 36 (Declarative Equivalence is invariant under bijections)**.** *Suppose μ is a bijection $\mu : vars_1 \leftrightarrow vars_2$, then*

+ *$P_1 \simeq^D P_2$ implies $[\mu]P_1 \simeq^D [\mu]P_2$, and there exists an inference tree of $[\mu]P_1 \simeq^D [\mu]P_2$ with the same shape as the one inferring $P_1 \simeq^D P_2$;*

− *$N_1 \simeq^D N_2$ implies $[\mu]N_1 \simeq^D [\mu]N_2$, and there exists an inference tree of $[\mu]N_1 \simeq^D [\mu]N_2$ with the same shape as the one inferring $N_1 \simeq^D N_2$.*

Proof. We prove it by induction on $P_1 \simeq^D P_2$ and mutually, on $N_1 \simeq^D N_2$. Let us consider the last rule used in the derivation.

Case 1. Rule $(\forall^{\simeq^D})$

Then we decompose N_1 as $\forall\overrightarrow{\alpha^+}_1.M_1$ and N_2 as $\forall\overrightarrow{\alpha^+}_2.M_2$, where M_1 and M_2 do not start with $\forall$-quantifiers. where $|\overrightarrow{\alpha^+}_1|+|\overrightarrow{\alpha^+}_2| > 0$. By convention, let us assume that $\overrightarrow{\alpha^+}_1$ and $\overrightarrow{\alpha^+}_2$ are disjoint form $vars_2$ and $vars_1$.

By inversion, $\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\,M_2 = \varnothing$ and $M_1 \simeq^D [\mu']M_2$ for some bijection $\mu' : (\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\,M_2) \leftrightarrow (\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\,M_1)$. Then let us apply the induction hypothesis to $M_1 \simeq^D [\mu']M_2$ to obtain $[\mu]M_1 \simeq^D [\mu][\mu']M_2$ inferred by the tree of the same shape as $M_1 \simeq^D [\mu']M_2$.

Notice that $[\mu]M_1$ and $[\mu]M_2$ do not start with $\forall$, That is $[\mu]\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D [\mu]\forall\overrightarrow{\alpha^+}_2.M_2$, rewritten as $\forall\overrightarrow{\alpha^+}_1.[\mu]M_1 \simeq^D \forall\overrightarrow{\alpha^+}_2.[\mu]M_2$, can be inferred by Rule $(\forall^{\simeq^D})$:

1. $\overrightarrow{\alpha^+}_1$ is disjoint from $vars_2 \cup \mathbf{fv}\, M_2 \subseteq \mathbf{fv}\,[\mu]M_2$;
2. $[\mu]M_1 \simeq^D [\mu'][\mu]M_2$ because $[\mu'][\mu]M_2 = [\mu][\mu']M_2$ (by corollary 5: $\mu' : (\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, M_2) \leftrightarrow (\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\, M_1)$, $\mu : vars_1 \leftrightarrow vars_2$, $vars_1$ is disjoint from $\overrightarrow{\alpha^+}_2$ and $\overrightarrow{\alpha^+}_1$; $\overrightarrow{\alpha^+}_2$ is disjoint from $vars_1$ and $vars_2$)

Notice that it is the same rule as the one inferring $N_1 \simeq^D N_2$, and thus, the shapes of the trees are the same.

Case 2. Rule ($\text{Var}^{-\simeq^D}$)
Then $N_1 = N_2 = \alpha^-$, and the required $[\mu]\alpha^- = [\mu]\alpha^-$ is also inferred by Rule ($\text{Var}^{-\simeq^D}$), since $[\mu]\alpha^-$ is a variable.

Case 3. Rule ($\rightarrow^{\simeq^D}$)
Then we are proving that $P_1 \rightarrow M_1 \simeq^D P_2 \rightarrow M_2$ implies $[\mu](P_1 \rightarrow M_1) \simeq^D [\mu](P_2 \rightarrow M_2)$ (preserving the tree structure).

By inversion, we have $P_1 \simeq^D P_2$ and $M_1 \simeq^D M_2$, and thus, by the induction hypothesis, $[\mu]P_1 \simeq^D [\mu]P_2$ and $[\mu]M_1 \simeq^D [\mu]M_2$. Then $[\mu](P_1 \rightarrow M_1) \simeq^D [\mu](P_2 \rightarrow M_2)$, or in other words, $[\mu]P_1 \rightarrow [\mu]M_1 \simeq^D [\mu]P_2 \rightarrow [\mu]M_2$, is inferred by the same rule—Rule ($\rightarrow^{\simeq^D}$).

Case 4. Rule ($\uparrow^{\simeq^D}$) This case is done by similar congruent arguments as the previous one.

Case 5. The positive cases are proved symmetrically.

□

Lemma 37 (Declarative equivalence is transitive)**.**

+ *if* $P_1 \simeq^D P_2$ *and* $P_2 \simeq^D P_3$ *then* $P_1 \simeq^D P_3$,

− *if* $N_1 \simeq^D N_2$ *and* $N_2 \simeq^D N_3$ *then* $N_1 \simeq^D N_3$.

Proof. We prove it by $\mathsf{size}(P_1 \simeq^D P_2) + \mathsf{size}(P_2 \simeq^D P_3)$ and mutually, $\mathsf{size}(N_1 \simeq^D N_2) + \mathsf{size}(N_2 \simeq^D N_3)$, where by size, we mean the size of the nodes in the corresponding inference tree.

Case 1. First, let us consider the case when either $N_1 \simeq^D N_2$ or $N_2 \simeq^D N_3$ is inferred by Rule ($\forall^{\simeq^D}$). Let us decompose N_1, N_2, and N_3 as follows: $N_1 = \forall\overrightarrow{\alpha^+}_1.M_1$, $N_2 = \forall\overrightarrow{\alpha^+}_2.M_2$, and $N_3 = \forall\overrightarrow{\alpha^+}_3.M_3$.

Then by inversion of $\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D \forall\overrightarrow{\alpha^+}_2.M_2$ (or if $\overrightarrow{\alpha^+}_1$ and $\overrightarrow{\alpha^+}_2$ are both empty, by assumption) :

1. $\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\, M_2 = \varnothing$ and
2. there exists a bijection on variables $\mu_1 : (\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, M_2) \leftrightarrow (\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\, M_1)$ such that $M_1 \simeq^D [\mu_1]M_2$.

Analogously, $\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D \forall\overrightarrow{\alpha^+}_2.M_2$ implies:

1. $\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, M_3 = \varnothing$ and
2. $M_2 \simeq^D [\mu_2]M_3$ for some bijection $\mu_2 : (\overrightarrow{\alpha^+}_3 \cap \mathbf{fv}\, M_3) \leftrightarrow (\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, M_2)$.

Notice that either $M_1 \simeq^D [\mu_1]M_2$ is inferred by a proper sub-tree of $\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D \forall\overrightarrow{\alpha^+}_2.M_2$ or $M_2 \simeq^D [\mu_2]M_3$ is inferred by a proper sub-tree of $\forall\overrightarrow{\alpha^+}_2.M_2 \simeq^D \forall\overrightarrow{\alpha^+}_3.M_3$.

Then by lemma 36, $[\mu_1]M_2 \simeq^D [\mu_1 \circ \mu_2]M_3$ and moreover, $\mathsf{size}([\mu_1]M_2 \simeq^D [\mu_1 \circ \mu_2]M_3) = \mathsf{size}(M_2 \simeq^D [\mu_2]M_3)$.

Since at least one of the trees inferring $M_1 \simeq^D [\mu_1]M_2$ and $M_2 \simeq^D [\mu_2]M_3$ is a proper sub-tree of the corresponding original tree, $\mathsf{size}(M_1 \simeq^D [\mu_1]M_2) + \mathsf{size}(M_2 \simeq^D [\mu_2]M_3) < \mathsf{size}(\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D \forall\overrightarrow{\alpha^+}_2.M_2) + \mathsf{size}(\forall\overrightarrow{\alpha^+}_2.M_2 \simeq^D \forall\overrightarrow{\alpha^+}_3.M_3)$, i.e., the induction hypothesis is applicable.

By the induction hypothesis, $M_1 \simeq^D [\mu_1 \circ \mu_2]M_3$. Where $\mu_1 \circ \mu_2$ is a bijection on variables $\mu_1 \circ \mu_2 : (\overrightarrow{\alpha^+}_3 \cap \mathbf{fv}\, M_3) \leftrightarrow (\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\, M_1)$. Then $\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D \forall\overrightarrow{\alpha^+}_3.M_3$ by Rule ($\forall^{\simeq^D}$).

Once this case has been considered, we can assume that neither $N_1 \simeq^D N_2$ nor $N_2 \simeq^D N_3$ is inferred by Rule ($\forall^{\simeq^D}$).

Case 2. $N_1 \simeq^D N_2$ is inferred by Rule ($\text{Var}^{-\simeq^D}$)
Then $N_1 = N_2 = \alpha^-$, and thus, $N_1 \simeq^D N_3$ holds since $N_2 \simeq^D N_3$.

Case 3. $N_1 \simeq^D N_2$ is inferred by Rule ($\rightarrow^{\simeq^D}$)
Then $N_1 = P_1 \rightarrow M_1$ and $N_2 = P_2 \rightarrow M_2$, and by inversion, $P_1 \simeq^D P_2$ and $M_1 \simeq^D M_2$.

Moreover, since N_3 does not start with $\forall$, $N_2 \simeq^D N_3$ is also inferred by Rule ($\rightarrow^{\simeq^D}$), which means that $N_3 = P_3 \rightarrow M_3$, $P_2 \simeq^D P_3$, and $M_2 \simeq^D M_3$.

Then by the induction hypothesis, $P_1 \simeq^D P_3$ and $M_1 \simeq^D M_3$, and thus, $P_1 \rightarrow M_1 \simeq^D P_3 \rightarrow M_3$ by Rule ($\rightarrow^{\simeq^D}$).

Case 4. $N_1 \simeq^D N_2$ is inferred by Rule $(\to^{\simeq^D})$
For this case, the reasoning is the same as for the previous one.

Case 5. The positive cases are proved symmetrically.

$\square$

Lemma 38 (Type well-formedness is invariant under equivalence)**.** *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,

− *if* $N \simeq^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. We prove it by induction on $P \simeq^D Q$ and mutually, on $N \simeq^D M$. Let us consider the last rule used in the derivation.

Case 1. Rule $(\mathrm{Var}^{-\simeq^D})$, that is $N \simeq^D M$ has shape $\alpha^- \simeq^D \alpha^-$.
Than $\Gamma \vdash P \iff \Gamma \vdash Q$ is rewritten as $\Gamma \vdash \alpha^- \iff \Gamma \vdash \alpha^-$, which holds trivially.

Case 2. Rule $(\uparrow^{\simeq^D})$, that is $N \simeq^D M$ has shape $\uparrow P \simeq^D \uparrow Q$.
By inversion, $P \simeq^D Q$, and by the induction hypothesis, $\Gamma \vdash P \iff \Gamma \vdash Q$. Also notice that $\Gamma \vdash \uparrow P \iff \Gamma \vdash P$ and $\Gamma \vdash \uparrow Q \iff \Gamma \vdash Q$ by inversion and Rule WFTShiftU. This way, $\Gamma \vdash \uparrow P \iff \Gamma \vdash P \iff \Gamma \vdash Q \iff \Gamma \vdash \uparrow Q$.

Case 3. Rule $(\to^{\simeq^D})$, that is $N \simeq^D M$ has shape $P \to N' \simeq^D Q \to M'$.
Then by inversion, $P \simeq^D Q$ and $N' \simeq^D M'$, and by the induction hypothesis, $\Gamma \vdash P \iff \Gamma \vdash Q$ and $\Gamma \vdash N' \iff \Gamma \vdash M'$.

$$\begin{aligned}
\Gamma \vdash P \to N' &\iff \Gamma \vdash P \text{ and } \Gamma \vdash N' && \text{by inversion and Rule WFTArrow}\\
&\iff \Gamma \vdash Q \text{ and } \Gamma \vdash M' && \text{as noted above}\\
&\iff \Gamma \vdash Q \to M' && \text{by Rule WFTArrow and inversion}
\end{aligned}$$

Case 4. Rule $(\forall^{\simeq^D})$, that is $N \simeq^D M$ has shape $\forall\overrightarrow{\alpha^+}.N' \simeq^D \forall\overrightarrow{\beta^+}.M'$.
By inversion, $\forall\overrightarrow{\alpha^+}.N' \simeq^D \forall\overrightarrow{\beta^+}.M'$ means that $\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing$ and that there exists a bijection on variables $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M') \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N')$ such that $N' \simeq^D [\mu]M'$.

By inversion and Rule WFTForall, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N'$ is equivalent to $\Gamma, \overrightarrow{\alpha^+} \vdash N'$, and by section 5.3, it is equivalent to $\Gamma, (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N') \vdash N'$, which, by the induction hypothesis, is equivalent to $\Gamma, (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N') \vdash [\mu]M'$.

Analogously, $\Gamma \vdash \forall\overrightarrow{\beta^+}.M'$ is equivalent to $\Gamma, (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M') \vdash M'$. By lemma 5, it implies $\Gamma, (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M') \vdash [\mu]M'$. And vice versa, $\Gamma, (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M') \vdash [\mu]M'$ implies $\Gamma, (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M') \vdash [\mu^{-1}][\mu]M'$.

This way, both $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N'$ and $\Gamma \vdash \forall\overrightarrow{\beta^+}.M'$ are equivalent to $\Gamma, (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N') \vdash [\mu]M'$.

Case 5. For the cases of the positive types, the proofs are symmetric.

$\square$

Corollary 16 (Normalization preserves well-formedness)**.**

+ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}\,(P)$,

− $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}\,(N)$

Proof. Immediately from lemmas 31 and 38. $\square$

Corollary 17 (Normalization preserves well-formedness of substitution)**.**
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}\,(\sigma) : \Gamma_1$

Proof. Let us prove the forward direction. Suppose that $\alpha^\pm \in \Gamma_1$. Let us show that $\Gamma_2 \vdash [\mathbf{nf}\,(\sigma)]\alpha^\pm$. By the definition of substitution normalization, $[\mathbf{nf}\,(\sigma)]\alpha^\pm = \mathbf{nf}\,([\sigma]\alpha^\pm)$. Then by corollary 16, to show $\Gamma_2 \vdash \mathbf{nf}\,([\sigma]\alpha^\pm)$, it suffices to show $\Gamma_2 \vdash [\sigma]\alpha^\pm$, which holds by the assumption $\Gamma_2 \vdash \sigma : \Gamma_1$.
The backward direction is proved analogously. $\square$

Lemma 39 (Normalization preserves substitution signature)**.** *Suppose that* σ *is a substitution,* Γ_1 *and* Γ_2 *are contexts. Then* $\Gamma_2 \vdash \sigma : \Gamma_1$ *implies* $\Gamma_2 \vdash \mathbf{nf}\,(\sigma) : \Gamma_1$.

Proof. Suppose that $\alpha^\pm \in \Gamma_1$. Then by corollary 16, $\Gamma_2 \vdash \mathbf{nf}\,([\sigma]\alpha^\pm) = [\mathbf{nf}\,(\sigma)]\alpha^\pm$ is equivalent to $\Gamma_2 \vdash [\sigma]\alpha^\pm$.
Suppose that $\alpha^\pm \notin \Gamma_1$. $\Gamma_2 \vdash \sigma : \Gamma_1$ means that $[\sigma]\alpha^\pm = \alpha^\pm$, and then $[\mathbf{nf}\,(\sigma)]\alpha^\pm = \mathbf{nf}\,([\sigma]\alpha^\pm) = \mathbf{nf}\,(\alpha^\pm) = \alpha^\pm$. $\square$

Lemma 40 (Soundness of equivalence)**.** *Declarative equivalence implies mutual subtyping.*

+ *if* $\Gamma \vdash P$, $\Gamma \vdash Q$, *and* $P \simeq^D Q$ *then* $\Gamma \vdash P \simeq^\leqslant Q$,

− *if* $\Gamma \vdash N$, $\Gamma \vdash M$, *and* $N \simeq^D M$ *then* $\Gamma \vdash N \simeq^\leqslant M$.

Proof. We prove it by mutual induction on $P \simeq^D Q$ and $N \simeq^D M$.

Case 1. $\alpha^- \simeq^D \alpha^-$
Then $\Gamma \vdash \alpha^- \leqslant \alpha^-$ by Rule (Var$^{-\leqslant}$), which immediately implies $\Gamma \vdash \alpha^- \simeq^\leqslant \alpha^-$ by Rule ($\simeq^\leqslant -$).

Case 2. $\uparrow P \simeq^D \uparrow Q$
Then by inversion of Rule ($\uparrow^\leqslant$), $P \simeq^D Q$, and from the induction hypothesis, $\Gamma \vdash P \simeq^\leqslant Q$, and (by symmetry) $\Gamma \vdash Q \simeq^\leqslant P$.

When Rule ($\uparrow^\leqslant$) is applied to $\Gamma \vdash P \simeq^\leqslant Q$, it gives us $\Gamma \vdash \uparrow P \leqslant \uparrow Q$; when it is applied to $\Gamma \vdash Q \simeq^\leqslant P$, we obtain $\Gamma \vdash \uparrow Q \leqslant \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \simeq^\leqslant \uparrow Q$.

Case 3. $P \to N \simeq^D Q \to M$
Then by inversion of Rule ($\to^\leqslant$), $P \simeq^D Q$ and $N \simeq^D M$. By the induction hypothesis, $\Gamma \vdash P \simeq^\leqslant Q$ and $\Gamma \vdash N \simeq^\leqslant M$, which means by inversion: (i) $\Gamma \vdash P \geqslant Q$, (ii) $\Gamma \vdash Q \geqslant P$, (iii) $\Gamma \vdash N \leqslant M$, (iv) $\Gamma \vdash M \leqslant N$. Applying Rule ($\to^\leqslant$) to (i) and (iii), we obtain $\Gamma \vdash P \to N \leqslant Q \to M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \to M \leqslant P \to N$. Together, it implies $\Gamma \vdash P \to N \simeq^\leqslant Q \to M$.

Case 4. $\forall \overrightarrow{\alpha^+}.N \simeq^D \forall \overrightarrow{\beta^+}.M$
Then by inversion, there exists bijection $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N)$, such that $N \simeq^D [\mu]M$. By the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq^\leqslant [\mu]M$. From lemma 47 and the fact that μ is bijective, we also have $\Gamma, \overrightarrow{\beta^+} \vdash [\mu^{-1}]N \simeq^\leqslant M$.

Let us construct a substitution $\overrightarrow{\alpha^+} \vdash \overrightarrow{P}/\overrightarrow{\beta^+} : \overrightarrow{\beta^+}$ by extending μ with arbitrary positive types on $\overrightarrow{\beta^+} \backslash \mathbf{fv}\, M$.

Notice that $[\mu]M = [\overrightarrow{P}/\overrightarrow{\beta^+}]M$, and therefore, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq^\leqslant [\mu]M$ implies $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{P}/\overrightarrow{\beta^+}]M \leqslant N$. Then by Rule ($\forall^\leqslant$), $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \leqslant \forall \overrightarrow{\alpha^+}.N$.

Analogously, we construct the substitution from μ^{-1}, and use it to instantiate $\overrightarrow{\alpha^+}$ in the application of Rule ($\forall^\leqslant$) to infer $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M$.

This way, $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \leqslant \forall \overrightarrow{\alpha^+}.N$ and $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M$ gives us $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \simeq^\leqslant \forall \overrightarrow{\alpha^+}.N$.

Case 5. For the cases of the positive types, the proofs are symmetric.

□

Corollary 18 (Normalization is sound w.r.t. subtyping-induced equivalence)**.**

+ *if* $\Gamma \vdash P$ *then* $\Gamma \vdash P \simeq^\leqslant \mathbf{nf}\,(P)$,

− *if* $\Gamma \vdash N$ *then* $\Gamma \vdash N \simeq^\leqslant \mathbf{nf}\,(N)$.

Proof. Immediately from lemmas 31 and 40 and corollary 16. □

Corollary 19 (Normalization preserves subtyping)**.** *Assuming all the types are well-formed in context* Γ,

+ $\Gamma \vdash P \geqslant Q \iff \Gamma \vdash \mathbf{nf}\,(P) \geqslant \mathbf{nf}\,(Q)$,

− $\Gamma \vdash N \leqslant M \iff \Gamma \vdash \mathbf{nf}\,(N) \leqslant \mathbf{nf}\,(M)$.

Proof.

+ $\Rightarrow$ Let us assume $\Gamma \vdash P \geqslant Q$. By corollary 18, $\Gamma \vdash P \simeq^\leqslant \mathbf{nf}\,(P)$ and $\Gamma \vdash Q \simeq^\leqslant \mathbf{nf}\,(Q)$, in particular, by inversion, $\Gamma \vdash \mathbf{nf}\,(P) \geqslant P$ and $\Gamma \vdash Q \geqslant \mathbf{nf}\,(Q)$. Then by the transitivity of subtyping (**??**), $\Gamma \vdash \mathbf{nf}\,(P) \geqslant \mathbf{nf}\,(Q)$.

  $\Leftarrow$ Let us assume $\Gamma \vdash \mathbf{nf}\,(P) \geqslant \mathbf{nf}\,(Q)$. Also by corollary 18 and inversion, $\Gamma \vdash P \geqslant \mathbf{nf}\,(P)$ and $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant Q$. Then by the transitivity, $\Gamma \vdash P \geqslant Q$.

− The negative case is proved symmetrically.

□

Lemma 41 (Subtyping induced by disjoint substitutions)**.** *If two disjoint substitutions induce subtyping, they are degenerate (so is the subtyping). Suppose that* $\Gamma \vdash \sigma_1 : \Gamma_1$ *and* $\Gamma \vdash \sigma_2 : \Gamma_1$, *where* $\Gamma_i \subseteq \Gamma$ *and* $\Gamma_1 \cap \Gamma_2 = \varnothing$. *Then*

− *assuming* $\Gamma \vdash N$, $\Gamma \vdash [\sigma_1]N \leqslant [\sigma_2]N$ *implies* $\Gamma \vdash \sigma_i \simeq^\leqslant \mathsf{id} : \mathbf{fv}\, N$

+ *assuming* $\Gamma \vdash P$, $\Gamma \vdash [\sigma_1]P \geqslant [\sigma_2]P$ *implies* $\Gamma \vdash \sigma_i \simeq^\leqslant \mathsf{id} : \mathbf{fv}\, P$

Proof. Proof by induciton on $\Gamma \vdash N$ (and mutually on $\Gamma \vdash P$).

Case 1. $N = \alpha^-$
Then $\Gamma \vdash [\sigma_1]N \leqslant [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\alpha^- \leqslant [\sigma_2]\alpha^-$. Let us consider the following cases:

a. $\alpha^- \notin \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then $\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \alpha^-$ holds immediately, since $[\sigma_i]\alpha^- = [\mathsf{id}]\alpha^- = \alpha^-$ and $\Gamma \vdash \alpha^- \simeq^{\leqslant} \alpha^-$.

b. $\alpha^- \in \Gamma_1$ and $\alpha^- \in \Gamma_2$
This case is not possible by assumption: $\Gamma_1 \cap \Gamma_2 = \varnothing$.

c. $\alpha^- \in \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then we have $\Gamma \vdash [\sigma_1]\alpha^- \leqslant \alpha^-$, which by corollary 8 means $\Gamma \vdash [\sigma_1]\alpha^- \simeq^{\leqslant} \alpha^-$, and hence, $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mathsf{id} : \alpha^-$.
$\Gamma \vdash \sigma_2 \simeq^{\leqslant} \mathsf{id} : \alpha^-$ holds since $[\sigma_2]\alpha^- = \alpha^-$, similarly to case 1.*a*.

d. $\alpha^- \notin \Gamma_1$ and $\alpha^- \in \Gamma_2$
Then we have $\Gamma \vdash \alpha^- \leqslant [\sigma_2]\alpha^-$, which by corollary 8 means $\Gamma \vdash \alpha^- \simeq^{\leqslant} [\sigma_2]\alpha^-$, and hence, $\Gamma \vdash \sigma_2 \simeq^{\leqslant} \mathsf{id} : \alpha^-$.
$\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mathsf{id} : \alpha^-$ holds since $[\sigma_1]\alpha^- = \alpha^-$, similarly to case 1.*a*.

Case 2. $N = \forall \overrightarrow{\alpha^+}.M$
Then by inversion, $\Gamma, \overrightarrow{\alpha^+} \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\forall\overrightarrow{\alpha^+}.M \leqslant [\sigma_2]\forall\overrightarrow{\alpha^+}.M$. By the congruence of substitution and by the inversion of Rule $(\forall^{\leqslant})$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{Q}/\overrightarrow{\alpha^+}][\sigma_1]M \leqslant [\sigma_2]M$, where $\Gamma, \overrightarrow{\alpha^+} \vdash Q_i$. Let us denote the (Kleisli) composition of σ_1 and $\overrightarrow{Q}/\overrightarrow{\alpha^+}$ as σ_1', noting that $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' : \Gamma_1, \overrightarrow{\alpha^+}$, and $(\Gamma_1, \overrightarrow{\alpha^+}) \cap \Gamma_2 = \varnothing$.

Let us apply the induction hypothesis to M and the substitutions σ_1' and σ_2 with $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_1']M \leqslant [\sigma_2]M$ to obtain:

$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{1}$$
$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{2}$$

Then $\Gamma \vdash \sigma_2 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\,\forall\overrightarrow{\alpha^+}.M$ holds by strengthening of 2: for any $\beta^\pm \in \mathbf{fv}\,\forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\,M\backslash\overrightarrow{\alpha^+}$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]\beta^\pm \simeq^{\leqslant} \beta^\pm$ is strengthened to $\Gamma \vdash [\sigma_2]\beta^\pm \simeq^{\leqslant} \beta^\pm$, because $\mathbf{fv}\,[\sigma_2]\beta^\pm = \mathbf{fv}\,\beta^\pm = \{\beta^\pm\} \subseteq \Gamma$.

To show that $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\,\forall\overrightarrow{\alpha^+}.M$, let us take an arbitrary $\beta^\pm \in \mathbf{fv}\,\forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\,M\backslash\overrightarrow{\alpha^+}$.

$$\begin{aligned}
\beta^\pm &= [\mathsf{id}]\beta^\pm && \text{by definition of } \mathsf{id}\\
&\simeq^{\leqslant} [\sigma_1']\beta^\pm && \text{by 1}\\
&= [\overrightarrow{Q}/\overrightarrow{\alpha^+}][\sigma_1]\beta^\pm && \text{by definition of } \sigma_1'\\
&= [\sigma_1]\beta^\pm && \text{because } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,[\sigma_1]\beta^\pm \subseteq \overrightarrow{\alpha^+} \cap \Gamma = \varnothing
\end{aligned}$$

This way, $\Gamma \vdash [\sigma_1]\beta^\pm \simeq^{\leqslant} \beta^\pm$ for any $\beta^\pm \in \mathbf{fv}\,\forall\overrightarrow{\alpha^+}.M$ and thus, $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\,\forall\overrightarrow{\alpha^+}.M$.

Case 3. $N = P \to M$
Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1](P \to M) \leqslant [\sigma_2](P \to M)$, then by congruence of substitution, $\Gamma \vdash [\sigma_1]P \to [\sigma_1]M \leqslant [\sigma_2]P \to [\sigma_2]M$, then by inversion $\Gamma \vdash [\sigma_1]P \geqslant [\sigma_2]P$ and $\Gamma \vdash [\sigma_1]M \leqslant [\sigma_2]M$.

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant [\sigma_2]P$ and to $\Gamma \vdash [\sigma_1]M \leqslant [\sigma_2]M$, we obtain (respectively):

$$\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, P \tag{3}$$
$$\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{4}$$

Noting that $\mathbf{fv}\,(P \to M) = \mathbf{fv}\,P \cup \mathbf{fv}\,M$, we combine eqs. (3) and (4) to conclude: $\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\,(P \to M)$.

Case 4. $N = \uparrow P$
Then by inversion, $\Gamma \vdash P$. $\Gamma \vdash [\sigma_1]N \leqslant [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]{\uparrow}P \leqslant [\sigma_2]{\uparrow}P$, then by congruence of substitution and by inversion, $\Gamma \vdash [\sigma_1]P \geqslant [\sigma_2]P$

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant [\sigma_2]P$, we obtain $\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\,P$. Since $\mathbf{fv}\,{\uparrow}P = \mathbf{fv}\,P$, we can conclude: $\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\,{\uparrow}P$.

Case 5. The positive cases are proved symmetrically.

∎

Corollary 20 (Substitution cannot induce proper subtypes or supertypes)**.** *Assuming all mentioned types are well-formed in* Γ *and* σ *is a substitution* $\Gamma \vdash \sigma : \Gamma$,

$$\begin{aligned}
\Gamma \vdash [\sigma]N \leqslant N \;&\Rightarrow\; \Gamma \vdash [\sigma]N \simeq^{\leqslant} N \text{ and } \Gamma \vdash \sigma \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, N \\
\Gamma \vdash N \leqslant [\sigma]N \;&\Rightarrow\; \Gamma \vdash N \simeq^{\leqslant} [\sigma]N \text{ and } \Gamma \vdash \sigma \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, N \\
\Gamma \vdash [\sigma]P \geqslant P \;&\Rightarrow\; \Gamma \vdash [\sigma]P \simeq^{\leqslant} P \text{ and } \Gamma \vdash \sigma \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, P \\
\Gamma \vdash P \geqslant [\sigma]P \;&\Rightarrow\; \Gamma \vdash P \simeq^{\leqslant} [\sigma]P \text{ and } \Gamma \vdash \sigma \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, P
\end{aligned}$$

Lemma 42. *Asssuming that the mentioned types (P, Q, N, and M) are well-formed in Γ and that the substitutions (σ_1 and σ_2) have signature $\Gamma \vdash \sigma_i : \Gamma$,*

+ *if* $\Gamma \vdash [\sigma_1]P \geqslant Q$ *and* $\Gamma \vdash [\sigma_2]Q \geqslant P$
  then there exists a bijection $\mu : \mathbf{fv}\, P \leftrightarrow \mathbf{fv}\, Q$ *such that* $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mu : \mathbf{fv}\, P$ *and* $\Gamma \vdash \sigma_2 \simeq^{\leqslant} \mu^{-1} : \mathbf{fv}\, Q$;

− *if* $\Gamma \vdash [\sigma_1]N \leqslant M$ *and* $\Gamma \vdash [\sigma_2]N \leqslant M$
  then there exists a bijection $\mu : \mathbf{fv}\, N \leftrightarrow \mathbf{fv}\, M$ *such that* $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mu : \mathbf{fv}\, N$ *and* $\Gamma \vdash \sigma_2 \simeq^{\leqslant} \mu^{-1} : \mathbf{fv}\, M$.

Proof.

+ Applying σ_2 to both sides of $\Gamma \vdash [\sigma_1]P \geqslant Q$ (by lemma 23), we have: $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant [\sigma_2]Q$. Composing it with $\Gamma \vdash [\sigma_2]Q \geqslant P$ by transitivity (lemma 24), we have $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant P$. Then by corollary 20, $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$. By a symmetric argument, we also have: $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, Q$.

  Now, we prove that $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$ and $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, Q$ implies that σ_1 and σ_1 are (equivalent to) mutually inverse bijections.

  To do so, it suffices to prove that

  (i) for any $\alpha^{\pm} \in \mathbf{fv}\, P$ there exists $\beta^{\pm} \in \mathbf{fv}\, Q$ such that $\Gamma \vdash [\sigma_1]\alpha^{\pm} \simeq^{\leqslant} \beta^{\pm}$ and $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq^{\leqslant} \alpha^{\pm}$; and

  (ii) for any $\beta^{\pm} \in \mathbf{fv}\, Q$ there exists $\alpha^{\pm} \in \mathbf{fv}\, P$ such that $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq^{\leqslant} \alpha^{\pm}$ and $\Gamma \vdash [\sigma_1]\alpha^{\pm} \simeq^{\leqslant} \beta^{\pm}$.

  Then these correspondences between $\mathbf{fv}\, P$ and $\mathbf{fv}\, Q$ are mutually inverse functions, since for any $\beta^{\pm}$ there can be at most one $\alpha^{\pm}$ such that $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq^{\leqslant} \alpha^{\pm}$ (and vice versa).

  (i) Let us take $\alpha^{\pm} \in \mathbf{fv}\, P$.

  (a) if $\alpha^{\pm}$ is positive ($\alpha^{\pm} = \alpha^{+}$), from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{+} \simeq^{\leqslant} \alpha^{+}$, by corollary 8, we have $[\sigma_2][\sigma_1]\alpha^{+} = \exists\overrightarrow{\beta^{-}}.\alpha^{+}$.
  What shape can $[\sigma_1]\alpha^{+}$ have? It cannot be $\exists\overrightarrow{\alpha^{-}}.{\downarrow}N$ (for potentially empty $\overrightarrow{\alpha^{-}}$), because the outer constructor $\downarrow$ would remain after the substitution σ_2, whereas $\exists\overrightarrow{\beta^{-}}.\alpha^{+}$ does not have $\downarrow$. The only case left is $[\sigma_1]\alpha^{+} = \exists\overrightarrow{\alpha^{-}}.\gamma^{+}$.
  Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.\gamma^{+} \simeq^{\leqslant} \gamma^{+}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{+} \simeq^{\leqslant} \gamma^{+}$. Also notice that $[\sigma_2]\exists\overrightarrow{\alpha^{-}}.\gamma^{+} = \exists\overrightarrow{\beta^{-}}.\alpha^{+}$ implies $\Gamma \vdash [\sigma_2]\gamma^{+} \simeq^{\leqslant} \alpha^{+}$.

  (b) if $\alpha^{\pm}$ is negative ($\alpha^{\pm} = \alpha^{-}$) from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{-} \simeq^{\leqslant} \alpha^{-}$, by corollary 8, we have $[\sigma_2][\sigma_1]\alpha^{-} = \forall\overrightarrow{\beta^{+}}.\alpha^{-}$.
  What shape can $[\sigma_1]\alpha^{-}$ have? It cannot be $\forall\overrightarrow{\alpha^{+}}.{\uparrow}P$ nor $\forall\overrightarrow{\alpha^{+}}.P \to M$ (for potentially empty $\overrightarrow{\alpha^{+}}$), because the outer constructor ($\to$ or $\uparrow$), remaining after the substitution σ_2, is however absent in the resulting $\forall\overrightarrow{\beta^{+}}.\alpha^{-}$. Hence, the only case left is $[\sigma_1]\alpha^{-} = \forall\overrightarrow{\alpha^{+}}.\gamma^{-}$ Notice that $\Gamma \vdash \gamma^{-} \simeq^{\leqslant} \forall\overrightarrow{\alpha^{+}}.\gamma^{-}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{-} \simeq^{\leqslant} \gamma^{-}$. Also notice that $[\sigma_2]\forall\overrightarrow{\alpha^{+}}.\gamma^{-} = \forall\overrightarrow{\beta^{+}}.\alpha^{-}$ implies $\Gamma \vdash [\sigma_2]\gamma^{-} \simeq^{\leqslant} \alpha^{-}$.

  (ii) The proof is symmetric: We swap P and Q, σ_1 and σ_2, and exploit $\Gamma \vdash [\sigma_1][\sigma_2]\alpha^{\pm} \simeq^{\leqslant} \alpha^{\pm}$ instead of $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{\pm} \simeq^{\leqslant} \alpha^{\pm}$.

− The proof is symmetric to the positive case.

□

Lemma 43 (Equivalent substitution act equivalently)**.** *Suppose that* $\Gamma' \vdash \sigma_1 : \Gamma$ *and* $\Gamma' \vdash \sigma_2 : \Gamma$ *are substitutions equivalent on their domain, that is* $\Gamma' \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \Gamma$. *Then*

+ *for any* $\Gamma \vdash P$, $\Gamma' \vdash [\sigma_1]P \simeq^{\leqslant} [\sigma_2]P$;

− *for any* $\Gamma \vdash N$, $\Gamma' \vdash [\sigma_1]N \simeq^{\leqslant} [\sigma_2]N$.

Proof. We prove it by induction on P (and mutually on N).

Case 1. $N = \alpha^{-}$
Then since by inversion, $\alpha^{-} \in \Gamma$, $\Gamma' \vdash [\sigma_1]\alpha^{-} \simeq^{\leqslant} [\sigma_2]\alpha^{-}$ holds by definition of $\Gamma' \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \Gamma$.

Case 2. $N = \uparrow P$

Then by inversion, $\Gamma \vdash P$. By the induction hypothesis, $\Gamma' \vdash [\sigma_1]P \simeq^{\leqslant} [\sigma_2]P$, Then by Rule $(\uparrow^{\leqslant})$, $\Gamma' \vdash \uparrow[\sigma_1]P \leqslant \uparrow[\sigma_2]P$, and symmetrically, $\Gamma' \vdash \uparrow[\sigma_2]P \leqslant \uparrow[\sigma_1]P$, together meaning that $\Gamma' \vdash \uparrow[\sigma_1]P \simeq^{\leqslant} \uparrow[\sigma_2]P$, or equivalently, $\Gamma' \vdash [\sigma_1]\uparrow P \simeq^{\leqslant} [\sigma_2]\uparrow P$.

Case 3. $N = P \to M$

Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. By the induction hypothesis, $\Gamma' \vdash [\sigma_1]P \simeq^{\leqslant} [\sigma_2]P$ and $\Gamma' \vdash [\sigma_1]M \simeq^{\leqslant} [\sigma_2]M$, that is $\Gamma' \vdash [\sigma_1]P \geqslant [\sigma_2]P$, $\Gamma' \vdash [\sigma_2]P \geqslant [\sigma_1]P$, $\Gamma' \vdash [\sigma_1]M \leqslant [\sigma_2]M$, and $\Gamma' \vdash [\sigma_2]M \leqslant [\sigma_1]M$. Then by Rule $(\to^{\leqslant})$, $\Gamma' \vdash [\sigma_1]P \to [\sigma_1]M \leqslant [\sigma_2]P \to [\sigma_2]M$, and again by Rule $(\to^{\leqslant})$, $\Gamma' \vdash [\sigma_2]P \to [\sigma_2]M \leqslant [\sigma_1]P \to [\sigma_1]M$. This way, $\Gamma' \vdash [\sigma_1]P \to [\sigma_1]M \simeq^{\leqslant} [\sigma_2]P \to [\sigma_2]M$, or equivalently, $\Gamma' \vdash [\sigma_1](P \to M) \simeq^{\leqslant} [\sigma_2](P \to M)$.

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$ We can assume that $\overrightarrow{\alpha^+}$ is disjoint from Γ and Γ'. By inversion, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.M$ implies $\Gamma, \overrightarrow{\alpha^+} \vdash M$. Notice that $\Gamma' \vdash \sigma_i : \Gamma$ and $\Gamma' \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \Gamma$ can be extended to $\Gamma', \overrightarrow{\alpha^+} \vdash \sigma_i : \Gamma, \overrightarrow{\alpha^+}$ and $\Gamma', \overrightarrow{\alpha^+} \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \Gamma, \overrightarrow{\alpha^+}$ by section 6.9. Then by the induction hypothesis, $\Gamma', \overrightarrow{\alpha^+} \vdash [\sigma_1]M \simeq^{\leqslant} [\sigma_2]M$, meaning by inversion that $\Gamma', \overrightarrow{\alpha^+} \vdash [\sigma_1]M \leqslant [\sigma_2]M$ and $\Gamma', \overrightarrow{\alpha^+} \vdash [\sigma_2]M \leqslant [\sigma_1]M$.

To infer $\Gamma' \vdash \forall\overrightarrow{\alpha^+}.[\sigma_1]M \leqslant \forall\overrightarrow{\alpha^+}.[\sigma_2]M$, we apply Rule $(\forall^{\leqslant})$ with the substitution $\Gamma', \overrightarrow{\alpha^+} \vdash \mathsf{id} : \overrightarrow{\alpha^+}$, noting that $\Gamma', \overrightarrow{\alpha^+} \vdash [\mathsf{id}][\sigma_1]M \leqslant [\sigma_2]M$ holds since $\Gamma', \overrightarrow{\alpha^+} \vdash [\sigma_1]M \leqslant [\sigma_2]M$, as noted above.

Symmetrically, we infer $\Gamma' \vdash \forall\overrightarrow{\alpha^+}.[\sigma_2]M \leqslant \forall\overrightarrow{\alpha^+}.[\sigma_1]M$, which together with $\Gamma' \vdash \forall\overrightarrow{\alpha^+}.[\sigma_1]M \leqslant \forall\overrightarrow{\alpha^+}.[\sigma_2]M$ means $\Gamma' \vdash \forall\overrightarrow{\alpha^+}.[\sigma_1]M \simeq^{\leqslant} \forall\overrightarrow{\alpha^+}.[\sigma_2]M$, or equivalently, $\Gamma' \vdash [\sigma_1]\forall\overrightarrow{\alpha^+}.M \simeq^{\leqslant} [\sigma_2]\forall\overrightarrow{\alpha^+}.M$.

Case 5. The positive cases are proved symmetrically.

□

Lemma 44 (Equivalence of polymorphic types)**.**

− *For* $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N$ *and* $\Gamma \vdash \forall\overrightarrow{\beta^+}.M$,
if $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \simeq^{\leqslant} \forall\overrightarrow{\beta^+}.M$ *then there exists a bijection* $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv}\,M \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$ *such that* $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq^{\leqslant} [\mu]M$,

+ *For* $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P$ *and* $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$,
if $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \simeq^{\leqslant} \exists\overrightarrow{\beta^-}.Q$ *then there exists a bijection* $\mu : \overrightarrow{\beta^-} \cap \mathbf{fv}\,Q \leftrightarrow \overrightarrow{\alpha^-} \cap \mathbf{fv}\,P$ *such that* $\Gamma, \overrightarrow{\beta^-} \vdash P \simeq^{\leqslant} [\mu]Q$.

Proof.

− First, by α-conversion, we ensure $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing$ and $\overrightarrow{\beta^+} \cap \mathbf{fv}\,N = \varnothing$. By inversion, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \simeq^{\leqslant} \forall\overrightarrow{\beta^+}.M$ implies

1. $\Gamma, \overrightarrow{\beta^+} \vdash [\sigma_1]N \leqslant M$ for $\Gamma, \overrightarrow{\beta^+} \vdash \sigma_1 : \overrightarrow{\alpha^+}$ and
2. $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]M \leqslant N$ for $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 : \overrightarrow{\beta^+}$.

To apply lemma 42, we weaken and rearrange the contexts, and extend the substitutions to act as identity outside of their initial domain:

1. $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \leqslant M$ for $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 : \Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$ and
2. $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \leqslant N$ for $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 : \Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$.

Then from lemma 42, there exists a bijection $\mu : \mathbf{fv}\,M \leftrightarrow \mathbf{fv}\,N$ such that $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq^{\leqslant} \mu : \mathbf{fv}\,M$ and $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 \simeq^{\leqslant} \mu^{-1} : \mathbf{fv}\,N$.

Let us show that $\mu|_{\overrightarrow{\beta^+}}$ is the appropriate candidate.

First, we show that if we restrict the domain of μ to $\overrightarrow{\beta^+}$, its range will be contained in $\overrightarrow{\alpha^+}$.

Let us take $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv}\,M$ and assume $[\mu]\gamma^+ \notin \overrightarrow{\alpha^+}$. Then since $\Gamma, \overrightarrow{\beta^+} \vdash \sigma_1 : \overrightarrow{\alpha^+}$, σ_1 acts as identity outside of $\overrightarrow{\alpha^+}$, i.e. $[\sigma_1][\mu]\gamma^+ = [\mu]\gamma^+$ (notice that γ^+ is in the domain of μ). Since $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 \simeq^{\leqslant} \mu^{-1} : \mathbf{fv}\,N$, application of σ_1 to $[\mu]\gamma^+ \in \mathbf{fv}\,N$ is equivalent to application of μ^{-1}, then $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}][\mu]\gamma^+ \simeq^{\leqslant} [\mu]\gamma^+$, i.e. $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \gamma^+ \simeq^{\leqslant} [\mu]\gamma^+$, which means $\gamma^+ \in \mathbf{fv}\,[\mu]\gamma^+ \subseteq \mathbf{fv}\,N$. By assumption, $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv}\,M$, i.e. $\overrightarrow{\beta^+} \cap \mathbf{fv}\,N \neq \varnothing$, hence contradiction.

Second, we will show $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$.

Since $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 : \overrightarrow{\beta^+}$ and $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq^{\leqslant} \mu : \mathbf{fv}\,M$, we have $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq^{\leqslant} \mu|_{\overrightarrow{\beta^+}} : \mathbf{fv}\,M$: for any $\alpha^{\pm} \in \mathbf{fv}\,M \backslash \overrightarrow{\beta^+}$, $[\sigma_2]\alpha^{\pm} = \alpha^{\pm}$ since $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 : \overrightarrow{\beta^+}$, and $[\mu|_{\overrightarrow{\beta^+}}]\alpha^{\pm} = \alpha^{\pm}$ by definition of substitution restriction; for $\beta^+ \in \overrightarrow{\beta^+}$, $[\mu|_{\overrightarrow{\beta^+}}]\beta^+ = [\mu]\beta^+$, and thus, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]\beta^+ \simeq^{\leqslant} [\mu]\beta^+$ can be rewritten to $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]\beta^+ \simeq^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]\beta^+$.

By lemma 43, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq^\leqslant \mu|_{\overrightarrow{\beta^+}} : \mathbf{fv}\, M$ implies $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \simeq^\leqslant [\mu|_{\overrightarrow{\beta^+}}]M$. By similar reasoning, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \simeq^\leqslant [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N$.

This way, by transitivity of subtyping (lemma 24),

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N \leqslant M \tag{5}$$

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu|_{\overrightarrow{\beta^+}}]M \leqslant N \tag{6}$$

By applying $\mu|_{\overrightarrow{\beta^+}}$ to both sides of 5 (lemma 23), we have $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu|_{\overrightarrow{\beta^+}}][\mu^{-1}|_{\overrightarrow{\alpha^+}}]N \leqslant [\mu|_{\overrightarrow{\beta^+}}]M$. By contracting $\mu^{-1}|_{\overrightarrow{\alpha^+}} \circ \mu|_{\overrightarrow{\beta^+}} = \mu|^{-1}_{\overrightarrow{\beta^+}} \circ \mu|_{\overrightarrow{\beta^+}}$ (notice that $\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing$), we have $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \leqslant [\mu|_{\overrightarrow{\beta^+}}]M$, which together with 6 means $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \simeq^\leqslant [\mu|_{\overrightarrow{\beta^+}}]M$, and by strengthening, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq^\leqslant [\mu|_{\overrightarrow{\beta^+}}]M$.

+ The proof is symmetric to the proof of the negative case.

□

Lemma 45 (Completeness of equivalence)**.** *Mutual subtyping implies declarative equivalence. Assuming all the types below are well-formed in* Γ*:*

+ *if* $\Gamma \vdash P \simeq^\leqslant Q$ *then* $P \simeq^D Q$*,*
− *if* $\Gamma \vdash N \simeq^\leqslant M$ *then* $N \simeq^D M$*.*

Proof. − Induction on the sum of sizes of N and M. By inversion, $\Gamma \vdash N \simeq^\leqslant M$ means $\Gamma \vdash N \leqslant M$ and $\Gamma \vdash M \leqslant N$. Let us consider the last rule that forms $\Gamma \vdash N \leqslant M$:

Case 1. Rule ($\text{Var}^{-\leqslant}$) i.e. $\Gamma \vdash N \leqslant M$ is of the form $\Gamma \vdash \alpha^- \leqslant \alpha^-$
Then $N \simeq^D M$ (i.e. $\alpha^- \simeq^D \alpha^-$) holds immediately by Rule ($\text{Var}^{-\simeq^D}$).

Case 2. Rule ($\uparrow^\leqslant$) i.e. $\Gamma \vdash N \leqslant M$ is of the form $\Gamma \vdash \uparrow P \leqslant \uparrow Q$
Then by inversion, $\Gamma \vdash P \simeq^\leqslant Q$, and by induction hypothesis, $P \simeq^D Q$. Then $N \simeq^D M$ (i.e. $\uparrow P \simeq^D \uparrow Q$) holds by Rule ($\uparrow^{\simeq^D}$).

Case 3. Rule ($\rightarrow^\leqslant$) i.e. $\Gamma \vdash N \leqslant M$ is of the form $\Gamma \vdash P \rightarrow N' \leqslant Q \rightarrow M'$
Then by inversion, $\Gamma \vdash P \geqslant Q$ and $\Gamma \vdash N' \leqslant M'$. Notice that $\Gamma \vdash M \leqslant N$ is of the form $\Gamma \vdash Q \rightarrow M' \leqslant P \rightarrow N'$, which by inversion means $\Gamma \vdash Q \geqslant P$ and $\Gamma \vdash M' \leqslant N'$.
This way, $\Gamma \vdash Q \simeq^\leqslant P$ and $\Gamma \vdash M' \simeq^\leqslant N'$. Then by induction hypothesis, $Q \simeq^D P$ and $M' \simeq^D N'$. Then $N \simeq^D M$ (i.e. $P \rightarrow N' \simeq^D Q \rightarrow M'$) holds by Rule ($\rightarrow^{\simeq^D}$).

Case 4. Rule ($\forall^\leqslant$) i.e. $\Gamma \vdash N \leqslant M$ is of the form $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.M'$
Then by case 4, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \simeq^\leqslant \forall\overrightarrow{\beta^+}.M'$ means that there exists a bijection $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv}\, M' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N'$ such that $\Gamma, \overrightarrow{\alpha^+} \vdash [\mu]M' \simeq^\leqslant N'$.
Notice that the application of bijection μ to M' does not change its size (which is less than the size of M), hence the induction hypothesis applies. This way, $[\mu]M' \simeq^D N'$ (and by symmetry, $N' \simeq^D [\mu]M'$) holds by induction. Then we apply Rule ($\forall^{\simeq^D}$) to get $\forall\overrightarrow{\alpha^+}.N' \simeq^D \forall\overrightarrow{\beta^+}.M'$, i.e. $N \simeq^D M$.

+ The proof is symmetric to the proof of the negative case.

□

Corollary 21 (Normalization is complete w.r.t. subtyping-induced equivalence)**.** *Assuming all the types below are well-formed in* Γ*:*

+ *if* $\Gamma \vdash P \simeq^\leqslant Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$*,*
− *if* $\Gamma \vdash N \simeq^\leqslant M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$*.*

Proof. Immediately from lemmas 33 and 45. □

Lemma 46 (Algorithmization of subtyping-induced equivalence)**.** *Mutual subtyping is equality of normal forms. Assuming all the types below are well-formed in* Γ*:*

+ $\Gamma \vdash P \simeq^\leqslant Q \iff \mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$,
− $\Gamma \vdash N \simeq^\leqslant M \iff \mathbf{nf}\,(N) = \mathbf{nf}\,(M)$.

Proof. Let us prove the positive case, the negative case is symmetric. We prove both directions of $\iff$ separately:

$\Rightarrow$ exactly corollary 21;

$\Leftarrow$ by lemmas 40 and 56.

$\square$

Lemma 47 (Substitution preserves declarative equivalence). *Suppose that σ is a substitution. Then*

+ $P \simeq^D Q$ *implies* $[\sigma]P \simeq^D [\sigma]Q$

− $N \simeq^D M$ *implies* $[\sigma]N \simeq^D [\sigma]M$

Proof.
$$\begin{aligned} P \simeq^D Q &\Rightarrow \mathbf{nf}\,(P) = \mathbf{nf}\,(Q) && \text{by lemma 46} \\ &\Rightarrow [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(Q) \\ &\Rightarrow \mathbf{nf}\,([\sigma]P) = \mathbf{nf}\,([\sigma]Q) && \text{by lemma 32} \\ &\Rightarrow [\sigma]P \simeq^D [\sigma]Q && \text{by lemma 46} \end{aligned}$$
$\square$

6 Properties of the Algorithmic Type System

6.1 Algorithmic Type Well-formedness

Lemma 48 (Soundness of algorithmic type well-formedness).

+ *if* $\Gamma; \Xi \vdash P$ *then* $\mathbf{fv}\,(P) \subseteq \Gamma$ *and* $\mathbf{uv}\,(P) \subseteq \Xi$*;*

− *if* $\Gamma; \Xi \vdash N$ *then* $\mathbf{fv}\,(N) \subseteq \Gamma$ *and* $\mathbf{uv}\,(N) \subseteq \Xi$.

Proof. The proof is analogous to lemma 3. The additional base case is when $\Gamma; \Xi \vdash P$ is derived by Rule WFATPUVar, and the symmetric negative case. In this case, $P = \widehat{\alpha}^+$, and $\mathbf{uv}\,(P) = \widehat{\alpha}^+ \subseteq \Xi$ by inversion; $\mathbf{fv}\,(P) = \varnothing \subseteq \Gamma$ vacuously. $\square$

Lemma 49 (Completeness of algorithmic type well-formedness). *In the well-formedness judgment, only used variables matter:*

+ *if* $\Gamma_1 \cap \mathbf{fv}\, P = \Gamma_2 \cap \mathbf{fv}\, P$ *and* $\Xi_1 \cap \mathbf{uv}\, P = \Xi_2 \cap \mathbf{uv}\, P$ *then* $\Gamma_1; \Xi_1 \vdash P \iff \Gamma_2; \Xi_2 \vdash P$*, and*

− *if* $\Gamma_1 \cap \mathbf{fv}\, N = \Gamma_2 \cap \mathbf{fv}\, N$ *and* $\Xi_1 \cap \mathbf{uv}\, N = \Xi_2 \cap \mathbf{uv}\, N$ *then* $\Gamma_1; \Xi_1 \vdash N \iff \Gamma_2; \Xi_2 \vdash N$.

Proof. By mutual structural induction on P and N. $\square$

Lemma 50 (Variable algorithmization agrees with well-formedness).

+ $\Gamma, \overrightarrow{\alpha^-} \vdash P$ *implies* $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P$*;*

− $\Gamma, \overrightarrow{\alpha^-} \vdash N$ *implies* $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]N$.

Proof. The proof is a structural induction on $\Gamma, \overrightarrow{\alpha^-} \vdash P$ and mutually, on $\Gamma, \overrightarrow{\alpha^-} \vdash N$. Notice that the substitutions commute with all the constructors, providing the step of the induction. $\square$

Lemma 51 (Variable dealgorithmization agrees with well-formedness).

+ $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash P$ *implies* $\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P$*;*

− $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash N$ *implies* $\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]N$.

Proof. As for lemma 50, the proof is a structural induction on $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash P$ and mutually, on $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash N$. $\square$

Corollary 22 (Well-formedness Algorithmic Context Weakening). *Suppose that* $\Gamma_1 \subseteq \Gamma_2$*, and* $\Xi_1 \subseteq \Xi_2$*. Then*

+ *if* $\Gamma_1; \Xi_1 \vdash P$ *implies* $\Gamma_2; \Xi_2 \vdash P$*,*

− *if* $\Gamma_1; \Xi_1 \vdash N$ *implies* $\Gamma_2; \Xi_2 \vdash N$.

Proof. By lemma 48, $\Gamma_1; \Xi_1 \vdash P$ implies $\mathbf{fv}\,(P) \subseteq \Gamma_1 \subseteq \Gamma_2$ and $\mathbf{uv}\,(P) \subseteq \Xi_1 \subseteq \Xi_2$, and thus, $\mathbf{fv}\,(P) = \mathbf{fv}\,(P) \cap \Gamma_1 = \mathbf{fv}\,(P) \cap \Gamma_2$, and $\mathbf{uv}\,(P) = \mathbf{uv}\,(P) \cap \Xi_1 = \mathbf{uv}\,(P) \cap \Xi_2$. Then by lemma 49, $\Gamma_2; \Xi_2 \vdash P$. The negative case is symmetric. $\square$

6.2 Substitution

Lemma 52 (Algorithmic Substitution Strengthening). *Restricting the substitution to the algorithmic variables of the substitution subject does not affect the result. Suppose that $\widehat{\sigma}$ is an algorithmic substitution, P and N are algorithmic types. Then*

+ $[\widehat{\sigma}]P = [\widehat{\sigma}|_{\mathbf{uv}\ P}]P$,

− $[\widehat{\sigma}]N = [\widehat{\sigma}|_{\mathbf{uv}\ N}]N$

Proof. The proof is analogous to the proof of lemma 6. □

Lemma 53 (Substitutions equal on the algorithmic variables). *Suppose that $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are normalized substitutions of signature $\Theta \vdash \widehat{\sigma}_i : \Xi$. Then*

+ *for a normalized type $\Gamma; \Xi \vdash P$, if $[\widehat{\sigma}_1]P = [\widehat{\sigma}_2]P$ then $\widehat{\sigma}_1|_{(\mathbf{uv}\ P)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ P)}$;*

− *for a normalized type $\Gamma; \Xi \vdash N$, if $[\widehat{\sigma}_1]N = [\widehat{\sigma}_2]N$ then $\widehat{\sigma}_1|_{(\mathbf{uv}\ N)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ N)}$.*

Proof. The proof is a simple structural induction on $\Gamma; \Xi \vdash P$ and mutually, on $\Gamma; \Xi \vdash N$. Let us consider the shape of N (the cases of P are symmetric).

Case 1. $\Gamma; \Xi \vdash \widehat{\alpha}^-$. Then $[\widehat{\sigma}_1]\widehat{\alpha}^- = [\widehat{\sigma}_2]\widehat{\alpha}^-$ implies $\widehat{\sigma}_1|_{(\mathbf{uv}\ \widehat{\alpha}^-)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ \widehat{\alpha}^-)}$ immediately.

Case 2. $\Gamma; \Xi \vdash \alpha^-$. Then $\mathbf{uv}\ \widehat{\alpha}^- = \varnothing$, and $\widehat{\sigma}_1|_{(\mathbf{uv}\ \widehat{\alpha}^-)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ \widehat{\alpha}^-)}$ holds vacuously.

Case 3. $\Gamma; \Xi \vdash \forall \overrightarrow{\alpha^+}. N$. Then we are proving that $[\widehat{\sigma}_1]\forall \overrightarrow{\alpha^+}. N = [\widehat{\sigma}_2]\forall \overrightarrow{\alpha^+}. N$ implies $\widehat{\sigma}_1|_{(\mathbf{uv}\ \forall \overrightarrow{\alpha^+}. N)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ \forall \overrightarrow{\alpha^+}. N)}$. By definition of substitution and Rule ($\forall^{\simeq^D}$), $[\widehat{\sigma}_1]N = [\widehat{\sigma}_2]N$ implies $\widehat{\sigma}_1|_{\mathbf{uv}\ N} = \widehat{\sigma}_2|_{\mathbf{uv}\ N}$. Since $\forall \overrightarrow{\alpha^+}. N$ is normalized, so is $\Gamma, \overrightarrow{\alpha^+}; \Xi \vdash N$, hence, the induction hypothesis is applicable and implies $\widehat{\sigma}_1|_{\mathbf{uv}\ N} = \widehat{\sigma}_2|_{\mathbf{uv}\ N}$, as required.

Case 4. $\Gamma; \Xi \vdash P \to N$. Then we are proving that $[\widehat{\sigma}_1](P \to N) = [\widehat{\sigma}_2](P \to N)$ implies $\widehat{\sigma}_1|_{(\mathbf{uv}\ P \to N)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ P \to N)}$. By definition of substitution and congruence of equality, $[\widehat{\sigma}_1](P \to N) = [\widehat{\sigma}_2](P \to N)$ means $[\widehat{\sigma}_1]P = [\widehat{\sigma}_2]P$ and $[\widehat{\sigma}_1]N = [\widehat{\sigma}_2]N$. Notice that P and N are normalized since $P \to N$ is normalized, and well-formed in the same contexts. This way, by the induction hypothesis, $\widehat{\sigma}_1|_{(\mathbf{uv}\ P)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ P)}$ and $\widehat{\sigma}_1|_{(\mathbf{uv}\ N)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ N)}$, which since $\mathbf{uv}\,(P \to N) = \mathbf{uv}\ P \cup \mathbf{uv}\ N$ implies $\widehat{\sigma}_1|_{(\mathbf{uv}\ P \to N)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ P \to N)}$.

Case 5. $\Gamma; \Xi \vdash \uparrow P$. The proof is similar to the previous case: we apply congruence of substitution, equality, and normalization, then the induction hypothesis, and then the fact that $\mathbf{uv}\,(\uparrow P) = \mathbf{uv}\ P$.

□

Corollary 23 (Substitutions equivalent on the algorithmic variables). *Suppose that $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are substitutions of signature $\Theta \vdash \widehat{\sigma}_i : \Xi$. Then*

+ *for a type $\Gamma; \Xi \vdash P$, if $\Theta \vdash \quad [\widehat{\sigma}_1]P \simeq^D [\widehat{\sigma}_2]P$ then $\Theta \vdash \widehat{\sigma}_1 \simeq^\leqslant \widehat{\sigma}_2 : \mathbf{uv}\ P$;*

− *for a type $\Gamma; \Xi \vdash N$, if $\Theta \vdash \quad [\widehat{\sigma}_1]N \simeq^D [\widehat{\sigma}_2]N$ then $\Theta \vdash \widehat{\sigma}_1 \simeq^\leqslant \widehat{\sigma}_2 : \mathbf{uv}\ N$.*

Proof. First, let us normalize the types and the substitutions, and show that the given equivalences and well-formedness properties are preserved. $\Gamma; \Xi \vdash P$ implies $\Gamma; \Xi \vdash \mathbf{nf}\,(P)$ by corollary 24. $\Theta \vdash \quad [\widehat{\sigma}_1]P \simeq^D [\widehat{\sigma}_2]P$ implies $\mathbf{nf}\,([\widehat{\sigma}_1]P) = \mathbf{nf}\,([\widehat{\sigma}_2]P)$ by lemma 46. Then $\mathbf{nf}\,([\widehat{\sigma}_1]P) = \mathbf{nf}\,([\widehat{\sigma}_2]P)$ implies $[\mathbf{nf}\,(\widehat{\sigma}_1)]\mathbf{nf}\,(P) = [\mathbf{nf}\,(\widehat{\sigma}_2)]\mathbf{nf}\,(P)$ by lemma 32. Notice that by corollary 25 $\Theta \vdash \widehat{\sigma}_i : \Xi$ implies $\Theta \vdash \mathbf{nf}\,(\widehat{\sigma}_i) : \Xi$.

This way, by lemma 53, $\Theta \vdash \quad [\widehat{\sigma}_1]P \simeq^D [\widehat{\sigma}_2]P$ implies $\mathbf{nf}\,(\widehat{\sigma}_1)|_{(\mathbf{uv}\,\mathbf{nf}\,(P))} = \mathbf{nf}\,(\widehat{\sigma}_2)|_{(\mathbf{uv}\,\mathbf{nf}\,(P))}$. Then by lemma 54, $\mathbf{nf}\,(\widehat{\sigma}_1)|_{(\mathbf{uv}\ P)} = \mathbf{nf}\,(\widehat{\sigma}_2)|_{(\mathbf{uv}\ P)}$, and by *corollary* : *subst* − *subt* − *equiv* − *algorithmization*, $\Theta \vdash \widehat{\sigma}_1 \simeq^\leqslant \widehat{\sigma}_2 : \mathbf{uv}\ P$.

Symmetrically, $\Theta \vdash \quad [\widehat{\sigma}_1]N \simeq^D [\widehat{\sigma}_2]N$ implies $\Theta \vdash \widehat{\sigma}_1 \simeq^\leqslant \widehat{\sigma}_2 : \mathbf{uv}\ N$. □

6.3 Normalization

Lemma 54. *Algorithmic variables are not changed by the normalization*

− $\mathbf{uv}\ N \equiv \mathbf{uv}\,\mathbf{nf}\,(N)$

+ $\mathbf{uv}\ P \equiv \mathbf{uv}\,\mathbf{nf}\,(P)$

Proof. By straightforward induction on N and mutually on P, similar to the proof of lemma 30. □

Lemma 55 (Soundness of normalization of algorithmic types).

− $N \simeq^D \mathbf{nf}\,(N)$

+ $P \simeq^D \mathbf{nf}\,(P)$

Proof. The proof coincides with the proof of lemma 31. □

6.4 Equivalence

Lemma 56 (Algorithmization of declarative equivalence). *Declarative equivalence is the equality of normal forms.*

+ $P \simeq^D Q \iff \mathbf{nf}(P) = \mathbf{nf}(Q)$,

− $N \simeq^D M \iff \mathbf{nf}(N) = \mathbf{nf}(M)$.

Proof.

+ Let us prove both directions separately.

  ⇒ exactly by lemma 33,

  ⇐ from lemma 31, we know $P \simeq^D \mathbf{nf}(P) = \mathbf{nf}(Q) \simeq^D Q$, then by transitivity (lemma 37), $P \simeq^D Q$.

− The proof is exactly the same.

□

Lemma 57 (Algorithmic type well-formedness is invariant under equivalence). *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq^D Q$ *then* $\Gamma; \Xi \vdash P \iff \Gamma; \Xi \vdash Q$,

− *if* $N \simeq^D M$ *then* $\Gamma; \Xi \vdash N \iff \Gamma; \Xi \vdash M$

Proof. The proof coincides with the proof of lemma 38, and adds two cases for equating two positive or two negative algorithmic variables, which must be equal by inversion, and thus, $\Gamma; \Xi \vdash \widehat{\alpha}^{\pm} \iff \Gamma; \Xi \vdash \widehat{\alpha}^{\pm}$ holds trivially. □

Corollary 24 (Normalization preserves well-formedness of algorithmic types). + $\Gamma; \Xi \vdash P \iff \Gamma; \Xi \vdash \mathbf{nf}(P)$,

− $\Gamma; \Xi \vdash N \iff \Gamma; \Xi \vdash \mathbf{nf}(N)$

Proof. Immediately from lemmas 55 and 57. □

Corollary 25 (Normalization preserves the signature of the algorithmic substitution). $\Theta \vdash \widehat{\sigma} : \Xi \iff \Theta \vdash \mathbf{nf}(\widehat{\sigma}) : \Xi$, $\Gamma \vdash \widehat{\sigma} : \Xi \iff \Gamma \vdash \mathbf{nf}(\widehat{\sigma}) : \Xi$.

Proof. The proof is analogous to corollary 17. □

Lemma 58 (Algorithmic substitution equivalence becomes equality after normalization). *Suppose that* $\Theta \vdash \widehat{\sigma}_1 : \Xi'$ *and* $\Theta \vdash \widehat{\sigma}_2 : \Xi'$ *are algorithmic substitutions and* $\Xi \subseteq \Xi'$. *Then* $\Theta \vdash \widehat{\sigma}_1 \simeq^{\leqslant} \widehat{\sigma}_2 : \Xi \iff \mathbf{nf}(\widehat{\sigma}_1)|_{\Xi} = \mathbf{nf}(\widehat{\sigma}_2)|_{\Xi}$.

Proof. Follows immediately from lemma 46:

⇒ If $\widehat{\alpha}^{\pm} \notin \Xi$, then $[\mathbf{nf}(\widehat{\sigma}_1)|_{\Xi}]\widehat{\alpha}^{\pm} = [\mathbf{nf}(\widehat{\sigma}_2)|_{\Xi}]\widehat{\alpha}^{\pm} = \widehat{\alpha}^{\pm}$ by definition. For any $\widehat{\alpha}^{\pm} \in \Xi$, $[\mathbf{nf}(\widehat{\sigma}_1)|_{\Xi}]\widehat{\alpha}^{\pm} = \mathbf{nf}([\widehat{\sigma}_1]\widehat{\alpha}^{\pm})$ and $[\mathbf{nf}(\widehat{\sigma}_2)|_{\Xi}]\widehat{\alpha}^{\pm} = \mathbf{nf}([\widehat{\sigma}_2]\widehat{\alpha}^{\pm})$; $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}_1]\widehat{\alpha}^{\pm} \simeq^{\leqslant} [\widehat{\sigma}_2]\widehat{\alpha}^{\pm}$ implies $\mathbf{nf}([\widehat{\sigma}_1]\widehat{\alpha}^{\pm}) = \mathbf{nf}([\widehat{\sigma}_2]\widehat{\alpha}^{\pm})$ by lemma 56.

⇐ If $\widehat{\alpha}^{\pm} \in \Xi$, then $\mathbf{nf}(\widehat{\sigma}_1)|_{\Xi} = \mathbf{nf}(\widehat{\sigma}_2)|_{\Xi}$ implies $\mathbf{nf}([\widehat{\sigma}_1]\widehat{\alpha}^{\pm}) = \mathbf{nf}([\widehat{\sigma}_2]\widehat{\alpha}^{\pm})$ by definition of substitution restriction and normalization. In turn, $\mathbf{nf}([\widehat{\sigma}_1]\widehat{\alpha}^{\pm}) = \mathbf{nf}([\widehat{\sigma}_2]\widehat{\alpha}^{\pm})$ means $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}_1]\widehat{\alpha}^{\pm} \simeq^{\leqslant} [\widehat{\sigma}_2]\widehat{\alpha}^{\pm}$ by lemma 56.

□

6.5 Unification Constraint Merge

Lemma 59 (Soundness of Unification Constraint Merge). *Suppose that* $\Theta \vdash UC_1$ *and* $\Theta \vdash UC_2$ *are normalized unification constraints. If* $\Theta \vdash UC_1 \& UC_2 = UC$ *is defined then* $UC = UC_1 \cup UC_2$.

Proof.

- $UC_1 \;\&\; UC_2 \subseteq UC_1 \cup UC_2$
  By definition, $UC_1 \;\&\; UC_2$ consists of three parts: entries of UC_1 that do not have matching entries of UC_2, entries of UC_2 that do not have matching entries of UC_1, and the merge of matching entries.

  If e is from the first or the second part, then $e \in UC_1 \cup UC_2$ holds immediately. If e is from the third part, then e is the merge of two matching entries $e_1 \in UC_1$ and $e_2 \in UC_2$. Since UC_1 and UC_2 are normalized unification , e_1 and e_2 have one of the following forms:

  – $\widehat{\alpha}^{+} :\approx P_1$ and $\widehat{\alpha}^{+} :\approx P_2$, where P_1 and P_2 are normalized, and then since $\Theta(\widehat{\alpha}^{+}) \vdash e_1 \& e_2 = e$ exists, Rule **??** was applied to infer it. It means that $e = e_1 = e_2$;

  – $\widehat{\alpha}^{-} :\approx N_1$ and $\widehat{\alpha}^{-} :\approx N_2$, then symmetrically, $\Theta(\widehat{\alpha}^{-}) \vdash e_1 \& e_2 = e = e_1 = e_2$

In both cases, $e \in UC_1 \cup UC_2$.

- $UC_1 \cup UC_2 \subseteq UC_1 \ \& \ UC_2$
  Let us take an arbitrary $e_1 \in UC_1$. Then since UC_1 is a unification constraint, e_1 has one of the following forms:
  - $\widehat{\alpha}^+ :\approx P$ where P is normalized. If $\widehat{\alpha}^+ \notin \mathbf{dom}\,(UC_2)$, then $e_1 \in UC_1 \ \& \ UC_2$. Otherwise, there is a normalized matching $e_2 = (\widehat{\alpha}^+ :\approx P') \in UC_2$ and then since $UC_1 \ \& \ UC_2$ exists, Rule **??** was applied to construct $e_1 \ \& \ e_2 \in UC_1 \ \& \ UC_2$. By inversion of Rule **??**, $e_1 \ \& \ e_2 = e_1$, and $\mathbf{nf}\,(P) = \mathbf{nf}\,(P')$, which since P and P' are normalized, implies that $P = P'$, that is $e_1 = e_2 \in UC_1 \ \& \ UC_2$.
  - $\widehat{\alpha}^- :\approx N$ where N is normalized. Then symmetrically, $e_1 = e_2 \in UC_1 \ \& \ UC_2$.

  Similarly, if we take an arbitrary $e_2 \in UC_2$, then $e_1 = e_2 \in UC_1 \ \& \ UC_2$.

□

Corollary 26. *Suppose that $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$ are normalized unification constraints. If $\Theta \vdash UC_1 \& UC_2 = UC$ is defined then*

1. *$\Theta \vdash UC$ is normalized unification constraint,*
2. *for any substitution $\Theta \vdash \widehat{\sigma} : \mathbf{dom}\,(UC)$, $\Theta \vdash \widehat{\sigma} : UC$ implies $\Theta \vdash \widehat{\sigma} : UC_1$ and $\Theta \vdash \widehat{\sigma} : UC_2$.*

Proof. It is clear that since $UC = UC_1 \cup UC_2$ (by lemma 59), and being normalized means that all entries are normalized, UC is a normalized unification constraint. Analogously, $\Theta \vdash UC = UC_1 \cup UC_2$ holds immediately, since $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$.

Let us take an arbitrary substitution $\Theta \vdash \widehat{\sigma} : \mathbf{dom}\,(UC)$ and assume that $\Theta \vdash \widehat{\sigma} : UC$. Then $\Theta \vdash \widehat{\sigma} : UC_i$ holds by definition: If $e \in UC_i \subseteq UC_1 \cup UC_2 = UC$ then $\Theta(\widehat{\alpha}^\pm) \vdash [\widehat{\sigma}]\widehat{\alpha}^\pm : e$ (where e restricts $\widehat{\alpha}^\pm$) holds since $\Theta \vdash \widehat{\sigma} : \mathbf{dom}\,(UC)$. □

Lemma 60 (Completeness of Unification Constraint Entry Merge)**.** *For a fixed context Γ, suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$ are matching constraint entries.*

- *for a type P such that $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$, $\Gamma \vdash e_1 \& e_2 = e$ is defined and $\Gamma \vdash P : e$.*
- *for a type N such that $\Gamma \vdash N : e_1$ and $\Gamma \vdash N : e_2$, $\Gamma \vdash e_1 \& e_2 = e$ is defined and $\Gamma \vdash N : e$.*

Proof. The proof repeats the one of lemma 80 and is done by the case analysis on the shape of e_1 and e_2. However, it only needs to consider two cases.

Case 1. e_1 is $\widehat{\alpha}^+ :\approx Q_1$ and e_2 is $\widehat{\alpha}^+ :\approx Q_2$.

Case 2. e_1 is $\widehat{\alpha}^- :\approx N_1$ and e_2 is $\widehat{\alpha}^- :\approx M_2$.

The proof of these cases is based only on lemma 46 and corollary 10, and does not require the properties of the least upper bound or subtyping. □

Lemma 61 (Completeness of Unification Constraint Merge)**.** *Suppose that $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$. Then for any $\Xi \supseteq \mathbf{dom}\,(UC_1) \cup \mathbf{dom}\,(UC_2)$ and substitution $\Theta \vdash \widehat{\sigma} : \Xi$ such that $\Theta \vdash \widehat{\sigma} : UC_1$ and $\Theta \vdash \widehat{\sigma} : UC_2$,*

1. *$\Theta \vdash UC_1 \& UC_2 = UC$ is defined and*
2. *$\Theta \vdash \widehat{\sigma} : UC$.*

Proof. The proof repeats the proof of lemma 81 but uses lemma 60 instead of lemma 80. □

6.6 Unification

Lemma 62 (Soundness of Unification)**.**

\+ *For normalized P and Q such that $\Gamma; \mathbf{dom}\,(\Theta) \vdash P$ and $\Gamma \vdash Q$,*
if $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC$ then $\Theta \vdash UC : \mathbf{uv}\ P$ and for any normalized $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : UC$, $[\widehat{\sigma}]P = Q$.

− *For normalized N and M such that $\Gamma; \mathbf{dom}\,(\Theta) \vdash N$ and $\Gamma \vdash M$,*
if $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$ then $\Theta \vdash UC : \mathbf{uv}\ N$ and for any normalized $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : UC$, $[\widehat{\sigma}]N = M$.

Proof. We prove by induction on the derivation of $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$ and mutually $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC$. Let us consider the last rule forming this derivation.

Case 1. Rule ($\mathrm{Var}^{-\overset{u}{\simeq}}$), then $N = \alpha^- = M$. The resulting unification constraint is empty: $UC = \cdot$. It satisfies $\Theta \vdash UC : \cdot$ vacuously, and $[\widehat{\sigma}]\alpha^- = \alpha^-$, that is $[\widehat{\sigma}]N = M$.

Case 2. Rule ($\uparrow\overset{u}{\simeq}$), then $N = \uparrow P$ and $M = \uparrow Q$. The algorithm makes a recursive call to $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC$ returning UC. By induction hypothesis, $\Theta \vdash UC : \mathbf{uv}\, P$ and thus, $\Theta \vdash UC : \mathbf{uv} \uparrow P$, and for any $\Theta \vdash \widehat{\sigma} : UC$, $[\widehat{\sigma}]N = [\widehat{\sigma}]\uparrow P = \uparrow[\widehat{\sigma}]P = \uparrow Q = M$, as required.

Case 3. Rule ($\rightarrow\overset{u}{\simeq}$), then $N = P \to N'$ and $M = Q \to M'$. The algorithm makes two recursive calls to $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC_1$ and $\Gamma; \Theta \models N' \overset{u}{\simeq} M' \dashv UC_2$ returning $\Theta \vdash UC_1 \& UC_2 = UC$ as the result.

It is clear that P, N', Q, and M' are normalized, and that $\Gamma; \mathbf{dom}\,(\Theta) \vdash P$, $\Gamma; \mathbf{dom}\,(\Theta) \vdash N'$, $\Gamma \vdash Q$, and $\Gamma \vdash M'$. This way, the induction hypothesis is applicable to both recursive calls.

By applying the induction hypothesis to $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC_1$, we have:

- $\Theta \vdash UC_1 : \mathbf{uv}\, P$,
- for any $(liftUC1)u'$, $[\widehat{\sigma}']P = Q$.

By applying it to $\Gamma; \Theta \models N' \overset{u}{\simeq} M' \dashv UC_2$, we have:

- $\Theta \vdash UC_2 : \mathbf{uv}\, N'$,
- for any $\Theta \vdash \widehat{\sigma}' : UC_2$, $[\widehat{\sigma}']N' = M'$.

Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : UC$. By the soundness of the constraint merge (lemma 79), $\Theta \vdash UC_1 \& UC_2 = UC$ implies $\Theta \vdash \widehat{\sigma} : UC_1$ and $\Theta \vdash \widehat{\sigma} : UC_2$.

Applying the induction hypothesis to $\Theta \vdash \widehat{\sigma} : UC_1$, we have $[\widehat{\sigma}]P = Q$; applying it to $\Theta \vdash \widehat{\sigma} : UC_2$, we have $[\widehat{\sigma}]N' = M'$. This way, $[\widehat{\sigma}]N = [\widehat{\sigma}]P \to [\widehat{\sigma}]N' = Q \to M' = M$.

Case 4. Rule ($\forall\overset{u}{\simeq}$), then $N = \forall\overrightarrow{\alpha^+}.N'$ and $M = \forall\overrightarrow{\alpha^+}.M'$. The algorithm makes a recursive call to $\Gamma, \overrightarrow{\alpha^+}; \Theta \models N' \overset{u}{\simeq} M' \dashv UC$ returning UC as the result.

The induction hypothesis is applicable: $\Gamma, \overrightarrow{\alpha^+}; \mathbf{dom}\,(\Theta) \vdash N'$ and $\Gamma, \overrightarrow{\alpha^+} \vdash M'$ hold by inversion, and N' and M' are normalized, since N and M are. Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : UC$. By the induction hypothesis, $[\widehat{\sigma}]N' = M'$. Then $[\widehat{\sigma}]N = [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N' = \forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N' = \forall\overrightarrow{\alpha^+}.M' = M$.

Case 5. Rule ($\mathrm{UVar}^{-}\overset{u}{\simeq}$), then $N = \widehat{\alpha}^-$, $\widehat{\alpha}^-\{\Delta\} \in \Theta$, and $\Delta \vdash M$. As the result, the algorithm returns $UC = (\widehat{\alpha}^- :\approx M)$.

It is clear that $\widehat{\alpha}^-\{\Delta\} \vdash (\widehat{\alpha}^- :\approx M)$, since $\Delta \vdash M$, meaning that $\Theta \vdash UC$.

Let us take an arbitrary $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : UC$. Since $UC = (\widehat{\alpha}^- :\approx M)$, $\Theta \vdash \widehat{\sigma} : UC$ implies $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]\widehat{\alpha}^- : (\widehat{\alpha}^- :\approx M)$. By inversion of Rule SATSCENEq, it means $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]\widehat{\alpha}^- \simeq^{\leqslant} M$. This way, $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M$. Notice that $\widehat{\sigma}$ and N are normalized, and by lemma 32, so is $[\widehat{\sigma}]N$. Since both sides of $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M$ are normalized, by lemma 46, we have $[\widehat{\sigma}]N = M$.

Case 6. The positive cases are proved symmetrically.

□

Lemma 63 (Completeness of Unification)**.**

+ *For normalized P and Q such that $\Gamma; \mathbf{dom}\,(\Theta) \vdash P$ and $\Gamma \vdash Q$, suppose that there exists $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P)$ such that $[\widehat{\sigma}]P = Q$, then $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC$ for some UC.*

− *For normalized N and M such that $\Gamma; \mathbf{dom}\,(\Theta) \vdash N$ and $\Gamma \vdash M$, suppose that there exists $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(N)$ such that $[\widehat{\sigma}]N = M$, then $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$ for some UC.*

Proof. We prove it by induction on the structure of P and mutually, N.

Case 1. $N = \widehat{\alpha}^-$
$\Gamma; \mathbf{dom}\,(\Theta) \vdash \widehat{\alpha}^-$ means that $\widehat{\alpha}^-\{\Delta\} \in \Theta$ for some Δ.

Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : \widehat{\alpha}^-$ such that $[\widehat{\sigma}]\widehat{\alpha}^- = M$. $\Theta \vdash \widehat{\sigma} : \widehat{\alpha}^-$ means that $\Delta \vdash M$. This way, Rule ($\mathrm{UVar}^{-}\overset{u}{\simeq}$) is applicable to infer $\Gamma; \Theta \models \widehat{\alpha}^- \overset{u}{\simeq} M \dashv (\widehat{\alpha}^- :\approx M)$.

Case 2. $N = \alpha^-$
Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(\alpha^-)$ such that $[\widehat{\sigma}]\alpha^- = M$. Since $\mathbf{uv}\,(\alpha^-) = \varnothing$, $[\widehat{\sigma}]\alpha^- = M$ means $M = \alpha^-$.

This way, Rule ($\mathrm{Var}^{-}\overset{u}{\simeq}$) infers $\Gamma; \Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot$, which is rewritten as $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \cdot$.

Case 3. $N = \uparrow P$
Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P)$ such that $[\widehat{\sigma}]\uparrow P = M$. The latter means $\uparrow[\widehat{\sigma}]P = M$, i.e. $M = \uparrow Q$ for some Q and $[\widehat{\sigma}]P = Q$.

Let us show that the induction hypothesis is applicable to $[\widehat{\sigma}]P = Q$. Notice that P is normalized, since $N = \uparrow P$ is normalized, $\Gamma; \mathbf{dom}\,(\Theta) \vdash P$ holds by inversion of $\Gamma; \mathbf{dom}\,(\Theta) \vdash \uparrow P$, and $\Gamma \vdash Q$ holds by inversion of $\Gamma \vdash \uparrow Q$.

This way, by the induction hypothesis there exists UC such that $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC$.

Case 4. $N = P \to N'$
Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P \to N')$ such that $[\widehat{\sigma}](P \to N') = M$. The latter means $[\widehat{\sigma}]P \to [\widehat{\sigma}]N' = M$, i.e. $M = Q \to M'$ for some Q and M', such that $[\widehat{\sigma}]P = Q$ and $[\widehat{\sigma}]N' = M'$.

Let us show that the induction hypothesis is applicable to $\Theta \vdash \widehat{\sigma}|_{\mathbf{uv}\,(P)} : \mathbf{uv}\,(P)$ and $[\widehat{\sigma}|_{\mathbf{uv}\,(P)}]P = Q$ (the latter holds since $[\widehat{\sigma}|_{\mathbf{uv}\,(P)}]P = [\widehat{\sigma}]P$ by lemma 52),

- P is normalized, since $N = P \to N'$ is normalized
- $\Gamma; \mathbf{dom}\,(\Theta) \vdash P$ follows from the inversion of $\Gamma; \mathbf{dom}\,(\Theta) \vdash P \to N'$,
- $\Gamma \vdash Q$.

Then by the induction hypothesis, $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC_1$. Analogously, the induction hypothesis is applicable to $[\widehat{\sigma}|_{\mathbf{uv}\,N'}]N' = M'$, and thus, $\Gamma; \Theta \models N' \overset{u}{\simeq} M' \dashv UC_2$.

To apply Rule $(\to^{\overset{u}{\simeq}})$ and infer the required $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$, it is left to show that $\Theta \vdash UC_1 \& UC_2 = UC$. It holds by completeness of the unification constraint merge (lemma 61) for $\Theta \vdash UC_1 : \mathbf{uv}\,P$, $\Theta \vdash UC_2 : \mathbf{uv}\,N'$ (which hold by soundness), and $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P) \cup \mathbf{uv}\,(N')$, which holds since $\mathbf{uv}\,(P) \cup \mathbf{uv}\,(N') = \mathbf{uv}\,(P \to N')$. Notice that by soundness, $\Theta \vdash \widehat{\sigma}|_{\mathbf{uv}\,(P)} : UC_1$, which implies $\Theta \vdash \widehat{\sigma} : UC_1$. Analogously, $\Theta \vdash \widehat{\sigma} : UC_2$.

Case 5. $N = \forall \overrightarrow{\alpha^+}. N'$
Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(N')$ such that $[\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N' = M$. The latter means $\forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N' = M$, i.e. $M = \forall\overrightarrow{\alpha^+}.M'$ for some M' such that $[\widehat{\sigma}]N' = M'$.

Let us show that the induction hypothesis is applicable to $[\widehat{\sigma}]N' = M'$. Notice that N' is normalized, since $N = \forall\overrightarrow{\alpha^+}.N'$ is normalized, $\Gamma, \overrightarrow{\alpha^+}; \mathbf{dom}\,(\Theta) \vdash N'$ follows from inversion of $\Gamma; \mathbf{dom}\,(\Theta) \vdash \forall\overrightarrow{\alpha^+}.N'$, $\Gamma, \overrightarrow{\alpha^+} \vdash M'$ follows from inversion of $\Gamma \vdash \forall\overrightarrow{\alpha^+}.M'$, and $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(N')$ by assumption.

This way, by the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+}; \Theta \models N' \overset{u}{\simeq} M' \dashv UC$ exists and moreover, $\Theta \vdash \widehat{\sigma} : UC$. Hence, Rule $(\forall^{\overset{u}{\simeq}})$ is applicable to infer $\Gamma; \Theta \models \forall\overrightarrow{\alpha^+}.N' \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M' \dashv UC$, that is $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$.

Case 6. The positive cases are proved symmetrically.

□

6.7 Anti-unification

Observation 1 (Anti-unification algorithm is deterministic)**.**

+ *If* $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ *and* $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$*, then* $\Xi = \Xi'$*,* $Q = Q'$*,* $\widehat{\tau}_1 = \widehat{\tau}'_1$*, and* $\widehat{\tau}_2 = \widehat{\tau}'_2$*.*

− *If* $\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ *and* $\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi', M', \widehat{\tau}'_1, \widehat{\tau}'_2)$*, then* $\Xi = \Xi'$*,* $M = M'$*,* $\widehat{\tau}_1 = \widehat{\tau}'_1$*, and* $\widehat{\tau}_2 = \widehat{\tau}'_2$*.*

Proof. By trivial induction on $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on $\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$. □

Observation 2. *Names of the anti-unification variables are uniquely defined by the types they are mapped to by the resulting substitutions.*

+ *Assuming* P_1 *and* P_2 *are normalized, if* $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ *then for any* $\widehat{\beta}^- \in \Xi$*,* $\widehat{\beta}^- = \widehat{\alpha}^-_{\{[\widehat{\tau}_1]\widehat{\beta}^-, [\widehat{\tau}_2]\widehat{\beta}^-\}}$

− *Assuming* N_1 *and* N_2 *are normalized, if* $\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ *then for any* $\widehat{\beta}^- \in \Xi$*,* $\widehat{\beta}^- = \widehat{\alpha}^-_{\{[\widehat{\tau}_1]\widehat{\beta}^-, [\widehat{\tau}_2]\widehat{\beta}^-\}}$

Proof. By simple induction on $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on $\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$. Let us consider tha last rule applied to infer this judgment.

Case 1. Rule ($\text{Var}^{+\overset{a}{\simeq}}$) or Rule ($\text{Var}^{-\overset{a}{\simeq}}$), then $\Xi = \cdot$, and the property holds vacuously.

Case 2. Rule (AU^-) Then $\Xi = \widehat{\alpha}^-_{\{N_1,N_2\}}$, $\widehat{\tau}_1 = \widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_1$, and $\widehat{\tau}_2 = \widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_2$. So the property holds trivially.

Case 3. Rule **??** In this case, $\Xi = \Xi' \cup \Xi''$, $\widehat{\tau}_1 = \widehat{\tau}_1' \cup \widehat{\tau}_1''$, and $\widehat{\tau}_2 = \widehat{\tau}_2' \cup \widehat{\tau}_2''$, where the property holds for (Ξ', $\widehat{\tau}_1'$, $\widehat{\tau}_2'$) and (Ξ'', $\widehat{\tau}_1''$, $\widehat{\tau}_2''$) by the induction hypothesis. Then since the union of solutions does not change the types the variables are mapped to, the required property holds for Ξ, $\widehat{\tau}_1$, and $\widehat{\tau}_2$.

Case 4. For the other rules, the resulting Ξ is taken from the recursive call and the required property holds immediately by the induction hypothesis.

□

Lemma 64 (Soundness of Anti-Unification)**.**

+ *Assuming* P_1 *and* P_2 *are normalized, if* $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ *then*

  1. $\Gamma; \Xi \vdash Q$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ *for* $i \in \{1, 2\}$ *are anti-unification solutions, and*
  3. $[\widehat{\tau}_i]Q = P_i$ *for* $i \in \{1, 2\}$.

− *Assuming* N_1 *and* N_2 *are normalized, if* $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ *then*

  1. $\Gamma; \Xi \vdash M$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ *for* $i \in \{1, 2\}$ *are anti-unification solutions, and*
  3. $[\widehat{\tau}_i]M = N_i$ *for* $i \in \{1, 2\}$.

Proof. We prove it by induction on $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually, $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$. Let us consider the last rule applied to infer this judgement.

Case 1. Rule ($\text{Var}^{-\overset{a}{\simeq}}$), then $N_1 = \alpha^- = N_2$, $\Xi = \cdot$, $M = \alpha^-$, and $\widehat{\tau}_1 = \widehat{\tau}_2 = \cdot$.

1. $\Gamma; \cdot \vdash \alpha^-$ follows from the assumption $\Gamma \vdash \alpha^-$,
2. $\Gamma; \cdot \vdash \cdot : \cdot$ holds trivially, and
3. $[\cdot]\alpha^- = \alpha^-$ holds trivially.

Case 2. Rule ($\uparrow^{\overset{a}{\simeq}}$), then $N_1 = \uparrow P_1$, $N_2 = \uparrow P_2$, and the algorithm makes the recursive call: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, returning $(\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ as the result.

Since $N_1 = \uparrow P_1$ and $N_2 = \uparrow P_2$ are normalized, so are P_1 and P_2, and thus, the induction hypothesis is applicable to $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$:

1. $\Gamma; \Xi \vdash Q$, and hence, $\Gamma; \Xi \vdash \uparrow Q$,
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ for $i \in \{1, 2\}$, and
3. $[\widehat{\tau}_i]Q = P_i$ for $i \in \{1, 2\}$, and then by the definition of the substitution, $[\widehat{\tau}_i]\uparrow Q = \uparrow P_i$ for $i \in \{1, 2\}$.

Case 3. Rule ($\rightarrow^{\overset{a}{\simeq}}$), then $N_1 = P_1 \rightarrow N_1'$, $N_2 = P_2 \rightarrow N_2'$, and the algorithm makes two recursive calls: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and $\Gamma \vDash N_1' \overset{a}{\simeq} N_2' \dashv (\Xi', M, \widehat{\tau}_1', \widehat{\tau}_2')$ and and returns $(\Xi \cup \Xi', Q \rightarrow M, \widehat{\tau}_1 \cup \widehat{\tau}_1', \widehat{\tau}_2 \cup \widehat{\tau}_2')$ as the result.

Notice that the induction hypothesis is applicable to $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$: P_1 and P_2 are normalized, since $N_1 = P_1 \rightarrow N_1'$ and $N_2 = P_2 \rightarrow N_2'$ are normalized. Similarly, the induction hypothesis is applicable to $\Gamma \vDash N_1' \overset{a}{\simeq} N_2' \dashv (\Xi', M, \widehat{\tau}_1', \widehat{\tau}_2')$.

This way, by the induction hypothesis:

1. $\Gamma; \Xi \vdash Q$ and $\Gamma; \Xi' \vdash M$. Then by weakening (corollary 22), $\Gamma; \Xi \cup \Xi' \vdash Q$ and $\Gamma; \Xi \cup \Xi' \vdash M$, which implies $\Gamma; \Xi \cup \Xi' \vdash Q \rightarrow M$;
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$ Then $\Gamma; \cdot \vdash \widehat{\tau}_i \cup \widehat{\tau}_i' : \Xi \cup \Xi'$ are well-defined anti-unification solution. Let us take an arbitrary $\widehat{\beta}^- \in \Xi \cup \Xi'$. If $\widehat{\beta}^- \in \Xi$. then $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ implies that $\widehat{\tau}_i$, and hence, $\widehat{\tau}_i \cup \widehat{\tau}_i'$ contains an entry well-formed in Γ. If $\widehat{\beta}^- \in \Xi'$, the reasoning is symmetric.

   $\widehat{\tau}_i \cup \widehat{\tau}_i'$ is a well-defined anti-unification solution: any anti-unification variable occurs uniquely $\widehat{\tau}_i \cup \widehat{\tau}_i'$, since by observation 2, the name of the variable is in one-to-one correspondence with the pair of types it is mapped to by $\widehat{\tau}_1$ and $\widehat{\tau}_2$, an is in one-to-one correspondence with the pair of types it is mapped to by $\widehat{\tau}_1'$ and $\widehat{\tau}_2'$ i.e. if $\widehat{\beta}^- \in \Xi \cap \Xi'$ then $[\widehat{\tau}_1]\widehat{\beta}^- = [\widehat{\tau}_1']\widehat{\beta}^-$, and $[\widehat{\tau}_2]\widehat{\beta}^- = [\widehat{\tau}_2']\widehat{\beta}^-$.

3. $[\widehat{\tau}_i]\,Q = P_i$ and $[\widehat{\tau}'_i]\,M = N'_i$. Since $\widehat{\tau}_i \cup \widehat{\tau}'_i$ restricted to Ξ is $\widehat{\tau}_i$, and $\widehat{\tau}_i \cup \widehat{\tau}'_i$ restricted to Ξ' is $\widehat{\tau}'_i$, we have $[\widehat{\tau}_i \cup \widehat{\tau}'_i]\,Q = P_i$ and $[\widehat{\tau}_i \cup \widehat{\tau}'_i]\,M = N'_i$, and thus, $[\widehat{\tau}_i \cup \widehat{\tau}'_i]\,Q \to M = P_1 \to N'_1$

Case 4. Rule ($\forall^{\overset{a}{\simeq}}$), then $N_1 = \forall\overrightarrow{\alpha^+}.N'_1$, $N_2 = \forall\overrightarrow{\alpha^+}.N'_2$, and the algorithm makes a recursive call: $\Gamma \models N'_1 \overset{a}{\simeq} N'_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ and returns $(\Xi, \forall\overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)$ as the result.

Similarly to case 2, we apply the induction hypothesis to $\Gamma \models N'_1 \overset{a}{\simeq} N'_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ to obtain:

1. $\Gamma; \Xi \vdash M$, and hence, $\Gamma; \Xi \vdash \forall\overrightarrow{\alpha^+}.M$;
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ for $i \in \{1, 2\}$, and
3. $[\widehat{\tau}_i]\,M = N'_i$ for $i \in \{1, 2\}$, and then by the definition of the substitution, $[\widehat{\tau}_i]\forall\overrightarrow{\alpha^+}.M = \forall\overrightarrow{\alpha^+}.N'_i$ for $i \in \{1, 2\}$.

Case 5. Rule (AU$^-$), which applies when other rules do not, and $\Gamma \vdash N_i$, returning as the result $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2) = (\widehat{\alpha}^-_{\{N_1,N_2\}}, \widehat{\alpha}^-_{\{N_1,N_2\}}, N_1), (\widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_2))$.

1. $\Gamma; \Xi \vdash M$ is rewritten as $\Gamma; \widehat{\alpha}^-_{\{N_1,N_2\}} \vdash \widehat{\alpha}^-_{\{N_1,N_2\}}$, which holds trivially;
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ is rewritten as $\Gamma; \cdot \vdash (\widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_i) : \widehat{\alpha}^-_{\{N_1,N_2\}}$, which holds since $\Gamma \vdash N_i$ by the premise of the rule;
3. $[\widehat{\tau}_i]\,M = N_i$ is rewritten as $[\widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_i]\widehat{\alpha}^-_{\{N_1,N_2\}} = N_i$, which holds trivially by the definition of substitution.

Case 6. Positive cases are proved symmetrically.

□

Lemma 65 (Completeness of Anti-Unification)**.**

\+ *Assume that P_1 and P_2 are normalized, and there exists $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ such that*

1. $\Gamma; \Xi' \vdash Q'$,
2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ *for* $i \in \{1, 2\}$ *are anti-unification solutions, and*
3. $[\widehat{\tau}'_i]\,Q' = P_i$ *for* $i \in \{1, 2\}$.

Then the anti-unification algorithm terminates, that is there exists $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$

− *Assume that N_1 and N_2 are normalized, and there exists $(\Xi', M', \widehat{\tau}'_1, \widehat{\tau}'_2)$ such that*

1. $\Gamma; \Xi' \vdash M'$,
2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ *for* $i \in \{1, 2\}$, *are anti-unification solutions, and*
3. $[\widehat{\tau}'_i]\,M' = N_i$ *for* $i \in \{1, 2\}$.

Then the anti-unification algorithm succeeds, that is there exists $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$.

Proof. We prove it by the induction on M' and mutually on Q'.

Case 1. $M' = \widehat{\alpha}^-$ Then since $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$, $\Gamma \vdash [\widehat{\tau}'_i]\,M' = N_i$. This way, Rule (AU$^-$) is always applicable if other rules are not.

Case 2. $M' = \alpha^-$ Then $\alpha^- = [\widehat{\tau}'_i]\alpha^- = N_i$, which means that Rule (Var$^{-\overset{a}{\simeq}}$) is applicable.

Case 3. $M' = {\uparrow}Q'$ Then ${\uparrow}[\widehat{\tau}'_i]\,Q' = [\widehat{\tau}'_i]{\uparrow}\,Q' = N_i$, that is N_1 and N_2 have form ${\uparrow}P_1$ and ${\uparrow}P_2$ respectively.
Moreover, $[\widehat{\tau}'_i]\,Q' = P_i$, which means that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2. Then by the induction hypothesis, there exists $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, and hence, $\Gamma \models {\uparrow}P_1 \overset{a}{\simeq} {\uparrow}P_2 \dashv (\Xi, {\uparrow}\,Q, \widehat{\tau}_1, \widehat{\tau}_2)$ by Rule (${\uparrow}^{\overset{a}{\simeq}}$).

Case 4. $M' = \forall\overrightarrow{\alpha^+}.M''$ This case is similar to the previous one: we consider $\forall\overrightarrow{\alpha^+}$ as a constructor. Notice that $\forall\overrightarrow{\alpha^+}.[\widehat{\tau}'_i]\,M'' = [\widehat{\tau}'_i]\forall\overrightarrow{\alpha^+}.M'' = N_i$, that is N_1 and N_2 have form $\forall\overrightarrow{\alpha^+}.N''_1$ and $\forall\overrightarrow{\alpha^+}.N''_2$ respectively.
Moreover, $[\widehat{\tau}'_i]\,M'' = N''_i$, which means that $(\Xi', M'', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of N''_1 and N''_2. Then by the induction hypothesis, there exists $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \models N''_1 \overset{a}{\simeq} N''_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$, and hence, $\Gamma \models \forall\overrightarrow{\alpha^+}.N''_1 \overset{a}{\simeq} \forall\overrightarrow{\alpha^+}.N''_2 \dashv (\Xi, \forall\overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)$ by Rule ($\forall^{\overset{a}{\simeq}}$).

Case 5. $M' = Q' \to M''$ Then $[\widehat{\tau}'_i]\,Q' \to [\widehat{\tau}'_i]\,M'' = [\widehat{\tau}'_i](Q' \to M'') = N_i$, that is N_1 and N_2 have form $P_1 \to N'_1$ and $P_2 \to N'_2$ respectively.
Moreover, $[\widehat{\tau}'_i]\,Q' = P_i$ and $[\widehat{\tau}'_i]\,M'' = N''_i$, which means that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2, and $(\Xi', M'', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of N''_1 and N''_2. Then by the induction hypothesis, $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and $\Gamma \models N''_1 \overset{a}{\simeq} N''_2 \dashv (\Xi_2, M, \widehat{\tau}_3, \widehat{\tau}_4)$ succeed. The result of the algorithm is $(\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}_3, \widehat{\tau}_2 \cup \widehat{\tau}_4)$.

Case 6. $Q' = \widehat{\alpha}^+$ This case if not possible, since $\Gamma; \Xi' \vdash Q'$ means $\widehat{\alpha}^+ \in \Xi'$, but Ξ' can only contain negative variables.

Case 7. Other positive cases are proved symmetrically to the corresponding negative ones.

$\square$

Lemma 66 (Initiality of Anti-Unification).

+ *Assume that P_1 and P_2 are normalized, and $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, then $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than any other sound anti-unifier $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$, i.e. if*

  1. *$\Gamma; \Xi' \vdash Q'$,*
  2. *$\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ for $i \in \{1, 2\}$ are anti-unification solutions , and*
  3. *$[\widehat{\tau}'_i]Q' = P_i$ for $i \in \{1, 2\}$*

  then there exists $\widehat{\rho}$ such that $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\ Q'})$ and $[\widehat{\rho}]Q' = Q$. Moreover, $[\widehat{\rho}]\widehat{\beta}^-$ can be uniquely determined by $[\widehat{\tau}'_1]\widehat{\beta}^-$, $[\widehat{\tau}'_2]\widehat{\beta}^-$, and Γ.

− *Assume that N_1 and N_2 are normalized, and $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$, then $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than any other sound anti-unifier $(\Xi', M', \widehat{\tau}'_1, \widehat{\tau}'_2)$, i.e. if*

  1. *$\Gamma; \Xi' \vdash M'$,*
  2. *$\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ for $i \in \{1, 2\}$ are anti-unification solutions, and*
  3. *$[\widehat{\tau}'_i]M' = N_i$ for $i \in \{1, 2\}$*

  then there exists $\widehat{\rho}$ such that $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\ M'})$ and $[\widehat{\rho}]M' = M$. Moreover, $[\widehat{\rho}]\widehat{\beta}^-$ can be uniquely determined by $[\widehat{\tau}'_1]\widehat{\beta}^-$, $[\widehat{\tau}'_2]\widehat{\beta}^-$, and Γ.

Proof. First, let us assume that M' is a algorithmic variable $\widehat{\alpha}^-$. Then we can take $\widehat{\rho} = \widehat{\alpha}^- \mapsto M$, which satisfies the required properties:

- $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\ M'})$ holds since $\Xi'|_{\mathbf{uv}\ M'} = \widehat{\alpha}^-$ and $\Gamma; \Xi \vdash M$ by the soundness of anti-unification (lemma 64);
- $[\widehat{\rho}]M' = M$ holds by construction
- $[\widehat{\rho}]\widehat{\alpha}^- = M$ is the anti-unifier of $N_1 = [\widehat{\tau}'_1]\widehat{\alpha}^-$ and $N_2 = [\widehat{\tau}'_2]\widehat{\alpha}^-$ in context Γ, and hence, it is uniquely determined by them (observation 1).

Now, we can assume that M' is not a algorithmic variable. We prove by induction on the derivation of $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on the derivation of $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$.

Since M' is not a algorithmic variable, the substitution acting on M' preserves its outer constructor. In other words, $[\widehat{\tau}'_i]M' = N_i$ means that M', N_1 and N_2 have the same outer constructor. Let us consider the algorithmic anti-unification rule corresponding to this constructor, and show that it was successfully applied to anti-unify N_1 and N_2 (or P_1 and P_2).

Case 1. Rule ($\text{Var}^{-\overset{a}{\simeq}}$), i.e. $N_1 = \alpha^- = N_2$. This rule is applicable since it has no premises.

Then $\Xi = \cdot$, $M = \alpha^-$, and $\widehat{\tau}_1 = \widehat{\tau}_2 = \cdot$. Since $[\widehat{\tau}'_i]M' = N_i = \alpha^-$ and M' is not a algorithmic variable, $M' = \alpha^-$. Then we can take $\widehat{\rho} = \cdot$, which satisfies the required properties:

- $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\ M'})$ holds vacuously since $\Xi'|_{\mathbf{uv}\ M'} = \varnothing$;
- $[\widehat{\rho}]M' = M$, that is $[\cdot]\alpha^- = \alpha^-$ holds by substitution properties;
- the unique determination of $[\widehat{\rho}]\widehat{\alpha}^-$ for $\widehat{\alpha}^- \in \Xi'|_{\mathbf{uv}\ M'} = \varnothing$ holds vacuously.

Case 2. Rule ($\uparrow^{\overset{a}{\simeq}}$), i.e. $N_1 = \uparrow P_1$ and $N_2 = \uparrow P_2$.

Then since $[\widehat{\tau}'_i]M' = N_i = \uparrow P_i$ and M' is not a algorithmic variable, $M' = \uparrow Q'$, where $[\widehat{\tau}'_i]Q' = P_i$. Let us show that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2.

1. $\Gamma; \Xi' \vdash Q'$ holds by inversion of $\Gamma; \Xi' \vdash \uparrow Q'$;
2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ holds by assumption;
3. $[\widehat{\tau}'_i]Q' = P_i$ holds by assumption.

This way, by the completeness of anti-unification (lemma 65), the anti-unification algorithm succeeds on P_1 and P_2: $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, which means that Rule $(\uparrow\overset{a}{\simeq})$ is applicable to infer $\Gamma \models \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$.

Moreover, by the induction hypothesis, $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$, which immediately implies that $(\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than $(\Xi', \uparrow Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ (we keep the same $\widehat{\rho}$).

Case 3. Rule $(\forall\overset{a}{\simeq})$, i.e. $N_1 = \forall\overrightarrow{\alpha^+}.N'_1$ and $N_2 = \forall\overrightarrow{\alpha^+}.N'_2$. The proof is symmetric to the previous case. Notice that the context Γ is not changed in Rule $(\forall\overset{a}{\simeq})$, as it represents the context in which the anti-unification variables must be instantiated, rather than the context forming the types that are being anti-unified.

Case 4. Rule $(\rightarrow\overset{a}{\simeq})$, i.e. $N_1 = P_1 \rightarrow N'_1$ and $N_2 = P_2 \rightarrow N'_2$.

Then since $[\widehat{\tau}'_i]M' = N_i = P_i \rightarrow N'_i$ and M' is not a algorithmic variable, $M' = Q' \rightarrow M''$, where $[\widehat{\tau}'_i]Q' = P_i$ and $[\widehat{\tau}'_i]M'' = N''_i$.

Let us show that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2.

1. $\Gamma; \Xi' \vdash Q'$ holds by inversion of $\Gamma; \Xi' \vdash Q' \rightarrow M''$;
2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ holds by assumption;
3. $[\widehat{\tau}'_i]Q' = P_i$ holds by assumption.

Similarly, $(\Xi', M'', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of N''_1 and N''_2.

Then by the completeness of anti-unification (lemma 65), the anti-unification algorithm succeeds on P_1 and P_2: $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2)$; and on N'_1 and N'_2: $\Gamma \models N''_1 \overset{a}{\simeq} N''_2 \dashv (\Xi_2, M''', \widehat{\tau}_3, \widehat{\tau}_4)$. Notice that $\widehat{\tau}_1$ & $\widehat{\tau}_3$ and $\widehat{\tau}_2$ & $\widehat{\tau}_4$ are defined, in other words, for any $\widehat{\beta}^- \in \Xi_1 \cap \Xi_2$, $[\widehat{\tau}_1]\widehat{\beta}^- = [\widehat{\tau}_2]\widehat{\beta}^-$ and $[\widehat{\tau}_3]\widehat{\beta}^- = [\widehat{\tau}_4]\widehat{\beta}^-$, which follows immediately from observation 2. This way, the algorithm proceeds by applying Rule $(\rightarrow\overset{a}{\simeq})$ and returns $(\Xi_1 \cup \Xi_2, Q \rightarrow M''', \widehat{\tau}_1 \cup \widehat{\tau}_3, \widehat{\tau}_2 \cup \widehat{\tau}_4)$.

It is left to construct $\widehat{\rho}$ such that $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\ M'})$ and $[\widehat{\rho}]M' = M$. By the induction hypothesis, there exist $\widehat{\rho}_1$ and $\widehat{\rho}_2$ such that $\Gamma; \Xi_1 \vdash \widehat{\rho}_1 : (\Xi'|_{\mathbf{uv}\ Q'})$, $\Gamma; \Xi_2 \vdash \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\ M''})$, $[\widehat{\rho}_1]Q' = Q$, and $[\widehat{\rho}_2]M'' = M'''$.

Let us show that $\widehat{\rho} = \widehat{\rho}_1 \cup \widehat{\rho}_2$ satisfies the required properties:

- $\Gamma; \Xi_1 \cup \Xi_2 \vdash \widehat{\rho}_1 \cup \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\ M'})$ holds since $\Xi'|_{\mathbf{uv}\ M'} = \Xi'|_{\mathbf{uv}\ Q' \rightarrow M''} = (\Xi'|_{\mathbf{uv}\ Q'}) \cup (\Xi'|_{\mathbf{uv}\ M''})$, $\Gamma; \Xi_1 \vdash \widehat{\rho}_1 : (\Xi'|_{\mathbf{uv}\ Q'})$ and $\Gamma; \Xi_2 \vdash \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\ M''})$;
- $[\widehat{\rho}]M' = [\widehat{\rho}](Q' \rightarrow M'') = [\widehat{\rho}|_{\mathbf{uv}\ Q'}]Q' \rightarrow [\widehat{\rho}|_{\mathbf{uv}\ M''}]M'' = [\widehat{\rho}_1]Q' \rightarrow [\widehat{\rho}_2]M'' = Q \rightarrow M''' = M$;
- Since $[\widehat{\rho}]\widehat{\beta}^-$ is either equal to $[\widehat{\rho}_1]\widehat{\beta}^-$ or $[\widehat{\rho}_2]\widehat{\beta}^-$, it inherits their property that it is uniquely determined by $[\widehat{\tau}'_1]\widehat{\beta}^-$, $[\widehat{\tau}'_2]\widehat{\beta}^-$, and Γ.

Case 5. $P_1 = P_2 = \alpha^+$. This case is symmetric to case 1.

Case 6. $P_1 = \downarrow N_1$ and $P_2 = \downarrow N_2$. This case is symmetric to case 2

Case 7. $P_1 = \exists\overrightarrow{\alpha^-}.P'_1$ and $P_2 = \exists\overrightarrow{\alpha^-}.P'_2$. This case is symmetric to case 3

$\square$

6.8 Upper Bounds

Lemma 67 (Characterization of the Supertypes)**.** *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \textit{for } \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-}, M', \textit{and } \overrightarrow{N} \textit{ s.t.} \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq^D \downarrow M \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Immediately from lemma 19

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$
Then if $\Gamma \vdash Q \geqslant \exists\overrightarrow{\beta^-}.P'$, then by **??**, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant P'$, and $\mathbf{fv}\, Q \cap \overrightarrow{\beta^-} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule ($\exists^{\geqslant}$). This way, $\{Q \mid \Gamma \vdash Q \geqslant \exists\overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant P' \text{ s.t. } \mathbf{fv}\,(Q) \cap \overrightarrow{\beta^-} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = \downarrow M$
Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\nexists}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists\overrightarrow{\beta^-}.Q'$
Then the last applied rule to infer $\Gamma \vdash Q \geqslant \downarrow M$ must be Rule ($\downarrow^{\geqslant}$), which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq^{\leqslant} M$, then by lemma 45 and Rule ($\downarrow^{\simeq^D}$), $\downarrow M' \simeq^D \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq^D \downarrow M\} = \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\nexists}$.
In the other direction, $\downarrow M' \simeq^D \downarrow M \Rightarrow \Gamma \vdash \downarrow M' \simeq^{\leqslant} \downarrow M$ by lemma 40, since $\Gamma \vdash \downarrow M'$ by lemma 38
$\Rightarrow \Gamma \vdash \downarrow M' \geqslant \downarrow M$ by inversion

b. $Q = \exists\overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)
Then the last rule applied to infer $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q' \geqslant \downarrow M$ must be Rule ($\exists^{\geqslant}$). Inversion of this rule gives us $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \geqslant \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.*a*, $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \simeq^D \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists\overrightarrow{\beta^-}.\downarrow M' \in \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).
In the other direction, $[\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq^D \downarrow M \Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq^{\leqslant} \downarrow M$ by lemma 40, since $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M'$ by lemma 38
$\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \geqslant \downarrow M$ by inversion
$\Rightarrow \Gamma \vdash \exists\overrightarrow{\beta^-}.\downarrow M' \geqslant \downarrow M$ by Rule ($\exists^{\geqslant}$)

□

Lemma 68 (Characterization of the Normalized Supertypes). *For a normalized positive type $P = \mathbf{nf}\,(P)$, let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-}, M', \textit{and } \overrightarrow{N} \textit{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \end{array} \right\}$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Then from lemma 67, $\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant \beta^+\} = \{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\beta^+) \mid \text{for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

$$\begin{aligned}
\mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') &= \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ not using } \overrightarrow{\beta^-} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothesis} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}\,(Q) = \mathbf{fv}\,Q \text{ by lemma 30} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{by lemma 67} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} && \text{by the definition of } \mathsf{UB} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant \exists\overrightarrow{\beta^-}.P'\} && \text{by lemma 67}
\end{aligned}$$

Case 3. $P = \downarrow M$ Let us prove the set equality by two inclusions.

$\subseteq$ Suppose that $\Gamma \vdash Q \geqslant {\downarrow}M$ and M is normalized.

By lemma 67, $Q \in \mathsf{UB}(\Gamma \vdash {\downarrow}M)$. Then by definition of UB, $Q = \exists\overrightarrow{\alpha^-}.{\downarrow}M'$ for some $\overrightarrow{\alpha^-}$, M', and $\Gamma \vdash \sigma : \overrightarrow{\alpha^-}$ s.t. $[\sigma]{\downarrow}M' \simeq^D {\downarrow}M$.

We need to show that $\mathbf{nf}\,(Q) \in \mathsf{NFUB}(\Gamma \vdash {\downarrow}M)$. Notice that $\mathbf{nf}\,(Q) = \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') = \exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0$, where $\mathbf{nf}\,(M') = M_0$ and $\mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M_0 = \overrightarrow{\alpha^-}_0$.

The belonging of $\exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0$ to $\mathsf{NFUB}(\Gamma \vdash {\downarrow}M)$ means that

1. $\mathbf{ord}\,\overrightarrow{\alpha^-}_0\,\mathbf{in}\,M_0 = \overrightarrow{\alpha^-}_0$ and
2. that there exists $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^-}_0$ such that $[\sigma_0]{\downarrow}M_0 = {\downarrow}M$.

The first requirement holds by corollary 13. To show the second requirement, we construct σ_0 as $\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})$. Let us show the required properties of σ_0:

1. $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^-}_0$. Notice that by lemma 7, $\Gamma \vdash \sigma|_{\mathbf{fv}\,(M')} : \overrightarrow{\alpha^-} \cap \mathbf{fv}\,(M')$, which we rewrite as $\Gamma \vdash \sigma|_{\mathbf{fv}\,(M')} : \overrightarrow{\alpha^-}_0$ (since by lemma 25 $\overrightarrow{\alpha^-}_0 = \overrightarrow{\alpha^-} \cap \mathbf{fv}\,M_0$ as sets, and $\mathbf{fv}\,(M_0) = \mathbf{fv}\,(M')$ by lemma 30). Then by lemma 39, $\Gamma \vdash \mathbf{nf}\,(\sigma|_{\mathbf{fv}\,(M')}) : \overrightarrow{\alpha^-}_0$, that is $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^-}_0$.
2. $[\sigma_0]{\downarrow}M_0 = {\downarrow}M$. $[\sigma]{\downarrow}M' \simeq^D {\downarrow}M$ means $[\sigma|_{\mathbf{fv}\,(M')}]{\downarrow}M' \simeq^D {\downarrow}M$ by lemma 6. Then by lemma 56, $\mathbf{nf}\,([\sigma|_{\mathbf{fv}\,(M')}]{\downarrow}M') = \mathbf{nf}\,({\downarrow}M)$, implying $[\sigma_0]{\downarrow}M_0 = \mathbf{nf}\,({\downarrow}M)$ by lemma 32, and further $[\sigma_0]{\downarrow}M_0 = {\downarrow}M$ by lemma 34 (since ${\downarrow}M$ is normal by assumption).

$\supseteq$ Suppose that a type belongs to $\mathsf{NFUB}(\Gamma \vdash {\downarrow}M)$ for a normalized ${\downarrow}M$. Then it must have shape $\exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0$ for some $\overrightarrow{\alpha^-}_0$, M_0, and $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^-}_0$ such that $\mathbf{ord}\,\overrightarrow{\alpha^-}_0\,\mathbf{in}\,M_0 = \overrightarrow{\alpha^-}_0$ and $[\sigma_0]{\downarrow}M_0 = {\downarrow}M$. It is suffices to show that 1. $\exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0$ is normalized itself, and 2. $\Gamma \vdash \exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0 \geqslant {\downarrow}M$.

1. By definition, $\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0) = \exists\overrightarrow{\alpha^-}_1.{\downarrow}M_1$, where $M_1 = \mathbf{nf}\,(M_0)$ and $\mathbf{ord}\,\overrightarrow{\alpha^-}_0\,\mathbf{in}\,M_1 = \overrightarrow{\alpha^-}_1$. First, notice that by lemmas 28 and 31, $\mathbf{ord}\,\overrightarrow{\alpha^-}_0\,\mathbf{in}\,M_1 = \mathbf{ord}\,\overrightarrow{\alpha^-}_0\,\mathbf{in}\,M_0 = \overrightarrow{\alpha^-}_0$. This way, $\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0) = \exists\overrightarrow{\alpha^-}_0.{\downarrow}\mathbf{nf}\,(M_0)$. Second, M_0 is normalized by lemma 35, since $[\sigma_0]{\downarrow}M_0 = {\downarrow}M$ is normal. As such, $\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0) = \exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0$, in other words, $\exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0$ is normalized.
2. $\Gamma \vdash \exists\overrightarrow{\alpha^-}_0.{\downarrow}M_0 \geqslant {\downarrow}M$ holds immediately by Rule ($\exists^{\geqslant}$) with the substitution σ_0. Notice that $\Gamma \vdash [\sigma_0]{\downarrow}M_0 \geqslant {\downarrow}M$ follows from $[\sigma_0]{\downarrow}M_0 = {\downarrow}M$ by reflexivity of subtyping (lemma 22).

$\square$

Lemma 69. *Upper bounds of a type do not depend on the context as soon as the type is well-formed in it.*
If $\Gamma_1 \vdash P$ *and* $\Gamma_2 \vdash P$ *then* $\mathsf{UB}(\Gamma_1 \vdash P) = \mathsf{UB}(\Gamma_2 \vdash P)$ *and* $\mathsf{NFUB}(\Gamma_1 \vdash P) = \mathsf{NFUB}(\Gamma_2 \vdash P)$

Proof. We prove both inclusions by structural induction on P.

Case 1. $P = \beta^+$ Then $\mathsf{UB}(\Gamma_1 \vdash \beta^+) = \mathsf{UB}(\Gamma_2 \vdash \beta^+) = \{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \text{for some } \overrightarrow{\alpha^-}\}$. $\mathsf{NFUB}(\Gamma_1 \vdash \beta^+) = \mathsf{NFUB}(\Gamma_2 \vdash \beta^+) = \{\beta^+\}$.

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$. Then $\mathsf{UB}(\Gamma_1 \vdash \exists\overrightarrow{\beta^-}.P') = \mathsf{UB}(\Gamma_1, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$. $\mathsf{UB}(\Gamma_2 \vdash \exists\overrightarrow{\beta^-}.P') = \mathsf{UB}(\Gamma_2, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$. By the induction hypothesis, $\mathsf{UB}(\Gamma_1, \overrightarrow{\beta^-} \vdash P') = \mathsf{UB}(\Gamma_2, \overrightarrow{\beta^-} \vdash P')$, and if we restrict these sets to the same domain, they stay equal. Analogously, $\mathsf{NFUB}(\Gamma_1 \vdash \exists\overrightarrow{\beta^-}.P') = \mathsf{NFUB}(\Gamma_2 \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = {\downarrow}M$. Suppose that $\exists\overrightarrow{\alpha^-}.{\downarrow}M' \in \mathsf{UB}(\Gamma_1 \vdash {\downarrow}M)$. It means that $\Gamma_1, \overrightarrow{\alpha^-} \vdash M'$ and there exist $\Gamma_1 \vdash \overrightarrow{N}$ s.t. $[\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' \simeq^D {\downarrow}M$, or in other terms, there exists $\Gamma_1 \vdash \sigma : \overrightarrow{\alpha^-}$ such that $[\sigma]{\downarrow}M' \simeq^D {\downarrow}M$.

We need to show that $\exists\overrightarrow{\alpha^-}.{\downarrow}M' \in UB(\Gamma_2 \vdash {\downarrow}M)$, in other words, $\Gamma_2, \overrightarrow{\alpha^-} \vdash M'$ and there exists $\Gamma_2 \vdash \sigma_0 : \overrightarrow{\alpha^-}$ such that $[\sigma_0]{\downarrow}M' \simeq^D {\downarrow}M$.

First, let us show $\Gamma_2, \overrightarrow{\alpha^-} \vdash M'$. Notice that $[\sigma]{\downarrow}M' \simeq^D {\downarrow}M$ implies $\mathbf{fv}\,([\sigma]M') = \mathbf{fv}\,({\downarrow}M)$ by lemma 29. By lemma 15, $\mathbf{fv}\,(M') \backslash \overrightarrow{\alpha^-} \subseteq \mathbf{fv}\,([\sigma]M')$. This way, $\mathbf{fv}\,(M') \backslash \overrightarrow{\alpha^-} \subseteq \mathbf{fv}\,(M)$, implying $\mathbf{fv}\,(M') \subseteq \mathbf{fv}\,(M) \cup \overrightarrow{\alpha^-}$. By lemma 3, $\Gamma_2 \vdash {\downarrow}M$ implies $\mathbf{fv}\,M \subseteq \Gamma_2$, hence, $\mathbf{fv}\,M' \subseteq (\Gamma_2, \overrightarrow{\alpha^-})$, which by corollary 1 means $\Gamma_2, \overrightarrow{\alpha^-} \vdash M'$.

Second, let us construct the required σ_0 in the following way:

$$\begin{cases} [\sigma_0]\alpha_i^- = [\sigma]\alpha_i^- & \text{for } \alpha_i^- \in \overrightarrow{\alpha^-} \cap \mathbf{fv}\,(M') \\ [\sigma_0]\alpha_i^- = \forall\gamma^+.{\uparrow}\gamma^+ & \text{for } \alpha_i^- \in \overrightarrow{\alpha^-} \backslash \mathbf{fv}\,(M') \\ [\sigma_0]\gamma^\pm = \gamma^\pm & \text{for any other } \gamma^\pm \end{cases}$$

This construction of a substitution coincides with the one from the proof of lemma 20. This way, for σ_0, hold the same properties:

1. $[\sigma_0]M' = [\sigma]M'$, which in particular, implies $[\sigma_0]{\downarrow}M = [\sigma]{\downarrow}M$, and thus, $[\sigma]{\downarrow}M' \simeq^D {\downarrow}M$ can be rewritten to $[\sigma_0]{\downarrow}M' \simeq^D {\downarrow}M$; and
2. $\mathbf{fv}\,([\sigma]M') \vdash \sigma_0 : \overrightarrow{\alpha^-}$, which, as noted above, can be rewritten to $\mathbf{fv}\,(M) \vdash \sigma_0 : \overrightarrow{\alpha^-}$, and since $\mathbf{fv}\,M \subseteq \Gamma_2$, weakened to $\Gamma_2 \vdash \sigma_0 : \overrightarrow{\alpha^-}$.

The proof of $\mathsf{NFUB}(\Gamma_1 \vdash {\downarrow}M) \subseteq \mathsf{NFUB}(\Gamma_2 \vdash {\downarrow}M)$ is analogous. The differences are:

1. $\mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}$ holds by assumption,
2. $[\sigma]{\downarrow}M' = {\downarrow}M$ implies $\mathbf{fv}\,([\sigma]M') = \mathbf{fv}\,({\downarrow}M)$ by rewriting,
3. $[\sigma]{\downarrow}M' = {\downarrow}M$ and $[\sigma_0]{\downarrow}M = [\sigma]{\downarrow}M$ imply $[\sigma_0]{\downarrow}M' = {\downarrow}M$ by rewriting.

□

Lemma 70 (Soundness of the Least Upper Bound). *For types $\Gamma \vdash P_1$, and $\Gamma \vdash P_2$, if $\Gamma \vDash P_1 \vee P_2 = Q$ then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant P_1$ *and* $\Gamma \vdash Q \geqslant P_2$

Proof. Induction on $\Gamma \vDash P_1 \vee P_2 = Q$.

Case 1. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$
Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant \alpha^+$ by Rule $(\mathrm{Var}^{+\geqslant})$.

Case 2. $\Gamma \vDash \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q$
Then by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applies to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q$. Then

(i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,
(ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant P_1$,
(iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}\,(Q) \cap (\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}) = \mathbf{fv}\,(Q) \cap \Gamma$ (and then apply section 5.3). The inclusion right-to-left is self-evident. To show $\mathbf{fv}\,(Q) \cap (\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}) \subseteq \mathbf{fv}\,(Q) \cap \Gamma$, we prove that $\mathbf{fv}\,(Q) \subseteq \Gamma$

$\mathbf{fv}\,(Q) \subseteq \mathbf{fv}\,P_1 \cap \mathbf{fv}\,P_2$ by lemma 17

$\subseteq ((\Gamma, \overrightarrow{\alpha^-})\backslash\overrightarrow{\beta^-}) \cap ((\Gamma, \overrightarrow{\beta^-})\backslash\overrightarrow{\alpha^-})$ since $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_1$, $\mathbf{fv}\,(P_1) \subseteq (\Gamma, \overrightarrow{\alpha^-}) = (\Gamma, \overrightarrow{\alpha^-})\backslash\overrightarrow{\beta^-}$ (the latter is because by the Barendregt's convention, $(\Gamma, \overrightarrow{\alpha^-}) \cap \overrightarrow{\beta^-} = \varnothing$); similarly, $\mathbf{fv}\,(P_2) \subseteq (\Gamma, \overrightarrow{\beta^-})\backslash\overrightarrow{\alpha^-}$

$\subseteq \Gamma$

To show $\Gamma \vdash Q \geqslant \exists\overrightarrow{\alpha^-}.P_1$, we apply Rule $(\exists^{\geqslant})$. Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant \exists\overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \vDash {\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$. By the inversion, $\Gamma, \cdot \vDash \mathbf{nf}\,({\downarrow}N) \overset{a}{\simeq} \mathbf{nf}\,({\downarrow}M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$. Then by the soundness of anti-unification (**??**),

(i) $\Gamma; \Xi \vdash P$, then by lemma 51,

$$\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]P \tag{7}$$

(ii) $\Gamma; \cdot \vdash \widehat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_2 : \Xi$. Assuming that $\Xi = \widehat{\beta}^-_1, .., \widehat{\beta}^-_n$, the antiunification solutions $\widehat{\tau}_1$ and $\widehat{\tau}_2$ can be put explicitly as $\widehat{\tau}_1 = (\widehat{\beta}^-_1 :\approx N_1, .., \widehat{\beta}^-_n :\approx N_n)$, and $\widehat{\tau}_2 = (\widehat{\beta}^-_1 :\approx M_1, .., \widehat{\beta}^-_n :\approx M_n)$. Then

$$\widehat{\tau}_1 = (\overrightarrow{N}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \tag{8}$$

$$\widehat{\tau}_2 = (\overrightarrow{M}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \tag{9}$$

(iii) $[\widehat{\tau}_1]\,Q = P_1$ and $[\widehat{\tau}_2]\,Q = P_1$, which, by 8 and 9, means

$$[\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]\,P = \mathbf{nf}\,(\downarrow N) \tag{10}$$

$$[\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]\,P = \mathbf{nf}\,(\downarrow M) \tag{11}$$

Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P$ follows directly from 7.

To show $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P \geqslant \downarrow N$, we apply Rule ($\exists^{\geqslant}$), instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{N}$. Then $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]\,P \geqslant \downarrow N$ follows from 10 and since $\Gamma \vdash \mathbf{nf}\,(\downarrow N) \geqslant \downarrow N$ (by corollary 18).

Analogously, instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{M}$, gives us $\Gamma \vdash [\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]\,P \geqslant \downarrow M$ (from 11), and hence, $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P \geqslant \downarrow M$.

□

Lemma 71 (Completeness and Initiality of the Least Upper Bound)**.** *For types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q$ such that $\Gamma \vdash Q \geqslant P_1$ and $\Gamma \vdash Q \geqslant P_2$, there exists Q' s.t. $\Gamma \vDash P_1 \vee P_2 = Q'$ and $\Gamma \vdash Q \geqslant Q'$.*

Proof. Induction on the pair (P_1, P_2). From lemma 68, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$. Let us consider the cases of what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_i$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$, where either $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then $Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_2.Q_2)$

$$\begin{aligned}
&\subseteq \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_2 \vdash Q_2) && \text{from the definition of UB}\\
&= \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_2) && \text{by lemma 69, weakening and exchange}\\
&= \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant Q_1\} \cap \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant Q_2\} && \text{by lemma 67,}
\end{aligned}$$

meaning that $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant Q_2$. Then the next step of the algorithm—the recursive call $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vDash Q_1 \vee Q_2 = Q'$ terminates by the induction hypothesis, and moreover, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant Q'$. This way, the result of the algorithm is Q', i.e. $\Gamma \vDash P_1 \vee P_2 = Q'$.

Since both Q and Q' are sound upper bounds, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant Q'$ can be strengthened to $\Gamma \vdash Q \geqslant Q'$ by lemma 20.

Case 2. $P_1 = \alpha^+$ and $P_2 = \downarrow N$
Then the set of common upper bounds of $\downarrow N$ and α^+ is empty, and thus, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$ gives a contradiction:
$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \downarrow N)$

$$\begin{aligned}
&= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\downarrow M' \mid \cdots\} && \text{by the definition of UB}\\
&= \varnothing && \text{since } \alpha^+ \neq \downarrow M' \text{ for any } M'
\end{aligned}$$

Case 3. $P_1 = \downarrow N$ and $P_2 = \alpha^+$
Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)
Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \beta^+)$

$$\begin{aligned}
&= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\beta^+ \mid \cdots\} && \text{by the definition of UB}\\
&= \varnothing && \text{since } \alpha^+ \neq \beta^+
\end{aligned}$$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$
Then the algorithm terminates in one step (Rule ($\mathrm{Var}^{\vee}$)) and the result is α^+, i.e. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$.

Since $Q \in \mathsf{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists\overrightarrow{\alpha^-}.\alpha^+$. Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant \alpha^+$ by Rule ($\exists^{\geqslant}$): $\overrightarrow{\alpha^-}$ can be instantiated with arbitrary negative types (for example $\forall\beta^+.\uparrow\beta^+$), since the substitution for unused variables does not change the term $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geqslant \alpha^+$ by Rule ($\mathrm{Var}^{+\geqslant}$).

Case 6. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$
Then on the next step, the algorithm tries to anti-unify $\mathbf{nf}\,(\downarrow M_1)$ and $\mathbf{nf}\,(\downarrow M_2)$. By **??**, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$$\begin{aligned}
\mathbf{nf}\,(Q) &\in \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,(\downarrow M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,(\downarrow M_2))\\
&= \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}\,(M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}\,(M_2))\\
&= \begin{array}{l}\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l}\text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-},\\ \Gamma \vdash N_i,\, \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1)\end{array} \right\}\\ \cap \\ \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l}\text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-},\\ \Gamma \vdash \overrightarrow{N}_1,\, \Gamma \vdash \overrightarrow{N}_2,\, \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2)\end{array} \right\}\end{array}\\
&= \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l}\text{for } \overrightarrow{\alpha^-},\, M',\, \overrightarrow{N}_1 \text{ and } \overrightarrow{N}_2 \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-},\\ \Gamma \vdash \overrightarrow{N}_1,\, \Gamma \vdash \overrightarrow{N}_2,\, \Gamma, \overrightarrow{\alpha^-} \vdash M',\, [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1), \text{ and } [\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2)\end{array} \right\}
\end{aligned}$$

The fact that the latter set is non-empty means that there exist $\overrightarrow{\alpha^-}$, M', $\overrightarrow{N}_1$ and $\overrightarrow{N}_2$ such that

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$ (notice that M' is normal)

(ii) $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma \vdash \overrightarrow{N}_1$,

(iii) $[\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1)$ and $[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2)$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative anti-unification variable $\widehat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\widehat{\alpha}^-}$. Let us show that $(\overrightarrow{\widehat{\alpha}^-},\ [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M',\ \overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-},\ \overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-})$ is a sound anti-unifier of $\mathbf{nf}\,(\downarrow M_1)$ and $\mathbf{nf}\,(\downarrow M_2)$ in context Γ:

- $\overrightarrow{\widehat{\alpha}^-}$ is negative by construction,
- $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M'$ because $\Gamma, \overrightarrow{\alpha^-} \vdash \downarrow M'$ `Ilya: lemma!`,
- $\Gamma; \cdot \vdash (\overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma; \cdot \vdash (\overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N}_2$,
- $[\overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1) = \mathbf{nf}\,(\downarrow M_1)$.
- $[\overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2) = \mathbf{nf}\,(\downarrow M_2)$.

Then by the completeness of the anti-unification (lemma 65), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it, i.e. $Q' = \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, where $(\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$ is the result of the anti-unification of $\mathbf{nf}\,(\downarrow M_1)$ and $\mathbf{nf}\,(\downarrow M_2)$ in context Γ.

Moreover, lemma 65 also says that the found anti-unification solution is initial, i.e. there exists $\widehat{\tau}$ such that $\Gamma; \Xi \vdash \widehat{\tau} : \overrightarrow{\widehat{\alpha}^-}$ and $[\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}$, (ii) $\widehat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]\downarrow M' = [\overrightarrow{\beta^-}/\Xi][\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{\beta^-}/\Xi]P$. In particular, from the reflexivity of subtyping: $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]\downarrow M' \geqslant [\overrightarrow{\beta^-}/\Xi]P$.

It allows us to show $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant Q'$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\downarrow M' \geqslant \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, by applying Rule ($\exists^{\geqslant}$), instantiating $\overrightarrow{\alpha^-}$ with respect to σ. Finally, $\Gamma \vdash Q \geqslant Q'$ by transitively combining $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant Q'$ and $\Gamma \vdash Q \geqslant \mathbf{nf}\,(Q)$ (holds by corollary 18 and inversion).

□

6.9 Upgrade

Let us consider a type P well-formed in Γ. Some of its Γ-supertypes are also well-formed in a smaller context $\Delta \subseteq \Gamma$. The upgrade is the operation that returns the least of such supertypes.

Lemma 72 (Soundness of Upgrade). *Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^\pm}$, if $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ then*

1. $\Delta \vdash Q$
2. $\Gamma \vdash Q \geqslant P$

Proof. By inversion, $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ means that for fresh $\overrightarrow{\beta^\pm}$ and $\overrightarrow{\gamma^\pm}$, $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vDash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q$. Then by the soundness of the least upper bound (lemma 70),

1. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q$,

2. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$, and

3. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P$.

$$
\begin{aligned}
\mathbf{fv}\, Q &\subseteq \mathbf{fv}\, [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \cap \mathbf{fv}\, [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P && \text{Since by lemma 17, } \mathbf{fv}\, Q \subseteq \mathbf{fv}\, [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \text{ and } \mathbf{fv}\, Q \subseteq \mathbf{fv}\, [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P \\
&\subseteq ((\mathbf{fv}\, P \backslash \overrightarrow{\alpha^\pm}) \cup \overrightarrow{\beta^\pm}) \cap ((\mathbf{fv}\, P \backslash \overrightarrow{\alpha^\pm}) \cup \overrightarrow{\gamma^\pm}) \\
&= (\mathbf{fv}\, P \backslash \overrightarrow{\alpha^\pm}) \cap (\mathbf{fv}\, P \backslash \overrightarrow{\alpha^\pm}) && \text{since } \overrightarrow{\beta^\pm} \text{ and } \overrightarrow{\gamma^\pm} \text{ are fresh} \\
&= \mathbf{fv}\, P \backslash \overrightarrow{\alpha^\pm} \\
&\subseteq \Gamma \backslash \overrightarrow{\alpha^\pm} && \text{since } P \text{ is well-formed in } \Gamma \\
&\subseteq \Delta
\end{aligned}
$$

This way, by section 5.3, $\Delta \vdash Q$.

Let us apply $\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}$—the inverse of the substitution $\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}$ to both sides of $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$ and by lemma 23 (since $\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}$ can be specified as $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash \overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm} : \Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm}$ by) obtain $\Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm} \vdash [\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}]Q \geqslant P$. Notice that $\Delta \vdash Q$ implies that $\mathbf{fv}\, Q \cap \overrightarrow{\beta^\pm} = \varnothing$, then by corollary 4, $[\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}]Q = Q$, and thus $\Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant P$. By context strengthening, $\Delta, \overrightarrow{\alpha^\pm} \vdash Q \geqslant P$. □

Lemma 73 (Completeness and Initiality of Upgrade)**.** *The upgrade returns the least Γ-supertype of P well-formed in Δ. Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^\pm}$,*
For any Q' such that

1. *$\Delta \vdash Q'$ and*

2. *$\Gamma \vdash Q' \geqslant P$,*

*The result of the upgrade algorithm Q exists (**upgrade** $\Gamma \vdash P$ **to** $\Delta = Q$) and satisfies $\Delta \vdash Q' \geqslant Q$.*

Proof. Let us consider fresh (not intersecting with Γ) $\overrightarrow{\beta^\pm}$ and $\overrightarrow{\gamma^\pm}$.

If we apply substitution $\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}$ to both sides of $\Delta, \overrightarrow{\alpha^\pm} \vdash Q' \geqslant P$, we have $\Delta, \overrightarrow{\beta^\pm} \vdash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]Q' \geqslant [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$, which by corollary 4, since $\overrightarrow{\alpha^\pm}$ is disjoint from $\mathbf{fv}\,(Q')$ (because $\Delta \vdash Q'$), simplifies to $\Delta, \overrightarrow{\beta^\pm} \vdash Q' \geqslant [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$.

Analogously, if we apply substitution $\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}$ to both sides of $\Delta, \overrightarrow{\alpha^\pm} \vdash Q' \geqslant P$, we have $\Delta, \overrightarrow{\gamma^\pm} \vdash Q' \geqslant [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P$.

This way, Q' is a common supertype of $[\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$ and $[\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P$ in context $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm}$. It means that we can apply the completeness of the least upper bound (lemma 71):

1. there exists Q s.t. $\Gamma \vDash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q$

2. $\Gamma \vdash Q' \geqslant Q$.

The former means that the upgrade algorithm terminates and returns Q. The latter means that since both Q' and Q are well-formed in Δ, by **??**, $\Delta \vdash Q' \geqslant Q$. □

6.10 Constraint Satisfaction

Lemma 74 (Any constraint is satisfiable)**.** *Suppose that $\Theta \vdash SC$ and Ξ is a set such that $\mathbf{dom}\,(SC) \subseteq \Xi \subseteq \mathbf{dom}\,(\Theta)$. Then there exists $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : \Xi$ and $\Theta \vdash \widehat{\sigma} : SC$.*

Proof. Let us define $\widehat{\sigma}$ on $\mathbf{dom}\,(SC)$ in the following way:

$$
[\widehat{\sigma}]\widehat{\alpha}^\pm = \begin{cases} P & \text{if } (\widehat{\alpha}^\pm :\approx P) \in SC \\ P & \text{if } (\widehat{\alpha}^\pm :\geqslant P) \in SC \\ N & \text{if } (\widehat{\alpha}^\pm :\approx N) \in SC \\ \exists \beta^-. {\downarrow} \beta^- & \text{if } \widehat{\alpha}^\pm = \widehat{\alpha}^+ \in \Xi \backslash \mathbf{dom}\,(SC) \\ \forall \beta^+. {\uparrow} \beta^+ & \text{if } \widehat{\alpha}^\pm = \widehat{\alpha}^- \in \Xi \backslash \mathbf{dom}\,(SC) \end{cases}
$$

Then $\Theta \vdash \widehat{\sigma} : SC$ follows immediately from the reflexivity of equivalence and subtyping (lemma 22) and the corresponding rules Rule SATSCEPEq, Rule SATSCENEq, and Rule SATSCESup. □

Lemma 75 (Constraint Entry Satiisfaction is Stable under Equivalence)**.** − *If $\Gamma \vdash N_1 : e$ and $\Gamma \vdash N_1 \simeq^\leqslant N_2$ then $\Gamma \vdash N_2 : e$.*

\+ *If $\Gamma \vdash P_1 : e$ and $\Gamma \vdash P_1 \simeq^\leqslant P_2$ then $\Gamma \vdash P_2 : e$.*

Proof. − Then e has form $(\widehat{\alpha}^- :\approx M)$, and by inversion, $\Gamma \vdash N_1 \simeq^\leqslant M$. Then by transitivity, $\Gamma \vdash N_2 \simeq^\leqslant M$, meaning $\Gamma \vdash N_2 : e$.

+ Let us consider what form e has.

Case 1. $e = (\widehat{\alpha}^+ :\approx Q)$. Then $\Gamma \vdash P_1 \simeq^{\leqslant} Q$, and hence, $\Gamma \vdash P_2 \simeq^{\leqslant} Q$ by transitivity. Then $\Gamma \vdash P_2 : e$.

Case 2. $e = (\widehat{\alpha}^+ :\geqslant Q)$. Then $\Gamma \vdash P_1 \geqslant Q$, and hence, $\Gamma \vdash P_2 \geqslant Q$ by transitivity. Then $\Gamma \vdash P_2 : e$.

□

Corollary 27 (Constraint Satisfaction is stable under Equivalence)**.** *If* $\Theta \vdash \widehat{\sigma}_1 : SC$ *and* $\Theta \vdash \widehat{\sigma}_1 \simeq^{\leqslant} \widehat{\sigma}_2 : \mathbf{dom}\,(SC)$ *then* $\Theta \vdash \widehat{\sigma}_2 : SC$*;*
if $\Theta \vdash \widehat{\sigma}_1 : UC$ *and* $\Theta \vdash \widehat{\sigma}_1 \simeq^{\leqslant} \widehat{\sigma}_2 : \mathbf{dom}\,(SC)$ *then* $\Theta \vdash \widehat{\sigma}_2 : UC$*.*

Corollary 28 (Normalization preserves Constraint Satisfaction)**.** *If* $\Theta \vdash \widehat{\sigma} : SC$ *then* $\Theta \vdash \mathbf{nf}\,(\widehat{\sigma}) : SC$*;*
if $\Theta \vdash \widehat{\sigma} : UC$ *then* $\Theta \vdash \mathbf{nf}\,(\widehat{\sigma}) : UC$*.*

6.11 Positive Subtyping

Lemma 76 (Soundness of the Positive Subtyping)**.** *If* $\Gamma \vdash^{\supseteq} \Theta$*,* $\Gamma \vdash Q$*,* $\Gamma; \mathbf{dom}\,(\Theta) \vdash P$*, and* $\Gamma; \Theta \models P \geqslant Q \dashv SC$*, then* $\Theta \vdash SC : \mathbf{uv}\,P$ *and for any normalized* $\widehat{\sigma}$ *such that* $\Theta \vdash \widehat{\sigma} : SC$*,* $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$*.*

Proof. We prove it by induction on $\Gamma; \Theta \models P \geqslant Q \dashv SC$. Let us consider the last rule to infer this judgment.

Case 1. Rule (UVar$^{\geqslant}$) then $\Gamma; \Theta \models P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \models \widehat{\alpha}^+ \geqslant P' \dashv (\widehat{\alpha}^+ :\geqslant Q')$ where $\widehat{\alpha}^+\{\Delta\} \in \Theta$ and $\mathbf{upgrade}\,\Gamma \vdash P'\,\mathbf{to}\,\Delta = Q'$.

Notice that $\widehat{\alpha}^+\{\Delta\} \in \Theta$ and $\Gamma \vdash^{\supseteq} \Theta$ implies $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$ for some $\overrightarrow{\alpha^{\pm}}$, hence, the soundness of upgrade (lemma 72) is applicable:

1. $\Delta \vdash Q'$ and
2. $\Gamma \vdash Q' \geqslant P$.

Since $\widehat{\alpha}^+\{\Delta\} \in \Theta$ and $\Delta \vdash Q'$, it is clear that $\Theta \vdash (\widehat{\alpha}^+ :\geqslant Q') : \widehat{\alpha}^+$.

It is left to show that $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant P'$ for any normalized $\widehat{\sigma}$ s.t. $\Theta \vdash \widehat{\sigma} : (\widehat{\alpha}^+ :\geqslant Q')$. The latter means that $\Theta(\widehat{\alpha}^+) \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant Q'$, i.e. $\Delta \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant Q'$. By weakening the context to Γ and combining this judgment transitively with $\Gamma \vdash Q' \geqslant P$, we have $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant P$, as required.

Case 2. Rule (Var$^{+\geqslant}$) then $\Gamma; \Theta \models P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot$. Then $\mathbf{uv}\,\alpha^+ = \varnothing$, and $SC = \cdot$ satisfies $\Theta \vdash SC : \cdot$. Since $\mathbf{uv}\,\alpha^+ = \varnothing$, application of any substitution $\widehat{\sigma}$ does not change α^+, i.e. $[\widehat{\sigma}]\alpha^+ = \alpha^+$. Therefore, $\Gamma \vdash [\widehat{\sigma}]\alpha^+ \geqslant \alpha^+$ holds by Rule (Var$^{-\leqslant}$).

Case 3. Rule ($\downarrow^{\geqslant}$) then $\Gamma; \Theta \models P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \models \downarrow N \geqslant \downarrow M \dashv SC$.
Then the next step of the algorithm is the unification of $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$, and it returns the resulting unification constraint $UC = SC$ as the result. By the soundness of unification (lemma 62), $\Theta \vdash SC : \mathbf{uv}\,(N)$ and for any normalized $\widehat{\sigma}$, $\Theta \vdash \widehat{\sigma} : SC$ implies $[\widehat{\sigma}]\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$, then we rewrite the left-hand side by lemma 32: $\mathbf{nf}\,([\widehat{\sigma}]N) = \mathbf{nf}\,(M)$ and apply lemma 46: $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M$, then by Rule ($\uparrow^{\leqslant}$), $\Gamma \vdash \downarrow[\widehat{\sigma}]N \geqslant \downarrow M$.

Case 4. Rule ($\exists^{\geqslant}$) then $\Gamma; \Theta \models P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \models \exists\overrightarrow{\alpha^-}.P' \geqslant \exists\overrightarrow{\beta^-}.Q' \dashv SC$ s.t. either $\overrightarrow{\alpha^-}$ or $\overrightarrow{\beta^-}$ is not empty.
Then the algorithm creates fresh unification variables $\overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\}$, substitutes the old $\overrightarrow{\alpha^-}$ with them in P', and makes the recursive call: $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P' \geqslant Q' \dashv SC'$, returning as the result $SC = SC' \backslash \overrightarrow{\widehat{\alpha}^-}$.

Let us take an arbitrary normalized $\widehat{\sigma}$ s.t. $\Theta \vdash \widehat{\sigma} : SC' \backslash \overrightarrow{\widehat{\alpha}^-}$. We wish to show $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\widehat{\sigma}]P' \geqslant \exists\overrightarrow{\beta^-}.Q'$. To do that, we apply Rule ($\exists^{\geqslant}$), and what is left to show is $\Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]P' \geqslant Q'$ for some $\overrightarrow{N}$. If we construct a normalized $\widehat{\sigma}'$ such that $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}' : SC'$ and for some $\overrightarrow{N}$, $[\overrightarrow{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]P' = [\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$, we can apply the induction hypothesis to $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv SC'$ and infer the required subtyping.

Let us construct such $\widehat{\sigma}'$ by extending $\widehat{\sigma}$ with $\overrightarrow{\widehat{\alpha}^-}$ mapped to the corresponding types in SC':

$$[\widehat{\sigma}']\widehat{\beta}^{\pm} = \begin{cases} [\widehat{\sigma}]\widehat{\beta}^{\pm} & \text{if } \widehat{\beta}^{\pm} \in \mathbf{dom}\,(SC') \backslash \overrightarrow{\widehat{\alpha}^-} \\ \mathbf{nf}\,(N) & \text{if } \widehat{\beta}^{\pm} \in \overrightarrow{\widehat{\alpha}^-} \text{ and } (\widehat{\beta}^{\pm} :\approx N) \in SC' \end{cases}$$

It is easy to see that $\widehat{\sigma}'$ is normalized: it inherits this property from $\widehat{\sigma}$. Let us show that $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}' : SC'$. Let us take an arbitrary entry e from SC' restricting a variable $\widehat{\beta}^{\pm}$. Suppose $\widehat{\beta}^{\pm} \in \mathbf{dom}\,(SC') \backslash \overrightarrow{\widehat{\alpha}^-}$. Then $(\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\})(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}']\widehat{\beta}^{\pm} : e$ is rewritten as $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\beta}^{\pm} : e$, which holds since $\Theta \vdash \widehat{\sigma} : SC'$. Suppose $\widehat{\beta}^{\pm} = \widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-}$. Then $e = (\widehat{\alpha_i}^- :\approx N)$ for some

N, $[\hat{\sigma}']\widehat{\alpha_i}^- = \mathbf{nf}(N)$ by the definition, and $\Gamma, \overrightarrow{\beta^-} \vdash \mathbf{nf}(N) : (\widehat{\alpha_i}^- :\approx N)$ by Rule SATSCENEq, since $\Gamma \vdash \mathbf{nf}(N) \simeq^{\leqslant} N$ by lemma 46.

Finally, let us show that $[\overrightarrow{N}/\overrightarrow{\alpha^-}][\hat{\sigma}]P' = [\hat{\sigma}'][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$. For N_i, we take the *normalized* type restricting $\widehat{\alpha_i}^-$ in SC'. Let us take an arbitrary variable from P.

1. If this variable is a unification variable $\hat{\beta}^\pm$, then $[\overrightarrow{N}/\overrightarrow{\alpha^-}][\hat{\sigma}]\hat{\beta}^\pm = [\hat{\sigma}]\hat{\beta}^\pm$, since $\Theta \vdash \hat{\sigma} : SC' \backslash \overrightarrow{\hat{\alpha}^-}$ and $\mathbf{dom}(\Theta) \cap \overrightarrow{\alpha^-} = \varnothing$. Notice that $\hat{\beta}^\pm \in \mathbf{dom}(\Theta)$, which is disjoint from $\overrightarrow{\hat{\alpha}^-}$, that is $\hat{\beta}^\pm \in \mathbf{dom}(SC') \backslash \overrightarrow{\hat{\alpha}^-}$. This way, $[\hat{\sigma}'][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\hat{\beta}^\pm = [\hat{\sigma}']\hat{\beta}^\pm = [\hat{\sigma}]\hat{\beta}^\pm$ by the definition of $\hat{\sigma}'$,
2. If this variable is a regular variable $\beta^\pm \notin \overrightarrow{\alpha^-}$, then $[\overrightarrow{N}/\overrightarrow{\alpha^-}][\hat{\sigma}]\beta^\pm = \beta^\pm$ and $[\hat{\sigma}'][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\beta^\pm = \beta^\pm$.
3. If this variable is a regular variable $\alpha_i^- \in \overrightarrow{\alpha^-}$, then $[\overrightarrow{N}/\overrightarrow{\alpha^-}][\hat{\sigma}]\alpha_i^- = N_i = \mathbf{nf}(N_i)$ (the latter equality holds since N_i is normalized) and $[\hat{\sigma}'][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\alpha_i^- = [\hat{\sigma}']\widehat{\alpha_i}^- = \mathbf{nf}(N_i)$.

□

Lemma 77 (Completeness of the Positive Subtyping)**.** *Suppose that* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash Q$ *and* $\Gamma; \mathbf{dom}(\Theta) \vdash P$. *Then for any* $\Theta \vdash \hat{\sigma} : \mathbf{uv}(P)$ *such that* $\Gamma \vdash [\hat{\sigma}]P \geqslant Q$, *there exists* $\Gamma;\, \Theta \models P \geqslant Q \dashv SC$ *and moreover,* $\Theta \vdash \hat{\sigma} : SC$.

Proof. Let us prove this lemma by induction on $\Gamma \vdash [\hat{\sigma}]P \geqslant Q$. Let us consider the last rule used in the derivation, but first, consider the base case for the substitution $[\hat{\sigma}]P$:

Case 1. $P = \exists\overrightarrow{\beta^-}.\hat{\alpha}^+$ (for potentially empty $\overrightarrow{\beta^-}$)

Then by assumption, $\Gamma \vdash \exists\overrightarrow{\beta^-}.[\hat{\sigma}]\hat{\alpha}^+ \geqslant Q$ (where $\overrightarrow{\beta^-} \cap \mathbf{fv}\,[\hat{\sigma}]\hat{\alpha}^+ = \varnothing$). Let us decompose Q as $Q = \exists\overrightarrow{\gamma^-}.Q_0$, where Q_0 does not start with $\exists$.

By inversion, $\Gamma; \mathbf{dom}(\Theta) \vdash \exists\overrightarrow{\beta^-}.\hat{\alpha}^+$ implies $\hat{\alpha}^+\{\Delta\} \in \Theta$ for some $\Delta \subseteq \Gamma$.

By lemma 18 applied twice, $\Gamma \vdash \exists\overrightarrow{\beta^-}.[\hat{\sigma}]\hat{\alpha}^+ \geqslant \exists\overrightarrow{\gamma^-}.Q_0$ implies $\Gamma, \overrightarrow{\gamma^-} \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}][\hat{\sigma}]\hat{\alpha}^+ \geqslant Q_0$ for some N, and since $\overrightarrow{\beta^-} \cap \mathbf{fv}([\hat{\sigma}]\hat{\alpha}^+) \subseteq \overrightarrow{\beta^-} \cap \Theta(\hat{\alpha}^+) \subseteq \overrightarrow{\beta^-} \cap \Gamma = \varnothing$, $[\overrightarrow{N}/\overrightarrow{\beta^-}][\hat{\sigma}]\hat{\alpha}^+ = [\hat{\sigma}]\hat{\alpha}^+$, that is $\Gamma, \overrightarrow{\gamma^-} \vdash [\hat{\sigma}]\hat{\alpha}^+ \geqslant Q_0$.

When algorithm tires to infer the subtyping $\Gamma;\, \Theta \models \exists\overrightarrow{\beta^-}.\hat{\alpha}^+ \geqslant \exists\overrightarrow{\gamma^-}.Q_0 \dashv SC$, it applies Rule ($\exists^{\geqslant}$), which reduces the problem to $\Gamma, \overrightarrow{\gamma^-};\, \Theta, \overrightarrow{\hat{\beta}^-}\{\Gamma, \overrightarrow{\gamma^-}\} \models [\overrightarrow{\hat{\beta}^-}/\overrightarrow{\beta^-}]\hat{\alpha}^+ \geqslant Q_0 \dashv SC$, which is equivalent to $\Gamma, \overrightarrow{\gamma^-};\, \Theta, \overrightarrow{\hat{\beta}^-}\{\Gamma, \overrightarrow{\gamma^-}\} \models \hat{\alpha}^+ \geqslant Q_0 \dashv SC$.

Next, the algorithm tries to apply Rule (UVar$^{\geqslant}$) and the resulting restriction is $SC = (\hat{\alpha}^+ :\geqslant Q_0')$ where **upgrade** $\Gamma, \overrightarrow{\gamma^-} \vdash Q_0$ **to** $\Delta = Q_0'$.

Why does the upgrade procedure terminate? Because $[\hat{\sigma}]\hat{\alpha}^+$ satisfies the pre-conditions of the completeness of the upgrade (lemma 73):

1. $\Delta \vdash [\hat{\sigma}]\hat{\alpha}^+$ because $\Theta \vdash \hat{\sigma} : \hat{\alpha}^+$ and $\hat{\alpha}^+\{\Delta\} \in \Theta$,
2. $\Gamma, \overrightarrow{\gamma^-} \vdash [\hat{\sigma}]\hat{\alpha}^+ \geqslant Q_0$ as noted above

Moreover, the completeness of upgrade also says that Q_0' is *the least* supertype of Q_0 among types well-formed in Δ, that is $\Delta \vdash [\hat{\sigma}]\hat{\alpha}^+ \geqslant Q_0'$, which means $\Theta \vdash \hat{\sigma} : (\hat{\alpha}^+ :\geqslant Q_0')$, that is $\Theta \vdash \hat{\sigma} : SC$.

Case 2. $\Gamma \vdash [\hat{\sigma}]P \geqslant Q$ is derived by Rule (Var$^{+\geqslant}$)

Then $P = [\hat{\sigma}]P = \alpha^+ = Q$, where the first equality holds because P is not a unification variable: it has been covered by case 1; and the second equality hold because Rule (Var$^{+\geqslant}$) was applied.

The algorithm applies Rule (Var$^{+\geqslant}$) and infers $SC = \cdot$, i.e. $\Gamma;\, \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot$. Then $\Theta \vdash \hat{\sigma} : \cdot$ holds trivially.

Case 3. $\Gamma \vdash [\hat{\sigma}]P \geqslant Q$ is derived by Rule ($\downarrow^{\geqslant}$),

Then $P = \downarrow N$, since the substitution $[\hat{\sigma}]P$ must preserve the top-level constructor of $P \neq \hat{\alpha}^+$ (the case $P = \hat{\alpha}^+$ has been covered by case 1), and $Q = \downarrow M$, and by inversion, $\Gamma \vdash [\hat{\sigma}]N \simeq^{\leqslant} M$.

Since both types start with $\downarrow$, the algorithm tries to apply Rule ($\downarrow^{\geqslant}$): $\Gamma;\, \Theta \models \downarrow N \geqslant \downarrow M \dashv SC$. The premise of this rule is the unification of $\mathbf{nf}(N)$ and $\mathbf{nf}(M)$: $\Gamma; \Theta \models \mathbf{nf}(N) \overset{u}{\simeq} \mathbf{nf}(M) \dashv UC$. And the algorithm returns it as a subtyping constraint $SC = UC$.

To demonstrate that the unification terminates ant $\hat{\sigma}$ satisfies the resulting constraints, we apply the completeness of the unification algorithm (lemma 63). In order to do that, we need to provide a substitution unifying $\mathbf{nf}(N)$ and $\mathbf{nf}(M)$. Let us show that $\mathbf{nf}(\hat{\sigma})$ is such a substitution.

- $\mathbf{nf}(N)$ and $\mathbf{nf}(M)$ are normalized
- $\Gamma; \mathbf{dom}(\Theta) \vdash \mathbf{nf}(N)$ because $\Gamma; \mathbf{dom}(\Theta) \vdash N$ (corollary 24)

- $\Gamma \vdash \mathbf{nf}(M)$ because $\Gamma \vdash M$ (corollary 16)
- $\Theta \vdash \mathbf{nf}(\widehat{\sigma}) : \mathbf{uv}(P)$ because $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(P)$ (corollary 25)
- $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M \Rightarrow [\widehat{\sigma}]N \simeq^{D} M$ by lemma 45
  $\Rightarrow \mathbf{nf}([\widehat{\sigma}]N) = \mathbf{nf}(M)$ by lemma 33
  $\Rightarrow [\mathbf{nf}(\widehat{\sigma})]\mathbf{nf}(N) = \mathbf{nf}(M)$ by lemma 32

By the completeness of the unification, $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$ exists, and $\Theta \vdash \mathbf{nf}(\widehat{\sigma}) : UC$, and by corollary 27, $\Theta \vdash \widehat{\sigma} : UC$.

Case 4. $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$ is derived by Rule $(\exists^{\geqslant})$.

We should only consider the case when the substitution $[\widehat{\sigma}]P$ results in the existential type $\exists\overrightarrow{\alpha^-}.P''$ (for $P'' \neq \exists \ldots$) by congruence, i.e. $P = \exists\overrightarrow{\alpha^-}.P'$ (for $P' \neq \exists \ldots$) and $[\widehat{\sigma}]P' = P''$. This is because the case when $P = \exists\overrightarrow{\beta^-}.\widehat{\alpha}^+$ has been covered (case 1), and thus, the substitution $\widehat{\sigma}$ must preserve all the outer quantifiers of P and does not generate any new ones.

This way, $P = \exists\overrightarrow{\alpha^-}.P'$, $[\widehat{\sigma}]P = \exists\overrightarrow{\alpha^-}.[\widehat{\sigma}]P'$ (assuming $\overrightarrow{\alpha^-}$ does not intersect with the range of $\widehat{\sigma}$) and $Q = \exists\overrightarrow{\beta^-}.Q'$, where either $\overrightarrow{\alpha^-}$ or $\overrightarrow{\beta^-}$ is not empty.

By inversion, $\Gamma \vdash [\sigma][\widehat{\sigma}]P' \geqslant Q'$ for some $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$. Since σ and $\widehat{\sigma}$ have disjoint domains, and the range of one does not intersect with the domain of the other, they commute, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}][\sigma]P' \geqslant Q'$ (notice that the tree inferring this judgement is a proper subtree of the tree inferring $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$).

At the next step, the algorithm creates fresh (disjoint with $\mathbf{uv}\, P'$) unification variables $\overrightarrow{\widehat{\alpha}^-}$, replaces $\overrightarrow{\alpha^-}$ with them in P', and makes the recursive call: $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \models P_0 \geqslant Q' \dashv SC_1$, (where $P_0 = [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$), returning $SC_1 \backslash \overrightarrow{\widehat{\alpha}^-}$ as the result.

To show that the recursive call terminates and that $\Theta \vdash \widehat{\sigma} : SC_1 \backslash \overrightarrow{\widehat{\alpha}^-}$, it suffices to build $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}_0 : \mathbf{uv}(P_0)$—an extension of $\widehat{\sigma}$ with $\overrightarrow{\widehat{\alpha}^-} \cap \mathbf{uv}(P_0)$ such that $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant Q$. Then by the induction hypothesis, $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}_0 : SC_1$, and hence, $\Theta \vdash \widehat{\sigma} : SC_1 \backslash \overrightarrow{\widehat{\alpha}^-}$, as required.

Let us construct such a substitution $\widehat{\sigma}_0$:

$$[\widehat{\sigma}_0]\widehat{\beta}^{\pm} = \begin{cases} [\sigma]\alpha_i^- & \text{if } \widehat{\beta}^{\pm} = \widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-} \cap \mathbf{uv}(P_0) \\ [\widehat{\sigma}]\widehat{\beta}^{\pm} & \text{if } \widehat{\beta}^{\pm} \in \mathbf{uv}(P') \end{cases}$$

It is easy to see $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}_0 : \mathbf{uv}(P_0)$: $\mathbf{uv}(P_0) = \mathbf{uv}([\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P') = \overrightarrow{\widehat{\alpha}^-} \cap \mathbf{uv}(P_0) \cup \mathbf{uv}(P')$. Then

1. for $\widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-} \cap \mathbf{uv}(P_0)$, $(\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\})(\widehat{\alpha_i}^-) \vdash [\widehat{\sigma}_0]\widehat{\alpha_i}^-$, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]\alpha_i^-$ holds since $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$,
2. for $\widehat{\beta}^{\pm} \in \mathbf{uv}(P') \subseteq \mathbf{dom}(\Theta)$, $(\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\})(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}_0]\widehat{\beta}^{\pm}$, i.e. $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\beta}^{\pm}$ holds since $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(P)$ and $\widehat{\beta}^{\pm} \in \mathbf{uv}(P') = \mathbf{uv}(P)$.

Now, let us show that $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant Q$. To do that, we notice that $[\widehat{\sigma}_0]P_0 = [\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P_0$: let us consider an arbitrary variable appearing freely in P_0:

1. if this variable is a algorithmic variable $\widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-}$, then $[\widehat{\sigma}_0]\widehat{\alpha_i}^- = [\sigma]\alpha_i^-$ and $[\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]\widehat{\alpha_i}^- = [\widehat{\sigma}][\sigma]\alpha_i^- = [\sigma]\alpha_i^-$,
2. if this variable is a algorithmic variable $\widehat{\beta}^{\pm} \in \mathbf{uv}(P_0) \backslash \overrightarrow{\widehat{\alpha}^-} = \mathbf{uv}(P')$, then $[\widehat{\sigma}_0]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$ and $[\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]\widehat{\beta}^{\pm} = [\widehat{\sigma}][\sigma]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$,
3. if this variable is a regular variable from $\mathbf{fv}(P_0)$, both substitutions do not change it: $\widehat{\sigma}_0$, $\widehat{\sigma}$ and $\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}$ act on algorithmic variables, and σ is defined on $\overrightarrow{\alpha^-}$, however, $\overrightarrow{\alpha^-} \cap \mathbf{fv}(P_0) = \varnothing$.

This way, $[\widehat{\sigma}_0]P_0 = [\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P_0 = [\widehat{\sigma}][\sigma]P'$, and thus, $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant Q'$.

□

6.12 Subtyping Constraint Merge

Lemma 78 (Soundness of Constraint Entry Merge)**.** *For a fixed context* Γ*, suppose that* $\Gamma \vdash e_1$ *and* $\Gamma \vdash e_2$*. If* $\Gamma \vdash e_1 \,\&\, e_2 = e$ *is defined then*

1. $\Gamma \vdash e$
2. *For any* $\Gamma \vdash P$*,* $\Gamma \vdash P : e$ *implies* $\Gamma \vdash P : e_1$ *and* $\Gamma \vdash P : e_2$

Proof. Let us consider the rule forming $\Gamma \vdash e_1 \,\&\, e_2 = e$.

Case 1. Rule ($\simeq \&^+ \simeq$), i.e. $\Gamma \vdash e_1 \mathbin{\&} e_2 = e$ has form $\Gamma \vdash (\widehat{\alpha}^+ :\approx Q) \mathbin{\&} (\widehat{\alpha}^+ :\approx Q') = (\widehat{\alpha}^+ :\approx Q)$ and $\mathbf{nf}\,(Q) = \mathbf{nf}\,(Q')$. The latter implies $\Gamma \vdash Q \simeq^{\leqslant} Q'$ by lemma 46. Then

1. $\Gamma \vdash e$, i.e. $\Gamma \vdash \widehat{\alpha}^+ :\approx Q$ holds by assumption;
2. by inversion, $\Gamma \vdash P : (\widehat{\alpha}^+ :\approx Q)$ means $\Gamma \vdash P \simeq^{\leqslant} Q$, and by transitivity of equivalence (corollary 10), $\Gamma \vdash P \simeq^{\leqslant} Q'$. Thus, $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$ hold by Rule SATSCEPEq.

Case 2. Rule ($\simeq \&^- \simeq$) the negative case is proved in exactly the same way as the positive one.

Case 3. Rule ($\geqslant \&^+ \geqslant$) Then e_1 is $\widehat{\alpha}^+ :\geqslant Q_1$, e_2 is $\widehat{\alpha}^+ :\geqslant Q_2$, and $e_1 \mathbin{\&} e_2 = e$ is $\widehat{\alpha}^+ :\geqslant Q$ where Q is the least upper bound of Q_1 and Q_2. Then by lemma 70,

- $\Gamma \vdash Q$,
- $\Gamma \vdash Q \geqslant Q_1$,
- $\Gamma \vdash Q \geqslant Q_2$.

Let us show the required properties.

- $\Gamma \vdash e$ holds from $\Gamma \vdash Q$,
- Assuming $\Gamma \vdash P : e$, by inversion, we have $\Gamma \vdash P \geqslant Q$. Combining it transitively with $\Gamma \vdash Q \geqslant Q_1$, we have $\Gamma \vdash P \geqslant Q_1$. Analogously, $\Gamma \vdash P \geqslant Q_2$. Then $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$ hold by Rule SATSCESup.

Case 4. Rule ($\geqslant \&^+ \simeq$) Then e_1 is $\widehat{\alpha}^+ :\geqslant Q_1$, e_2 is $\widehat{\alpha}^+ :\approx Q_2$, where $\Gamma;\cdot \models Q_2 \geqslant Q_1 \dashv \cdot$, and the resulting $e_1 \mathbin{\&} e_2 = e$ is equal to e_2, that is $\widehat{\alpha}^+ :\approx Q_2$.

Let us show the required properties.

- By assumption, $\Gamma \vdash Q$, and hence $\Gamma \vdash e$.
- Since $\mathbf{uv}\,(Q_2) = \varnothing$, $\Gamma;\cdot \models Q_2 \geqslant Q_1 \dashv \cdot$ implies $\Gamma \vdash Q_2 \geqslant Q_1$ by the soundness of positive subtyping (lemma 76). Then let us take an arbitrary $\Gamma \vdash P$ such that $\Gamma \vdash P : e$. Since $e_2 = e$, $\Gamma \vdash P : e_2$ holds immediately.
  By inversion, $\Gamma \vdash P : (\widehat{\alpha}^+ :\approx Q_2)$ means $\Gamma \vdash P \simeq^{\leqslant} Q_2$, and then by transitivity of subtyping (lemma 24), $\Gamma \vdash P \geqslant Q_1$. Then $\Gamma \vdash P : e_1$ holds by Rule SATSCESup.

Case 5. Rule ($\simeq \&^+ \geqslant$) Thee proof is analogous to the previous case.

□

Lemma 79 (Soundness of Constraint Merge)**.** *Suppose that $\Theta \vdash SC_1 : \Xi_1$ and $\Theta \vdash SC_2 : \Xi_2$ and $\Theta \vdash SC_1 \& SC_2 = SC$ is defined. Then*

1. *$\Theta \vdash SC : \Xi_1 \cup \Xi_2$,*
2. *for any substitution $\Theta \vdash \widehat{\sigma} : \Xi_1 \cup \Xi_2$, $\Theta \vdash \widehat{\sigma} : SC$ implies $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$.*

Proof. By definition, $\Theta \vdash SC_1 \& SC_2 = SC$ consists of three parts: entries of SC_1 that do not have matching entries of SC_2, entries of SC_2 that do not have matching entries of SC_1, and the merge of matching entries.

Notice that $\widehat{\alpha}^{\pm} \in \Xi_1 \backslash \Xi_2$ if and only if there is an entry e in SC_1 restricting $\widehat{\alpha}^{\pm}$, but there is no such entry in SC_2. Therefore, for any $\widehat{\alpha}^{\pm} \in \Xi_1 \backslash \Xi_2$, there is an entry e in SC restricting $\widehat{\alpha}^{\pm}$. Notice that $\Theta(\widehat{\alpha}^{\pm}) \vdash e$ holds since $\Theta \vdash SC_1 : \Xi_1$.

Analogously, for any $\widehat{\beta}^{\pm} \in \Xi_2 \backslash \Xi_1$, there is an entry e in SC restricting $\widehat{\beta}^{\pm}$. Notice that $\Theta(\widehat{\beta}^{\pm}) \vdash e$ holds since $\Theta \vdash SC_2 : \Xi_2$.

Finally, for any $\widehat{\gamma}^{\pm} \in \Xi_1 \cap \Xi_2$, there is an entry e_1 in SC_1 restricting $\widehat{\gamma}^{\pm}$ and an entry e_2 in SC_2 restricting $\widehat{\gamma}^{\pm}$. Since $\Theta \vdash SC_1 \& SC_2 = SC$ is defined, $\Theta(\widehat{\gamma}^{\pm}) \vdash e_1 \mathbin{\&} e_2 = e$ restricting $\widehat{\gamma}^{\pm}$ is defined and belongs to SC, moreover, $\Theta(\widehat{\gamma}^{\pm}) \vdash e$ by lemma 78. This way, $\Theta \vdash SC : \Xi_1 \cup \Xi_2$.

Let us show the second property. We take an arbitrary $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : \Xi_1 \cup \Xi_2$ and $\Theta \vdash \widehat{\sigma} : SC$. To prove $\Theta \vdash \widehat{\sigma} : SC_1$, we need to show that for any $e_1 \in SC_1$, restricting $\widehat{\alpha}^{\pm}$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$ holds.

Let us assume that $\widehat{\alpha}^{\pm} \notin \mathbf{dom}\,(SC_2)$. It means that $SC \ni e_1$, and then since $\Theta \vdash \widehat{\sigma} : SC$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$.

Otherwise, SC_2 contains an entry e_2 restricting $\widehat{\alpha}^{\pm}$, and $SC \ni e$ where $\Theta(\widehat{\alpha}^{\pm}) \vdash e_1 \mathbin{\&} e_2 = e$. Then since $\Theta \vdash \widehat{\sigma} : SC$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$, and by lemma 78, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$.

The proof of $\Theta \vdash \widehat{\sigma} : SC_2$ is symmetric. □

Lemma 80 (Completeness of Constraint Entry Merge)**.** *For a fixed context Γ, suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$ are matching constraint entries.*

- *for a type P such that $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$, $\Gamma \vdash e_1 \mathbin{\&} e_2 = e$ is defined and $\Gamma \vdash P : e$.*
- *for a type N such that $\Gamma \vdash N : e_1$ and $\Gamma \vdash N : e_2$, $\Gamma \vdash e_1 \mathbin{\&} e_2 = e$ is defined and $\Gamma \vdash N : e$.*

Proof. Let us consider the shape of e_1 and e_2.

Case 1. e_1 is $\widehat{\alpha}^+ :\approx Q_1$ and e_2 is $\widehat{\alpha}^+ :\approx Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \simeq^\leqslant Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \simeq^\leqslant Q_2$. Then by transitivity of equivalence (corollary 10), $\Gamma \vdash Q_1 \simeq^\leqslant Q_2$, which means $\mathbf{nf}(Q_1) = \mathbf{nf}(Q_2)$ by lemma 46. Hence, Rule ($\simeq \&^+ \simeq$) applies to infer $\Gamma \vdash e_1 \mathbin{\&} e_2 = e_2$, and $\Gamma \vdash P : e_2$ holds by assumption.

Case 2. e_1 is $\widehat{\alpha}^+ :\approx Q_1$ and e_2 is $\widehat{\alpha}^+ :\geqslant Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \simeq^\leqslant Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \geqslant Q_2$. Then by transitivity of subtyping, $\Gamma \vdash Q_1 \geqslant Q_2$, which means $\Gamma; \cdot \vDash Q_1 \geqslant Q_2 \dashv \cdot$ by lemma 77. This way, Rule ($\simeq \&^+ \geqslant$) applies to infer $\Gamma \vdash e_1 \mathbin{\&} e_2 = e_1$, and $\Gamma \vdash P : e_1$ holds by assumption.

Case 3. e_1 is $\widehat{\alpha}^+ :\geqslant Q_1$ and e_2 is $\widehat{\alpha}^+ :\geqslant Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \geqslant Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \geqslant Q_2$. By the completeness of the least upper bound (lemma 71), $\Gamma \vDash Q_1 \vee Q_2 = Q$, and $\Gamma \vdash P \geqslant Q$. This way, Rule ($\geqslant \&^+ \geqslant$) applies to infer $\Gamma \vdash e_1 \mathbin{\&} e_2 = (\widehat{\alpha}^+ :\geqslant Q)$, and $\Gamma \vdash P : (\widehat{\alpha}^+ :\geqslant Q)$ holds by Rule SATSCESup.

Case 4. The negative cases are proved symmetrically.

□

Lemma 81 (Completeness of Constraint Merge)**.** *Suppose that $\Theta \vdash SC_1 : \Xi_1$ and $\Theta \vdash SC_2 : \Xi_2$. If there exists a substitution $\Theta \vdash \widehat{\sigma} : \Xi_1 \cup \Xi_2$ such that $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$ then $\Theta \vdash SC_1 \& SC_2 = SC$ is defined.*

Proof. By definition, $SC_1 \mathbin{\&} SC_2$ is a union of

1. entries of SC_1, which do not have matching entries in SC_2,
2. entries of SC_2, which do not have matching entries in SC_1, and
3. the merge of matching entries.

This way, to show that $\Theta \vdash SC_1 \& SC_2 = SC$ is defined, we need to demonstrate that each of these components is defined and satisfies the required property (that the result of $\widehat{\sigma}$ satisfies the corresponding constraint entry).

It is clear that the first two components of this union exist. Moreover, if e is an entry of SC_i restricting $\widehat{\alpha}^\pm \notin \mathbf{dom}(SC_2)$, then $\Theta \vdash \widehat{\sigma} : SC_i$ implies $\Theta(\widehat{\alpha}^\pm) \vdash [\widehat{\sigma}]\widehat{\alpha}^\pm : e$,

Let us show that the third component exists. Let us take two entries $e_1 \in SC_1$ and $e_2 \in SC_2$ restricting the same variable $\widehat{\alpha}^\pm$. $\Theta \vdash \widehat{\sigma} : SC_1$ means that $\Theta(\widehat{\alpha}^\pm) \vdash [\widehat{\sigma}]\widehat{\alpha}^\pm : e_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$ means $\Theta(\widehat{\alpha}^\pm) \vdash [\widehat{\sigma}]\widehat{\alpha}^\pm : e_2$. Then by lemma 80, $\Theta(\widehat{\alpha}^\pm) \vdash e_1 \mathbin{\&} e_2 = e$ is defined and $\Theta(\widehat{\alpha}^\pm) \vdash [\widehat{\sigma}]\widehat{\alpha}^\pm : e$.

□

Corollary 29 (Completeness of Unification Constraint Merge)**.** *Suppose that $\Theta \vdash UC_1 : \Xi_1$ and $\Theta \vdash UC_2 : \Xi_2$. If there exists a substitution $\Theta \vdash \widehat{\sigma} : \Xi_1 \cup \Xi_2$ such that $\Theta \vdash \widehat{\sigma} : UC_1$ and $\Theta \vdash \widehat{\sigma} : UC_2$ then $\Theta \vdash UC_1 \& UC_2 = UC$ is defined.*

6.13 Negative Subtyping

Lemma 82 (Soundness of Negative Subtyping)**.** *If $\Gamma \vdash^\supseteq \Theta$, $\Gamma \vdash M$, $\Gamma; \mathbf{dom}(\Theta) \vdash N$ and $\Gamma; \Theta \vDash N \leqslant M \dashv SC$, then $\Theta \vdash SC : \mathbf{uv}(N)$ and for any normalized $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : SC$, $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$.*

Proof. We prove it by induction on $\Gamma; \Theta \vDash N \leqslant M \dashv SC$.

Suppose that $\widehat{\sigma}$ is normalized and $\Theta \vdash \widehat{\sigma} : SC$, Let us consider the last rule to infer this judgment.

Case 1. Rule ($\to^\leqslant$). Then $\Gamma; \Theta \vDash N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \vDash P \to N' \leqslant Q \to M' \dashv SC$
On the next step, the the algorithm makes two recursive calls: $\Gamma; \Theta \vDash P \geqslant Q \dashv SC_1$ and $\Gamma; \Theta \vDash N' \leqslant M' \dashv SC_2$ and returns $\Theta \vdash SC_1 \& SC_2 = SC$ as the result.

By the soundness of constraint merge (lemma 79), $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$. Then by the soundness of positive subtyping (lemma 76), $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$; and by the induction hypothesis, $\Gamma \vdash [\widehat{\sigma}]N' \leqslant M'$. This way, by Rule ($\to^\leqslant$), $\Gamma \vdash [\widehat{\sigma}](P \to N') \leqslant Q \to M'$.

Case 2. Rule ($\text{Var}^{-\leqslant}$), and then $\Gamma; \Theta \vDash N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \vDash \alpha^- \leqslant \alpha^- \dashv \cdot$
This case is symmetric to case 2 of lemma 76.

Case 3. Rule ($\uparrow^\leqslant$), and then $\Gamma; \Theta \vDash N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \vDash \uparrow P \leqslant \uparrow Q \dashv SC$
This case is symmetric to case 3 of lemma 76.

Case 4. Rule ($\forall^\leqslant$), and then $\Gamma; \Theta \vDash N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \vDash \forall \overrightarrow{\alpha^+}. N' \leqslant \forall \overrightarrow{\beta^+}. M' \dashv SC$ s.t. either $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is not empty
This case is symmetric to case 4 of lemma 76.

$\square$

Lemma 83 (Completeness of the Negative Subtyping). *Suppose that $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash M$, $\Gamma; \mathbf{dom}(\Theta) \vdash N$, and N does not contain negative unification variables ($\widehat{\alpha}^- \notin \mathbf{uv}\, N$). Then for any $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(N)$ such that $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$, there exists $\Gamma; \Theta \models N \leqslant M \dashv SC$ and moreover, $\Theta \vdash \widehat{\sigma} : SC$.*

Proof. We prove it by induction on $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$. Let us consider the last rule used in the derivation of $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$.

Case 1. $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$ is derived by Rule $(\uparrow^{\leqslant})$
Then $N = \uparrow P$, since the substitution $[\widehat{\sigma}]N$ must preserve the top-level constructor of $N \neq \widehat{\alpha}^-$ (since by assumption, $\widehat{\alpha}^- \notin \mathbf{uv}\, N$), and $Q = \downarrow M$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M$. The rest of the proof is symmetric to case 3 of lemma 77: notice that the algorithm does not make a recursive call, and the difference in the induction statement for the positive and the negative case here does not matter.

Case 2. $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$ is derived by Rule $(\rightarrow^{\leqslant})$, i.e. $[\widehat{\sigma}]N = [\widehat{\sigma}]P \rightarrow [\widehat{\sigma}]N'$ and $M = Q \rightarrow M'$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$ and $\Gamma \vdash [\widehat{\sigma}]N' \leqslant M'$.

The algorithm makes two recursive calls: $\Gamma; \Theta \models P \geqslant Q \dashv SC_1$ and $\Gamma; \Theta \models N' \leqslant M' \dashv SC_2$, and then returns $\Theta \vdash SC_1 \& SC_2 = SC$ as the result. Let us show that these recursive calls are successful and the returning constraints are fulfilled by $\widehat{\sigma}$.

Notice that from the inversion of $\Gamma \vdash M$, we have: $\Gamma \vdash Q$ and $\Gamma \vdash M'$; from the inversion of $\Gamma; \mathbf{dom}(\Theta) \vdash N$, we have: $\Gamma; \mathbf{dom}(\Theta) \vdash P$ and $\Gamma; \mathbf{dom}(\Theta) \vdash N'$; and since N does not contain negative unification variables, N' does not contain negative unification variables either.

This way, we can apply the induction hypothesis to $\Gamma \vdash [\widehat{\sigma}]N' \leqslant M'$ to obtain $\Gamma; \Theta \models N' \leqslant M' \dashv SC_2$ such that $\Theta \vdash SC_2 : \mathbf{uv}(N')$ and $\Theta \vdash \widehat{\sigma} : SC_2$. Also, we can apply the completeness of the positive subtyping (lemma 77) to $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$ to obtain $\Gamma; \Theta \models P \geqslant Q \dashv SC_1$ such that $\Theta \vdash SC_1 : \mathbf{uv}(P)$ and $\Theta \vdash \widehat{\sigma} : SC_1$.

Finally, we need to show that the merge of SC_1 and SC_2 is successful and satisfies the required properties. To do so, we apply the completeness of subtyping constraint merge (lemma 81) (notice that $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(P \rightarrow N')$ means $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(P) \cup \mathbf{uv}(N')$). This way, $\Theta \vdash SC_1 \& SC_2 = SC$ is defined and $\Theta \vdash \widehat{\sigma} : SC$ holds.

Case 3. $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$ is derived by Rule $(\forall^{\leqslant})$. Since N does not contain negative unification variables, N must be of the form $\forall \overrightarrow{\alpha^+}.N'$, such that $[\widehat{\sigma}]N = \forall \overrightarrow{\alpha^+}.[\widehat{\sigma}]N'$ and $[\widehat{\sigma}]N' \neq \forall \ldots$ (assuming $\overrightarrow{\alpha^+}$ does not intersect with the range of $\widehat{\sigma}$). Also, $M = \forall \overrightarrow{\beta^+}.M'$ and either $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is non-empty.

The rest of the proof is symmetric to case 4 of lemma 77. To apply the induction hypothesis, we need to show additionally that there are no negative unification variables in $N_0 = [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N'$. This is because $\mathbf{uv}\, N_0 \subseteq \mathbf{uv}\, N \cup \overrightarrow{\widehat{\alpha}^+}$, and N is free of negative unification variables by assumption.

Case 4. $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$ is derived by Rule $(\mathrm{Var}^{-\leqslant})$.
Then $N = [\widehat{\sigma}]N = \alpha^- = M$. Here the first equality holds because N is not a unification variable: by assumption, N is free of negative unification variables. The second and the third equations hold because Rule $(\mathrm{Var}^{-\leqslant})$ was applied.

The rest of the proof is symmetric to case 2 of lemma 77.

$\square$

7 Properties of the Declarative Typing

Lemma 84. *If $\Gamma; \Phi \vdash N_1 \bullet \overrightarrow{v} \Rightarrow\!\!\!\!> M$ and $\Gamma \vdash N_1 \simeq^{\leqslant} N_2$ then $\Gamma; \Phi \vdash N_2 \bullet \overrightarrow{v} \Rightarrow\!\!\!\!> M$.*

Proof. By lemma 45, $\Gamma \vdash N_1 \simeq^{\leqslant} N_2$ implies $N_1 \simeq^{D} N_2$. Let us prove the required judgement by induction on $N_1 \simeq^{D} N_2$. Let us consider the last rule used in the derivation.

Case 1. Rule $(\mathrm{Var}^{-\simeq^{D}})$. It means that N_1 is α^- and N_2 is α^-. Then the required property coincides with the assumption.

Case 2. Rule $(\uparrow^{\simeq^{D}})$. It means that N_1 is $\uparrow P_1$ and N_2 is $\uparrow P_2$. where $P_1 \simeq^{D} P_2$.

Then the only rule applicable to infer $\Gamma; \Phi \vdash \uparrow P_1 \bullet \overrightarrow{v} \Rightarrow\!\!\!\!> M$ is Rule DTEmptyApp, meaning that $\overrightarrow{v} = \cdot$ and $\Gamma \vdash \uparrow P_1 \simeq^{\leqslant} M$. Then by transitivity of equivalence corollary 10, $\Gamma \vdash \uparrow P_2 \simeq^{\leqslant} M$, and then Rule DTEmptyApp is applicable to infer $\Gamma; \Phi \vdash \uparrow P_2 \bullet \cdot \Rightarrow\!\!\!\!> M$.

Case 3. Rule $(\rightarrow^{\simeq^{D}})$. Then we are proving that $\Gamma; \Phi \vdash (Q_1 \rightarrow N_1) \bullet v, \overrightarrow{v} \Rightarrow\!\!\!\!> M$ and $Q_1 \rightarrow N_1 \simeq^{D} Q_2 \rightarrow N_2$ imply $\Gamma; \Phi \vdash (Q_2 \rightarrow N_2) \bullet v, \overrightarrow{v} \Rightarrow\!\!\!\!> M$.

By inversion, $(Q_1 \rightarrow N_1) \simeq^{D} (Q_2 \rightarrow N_2)$ means $Q_1 \simeq^{D} Q_2$ and $N_1 \simeq^{D} N_2$.

By inversion of $\Gamma; \Phi \vdash (Q_1 \rightarrow N_1) \bullet v, \overrightarrow{v} \Rightarrow\!\!\!\!> M$:

1. $\Gamma; \Phi \vdash v : P$
2. $\Gamma \vdash Q_1 \geqslant P$, and then by transitivity **??**, $\Gamma \vdash Q_2 \geqslant P$;
3. $\Gamma; \Phi \vdash N_1 \bullet \vec{v} \Longrightarrow M$, and then by induction hypothesis, $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Longrightarrow M$.

Since we have $\Gamma; \Phi \vdash v : P$, $\Gamma \vdash Q_2 \geqslant P$ and $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Longrightarrow M$, we can apply Rule DTArrowApp to infer $\Gamma; \Phi \vdash (Q_2 \to N_2) \bullet v, \vec{v} \Longrightarrow M$.

Case 4. Rule ($\forall^{\simeq^D}$) Then we are proving that $\Gamma; \Phi \vdash \forall \overrightarrow{\alpha^+}_1 . N_1' \bullet \vec{v} \Longrightarrow M$ and $\forall \overrightarrow{\alpha^+}_1 . N_1' \simeq^D \forall \overrightarrow{\alpha^+}_2 . N_2'$ imply $\Gamma; \Phi \vdash \forall \overrightarrow{\alpha^+}_2 . N_2' \bullet \vec{v} \Rightarrow > M$.

By inversion of $\forall \overrightarrow{\alpha^+}_1 . N_1' \simeq^D \forall \overrightarrow{\alpha^+}_2 . N_2'$:

1. $\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, N_1 = \varnothing$,
2. there exists a bijection $\mu : (\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, N_2') \leftrightarrow (\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\, N_1')$ such that $N_1' \simeq^D [\mu] N_2'$.

By inversion of $\Gamma; \Phi \vdash \forall \overrightarrow{\alpha^+}_1 . N_1' \bullet \vec{v} \Longrightarrow M$:

1. $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}_1$
2. $\Gamma; \Phi \vdash [\sigma] N_1' \bullet \vec{v} \Longrightarrow M$
3. $\vec{v} \neq \cdot$

Let us construct $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^+}_2$ in the following way:

$$\begin{cases} [\sigma_0]\alpha^+ = [\sigma][\mu]\alpha^+ & \text{if } \alpha^+ \in \overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, N_2' \\ [\sigma_0]\alpha^+ = \exists \beta^- . {\downarrow} \beta^- & \text{otherwise (the type does not matter here)} \end{cases}$$

Then to infer $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Longrightarrow M$, we apply Rule DTArrowApp with σ_0. Let us show the required premises:

1. $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^+}_2$ by construction;
2. $\vec{v} \neq \cdot$ as noted above;
3. To show $\Gamma; \Phi \vdash [\sigma_0] N_2' \bullet \vec{v} \Longrightarrow M$, Notice that $[\sigma_0] N_2' = [\sigma][\mu] N_2'$ and since $[\mu] N_2' \simeq^D N_1'$, $[\sigma][\mu] N_2' \simeq^D [\sigma] N_1'$. This way, by lemma 40, $\Gamma \vdash [\sigma] N_1' \simeq^{\leqslant} [\sigma_0] N_2'$. Then the required judgement holds by the induction hypothesis applied to $\Gamma; \Phi \vdash [\sigma] N_1' \bullet \vec{v} \Longrightarrow M$.

□

Definition 26 (Number of prenex quantifiers). *Let us define* $\mathsf{npq}(N)$ *and* $\mathsf{npq}(P)$ *as the number of prenex quantifiers in these types, i.e.*

+ $\mathsf{npq}(\exists \overrightarrow{\alpha^-} . P) = |nas|$, *if* $P \neq \exists \overrightarrow{\beta^-} . P'$,
− $\mathsf{npq}(\forall \overrightarrow{\alpha^+} . N) = |pas|$, *if* $N \neq \forall \overrightarrow{\beta^+} . N'$.

Definition 27 (Size of a Declarative Judgement). *For a declarative typing judgement J let us define a metrics* $\mathsf{size}(J)$ *as a pair of numbers in the following way:*

+ $\mathsf{size}(\Gamma; \Phi \vdash v : P) = (\mathsf{size}(v), 0)$;
− $\mathsf{size}(\Gamma; \Phi \vdash c : N) = (\mathsf{size}(c), 0)$;
• $\mathsf{size}(\Gamma; \Phi \vdash N \bullet \vec{v} \Longrightarrow M) = (\mathsf{size}(\vec{v}), \mathsf{npq}(N)))$

where $\mathsf{size}(v)$ *or* $\mathsf{size}(c)$ *is the size of the syntax tree of the term v or c and* $\mathsf{size}(\vec{v})$ *is the sum of sizes of the terms in* $\vec{v}$.

Definition 28 (Number of Equivalence Nodes). *For a tree T inferring a declarative typing judgement, let us a function* $\mathsf{eq\_nodes}(T)$ *as the number of nodes in T labeled with Rule DTPEquiv or Rule DTNEquiv.*

Definition 29 (Metric). *For a tree T inferring a declarative typing judgement J, let us define a metric* $\mathsf{metric}(T)$ *as a pair* $(\mathsf{size}(J), \mathsf{eq\_nodes}(T))$.

Lemma 85 (Declarative typing is preserved under context equivalence). *Assuming $\Gamma \vdash \Phi_1$, $\Gamma \vdash \Phi_2$, and $\Gamma \vdash \Phi_1 \simeq^{\leqslant} \Phi_2$:*

+ *for any tree T_1 inferring $\Gamma; \Phi_1 \vdash v : P$, there exists a tree T_2 inferring $\Gamma; \Phi_2 \vdash v : P$.*
− *for any tree T_1 inferring $\Gamma; \Phi_1 \vdash c : N$, there exists a tree T_2 inferring $\Gamma; \Phi_2 \vdash c : N$.*

- *for any tree T_1 inferring $\Gamma; \Phi_1 \vdash N \bullet \vec{v} \Rightarrow\!\!\!> M$, there exists a tree T_2 inferring $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rightarrow\!\!\!> M$.*

Proof. Let us prove it by induction on the $\mathsf{metric}(T_1)$. Let us consider the last rule applied in T_1 (i.e., its root node).

Case 1. Rule DTVar
Then we are proving that $\Gamma; \Phi_1 \vdash x\colon P$ implies $\Gamma; \Phi_2 \vdash x\colon P$. By inversion, $x : P \in \Phi_1$, and since $\Gamma \vdash \Phi_1 \simeq^{\leqslant} \Phi_2$, $x : P' \in \Phi_2$ for some P' such that $\Gamma \vdash P \simeq^{\leqslant} P'$. Then we infer $\Gamma; \Phi_2 \vdash x\colon P'$ by Rule DTVar, and next, $\Gamma; \Phi_2 \vdash x\colon P$ by Rule DTPEquiv.

Case 2. For Rule DTThunk, Rule DTPAnnot, Rule DTTLam, Rule DTReturn, and Rule DTNAnnot the proof is analogous. We apply the induction hypothesis to the premise of the rule to substitute Φ_1 for Φ_2 in it. The induction is applicable because the metric of the premises is less than the metric of the conclusion: the term in the premise is a syntactic subterm of the term in the conclusion.

And after that, we apply the same rule to infer the required judgement.

Case 3. Rule DTPEquiv and Rule DTNEquiv In these cases, the induction hypothesis is also applicable to the premise: although the first component of the metric is the same for the premise and the conclusion: $\mathsf{size}(\Gamma; \Phi \vdash c\colon N') = \mathsf{size}(\Gamma; \Phi \vdash c\colon N) = \mathsf{size}(c)$, the second component of the metric is less for the premise by one, since the equivalence rule was applied to turn the premise tree into $T1$. Having made this note, we continue the proof in the same way as in the previous case.

Case 4. Rule DTtLam Then we are proving that $\Gamma; \Phi_1 \vdash \lambda x : P.c\colon P \to N$ implies $\Gamma; \Phi_2 \vdash \lambda x : P.c\colon P \to N$. Analogously to the previous cases, we apply the induction hypothesis to the equivalent contexts $\Gamma \vdash \Phi_1, x : P \simeq^{\leqslant} \Phi_2, x : P$ and the premise $\Gamma; \Phi_1, x : P \vdash c\colon N$ to obtain $\Gamma; \Phi_2, x : P \vdash c\colon N$. Notice that c is a subterm of $\lambda x : P.c$, i.e., the metric of the premise tree is less than the metric of the conclusion, and the induction hypothesis is applicable. Then we infer $\Gamma; \Phi_2 \vdash \lambda x : P.c\colon P \to N$ by Rule DTtLam.

Case 5. Rule DTVarLet Then we are proving that $\Gamma; \Phi_1 \vdash \textbf{let}\, x = v; c\colon N$ implies $\Gamma; \Phi_2 \vdash \textbf{let}\, x = v; c\colon N$. First, we apply the induction hypothesis to $\Gamma; \Phi_1 \vdash v\colon P$ to obtain $\Gamma; \Phi_2 \vdash v\colon P$ of the same pure size.

Then we apply the induction hypothesis to the equivalent contexts $\Gamma \vdash \Phi_1, x : P \simeq^{\leqslant} \Phi_2, x : P$ and the premise $\Gamma; \Phi_1, x : P \vdash c\colon N$ to obtain $\Gamma; \Phi_2, x : P \vdash c\colon N$. Then we infer $\Gamma; \Phi_2 \vdash \textbf{let}\, x = v; c\colon N$ by Rule DTVarLet.

Case 6. Rule DTAppLet Then we are proving that $\Gamma; \Phi_1 \vdash \textbf{let}\, x = v(\vec{v}); c\colon N$ implies $\Gamma; \Phi_2 \vdash \textbf{let}\, x = v(\vec{v}); c\colon N$.

We apply the induction hypothesis to each of the premises. to rewrite:

- $\Gamma; \Phi_1 \vdash v\colon \downarrow M$ into $\Gamma; \Phi_2 \vdash v\colon \downarrow M$,
- $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rightarrow\!\!\!> \uparrow Q$ into $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rightarrow\!\!\!> \uparrow Q$.
- $\Gamma; \Phi_1, x : Q \vdash c\colon N$ into $\Gamma; \Phi_2, x : Q \vdash c\colon N$ (notice that $\Gamma \vdash \Phi_1, x : Q \simeq^{\leqslant} \Phi_2, x : Q$).

It is left to show the uniqueness of $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rightarrow\!\!\!> \uparrow Q$. Let us assume that this judgement holds for other Q', i.e. there exists a tree T_0 inferring $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rightarrow\!\!\!> \uparrow Q'$. Then notice that the induction hypothesis is applicable to T_0: the first component of the first component of $\mathsf{metric}(T_0)$ is $S = \sum_{v \in \vec{v}} \mathsf{size}(v)$, and it is less than the corresponding component of $\mathsf{metric}(T_1)$, which is $\mathsf{size}(\textbf{let}\, x = v(\vec{v}); c) = 1 + \mathsf{size}(v) + \mathsf{size}(c) + S$. This way, $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rightarrow\!\!\!> \uparrow Q'$ holds by the induction hypothesis, but since $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rightarrow\!\!\!> \uparrow Q$ unique, we have $\Gamma \vdash Q' \simeq^{\leqslant} Q$.

Then we infer $\Gamma; \Phi_2 \vdash \textbf{let}\, x = v(\vec{v}); c\colon N$ by Rule DTAppLet.

Case 7. Rule DTAppLetAnn Then we are proving that $\Gamma; \Phi_1 \vdash \textbf{let}\, x : P = v(\vec{v}); c\colon N$ implies $\Gamma; \Phi_2 \vdash \textbf{let}\, x : P = v(\vec{v}); c\colon N$.

As in the previous case, we apply the induction hypothesis to each of the premises and rewrite:

- $\Gamma; \Phi_1 \vdash v\colon \downarrow M$ into $\Gamma; \Phi_2 \vdash v\colon \downarrow M$,
- $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rightarrow\!\!\!> \uparrow Q$ into $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rightarrow\!\!\!> \uparrow Q$, and
- $\Gamma; \Phi_1, x : P \vdash c\colon N$ into $\Gamma; \Phi_2, x : P \vdash c\colon N$ (notice that $\Gamma \vdash \Phi_1, x : P \simeq^{\leqslant} \Phi_2, x : P$).

Notice that $\Gamma \vdash P$ and $\Gamma \vdash \uparrow Q \leqslant \uparrow P$ do not depend on the variable context, and hold by assumption. Then we infer $\Gamma; \Phi_2 \vdash \textbf{let}\, x : P = v(\vec{v}); c\colon N$ by Rule DTAppLetAnn.

Case 8. Rule DTUnpack, and Rule DTNAnnot are proved in the same way.

Case 9. Rule DTEmptyApp Then we are proving that $\Gamma; \Phi_1 \vdash N \bullet \cdot \Rightarrow\!\!\!> N'$ (inferred by Rule DTEmptyApp) implies $\Gamma; \Phi_2 \vdash N \bullet \cdot \Rightarrow\!\!\!> N'$.

To infer $\Gamma; \Phi_2 \vdash N \bullet \cdot \Rightarrow\!\!\!> N'$, we apply Rule DTEmptyApp, noting that $\Gamma \vdash N \simeq^{\leqslant} N'$ holds by assumption.

Case 10. Rule DTArrowApp Then we are proving that $\Gamma;\Phi_1 \vdash Q \to N \bullet v, \overrightarrow{v} \Rrightarrow M$ (inferred by Rule DTArrowApp) implies $\Gamma;\Phi_2 \vdash Q \to N \bullet v, \overrightarrow{v} \Rrightarrow M$. And uniqueness of the M in the first case implies uniqueness in the second case.

By induction, we rewrite $\Gamma;\Phi_1 \vdash v\colon P$ into $\Gamma;\Phi_2 \vdash v\colon P$, and $\Gamma;\Phi_1 \vdash N \bullet \overrightarrow{v} \Rrightarrow M$ into $\Gamma;\Phi_2 \vdash N \bullet \overrightarrow{v} \Rrightarrow M$. Then we infer $\Gamma;\Phi_2 \vdash Q \to N \bullet v, \overrightarrow{v} \Rrightarrow M$ by Rule DTArrowApp.

Now, let us show the uniqueness. The only rule that can infer $\Gamma;\Phi_1 \vdash Q \to N \bullet v, \overrightarrow{v} \Rrightarrow M$ is Rule DTArrowApp. Then by inversion, uniqueness of $\Gamma;\Phi_1 \vdash Q \to N \bullet v, \overrightarrow{v} \Rrightarrow M$ implies uniqueness of $\Gamma;\Phi_1 \vdash N \bullet \overrightarrow{v} \Rrightarrow M$. By the induction hypothesis, it implies the uniqueness of $\Gamma;\Phi_2 \vdash N \bullet \overrightarrow{v} \Rrightarrow M$.

Suppose that $\Gamma;\Phi_2 \vdash Q \to N \bullet v, \overrightarrow{v} \Rrightarrow M'$. By inversion, $\Gamma;\Phi_2 \vdash N \bullet \overrightarrow{v} \Rrightarrow M'$, which by uniqueness of $\Gamma;\Phi_2 \vdash N \bullet \overrightarrow{v} \Rrightarrow M$ implies $\Gamma \vdash M \simeq^{\leqslant} M'$.

Case 11. Rule DTForallApp Then we are proving that $\Gamma;\Phi_1 \vdash \forall\overrightarrow{\alpha^+}.N \bullet \overrightarrow{v} \Rrightarrow M$ (inferred by Rule DTForallApp) implies $\Gamma;\Phi_2 \vdash \forall\overrightarrow{\alpha^+}.N \bullet \overrightarrow{v} \Rrightarrow M$.

By inversion, we have σ such that $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}$ and $\Gamma;\Phi_1 \vdash [\sigma]N \bullet \overrightarrow{v} \Rrightarrow M$ is inferred. Let us denote the inference tree as T_1'. Notice that the induction hypothesis is applicable to T_1': $\mathsf{metric}(T_1') = ((\mathsf{size}(\overrightarrow{v}), 0), x)$ is less than $\mathsf{metric}(T_1) = ((\mathsf{size}(\overrightarrow{v}), |\overrightarrow{\alpha^+}|), y)$ for any x and y, since $|\overrightarrow{\alpha^+}| > 0$ by inversion.

This way, by the induction hypothesis, there exists a tree T_2' inferring $\Gamma;\Phi_2 \vdash [\sigma]N \bullet \overrightarrow{v} \Rrightarrow M$. Notice that the premises $\overrightarrow{v} \neq \cdot$, $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}$, and $\overrightarrow{\alpha^+} \neq \cdot$ do not depend on the variable context, and hold by inversion. Then we infer $\Gamma;\Phi_2 \vdash \forall\overrightarrow{\alpha^+}.N \bullet \overrightarrow{v} \Rrightarrow M$ by Rule DTForallApp.

□

8 Properties of the Algorithmic Typing

8.1 Singularity

Lemma 86 (Soundness of Entry Singularity). + *Suppose* e **singular with** P *for* P *well-formed in* Γ*. Then* $\Gamma \vdash P : e$ *and for any* $\Gamma \vdash P'$ *such that* $\Gamma \vdash P' : e$*,* $\Gamma \vdash P' \simeq^{\leqslant} P$*;*

– *Suppose* e **singular with** N *for* N *well-formed in* Γ*. Then* $\Gamma \vdash N : e$ *and for any* $\Gamma \vdash N'$ *such that* $\Gamma \vdash N' : e$*,* $\Gamma \vdash N' \simeq^{\leqslant} N$*.*

Proof. Let us consider how e **singular with** P or e **singular with** N is formed.

Case 1. Rule SINGNEq, that is $e = \widehat{\alpha}^- :\approx N_0$. and N is $\mathbf{nf}\,(N_0)$. Then $\Gamma \vdash N' : e$ means $\Gamma \vdash N' \simeq^{\leqslant} N_0$, (by inversion of Rule SATSCENEq), which by transitivity, using corollary 18, means $\Gamma \vdash N' \simeq^{\leqslant} \mathbf{nf}\,(N_0)$, as required.

Case 2. Rule SINGPEq. This case is symmetric to the previous one.

Case 3. Rule SINGSupVar, that is $e = \widehat{\alpha}^+ :\geqslant \exists\overrightarrow{\alpha^-}.\beta^+$, and $P = \beta^+$.

Since $\Gamma \vdash \beta^+ \geqslant \exists\overrightarrow{\alpha^-}.\beta^+$, we have $\Gamma \vdash \beta^+ : e$, as required.

Notice that $\Gamma \vdash P' : e$ means $\Gamma \vdash P' \geqslant \exists\overrightarrow{\alpha^-}.\beta^+$. Let us show that it implies $\Gamma \vdash P' \simeq^{\leqslant} \beta^+$. By applying lemma 67 once, we have $\Gamma, \overrightarrow{\alpha^-} \vdash P' \geqslant \beta^+$. By applying it again, we notice that $\Gamma, \overrightarrow{\alpha^-} \vdash P' \geqslant \beta^+$ implies $P_i = \exists\overrightarrow{\alpha^-}'.\beta^+$. Finally, it is easy to see that $\Gamma \vdash \exists\overrightarrow{\alpha^-}'.\beta^+ \simeq^{\leqslant} \beta^+$

Case 4. Rule SINGSupShift, that is $e = \widehat{\alpha}^+ :\geqslant \exists\overrightarrow{\beta^-}.{\downarrow}N_1$, where $N_1 \simeq^{D} \beta_j^-$, and $P = \exists\alpha^-.{\downarrow}\alpha^-$.

Since $\Gamma \vdash \exists\alpha^-.{\downarrow}\alpha^- \geqslant \exists\overrightarrow{\beta^-}.{\downarrow}N_1$ (by Rule $(\exists^{\geqslant})$, with substitution N_1/α^-), we have $\Gamma \vdash \exists\alpha^-.{\downarrow}\alpha^- : e$, as required.

Notice $\Gamma \vdash P' : e$ means $\Gamma \vdash P' \geqslant \exists\overrightarrow{\beta^-}.{\downarrow}N_1$. Let us show that it implies $\Gamma \vdash P' \simeq^{\leqslant} \exists\alpha^-.{\downarrow}\alpha^-$.

$$
\begin{aligned}
[h]\Gamma \vdash P' \geqslant \exists\overrightarrow{\beta^-}.{\downarrow}N_1 &\Rightarrow \Gamma \vdash \mathbf{nf}\,(P') \geqslant \exists\overrightarrow{\beta^-}'.{\downarrow}\mathbf{nf}\,(N_1) \text{ where } \mathbf{ord}\,\overrightarrow{\beta^-}\,\mathbf{in}\,N' = \overrightarrow{\beta^-}' && \text{by corollary 19}\\
&\Rightarrow \Gamma \vdash \mathbf{nf}\,(P') \geqslant \exists\overrightarrow{\beta^-}'.{\downarrow}\mathbf{nf}\,(\beta_j^-) && \text{by lemma 33}\\
&\Rightarrow \Gamma \vdash \mathbf{nf}\,(P') \geqslant \exists\overrightarrow{\beta^-}'.{\downarrow}\beta_n^- && \text{by definition of normalization}\\
&\Rightarrow \Gamma \vdash \mathbf{nf}\,(P') \geqslant \exists\beta_j^-.{\downarrow}\beta_j^- && \text{since } \mathbf{ord}\,\overrightarrow{\beta^-}\,\mathbf{in}\,\mathbf{nf}\,(N_1) = \beta_j^-\\
&\Rightarrow \Gamma, \beta_j^- \vdash \mathbf{nf}\,(P') \geqslant {\downarrow}\beta_j^- \text{ and } \beta_j^- \notin \mathbf{fv}\,(\mathbf{nf}\,(P')) && \text{by lemma 68}
\end{aligned}
$$

By lemma 68, the last subtyping means that $\mathbf{nf}\,(P') = \exists\overrightarrow{\alpha^-}.{\downarrow}N'$, such that

1. $\Gamma, \beta_j^-, \overrightarrow{\alpha^-} \vdash N'$

2. $\mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,N' = \overrightarrow{\alpha^-}$
3. for some substitution $\Gamma, \beta_j^- \vdash \sigma : \overrightarrow{\alpha^-}$, $[\sigma]N' = \beta_j^-$.

Since $\beta_j^- \notin \mathbf{fv}\,(\mathbf{nf}\,(P'))$, the latter means that $N' = \alpha^-$, and then $\mathbf{nf}\,(P') = \exists\alpha^-.{\downarrow}\alpha^-$ for some α^-. Finally, notice that all the types of shape $\exists\alpha^-.{\downarrow}\alpha^-$ are equal.

□

Lemma 87 (Completeness of Entry Singularity)**.**

− *Suppose that there exists N well-formed in Γ such that for any N' well-formed in Γ, $\Gamma \vdash N' : e$ implies $\Gamma \vdash N' \simeq^{\leqslant} N$. Then* $e\,\mathbf{singular\ with\ nf}\,(N)$.

+ *Suppose that there exists P well-formed in Γ such that for any P' well-formed in Γ, $\Gamma \vdash P' : e$ implies $\Gamma \vdash P' \simeq^{\leqslant} P$. Then* $e\,\mathbf{singular\ with\ nf}\,(P)$.

Proof.

− By lemma 74, there exists $\Gamma \vdash N' : e$. Since N' is negative, by inversion of $\Gamma \vdash N' : e$, e has shape $\widehat{\alpha}^- :\approx M$, where $\Gamma \vdash N' \simeq^{\leqslant} M$, and transitively, $\Gamma \vdash N \simeq^{\leqslant} M$. Then $\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$, and $e\,\mathbf{singular\ with\ nf}\,(M)$ (by Rule SINGNEq) is rewritten as $e\,\mathbf{singular\ with\ nf}\,(N)$.

+ By lemma 74, there exists $\Gamma \vdash P' : e$, then by assumption, $\Gamma \vdash P' \simeq^{\leqslant} P$, which by lemma 75 implies $\Gamma \vdash P : e$.

  Let us consider the shape of e:

  Case 1. $e = (\widehat{\alpha}^+ :\approx Q)$ then inversion of $\Gamma \vdash P : e$ implies $\Gamma \vdash P \simeq^{\leqslant} Q$, and hence, $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$ (by lemma 46). Then $e\,\mathbf{singular\ with\ nf}\,(Q)$, which holds by Rule SINGPEq, is rewritten as $e\,\mathbf{singular\ with\ nf}\,(P)$.

  Case 2. $e = (\widehat{\alpha}^+ :\geqslant Q)$. Then the inversion of $\Gamma \vdash P : e$ implies $\Gamma \vdash P \geqslant Q$. Let us consider the shape of Q:

  a. $Q = \exists\overrightarrow{\beta^-}.\beta^+$ (for potentially empty $\overrightarrow{\beta^-}$). Then $\Gamma \vdash P \geqslant \exists\overrightarrow{\beta^-}.\beta^+$ implies $\Gamma \vdash P \simeq^{\leqslant} \beta^+$ by lemma 67, as was noted in the proof of lemma 86, and hence, $\mathbf{nf}\,(P) = \beta^+$.
  Then $e\,\mathbf{singular\ with}\,\beta^+$, which holds by Rule SINGSupVar, can be rewritten as $e\,\mathbf{singular\ with\ nf}\,(P)$.

  b. $Q = \exists\overrightarrow{\beta^-}.{\downarrow}N$ (for potentially empty $\overrightarrow{\beta^-}$). Notice that $\Gamma \vdash \exists\gamma^-.{\downarrow}\gamma^- \geqslant \exists\overrightarrow{\beta^-}.{\downarrow}N$ (by Rule ($\exists^{\geqslant}$), with substitution N/γ^-), and thus, $\Gamma \vdash \exists\gamma^-.{\downarrow}\gamma^- : e$ by Rule SATSCESup.
  Then by assumption, $\Gamma \vdash \exists\gamma^-.{\downarrow}\gamma^- \simeq^{\leqslant} P$, that is $\mathbf{nf}\,(P) = \exists\gamma^-.{\downarrow}\gamma^-$. To apply Rule SINGSupShift to infer $(\widehat{\alpha}^+ :\geqslant \exists\overrightarrow{\beta^-}.{\downarrow}N)\,\mathbf{singular\ with}\,\exists\gamma^-.{\downarrow}\gamma^-$, it is left to show that $N \simeq^{D} \beta_i^-$ for some i.
  Since $\Gamma \vdash Q : e$, by assumption, $\Gamma \vdash Q \simeq^{\leqslant} P$, and by transitivity, $\Gamma \vdash Q \simeq^{\leqslant} \exists\gamma^-.{\downarrow}\gamma^-$. It implies $\mathbf{nf}\,(\exists\overrightarrow{\beta^-}.{\downarrow}N) = \exists\gamma^-.{\downarrow}\gamma^-$ (by lemma 46), which by definition of normalization means $\exists\overrightarrow{\beta^-}'.{\downarrow}\mathbf{nf}\,(N) = \exists\gamma^-.{\downarrow}\gamma^-$, where $\mathbf{ord}\,\overrightarrow{\beta^-}\,\mathbf{in}\,N' = \overrightarrow{\beta^-}'$. This way, $\overrightarrow{\beta^-}'$ is a variable β^-, and $\mathbf{nf}\,(N) = \beta^-$. Notice that $\beta^- \in \overrightarrow{\beta^-}' \subseteq \overrightarrow{\beta^-}$ by lemma 25. This way, $N \simeq^{D} \beta^-$ for $\beta^- \in \overrightarrow{\beta^-}$ (by lemma 46),

□

Lemma 88 (Soundness of Singularity)**.** *Suppose $\Theta \vdash SC : \Xi$, and* $SC\,\mathbf{singular\ with}\,\widehat{\sigma}$. *Then $\Theta \vdash \widehat{\sigma} : \Xi$, $\Theta \vdash \widehat{\sigma} : SC$ and for any $\widehat{\sigma}'$ such that $\Theta \vdash \widehat{\sigma} : SC$, $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma} : \Xi$.*

Proof. Suppose that $(SC)u'$. It means that for every $e \in SC$ restricting $\widehat{\alpha}^{\pm}$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}']\widehat{\alpha}^{\pm} : e$ holds. $SC\,\mathbf{singular\ with}\,\widehat{\sigma}$ means $e\,\mathbf{singular\ with}\,[\widehat{\sigma}]\widehat{\alpha}^{\pm}$, and hence, by lemma 87, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}']\widehat{\alpha}^{\pm} \simeq^{\leqslant} [\widehat{\sigma}]\widehat{\alpha}^{\pm}$ holds.

Since the uniqueness holds for every variable from $\mathbf{dom}\,(SC)$, $\widehat{\sigma}$ is equivalent to $\widehat{\sigma}'$ on this set. □

Lemma 89 (Completeness of Singularity)**.** *For a given $\Theta \vdash SC$, suppose that all the substitutions satisfying SC are equivalent on $\Xi \supseteq$* $\mathbf{dom}\,(SC)$. *In other words, suppose that there exists $\Theta \vdash \widehat{\sigma}_1 : \Xi$ such that for any $\Theta \vdash \widehat{\sigma} : \Xi$, $\Theta \vdash \widehat{\sigma} : SC$ implies $\Theta \vdash \widehat{\sigma} \simeq^{\leqslant} \widehat{\sigma}_1 : \Xi$. Then*

- $SC\,\mathbf{singular\ with}\,\widehat{\sigma}_0$ *for some $\widehat{\sigma}_0$ and*
- $\Xi = \mathbf{dom}\,(SC)$.

Proof. First, let us assume $\Xi \neq \mathbf{dom}\,(SC)$. Then there exists $\widehat{\alpha}^{\pm} \in \Xi \backslash \mathbf{dom}\,(SC)$. Let us take $\Theta \vdash \widehat{\sigma}_1 : \Xi$ such that any other substitution $\Theta \vdash \widehat{\sigma} : \Xi$ satisfying SC is equivalent to $\widehat{\sigma}_1$ on Ξ.

Notice that $\Theta \vdash \widehat{\sigma}_1 : SC$: by lemma 74, there exists $\widehat{\sigma}'$ such that $\Theta \vdash \widehat{\sigma}' : \Xi$ and $\Theta \vdash \widehat{\sigma}' : SC$, and by assumption, $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma}_1 : \Xi$, implying $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma}_1 : \mathbf{dom}\,(SC)$.

Let us construct $\widehat{\sigma}_2$ such that $\Theta \vdash \widehat{\sigma}_2 : \Xi$ as follows:

$$\begin{cases} [\widehat{\sigma}_2]\widehat{\beta}^\pm = [\widehat{\sigma}_1]\widehat{\beta}^\pm & \text{if } \widehat{\beta}^\pm \neq \widehat{\alpha}^\pm \\ [\widehat{\sigma}_2]\widehat{\alpha}^\pm = T & \text{where } T \text{ is any closed type not equivalent to } [\widehat{\sigma}_1]\widehat{\alpha}^\pm \end{cases}$$

It is easy to see that $\Theta \vdash \widehat{\sigma}_2 : SC$ since $\widehat{\sigma}_1|_{\mathbf{dom}\,(SC)} = \widehat{\sigma}_2|_{\mathbf{dom}\,(SC)}$, and $\Theta \vdash \widehat{\sigma}_1 : SC$. However, $\Theta \vdash \widehat{\sigma}_2 \simeq^{\leqslant} \widehat{\sigma}_1 : \Xi$ does not hold because by construction, $\Theta(\widehat{\alpha}^\pm) \vdash [\widehat{\sigma}_2]\widehat{\alpha}^\pm \simeq^{\leqslant} [\widehat{\sigma}_1]\widehat{\alpha}^\pm$ does not hold. This way, we have a contradiction.

Second, let us show SC **singular with** $\widehat{\sigma}_0$. Let us take arbitrary $e \in SC$ restricting $\widehat{\beta}^\pm$. We need to show that e is singular. Notice that $\Theta \vdash \widehat{\sigma}_1 : SC$ implies $\Theta(\widehat{\beta}^\pm) \vdash [\widehat{\sigma}_1]\widehat{\beta}^\pm$ and $\Theta(\widehat{\beta}^\pm) \vdash [\widehat{\sigma}_1]\widehat{\beta}^\pm : e$. We will show that any other type satisfying e is equivalent to $[\widehat{\sigma}_1]\widehat{\beta}^\pm$, then by lemma 87, e **singular with** $[\widehat{\sigma}_1]\widehat{\beta}^\pm$.

- if $\widehat{\beta}^\pm$ is positive, let us take any type $\Theta(\widehat{\beta}^\pm) \vdash P'$ and assume $\Theta(\widehat{\beta}^\pm) \vdash P' : e$. We will show that $\Theta(\widehat{\beta}^\pm) \vdash P' \simeq^{\leqslant} [\widehat{\sigma}_1]\widehat{\beta}^\pm$, which by lemma 46 will imply e **singular with nf** $([\widehat{\sigma}_1]\widehat{\beta}^\pm)$.

  Let us construct $\widehat{\sigma}_2$ such that $\Theta \vdash \widehat{\sigma}_2 : \Xi$ as follows:

$$\begin{cases} [\widehat{\sigma}_2]\widehat{\gamma}^\pm = [\widehat{\sigma}_1]\widehat{\gamma}^\pm & \text{if } \widehat{\gamma}^\pm \neq \widehat{\beta}^\pm \\ [\widehat{\sigma}_2]\widehat{\beta}^\pm = P' \end{cases}$$

  It is easy to see that $\Theta \vdash \widehat{\sigma}_2 : SC$: for e, $\Theta(\widehat{\beta}^\pm) \vdash [\widehat{\sigma}_2]\widehat{\beta}^\pm : e$ by construction, since $\Theta(\widehat{\beta}^\pm) \vdash P' : e$; for any other $e' \in SC$ restricting $\widehat{\gamma}^\pm$, $[\widehat{\sigma}_2]\widehat{\gamma}^\pm = [\widehat{\sigma}_1]\widehat{\gamma}^\pm$, and $\Theta(\widehat{\gamma}^\pm) \vdash [\widehat{\sigma}_1]\widehat{\gamma}^\pm : e'$ since $\Theta \vdash \widehat{\sigma}_1 : SC$.

  Then by assumption, $\Theta \vdash \widehat{\sigma}_2 \simeq^{\leqslant} \widehat{\sigma}_1 : \Xi$, which in particular means $\Theta(\widehat{\beta}^\pm) \vdash [\widehat{\sigma}_2]\widehat{\beta}^\pm \simeq^{\leqslant} [\widehat{\sigma}_1]\widehat{\beta}^\pm$, that is $\Theta(\widehat{\beta}^\pm) \vdash P' \simeq^{\leqslant} [\widehat{\sigma}_1]\widehat{\beta}^\pm$.

- if $\widehat{\beta}^\pm$ is negative, the proof is analogous.

$\square$

8.2 Correctness of the Typing Algorithm

Let us extend the declarative typing metric (definition 29) to the algorithmic typing.

Definition 30 (Size of an Algorithmic Judgement)**.** *For an algorithmic typing judgement J let us define a metrics $\mathsf{size}(J)$ as a pair of numbers in the following way:*

+ $\mathsf{size}(\Gamma; \Phi \vDash v \colon P) = (\mathsf{size}(v), 0)$*;*

− $\mathsf{size}(\Gamma; \Phi \vDash c \colon N) = (\mathsf{size}(c), 0)$*;*

- $\mathsf{size}(\Gamma; \Phi; \Theta \vDash N \bullet \vec{v} \Rightarrow\!\!\!\!> M \dashv \Theta'; SC) = (\mathsf{size}(\vec{v}), \mathsf{npq}(N)))$

Definition 31 (Metric)**.** *We extend the metric from definition 29 to the algorithmic typing in the following way. For a tree T inferring an algorithmic typing judgement J, we define $\mathsf{size}(T)$ as $(\mathsf{size}(J), 0)$.*

Soundness and the completeness are proved by mutual induction on the metric of the inference tree.

Lemma 90 (Soundness of typing)**.** *Suppose that $\Gamma \vdash \Phi$. For an inference tree T_1,*

+ *If T_1 infers $\Gamma; \Phi \vDash v \colon P$ then $\Gamma \vdash P$ and $\Gamma; \Phi \vdash v \colon P$*

− *If T_1 infers $\Gamma; \Phi \vDash c \colon N$ then $\Gamma \vdash N$ and $\Gamma; \Phi \vdash c \colon N$*

- *If T_1 infers $\Gamma; \Phi; \Theta \vDash N \bullet \vec{v} \Rightarrow\!\!\!\!> M \dashv \Theta'; SC$ for $\Gamma \vdash^{\supseteq} \Theta$ and $\Gamma; \mathbf{dom}\,(\Theta) \vdash N$ free from negative algorithmic variables, then*
  1. *$\Gamma \vdash^{\supseteq} \Theta'$*
  2. *$\Theta \subseteq \Theta'$*
  3. *$\Gamma; \mathbf{dom}\,(\Theta') \vdash M$*
  4. *$\mathbf{dom}\,(\Theta) \cap \mathbf{uv}\,(M) \subseteq \mathbf{uv}\,N$*
  5. *M is normalized and free from negative algorithmic variables*
  6. *$\Theta'|_{\mathbf{uv}\,N \cup \mathbf{uv}\,M} \vdash SC$*
  7. *for any $\Theta' \vdash \widehat{\sigma} : \mathbf{uv}\,N \cup \mathbf{uv}\,M$, $\Theta' \vdash \widehat{\sigma} : SC$ implies $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \vec{v} \Rightarrow\!\!\!\!> [\widehat{\sigma}]M$*

Proof. We prove it by induction on $\mathsf{metric}(T_1)$, mutually with the completeness of typing (lemma 90). Let us consider the last rule used to infer the derivation.

Case 1. Rule ATVar We are proving that if $\Gamma; \Phi \models x\colon \mathbf{nf}(P)$ then $\Gamma \vdash \mathbf{nf}(P)$ and $\Gamma; \Phi \vdash x\colon \mathbf{nf}(P)$.

By inversion, $x : P \in \Phi$. Since $\Gamma \vdash \Phi$, we have $\Gamma \vdash P$, and by corollary 16, $\Gamma \vdash \mathbf{nf}(P)$.

By applying Rule DTVar to $x : P \in \Phi$, we infer $\Gamma; \Phi \vdash x\colon P$. Finally, by Rule DTPEquiv, since $\Gamma \vdash P \simeq^{\leqslant} \mathbf{nf}(P)$ (corollary 18), we have $\Gamma; \Phi \vdash x\colon \mathbf{nf}(P)$.

Case 2. Rule ATThunk

We are proving that if $\Gamma; \Phi \models \{c\}\colon \downarrow N$ then $\Gamma \vdash \downarrow N$ and $\Gamma; \Phi \vdash \{c\}\colon \downarrow N$.

By inversion of $\Gamma; \Phi \models \{c\}\colon \downarrow N$, we have $\Gamma; \Phi \models c\colon N$. By the induction hypothesis applied to $\Gamma; \Phi \models c\colon N$, we have

1. $\Gamma \vdash N$, and hence, $\Gamma \vdash \downarrow N$;
2. $\Gamma; \Phi \vdash c\colon N$, which by Rule DTThunk implies $\Gamma; \Phi \vdash \{c\}\colon \downarrow N$.

Case 3. Rule ATReturn The proof is symmetric to the previous case (case 2).

Case 4. Rule ATPAnnot We are proving that if $\Gamma; \Phi \models (v : Q)\colon \mathbf{nf}(Q)$ then $\Gamma \vdash \mathbf{nf}(Q)$ and $\Gamma; \Phi \vdash (v : Q)\colon \mathbf{nf}(Q)$.

By inversion of $\Gamma; \Phi \models (v : Q)\colon \mathbf{nf}(Q)$, we have:

1. $\Gamma \vdash (v : Q)$, hence, $\Gamma \vdash Q$, and by corollary 16, $\Gamma \vdash \mathbf{nf}(Q)$;
2. $\Gamma; \Phi \models v\colon P$, which by the induction hypothesis implies $\Gamma \vdash P$ and $\Gamma; \Phi \vdash v\colon P$;
3. $\Gamma; \cdot \models Q \geqslant P \dashv \cdot$, which by lemma 76 implies $\Gamma \vdash [\cdot] Q \geqslant P$, that is $\Gamma \vdash Q \geqslant P$.

To infer $\Gamma; \Phi \vdash (v : Q)\colon Q$, we apply Rule DTPAnnot to $\Gamma; \Phi \vdash v\colon P$ and $\Gamma \vdash Q \geqslant P$. Then by Rule DTPEquiv, $\Gamma; \Phi \vdash (v : Q)\colon \mathbf{nf}(Q)$.

Case 5. Rule ATNAnnot The proof is symmetric to the previous case (case 4).

Case 6. Rule ATtLam We are proving that if $\Gamma; \Phi \models \lambda x : P.c\colon \mathbf{nf}(P \to N)$ then $\Gamma \vdash \mathbf{nf}(P \to N)$ and $\Gamma; \Phi \vdash \lambda x : P.c\colon \mathbf{nf}(P \to N)$.

By inversion of $\Gamma; \Phi \models \lambda x : P.c\colon \mathbf{nf}(P \to N)$, we have $\Gamma \vdash \lambda x : P.c$, which implies $\Gamma \vdash P$.

Also by inversion of $\Gamma; \Phi \models \lambda x : P.c\colon \mathbf{nf}(P \to N)$, we have $\Gamma; \Phi, x : P \models c\colon N$, applying induction hypothesis to which gives us:

1. $\Gamma \vdash N$, thus $\Gamma \vdash P \to N$, and by corollary 16, $\Gamma \vdash \mathbf{nf}(P \to N)$;
2. $\Gamma; \Phi, x : P \vdash c\colon N$, which by Rule DTtLam implies $\Gamma; \Phi \vdash \lambda x : P.c\colon P \to N$, and by Rule DTPEquiv, $\Gamma; \Phi \vdash \lambda x : P.c\colon \mathbf{nf}(P \to N)$.

Case 7. Rule ATTLam We are proving that if $\Gamma; \Phi \models \Lambda\alpha^+.c\colon \mathbf{nf}(\forall\alpha^+.N)$ then $\Gamma; \Phi \vdash \Lambda\alpha^+.c\colon \mathbf{nf}(\forall\alpha^+.N)$ and $\Gamma \vdash \mathbf{nf}(\forall\alpha^+.N)$.

By inversion of $\Gamma, \alpha^+; \Phi \models c\colon N$, we have $\Gamma \vdash \Lambda\alpha^+.c$, which implies $\Gamma, \alpha^+ \vdash c$.

Also by inversion of $\Gamma, \alpha^+; \Phi \models c\colon N$, we have $\Gamma, \alpha^+; \Phi \models c\colon N$. Obtaining the induction hypothesis to $\Gamma, \alpha^+; \Phi \models c\colon N$, we have:

1. $\Gamma, \alpha^+ \vdash N$, thus $\Gamma \vdash \forall\alpha^+.N$, and by corollary 16, $\Gamma \vdash \mathbf{nf}(\forall\alpha^+.N)$;
2. $\Gamma, \alpha^+; \Phi \vdash c\colon N$, which by Rule DTTLam implies $\Gamma; \Phi \vdash \Lambda\alpha^+.c\colon \forall\alpha^+.N$, and by Rule DTPEquiv, $\Gamma; \Phi \vdash \Lambda\alpha^+.c\colon \mathbf{nf}(\forall\alpha^+.N)$

Case 8. Rule ATVarLet We are proving that if $\Gamma; \Phi \models \mathbf{let}\, x = v; c\colon N$ then $\Gamma; \Phi \vdash \mathbf{let}\, x = v; c\colon N$ and $\Gamma \vdash N$.

By inversion of $\Gamma; \Phi \models \mathbf{let}\, x = v; c\colon N$, we have:

1. $\Gamma \vdash \mathbf{let}\, x = v; c$, which gives us $\Gamma \vdash v$ and $\Gamma \vdash c$.
2. $\Gamma; \Phi \models v\colon P$, which by the induction hypothesis implies $\Gamma \vdash P$ (and thus, $\Gamma \vdash \Phi, x : P$) and $\Gamma; \Phi \vdash v\colon P$;
3. $\Gamma; \Phi, x : P \models c\colon N$, which by the induction hypothesis implies $\Gamma \vdash N$ and $\Gamma; \Phi, x : P \vdash c\colon N$.

This way, $\Gamma; \Phi \vdash \mathbf{let}\, x = v; c\colon N$ holds by Rule DTVarLet.

Case 9. Rule ATAppLetAnn We are proving that if $\Gamma; \Phi \models \mathbf{let}\, x : P = v(\vec{v}); c'\colon N$ then $\Gamma; \Phi \vdash \mathbf{let}\, x : P = v(\vec{v}); c'\colon N$ and $\Gamma \vdash N$.

By inversion, we have:

1. $\Gamma \vdash P$, hence, $\Gamma \vdash \Phi, x : P$
2. $\Gamma; \Phi \models v\colon \downarrow M$
3. $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow\!\!\!> \uparrow Q \dashv \Theta; SC_1$

4. $\Gamma; \Theta \models \uparrow Q \leqslant \uparrow P \dashv SC_2$
5. $\Theta \vdash SC_1 \& SC_2 = SC$
6. $\Gamma; \Phi, x : P \models c' \colon N$

By the induction hypothesis applied to $\Gamma; \Phi \models v \colon \downarrow M$, we have $\Gamma; \Phi \vdash v \colon \downarrow M$ and $\Gamma \vdash \downarrow M$ (and hence, $\Gamma; \mathbf{dom}(\Theta) \vdash M$).

By the induction hypothesis applied to $\Gamma; \Phi, x : P \models c' \colon N$, we have $\Gamma; \Phi, x : P \vdash c' \colon N$ and $\Gamma \vdash N$.

By the induction hypothesis applied to $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rrightarrow \uparrow Q \dashv \Theta; SC_1$, we have:

1. $\Gamma \vdash^{\supseteq} \Theta$,
2. $\Gamma; \mathbf{dom}(\Theta) \vdash \uparrow Q$,
3. $\Theta'|_{\mathbf{uv}\, M \cup \mathbf{uv}\, Q} \vdash SC_1$, and thus, $\mathbf{dom}(SC_1) \subseteq \mathbf{uv}\, M \cup \mathbf{uv}\, Q$.
4. for any $\Theta' \vdash \widehat{\sigma} : SC_1$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}] M \bullet \vec{v} \Rrightarrow [\widehat{\sigma}] \uparrow Q$.

By soundness of negative subtyping (**??**) applied to $\Gamma; \Theta \models \uparrow Q \leqslant \uparrow P \dashv SC_2$, we have $\Theta \vdash SC_2 : \mathbf{uv}(\uparrow Q)$, and thus, $\mathbf{uv}(\uparrow Q) = \mathbf{dom}(SC_2)$.

By soundness of constraint merge (lemma 79), $\mathbf{dom}(SC) = \mathbf{dom}(SC_1) \cup \mathbf{dom}(SC_2) \subseteq \mathbf{uv}\, M \cup \mathbf{uv}\, Q$ Then by lemma 74, let us take $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(M) \cup \mathbf{uv}(Q)$ and $\Theta \vdash \widehat{\sigma} : SC$. By the soundness of constraint merge, $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$, and by weakening, $\Theta' \vdash \widehat{\sigma} : SC_1$ and $\Theta' \vdash \widehat{\sigma} : SC_2$.

Then as noted above (4), $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow [\widehat{\sigma}] \uparrow Q$ And again, by soundness of negative subtyping (**??**) applied to $\Gamma; \Theta \models \uparrow Q \leqslant \uparrow P \dashv SC_2$, we have $\Gamma \vdash [\widehat{\sigma}] \uparrow Q \leqslant \uparrow P$.

To infer $\Gamma; \Phi \vdash \mathbf{let}\, x : P = v(\vec{v}); c' \colon N$, we apply the corresponding declarative rule Rule DTAppLetAnn, where Q is $[\widehat{\sigma}]\, Q$. Notice that all the premises were already shown to hold above:

1. $\Gamma \vdash P$ and $\Gamma; \Phi \vdash v \colon \downarrow M$ from the assumption,
2. $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow \uparrow [\widehat{\sigma}]\, Q$ holds since $[\widehat{\sigma}] \uparrow Q = \uparrow [\widehat{\sigma}]\, Q$,
3. $\Gamma \vdash \uparrow [\widehat{\sigma}]\, Q \leqslant \uparrow P$ by soundness of negative subtyping,
4. $\Gamma; \Phi, x : P \vdash c' \colon N$ from the the induction hypothesis.

Case 10. Rule ATAppLet We are proving that if $\Gamma; \Phi \models \mathbf{let}\, x = v(\vec{v}); c' \colon N$ then $\Gamma; \Phi \vdash \mathbf{let}\, x = v(\vec{v}); c' \colon N$ and $\Gamma \vdash N$.

By the inversion, we have:

1. $\Gamma; \Phi \models v \colon \downarrow M$
2. $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rrightarrow \uparrow Q \dashv \Theta; SC$
3. $\mathbf{uv}\, Q \subseteq \mathbf{dom}(SC)$
4. $SC|_{\mathbf{uv}(Q)}$ **singular with** $\widehat{\sigma}_3$
5. $\Gamma; \Phi, x : [\widehat{\sigma}_3]\, Q \models c' \colon N$

By the induction hypothesis applied to $\Gamma; \Phi \models v \colon \downarrow M$, we have $\Gamma; \Phi \vdash v \colon \downarrow M$ and $\Gamma \vdash \downarrow M$ (and thus, $\Gamma; \mathbf{dom}(\Theta) \vdash M$).

By the induction hypothesis applied to $\Gamma; \Phi, x : [\widehat{\sigma}_3]\, Q \models c' \colon N$, we have $\Gamma \vdash N$ and $\Gamma; \Phi, x : [\widehat{\sigma}_3]\, Q \vdash c' \colon N$.

By the induction hypothesis applied to $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rrightarrow \uparrow Q \dashv \Theta; SC$, we have:

1. $\Gamma \vdash^{\supseteq} \Theta$
2. $\Gamma; \mathbf{dom}(\Theta) \vdash \uparrow Q$
3. $\Theta|_{\mathbf{uv}\, M \cup \mathbf{uv}\, Q} \vdash SC$ (and thus, $\mathbf{dom}(SC) \subseteq \mathbf{uv}\, M \cup \mathbf{uv}\, Q$)
4. for any $\Theta \vdash \widehat{\sigma} : SC$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}] M \bullet \vec{v} \Rrightarrow [\widehat{\sigma}] \uparrow Q$, which, since M is ground means $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow \uparrow [\widehat{\sigma}]\, Q$.

To infer $\Gamma; \Phi \vdash \mathbf{let}\, x = v(\vec{v}); c' \colon N$, we apply the corresponding declarative rule Rule DTAppLet. Let us show that the premises hold:

- $\Gamma; \Phi \vdash v \colon \downarrow M$ holds by the induction hypothesis;
- $\Gamma; \Phi, x : [\widehat{\sigma}_3]\, Q \vdash c' \colon N$ also holds by the induction hypothesis, as noted above;
- Let us take an arbitrary substitution $\widehat{\sigma}$ $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\, M \cup \mathbf{uv}\, Q$ satisfying $\Theta \vdash \widehat{\sigma} : SC$ (it exists by lemma 74). Then $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow \uparrow [\widehat{\sigma}]\, Q$ holds, as noted above;
- To show the uniqueness of $\uparrow [\widehat{\sigma}]\, Q$, we assume that for some other type K holds $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow K$, that is $\Gamma; \Phi \vdash [\cdot] M \bullet \vec{v} \Rrightarrow K$. Then by the completeness of typing (lemma 91), there exist N', Θ', and SC' such that

1. $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow\!\!\!> N' \dashv \Theta'; SC'$ and
2. there exists a substitution $\Theta' \vdash \widehat{\sigma}' : SC'$ such that $\Gamma \vdash [\widehat{\sigma}']N' \simeq^{\leqslant} K$.

By determinicity of the typing algorithm (??), $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow\!\!\!> N' \dashv \Theta'; SC'$, means that SC' is SC, Θ' is Θ, and N' is $\uparrow Q$. This way, $\Gamma \vdash [\widehat{\sigma}']\uparrow Q \simeq^{\leqslant} K$ for substitution $\Theta \vdash \widehat{\sigma}' : SC$.

It is left to show that $\Gamma \vdash [\widehat{\sigma}']\uparrow Q \simeq^{\leqslant} [\widehat{\sigma}]\uparrow Q$, then by transitivity of equivalence, we will have $\Gamma \vdash [\widehat{\sigma}]\uparrow Q \simeq^{\leqslant} K$. Since $\Theta \vdash \widehat{\sigma} : SC|_{\mathbf{uv}\,(Q)}$ and $(SC|uv(uQ))u'$, and $SC|_{\mathbf{uv}\,(Q)}$ **singular with** $\widehat{\sigma}_3$, we have $\Theta \vdash \widehat{\sigma} \simeq^{\leqslant} \widehat{\sigma}_3 : \mathbf{dom}\,(SC) \cap \mathbf{uv}\,(Q)$ and $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma}_3 : \mathbf{dom}\,(SC) \cap \mathbf{uv}\,(Q)$. Then since $\mathbf{uv}\,(Q) \subseteq \mathbf{dom}\,(SC)$, we have $\mathbf{dom}\,(SC) \cap \mathbf{uv}\,(Q) = \mathbf{uv}\,(Q)$. This way, by transitivity and symmetry of the equivalence, $\Theta \vdash \widehat{\sigma} \simeq^{\leqslant} \widehat{\sigma}' : \mathbf{uv}\,(Q)$, which implies $\Gamma \vdash [\widehat{\sigma}']\uparrow Q \simeq^{\leqslant} [\widehat{\sigma}]\uparrow Q$.

Case 11. Rule ATUnpack We are proving that if $\Gamma; \Phi \models \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c' : N$ then $\Gamma; \Phi \vdash \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c' : N$ and $\Gamma \vdash N$. By the inversion, we have:

1. $\Gamma; \Phi \models v : \exists \overrightarrow{\alpha^-}.P$
2. $\Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \models c' : N$
3. $\Gamma \vdash N$

By the induction hypothesis applied to $\Gamma; \Phi \models v : \exists \overrightarrow{\alpha^-}.P$, we have $\Gamma; \Phi \vdash v : \exists \overrightarrow{\alpha^-}.P$ and $\exists \overrightarrow{\alpha^-}.P$ is normalized. By the induction hypothesis applied to $\Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \models c' : N$, we have $\Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \vdash c' : N$.

To show $\Gamma; \Phi \vdash \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c' : N$, we apply the corresponding declarative rule Rule DTUnpack. Let us show that the premises hold:

1. $\Gamma; \Phi \vdash v : \exists \overrightarrow{\alpha^-}.P$ holds by the induction hypothesis, as noted above,
2. $\mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.P) = \exists \overrightarrow{\alpha^-}.P$ holds since $\exists \overrightarrow{\alpha^-}.P$ is normalized,
3. $\Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \vdash c' : N$ also holds by the induction hypothesis,
4. $\Gamma \vdash N$ holds by the inversion, as noted above.

Case 12. Rule ATEmptyApp Then by assumption:

- $\Gamma \vdash^{\supseteq} \Theta$,
- $\Gamma; \mathbf{dom}\,(\Theta) \vdash N$ is free from negative algorithmic variables,
- $\Gamma; \Phi; \Theta \models N \bullet \cdot \Rightarrow\!\!\!> \mathbf{nf}\,(N) \dashv \Theta; \cdot$.

Let us show the required properties:

1. $\Gamma \vdash^{\supseteq} \Theta$ holds by assumption,
2. $\Theta \subseteq \Theta$ holds trivially,
3. $\mathbf{nf}\,(N)$ is evidently normalized, $\Gamma; \mathbf{dom}\,(\Theta) \vdash N$ implies $\Gamma; \mathbf{dom}\,(\Theta) \vdash \mathbf{nf}\,(N)$ by corollary 24, and lemma 30 means that $\mathbf{nf}\,(N)$ is inherently free from negative algorithmic variables,
4. $\mathbf{dom}\,(\Theta) \cap \mathbf{uv}\,(\mathbf{nf}\,(N)) \subseteq \mathbf{uv}\, N$ holds since $\mathbf{uv}\,(\mathbf{nf}\,(N)) = \mathbf{uv}\,(N)$,
5. $\Theta|_{\mathbf{uv}\, N \cup \mathbf{uv}\, \mathbf{nf}\,(N)} \vdash \cdot$ holds trivially,
6. suppose that $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\, N \cup \mathbf{uv}\, \mathbf{nf}\,(N)$. To show $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \cdot \Rightarrow\!\!\!> [\widehat{\sigma}]\mathbf{nf}\,(N)$, we apply the corresponding declarative rule Rule DTEmptyApp. To show $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} [\widehat{\sigma}]\mathbf{nf}\,(N)$, we apply the following sequence: $N \simeq^{D} \mathbf{nf}\,(N)$ by lemma 31, then $[\widehat{\sigma}]N \simeq^{D} [\widehat{\sigma}]\mathbf{nf}\,(N)$ by lemma 47, then $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} [\widehat{\sigma}]\mathbf{nf}\,(N)$ by lemma 40.

Case 13. Rule ATArrowApp By assumption:

1. $\Gamma \vdash^{\supseteq} \Theta$,
2. $\Gamma; \mathbf{dom}\,(\Theta) \vdash Q \to N$ is free from negative algorithmic variables, and hence, so are Q and N,
3. $\Gamma; \Phi; \Theta \models Q \to N \bullet v, \vec{v} \Rightarrow\!\!\!> M \dashv \Theta'; SC$, and by inversion:
   (a) $\Gamma; \Phi \models v : P$, and by the induction hypothesis applied to this judgment, we have $\Gamma; \Phi \vdash v : P$, and $\Gamma \vdash P$;
   (b) $\Gamma;\ \Theta \models Q \geqslant P \dashv SC_1$, and by the soundness of subtyping: $\Theta \vdash SC_1 : \mathbf{uv}\, Q$ (and thus, $\mathbf{dom}\,(SC_1) = \mathbf{uv}\, Q$), and for any
   $\Theta \vdash \widehat{\sigma} : SC_1$, we have $\Gamma \vdash [\widehat{\sigma}]Q \geqslant P$;
   (c) $\Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rightarrow\!\!\!> M \dashv \Theta'; SC_2$, and by the induction hypothesis applied to this judgment,
      i. $\Gamma \vdash^{\supseteq} \Theta'$,
      ii. $\Theta \subseteq \Theta'$,

iii. $\Gamma; \mathbf{dom}(\Theta') \vdash M$ is normalized and free from negative algorithmic variables,

iv. $\mathbf{dom}(\Theta) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}\, N$,

v. $\Theta'|_{\mathbf{uv}(M) \cup \mathbf{uv}(N)} \vdash SC_2$, and thus, $\mathbf{dom}(SC_2) \subseteq \mathbf{uv}(M) \cup \mathbf{uv}(N)$,

vi. for any $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(M) \cup \mathbf{uv}(N)$ and $\Theta' \vdash \widehat{\sigma} : SC_2$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \overrightarrow{v} \Rightarrow\!\!\!> [\widehat{\sigma}]M$;

(d) $\Theta \vdash SC_1 \& SC_2 = SC$, which by lemma 79 implies $\mathbf{dom}(SC) = \mathbf{dom}(SC_1) \cup \mathbf{dom}(SC_2) \subseteq \mathbf{uv}\, Q \cup \mathbf{uv}\, M \cup \mathbf{uv}\, N$.

Let us show the required properties:

1. $\Gamma \vdash^{\supseteq} \Theta'$ is shown above,
2. $\Theta \subseteq \Theta'$ is shown above,
3. $\Gamma; \mathbf{dom}(\Theta') \vdash M$ is normalized and free from negative algorithmic variables, as shown above,
4. $\mathbf{dom}(\Theta) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}\, N \subseteq \mathbf{uv}(Q \to N)$ (the first inclusion is shown above, the second one is by definition),
5. To show $\Theta'|_{\mathbf{uv}(Q) \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)} \vdash SC$, first let us notice that $\mathbf{uv}(Q) \cup \mathbf{uv}(N) \cup \mathbf{uv}(M) \subseteq \mathbf{dom}(SC)$, as mentioned above. Then we demonstrate $\Theta' \vdash SC$: $\Theta \vdash SC_1$ and $\Theta \subseteq \Theta'$ imply $\Theta' \vdash SC_1$, by the soundness of constraint merge (lemma 79) applied to $\Theta' \vdash SC_1 \& SC_2 = SC$:
   (a) $\Theta' \vdash SC$,
   (b) for any $\Theta' \vdash \widehat{\sigma} : SC$, $\Theta' \vdash \widehat{\sigma} : SC_i$ holds;
6. Suppose that $\Theta' \vdash \widehat{\sigma} : \mathbf{uv}(Q) \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$ and $\Theta' \vdash \widehat{\sigma} : SC$. To show $\Gamma; \Phi \vdash [\widehat{\sigma}](Q \to N) \bullet v, \overrightarrow{v} \Rightarrow\!\!\!> [\widehat{\sigma}]M$, that is $\Gamma; \Phi \vdash [\widehat{\sigma}]Q \to [\widehat{\sigma}]N \bullet v, \overrightarrow{v} \Rightarrow\!\!\!> [\widehat{\sigma}]M$, we apply the corresponding declarative rule Rule DTArrowApp. Let us show the required premises:
   (a) $\Gamma; \Phi \vdash v : P$ holds as shown above,
   (b) $\Gamma \vdash [\widehat{\sigma}]Q \geqslant P$ holds since $\Gamma \vdash [\widehat{\sigma}|_{\mathbf{uv}(Q)}]Q \geqslant P$ by the soundness of subtyping as noted above: since $\Theta' \vdash \widehat{\sigma} : SC$ implies $\Theta' \vdash \widehat{\sigma}|_{\mathbf{uv}(Q)} : SC_1$, which we strengthen to $\Theta \vdash \widehat{\sigma}|_{\mathbf{uv}(Q)} : SC_1$,
   (c) $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \overrightarrow{v} \Rightarrow\!\!\!> [\widehat{\sigma}]M$ holds by the induction hypothesis as shown above, since $\Theta' \vdash \widehat{\sigma} : SC$ implies $\Theta' \vdash \widehat{\sigma} : SC_2$, and then $\Theta' \vdash \widehat{\sigma}|_{\mathbf{uv}(N) \cup \mathbf{uv}(M)} : SC_2$ and $\Theta \vdash \widehat{\sigma}|_{\mathbf{uv}(N) \cup \mathbf{uv}(M)} : \mathbf{uv}(N) \cup \mathbf{uv}(M)$.

Case 14. Rule ATForallApp

By assumption:

1. $\Gamma \vdash^{\supseteq} \Theta$,
2. $\Gamma; \mathbf{dom}(\Theta) \vdash \forall \overrightarrow{\alpha^+}. N$ is free from negative algorithmic variables,
3. $\Gamma; \Phi; \Theta \models \forall \overrightarrow{\alpha^+}. N \bullet \overrightarrow{v} \Rightarrow\!\!\!> M \dashv \Theta'; SC$, which by inversion means $\overrightarrow{v} \neq \cdot$, $\overrightarrow{\alpha^+} \neq \cdot$, and $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \overrightarrow{v} \Rightarrow\!\!\!> M \dashv \Theta'; SC$. It is easy to see that the induction hypothesis is applicable to the latter judgment:
   - $\Gamma \vdash^{\supseteq} \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\}$ holds by $\Gamma \vdash^{\supseteq} \Theta$,
   - $\Gamma; \mathbf{dom}(\Theta), \overrightarrow{\widehat{\alpha}^+} \vdash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N$ holds since $\Gamma; \mathbf{dom}(\Theta) \vdash \forall \overrightarrow{\alpha^+}. N$ $[\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N$ is normalized and free from negative algorithmic variables since so is N;

   This way, by the inductive hypothesis applied to $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \overrightarrow{v} \Rightarrow\!\!\!> M \dashv \Theta'; SC$, we have:
   (a) $\Gamma \vdash^{\supseteq} \Theta'$,
   (b) $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \subseteq \Theta'$,
   (c) $\Gamma; \mathbf{dom}(\Theta') \vdash M$ is normalized and free from negative algorithmic variables,
   (d) $\mathbf{dom}(\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\}) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}([\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N)$,
   (e) $\Theta'|_{\Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)} \vdash SC$, where Ξ denotes $\mathbf{uv}([\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N) \cap \overrightarrow{\widehat{\alpha}^+}$, that is the algorithmization of the $\forall$-variables that are actually used in N.
   (f) for any $\widehat{\sigma}$ such that $\Theta' \vdash \widehat{\sigma} : \Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$ and $\Theta' \vdash \widehat{\sigma} : SC$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \overrightarrow{v} \Rightarrow\!\!\!> [\widehat{\sigma}]M$.

Let us show the required properties:

1. $\Gamma \vdash^{\supseteq} \Theta'$ is shown above;
2. $\Theta \subseteq \Theta'$ since $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \subseteq \Theta'$;
3. $\Gamma; \mathbf{dom}(\Theta') \vdash M$ is normalized and free from negative algorithmic variables, as shown above;
4. $\mathbf{dom}(\Theta) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}(N)$ since $\mathbf{dom}(\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\}) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}([\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N)$ implies $(\mathbf{dom}(\Theta) \cup \overrightarrow{\widehat{\alpha}^+}) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}(N) \cup \overrightarrow{\widehat{\alpha}^+}$, thus, $\mathbf{dom}(\Theta) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}(N) \cup \overrightarrow{\widehat{\alpha}^+}$, and since $\mathbf{dom}(\Theta)$ is disjoint with $\overrightarrow{\widehat{\alpha}^+}$, $\mathbf{dom}(\Theta) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}(N)$;

5. $\Theta'|_{\mathbf{uv}(N)\cup\mathbf{uv}(M)} \vdash SC|_{\mathbf{uv}(N)\cup\mathbf{uv}(M)}$ follows from $\Theta'|_{\Xi\cup\mathbf{uv}(N)\cup\mathbf{uv}(M)} \vdash SC$ if we restrict both sides to $\mathbf{uv}(N)\cup\mathbf{uv}(M)$.

6. Let us assume $\Theta' \vdash \widehat{\sigma} : \mathbf{uv}(N) \cup \mathbf{uv}(M)$ and $\Theta' \vdash \widehat{\sigma} : SC|_{\mathbf{uv}(N)\cup\mathbf{uv}(M)}$. Then to show $\Gamma; \Phi \vdash [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N \bullet \vec{v} \Rightarrow\!\!\!> [\widehat{\sigma}]M$, that is $\Gamma; \Phi \vdash \forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N \bullet \vec{v} \Rightarrow\!\!\!> [\widehat{\sigma}]M$, we apply the corresponding declarative rule Rule DTForallApp. To do so, we need to provide a substitution for $\overrightarrow{\alpha^+}$, i.e. $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^+}$ such that $\Gamma; \Phi \vdash [\sigma_0][\widehat{\sigma}]N \bullet \vec{v} \Rightarrow\!\!\!> [\widehat{\sigma}]M$.

   By lemma 74, we construct $\widehat{\sigma}_0$ such that $\Theta' \vdash \widehat{\sigma}_0 : \overrightarrow{\widehat{\alpha}^+}$ and $\Theta' \vdash \widehat{\sigma}_0 : SC|_{\overrightarrow{\widehat{\alpha}^+}}$.

   Then σ_0 is defined as $\widehat{\sigma}_0 \circ \widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^+}} \circ \overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}$.

   Let us show that the premises of Rule DTForallApp hold:

   - To show $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^+}$, let us take $\alpha_i^+ \in \overrightarrow{\alpha^+}$. If $\widehat{\alpha_i}^+ \in \mathbf{uv}(M)$ then $[\sigma_0]\alpha_i^+ = [\widehat{\sigma}]\widehat{\alpha_i}^+$, and $\Theta' \vdash \widehat{\sigma} : \mathbf{uv}(N) \cup \mathbf{uv}(M)$ implies $\Theta'(\widehat{\alpha}^+) \vdash [\widehat{\sigma}]\widehat{\alpha}^+$. Analogously, if $\widehat{\alpha_i}^+ \in \overrightarrow{\widehat{\alpha}^+}\backslash\mathbf{uv}(M)$ then $[\sigma_0]\alpha_i^+ = [\widehat{\sigma}_0]\widehat{\alpha_i}^+$, and $\Theta' \vdash \widehat{\sigma}_0 : \overrightarrow{\widehat{\alpha}^+}$ implies $\Theta'(\widehat{\alpha_i}^+) \vdash [\widehat{\sigma}_0]\widehat{\alpha_i}^+$. In any case, $\Theta'(\widehat{\alpha_i}^+) \vdash [\sigma]\alpha_i^+$ can be weakened to $\Gamma \vdash [\sigma_0]\alpha_i^+$, since $\Gamma \vdash^{\supseteq} \Theta'$.
   - Let us show $\Gamma; \Phi \vdash [\sigma_0][\widehat{\sigma}]N \bullet \vec{v} \Rightarrow\!\!\!> [\widehat{\sigma}]M$. It suffices to construct $\widehat{\sigma}_1$ such that
     (a) $\Theta' \vdash \widehat{\sigma}_1 : \Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$,
     (b) $\Theta' \vdash \widehat{\sigma}_1 : SC$,
     (c) $[\sigma_0][\widehat{\sigma}]N = [\widehat{\sigma}_1][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N$, and
     (d) $[\widehat{\sigma}]M = [\widehat{\sigma}_1]M$,

     because then we can apply the induction hypothesis (3f) to $\widehat{\sigma}_1$, rewrite the conclusion by $[\widehat{\sigma}_1][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N = [\sigma_0][\widehat{\sigma}]N$ and $[\widehat{\sigma}_1]M = [\widehat{\sigma}]M$, and infer the required judgement.
     Let us take $\widehat{\sigma}_1 = (\widehat{\sigma}_0 \circ \widehat{\sigma})|_{\Xi\cup\mathbf{uv}(N)\cup\mathbf{uv}(M)}$, then

     (a) $\Theta' \vdash \widehat{\sigma}_1 : \Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$, since $\Theta' \vdash \widehat{\sigma}_0 : \overrightarrow{\widehat{\alpha}^+}$ and $\Theta' \vdash \widehat{\sigma} : \mathbf{uv}(N) \cup \mathbf{uv}(M)$, we have $\Theta' \vdash \widehat{\sigma}_0 \circ \widehat{\sigma} : \overrightarrow{\widehat{\alpha}^+} \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$, which we restrict to $\Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$.

     (b) $\Theta' \vdash \widehat{\sigma}_1 : SC$, Let us take any constraint $e \in SC$ restricting variable $\widehat{\beta}^\pm$. $\Theta'|_{\Xi\cup\mathbf{uv}(N)\cup\mathbf{uv}(M)} \vdash SC$ implies that $\widehat{\beta}^\pm \in \Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$.
     If $\widehat{\beta}^\pm \in \mathbf{uv}(N) \cup \mathbf{uv}(M)$ then $[\widehat{\sigma}_1]\widehat{\beta}^\pm = [\widehat{\sigma}]\widehat{\beta}^\pm$. Additionally, $e \in SC|_{\mathbf{uv}(N)\cup\mathbf{uv}(M)}$, which, since $\Theta' \vdash \widehat{\sigma} : SC|_{\mathbf{uv}(N)\cup\mathbf{uv}(M)}$, means $\Theta'(\widehat{\beta}^\pm) \vdash [\widehat{\sigma}]\widehat{\beta}^\pm : e$.
     If $\widehat{\beta}^\pm \in \Xi\backslash(\mathbf{uv}(N) \cup \mathbf{uv}(M))$ then $[\widehat{\sigma}_1]\widehat{\beta}^\pm = [\widehat{\sigma}_0]\widehat{\beta}^\pm$. Additionally, $e \in SC|_{\overrightarrow{\widehat{\alpha}^+}}$, which, since $\Theta' \vdash \widehat{\sigma}_0 : SC|_{\overrightarrow{\widehat{\alpha}^+}}$, means $\Theta'(\widehat{\beta}^\pm) \vdash [\widehat{\sigma}_0]\widehat{\beta}^\pm : e$.

     (c) Let us prove $[\sigma_0][\widehat{\sigma}]N = [\widehat{\sigma}_1][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N$ by the following reasoning

$$
\begin{aligned}
[\sigma_0][\widehat{\sigma}]N &= [\widehat{\sigma}_0][\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^+}}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}][\widehat{\sigma}]N && \text{by definition of } \sigma_0 \\
&= [\widehat{\sigma}_0][\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^+}}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}][\widehat{\sigma}|_{\mathbf{uv}(N)}]N && \text{by lemma 52} \\
&= [\widehat{\sigma}_0][\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^+}}][\widehat{\sigma}|_{\mathbf{uv}(N)}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N && \mathbf{uv}(N) \cap \overrightarrow{\widehat{\alpha}^+} = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma = \varnothing \\
&= [\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^+}}][\widehat{\sigma}|_{\mathbf{uv}(N)}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N && [\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^+}}][\widehat{\sigma}|_{\mathbf{uv}(N)}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \text{ is ground} \\
&= [\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^+}\cup\mathbf{uv}(N)}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \\
&= [\widehat{\sigma}|_{\Xi\cup\mathbf{uv}(N)}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N && \text{by lemma 52: } \mathbf{uv}([\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N) = \Xi \cup \mathbf{uv}(N) \\
&= [\widehat{\sigma}|_{\Xi\cup\mathbf{uv}(N)\cup\mathbf{uv}(M)}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N && \text{also by lemma 52} \\
&= [(\widehat{\sigma}_0 \circ \widehat{\sigma})|_{\Xi\cup\mathbf{uv}(N)\cup\mathbf{uv}(M)}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N && [\widehat{\sigma}|_{\Xi\cup\mathbf{uv}(N)\cup\mathbf{uv}(M)}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \text{ is ground} \\
&= [\widehat{\sigma}_1][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N && \text{by definition of } \widehat{\sigma}_1
\end{aligned}
$$

     (d) $[\widehat{\sigma}]M = [\widehat{\sigma}_1]M$ By definition of $\widehat{\sigma}_1$, $[\widehat{\sigma}_1]M$ is equal to $[(\widehat{\sigma}_0 \circ \widehat{\sigma})|_{\Xi\cup\mathbf{uv}(N)\cup\mathbf{uv}(M)}]M$, which by lemma 52 is equal to $[\widehat{\sigma}_0 \circ \widehat{\sigma}]M$, that is $[\widehat{\sigma}_0][\widehat{\sigma}]M$, and since $[\widehat{\sigma}]M$ is ground, $[\widehat{\sigma}_0][\widehat{\sigma}]M = [\widehat{\sigma}]M$.
   - $\overrightarrow{\alpha^+} \neq \cdot$ and $\vec{v} \neq \cdot$ hold by assumption.

□

Lemma 91 (Completeness of Typing)**.** *Suppose that $\Gamma \vdash \Phi$. For an inference tree T_1,*

+ *If T_1 infers $\Gamma; \Phi \vdash v : P$ then $\Gamma; \Phi \models v : \mathbf{nf}(P)$*

− *If T_1 infers $\Gamma; \Phi \vdash c : N$ then $\Gamma; \Phi \models c : \mathbf{nf}(N)$*

- *If T_1 infers $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \vec{v} \Rightarrow\!\!\!> M$ and*
  1. *$\Gamma \vdash^{\supseteq} \Theta$,*
  2. *$\Gamma \vdash M$,*
  3. *$\Gamma; \mathbf{dom}(\Theta) \vdash N$ (free from negative algorithmic variables, that is $\widehat{\alpha}^- \notin \mathbf{uv}\, N$), and*
  4. *$\Theta \vdash \widehat{\sigma} : \mathbf{uv}(N)$,*

  then there exist M', Θ', and SC such that
  1. *$\Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rightarrow\!\!\!> M' \dashv \Theta'; SC$ and*
  2. *for any $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(N)$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \vec{v} \Rightarrow\!\!\!> M$, there exists $\widehat{\sigma}'$ such that*
     (a) $\Theta' \vdash \widehat{\sigma}' : \mathbf{uv}\, N \cup \mathbf{uv}\, M$ and $\Theta' \vdash \widehat{\sigma}' : SC$,
     (b) $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}\, N$, and
     (c) $\Gamma \vdash [\widehat{\sigma}']M' \simeq^{\leqslant} M$.

Proof. We prove it by induction on $\mathsf{metric}(T_1)$, mutually with the soundness of typing (lemma 90). Let us consider the last rule applied to infer the derivation.

Case 1. Rule DTThunk
Then we are proving that if $\Gamma; \Phi \vdash \{c\}\colon \downarrow N$ (inferred by Rule DTThunk) then $\Gamma; \Phi \models \{c\}\colon \mathbf{nf}(\downarrow N)$. By inversion of $\Gamma; \Phi \vdash \{c\}\colon \downarrow N$, we have $\Gamma; \Phi \vdash c\colon N$, which we apply the induction hypothesis to to obtain $\Gamma; \Phi \models c\colon \mathbf{nf}(N)$. Then by Rule ATThunk, we have $\Gamma; \Phi \models \{c\}\colon \downarrow \mathbf{nf}(N)$. It is left to notice that $\downarrow \mathbf{nf}(N) = \mathbf{nf}(\downarrow N)$.

Case 2. Rule DTReturn
The proof is symmetric to the previous case (case 1).

Case 3. Rule DTPAnnot
Then we are proving that if $\Gamma; \Phi \vdash (v : Q)\colon Q$ is inferred by Rule DTPAnnot then $\Gamma; \Phi \models (v : Q)\colon \mathbf{nf}(Q)$. By inversion, we have:

1. $\Gamma \vdash Q$;
2. $\Gamma; \Phi \vdash v\colon P$, which by the induction hypothesis implies $\Gamma; \Phi \models v\colon \mathbf{nf}(P)$;
3. $\Gamma \vdash Q \geqslant P$, and by transitivity, $\Gamma \vdash Q \geqslant \mathbf{nf}(P)$; Since Q is ground, we have $\Gamma; \cdot \vdash Q$ and $\Gamma \vdash [\cdot]Q \geqslant \mathbf{nf}(P)$. Then by the completeness of subtyping (**??**), we have $\Gamma; \cdot \models Q \geqslant \mathbf{nf}(P) \dashv SC$, where $\cdot \vdash SC$ (implying $SC = \cdot$). This way, $\Gamma; \cdot \models Q \geqslant \mathbf{nf}(P) \dashv \cdot$.

Then we can apply Rule ATPAnnot to $\Gamma \vdash Q$, $\Gamma; \Phi \models v\colon \mathbf{nf}(P)$ and $\Gamma; \cdot \models Q \geqslant \mathbf{nf}(P) \dashv \cdot$ to infer $\Gamma; \Phi \models (v : Q)\colon \mathbf{nf}(Q)$.

Case 4. Rule DTNAnnot
The proof is symmetric to the previous case (case 3).

Case 5. Rule DTtLam
Then we are proving that if $\Gamma; \Phi \vdash \lambda x : P.c\colon P \to N$ is inferred by Rule DTtLam, then $\Gamma; \Phi \models \lambda x : P.c\colon \mathbf{nf}(P \to N)$.

By inversion of $\Gamma; \Phi \vdash \lambda x : P.c\colon P \to N$, we have $\Gamma \vdash P$ and $\Gamma; \Phi, x : P \vdash c\colon N$. Then by the induction hypothesis, $\Gamma; \Phi, x : P \models c\colon \mathbf{nf}(N)$. By Rule ATtLam, we infer $\Gamma; \Phi \models \lambda x : P.c\colon \mathbf{nf}(P \to \mathbf{nf}(N))$. By idempotence of normalization (lemma 34), $\mathbf{nf}(P \to \mathbf{nf}(N)) = \mathbf{nf}(P \to N)$, which concludes the proof for this case.

Case 6. Rule DTTLam
Then we are proving that if $\Gamma; \Phi \vdash \Lambda\alpha^+.c\colon \forall\alpha^+.N$ is inferred by Rule DTTLam, then $\Gamma; \Phi \models \Lambda\alpha^+.c\colon \mathbf{nf}(\forall\alpha^+.N)$. Similar to the previous case, by inversion of $\Gamma; \Phi \vdash \Lambda\alpha^+.c\colon \forall\alpha^+.N$, we have $\Gamma, \alpha^+; \Phi \vdash c\colon N$, and then by the induction hypothesis, $\Gamma, \alpha^+; \Phi \models c\colon \mathbf{nf}(N)$. After that, application of Rule ATTLam, gives as $\Gamma; \Phi \models \Lambda\alpha^+.c\colon \mathbf{nf}(\forall\alpha^+.\mathbf{nf}(N))$.

It is left to show that $\mathbf{nf}(\forall\alpha^+.\mathbf{nf}(N)) = \mathbf{nf}(\forall\alpha^+.N)$. Assume $N = \forall\overrightarrow{\beta^+}.M$ (where M does not start with $\forall$).

- Then by definition, $\mathbf{nf}(\forall\alpha^+.N) = \mathbf{nf}(\forall\alpha^+, \overrightarrow{\beta^+}.M) = \forall\overrightarrow{\gamma^+}.\mathbf{nf}(M)$, where $\mathbf{ord}\,\alpha^+, \overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}(M) = \overrightarrow{\gamma^+}$.
- On the other hand, $\mathbf{nf}(N) = \forall\overrightarrow{\gamma^+}'.\mathbf{nf}(M)$, where $\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}(M) = \overrightarrow{\gamma^+}'$, and thus, $\mathbf{nf}(\forall\alpha^+.\mathbf{nf}(N)) = \mathbf{nf}(\forall\alpha^+, \overrightarrow{\gamma^+}'.\mathbf{nf}(M))$ $\forall\overrightarrow{\gamma^+}''.\mathbf{nf}(\mathbf{nf}(M)) = \forall\overrightarrow{\gamma^+}''.\mathbf{nf}(M)$, where $\mathbf{ord}\,\alpha^+, \overrightarrow{\gamma^+}'\,\mathbf{in}\,\mathbf{nf}(\mathbf{nf}(M)) = \overrightarrow{\gamma^+}''$.

It is left to show that $\overrightarrow{\gamma^+}'' = \overrightarrow{\gamma^+}$.

$$
\begin{aligned}
\overrightarrow{\gamma^+}'' &= \mathbf{ord}\,\alpha^+, \overrightarrow{\gamma^+}'\,\mathbf{in}\,\mathbf{nf}\,(\mathbf{nf}\,(M)) & \\
&= \mathbf{ord}\,\alpha^+, \overrightarrow{\gamma^+}'\,\mathbf{in}\,\mathbf{nf}\,(M) & \text{by idempotence (lemma 34)}\\
&= \mathbf{ord}\,\alpha^+ \cup \overrightarrow{\beta^+} \cap \mathbf{fv}\,\mathbf{nf}\,(M)\,\mathbf{in}\,\mathbf{nf}\,(M) & \text{by definition of } \overrightarrow{\gamma^+}' \text{ and lemma 25}\\
&= \mathbf{ord}\,(\alpha^+ \cup \overrightarrow{\beta^+} \cap \mathbf{fv}\,\mathbf{nf}\,(M)) \cap \mathbf{fv}\,\mathbf{nf}\,(M)\,\mathbf{in}\,\mathbf{nf}\,(M) & \text{by corollary 12}\\
&= \mathbf{ord}\,(\alpha^+ \cup \overrightarrow{\beta^+}) \cap \mathbf{fv}\,\mathbf{nf}\,(M)\,\mathbf{in}\,\mathbf{nf}\,(M) & \text{by set properties}\\
&= \mathbf{ord}\,\alpha^+, \overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M) & \\
&= \overrightarrow{\gamma^+} &
\end{aligned}
$$

Case 7. Rule DTUnpack

Then we are proving that if $\Gamma; \Phi \vdash \mathbf{let}^\exists(\overrightarrow{\alpha^-}, x) = v; c \colon N$ is inferred by Rule DTUnpack, then $\Gamma; \Phi \models \mathbf{let}^\exists(\overrightarrow{\alpha^-}, x) = v; c \colon \mathbf{nf}\,(N)$.

By inversion of $\Gamma; \Phi \vdash \mathbf{let}^\exists(\overrightarrow{\alpha^-}, x) = v; c \colon N$, we have

1. $\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}.P$,
2. $\Gamma; \Phi \vdash v \colon \exists\overrightarrow{\alpha^-}.P$, which by the induction hypothesis implies $\Gamma; \Phi \models v \colon \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P)$, and hence, $\Gamma; \Phi \models v \colon \exists\overrightarrow{\alpha^-}.P$.
3. $\Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \vdash c \colon N$, and by the induction hypothesis, $\Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \models c \colon \mathbf{nf}\,(N)$.
4. $\Gamma \vdash N$.

This way, we can apply Rule ATUnpack to infer $\Gamma; \Phi \models \mathbf{let}^\exists(\overrightarrow{\alpha^-}, x) = v; c \colon \mathbf{nf}\,(N)$.

Case 8. Rule DTPEquiv

Then we are proving that if $\Gamma; \Phi \vdash v \colon P'$ is inferred by Rule DTPEquiv, then $\Gamma; \Phi \models v \colon \mathbf{nf}\,(P')$. By inversion, $\Gamma; \Phi \vdash v \colon P$ and $\Gamma \vdash P \simeq^\leqslant P'$, and the metric of the tree inferring $\Gamma; \Phi \vdash v \colon P$ is less than the one inferring $\Gamma; \Phi \vdash v \colon P'$. Then by the induction hypothesis, $\Gamma; \Phi \models v \colon \mathbf{nf}\,(P)$.

By lemma 46 $\Gamma \vdash P \simeq^\leqslant P'$ implies $\mathbf{nf}\,(P) = \mathbf{nf}\,(P')$, and thus, $\Gamma; \Phi \models v \colon \mathbf{nf}\,(P)$ can be rewritten to $\Gamma; \Phi \models v \colon \mathbf{nf}\,(P')$.

Case 9. Rule DTVar

Then we are proving that $\Gamma; \Phi \vdash x \colon P$ implies $\Gamma; \Phi \models x \colon \mathbf{nf}\,(P)$. By inversion of $\Gamma; \Phi \vdash x \colon P$, we have $x : P \in \Phi$. Then Rule ATVar applies to infer $\Gamma; \Phi \models x \colon \mathbf{nf}\,(P)$.

Case 10. Rule DTVarLet

Then we are proving that $\Gamma; \Phi \vdash \mathbf{let}\, x = v(\vec{v}); c \colon N$ implies $\Gamma; \Phi \models \mathbf{let}\, x = v(\vec{v}); c \colon \mathbf{nf}\,(N)$.

By inversion of $\Gamma; \Phi \vdash \mathbf{let}\, x = v(\vec{v}); c \colon N$, we have

1. $\Gamma; \Phi \vdash v \colon P$, and by the induction hypothesis, $\Gamma; \Phi \models v \colon \mathbf{nf}\,(P)$.
2. $\Gamma; \Phi, x : P \vdash c \colon N$, and by lemma 85, since $\Gamma \vdash P \simeq^\leqslant \mathbf{nf}\,(P)$, we have $\Gamma; \Phi, x : \mathbf{nf}\,(P) \vdash c \colon N$. Then by the induction hypothesis, $\Gamma; \Phi, x : \mathbf{nf}\,(P) \models c \colon \mathbf{nf}\,(N)$.

Together, $\Gamma; \Phi \models v \colon \mathbf{nf}\,(P)$ and $\Gamma; \Phi, x : \mathbf{nf}\,(P) \models c \colon \mathbf{nf}\,(N)$ imply $\Gamma; \Phi \models \mathbf{let}\, x = v(\vec{v}); c \colon \mathbf{nf}\,(N)$ by Rule ATVarLet.

Case 11. Rule DTAppLetAnn

Then we are proving that $\Gamma; \Phi \vdash \mathbf{let}\, x : P = v(\vec{v}); c \colon N$ implies $\Gamma; \Phi \models \mathbf{let}\, x : P = v(\vec{v}); c \colon \mathbf{nf}\,(N)$.

By inversion of $\Gamma; \Phi \vdash \mathbf{let}\, x : P = v(\vec{v}); c \colon N$, we have

1. $\Gamma \vdash P$
2. $\Gamma; \Phi \vdash v \colon \downarrow M$ for some ground M, which by the induction hypothesis means $\Gamma; \Phi \models v \colon \downarrow\mathbf{nf}\,(M)$
3. $\Gamma; \Phi \vdash M \bullet \vec{v} \implies \uparrow Q$. By lemma 84, since $\Gamma \vdash M \simeq^\leqslant \mathbf{nf}\,(M)$, we have $\Gamma; \Phi \vdash [\cdot]\mathbf{nf}\,(M) \bullet \vec{v} \implies \uparrow Q$, which by the induction hypothesis means that there exist normalized M', Θ, and SC_1 such that (noting that M is ground):
   (a) $\Gamma; \Phi; \cdot \models \mathbf{nf}\,(M) \bullet \vec{v} \implies M' \dashv \Theta; SC_1$, where by the soundness, $\Gamma; \mathbf{dom}\,(\Theta) \vdash M'$ and $\Theta \vdash SC_1$.
   (b) for any $\Gamma \vdash M''$ such that $\Gamma; \Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \implies M''$ there exists $\widehat{\sigma}$ such that
      i. $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\, M'$, $\Theta \vdash \widehat{\sigma} : SC_1$, and
      ii. $\Gamma \vdash [\widehat{\sigma}]M' \simeq^\leqslant M''$,

   In particular, there exists $\widehat{\sigma}_0$ such that $\Theta \vdash \widehat{\sigma}_0 : \mathbf{uv}\, M'$, $\Theta \vdash \widehat{\sigma}_0 : SC_1$, $\Gamma \vdash [\widehat{\sigma}_0]M' \simeq^\leqslant \uparrow Q$. Since uM' is normalized and free of negative algorithmic variables, the latter equivalence means $M' = \uparrow Q_0$ for some Q_0, and $\Gamma \vdash [\widehat{\sigma}_0]\, Q_0 \simeq^\leqslant Q$.

4. $\Gamma \vdash {\uparrow}Q \leqslant {\uparrow}P$, and by transitivity, since $\Gamma \vdash [\widehat{\sigma}_0]{\uparrow}Q_0 \simeq^{\leqslant} {\uparrow}Q$, we have $\Gamma \vdash [\widehat{\sigma}_0]{\uparrow}Q_0 \leqslant {\uparrow}P$.
   Let us apply lemma 83 to $\Gamma \vdash [\widehat{\sigma}_0]{\uparrow}Q_0 \leqslant {\uparrow}P$ and obtain $\Theta \vdash SC_2$ such that
   (a) $\Gamma;\ \Theta \models {\uparrow}Q_0 \leqslant {\uparrow}P \dashv SC_2$ and
   (b) $\Theta \vdash \widehat{\sigma}_0 : SC_2$.
5. $\Gamma; \Phi, x : P \vdash c \colon N$, and by the induction hypothesis, $\Gamma; \Phi, x : P \models c \colon \mathbf{nf}\,(N)$.

To infer $\Gamma; \Phi \models \mathbf{let}\, x : P = v(\vec{v}); c \colon \mathbf{nf}\,(N)$, we apply the corresponding algorithmic rule Rule ATAppLetAnn. Let us show that the premises hold:

1. $\Gamma \vdash P$,
2. $\Gamma; \Phi \models v \colon {\downarrow}\mathbf{nf}\,(M)$,
3. $\Gamma; \Phi; \cdot \models \mathbf{nf}\,(M) \bullet \vec{v} \Rrightarrow {\uparrow}Q_0 \dashv \Theta; SC_1$,
4. $\Gamma;\ \Theta \models {\uparrow}Q_0 \leqslant {\uparrow}P \dashv SC_2$, and
5. $\Gamma; \Phi, x : P \models c \colon \mathbf{nf}\,(N)$ hold as noted above;
6. $\Theta \vdash SC_1 \& SC_2 = SC$ is defined by lemma 81, since $\Theta \vdash \widehat{\sigma}_0 : SC_1$ and $\Theta \vdash \widehat{\sigma}_0 : SC_2$.

Case 12. Rule DTAppLet
By assumption, c is $\mathbf{let}\, x = v(\vec{v}); c'$. Then by inversion of $\Gamma; \Phi \vdash \mathbf{let}\, x = v(\vec{v}); c' \colon N$:

- $\Gamma; \Phi \vdash v \colon {\downarrow}M$, which by the induction hypothesis means $\Gamma; \Phi \models v \colon {\downarrow}\mathbf{nf}\,(M)$;
- $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow {\uparrow}Q$ unique. Then by lemma 84, since $\Gamma \vdash M \simeq^{\leqslant} \mathbf{nf}\,(M)$, we have $\Gamma; \Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rrightarrow {\uparrow}Q$ and moreover, $\Gamma; \Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rrightarrow {\uparrow}Q$ unique (since symmetrically, $\mathbf{nf}\,(M)$ can be replaced back by M). Then the induction hypothesis applied to $\Gamma; \Phi \vdash [\cdot]\mathbf{nf}\,(M) \bullet \vec{v} \Rrightarrow {\uparrow}Q$ implies that there exist M', Θ, and SC such that (considering M is ground):
  1. $\Gamma; \Phi; \cdot \models \mathbf{nf}\,(M) \bullet \vec{v} \Rrightarrow M' \dashv \Theta; SC$, which, by the soundness, implies, in particular that
     (a) $\Gamma; \mathbf{dom}\,(\Theta) \vdash M'$ is normalized and free of negative algorithmic variables,
     (b) $\Theta|_{\mathbf{uv}\,(M')} \vdash SC$, which means $\mathbf{dom}\,(SC) \subseteq \mathbf{uv}\,(M')$,
     (c) for any $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\, M'$ such that $\Theta \vdash \widehat{\sigma} : SC$, we have $\Gamma; \Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]M'$, which, since $\Gamma; \Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rrightarrow {\uparrow}Q$ unique, means $\Gamma \vdash [\widehat{\sigma}]M' \simeq^{\leqslant} {\uparrow}Q$.

     and
  2. for any $\Gamma \vdash M''$ such that $\Gamma; \Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rrightarrow M''$, (and in particular, for $\Gamma \vdash {\uparrow}Q$) there exists $\widehat{\sigma}_1$ such that
     (a) $\Theta \vdash \widehat{\sigma}_1 : \mathbf{uv}\, M'$, $\Theta \vdash \widehat{\sigma}_1 : SC$, and
     (b) $\Gamma \vdash [\widehat{\sigma}_1]M' \simeq^{\leqslant} M''$, and in particular, $\Gamma \vdash [\widehat{\sigma}_1]M' \simeq^{\leqslant} {\uparrow}Q$. Since M' is normalized and free of negative algorithmic variables, it means that $M' = {\uparrow}P$ for some P ($\Gamma; \mathbf{dom}\,(\Theta) \vdash P$) that is $\Gamma \vdash [\widehat{\sigma}_1]P \simeq^{\leqslant} Q$.
- $\Gamma; \Phi, x : Q \vdash c' \colon N$

To infer $\Gamma; \Phi \models \mathbf{let}\, x = v(\vec{v}); c' \colon \mathbf{nf}\,(N)$, let us apply the corresponding algorithmic rule (Rule ATAppLet):

1. $\Gamma; \Phi \models v \colon {\downarrow}\mathbf{nf}\,(M)$ holds as noted above;
2. $\Gamma; \Phi; \cdot \models \mathbf{nf}\,(M) \bullet \vec{v} \Rrightarrow {\uparrow}P \dashv \Theta; SC$ holds as noted above;
3. To show $\mathbf{uv}\, P = \mathbf{dom}\,(SC)$ and SC **singular with** $\widehat{\sigma}_0$ for some $\widehat{\sigma}_0$, we apply lemma 89 with $\Xi = \mathbf{uv}\, P = \mathbf{uv}\,(M')$ (as noted above, $\mathbf{dom}\,(SC) \subseteq \mathbf{uv}\,(M') = \Xi$).
   Now we will show that any substitution satisfying SC is equivalent to $\widehat{\sigma}_1$. As noted in 1c, for any substitution $\Theta \vdash \widehat{\sigma} : \Xi$, $\Theta \vdash \widehat{\sigma} : SC$ implies $\Gamma \vdash [\widehat{\sigma}]M' \simeq^{\leqslant} {\uparrow}Q$, which is rewritten as $\Gamma \vdash [\widehat{\sigma}]P \simeq^{\leqslant} Q$. And since $\Gamma \vdash [\widehat{\sigma}_1]P \simeq^{\leqslant} Q$, we have $\Gamma \vdash [\widehat{\sigma}]P \simeq^{\leqslant} [\widehat{\sigma}_1]P$, which implies $\Theta \vdash \widehat{\sigma} \simeq^{\leqslant} \widehat{\sigma}_1 : \Xi$ by corollary 23.
4. Let us show $\Gamma; \Phi, x : [\widehat{\sigma}_0]P \models c' \colon \mathbf{nf}\,(N)$. By the soundness of singularity (lemma 88), we have $\Theta \vdash \widehat{\sigma}_0 : SC$, which by 1c means $\Gamma \vdash [\widehat{\sigma}_0]M' \simeq^{\leqslant} {\uparrow}Q$, that is $\Gamma \vdash [\widehat{\sigma}_0]P \simeq^{\leqslant} Q$, and thus, $\Gamma \vdash \Phi, x : Q \simeq^{\leqslant} \Phi, x : [\widehat{\sigma}_0]P$.
   Then by lemma 85, $\Gamma; \Phi, x : Q \vdash c' \colon N$ can be rewritten as $\Gamma; \Phi, x : [\widehat{\sigma}_0]P \vdash c' \colon N$. Then by the induction hypothesis applied to it, $\Gamma; \Phi, x : [\widehat{\sigma}_0]P \models c' \colon \mathbf{nf}\,(N)$ holds.

Case 13. Rule DTForallApp
Since N cannot be a algorithmic variable, if $[\widehat{\sigma}]N$ starts with $\forall$, so does N. This way, $N = \forall\overrightarrow{\alpha^+}.N_1$. Then by assumption:

1. $\Gamma \vdash^{\supseteq} \Theta$
2. $\Gamma; \mathbf{dom}\,(\Theta) \vdash \forall\overrightarrow{\alpha^+}.N_1$ is free from negative algorithmic variables, and then $\Gamma, \overrightarrow{\alpha^+}; \mathbf{dom}\,(\Theta) \vdash N_1$ is free from negative algorithmic variables too;

3. $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\, N_1$;

4. $\Gamma \vdash M$;

5. $\Gamma; \Phi \vdash [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}. N_1 \bullet \overrightarrow{v} \Longrightarrow M$, that is $\Gamma; \Phi \vdash (\forall\overrightarrow{\alpha^+}.[\widehat{\sigma}] N_1) \bullet \overrightarrow{v} \Longrightarrow M$. Then by inversion there exists σ such that

   (a) $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}$;

   (b) $\overrightarrow{v} \neq \cdot$ and $\overrightarrow{\alpha^+} \neq \cdot$; and

   (c) $\Gamma; \Phi \vdash [\sigma][\widehat{\sigma}] N_1 \bullet \overrightarrow{v} \Longrightarrow M$. Notice that σ and $\widehat{\sigma}$ commute because the codomain of σ does not contain algorithmic variables (and thus, does not intersect with the domain of $\widehat{\sigma}$), and the codomain of $\widehat{\sigma}$ is Γ and does not intersect with $\overrightarrow{\alpha^+}$—the domain of σ.
   Let us construct $N_0 = [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}] N_1$ and $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}_0 : \mathbf{uv}\,(N_0)$ defined as

$$\begin{cases} [\widehat{\sigma}_0]\widehat{\alpha_i}^+ = [\sigma]\alpha_i^+ & \text{for } \widehat{\alpha_i}^+ \in \overrightarrow{\widehat{\alpha}^+} \cap \mathbf{uv}\, N_0 \\ [\widehat{\sigma}_0]\widehat{\beta}^\pm = [\widehat{\sigma}]\widehat{\beta}^\pm & \text{for } \widehat{\beta}^\pm \in \mathbf{uv}\, N_1 \end{cases}$$

   Then it is easy to see that $[\widehat{\sigma}_0][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}] N_1 = [\sigma][\widehat{\sigma}] N_1$ because this substitution compositions coincide on $\mathbf{uv}\,(N_1) \cup \mathbf{fv}\,(N_1)$. In other words, $[\widehat{\sigma}_0] N_0 = [\sigma][\widehat{\sigma}] N_1$.
   Then let us apply the induction hypothesis to $\Gamma; \Phi \vdash [\widehat{\sigma}_0] N_0 \bullet \overrightarrow{v} \Longrightarrow M$ and obtain M', Θ', and SC such that

   - $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash N_0 \bullet \overrightarrow{v} \Longrightarrow M' \dashv \Theta'; SC$ and
   - for any $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}_0 : \mathbf{uv}\,(N_0)$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\widehat{\sigma}_0] N_0 \bullet \overrightarrow{v} \Longrightarrow M$, there exists $\widehat{\sigma}'_0$ such that

     i. $\Theta' \vdash \widehat{\sigma}'_0 : \mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M')$, $\Theta' \vdash \widehat{\sigma}'_0 : SC$,

     ii. $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}'_0 \simeq^\leqslant \widehat{\sigma}_0 : \mathbf{uv}\, N_0$, and

     iii. $\Gamma \vdash [\widehat{\sigma}'_0] M' \simeq^\leqslant M$.

Let us take M', Θ', and SC from the induction hypothesis (5c) (from SC we subtract entries restricting $\overrightarrow{\widehat{\alpha}^+}$) and show they satisfy the required properties

1. To infer $\Gamma; \Phi; \Theta \vDash \forall\overrightarrow{\alpha^+}. N_1 \bullet \overrightarrow{v} \Longrightarrow M' \dashv \Theta'; SC\backslash\overrightarrow{\widehat{\alpha}^+}$ we apply the corresponding algorithmic rule Rule ATForallApp. As noted above, the required premises hold:

   (a) $\overrightarrow{v} \neq \cdot$, $\overrightarrow{\alpha^+} \neq \cdot$; and

   (b) $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}] N_1 \bullet \overrightarrow{v} \Longrightarrow M' \dashv \Theta'; SC$ is obtained by unfolding the definition of N_0 in $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash N_0 \bullet \overrightarrow{v} \Longrightarrow M' \dashv \Theta'; SC$ (5c).

2. Let us take and arbitrary $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\, N_1$ and $\Gamma \vdash M$ and assume $\Gamma; \Phi \vdash [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}. N_1 \bullet \overrightarrow{v} \Longrightarrow M$. Then the same reasoning as in 5c applies. In particular, we construct $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}_0 : \mathbf{uv}\,(N_0)$ as an extension of $\widehat{\sigma}$ and obtain $\Gamma; \Phi \vdash [\widehat{\sigma}_0] N_0 \bullet \overrightarrow{v} \Longrightarrow M$.
   It means we can apply the property inferred from the induction hypothesis (5c) to obtain $\widehat{\sigma}'_0$ such that

   (a) $\Theta' \vdash \widehat{\sigma}'_0 : \mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M')$ and $\Theta' \vdash \widehat{\sigma}'_0 : SC$,

   (b) $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}'_0 \simeq^\leqslant \widehat{\sigma}_0 : \mathbf{uv}\, N_0$, and

   (c) $\Gamma \vdash [\widehat{\sigma}'_0] M' \simeq^\leqslant M$.

   Let us show that $\widehat{\sigma}'_0|_{(\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))}$ satisfies the required properties.

   (a) $\Theta' \vdash \widehat{\sigma}'_0|_{(\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))} : (\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))$ holds since $\Theta' \vdash \widehat{\sigma}'_0 : \mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M')$ and $\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M') \subseteq \mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M')$; $\Theta' \vdash \widehat{\sigma}'_0|_{(\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))} : SC\backslash\overrightarrow{\widehat{\alpha}^+}$ holds since $\Theta' \vdash \widehat{\sigma}'_0 : SC$, $\Theta' \vdash \widehat{\sigma}'_0 : \mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M')$, and $(\mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M'))\backslash\overrightarrow{\widehat{\alpha}^+} = \mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M')$.

   (b) $\Gamma \vdash [\widehat{\sigma}'_0] M' \simeq^\leqslant M$ holds as shown, and hence it holds for $\widehat{\sigma}'_0|_{(\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))}$;

   (c) We show $\Theta \vdash \widehat{\sigma}'_0 \simeq^\leqslant \widehat{\sigma} : \mathbf{uv}\, N_1$, from which it follows that it holds for $\widehat{\sigma}'_0|_{(\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))}$. Let us take an arbitrary $\widehat{\beta}^\pm \in \mathbf{dom}\,(\Theta) \subseteq \mathbf{dom}\,(\Theta) \cup \overrightarrow{\widehat{\alpha}^+}$. Then since $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}'_0 \simeq^\leqslant \widehat{\sigma}_0 : \mathbf{uv}\, N_0$, we have $\Theta(\widehat{\beta}^\pm) \vdash [\widehat{\sigma}'_0]\widehat{\beta}^\pm \simeq^\leqslant [\widehat{\sigma}_0]\widehat{\beta}^\pm$ and by definition of $\widehat{\sigma}_0$, $[\widehat{\sigma}_0]\widehat{\beta}^\pm = [\widehat{\sigma}]\widehat{\beta}^\pm$.

Case 14. Rule DTArrowApp
Since N cannot be a algorithmic variable, if the shape of $[\widehat{\sigma}]N$ is an arrow, so is the shape of N. This way, $N = Q \to N_1$. Then by assumption:

1. $\Gamma \vdash^{\supseteq} \Theta$;

2. $\Gamma; \mathbf{dom}\,(\Theta) \vdash Q \to N_1$ is free from negative algorithmic variables;

3. $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\, Q \cup \mathbf{uv}\, N_1$;

4. $\Gamma \vdash M$;
5. $\Gamma; \Phi \vdash [\widehat{\sigma}](Q \to N_1) \bullet v, \vec{v} \Rrightarrow M$, that is $\Gamma; \Phi \vdash ([\widehat{\sigma}]Q \to [\widehat{\sigma}]N_1) \bullet v, \vec{v} \Rrightarrow M$, and by inversion:
   (a) $\Gamma; \Phi \vdash v : P$, and by the induction hypothesis, $\Gamma; \Phi \vDash v : \mathbf{nf}(P)$;
   (b) $\Gamma \vdash [\widehat{\sigma}]Q \geqslant P$, which by transitivity (lemma 24) means $\Gamma \vdash [\widehat{\sigma}]Q \geqslant \mathbf{nf}(P)$, and then by completeness of subtyping (lemma 77), $\Gamma; \Theta \vDash Q \geqslant \mathbf{nf}(P) \dashv SC_1$, for some $\Theta \vdash SC_1 : \mathbf{uv}(Q)$, and moreover, $\Theta \vdash \widehat{\sigma} : SC_1$;
   (c) $\Gamma; \Phi \vdash [\widehat{\sigma}]N_1 \bullet \vec{v} \Rrightarrow M$. Notice that the induction hypothesis is applicable to this case: $\Gamma; \mathbf{dom}(\Theta) \vdash N_1$ is free from negative algorithmic variables because so is $Q \to N_1$. This way, there exist M', Θ', and SC_2 such that
      i. $\Gamma; \Phi; \Theta \vDash N_1 \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC_2$ and then by soundness of typing (i.e. the induction hypothesis),
         A. $\Theta \subseteq \Theta'$
         B. $\Gamma; \mathbf{dom}(\Theta') \vdash M'$
         C. $\mathbf{dom}(\Theta) \cap \mathbf{uv}(M') \subseteq \mathbf{uv}\, N_1$
         D. $\Theta'|_{\mathbf{uv}\, N_1 \cup \mathbf{uv}\, M'} \vdash SC_2$
      ii. for any $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(N_1)$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\widehat{\sigma}]N_1 \bullet \vec{v} \Rrightarrow M$, there exists $\widehat{\sigma}'$ such that
         A. $\Theta' \vdash \widehat{\sigma}' : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ and $\Theta' \vdash \widehat{\sigma}' : SC_2$,
         B. $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(N_1)$, and
         C. $\Gamma \vdash [\widehat{\sigma}']M' \simeq^{\leqslant} M$.

We need to show that there exist M', Θ', and SC such that $\Gamma; \Phi; \Theta \vDash Q \to N_1 \bullet v, \vec{v} \Rrightarrow M' \dashv \Theta'; SC$ and the initiality property holds. We take M' and Θ' from the induction hypothesis (5c), and SC as a merge of SC_1 and SC_2. To show that $\Theta' \vdash SC_1 \& SC_2 = SC$ exists, we apply lemma 81. To do so, we need to provide a substitution satisfying both SC_1 and SC_2.

Notice that $\mathbf{dom}(SC_1) = \mathbf{uv}(Q)$ and $\mathbf{dom}(SC_2) \subseteq \mathbf{uv}\, N_1 \cup \mathbf{uv}\, M'$. This way, it suffices to construct $\Theta' \vdash \widehat{\sigma}'' : \mathbf{uv}(Q) \cup \mathbf{uv}\, N_1 \cup \mathbf{uv}\, M'$ such that $\Theta' \vdash \widehat{\sigma}'' : SC_1$ and $\Theta' \vdash \widehat{\sigma}'' : SC_2$.

By the induction hypothesis (5(c)ii), $\widehat{\sigma}|_{\mathbf{uv}(N_1)}$ can be extended to $\widehat{\sigma}'$ such that

1. $\Theta' \vdash \widehat{\sigma}' : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ and $\Theta' \vdash \widehat{\sigma}' : SC_2$,
2. $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(N_1)$, and
3. $\Gamma \vdash [\widehat{\sigma}']M' \simeq^{\leqslant} M$.

Let us extend $\widehat{\sigma}'$ to $\widehat{\sigma}''$ defined on $\mathbf{uv}(Q) \cup \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ with values of $\widehat{\sigma}$ as follows:

$$\begin{cases} [\widehat{\sigma}'']\widehat{\beta}^{\pm} = [\widehat{\sigma}']\widehat{\beta}^{\pm} & \text{for } \widehat{\beta}^{\pm} \in \mathbf{uv}(N_1) \cup \mathbf{uv}(M') \\ [\widehat{\sigma}'']\widehat{\gamma}^{\pm} = [\widehat{\sigma}]\widehat{\gamma}^{\pm} & \text{for } \widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \backslash (\mathbf{uv}(N_1) \cup \mathbf{uv}(M')) \end{cases}$$

First, notice that $\Theta' \vdash \widehat{\sigma}'' \simeq^{\leqslant} \widehat{\sigma}' : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ by definition. Then since $\Theta' \vdash \widehat{\sigma}' : SC_2$ and $\Theta' \vdash SC_2 : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$, we have $\Theta' \vdash \widehat{\sigma}'' : SC_2$.

Second, notice that $\Theta \vdash \widehat{\sigma}'' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(N_1) \cup \mathbf{uv}(Q)$:

- if $\widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \backslash (\mathbf{uv}(N_1) \cup \mathbf{uv}(M'))$ then $[\widehat{\sigma}'']\widehat{\gamma}^{\pm} = [\widehat{\sigma}]\widehat{\gamma}^{\pm}$ by definition of $\widehat{\sigma}''$;
- if $\widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \cap \mathbf{uv}(N_1)$ then $[\widehat{\sigma}'']\widehat{\gamma}^{\pm} = [\widehat{\sigma}']\widehat{\gamma}^{\pm}$, and $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(N_1)$, as noted above;
- if $\widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \cap \mathbf{uv}(M')$ then since $\Gamma; \mathbf{dom}(\Theta) \vdash Q$, we have $\mathbf{uv}(Q) \subseteq \mathbf{dom}(\Theta)$, implying $\widehat{\gamma}^{\pm} \in \mathbf{dom}(\Theta) \cap \mathbf{uv}(M') \subseteq \mathbf{uv}(N_1)$. This way, $\widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \cap \mathbf{uv}(N_1)$, and this case is covered by the previous one.

In particular, $\Theta \vdash \widehat{\sigma}'' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(Q)$. Then since $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash SC_1 : \mathbf{uv}(Q)$, we have $\Theta \vdash \widehat{\sigma}'' : SC_1$.

This way, $\widehat{\sigma}'$ satisfies both SC_1 and SC_2, and by the completeness of constraint merge (lemma 81), $\Theta' \vdash SC_1 \& SC_2 = SC$ exists.

Finally, to show the required properties, we take M' and Θ' from the induction hypothesis (5(c)ii), and SC defined above. Then

1. $\Gamma; \Phi; \Theta \vDash Q \to N_1 \bullet v, \vec{v} \Rrightarrow M' \dashv \Theta'; SC$ is inferred by Rule ATArrowApp. As noted above:
   (a) $\Gamma; \Phi \vDash v : \mathbf{nf}(P)$,
   (b) $\Gamma; \Theta \vDash Q \geqslant \mathbf{nf}(P) \dashv SC_1$,
   (c) $\Gamma; \Phi; \Theta \vDash N_1 \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC_2$, and
   (d) $\Theta' \vdash SC_1 \& SC_2 = SC$.
2. let us take an arbitrary $\Theta \vdash \widehat{\sigma}_0 : \mathbf{uv}\, Q \cup \mathbf{uv}\, N_1$; and $\Gamma \vdash M_0$; such that $\Gamma; \Phi \vdash [\widehat{\sigma}_0](Q \to N_1) \bullet v, \vec{v} \Rrightarrow M_0$. Then by inversion of $\Gamma; \Phi \vdash [\widehat{\sigma}_0]Q \to [\widehat{\sigma}_0]N_1 \bullet v, \vec{v} \Rrightarrow M_0$, we have the same properties as in 5. In particular,
   - $\Gamma \vdash [\widehat{\sigma}_0]Q \geqslant \mathbf{nf}(P)$ and by the completeness of subtyping (lemma 77), $\Theta \vdash \widehat{\sigma}_0 : SC_1$.

- $\Gamma; \Phi \vdash [\widehat{\sigma}_0] N_1 \bullet \vec{v} \Rrightarrow M_0$. Then by 5(c)ii, there exists $\widehat{\sigma}'_0$ such that
  (a) $\Theta' \vdash \widehat{\sigma}'_0 : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ and $\Theta' \vdash \widehat{\sigma}'_0 : SC_2$,
  (b) $\Theta \vdash \widehat{\sigma}'_0 \simeq^{\leqslant} \widehat{\sigma}_0 : \mathbf{uv}(N_1)$, and
  (c) $\Gamma \vdash [\widehat{\sigma}'_0] M' \simeq^{\leqslant} M_0$.

Let us extend $\widehat{\sigma}'_0$ to be defined on $\mathbf{uv}(Q) \cup \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ with the values of $\widehat{\sigma}_0$. We define $\widehat{\sigma}''_0$ as follows:

$$\begin{cases} [\widehat{\sigma}''_0]\widehat{\gamma}^{\pm} = [\widehat{\sigma}'_0]\widehat{\gamma}^{\pm} & \text{for } \widehat{\gamma}^{\pm} \in \mathbf{uv}(N_1) \cup \mathbf{uv}(M') \\ [\widehat{\sigma}''_0]\widehat{\gamma}^{\pm} = [\widehat{\sigma}_0]\widehat{\gamma}^{\pm} & \text{for } \widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \backslash (\mathbf{uv}(N_1) \cup \mathbf{uv}(M')) \end{cases}$$

This way,

- $\Theta' \vdash \widehat{\sigma}''_0 : \mathbf{uv}(Q) \cup \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$,
- $\Theta' \vdash \widehat{\sigma}''_0 : SC$, since $\Theta' \vdash \widehat{\sigma}''_0 : SC_1$ and $\Theta' \vdash \widehat{\sigma}''_0 : SC_2$, which is proved similarly to $\Theta' \vdash \widehat{\sigma}'' : SC_1$ and $\Theta' \vdash \widehat{\sigma}'' : SC_2$ above;
- $\Theta \vdash \widehat{\sigma}''_0 \simeq^{\leqslant} \widehat{\sigma}_0 : \mathbf{uv}(N_1) \cup \mathbf{uv}(Q)$: the proof is analogous to $\Theta \vdash \widehat{\sigma}'' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(N_1) \cup \mathbf{uv}(Q)$ above.
- $\Gamma \vdash [\widehat{\sigma}''_0] M' \simeq^{\leqslant} M_0$ Notice that $\Theta' \vdash \widehat{\sigma}''_0 \simeq^{\leqslant} \widehat{\sigma}'_0 : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$, which is proved analogously to $\Theta' \vdash \widehat{\sigma}'' \simeq^{\leqslant} \widehat{\sigma}' : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ above. Then $\Gamma \vdash [\widehat{\sigma}'_0] M' \simeq^{\leqslant} M_0$ can be rewritten to $\Gamma \vdash [\widehat{\sigma}''_0] M' \simeq^{\leqslant} M_0$.

Case 15. Rule DTEmptyApp
By assumption:

1. $\Gamma \vdash^{\supseteq} \Theta$,
2. $\Gamma \vdash N'$,
3. $\Gamma; \mathbf{dom}(\Theta) \vdash N$ and N is free from negative variables,
4. $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(N)$,
5. $\Gamma; \Phi \vdash [\widehat{\sigma}] N \bullet \cdot \Rrightarrow N'$, and by inversion, $\Gamma \vdash [\widehat{\sigma}] N \simeq^{\leqslant} N'$.

Then we can apply the corresponding algorithmic rule Rule ATEmptyApp to infer $\Gamma; \Phi; \Theta \models N \bullet \cdot \Rrightarrow \mathbf{nf}(N) \dashv \Theta; \cdot$. Let us show the required properties. Let us take an arbitrary $\Theta \vdash \widehat{\sigma}_0 : \mathbf{uv}(N)$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\widehat{\sigma}_1] N \bullet \cdot \Rrightarrow M$. Then we can take $\widehat{\sigma}_0$ as the required substitution:

1. $\Theta \vdash \widehat{\sigma}_0 : \mathbf{uv}(N) \cup \mathbf{uv}(\mathbf{nf}(N))$, since $\mathbf{uv}(\mathbf{nf}(N)) = \mathbf{uv}(N)$, and thus, $\mathbf{uv}(N) \cup \mathbf{uv}(\mathbf{nf}(N)) = \mathbf{uv}(N)$;
2. $\Theta \vdash \widehat{\sigma}_0 : \cdot$ vacuously;
3. $\Theta \vdash \widehat{\sigma}_0 \simeq^{\leqslant} \widehat{\sigma}_0 : \mathbf{uv}(N)$ by reflexivity;
4. Let us show $\Gamma \vdash [\widehat{\sigma}_0]\mathbf{nf}(N) \simeq^{\leqslant} M$. Notice that $\Gamma; \Phi \vdash [\widehat{\sigma}_0] N \bullet \cdot \Rrightarrow M$ can only be inferred by Rule DTEmptyApp, and thus, $\Gamma \vdash [\widehat{\sigma}_0] N \simeq^{\leqslant} M$. By corollary 19, $\Gamma \vdash [\widehat{\sigma}_0] N \simeq^{\leqslant} [\widehat{\sigma}_0]\mathbf{nf}(N)$, and then by transitivity, $\Gamma \vdash [\widehat{\sigma}_0]\mathbf{nf}(N) \simeq^{\leqslant} M$, that is $\Gamma \vdash [\widehat{\sigma}_0]\mathbf{nf}(N) \simeq^{\leqslant} M$.

□