α, β, α, β, γ, δ	type variables
n, m, i, j	index variables

$n,\ m$	$::=$		cohort index
	$\mid$	0	
	$\mid$	$n+1$	
$\alpha^+,\ \beta^+,\ \gamma^+,\ \delta^+$	$::=$		positive variable
	$\mid$	α^+	
	$\mid$	α^{+n}	
	$\mid$	$\beta^{\pm}$	
$\alpha^-,\ \beta^-,\ \gamma^-,\ \delta^-$	$::=$		negative variable
	$\mid$	α^-	
	$\mid$	α^{-n}	
	$\mid$	$\alpha^{\pm}$	
$\alpha^{\pm},\ \beta^{\pm}$	$::=$		positive or negative variable
	$\mid$	$\alpha^{\pm}$	
	$\mid$	$\alpha^{\pm n}$	
σ	$::=$		substitution
	$\mid$	$\cdot$	
	$\mid$	id	
	$\mid$	P/α^+	
	$\mid$	N/α^-	
	$\mid$	$\vec{P}/\overrightarrow{\alpha^+}$	
	$\mid$	$\vec{N}/\overrightarrow{\alpha^-}$	
	$\mid$	$\mathbf{pmas}\,/\overrightarrow{\alpha^+}$	
	$\mid$	$\mathbf{nmas}\,/\overrightarrow{\alpha^-}$	
	$\mid$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$	
	$\mid$	$\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}$	
	$\mid$	$\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha^-}}$	
	$\mid$	μ	
	$\mid$	$\widehat{\sigma}$	
	$\mid$	$\sigma_1 \circ \sigma_2$	
	$\mid$	$\vec{\alpha}_1/\vec{\alpha}_2$	
	$\mid$	(σ)	S
	$\mid$	$\overline{\sigma_i}^{\,i}$	concatenate
	$\mid$	$\mathbf{nf}\,(\sigma')$	M
	$\mid$	$\sigma'\vert_{vars}$	M
e	$::=$		entry of a unification solution
	$\mid$	$\widehat{\alpha}^+ :\approx P$	
	$\mid$	$\widehat{\alpha}^- :\approx N$	
	$\mid$	$\widehat{\alpha}^+ :\geqslant P$	
	$\mid$	(e)	S
	$\mid$	$\widehat{\sigma}(\widehat{\alpha}^+)$	M
	$\mid$	$\widehat{\sigma}(\widehat{\alpha}^-)$	M
	$\mid$	$e_1\ \&\ e_2$	M
$\widehat{\sigma}$	$::=$		unification solution (substitution)

	|	$\cdot$		
	|	e		
	|	$\widehat{\sigma} \backslash vars$		
	|	$\widehat{\sigma} \vert vars$		
	|	$\widehat{\sigma}_1 \cup \widehat{\sigma}_2$		
	|	$\overline{\widehat{\sigma}_i}^{\,i}$		concatenate
	|	$(\widehat{\sigma})$	S	
	|	$\mathbf{nf}\,(\widehat{\sigma}')$	M	
	|	$\widehat{\sigma}' \vert_{vars}$	M	
	|	$\widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2$	M	
$\widehat{\tau}$	::=			anti-unification substitution
	|	$\cdot$		
	|	$\widehat{\alpha}^- :\approx N$		
	|	$\widehat{\alpha}^- :\approx N$		
	|	$\overrightarrow{\alpha^-} / \overrightarrow{\widehat{\alpha}^-}$		
	|	$\overrightarrow{N} / \overrightarrow{\widehat{\alpha}^-}$		
	|	$\widehat{\tau}_1 \cup \widehat{\tau}_2$		
	|	$\overline{\widehat{\tau}_i}^{\,i}$		concatenate
	|	$(\widehat{\tau})$	S	
$P,\ Q$	::=			positive types
	|	α^+		
	|	$\downarrow N$		
	|	$\exists \alpha^-.P$		
	|	$[\sigma]P$	M	
$N,\ M$	::=			negative types
	|	α^-		
	|	$\uparrow P$		
	|	$\forall \alpha^+.N$		
	|	$P \to N$		
	|	$[\sigma]N$	M	
$\overrightarrow{\alpha^+},\ \overrightarrow{\beta^+},\ \overrightarrow{\gamma^+},\ \overrightarrow{\delta^+}$	::=			positive variable list
	|	$\cdot$		empty list
	|	α^+		a variable
	|	$\overrightarrow{\alpha^+}$		a variable
	|	$\overline{\overrightarrow{\alpha^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^-},\ \overrightarrow{\beta^-},\ \overrightarrow{\gamma^-},\ \overrightarrow{\delta^-}$	::=			negative variables
	|	$\cdot$		empty list
	|	α^-		a variable
	|	$\overrightarrow{\alpha^-}$		variables
	|	$\overline{\overrightarrow{\alpha^-}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^\pm},\ \overrightarrow{\beta^\pm},\ \overrightarrow{\gamma^\pm},\ \overrightarrow{\delta^\pm}$	::=			positive or negative variable list
	|	$\cdot$		empty list
	|	$\alpha^\pm$		a variable

	|	$\overrightarrow{\mathbf{pa}}$		variables
	|	$\overline{\overrightarrow{\alpha^{\pm}}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$			multi-quantified positive types
	|	α^{+}		
	|	$\downarrow N$		
	|	$\exists \overrightarrow{\alpha^{-}}.P$		$P \neq \exists \ldots$
	|	$[\sigma]P$	M	
	|	$[\widehat{\tau}]P$	M	
	|	$[\widehat{\sigma}]P$	M	
	|	$[\mu]P$	M	
	|	(P)	S	
	|	$P_1 \vee P_2$	M	
	|	$\mathbf{nf}\,(P')$	M	
$N,\ M$	$::=$			multi-quantified negative types
	|	α^{-}		
	|	$\uparrow P$		
	|	$P \to N$		
	|	$\forall \overrightarrow{\alpha^{+}}.N$		$N \neq \forall \ldots$
	|	$[\sigma]N$	M	
	|	$[\mu]N$	M	
	|	$[\widehat{\sigma}]N$	M	
	|	(N)	S	
	|	$\mathbf{nf}\,(N')$	M	
$\vec{P},\ \vec{Q}$	$::=$			list of positive types
	|	$\cdot$		empty list
	|	P		a singel type
	|	$\overline{\vec{P}_i}^{\,i}$		concatenate lists
	|	$\mathbf{nf}\,(\vec{P}')$	M	
$\vec{N},\ \vec{M}$	$::=$			list of negative types
	|	$\cdot$		empty list
	|	N		a singel type
	|	$\overline{\vec{N}_i}^{\,i}$		concatenate lists
	|	$\mathbf{nf}\,(\vec{N}')$	M	
$\Delta,\ \Gamma$	$::=$			declarative type context
	|	$\cdot$		empty context
	|	$\overrightarrow{\alpha^{+}}$		list of variables
	|	$\overrightarrow{\alpha^{-}}$		list of variables
	|	$\overrightarrow{\alpha^{\pm}}$		list of variables
	|	$vars$		
	|	$\overline{\Gamma_i}^{\,i}$		concatenate contexts
	|	(Γ)	S	
	|	$\Theta(\widehat{\alpha}^{+})$	M	
	|	$\Theta(\widehat{\alpha}^{-})$	M	

Θ	::=			unification type variable context
	|	$\cdot$		empty context
	|	$\overrightarrow{\widetilde{\alpha}^{+}}$		list of variables
	|	$\overrightarrow{\widehat{\alpha}^{-}}$		list of variables
	|	$vars$		
	|	$\overline{\Theta_i}^{\,i}$		concatenate contexts
	|	(Θ)	S	
	|	$\Theta\|_{vars}$		leave only those variables that are in the set
	|	$\Theta_1 \cup \Theta_2$		
Ξ	::=			anti-unification type variable context
	|	$\cdot$		empty context
	|	$\overrightarrow{\widetilde{\alpha}^{+}}$		list of variables
	|	$\overrightarrow{\widehat{\alpha}^{-}}$		list of variables
	|	$\overline{\Xi_i}^{\,i}$		concatenate contexts
	|	(Ξ)	S	
	|	$\Xi_1 \cup \Xi_2$		
$\vec{\alpha},\ \vec{\beta}$	::=			ordered positive or negative variables
	|	$\cdot$		empty list
	|	$\overrightarrow{\alpha^{+}}$		list of variables
	|	$\overrightarrow{\alpha^{-}}$		list of variables
	|	$\overrightarrow{\alpha^{\pm}}$		list of variables
	|	$\overrightarrow{\widetilde{\alpha}^{+}}$		list of variables
	|	$\overrightarrow{\widehat{\alpha}^{-}}$		list of variables
	|	$\vec{\alpha}_1 \backslash vars$		setminus
	|	Γ		context
	|	$vars$		
	|	$\overline{\vec{\alpha}_i}^{\,i}$		concatenate contexts
	|	$(\vec{\alpha})$	S	parenthesis
	|	$[\mu]\vec{\alpha}$	M	apply moving to list
	|	$\mathbf{ord}\ vars\ \mathbf{in}\ P$	M	
	|	$\mathbf{ord}\ vars\ \mathbf{in}\ N$	M	
	|	$\mathbf{ord}\ vars\ \mathbf{in}\ P$	M	
	|	$\mathbf{ord}\ vars\ \mathbf{in}\ N$	M	
$vars$	::=			set of variables
	|	$\varnothing$		empty set
	|	$\mathbf{fv}\ P$		free variables
	|	$\mathbf{fv}\ N$		free variables
	|	$\mathbf{fv}\ \mathbf{imP}$		free variables
	|	$\mathbf{fv}\ \mathbf{imN}$		free variables
	|	$vars_1 \cap vars_2$		set intersection
	|	$vars_1 \cup vars_2$		set union
	|	$vars_1 \backslash vars_2$		set complement
	|	$\mathbf{mv}\ \mathbf{imP}$		movable variables
	|	$\mathbf{mv}\ \mathbf{imN}$		movable variables
	|	$\mathbf{uv}\ N$		unification variables
	|	$\mathbf{uv}\ P$		unification variables
	|	$\mathbf{fv}\ N$		free variables

	$\mid$	$\mathbf{fv}\ P$		free variables
	$\mid$	$(vars)$	S	parenthesis
	$\mid$	$\overrightarrow{\alpha}$		ordered list of variables
	$\mid$	$[\mu]vars$	M	apply moving to varset
	$\mid$	$\mathbf{dom}\,(\widehat{\sigma})$	M	
	$\mid$	$\mathbf{dom}\,(\Theta)$	M	
μ	$::=$			
	$\mid$	$\cdot$		empty moving
	$\mid$	$pma1 \mapsto pma2$		Positive unit substitution
	$\mid$	$nma1 \mapsto nma2$		Positive unit substitution
	$\mid$	$\mu_1 \cup \mu_2$	M	Set-like union of movings
	$\mid$	$\mu_1 \circ \mu_2$	M	Composition
	$\mid$	$\overline{\mu_i}^{\,i}$		concatenate movings
	$\mid$	$\mu\vert_{vars}$	M	restriction on a set
	$\mid$	μ^{-1}	M	inversion
	$\mid$	$\mathbf{nf}\,(\mu')$	M	
$\widehat{\alpha}^{\pm}$	$::=$			positive/negative unification variable
	$\mid$	$\widehat{\alpha}^{\pm}$		
$\widehat{\alpha}^{+}$	$::=$			positive unification variable
	$\mid$	$\widehat{\alpha}^{+}$		
	$\mid$	$\widehat{\alpha}^{+}\{\Delta\}$		
	$\mid$	$\widehat{\alpha}^{\pm}$		
$\widehat{\alpha}^{-},\ \widehat{\beta}^{-}$	$::=$			negative unification variable
	$\mid$	$\widehat{\alpha}^{-}$		
	$\mid$	$\widehat{\alpha}^{-}_{\{N,M\}}$		
	$\mid$	$\widehat{\alpha}^{-}\{\Delta\}$		
	$\mid$	$\widehat{\alpha}^{\pm}$		
$\overrightarrow{\widetilde{\alpha}^{+}},\ \overrightarrow{\widetilde{\beta}^{+}}$	$::=$			positive unification variable list
	$\mid$	$\cdot$		empty list
	$\mid$	$\widehat{\alpha}^{+}$		a variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^{+}\{\Delta\}}$		from a normal variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^{+}}$		from a normal variable, context unspecified
	$\mid$	$\overline{\overrightarrow{\widetilde{\alpha}^{+}}_i}^{\,i}$		concatenate lists
$\overrightarrow{\overset{\frown}{\alpha}^{-}},\ \overrightarrow{\overset{\frown}{\beta}^{-}}$	$::=$			negative unification variable list
	$\mid$	$\cdot$		empty list
	$\mid$	$\widehat{\alpha}^{-}$		a variable
	$\mid$	Ξ		from an antiunification context
	$\mid$	$\overrightarrow{\widehat{\alpha}^{-}\{\Delta\}}$		from a normal variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^{-}}$		from a normal variable, context unspecified
	$\mid$	$\overline{\overrightarrow{\overset{\frown}{\alpha}^{-}}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$			a positive algorithmic type (potentially with metavariables)
	$\mid$	α^{+}		

	$\mid$	**pma**		
	$\mid$	$\widehat{\alpha}^{+}$		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists \overrightarrow{\alpha^{-}}. P$		
	$\mid$	$[\sigma] P$	M	
	$\mid$	$[\widehat{\tau}] P$	M	
	$\mid$	$[\mu] P$	M	
	$\mid$	(P)	S	
	$\mid$	$\mathbf{nf}\,(P')$	M	
N, M	$::=$			a negative algorithmic type (potentially with metavariables)
	$\mid$	α^{-}		
	$\mid$	$\widehat{\alpha}^{-}$		
	$\mid$	$\uparrow P$		
	$\mid$	$P \to N$		
	$\mid$	$\forall \overrightarrow{\alpha^{+}}. N$		
	$\mid$	$[\sigma] N$	M	
	$\mid$	$[\mu] N$	M	
	$\mid$	(N)	S	
	$\mid$	$\mathbf{nf}\,(N')$	M	
auSol	$::=$			
	$\mid$	$(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$		
terminals	$::=$			
	$\mid$	$\exists$		
	$\mid$	$\forall$		
	$\mid$	$\uparrow$		
	$\mid$	$\downarrow$		
	$\mid$	$\rightarrow$		
	$\mid$	$\leftrightarrow$		
	$\mid$	$\in$		
	$\mid$	$\notin$		
	$\mid$	$\cdot$		
	$\mid$	$\vdash$		
	$\mid$	$\leqslant$		
	$\mid$	$\geqslant$		
	$\mid$	$\simeq$		
	$\mid$	$\cup$		
	$\mid$	$\cap$		
	$\mid$	$\setminus$		
	$\mid$	$\subseteq$		
	$\mid$	$\mapsto$		
	$\mid$	$\stackrel{u}{\simeq}$		
	$\mid$	$\stackrel{a}{\simeq}$		
	$\mid$	$\varnothing$		
	$\mid$	$\circ$		
	$\mid$	$\Rightarrow$		
	$\mid$	$\models$		
	$\mid$	$\Dashv$		

| $\neq$
| $\equiv_n$
| $\vee$
| $\Downarrow$
| $:\geqslant$
| $:\simeq$

formula ::=
| *judgement*
| $formula_1 \;\; .. \;\; formula_n$
| $\mu : vars_1 \leftrightarrow vars_2$
| μ **is bijective**
| $\widehat{\sigma}$ **is functional**
| $\widehat{\sigma}_1 \in \widehat{\sigma}_2$
| $\widehat{\sigma}_1 \subseteq \widehat{\sigma}_2$
| $vars_1 \subseteq vars_2$
| $vars_1 = vars_2$
| $vars$ **is fresh**
| $\alpha^- \notin vars$
| $\alpha^+ \notin vars$
| $\alpha^- \in vars$
| $\alpha^+ \in vars$
| $\widehat{\alpha}^- \in \Theta$
| $\widehat{\alpha}^+ \in \Theta$
| if any other rule is not applicable
| $\vec{\alpha}_1 = \vec{\alpha}_2$
| $e_1 = e_2$
| $N \neq M$
| $P \neq Q$

A ::=
| $\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}$ — Negative subtyping
| $\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}$ — Positive supertyping

AU ::=
| $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$
| $\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$

E1 ::=
| $N \simeq_1^D M$ — Negative multi-quantified type equivalence
| $P \simeq_1^D Q$ — Positive multi-quantified type equivalence
| $P \simeq Q$

D1 ::=
| $\Gamma \vdash N \simeq_1^{\leqslant} M$ — Negative equivalence on MQ types
| $\Gamma \vdash P \simeq_1^{\leqslant} Q$ — Positive equivalence on MQ types
| $\Gamma \vdash N \leqslant_1 M$ — Negative subtyping
| $\Gamma \vdash P \geqslant_1 Q$ — Positive supertyping
| $\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1$ — Equivalence of substitutions

D0	::=		
	|	$\Gamma \vdash N \simeq^{\leqslant}_0 M$	Negative equivalence
	|	$\Gamma \vdash P \simeq^{\leqslant}_0 Q$	Positive equivalence
	|	$\Gamma \vdash N \leqslant_0 M$	Negative subtyping
	|	$\Gamma \vdash P \geqslant_0 Q$	Positive supertyping
EQ	::=		
	|	$N = M$	Negative type equality (alpha-equivalence)
	|	$P = Q$	Positive type equuality (alphha-equivalence)
	|	$P = Q$	
LUBF	::=		
	|	$P_1 \vee P_2 === Q$	
	|	$\textbf{ord}\, vars\, \textbf{in}\, P === \vec{\alpha}$	
	|	$\textbf{ord}\, vars\, \textbf{in}\, N === \vec{\alpha}$	
	|	$\textbf{ord}\, vars\, \textbf{in}\, P === \vec{\alpha}$	
	|	$\textbf{ord}\, vars\, \textbf{in}\, N === \vec{\alpha}$	
	|	$\textbf{nf}\,(N') === N$	
	|	$\textbf{nf}\,(P') === P$	
	|	$\textbf{nf}\,(N') === N$	
	|	$\textbf{nf}\,(P') === P$	
	|	$\textbf{nf}\,(\vec{N}') === \vec{N}$	
	|	$\textbf{nf}\,(\vec{P}') === \vec{P}$	
	|	$\textbf{nf}\,(\sigma') === \sigma$	
	|	$\textbf{nf}\,(\mu') === \mu$	
	|	$\textbf{nf}\,(\widehat{\sigma}') === \widehat{\sigma}$	
	|	$\sigma'\|_{vars}$	
	|	$\widehat{\sigma}'\|_{vars}$	
	|	$e_1 \;\&\; e_2$	
	|	$\widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2$	
	|	$\textbf{dom}\,(\widehat{\sigma}) === vars$	
	|	$\textbf{dom}\,(\Theta) === vars$	
LUB	::=		
	|	$\Gamma \vDash P_1 \vee P_2 = Q$	Least Upper Bound (Least Common Supertype)
	|	$\textbf{upgrade}\,\Gamma \vdash P\, \textbf{to}\, \Delta = Q$	
Nrm	::=		
	|	$\textbf{nf}\,(N) = M$	
	|	$\textbf{nf}\,(P) = Q$	
	|	$\textbf{nf}\,(N) = M$	
	|	$\textbf{nf}\,(P) = Q$	
Order	::=		
	|	$\textbf{ord}\, vars\, \textbf{in}\, N = \vec{\alpha}$	
	|	$\textbf{ord}\, vars\, \textbf{in}\, P = \vec{\alpha}$	
	|	$\textbf{ord}\, vars\, \textbf{in}\, N = \vec{\alpha}$	
	|	$\textbf{ord}\, vars\, \textbf{in}\, P = \vec{\alpha}$	

SM	::=		
	|	$\Gamma \vdash e_1 \ \& \ e_2 = e_3$	Unification Solution Entry Merge
	|	$\Theta \vdash \widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3$	Merge unification solutions
SImp	::=		
	|	$\Gamma \vdash e_1 \Rightarrow e_2$	Weakening of unification solution entries
	|	$\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2$	Weakening of unification solutions
	|	$\Gamma \vdash e_1 \simeq e_2$	
	|	$\Theta \vdash \widehat{\sigma}_1 \simeq \widehat{\sigma}_2$	
U	::=		
	|	$\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}$	Negative unification
	|	$\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}$	Positive unification
WF	::=		
	|	$\Gamma \vdash N$	Negative type well-formedness
	|	$\Gamma \vdash P$	Positive type well-formedness
	|	$\Gamma \vdash N$	Negative type well-formedness
	|	$\Gamma \vdash P$	Positive type well-formedness
	|	$\Gamma \vdash \overrightarrow{N}$	Negative type list well-formedness
	|	$\Gamma \vdash \overrightarrow{P}$	Positive type list well-formedness
	|	$\Gamma; \Theta \vdash N$	Negative unification type well-formedness
	|	$\Gamma; \Theta \vdash P$	Positive unifiication type well-formedness
	|	$\Gamma; \Xi \vdash P$	Positive anti-unification type well-formedness
	|	$\Gamma; \Xi_2 \vdash \widehat{\tau} : \Xi_1$	Antiunification substitution well-formedness
	|	$\widehat{\sigma} : \Theta$	Unification substitution well-formedness
	|	$\Gamma \vdash^{\supseteq} \Theta$	Unification context well-formedness
	|	$\Gamma_1 \vdash \sigma : \Gamma_2$	Substitution well-formedness
	|	$\Gamma \vdash e$	Unification solution entry well-formedness
judgement	::=		
	|	*A*	
	|	*AU*	
	|	*E1*	
	|	*D1*	
	|	*D0*	
	|	*EQ*	
	|	*LUB*	
	|	*Nrm*	
	|	*Order*	
	|	*SM*	
	|	*SImp*	
	|	*U*	
	|	*WF*	
user_syntax	::=		
	|	α	
	|	n	
	|	n	
	|	α^+	

| α^-
| $\alpha^\pm$
| σ
| e
| $\widehat{\sigma}$
| $\widehat{\tau}$
| P
| N
| $\overrightarrow{\alpha^+}$
| $\overrightarrow{\alpha^-}$
| $\overrightarrow{\alpha^\pm}$
| P
| N
| $\overrightarrow{P}$
| $\overrightarrow{N}$
| Γ
| Θ
| Ξ
| $\overrightarrow{\alpha}$
| $vars$
| μ
| $\widehat{\alpha}^\pm$
| $\widehat{\alpha}^+$
| $\widehat{\alpha}^-$
| $\overrightarrow{\widetilde{\alpha^+}}$
| $\overrightarrow{\widehat{\alpha^-}}$
| P
| N
| $auSol$
| $terminals$
| $formula$

$\boxed{\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma; \Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot} \quad \text{ANVar}$$

$$\frac{\Gamma; \Theta \models \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma; \Theta \models \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad \text{AShiftU}$$

$$\frac{\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2} \quad \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}] N \leqslant M \dashv \widehat{\sigma}}{\Gamma; \Theta \models \forall \overrightarrow{\alpha^+}. N \leqslant \forall \overrightarrow{\beta^+}. M \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^+}} \quad \text{AForall}$$

$\boxed{\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma; \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad \text{APVar}$$

$$\frac{\Gamma; \Theta \models \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma; \Theta \models \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \quad \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vDash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^-}} \text{ AEXISTS}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \textbf{upgrade}\, \Gamma \vdash P\, \textbf{to}\, \Delta = Q}{\Gamma; \Theta \vDash \widehat{\alpha}^+ \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant Q)} \text{ APUVAR}$$

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \text{ AUPVAR}$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \dashv (\Xi, \downarrow M, \widehat{\tau}_1, \widehat{\tau}_2)} \text{ AUPSHIFT}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \exists\overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists\overrightarrow{\alpha^-}.Q, \widehat{\tau}_1, \widehat{\tau}_2)} \text{ AUPEXISTS}$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)} \text{ AUNVAR}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)} \text{ AUNSHIFT}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}_1', \widehat{\tau}_2')}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}_1', \widehat{\tau}_2 \cup \widehat{\tau}_2')} \text{ AUNARROW}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \text{ AUNAU}$$

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-} \text{ E1NVAR}$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \text{ E1SHIFTU}$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M} \text{ E1ARROW}$$

$$\frac{\overrightarrow{\alpha^+} \cap \textbf{fv}\, M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \textbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \textbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M} \text{ E1FORALL}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \text{ E1PVAR}$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \text{ E1SHIFTD}$$

$$\frac{\overrightarrow{\alpha^-} \cap \textbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \textbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \textbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists\overrightarrow{\alpha^-}.P \simeq_1^D \exists\overrightarrow{\beta^-}.Q} \text{ E1EXISTS}$$

$\boxed{P \simeq Q}$

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M}\ \textsc{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q}\ \textsc{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-}\ \textsc{D1NVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q}\ \textsc{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M}\ \textsc{D1Arrow}$$

$$\frac{\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M}\ \textsc{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+}\ \textsc{D1PVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M}\ \textsc{D1ShiftD}$$

$$\frac{\mathbf{fv}\, P \cap \overrightarrow{\beta^-} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q}\ \textsc{D1Exists}$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1}$ Equivalence of substitutions

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M}\ \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q}\ \textsc{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-}\ \textsc{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q}\ \textsc{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M}\ \textsc{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M}\ \textsc{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \quad \text{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \quad \text{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \quad \text{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \quad \text{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \quad \text{D0ExistsR}$$

$\boxed{N = M}$ Negative type equality (alpha-equivalence)

$\boxed{P = Q}$ Positive type equuality (alphha-equivalence)

$\boxed{P = Q}$

$\boxed{P_1 \vee P_2}$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P}$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N}$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P}$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N}$

$\boxed{\mathbf{nf}\,(N')}$

$\boxed{\mathbf{nf}\,(P')}$

$\boxed{\mathbf{nf}\,(N')}$

$\boxed{\mathbf{nf}\,(P')}$

$\boxed{\mathbf{nf}\,(\overrightarrow{N}')}$

$\boxed{\mathbf{nf}\,(\overrightarrow{P'})}$

$\boxed{\mathbf{nf}\,(\sigma')}$

$\boxed{\mathbf{nf}\,(\mu')}$

$\boxed{\mathbf{nf}\,(\widehat{\sigma}')}$

$\boxed{\sigma'|_{vars}}$

$\boxed{\widehat{\sigma}'|_{vars}}$

$\boxed{e_1 \ \&\ e_2}$

$\boxed{\widehat{\sigma}_1 \ \&\ \widehat{\sigma}_2}$

$\boxed{\mathbf{dom}\,(\widehat{\sigma})}$

$\boxed{\mathbf{dom}\,(\Theta)}$

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+}\ \textsc{LUBVar}$$

$$\frac{\Gamma, \cdot \vDash \mathbf{nf}\,(\downarrow N) \overset{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\, P}\ \textsc{LUBShift}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q}\ \textsc{LUBExists}$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

$$\frac{\begin{array}{c}\Gamma = \Delta, \overrightarrow{\alpha^\pm} \quad \overrightarrow{\beta^\pm}\,\mathbf{is\,fresh} \quad \overrightarrow{\gamma^\pm}\,\mathbf{is\,fresh} \\ \Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vDash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q\end{array}}{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}\ \textsc{LUBUpgrade}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\ \textsc{NrmNVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\ \textsc{NrmShiftU}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(P \to N) = Q \to M}\ \textsc{NrmArrow}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\forall \overrightarrow{\alpha^+}.N) = \forall \overrightarrow{\alpha^+}'.N'}\ \textsc{NrmForall}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\ \textsc{NrmPVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\ \textsc{NrmShiftD}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.P) = \exists \overrightarrow{\alpha^-}'.P'}\ \textsc{NrmExists}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\ \textsc{NrmNUVar}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\ \textsc{NrmPUVar}$$

$\boxed{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}}$

$$\frac{\alpha^- \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\alpha^- = \alpha^-}\ \textsc{ONVarIn}$$

$$\frac{\alpha^- \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\alpha^- = \cdot}\ \textsc{ONVarNIn}$$

$$\frac{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\uparrow P = \vec{\alpha}}\ \textsc{OShiftU}$$

$$\frac{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}_1 \quad \mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}_2}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)}\ \textsc{OArrow}$$

$$\frac{\mathit{vars} \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\forall \overrightarrow{\alpha^+}.N = \vec{\alpha}}\ \textsc{OForall}$$

$\boxed{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}}$

$$\frac{\alpha^+ \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\alpha^+ = \alpha^+}\ \textsc{OPVarIn}$$

$$\frac{\alpha^+ \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,\alpha^+ = \cdot}\ \textsc{OPVarNIn}$$

$$\frac{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\,\mathbf{in}\downarrow N = \vec{\alpha}}\ \textsc{OShiftD}$$

$$\frac{vars \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\ vars\,\mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\,\mathbf{in}\ \exists\overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad \text{OEXISTS}$$

$\boxed{\mathbf{ord}\ vars\,\mathbf{in}\ N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\,\mathbf{in}\ \widehat{\alpha}^- = \cdot} \quad \text{ONUVAR}$$

$\boxed{\mathbf{ord}\ vars\,\mathbf{in}\ P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\,\mathbf{in}\ \widehat{\alpha}^+ = \cdot} \quad \text{OPUVAR}$$

$\boxed{\Gamma \vdash e_1 \ \&\ e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{\Gamma \vDash P_1 \vee P_2 = Q}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P_1) \ \&\ (\widehat{\alpha}^+ :\geqslant P_2) = (\widehat{\alpha}^+ :\geqslant Q)} \quad \text{SMESUPSUP}$$

$$\frac{\Gamma; \cdot \vDash P \geqslant Q \dashv \widehat{\sigma}'}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \ \&\ (\widehat{\alpha}^+ :\geqslant Q) = (\widehat{\alpha}^+ :\approx P)} \quad \text{SMEEQSUP}$$

$$\frac{\Gamma; \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P) \ \&\ (\widehat{\alpha}^+ :\approx Q) = (\widehat{\alpha}^+ :\approx Q)} \quad \text{SMESUPEQ}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \ \&\ (\widehat{\alpha}^+ :\approx P') = (\widehat{\alpha}^+ :\approx P)} \quad \text{SMEPEQEQ}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^- :\approx N_1) \ \&\ (\widehat{\alpha}^- :\approx N') = (\widehat{\alpha}^- :\approx N)} \quad \text{SMENEQEQ}$$

$\boxed{\Theta \vdash \widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$\boxed{\Gamma \vdash e_1 \Rightarrow e_2}$ Weakening of unification solution entries

$$\frac{\Gamma \vdash P_1 \geqslant_1 P_2}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P_1) \Rightarrow (\widehat{\alpha}^+ :\geqslant P_2)} \quad \text{SIMPESUPSUP}$$

$$\frac{\Gamma \vdash P_1 \geqslant_1 P_2}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P_1) \Rightarrow (\widehat{\alpha}^+ :\geqslant P_2)} \quad \text{SIMPEEQSUP}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P_1) \Rightarrow (\widehat{\alpha}^+ :\approx P_2)} \quad \text{SIMPEPEQEQ}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^- :\approx N_1) \Rightarrow (\widehat{\alpha}^- :\approx N_2)} \quad \text{SIMPENEQEQ}$$

$\boxed{\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2}$ Weakening of unification solutions

$\boxed{\Gamma \vdash e_1 \simeq e_2}$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P_1) \simeq (\widehat{\alpha}^+ :\geqslant P_2)} \quad \text{SIMPEEQSUPSUP}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P_1) \simeq (\widehat{\alpha}^+ :\approx P_2)} \quad \text{SIMPEEQPEQEQ}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^- :\approx N_1) \simeq (\widehat{\alpha}^- :\approx N_2)} \quad \text{SIMPEEQNEQEQ}$$

$\boxed{\Theta \vdash \widehat{\sigma}_1 \simeq \widehat{\sigma}_2}$

$\boxed{\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Gamma;\Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\quad \text{UNVAR}$$

$$\frac{\Gamma;\Theta \models \boxed{P} \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma;\Theta \models \uparrow\boxed{P} \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\quad \text{USHIFTU}$$

$$\frac{\Gamma;\Theta \models \boxed{P} \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Gamma;\Theta \models \boxed{N} \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Gamma;\Theta \models \boxed{P} \to \boxed{N} \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\quad \text{UARROW}$$

$$\frac{\Gamma, \overrightarrow{\alpha^+};\Theta \models \boxed{N} \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma;\Theta \models \forall\overrightarrow{\alpha^+}.\boxed{N} \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\quad \text{UFORALL}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma;\Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\widehat{\alpha}^- :\approx N)}\quad \text{UNUVAR}$$

$\boxed{\Gamma;\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Gamma;\Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\quad \text{UPVAR}$$

$$\frac{\Gamma;\Theta \models \boxed{N} \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma;\Theta \models \downarrow\boxed{N} \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\quad \text{USHIFTD}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-};\Theta \models \boxed{P} \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma;\Theta \models \exists\overrightarrow{\alpha^-}.\boxed{P} \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\quad \text{UEXISTS}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma;\Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\widehat{\alpha}^+ :\approx P)}\quad \text{UPUVAR}$$

$\boxed{\Gamma \vdash N}$ Negative type well-formedness
$\boxed{\Gamma \vdash P}$ Positive type well-formedness
$\boxed{\Gamma \vdash N}$ Negative type well-formedness
$\boxed{\Gamma \vdash P}$ Positive type well-formedness
$\boxed{\Gamma \vdash \overrightarrow{N}}$ Negative type list well-formedness
$\boxed{\Gamma \vdash \overrightarrow{P}}$ Positive type list well-formedness
$\boxed{\Gamma;\Theta \vdash N}$ Negative unification type well-formedness
$\boxed{\Gamma;\Theta \vdash P}$ Positive unifiication type well-formedness
$\boxed{\Gamma;\Xi \vdash P}$ Positive anti-unification type well-formedness
$\boxed{\Gamma;\Xi_2 \vdash \widehat{\tau} : \Xi_1}$ Antiunification substitution well-formedness
$\boxed{\widehat{\sigma} : \Theta}$ Unification substitution well-formedness
$\boxed{\Gamma \vdash^{\supseteq} \Theta}$ Unification context well-formedness
$\boxed{\Gamma_1 \vdash \sigma : \Gamma_2}$ Substitution well-formedness
$\boxed{\Gamma \vdash e}$ Unification solution entry well-formedness

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Definition rules:        73 good    14 bad
Definition rule clauses: 142 good    14 bad
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