

$\alpha, \beta, \alpha, \beta, \gamma, \delta$	type variables
n, m, i, j	index variables
x, y, z	term variables

UC	$::=$ \cdot e $UC \backslash vars$ $UC vars$ $UC_1 \cup UC_2$ $\overline{UC_i}^i$ (UC) S $UC' vars$ M $UC_1 \ \& \ UC_2$ M $UC_1 \cup UC_2$ M $ SC $ M	unification constraint
SC	$::=$ \cdot e $SC \backslash vars$ $SC vars$ $SC_1 \cup SC_2$ UC $\overline{SC_i}^i$ (SC) S $SC' vars$ M $SC_1 \ \& \ SC_2$ M	subtyping constraint
$\hat{\sigma}$	$::=$ \cdot $P/\hat{\alpha}^+$ $N/\hat{\alpha}^-$ $\vec{P}/\vec{\hat{\alpha}}^+$ $\vec{N}/\vec{\hat{\alpha}}^-$ $(\hat{\sigma})$ S $\hat{\sigma}_1 \circ \hat{\sigma}_2$ $\overline{\hat{\sigma}_i}^i$ $\mathbf{nf}(\hat{\sigma}')$ M $\hat{\sigma}' vars$ M	unification substitution
$\hat{\tau}, \hat{\rho}$	$::=$ \cdot $\hat{\alpha}^- : \approx N$ $\hat{\alpha}^- : \approx \textcolor{gray}{N}$ $\vec{\alpha}^- / \vec{\hat{\alpha}}^-$ $\vec{N} / \vec{\hat{\alpha}}^-$ $\hat{\tau}_1 \cup \hat{\tau}_2$ $\overline{\hat{\tau}_i}^i$ $(\hat{\tau})$ S $\hat{\tau}' vars$ M $\hat{\tau}_1 \ \& \ \hat{\tau}_2$ M	anti-unification substitution
$\vec{\alpha}^+, \vec{\beta}^+, \vec{\gamma}^+, \vec{\delta}^+$	$::=$	positive variable list

		\cdot	empty list
		α^+	a variable
		$\overrightarrow{\alpha^+}$	a variable
		$\overrightarrow{\overrightarrow{\alpha^+}}^i$	concatenate lists
$\overrightarrow{\alpha^-}, \overrightarrow{\beta^-}, \overrightarrow{\gamma^-}, \overrightarrow{\delta^-}$::=		negative variables
		\cdot	empty list
		α^-	a variable
		$\overrightarrow{\alpha^-}$	variables
		$\overrightarrow{\overrightarrow{\alpha^-}}^i$	concatenate lists
$\overrightarrow{\alpha^\pm}, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm}, \overrightarrow{\delta^\pm}$::=		positive or negative variable list
		\cdot	empty list
		α^\pm	a variable
		$\overrightarrow{\alpha^\pm}$	variables
		$\overrightarrow{\overrightarrow{\alpha^\pm}}^i$	concatenate lists
P, Q, R	::=		positive declarative types
		α^+	
		$\downarrow N$	
		$\exists \alpha^-. P$	
		$[\sigma]P$	M
		$[\hat{\tau}]P$	M
		$[\hat{\sigma}]P$	M
		$[\mu]P$	M
		(P)	S
		$P_1 \vee P_2$	M
		$\mathbf{nf}(P')$	M
N, M, K	::=		negative declarative types
		α^-	
		$\uparrow P$	
		$P \rightarrow N$	
		$\forall \alpha^+. N$	
		$[\sigma]N$	M
		$[\hat{\tau}]N$	M
		$[\mu]N$	M
		$[\hat{\sigma}]N$	M
		(N)	S
		$\mathbf{nf}(N')$	M
\vec{P}, \vec{Q}	::=		list of positive types
		\cdot	empty list
		P	a singel type
		$[\sigma]\vec{P}$	M
		$\overrightarrow{\vec{P}}^i$	concatenate lists
		(\vec{P})	S
		$\mathbf{nf}(\vec{P}')$	M

\vec{N}, \vec{M}	$::=$		list of negative types
	\cdot		empty list
	N		a singel type
	$[\sigma]\vec{N}$	M	
	\vec{N}_i^i		concatenate lists
	(\vec{N})	S	
	$\mathbf{nf}(\vec{N}')$	M	
Δ, Γ	$::=$		declarative type context
	\cdot		empty context
	$\vec{\alpha}^+$		list of variables
	$\vec{\alpha}^-$		list of variables
	$\{\alpha^\pm\}$		list of variables
	$\vec{\Gamma}_i^i$		concatenate contexts
	(Γ)	S	
	$\Gamma, \vec{\alpha}^+$	M	append a list of variables
	$\Gamma, \vec{\alpha}^-$	M	append a list of variables
	Γ, α^\pm	M	append a list of variables
	$\Theta(\hat{\alpha}^+)$	M	
	$\Theta(\hat{\alpha}^-)$	M	
	$\Gamma_1 \cup \Gamma_2$		
	$\Gamma_1 \cap vars$		
	$\Gamma_1 \cup \Gamma_2$	M	
	$\mathbf{fv} N$	M	
	$\mathbf{fv} P$	M	
	$\mathbf{fv} P$	M	
	$\mathbf{fv} N$	M	
Θ	$::=$		algorithmic variable context
	\cdot		empty context
	$\vec{\alpha}\{\Delta\}$		from an ordered list of variables
	$\hat{\alpha}^+\{\Delta\}$		from a variable to a list
	$\vec{\Theta}_i^i$		concatenate contexts
	(Θ)	S	
	$\Theta _{vars}$		leave only those variables that are in the set
	$\Theta_1 \cup \Theta_2$		
Ξ	$::=$		anti-unification type variable context
	\cdot		empty context
	$\vec{\hat{\alpha}}^+$		list of positive variables
	$\vec{\hat{\alpha}}^-$		list of negative variables
	$\Xi, \vec{\hat{\alpha}}^+$	M	append a list of variables
	$\Xi, \vec{\hat{\alpha}}^-$	M	append a list of variables
	$\vec{\Xi}_i^i$		concatenate contexts
	(Ξ)	S	
	$\Xi_1 \cup \Xi_2$		
	$\Xi_1 \cap vars$		
	$\Xi' _{vars}$	M	
	$\mathbf{dom}(UC)$	M	
	$\mathbf{dom}(SC)$	M	

		dom ($\hat{\sigma}$)	M	
		dom ($\hat{\tau}$)	M	
		dom (Θ)	M	
		uv N	M	
		uv P	M	
$\vec{\alpha}, \vec{\beta}$::=			ordered positive or negative variables
		\cdot		empty list
		$\vec{\alpha}^+$		list of variables
		$\vec{\alpha}^-$		list of variables
		$\vec{\alpha}^\pm$		list of variables
		$\vec{\hat{\alpha}}^+$		list of variables
		$\vec{\hat{\alpha}}^-$		list of variables
		$\vec{\alpha}_1 \setminus vars$		setminus
		$vars$		
		$\vec{\alpha}_i^i$		concatenate contexts
		$(\vec{\alpha})$	S	parenthesis
		$[\mu]\vec{\alpha}$	M	apply moving to list
		$[\vec{\mu}]\vec{\alpha}$	M	apply umoving to list
		ord $vars$ in P	M	
		ord $vars$ in N	M	
		ord $vars$ in P	M	
		ord $vars$ in N	M	
$vars$::=			set of variables
		\emptyset		empty set
		$vars_1 \cap vars_2$		set intersection
		$vars_1 \cup vars_2$		set union
		$vars_1 \setminus vars_2$		set complement
		$(vars)$	S	parenthesis
		$\vec{\alpha}$		ordered list of variables
		$[\mu]vars$	M	apply moving to varset
		Ξ		algorithmic type context
		Γ		declarative type context
μ	::=			
		\cdot		empty moving
		$pma1 \mapsto pma2$		Positive unit substitution
		$nma1 \mapsto nma2$		Positive unit substitution
		$\mu_1 \cup \mu_2$	M	Set-like union of movings
		$\mu_1 \circ \mu_2$	M	Composition
		$\vec{\mu}_i^i$		concatenate movings
		$\mu _{vars}$	M	restriction on a set
		μ^{-1}	M	inversion
		nf (μ')	M	
$\vec{\mu}$::=			
		\cdot		empty moving
		$\vec{\hat{\alpha}}^+ / \vec{\alpha}^+$		
		$\vec{\hat{\alpha}}^- / \vec{\alpha}^-$		

$\hat{\alpha}^\pm$	$::=$ $\hat{\alpha}^\pm$	positive/negative unification variable
$\hat{\alpha}^+$	$::=$ $\hat{\alpha}^+$ $\hat{\alpha}^+\{\Delta\}$ $\hat{\alpha}^\pm$	positive unification variable
$\hat{\alpha}^-, \hat{\beta}^-$	$::=$ $\hat{\alpha}^-$ $\hat{\alpha}^-_{\{N,M\}}$ $\hat{\alpha}^-\{\Delta\}$ $\hat{\alpha}^\pm$	negative unification variable
$\overrightarrow{\hat{\alpha}^+}, \overrightarrow{\hat{\beta}^+}$	$::=$ \cdot $\hat{\alpha}^+$ $\overrightarrow{\hat{\alpha}^+}$ $\overrightarrow{\hat{\alpha}^+}_i$	positive unification variable list empty list a variable from a normal variable, context unspecified concatenate lists
$\overrightarrow{\hat{\alpha}^-}, \overrightarrow{\hat{\beta}^-}$	$::=$ \cdot $\hat{\alpha}^-$ Ξ $\overrightarrow{\hat{\alpha}^-\{\Delta\}}$ $\overrightarrow{\hat{\alpha}^-}$ $\overrightarrow{\hat{\alpha}^-}_i$	negative unification variable list empty list a variable from an antiunification context from a normal variable from a normal variable, context unspecified concatenate lists
P, Q	$::=$ $\hat{\alpha}^+$ α^+ $\downarrow N$ $\exists \alpha^-. P$ $[\sigma] P$ M $[\hat{\tau}] P$ M $[\mu] P$ M $[\hat{\sigma}] P$ M $[\vec{\mu}] P$ M (P) S $\mathbf{nf}(P')$ M	a positive algorithmic type (potentially with algorithmic variables)
N, M	$::=$ $\hat{\alpha}^-$ α^- $\uparrow P$ $P \rightarrow N$ $\forall \alpha^+. N$ $[\sigma] N$ M $[\hat{\tau}] N$ M	a negative algorithmic type (potentially with algorithmic variables)

		$[\mu]N$	M
		$[\hat{\sigma}]N$	M
		$[\vec{\mu}]N$	M
		(N)	S
		$\mathbf{nf}(N')$	M
$auSol$	$::=$		
		$(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$	
		$(\Xi, N, \hat{\tau}_1, \hat{\tau}_2)$	
$terminals$	$::=$		
		\exists	
		\forall	
		\uparrow	
		\downarrow	
		\rightarrow	
		\leftrightarrow	
		\in	
		\notin	
		\cdot	
		\perp	
		\preceq	
		\succcurlyeq	
		\wr	
		\subset	
		\supset	
		\diagdown	
		\sqcup	
		\mapsto	
		\wr^u	
		\wr^a	
		\emptyset	
		\circ	
		\Rightarrow	
		Π	
		\equiv	
		\neq	
		\equiv_n	
		\prec	
		\Downarrow	
		$\colon\geq$	
		$\colon\wr$	
		Λ	
		λ	
		\mathbf{let}^\exists	
		\bullet	
		$\Rightarrow\Rightarrow$	
		$\Leftarrow\Leftarrow$	
v, w	$::=$	value terms	

	$ \begin{array}{ l} x \\ \{c\} \\ (v : P) \\ (v) \end{array} $	M
\vec{v}	$ \begin{array}{ l} ::= \\ \cdot \\ v \\ \overrightarrow{v}_i^i \end{array} $	list of arguments concatenate
c, d	$ \begin{array}{ l} ::= \\ (c : N) \\ \lambda x : P. c \\ \Lambda \alpha^+. c \\ \mathbf{return} \ v \\ \mathbf{let} \ x = v; c \\ \mathbf{let} \ x : P = v(\vec{v}); c \\ \mathbf{let} \ x = v(\vec{v}); c \\ \mathbf{let}^{\exists}(\alpha^-, x) = v; c \end{array} $	computation terms
$vctx, \Phi$	$ \begin{array}{ l} ::= \\ \cdot \\ x : P \\ \overline{\Phi}_i^i \end{array} $	variable context concatenate contexts
<i>formula</i>	$ \begin{array}{ l} ::= \\ judgement \\ judgement \text{ unique} \\ formula_1 \ .. \ formula_n \\ \mu : vars_1 \leftrightarrow vars_2 \\ \mu \text{ is bijective} \\ x : P \in \Phi \\ UC_1 \subseteq UC_2 \\ UC_1 = UC_2 \\ SC_1 \subseteq SC_2 \\ e \in SC \\ e \in UC \\ vars_1 \subseteq vars_2 \\ vars_1 \subseteq vars_2 \subseteq vars_3 \\ vars_1 = vars_2 \\ vars \text{ are fresh} \\ \alpha^- \notin vars \\ \alpha^+ \notin vars \\ \alpha^- \in vars \\ \alpha^+ \in vars \\ \hat{\alpha}^+ \in vars \\ \hat{\alpha}^- \in vars \\ \hat{\alpha}^- \in \Theta \\ \hat{\alpha}^+ \in \Theta \\ \hat{\alpha}^- \notin vars \end{array} $	

	$\hat{\alpha}^+ \notin vars$ $\hat{\alpha}^- \notin \Theta$ $\hat{\alpha}^+ \notin \Theta$ $\hat{\alpha}^- \in \Xi$ $\hat{\alpha}^- \notin \Xi$ $\hat{\alpha}^+ \in \Xi$ $\hat{\alpha}^+ \notin \Xi$ if any other rule is not applicable $\vec{\alpha}_1 = \vec{\alpha}_2$ $e_1 = e_2$ $e_1 = e_2$ $\hat{\sigma}_1 = \hat{\sigma}_2$ $\boxed{N} = \boxed{M}$ $\Theta \subseteq \Theta'$ $\vec{v}_1 = \vec{v}_2$ $\mathbf{N} \neq \mathbf{M}$ $\mathbf{P} \neq \mathbf{Q}$ $N \neq M$ $P \neq Q$ $\boxed{P} \neq \boxed{Q}$ $\boxed{N} \neq \boxed{M}$ $\vec{v}_1 \neq \vec{v}_2$ $\vec{\alpha}_1^+ \neq \vec{\alpha}_2^+$ $ \vec{\alpha}^- + \vec{\beta}^- > 0$ $ \vec{\alpha}^+ + \vec{\beta}^+ > 0$	
A	$::=$ $\Gamma; \Theta \models \boxed{N} \leq M \Rightarrow SC$ $\Gamma; \Theta \models \boxed{P} \geq Q \Rightarrow SC$	Negative subtyping Positive supertyping
AT	$::=$ $\Gamma; \Phi \models v : P$ $\Gamma; \Phi \models c : N$ $\Gamma; \Phi; \Theta_1 \models \boxed{N} \bullet \vec{v} \Rightarrow \boxed{M} \Rightarrow \Theta_2; SC$	Positive type inference Negative type inference Application type inference
AU	$::=$ $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \boxed{Q}, \hat{\tau}_1, \hat{\tau}_2)$ $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \boxed{M}, \hat{\tau}_1, \hat{\tau}_2)$	
SCM	$::=$ $\Gamma \vdash e_1 \& e_2 = e_3$ $\Theta \vdash SC_1 \& SC_2 = SC_3$	Subtyping Constraint Entry Merge Merge of subtyping constraints
UCM	$::=$ $\Gamma \vdash e_1 \& e_2 = e_3$ $\Theta \vdash UC_1 \& UC_2 = UC_3$	Merge of unification constraints
$SATSCE$	$::=$ $\Gamma \vdash P : e$	Positive type satisfies with the subtyping constraints

	$\Gamma \vdash N : e$	Negative type satisfies with the subtyping constraint entry
<i>SING</i>	$::=$ e_1 singular with P e_1 singular with N SC singular with \hat{o}	Positive Subtyping Constraint Entry Is Singular Negative Subtyping Constraint Entry Is Singular Subtyping Constraint Is Singular
<i>E1</i>	$::=$ $N \simeq^D M$ $P \simeq^D Q$ $P \simeq^D Q$ $N \simeq^D M$	Negative type equivalence Positive type equivalence Positive unification type equivalence Positive unification type equivalence
<i>D1</i>	$::=$ $\Gamma \vdash N \simeq^{\leq} M$ $\Gamma \vdash P \simeq^{\leq} Q$ $\Gamma \vdash N \leq M$ $\Gamma \vdash P \geq Q$	Negative subtyping-induced equivalence Positive subtyping-induced equivalence Negative subtyping Positive supertyping
<i>D1S</i>	$::=$ $\Gamma_2 \vdash \sigma_1 \simeq^{\leq} \sigma_2 : \Gamma_1$ $\Gamma \vdash \sigma_1 \simeq^{\leq} \sigma_2 : vars$ $\Theta \vdash \hat{\sigma}_1 \simeq^{\leq} \hat{\sigma}_2 : vars$ $\Gamma \vdash \hat{\sigma}_1 \simeq^{\leq} \hat{\sigma}_2 : vars$	Equivalence of substitutions Equivalence of substitutions Equivalence of unification substitutions Equivalence of unification substitutions
<i>D1C</i>	$::=$ $\Gamma \vdash \Phi_1 \simeq^{\leq} \Phi_2$	Equivalence of contexts
<i>DT</i>	$::=$ $\Gamma; \Phi \vdash v : P$ $\Gamma; \Phi \vdash c : N$ $\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M$	Positive type inference Negative type inference Application type inference
<i>EQ</i>	$::=$ $N = M$ $P = Q$ $P = Q$	Negative type equality (alpha-equivalence) Positive type equality (alpha-equivalence)
<i>LUBF</i>	$::=$ $P_1 \vee P_2 === Q$ ord <i>vars</i> in $P === \vec{\alpha}$ ord <i>vars</i> in $N === \vec{\alpha}$ ord <i>vars</i> in $P === \vec{\alpha}$ ord <i>vars</i> in $N === \vec{\alpha}$ nf $(N') === N$ nf $(P') === P$ nf $(N') === N$ nf $(P') === P$ nf $(\vec{N}') === \vec{N}$	

	$\text{nf}(\vec{P}') === \vec{P}$ $\text{nf}(\sigma') === \sigma$ $\text{nf}(\hat{\sigma}') === \hat{\sigma}$ $\text{nf}(\mu') === \mu$ $\sigma' _{vars}$ $\hat{\sigma}' _{vars}$ $\hat{\tau}' _{vars}$ $\Xi' _{vars}$ $SC' _{vars}$ $UC' _{vars}$ $e_1 \ \& \ e_2$ $e_1 \ \& \ e_2$ $UC_1 \ \& \ UC_2$ $UC_1 \cup UC_2$ $\Gamma_1 \cup \Gamma_2$ $SC_1 \ \& \ SC_2$ $\hat{\tau}_1 \ \& \ \hat{\tau}_2$ $\text{dom}(UC) === \Xi$ $\text{dom}(SC) === \Xi$ $\text{dom}(\hat{\sigma}) === \Xi$ $\text{dom}(\hat{\tau}) === \Xi$ $\text{dom}(\Theta) === \Xi$ $ SC === UC$ $\text{fv } N === \Gamma$ $\text{fv } P === \Gamma$ $\text{fv } P === \Gamma$ $\text{fv } N === \Gamma$ $\text{uv } N === \Xi$ $\text{uv } P === \Xi$	
<i>LUB</i>	$::=$ $\Gamma \models P_1 \vee P_2 = Q$ $\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q$	Least Upper Bound (Least Common Supertype)
<i>Nrm</i>	$::=$ $\text{nf}(N) = M$ $\text{nf}(P) = Q$ $\text{nf}(N) = M$ $\text{nf}(P) = Q$	
<i>Order</i>	$::=$ $\text{ord } vars \text{ in } N = \vec{\alpha}$ $\text{ord } vars \text{ in } P = \vec{\alpha}$ $\text{ord } vars \text{ in } N = \vec{\alpha}$ $\text{ord } vars \text{ in } P = \vec{\alpha}$	
<i>U</i>	$::=$ $\Gamma; \Theta \models N \overset{u}{\simeq} M = UC$ $\Gamma; \Theta \models P \overset{u}{\simeq} Q = UC$	Negative unification Positive unification

WFT	$::=$		
		$\Gamma \vdash N$	Negative type well-formedness
		$\Gamma \vdash P$	Positive type well-formedness
$WFAT$	$::=$		
		$\Gamma; \Xi \vdash N$	Negative algorithmic type well-formedness
		$\Gamma; \Xi \vdash P$	Positive algorithmic type well-formedness
$WFALL$	$::=$		
		$\Gamma \vdash \vec{N}$	Negative type list well-formedness
		$\Gamma \vdash \vec{P}$	Positive type list well-formedness
		$\Gamma; \Xi_2 \vdash \hat{\tau} : \Xi_1$	Antiunification substitution well-formedness
		$\Gamma \vdash^\exists \Theta$	Unification context well-formedness
		$\Gamma_1 \vdash \sigma : \Gamma_2$	Substitution signature
		$\Theta \vdash \hat{\sigma} : \Xi$	Unification substitution signature
		$\Gamma \vdash \hat{\sigma} : \Xi$	Unification substitution general signature
		$\Theta \vdash \hat{\sigma} : UC$	Unification substitution satisfies unification constraint
		$\Theta \vdash \hat{\sigma} : SC$	Unification substitution satisfies subtyping constraint
		$\Gamma \vdash e$	Unification constraint entry well-formedness
		$\Gamma \vdash e$	Subtyping constraint entry well-formedness
		$\Gamma \vdash P : e$	Positive type satisfies unification constraint
		$\Gamma \vdash N : e$	Negative type satisfies unification constraint
		$\Gamma \vdash P : e$	Positive type satisfies subtyping constraint
		$\Gamma \vdash N : e$	Negative type satisfies subtyping constraint
		$\Theta \vdash UC : \Xi$	Unification constraint well-formedness with specified domain
		$\Theta \vdash SC : \Xi$	Subtyping constraint well-formedness with specified domain
		$\Theta \vdash UC$	Unification constraint well-formedness
		$\Theta \vdash SC$	Subtyping constraint well-formedness
		$\Gamma \vdash \vec{v}$	Argument List well-formedness
		$\Gamma \vdash \Phi$	Context well-formedness
		$\Gamma \vdash v$	Value well-formedness
		$\Gamma \vdash c$	Computation well-formedness
$judgement$	$::=$		
		A	
		AT	
		AU	
		SCM	
		UCM	
		$SATSCE$	
		$SING$	
		$E1$	
		$D1$	
		$D1S$	
		$D1C$	
		DT	
		EQ	
		LUB	
		Nrm	
		$Order$	

		U
		WFT
		$WFAT$
		$WFALL$
$user_syntax$	$::=$	
		α
		n
		x
		n
		α^+
		α^-
		α^\pm
		σ
		e
		e
		UC
		SC
		$\hat{\sigma}$
		$\hat{\tau}$
		$\overrightarrow{\alpha^+}$
		$\overrightarrow{\alpha^-}$
		$\overrightarrow{\alpha^\pm}$
		P
		N
		\vec{P}
		\vec{N}
		Γ
		Θ
		Ξ
		$\vec{\alpha}$
		$vars$
		μ
		$\vec{\mu}$
		$\hat{\alpha}^\pm$
		$\hat{\alpha}^+$
		$\hat{\alpha}^-$
		$\overrightarrow{\hat{\alpha}^+}$
		$\overrightarrow{\hat{\alpha}^-}$
		P
		N
		$auSol$
		$terminals$
		v
		\vec{v}
		c
		$vctx$
		$formula$

$\boxed{\Gamma; \Theta \models N \leqslant M \Rightarrow SC}$ Negative subtyping

$$\begin{array}{c}
\frac{}{\Gamma; \Theta \models \alpha^- \leq \alpha^- \Rightarrow} \text{ANVAR} \\
\frac{\Gamma; \Theta \models \mathbf{nf}(P) \stackrel{u}{\simeq} \mathbf{nf}(Q) \Rightarrow UC}{\Gamma; \Theta \models \uparrow P \leq \uparrow Q \Rightarrow UC} \text{AShiftU} \\
\frac{\Gamma; \Theta \models P \geq Q \Rightarrow SC_1 \quad \Gamma; \Theta \models N \leq M \Rightarrow SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC}{\Gamma; \Theta \models P \rightarrow N \leq Q \rightarrow M \Rightarrow SC} \text{AArrow} \\
\frac{\text{<<multiple parses>>}}{\Gamma; \Theta \models \forall \alpha^+. N \leq \forall \beta^+. M \Rightarrow SC \setminus \hat{\alpha}^+} \text{AForall} \\
\boxed{\Gamma; \Theta \models P \geq Q \Rightarrow SC} \quad \text{Positive supertyping} \\
\frac{}{\Gamma; \Theta \models \alpha^+ \geq \alpha^+ \Rightarrow} \text{APVAR} \\
\frac{\Gamma; \Theta \models \mathbf{nf}(N) \stackrel{u}{\simeq} \mathbf{nf}(M) \Rightarrow UC}{\Gamma; \Theta \models \downarrow N \geq \downarrow M \Rightarrow UC} \text{AShiftD} \\
\frac{\hat{\alpha}^- \text{ are fresh} \quad \Gamma, \beta^-; \Theta, \hat{\alpha}^- \{ \Gamma, \beta^- \} \models [\hat{\alpha}^- / \alpha^-] P \geq Q \Rightarrow SC}{\Gamma; \Theta \models \exists \alpha^-. P \geq \exists \beta^-. Q \Rightarrow SC \setminus \hat{\alpha}^-} \text{AExists} \\
\frac{\text{upgrade } \Gamma \vdash P \text{ to } \Theta(\hat{\alpha}^+) = Q}{\Gamma; \Theta \models \hat{\alpha}^+ \geq P \Rightarrow (\hat{\alpha}^+ : \geq Q)} \text{APUVar} \\
\boxed{\Gamma; \Phi \models v : P} \quad \text{Positive type inference} \\
\frac{x : P \in \Phi}{\Gamma; \Phi \models x : \mathbf{nf}(P)} \text{ATVar} \\
\frac{\Gamma; \Phi \models c : N}{\Gamma; \Phi \models \{c\} : \downarrow N} \text{ATThunk} \\
\frac{\Gamma \vdash Q \quad \Gamma; \Phi \models v : P \quad \Gamma; \cdot \models Q \geq P \Rightarrow \cdot}{\Gamma; \Phi \models (v : Q) : \mathbf{nf}(Q)} \text{ATPAnnot} \\
\boxed{\Gamma; \Phi \models c : N} \quad \text{Negative type inference} \\
\frac{\Gamma \vdash M \quad \Gamma; \Phi \models c : N \quad \Gamma; \cdot \models N \leq M \Rightarrow \cdot}{\Gamma; \Phi \models (c : M) : \mathbf{nf}(M)} \text{ATNAnnot} \\
\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \lambda x : P. c : \mathbf{nf}(P \rightarrow N)} \text{ATTLam} \\
\frac{\Gamma, \alpha^+; \Phi \models c : N}{\Gamma; \Phi \models \Lambda \alpha^+. c : \mathbf{nf}(\forall \alpha^+. N)} \text{ATTLam} \\
\frac{\Gamma; \Phi \models v : P}{\Gamma; \Phi \models \mathbf{return} v : \uparrow P} \text{ATReturn} \\
\frac{\Gamma; \Phi \models v : P \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \mathbf{let} x = v; c : N} \text{ATVarLet} \\
\frac{\Gamma \vdash P \quad \Gamma; \Phi \models v : \downarrow M \quad \Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow \uparrow Q \Rightarrow \Theta; SC_1 \quad \Gamma; \Theta \models \uparrow Q \leq \uparrow P \Rightarrow SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \mathbf{let} x : P = v(\vec{v}); c : N} \text{ATAppLetAnn}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma; \Phi \models v : \downarrow M \quad \Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow \uparrow Q \equiv \Theta; SC \quad \text{\textcolor{red}{<<multiple parses>>}} \quad \Gamma; \Phi, x : [\hat{\sigma}] Q \models c : N}{\Gamma; \Phi \models \text{let } x = v(\vec{v}); c : N} \text{ ATAPPLET} \\
\\
\frac{\Gamma; \Phi \models v : \exists \alpha^-. P \quad \Gamma, \alpha^-; \Phi, x : P \models c : N \quad \Gamma \vdash N}{\Gamma; \Phi \models \text{let}^\exists(\alpha^-, x) = v; c : N} \text{ ATUNPACK} \\
\\
\boxed{\Gamma; \Phi; \Theta_1 \models N \bullet \vec{v} \Rightarrow M \equiv \Theta_2; SC} \quad \text{Application type inference} \\
\\
\frac{}{\Gamma; \Phi; \Theta \models N \bullet \cdot \Rightarrow \text{nf}(N) \equiv \Theta; \cdot} \text{ AEMPTYAPP} \\
\\
\frac{\Gamma; \Phi \models v : P \quad \Gamma; \Theta \models Q \succcurlyeq P \equiv SC_1 \quad \Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rightarrow M \equiv \Theta'; SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC}{\Gamma; \Phi; \Theta \models Q \rightarrow N \bullet v, \vec{v} \Rightarrow M \equiv \Theta'; SC} \text{ ATARROWAPP} \\
\\
\frac{\text{\textcolor{red}{<<multiple parses>>}} \quad \hat{\alpha}^+ \text{ are fresh} \quad \vec{v} \neq \cdot \quad \alpha^+ \neq \cdot}{\text{\textcolor{red}{<<multiple parses>>}}} \text{ ATFORALLAPP} \\
\\
\boxed{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \equiv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)} \\
\\
\frac{}{\Gamma \models \alpha^+ \stackrel{a}{\simeq} \alpha^+ \equiv (\cdot, \alpha^+, \cdot, \cdot)} \text{ AUPVAR} \\
\\
\frac{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \equiv (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N_1 \stackrel{a}{\simeq} \downarrow N_2 \equiv (\Xi, \downarrow M, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUSHIFTD} \\
\\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\Gamma \models \exists \alpha^-. P_1 \stackrel{a}{\simeq} \exists \alpha^-. P_2 \equiv (\Xi, \exists \alpha^-. Q, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUEXISTS} \\
\\
\boxed{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \equiv (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)} \\
\\
\frac{}{\Gamma \models \alpha^- \stackrel{a}{\simeq} \alpha^- \equiv (\cdot, \alpha^-, \cdot, \cdot)} \text{ AUNVAR} \\
\\
\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \equiv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \uparrow P_1 \stackrel{a}{\simeq} \uparrow P_2 \equiv (\Xi, \uparrow Q, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUSHIFTU} \\
\\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\Gamma \models \forall \alpha^+. N_1 \stackrel{a}{\simeq} \forall \alpha^+. N_2 \equiv (\Xi, \forall \alpha^+. M, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUFORALL} \\
\\
\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \equiv (\Xi_1, Q, \hat{\tau}_1, \hat{\tau}_2) \quad \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \equiv (\Xi_2, M, \hat{\tau}'_1, \hat{\tau}'_2)}{\Gamma \models P_1 \rightarrow N_1 \stackrel{a}{\simeq} P_2 \rightarrow N_2 \equiv (\Xi_1 \cup \Xi_2, Q \rightarrow M, \hat{\tau}_1 \cup \hat{\tau}'_1, \hat{\tau}_2 \cup \hat{\tau}'_2)} \text{ AUARROW} \\
\\
\frac{\text{if other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \stackrel{a}{\simeq} M \equiv (\hat{\alpha}^-_{\{N,M\}}, \hat{\alpha}^-_{\{N,M\}}, (\hat{\alpha}^-_{\{N,M\}} \approx N), (\hat{\alpha}^-_{\{N,M\}} \approx M))} \text{ AUAU} \\
\\
\boxed{\Gamma \vdash e_1 \& e_2 = e_3} \quad \text{Subtyping Constraint Entry Merge} \\
\\
\frac{\Gamma \models P_1 \vee P_2 = Q}{\Gamma \vdash (\hat{\alpha}^+ \succcurlyeq P_1) \& (\hat{\alpha}^+ \succcurlyeq P_2) = (\hat{\alpha}^+ \succcurlyeq Q)} \text{ SCMESUPSUP} \\
\\
\frac{\Gamma; \cdot \models P \succcurlyeq Q \equiv \cdot}{\Gamma \vdash (\hat{\alpha}^+ \preceq P) \& (\hat{\alpha}^+ \succcurlyeq Q) = (\hat{\alpha}^+ \preceq P)} \text{ SCMEEQSUP}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma; \cdot \models Q \geq P = \cdot}{\Gamma \vdash (\hat{\alpha}^+ : \geq P) \& (\hat{\alpha}^+ : \simeq Q) = (\hat{\alpha}^+ : \simeq Q)} \text{SCMESUP} \text{Eq} \\
\\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\Gamma \vdash (\hat{\alpha}^+ : \simeq P) \& (\hat{\alpha}^+ : \simeq P') = (\hat{\alpha}^+ : \simeq P)} \text{SCMEPEQ} \text{Eq} \\
\\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\Gamma \vdash (\hat{\alpha}^- : \simeq N) \& (\hat{\alpha}^- : \simeq N') = (\hat{\alpha}^- : \simeq N)} \text{SCMENEQ} \text{Eq} \\
\\
\boxed{\Theta \vdash SC_1 \& SC_2 = SC_3} \quad \text{Merge of subtyping constraints} \\
\boxed{\Gamma \vdash e_1 \& e_2 = e_3} \\
\\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\Gamma \vdash (\hat{\alpha}^+ : \simeq P) \& (\hat{\alpha}^+ : \simeq P') = (\hat{\alpha}^+ : \simeq P)} \text{UCMEPEQ} \text{Eq} \\
\\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\Gamma \vdash (\hat{\alpha}^- : \simeq N) \& (\hat{\alpha}^- : \simeq N') = (\hat{\alpha}^- : \simeq N)} \text{UCMENEQ} \text{Eq} \\
\\
\boxed{\Theta \vdash UC_1 \& UC_2 = UC_3} \quad \text{Merge of unification constraints} \\
\boxed{\Gamma \vdash P : e} \quad \text{Positive type satisfies with the subtyping constraint entry} \\
\\
\frac{\Gamma \vdash P \geq Q}{\Gamma \vdash P : (\hat{\alpha}^+ : \geq Q)} \text{SATSCESUP} \\
\\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\Gamma \vdash P : (\hat{\alpha}^+ : \simeq Q)} \text{SATSCEPEQ} \\
\\
\boxed{\Gamma \vdash N : e} \quad \text{Negative type satisfies with the subtyping constraint entry} \\
\\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\Gamma \vdash N : (\hat{\alpha}^- : \simeq M)} \text{SATSCENEQ} \\
\\
\boxed{e_1 \text{ singular with } P} \quad \text{Positive Subtyping Constraint Entry Is Singular} \\
\\
\frac{}{\hat{\alpha}^+ : \simeq P \text{ singular with nf}(P)} \text{SINGPEQ} \\
\\
\frac{}{\hat{\alpha}^+ : \geq \exists \alpha^-. \alpha^+ \text{ singular with } \alpha^+} \text{SINGSUPVAR} \\
\\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\hat{\alpha}^+ : \geq \exists \alpha^-. \downarrow N \text{ singular with } \exists \alpha^-. \downarrow \alpha^-} \text{SINGSUPSHIFT} \\
\\
\boxed{e_1 \text{ singular with } N} \quad \text{Negative Subtyping Constraint Entry Is Singular} \\
\\
\frac{}{\hat{\alpha}^- : \simeq N \text{ singular with nf}(N)} \text{SINGNEQ} \\
\\
\boxed{SC \text{ singular with } \hat{\sigma}} \quad \text{Subtyping Constraint Is Singular} \\
\boxed{N \simeq^D M} \quad \text{Negative type equivalence} \\
\\
\frac{}{\alpha^- \simeq^D \alpha^-} \text{E1NVAR} \\
\\
\frac{P \simeq^D Q}{\uparrow P \simeq^D \uparrow Q} \text{E1SHIFTU} \\
\\
\frac{P \simeq^D Q \quad N \simeq^D M}{P \rightarrow N \simeq^D Q \rightarrow M} \text{E1ARROW}
\end{array}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\forall \alpha^+. N \simeq^D \forall \beta^+. M} \quad \text{E1FORALL}$$

$\boxed{P \simeq^D Q}$ Positive type equivalence

$$\frac{}{\alpha^+ \simeq^D \alpha^+} \quad \text{E1PVAR}$$

$$\frac{N \simeq^D M}{\downarrow N \simeq^D \downarrow M} \quad \text{E1SHIFTD}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\exists \alpha^-. P \simeq^D \exists \beta^-. Q} \quad \text{E1EXISTS}$$

$\boxed{P \simeq^D Q}$ Positive unification type equivalence

$\boxed{N \simeq^D M}$ Positive unification type equivalence

$\boxed{\Gamma \vdash N \simeq^{\leq} M}$ Negative subtyping-induced equivalence

$$\frac{\Gamma \vdash N \leq M \quad \Gamma \vdash M \leq N}{\Gamma \vdash N \simeq^{\leq} M} \quad \text{D1NDEF}$$

$\boxed{\Gamma \vdash P \simeq^{\leq} Q}$ Positive subtyping-induced equivalence

$$\frac{\Gamma \vdash P \geq Q \quad \Gamma \vdash Q \geq P}{\Gamma \vdash P \simeq^{\leq} Q} \quad \text{D1PDEF}$$

$\boxed{\Gamma \vdash N \leq M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leq \alpha^-} \quad \text{D1NVAR}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\Gamma \vdash \uparrow P \leq \uparrow Q} \quad \text{D1SHIFTU}$$

$$\frac{\Gamma \vdash P \geq Q \quad \Gamma \vdash N \leq M}{\Gamma \vdash P \rightarrow N \leq Q \rightarrow M} \quad \text{D1ARROW}$$

$$\frac{\Gamma, \vec{\beta}^+ \vdash \sigma : \vec{\alpha}^+ \quad \Gamma, \vec{\beta}^+ \vdash [\sigma]N \leq M}{\Gamma \vdash \forall \alpha^+. N \leq \forall \beta^+. M} \quad \text{D1FORALL}$$

$\boxed{\Gamma \vdash P \geq Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geq \alpha^+} \quad \text{D1PVAR}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\Gamma \vdash \downarrow N \geq \downarrow M} \quad \text{D1SHIFTD}$$

$$\frac{\Gamma, \vec{\beta}^- \vdash \sigma : \vec{\alpha}^- \quad \Gamma, \vec{\beta}^- \vdash [\sigma]P \geq Q}{\Gamma \vdash \exists \alpha^-. P \geq \exists \beta^-. Q} \quad \text{D1EXISTS}$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq^{\leq} \sigma_2 : \Gamma_1}$ Equivalence of substitutions

$\boxed{\Gamma \vdash \sigma_1 \simeq^{\leq} \sigma_2 : vars}$ Equivalence of substitutions

$\boxed{\Theta \vdash \hat{\sigma}_1 \simeq^{\leq} \hat{\sigma}_2 : vars}$ Equivalence of unification substitutions

$\boxed{\Gamma \vdash \hat{\sigma}_1 \simeq^{\leq} \hat{\sigma}_2 : vars}$ Equivalence of unification substitutions

$\boxed{\Gamma \vdash \Phi_1 \simeq^{\leq} \Phi_2}$ Equivalence of contexts

$\boxed{\Gamma; \Phi \vdash v : P}$ Positive type inference

$$\frac{x : P \in \Phi}{\Gamma; \Phi \vdash x : P} \quad \text{DTVAR}$$

$\frac{\Gamma; \Phi \vdash c : N}{\Gamma; \Phi \vdash \{c\} : \downarrow N} \text{DTTHUNK}$	
$\frac{\Gamma \vdash Q \quad \Gamma; \Phi \vdash v : P \quad \Gamma \vdash Q \geq P}{\Gamma; \Phi \vdash (v : Q) : Q} \text{DTPANNOT}$	
$\frac{\text{<<multiple parses>>}}{\Gamma; \Phi \vdash v : P'} \text{DTPEQUIV}$	
$\boxed{\Gamma; \Phi \vdash c : N}$	Negative type inference
$\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \lambda x : P. c : P \rightarrow N} \text{DTTLAM}$	
$\frac{\Gamma, \alpha^+; \Phi \vdash c : N}{\Gamma; \Phi \vdash \Lambda \alpha^+. c : \forall \alpha^+. N} \text{DTTLAM}$	
$\frac{\Gamma; \Phi \vdash v : P}{\Gamma; \Phi \vdash \text{return } v : \uparrow P} \text{DTRETURN}$	
$\frac{\Gamma; \Phi \vdash v : P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \text{let } x = v; c : N} \text{DTVARLET}$	
$\frac{\Gamma; \Phi \vdash v : \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q \text{ unique} \quad \Gamma; \Phi, x : Q \vdash c : N}{\Gamma; \Phi \vdash \text{let } x = v(\vec{v}); c : N} \text{DTAPPLET}$	
$\frac{\Gamma \vdash P \quad \Gamma; \Phi \vdash v : \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q \quad \Gamma \vdash \uparrow Q \leq \uparrow P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \text{let } x : P = v(\vec{v}); c : N} \text{DTAPPLETANN}$	
$\frac{\text{<<multiple parses>>}}{\Gamma; \Phi \vdash \text{let}^3(\vec{\alpha^-}, x) = v; c : N} \text{DTUNPACK}$	
$\frac{\text{<<multiple parses>>}}{\Gamma; \Phi \vdash (c : M) : M} \text{DTNANNOT}$	
$\frac{\text{<<multiple parses>>}}{\Gamma; \Phi \vdash c : N'} \text{DTNEQUIV}$	
$\boxed{\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M}$	Application type inference
$\frac{\text{<<multiple parses>>}}{\Gamma; \Phi \vdash N \bullet \cdot \Rightarrow N'} \text{DTEMPTYPAPP}$	
$\frac{\Gamma; \Phi \vdash v : P \quad \Gamma \vdash Q \geq P \quad \Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M}{\Gamma; \Phi \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M} \text{DTARROWAPP}$	
$\frac{\Gamma \vdash \sigma : \vec{\alpha^+} \quad \Gamma; \Phi \vdash [\sigma] N \bullet \vec{v} \Rightarrow M \quad \vec{v} \neq \cdot \quad \vec{\alpha^+} \neq \cdot}{\Gamma; \Phi \vdash \forall \alpha^+. N \bullet \vec{v} \Rightarrow M} \text{DTFORALLAPP}$	
$\boxed{N = M}$	Negative type equality (alpha-equivalence)
$\boxed{P = Q}$	Positive type equality (alpha-equivalence)
$\boxed{P = Q}$	
$\boxed{P_1 \vee P_2}$	

$$\mathbf{ord} \, vars \mathbf{in} \, P$$

$$\mathbf{ord} \, vars \mathbf{in} \, N$$

$$\mathbf{ord} \, vars \mathbf{in} \, P$$

$$\mathbf{ord} \, vars \mathbf{in} \, N$$

$$\mathbf{nf} \, (N')$$

$$\mathbf{nf} \, (P')$$

$$\mathbf{nf} \, (N')$$

$$\mathbf{nf} \, (P')$$

$$\mathbf{nf} \, (\vec{N}')$$

$$\mathbf{nf} \, (\vec{P}')$$

$$\mathbf{nf} \, (\sigma')$$

$$\mathbf{nf} \, (\hat{\sigma}')$$

$$\mathbf{nf} \, (\mu')$$

$$\sigma'|_{vars}$$

$$\hat{\sigma}'|_{vars}$$

$$\widehat{\tau}'|_{vars}$$

$$\Xi'|_{vars}$$

$$SC'|_{vars}$$

$$UC'|_{vars}$$

$$e_1 \ \& \ e_2$$

$$e_1 \ \& \ e_2$$

$$UC_1 \ \& \ UC_2$$

$$UC_1 \cup UC_2$$

$$\Gamma_1 \cup \Gamma_2$$

$$SC_1 \ \& \ SC_2$$

$$\widehat{\tau}_1 \ \& \ \widehat{\tau}_2$$

$$\mathbf{dom} \, (UC)$$

$$\mathbf{dom} \, (SC)$$

$$\mathbf{dom} \, (\widehat{\sigma})$$

$$\mathbf{dom} \, (\widehat{\tau})$$

$\boxed{\text{dom}(\Theta)}$

$\boxed{|SC|}$

$\boxed{\text{fv } N}$

$\boxed{\text{fv } P}$

$\boxed{\text{fv } P}$

$\boxed{\text{fv } N}$

$\boxed{\text{uv } N}$

$\boxed{\text{uv } P}$

$\boxed{\Gamma \models P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\begin{array}{c}
\overline{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVAR} \\
\frac{\Gamma, \cdot \models \mathbf{nf}(\downarrow N) \stackrel{a}{\cong} \mathbf{nf}(\downarrow M) \models (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N \vee \downarrow M = \exists \alpha^{\rightarrow}. [\alpha^{\rightarrow} / \Xi] P} \quad \text{LUBSHIFT} \\
\frac{\Gamma, \alpha^{\rightarrow}, \beta^{\rightarrow} \models P_1 \vee P_2 = Q}{\Gamma \models \exists \alpha^{\rightarrow}. P_1 \vee \exists \beta^{\rightarrow}. P_2 = Q} \quad \text{LUBEXISTS}
\end{array}$$

$\boxed{\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q}$

$$\frac{\begin{array}{c} \text{\textcolor{red}{<<multiple parses>>}} \\ \Delta, \beta^{\pm}, \gamma^{\pm} \models [\beta^{\pm} / \alpha^{\pm}] P \vee [\gamma^{\pm} / \alpha^{\pm}] P = Q \end{array}}{\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q} \quad \text{LUBUPGRADE}$$

$\boxed{\mathbf{nf}(N) = M}$

$$\begin{array}{c}
\overline{\mathbf{nf}(\alpha^-) = \alpha^-} \quad \text{NRMNVAR} \\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\mathbf{nf}(\uparrow P) = \uparrow Q} \quad \text{NRMSHIFTU} \\
\frac{\text{\textcolor{red}{<<multiple parses>>}}}{\mathbf{nf}(P \rightarrow N) = Q \rightarrow M} \quad \text{NRMARROW}
\end{array}$$

$$\begin{array}{c}
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(\overrightarrow{\forall \alpha^+}.N) = \overrightarrow{\forall \alpha^{+'}}.N'} \quad \text{NRMFORALL} \\
\boxed{\mathbf{nf}(P) = Q} \\
\frac{}{\overline{\mathbf{nf}(\alpha^+) = \alpha^+}} \quad \text{NRMPVAR} \\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(\downarrow N) = \downarrow M} \quad \text{NRMSHIFTD} \\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(\overrightarrow{\exists \alpha^-}.P) = \overrightarrow{\exists \alpha^{-'}}.P'} \quad \text{NRME EXISTS} \\
\boxed{\mathbf{nf}(N) = M} \\
\frac{}{\overline{\mathbf{nf}(\hat{\alpha}^-) = \hat{\alpha}^-}} \quad \text{NRMNUVAR} \\
\boxed{\mathbf{nf}(P) = Q} \\
\frac{}{\overline{\mathbf{nf}(\hat{\alpha}^+) = \hat{\alpha}^+}} \quad \text{NRMPUVAR} \\
\boxed{\mathbf{ord vars in } N = \vec{\alpha}} \\
\frac{\alpha^- \in \text{vars}}{\mathbf{ord vars in } \alpha^- = \alpha^-} \quad \text{ONVARIN} \\
\frac{\alpha^- \notin \text{vars}}{\langle\langle \text{multiple parses} \rangle\rangle} \quad \text{ONVARNIN} \\
\frac{\mathbf{ord vars in } P = \vec{\alpha}}{\mathbf{ord vars in } \uparrow P = \vec{\alpha}} \quad \text{OSHIFTU} \\
\frac{\mathbf{ord vars in } P = \vec{\alpha}_1 \quad \mathbf{ord vars in } N = \vec{\alpha}_2}{\mathbf{ord vars in } P \rightarrow N = \vec{\alpha}_1, (\vec{\alpha}_2 \setminus \vec{\alpha}_1)} \quad \text{OARROW} \\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{ord vars in } \overrightarrow{\forall \alpha^+}.N = \vec{\alpha}} \quad \text{OFORALL} \\
\boxed{\mathbf{ord vars in } P = \vec{\alpha}} \\
\frac{\alpha^+ \in \text{vars}}{\mathbf{ord vars in } \alpha^+ = \alpha^+} \quad \text{OPVARIN} \\
\frac{\alpha^+ \notin \text{vars}}{\langle\langle \text{multiple parses} \rangle\rangle} \quad \text{OPVARNIN} \\
\frac{\mathbf{ord vars in } N = \vec{\alpha}}{\mathbf{ord vars in } \downarrow N = \vec{\alpha}} \quad \text{OSHIFTD} \\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{ord vars in } \overrightarrow{\exists \alpha^-}.P = \vec{\alpha}} \quad \text{OEXISTS} \\
\boxed{\mathbf{ord vars in } N = \vec{\alpha}} \\
\frac{}{\langle\langle \text{multiple parses} \rangle\rangle} \quad \text{ONUVAR} \\
\boxed{\mathbf{ord vars in } P = \vec{\alpha}}
\end{array}$$

$$\begin{array}{c}
\frac{}{\text{<<multiple parses>>}} \text{OPUVar} \\
\boxed{\Gamma; \Theta \models N \overset{u}{\simeq} M \Rightarrow UC} \quad \text{Negative unification} \\
\\
\frac{}{\Gamma; \Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \Rightarrow \cdot} \text{UNVar} \\
\frac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \Rightarrow UC}{\Gamma; \Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \Rightarrow UC} \text{USHIFTU} \\
\frac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \Rightarrow UC_1 \quad \Gamma; \Theta \models N \overset{u}{\simeq} M \Rightarrow UC_2}{\Gamma; \Theta \models P \rightarrow N \overset{u}{\simeq} Q \rightarrow M \Rightarrow UC_1 \ \& \ UC_2} \text{UArrow} \\
\frac{\Gamma, \overrightarrow{\alpha^+}; \Theta \models N \overset{u}{\simeq} M \Rightarrow UC}{\Gamma; \Theta \models \forall \alpha^+. N \overset{u}{\simeq} \forall \alpha^+. M \Rightarrow UC} \text{UForall} \\
\frac{\Theta(\hat{\alpha}^-) \vdash N}{\Gamma; \Theta \models \hat{\alpha}^- \overset{u}{\simeq} N \Rightarrow (\hat{\alpha}^- : \simeq N)} \text{UNUVar}
\end{array}$$

$$\boxed{\Gamma; \Theta \models P \overset{u}{\simeq} Q \Rightarrow UC} \quad \text{Positive unification}$$

$$\begin{array}{c}
\frac{}{\Gamma; \Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \Rightarrow \cdot} \text{UPVar} \\
\frac{\Gamma; \Theta \models N \overset{u}{\simeq} M \Rightarrow UC}{\Gamma; \Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \Rightarrow UC} \text{USHIFTD} \\
\frac{\Gamma, \overrightarrow{\alpha^-}; \Theta \models P \overset{u}{\simeq} Q \Rightarrow UC}{\Gamma; \Theta \models \exists \alpha^-. P \overset{u}{\simeq} \exists \alpha^-. Q \Rightarrow UC} \text{UExists} \\
\frac{\Theta(\hat{\alpha}^+) \vdash P}{\Gamma; \Theta \models \hat{\alpha}^+ \overset{u}{\simeq} P \Rightarrow (\hat{\alpha}^+ : \simeq P)} \text{UPUVar}
\end{array}$$

$$\boxed{\Gamma \vdash N} \quad \text{Negative type well-formedness}$$

$$\begin{array}{c}
\frac{\alpha^- \in \Gamma}{\Gamma \vdash \alpha^-} \text{WFTNVar} \\
\frac{\Gamma \vdash P}{\Gamma \vdash \uparrow P} \text{WFTSHIFTU} \\
\frac{\Gamma \vdash P \quad \Gamma \vdash N}{\Gamma \vdash P \rightarrow N} \text{WFTArrow} \\
\frac{\Gamma, \overrightarrow{\alpha^+} \vdash N}{\Gamma \vdash \forall \alpha^+. N} \text{WFTForall}
\end{array}$$

$$\boxed{\Gamma \vdash P} \quad \text{Positive type well-formedness}$$

$$\begin{array}{c}
\frac{\alpha^+ \in \Gamma}{\Gamma \vdash \alpha^+} \text{WFTPVar} \\
\frac{\Gamma \vdash N}{\Gamma \vdash \downarrow N} \text{WFTSHIFTD}
\end{array}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-} \vdash P}{\Gamma \vdash \overrightarrow{\exists \alpha^-}.P} \quad \text{WFT}_{\text{EXISTS}}$$

$\boxed{\Gamma; \Xi \vdash N}$ Negative algorithmic type well-formedness

$$\frac{\alpha^- \in \Gamma}{\Gamma; \Xi \vdash \alpha^-} \quad \text{WFATN}_{\text{VAR}}$$

$$\frac{\hat{\alpha}^- \in \Xi}{\Gamma; \Xi \vdash \hat{\alpha}^-} \quad \text{WFATNU}_{\text{VAR}}$$

$$\frac{\Gamma; \Xi \vdash P}{\Gamma; \Xi \vdash \uparrow P} \quad \text{WFAT}_{\text{SHIFTU}}$$

$$\frac{\Gamma; \Xi \vdash P \quad \Gamma; \Xi \vdash N}{\Gamma; \Xi \vdash P \rightarrow N} \quad \text{WFAT}_{\text{ARROW}}$$

$$\frac{\Gamma, \overrightarrow{\alpha^+}; \Xi \vdash N}{\Gamma; \Xi \vdash \overrightarrow{\forall \alpha^+}.N} \quad \text{WFAT}_{\text{FORALL}}$$

$\boxed{\Gamma; \Xi \vdash P}$ Positive algorithmic type well-formedness

$$\frac{\alpha^+ \in \Gamma}{\Gamma; \Xi \vdash \alpha^+} \quad \text{WFATP}_{\text{VAR}}$$

$$\frac{\hat{\alpha}^+ \in \Xi}{\Gamma; \Xi \vdash \hat{\alpha}^+} \quad \text{WFATPU}_{\text{VAR}}$$

$$\frac{\Gamma; \Xi \vdash N}{\Gamma; \Xi \vdash \downarrow N} \quad \text{WFAT}_{\text{SHIFTD}}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}; \Xi \vdash P}{\Gamma; \Xi \vdash \overrightarrow{\exists \alpha^-}.P} \quad \text{WFAT}_{\text{EXISTS}}$$

$\boxed{\Gamma \vdash \overrightarrow{N}}$ Negative type list well-formedness

$\boxed{\Gamma \vdash \overrightarrow{P}}$ Positive type list well-formedness

$\boxed{\Gamma; \Xi_2 \vdash \hat{\tau} : \Xi_1}$ Antiunification substitution well-formedness

$\boxed{\Gamma \vdash \Xi \vdash \Theta}$ Unification context well-formedness

$\boxed{\Gamma_1 \vdash \sigma : \Gamma_2}$ Substitution signature

$\boxed{\Theta \vdash \hat{\sigma} : \Xi}$ Unification substitution signature

$\boxed{\Gamma \vdash \hat{\sigma} : \Xi}$ Unification substitution general signature

$\boxed{\Theta \vdash \hat{\sigma} : UC}$ Unification substitution satisfies unification constraint

$\boxed{\Theta \vdash \hat{\sigma} : SC}$ Unification substitution satisfies subtyping constraint

$\boxed{\Gamma \vdash e}$ Unification constraint entry well-formedness

$\boxed{\Gamma \vdash e}$ Subtyping constraint entry well-formedness

$\boxed{\Gamma \vdash P : e}$ Positive type satisfies unification constraint

$\boxed{\Gamma \vdash N : e}$ Negative type satisfies unification constraint

$\boxed{\Gamma \vdash P : e}$ Positive type satisfies subtyping constraint

$\boxed{\Gamma \vdash N : e}$ Negative type satisfies subtyping constraint

$\boxed{\Theta \vdash UC : \Xi}$ Unification constraint well-formedness with specified domain

$\boxed{\Theta \vdash SC : \Xi}$ Subtyping constraint well-formedness with specified domain

$\boxed{\Theta \vdash UC}$ Unification constraint well-formedness

$\boxed{\Theta \vdash SC}$ Subtyping constraint well-formedness

$\boxed{\Gamma \vdash \overrightarrow{v}}$ Argument List well-formedness

$\boxed{\Gamma \vdash \Phi}$ Context well-formedness

$\boxed{\Gamma \vdash v}$ Value well-formedness

$\overline{\Gamma \vdash x}$ WFALLVAR

$\boxed{\Gamma \vdash c}$ Computation well-formedness

$\frac{\Gamma \vdash v \quad \Gamma \vdash c \quad \Gamma \vdash \vec{v}}{\Gamma \vdash \mathbf{let} \ x = v(\vec{v}); c}$ WFALLAPPLET

Definition rules: 94 good 33 bad

Definition rule clauses: 208 good 34 bad