α, β, α, β, γ, δ	type variables
n, m, i, j	index variables

n, m	$::=$			cohort index
	$\vert$	0		
	$\vert$	$n+1$		
$\alpha^+, \beta^+, \gamma^+, \delta^+$	$::=$			positive variable
	$\vert$	α^+		
	$\vert$	α^{+n}		
	$\vert$	$\beta^\pm$		
$\alpha^-, \beta^-, \gamma^-, \delta^-$	$::=$			negative variable
	$\vert$	α^-		
	$\vert$	α^{-n}		
	$\vert$	$\alpha^\pm$		
$\alpha^\pm, \beta^\pm$	$::=$			positive or negative variable
	$\vert$	$\alpha^\pm$		
	$\vert$	$\alpha^{\pm n}$		
σ	$::=$			substitution
	$\vert$	$\cdot$		
	$\vert$	id		
	$\vert$	P/α^+		
	$\vert$	N/α^-		
	$\vert$	$\overrightarrow{P}/\overrightarrow{\alpha^+}$		
	$\vert$	$\overrightarrow{N}/\overrightarrow{\alpha^-}$		
	$\vert$	$\mathbf{pmas}\,/\overrightarrow{\alpha^+}$		
	$\vert$	$\mathbf{nmas}\,/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	$\vert$	$\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha^-}}$		
	$\vert$	μ		
	$\vert$	$\widehat{\sigma}$		
	$\vert$	$\sigma_1 \circ \sigma_2$		
	$\vert$	$\vec{\alpha}_1/\vec{\alpha}_2$		
	$\vert$	(σ)	S	
	$\vert$	$\overline{\sigma_i}^{\,i}$		concatenate
	$\vert$	$\mathbf{nf}\,(\sigma')$	M	
	$\vert$	$\sigma'\vert_{vars}$	M	
e	$::=$			entry of a unification solution
	$\vert$	$\widehat{\alpha}^+ :\approx P$		
	$\vert$	$\widehat{\alpha}^- :\approx N$		
	$\vert$	$\widehat{\alpha}^+ :\geqslant P$		
	$\vert$	(e)	S	
	$\vert$	$\widehat{\sigma}(\widehat{\alpha}^+)$	M	
	$\vert$	$\widehat{\sigma}(\widehat{\alpha}^-)$	M	
	$\vert$	$e_1 \,\&\, e_2$	M	
$\widehat{\sigma}$	$::=$			unification solution (substitution)

	$\vert$	$\cdot$		
	$\vert$	e		
	$\vert$	$\widehat{\sigma}\backslash vars$		
	$\vert$	$\widehat{\sigma}\vert vars$		
	$\vert$	$\widehat{\sigma}_1 \cup \widehat{\sigma}_2$		
	$\vert$	$\overline{\widehat{\sigma}_i}^{\,i}$		concatenate
	$\vert$	$(\widehat{\sigma})$	S	
	$\vert$	$\mathbf{nf}\,(\widehat{\sigma}')$	M	
	$\vert$	$\widehat{\sigma}'\vert_{vars}$	M	
	$\vert$	$\widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2$	M	
$\widehat{\tau},\ \widehat{\rho}$	$::=$			anti-unification substitution
	$\vert$	$\cdot$		
	$\vert$	$\widehat{\alpha}^- :\approx N$		
	$\vert$	$\widehat{\alpha}^- :\approx N$		
	$\vert$	$\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}$		
	$\vert$	$\overrightarrow{N}/\overrightarrow{\widehat{\alpha}^-}$		
	$\vert$	$\widehat{\tau}_1 \cup \widehat{\tau}_2$		
	$\vert$	$\overline{\widehat{\tau}_i}^{\,i}$		concatenate
	$\vert$	$(\widehat{\tau})$	S	
	$\vert$	$\widehat{\tau}'\vert_{vars}$	M	
	$\vert$	$\widehat{\tau}_1 \;\&\; \widehat{\tau}_2$	M	
$P,\ Q$	$::=$			positive types
	$\vert$	α^+		
	$\vert$	$\downarrow N$		
	$\vert$	$\exists \alpha^-.P$		
	$\vert$	$[\sigma]P$	M	
$N,\ M$	$::=$			negative types
	$\vert$	α^-		
	$\vert$	$\uparrow P$		
	$\vert$	$\forall \alpha^+.N$		
	$\vert$	$P \rightarrow N$		
	$\vert$	$[\sigma]N$	M	
$\overrightarrow{\alpha^+},\ \overrightarrow{\beta^+},\ \overrightarrow{\gamma^+},\ \overrightarrow{\delta^+}$	$::=$			positive variable list
	$\vert$	$\cdot$		empty list
	$\vert$	α^+		a variable
	$\vert$	$\overrightarrow{\alpha^+}$		a variable
	$\vert$	$\overline{\overrightarrow{\alpha^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^-},\ \overrightarrow{\beta^-},\ \overrightarrow{\gamma^-},\ \overrightarrow{\delta^-}$	$::=$			negative variables
	$\vert$	$\cdot$		empty list
	$\vert$	α^-		a variable
	$\vert$	$\overrightarrow{\alpha^-}$		variables
	$\vert$	$\overline{\overrightarrow{\alpha^-}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^\pm},\ \overrightarrow{\beta^\pm},\ \overrightarrow{\gamma^\pm},\ \overrightarrow{\delta^\pm}$	$::=$			positive or negative variable list

	$\mid$	$\cdot$		empty list
	$\mid$	$\alpha^{\pm}$		a variable
	$\mid$	$\overrightarrow{\mathbf{pa}}$		variables
	$\mid$	$\overline{\overrightarrow{\alpha^{\pm}}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$			multi-quantified positive types
	$\mid$	α^{+}		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists\overrightarrow{\alpha^{-}}.P$		$P \neq \exists \ldots$
	$\mid$	$[\sigma]P$	M	
	$\mid$	$[\widehat{\tau}]P$	M	
	$\mid$	$[\widehat{\sigma}]P$	M	
	$\mid$	$[\mu]P$	M	
	$\mid$	(P)	S	
	$\mid$	$P_1 \vee P_2$	M	
	$\mid$	$\mathbf{nf}\,(P')$	M	
$N,\ M$	$::=$			multi-quantified negative types
	$\mid$	α^{-}		
	$\mid$	$\uparrow P$		
	$\mid$	$P \rightarrow N$		
	$\mid$	$\forall\overrightarrow{\alpha^{+}}.N$		$N \neq \forall \ldots$
	$\mid$	$[\sigma]N$	M	
	$\mid$	$[\widehat{\tau}]N$	M	
	$\mid$	$[\mu]N$	M	
	$\mid$	$[\widehat{\sigma}]N$	M	
	$\mid$	(N)	S	
	$\mid$	$\mathbf{nf}\,(N')$	M	
$\vec{P},\ \vec{Q}$	$::=$			list of positive types
	$\mid$	$\cdot$		empty list
	$\mid$	P		a singel type
	$\mid$	$\overline{\vec{P}_i}^{\,i}$		concatenate lists
	$\mid$	$\mathbf{nf}\,(\vec{P}')$	M	
$\vec{N},\ \vec{M}$	$::=$			list of negative types
	$\mid$	$\cdot$		empty list
	$\mid$	N		a singel type
	$\mid$	$\overline{\vec{N}_i}^{\,i}$		concatenate lists
	$\mid$	$\mathbf{nf}\,(\vec{N}')$	M	
$\Delta,\ \Gamma$	$::=$			declarative type context
	$\mid$	$\cdot$		empty context
	$\mid$	$\overrightarrow{\alpha^{+}}$		list of variables
	$\mid$	$\overrightarrow{\alpha^{-}}$		list of variables
	$\mid$	$\overrightarrow{\alpha^{\pm}}$		list of variables
	$\mid$	$vars$		
	$\mid$	$\overline{\Gamma_i}^{\,i}$		concatenate contexts
	$\mid$	(Γ)	S	
	$\mid$	$\Theta(\widehat{\alpha}^{+})$	M	

	$\vert$	$\Theta(\widehat{\alpha}^-)$	M	
Θ	::=			unification type variable context
	$\vert$	$\cdot$		empty context
	$\vert$	$\overrightarrow{\widetilde{\alpha}^+}$		list of variables
	$\vert$	$\overrightarrow{\widehat{\alpha}^-}$		list of variables
	$\vert$	$vars$		
	$\vert$	$\overline{\Theta_i}^{\,i}$		concatenate contexts
	$\vert$	(Θ)	S	
	$\vert$	$\Theta\vert_{vars}$		leave only those variables that are in the set
	$\vert$	$\Theta_1 \cup \Theta_2$		
Ξ	::=			anti-unification type variable context
	$\vert$	$\cdot$		empty context
	$\vert$	$\overrightarrow{\widehat{\alpha}^-}$		list of variables
	$\vert$	$\overline{\Xi_i}^{\,i}$		concatenate contexts
	$\vert$	(Ξ)	S	
	$\vert$	$\Xi_1 \cup \Xi_2$		
	$\vert$	$\Xi_1 \cap \Xi_2$		
	$\vert$	$\Xi'\vert_{vars}$	M	
$\vec{\alpha},\ \vec{\beta}$	::=			ordered positive or negative variables
	$\vert$	$\cdot$		empty list
	$\vert$	$\overrightarrow{\alpha^+}$		list of variables
	$\vert$	$\overrightarrow{\alpha^-}$		list of variables
	$\vert$	$\overrightarrow{\alpha^\pm}$		list of variables
	$\vert$	$\overrightarrow{\widetilde{\alpha}^+}$		list of variables
	$\vert$	$\overrightarrow{\widehat{\alpha}^-}$		list of variables
	$\vert$	$\vec{\alpha}_1 \backslash vars$		setminus
	$\vert$	Γ		context
	$\vert$	$vars$		
	$\vert$	$\overline{\vec{\alpha}_i}^{\,i}$		concatenate contexts
	$\vert$	$(\vec{\alpha})$	S	parenthesis
	$\vert$	$[\mu]\vec{\alpha}$	M	apply moving to list
	$\vert$	$\mathbf{ord}\,vars\,\mathbf{in}\,P$	M	
	$\vert$	$\mathbf{ord}\,vars\,\mathbf{in}\,N$	M	
	$\vert$	$\mathbf{ord}\,vars\,\mathbf{in}\,P$	M	
	$\vert$	$\mathbf{ord}\,vars\,\mathbf{in}\,N$	M	
$vars$	::=			set of variables
	$\vert$	$\varnothing$		empty set
	$\vert$	$\mathbf{fv}\,P$		free variables
	$\vert$	$\mathbf{fv}\,N$		free variables
	$\vert$	$\mathbf{fv}\,\mathbf{imP}$		free variables
	$\vert$	$\mathbf{fv}\,\mathbf{imN}$		free variables
	$\vert$	$vars_1 \cap vars_2$		set intersection
	$\vert$	$vars_1 \cup vars_2$		set union
	$\vert$	$vars_1 \backslash vars_2$		set complement
	$\vert$	$\mathbf{mv}\,\mathbf{imP}$		movable variables
	$\vert$	$\mathbf{mv}\,\mathbf{imN}$		movable variables

	$\mid$	$\mathbf{uv}\ N$		unification variables
	$\mid$	$\mathbf{uv}\ P$		unification variables
	$\mid$	$\mathbf{fv}\ N$		free variables
	$\mid$	$\mathbf{fv}\ P$		free variables
	$\mid$	(vars)	S	parenthesis
	$\mid$	$\overrightarrow{\alpha}$		ordered list of variables
	$\mid$	$[\mu]\mathit{vars}$	M	apply moving to varset
	$\mid$	$\mathbf{dom}\,(\widehat{\sigma})$	M	
	$\mid$	$\mathbf{dom}\,(\widehat{\tau})$	M	
	$\mid$	$\mathbf{dom}\,(\Theta)$	M	
μ	$::=$			
	$\mid$	$\cdot$		empty moving
	$\mid$	$\mathit{pma}1 \mapsto \mathit{pma}2$		Positive unit substitution
	$\mid$	$\mathit{nma}1 \mapsto \mathit{nma}2$		Positive unit substitution
	$\mid$	$\mu_1 \cup \mu_2$	M	Set-like union of movings
	$\mid$	$\mu_1 \circ \mu_2$	M	Composition
	$\mid$	$\overline{\mu_i}^{\,i}$		concatenate movings
	$\mid$	$\mu\vert_{\mathit{vars}}$	M	restriction on a set
	$\mid$	μ^{-1}	M	inversion
	$\mid$	$\mathbf{nf}\,(\mu')$	M	
$\widehat{\alpha}^{\pm}$	$::=$			positive/negative unification variable
	$\mid$	$\widehat{\alpha}^{\pm}$		
$\widehat{\alpha}^{+}$	$::=$			positive unification variable
	$\mid$	$\widehat{\alpha}^{+}$		
	$\mid$	$\widehat{\alpha}^{+}\{\Delta\}$		
	$\mid$	$\widehat{\alpha}^{\pm}$		
$\widehat{\alpha}^{-},\ \widehat{\beta}^{-}$	$::=$			negative unification variable
	$\mid$	$\widehat{\alpha}^{-}$		
	$\mid$	$\widehat{\alpha}^{-}_{\{N,M\}}$		
	$\mid$	$\widehat{\alpha}^{-}\{\Delta\}$		
	$\mid$	$\widehat{\alpha}^{\pm}$		
$\overrightarrow{\widetilde{\alpha}^{+}},\ \overrightarrow{\widetilde{\beta}^{+}}$	$::=$			positive unification variable list
	$\mid$	$\cdot$		empty list
	$\mid$	$\widehat{\alpha}^{+}$		a variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^{+}\{\Delta\}}$		from a normal variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^{+}}$		from a normal variable, context unspecified
	$\mid$	$\overline{\overrightarrow{\widetilde{\alpha}^{+}}_{i}}^{\,i}$		concatenate lists
$\overrightarrow{\widehat{\alpha}^{-}},\ \overrightarrow{\widehat{\beta}^{-}}$	$::=$			negative unification variable list
	$\mid$	$\cdot$		empty list
	$\mid$	$\widehat{\alpha}^{-}$		a variable
	$\mid$	Ξ		from an antiunification context
	$\mid$	$\overrightarrow{\widehat{\alpha}^{-}\{\Delta\}}$		from a normal variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^{-}}$		from a normal variable, context unspecified

	|	$\overline{\overrightarrow{\widehat{\alpha}^{-}}_{i}}^{\,i}$		concatenate lists
P, Q	::=			a positive algorithmic type (potentially with metavariables)
	|	α^{+}		
	|	$\mathbf{pma}$		
	|	$\widehat{\alpha}^{+}$		
	|	$\downarrow N$		
	|	$\exists\overrightarrow{\alpha^{-}}.P$		
	|	$[\sigma]P$	M	
	|	$[\widehat{\tau}]P$	M	
	|	$[\mu]P$	M	
	|	(P)	S	
	|	$\mathbf{nf}\,(P')$	M	
N, M	::=			a negative algorithmic type (potentially with metavariables)
	|	α^{-}		
	|	$\widehat{\alpha}^{-}$		
	|	$\uparrow P$		
	|	$P \to N$		
	|	$\forall\overrightarrow{\alpha^{+}}.N$		
	|	$[\sigma]N$	M	
	|	$[\widehat{\tau}]N$	M	
	|	$[\mu]N$	M	
	|	(N)	S	
	|	$\mathbf{nf}\,(N')$	M	
auSol	::=			
	|	$(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$		
	|	$(\Xi, N, \widehat{\tau}_1, \widehat{\tau}_2)$		
terminals	::=			
	|	$\exists$		
	|	$\forall$		
	|	$\uparrow$		
	|	$\downarrow$		
	|	$\to$		
	|	$\leftrightarrow$		
	|	$\in$		
	|	$\notin$		
	|	$\cdot$		
	|	$\vdash$		
	|	$\leqslant$		
	|	$\geqslant$		
	|	$\simeq$		
	|	$\cup$		
	|	$\cap$		
	|	$\setminus$		
	|	$\subseteq$		
	|	$\mapsto$		
	|	$\overset{u}{\simeq}$		

| $\overset{a}{\simeq}$
| $\varnothing$
| $\circ$
| $\Rightarrow$
| $\vDash$
| $\dashv$
| $\neq$
| $\equiv_n$
| $\vee$
| $\Downarrow$
| $:\geqslant$
| $:\simeq$

$formula$ ::=
| $judgement$
| $formula_1 \quad .. \quad formula_n$
| $\mu : vars_1 \leftrightarrow vars_2$
| μ **is bijective**
| $\widehat{\sigma}$ **is functional**
| $\widehat{\sigma}_1 \in \widehat{\sigma}_2$
| $\widehat{\sigma}_1 \subseteq \widehat{\sigma}_2$
| $vars_1 \subseteq vars_2$
| $vars_1 = vars_2$
| $vars$ **is fresh**
| $\alpha^- \notin vars$
| $\alpha^+ \notin vars$
| $\alpha^- \in vars$
| $\alpha^+ \in vars$
| $\widehat{\alpha}^+ \in vars$
| $\widehat{\alpha}^- \in vars$
| $\widehat{\alpha}^- \in \Theta$
| $\widehat{\alpha}^+ \in \Theta$
| if any other rule is not applicable
| $\overrightarrow{\alpha}_1 = \overrightarrow{\alpha}_2$
| $e_1 = e_2$
| $N = M$
| $N \neq M$
| $P \neq Q$

A ::=
| $\Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}$ — Negative subtyping
| $\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}$ — Positive supertyping

AU ::=
| $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$
| $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$

$E1$ ::=
| $N \simeq_1^D M$ — Negative multi-quantified type equivalence
| $P \simeq_1^D Q$ — Positive multi-quantified type equivalence

	$\mid$	$P \simeq Q$	
D1	::=		
	$\mid$	$\Gamma \vdash N \simeq_1^{\leqslant} M$	Negative equivalence on MQ types
	$\mid$	$\Gamma \vdash P \simeq_1^{\leqslant} Q$	Positive equivalence on MQ types
	$\mid$	$\Gamma \vdash N \leqslant_1 M$	Negative subtyping
	$\mid$	$\Gamma \vdash P \geqslant_1 Q$	Positive supertyping
	$\mid$	$\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1$	Equivalence of substitutions
D0	::=		
	$\mid$	$\Gamma \vdash N \simeq_0^{\leqslant} M$	Negative equivalence
	$\mid$	$\Gamma \vdash P \simeq_0^{\leqslant} Q$	Positive equivalence
	$\mid$	$\Gamma \vdash N \leqslant_0 M$	Negative subtyping
	$\mid$	$\Gamma \vdash P \geqslant_0 Q$	Positive supertyping
EQ	::=		
	$\mid$	$N = M$	Negative type equality (alpha-equivalence)
	$\mid$	$P = Q$	Positive type equuality (alphha-equivalence)
	$\mid$	$P = Q$	
LUBF	::=		
	$\mid$	$P_1 \vee P_2 === Q$	
	$\mid$	$\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, P === \vec{\alpha}$	
	$\mid$	$\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, N === \vec{\alpha}$	
	$\mid$	$\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, P === \vec{\alpha}$	
	$\mid$	$\mathbf{ord}\, \mathit{vars}\, \mathbf{in}\, N === \vec{\alpha}$	
	$\mid$	$\mathbf{nf}\,(N') === N$	
	$\mid$	$\mathbf{nf}\,(P') === P$	
	$\mid$	$\mathbf{nf}\,(N') === N$	
	$\mid$	$\mathbf{nf}\,(P') === P$	
	$\mid$	$\mathbf{nf}\,(\vec{N}') === \vec{N}$	
	$\mid$	$\mathbf{nf}\,(\vec{P}') === \vec{P}$	
	$\mid$	$\mathbf{nf}\,(\sigma') === \sigma$	
	$\mid$	$\mathbf{nf}\,(\mu') === \mu$	
	$\mid$	$\mathbf{nf}\,(\widehat{\sigma}') === \widehat{\sigma}$	
	$\mid$	$\sigma'\|_{\mathit{vars}}$	
	$\mid$	$\widehat{\sigma}'\|_{\mathit{vars}}$	
	$\mid$	$\widehat{\tau}'\|_{\mathit{vars}}$	
	$\mid$	$\Xi'\|_{\mathit{vars}}$	
	$\mid$	$e_1\ \&\ e_2$	
	$\mid$	$\widehat{\sigma}_1\ \&\ \widehat{\sigma}_2$	
	$\mid$	$\widehat{\tau}_1\ \&\ \widehat{\tau}_2$	
	$\mid$	$\mathbf{dom}\,(\widehat{\sigma}) === \mathit{vars}$	
	$\mid$	$\mathbf{dom}\,(\widehat{\tau}) === \mathit{vars}$	
	$\mid$	$\mathbf{dom}\,(\Theta) === \mathit{vars}$	
LUB	::=		
	$\mid$	$\Gamma \vDash P_1 \vee P_2 = Q$	Least Upper Bound (Least Common Supertype)
	$\mid$	$\mathbf{upgrade}\, \Gamma \vdash P\, \mathbf{to}\, \Delta = Q$	

Nrm ::=
| $\mathbf{nf}\,(N) = M$
| $\mathbf{nf}\,(P) = Q$
| $\mathbf{nf}\,(N) = M$
| $\mathbf{nf}\,(P) = Q$

$Order$ ::=
| $\mathbf{ord}\; vars\,\mathbf{in}\; N = \vec{\alpha}$
| $\mathbf{ord}\; vars\,\mathbf{in}\; P = \vec{\alpha}$
| $\mathbf{ord}\; vars\,\mathbf{in}\; N = \vec{\alpha}$
| $\mathbf{ord}\; vars\,\mathbf{in}\; P = \vec{\alpha}$

SM ::=
| $\Gamma \vdash e_1 \;\&\; e_2 = e_3$ — Unification Solution Entry Merge
| $\Theta \vdash \widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3$ — Merge unification solutions

$SImp$ ::=
| $\Gamma \vdash e_1 \Rightarrow e_2$ — Weakening of unification solution entries
| $\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2$ — Weakening of unification solutions
| $\Gamma \vdash e_1 \simeq e_2$
| $\Theta \vdash \widehat{\sigma}_1 \simeq \widehat{\sigma}_2$

U ::=
| $\Gamma; \Theta \models N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}$ — Negative unification
| $\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}$ — Positive unification

WF ::=
| $\Gamma \vdash N$ — Negative type well-formedness
| $\Gamma \vdash P$ — Positive type well-formedness
| $\Gamma \vdash N$ — Negative type well-formedness
| $\Gamma \vdash P$ — Positive type well-formedness
| $\Gamma \vdash \vec{N}$ — Negative type list well-formedness
| $\Gamma \vdash \vec{P}$ — Positive type list well-formedness
| $\Gamma; \Theta \vdash N$ — Negative unification type well-formedness
| $\Gamma; \Theta \vdash P$ — Positive unifiication type well-formedness
| $\Gamma; \Xi \vdash N$ — Negative anti-unification type well-formedness
| $\Gamma; \Xi \vdash P$ — Positive anti-unification type well-formedness
| $\Gamma; \Xi_2 \vdash \widehat{\tau} : \Xi_1$ — Antiunification substitution well-formedness
| $\widehat{\sigma} : \Theta$ — Unification substitution well-formedness
| $\Gamma \vdash^{\supseteq} \Theta$ — Unification context well-formedness
| $\Gamma_1 \vdash \sigma : \Gamma_2$ — Substitution well-formedness
| $\Gamma \vdash e$ — Unification solution entry well-formedness

$judgement$ ::=
| A
| AU
| $E1$
| $D1$
| $D0$

$$
\begin{array}{lll}
 & | & EQ \\
 & | & LUB \\
 & | & Nrm \\
 & | & Order \\
 & | & SM \\
 & | & SImp \\
 & | & U \\
 & | & WF \\
 & & \\
user\_syntax & ::= & \\
 & | & \alpha \\
 & | & n \\
 & | & n \\
 & | & \alpha^{+} \\
 & | & \alpha^{-} \\
 & | & \alpha^{\pm} \\
 & | & \sigma \\
 & | & e \\
 & | & \widehat{\sigma} \\
 & | & \widehat{\tau} \\
 & | & P \\
 & | & N \\
 & | & \overrightarrow{\alpha^{+}} \\
 & | & \overrightarrow{\alpha^{-}} \\
 & | & \overrightarrow{\alpha^{\pm}} \\
 & | & P \\
 & | & N \\
 & | & \overrightarrow{P} \\
 & | & \overrightarrow{N} \\
 & | & \Gamma \\
 & | & \Theta \\
 & | & \Xi \\
 & | & \overrightarrow{\alpha} \\
 & | & vars \\
 & | & \mu \\
 & | & \widehat{\alpha}^{\pm} \\
 & | & \widehat{\alpha}^{+} \\
 & | & \widehat{\alpha}^{-} \\
 & | & \overrightarrow{\widetilde{\alpha}^{+}} \\
 & | & \overrightarrow{\widehat{\alpha}^{-}} \\
 & | & P \\
 & | & N \\
 & | & auSol \\
 & | & terminals \\
 & | & formula
\end{array}
$$

$\boxed{\Gamma ; \Theta \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$
\frac{}{\Gamma ; \Theta \models \alpha^{-} \leqslant \alpha^{-} \dashv \cdot} \quad \textsc{ANVar}
$$

$$\frac{\Gamma;\Theta \models \mathbf{nf}\,(P) \stackrel{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma;\ \Theta \models {\uparrow}P \leqslant {\uparrow}Q \dashv \widehat{\sigma}}\ \text{AShiftU}$$

$$\frac{\Gamma;\ \Theta \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma;\ \Theta \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma;\ \Theta \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1\ \&\ \widehat{\sigma}_2}\ \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+};\ \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma;\ \Theta \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}}\ \text{AForall}$$

$\boxed{\Gamma;\ \Theta \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma;\ \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \text{APVar}$$

$$\frac{\Gamma;\Theta \models \mathbf{nf}\,(N) \stackrel{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma;\ \Theta \models {\downarrow}N \geqslant {\downarrow}M \dashv \widehat{\sigma}}\ \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-};\ \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma;\ \Theta \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^-}}\ \text{AExists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{\Gamma;\ \Theta \models \widehat{\alpha}^+ \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant Q)}\ \text{APUVar}$$

$\boxed{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \stackrel{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)}\ \text{AUPVar}$$

$$\frac{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models {\downarrow}N_1 \stackrel{a}{\simeq} {\downarrow}N_2 \dashv (\Xi, {\downarrow}M, \widehat{\tau}_1, \widehat{\tau}_2)}\ \text{AUShiftD}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \stackrel{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists\overrightarrow{\alpha^-}.Q, \widehat{\tau}_1, \widehat{\tau}_2)}\ \text{AUExists}$$

$\boxed{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^- \stackrel{a}{\simeq} \alpha^- \dashv (\cdot, \alpha^-, \cdot, \cdot)}\ \text{AUNVar}$$

$$\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models {\uparrow}P_1 \stackrel{a}{\simeq} {\uparrow}P_2 \dashv (\Xi, {\uparrow}Q, \widehat{\tau}_1, \widehat{\tau}_2)}\ \text{AUShiftU}$$

$$\frac{\overrightarrow{\alpha^+} \cap \Gamma = \varnothing \quad \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \forall\overrightarrow{\alpha^+}.N_1 \stackrel{a}{\simeq} \forall\overrightarrow{\alpha^+}.N_2 \dashv (\Xi, \forall\overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)}\ \text{AUForall}$$

$$\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \models P_1 \to N_1 \stackrel{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)}\ \text{AUArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \stackrel{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))}\ \text{AUAU}$$

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-}\ \text{E1NVar}$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q}\ \textsc{E1ShiftU}$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M}\ \textsc{E1Arrow}$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M}\ \textsc{E1Forall}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+}\ \textsc{E1PVar}$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M}\ \textsc{E1ShiftD}$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q}\ \textsc{E1Exists}$$

$\boxed{P \simeq Q}$

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M}\ \textsc{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q}\ \textsc{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-}\ \textsc{D1NVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q}\ \textsc{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M}\ \textsc{D1Arrow}$$

$$\frac{\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M}\ \textsc{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+}\ \textsc{D1PVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M}\ \textsc{D1ShiftD}$$

$$\frac{\mathbf{fv}\, P \cap \overrightarrow{\beta^-} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q}\ \textsc{D1Exists}$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1}$ Equivalence of substitutions

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \quad \text{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \quad \text{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \quad \text{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \quad \text{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \quad \text{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \quad \text{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \quad \text{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \quad \text{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \quad \text{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \quad \text{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \quad \text{D0ExistsR}$$

$\boxed{N = M}$ Negative type equality (alpha-equivalence)

$\boxed{P = Q}$ Positive type equuality (alphha-equivalence)

$\boxed{P = Q}$

$\boxed{P_1 \vee P_2}$

$\boxed{\textbf{ord } \mathit{vars} \textbf{ in } P}$

$\boxed{\textbf{ord } \mathit{vars} \textbf{ in } N}$

$\boxed{\textbf{ord } \mathit{vars} \textbf{ in } P}$

$\boxed{\textbf{ord } \mathit{vars} \textbf{ in } N}$

$\boxed{\mathbf{nf}\,(N')}$

$\boxed{\mathbf{nf}\,(P')}$

$\boxed{\mathbf{nf}\,(N')}$

$\boxed{\mathbf{nf}\,(P')}$

$\boxed{\mathbf{nf}\,(\overrightarrow{N'})}$

$\boxed{\mathbf{nf}\,(\overrightarrow{P'})}$

$\boxed{\mathbf{nf}\,(\sigma')}$

$\boxed{\mathbf{nf}\,(\mu')}$

$\boxed{\mathbf{nf}\,(\widehat{\sigma}')}$

$\boxed{\sigma'|_{vars}}$

$\boxed{\widehat{\sigma}'|_{vars}}$

$\boxed{\widehat{\tau}'|_{vars}}$

$\boxed{\Xi'|_{vars}}$

$\boxed{e_1\ \&\ e_2}$

$\boxed{\widehat{\sigma}_1\ \&\ \widehat{\sigma}_2}$

$\boxed{\widehat{\tau}_1 \,\&\, \widehat{\tau}_2}$

$\boxed{\mathbf{dom}\,(\widehat{\sigma})}$

$\boxed{\mathbf{dom}\,(\widehat{\tau})}$

$\boxed{\mathbf{dom}\,(\Theta)}$

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVAR}$$

$$\frac{\Gamma, \cdot \vDash \mathbf{nf}\,(\downarrow N) \overset{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P} \quad \text{LUBSHIFT}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q} \quad \text{LUBEXISTS}$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

$$\frac{\Gamma = \Delta, \overrightarrow{\alpha^\pm} \quad \overrightarrow{\beta^\pm}\,\mathbf{is\ fresh} \quad \overrightarrow{\gamma^\pm}\,\mathbf{is\ fresh} \quad \Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vDash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q}{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q} \quad \text{LUBUPGRADE}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-} \quad \text{NRMNVAR}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\uparrow P) = \uparrow Q} \quad \text{NRMSHIFTU}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(P \to N) = Q \to M} \quad \text{NRMARROW}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\forall \overrightarrow{\alpha^+}.N) = \forall \overrightarrow{\alpha^+}'.N'} \quad \text{NRMFORALL}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+} \quad \text{NRMPVAR}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\downarrow N) = \downarrow M} \quad \text{NRMSHIFTD}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.P) = \exists \overrightarrow{\alpha^-}'.P'} \quad \text{NRMEXISTS}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\quad \text{NrmNUVar}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\quad \text{NrmPUVar}$$

$\boxed{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}}$

$$\frac{\alpha^- \in \mathit{vars}}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^- = \alpha^-}\quad \text{ONVarIn}$$

$$\frac{\alpha^- \notin \mathit{vars}}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^- = \cdot}\quad \text{ONVarNIn}$$

$$\frac{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,{\uparrow}P = \vec{\alpha}}\quad \text{OShiftU}$$

$$\frac{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}_1 \quad \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}_2}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)}\quad \text{OArrow}$$

$$\frac{\mathit{vars} \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.N = \vec{\alpha}}\quad \text{OForall}$$

$\boxed{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}}$

$$\frac{\alpha^+ \in \mathit{vars}}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^+ = \alpha^+}\quad \text{OPVarIn}$$

$$\frac{\alpha^+ \notin \mathit{vars}}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^+ = \cdot}\quad \text{OPVarNIn}$$

$$\frac{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,{\downarrow}N = \vec{\alpha}}\quad \text{OShiftD}$$

$$\frac{\mathit{vars} \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\exists\overrightarrow{\alpha^-}.P = \vec{\alpha}}\quad \text{OExists}$$

$\boxed{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\widehat{\alpha}^- = \cdot}\quad \text{ONUVar}$$

$\boxed{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\widehat{\alpha}^+ = \cdot}\quad \text{OPUVar}$$

$\boxed{\Gamma \vdash e_1 \,\&\, e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{\Gamma \vDash P_1 \vee P_2 = Q}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P_1) \,\&\, (\widehat{\alpha}^+ :\geqslant P_2) = (\widehat{\alpha}^+ :\geqslant Q)}\quad \text{SMESupSup}$$

$$\frac{\Gamma; \cdot \vDash P \geqslant Q \dashv \widehat{\sigma}'}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \,\&\, (\widehat{\alpha}^+ :\geqslant Q) = (\widehat{\alpha}^+ :\approx P)}\quad \text{SMEEqSup}$$

$$\frac{\Gamma; \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P) \,\&\, (\widehat{\alpha}^+ :\approx Q) = (\widehat{\alpha}^+ :\approx Q)}\quad \text{SMESupEq}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^{+} :\approx P) \;\&\; (\widehat{\alpha}^{+} :\approx P') = (\widehat{\alpha}^{+} :\approx P)} \quad \text{SMEPE}\textsc{q}\text{E}\textsc{q}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^{-} :\approx N_1) \;\&\; (\widehat{\alpha}^{-} :\approx N') = (\widehat{\alpha}^{-} :\approx N)} \quad \text{SMENE}\textsc{q}\text{E}\textsc{q}$$

$\boxed{\Theta \vdash \widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$\boxed{\Gamma \vdash e_1 \Rightarrow e_2}$ Weakening of unification solution entries

$$\frac{\Gamma \vdash P_1 \geqslant_1 P_2}{\Gamma \vdash (\widehat{\alpha}^{+} :\geqslant P_1) \Rightarrow (\widehat{\alpha}^{+} :\geqslant P_2)} \quad \text{S}\textsc{imp}\text{ES}\textsc{up}\text{S}\textsc{up}$$

$$\frac{\Gamma \vdash P_1 \geqslant_1 P_2}{\Gamma \vdash (\widehat{\alpha}^{+} :\approx P_1) \Rightarrow (\widehat{\alpha}^{+} :\geqslant P_2)} \quad \text{S}\textsc{imp}\text{EE}\textsc{q}\text{S}\textsc{up}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^{+} :\approx P_1) \Rightarrow (\widehat{\alpha}^{+} :\approx P_2)} \quad \text{S}\textsc{imp}\text{EPE}\textsc{q}\text{E}\textsc{q}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^{-} :\approx N_1) \Rightarrow (\widehat{\alpha}^{-} :\approx N_2)} \quad \text{S}\textsc{imp}\text{ENE}\textsc{q}\text{E}\textsc{q}$$

$\boxed{\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2}$ Weakening of unification solutions

$\boxed{\Gamma \vdash e_1 \simeq e_2}$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^{+} :\geqslant P_1) \simeq (\widehat{\alpha}^{+} :\geqslant P_2)} \quad \text{S}\textsc{imp}\text{EE}\textsc{q}\text{S}\textsc{up}\text{S}\textsc{up}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^{+} :\approx P_1) \simeq (\widehat{\alpha}^{+} :\approx P_2)} \quad \text{S}\textsc{imp}\text{EE}\textsc{q}\text{PE}\textsc{q}\text{E}\textsc{q}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash (\widehat{\alpha}^{-} :\approx N_1) \simeq (\widehat{\alpha}^{-} :\approx N_2)} \quad \text{S}\textsc{imp}\text{EE}\textsc{q}\text{NE}\textsc{q}\text{E}\textsc{q}$$

$\boxed{\Theta \vdash \widehat{\sigma}_1 \simeq \widehat{\sigma}_2}$

$\boxed{\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Gamma; \Theta \vDash \alpha^{-} \overset{u}{\simeq} \alpha^{-} \dashv \cdot} \quad \text{UNV}\textsc{ar}$$

$$\frac{\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash {\uparrow}P \overset{u}{\simeq} {\uparrow}Q \dashv \widehat{\sigma}} \quad \text{US}\textsc{hift}\text{U}$$

$$\frac{\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \vDash P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2} \quad \text{UA}\textsc{rrow}$$

$$\frac{\Gamma, \overrightarrow{\alpha^{+}}; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \forall \overrightarrow{\alpha^{+}}.N \overset{u}{\simeq} \forall \overrightarrow{\alpha^{+}}.M \dashv \widehat{\sigma}} \quad \text{UF}\textsc{orall}$$

$$\frac{\widehat{\alpha}^{-}\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma; \Theta \vDash \widehat{\alpha}^{-} \overset{u}{\simeq} N \dashv (\widehat{\alpha}^{-} :\approx N)} \quad \text{UNUV}\textsc{ar}$$

$\boxed{\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Gamma; \Theta \vDash \alpha^{+} \overset{u}{\simeq} \alpha^{+} \dashv \cdot} \quad \text{UPV}\textsc{ar}$$

$$\frac{\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash {\downarrow}N \overset{u}{\simeq} {\downarrow}M \dashv \widehat{\sigma}} \quad \text{US}\textsc{hift}\text{D}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}; \Theta \models \boxed{P} \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma; \Theta \models \exists\overrightarrow{\alpha^-}. \boxed{P} \stackrel{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}} \quad \text{UEXISTS}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma; \Theta \models \widehat{\alpha}^+ \stackrel{u}{\simeq} P \dashv (\widehat{\alpha}^+ :\approx P)} \quad \text{UPUVAR}$$

$\boxed{\Gamma \vdash N}$ Negative type well-formedness
$\boxed{\Gamma \vdash P}$ Positive type well-formedness
$\boxed{\Gamma \vdash N}$ Negative type well-formedness
$\boxed{\Gamma \vdash P}$ Positive type well-formedness
$\boxed{\Gamma \vdash \overrightarrow{N}}$ Negative type list well-formedness
$\boxed{\Gamma \vdash \overrightarrow{P}}$ Positive type list well-formedness
$\boxed{\Gamma; \Theta \vdash N}$ Negative unification type well-formedness
$\boxed{\Gamma; \Theta \vdash P}$ Positive unifiication type well-formedness
$\boxed{\Gamma; \Xi \vdash N}$ Negative anti-unification type well-formedness
$\boxed{\Gamma; \Xi \vdash P}$ Positive anti-unification type well-formedness
$\boxed{\Gamma; \Xi_2 \vdash \widehat{\tau} : \Xi_1}$ Antiunification substitution well-formedness
$\boxed{\widehat{\sigma} : \Theta}$ Unification substitution well-formedness
$\boxed{\Gamma \vdash^{\supseteq} \Theta}$ Unification context well-formedness
$\boxed{\Gamma_1 \vdash \sigma : \Gamma_2}$ Substitution well-formedness
$\boxed{\Gamma \vdash e}$ Unification solution entry well-formedness

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Definition rules:        74 good    14 bad
Definition rule clauses: 144 good    14 bad
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