α, β, α, β	type variables
n, m, i, j	index variables

$\alpha^+,\ \beta^+$	$::=$			positive variable
	$\vert$	α^+		
$\alpha^-,\ \beta^-$	$::=$			negative variable
	$\vert$	α^-		
σ	$::=$			substitution
	$\vert$	$\cdot$		
	$\vert$	P/α^+		
	$\vert$	N/α^-		
	$\vert$	$\vec{P}/\overrightarrow{\alpha^+}$		
	$\vert$	$\vec{N}/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^-}}/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	$\vert$	$\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}$		
	$\vert$	$\vec{\alpha}_1/\vec{\alpha}_2$		
	$\vert$	$\overline{\sigma_i}^{\,i}$		concatenate
e	$::=$			entry of a unification solution
	$\vert$	$\Gamma \vdash \widehat{\alpha}^+ :\approx P$		
	$\vert$	$\Gamma \vdash \widehat{\alpha}^- :\approx N$		
	$\vert$	$\Gamma \vdash \widehat{\alpha}^+ :\geqslant P$		
	$\vert$	(e)	S	
	$\vert$	$e_1 \ \&\ e_2$	M	
$\widehat{\sigma}$	$::=$			unification solution (substitution)
	$\vert$	$\cdot$		
	$\vert$	e		
	$\vert$	$\widehat{\sigma}\backslash\overrightarrow{\widetilde{\alpha^+}}$		
	$\vert$	$\widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha^-}}$		
	$\vert$	$\widehat{\sigma}\backslash\widehat{\alpha}^+$		
	$\vert$	$\widehat{\sigma}\backslash\widehat{\alpha}^-$		
	$\vert$	$\widehat{\sigma}_1 \cup \widehat{\sigma}_2$		
	$\vert$	$\overline{\widehat{\sigma}_i}^{\,i}$		concatenate
	$\vert$	$(\widehat{\sigma})$	S	
	$\vert$	$\widehat{\sigma}_1 \ \&\ \widehat{\sigma}_2$	M	
$P,\ Q$	$::=$			positive types
	$\vert$	α^+		
	$\vert$	$\downarrow N$		
	$\vert$	$\exists \alpha^-.P$		
	$\vert$	$[\sigma]P$	M	
$N,\ M$	$::=$			negative types
	$\vert$	α^-		
	$\vert$	$\uparrow P$		
	$\vert$	$\forall \alpha^+.N$		
	$\vert$	$P \to N$		

	$\mid$	$[\sigma]N$	M	
$\overrightarrow{\alpha^+},\ \overrightarrow{\beta^+}$	::=			positive variable list
	$\mid$	$\cdot$		empty list
	$\mid$	α^+		a variable
	$\mid$	$\overline{\overrightarrow{\alpha^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^-},\ \overrightarrow{\beta^-}$	::=			negative variables
	$\mid$	$\cdot$		empty list
	$\mid$	α^-		a variable
	$\mid$	$\overline{\overrightarrow{\alpha^-}_i}^{\,i}$		concatenate lists
$P,\ Q$	::=			multi-quantified positive types
	$\mid$	α^+		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists\overrightarrow{\alpha^-}.P$		$P \neq \exists \ldots$
	$\mid$	$[\sigma]P$	M	
	$\mid$	$[\mu]P$	M	
	$\mid$	(P)	S	
	$\mid$	$\mathbf{nf}\,(P')$	M	
$N,\ M$	::=			multi-quantified negative types
	$\mid$	α^-		
	$\mid$	$\uparrow P$		
	$\mid$	$P \to N$		
	$\mid$	$\forall\overrightarrow{\alpha^+}.N$		$N \neq \forall \ldots$
	$\mid$	$[\sigma]N$	M	
	$\mid$	$[\mu]N$	M	
	$\mid$	(N)	S	
	$\mid$	$\mathbf{nf}\,(N')$	M	
$\vec{P}$	::=			list of positive types
	$\mid$	$\cdot$		empty list
	$\mid$	P		a singel type
	$\mid$	$\overline{\vec{P}_i}^{\,i}$		concatenate lists
$\vec{N}$	::=			list of negative types
	$\mid$	$\cdot$		empty list
	$\mid$	N		a singel type
	$\mid$	$\overline{\vec{N}_i}^{\,i}$		concatenate lists
$\Delta,\ \Gamma$	::=			declarative type context
	$\mid$	$\cdot$		empty context
	$\mid$	$\overrightarrow{\alpha^+}$		list of variables
	$\mid$	$\overrightarrow{\alpha^-}$		list of variables
	$\mid$	$vars$		
	$\mid$	$\overline{\Gamma_i}^{\,i}$		concatenate contexts
	$\mid$	(Γ)	S	

$\Theta,\ \Xi$	$::=$			unification type variable context
	$\vert$	$\cdot$		empty context
	$\vert$	$\overrightarrow{\widetilde{\alpha}^{+}}$		list of variables
	$\vert$	$\overrightarrow{\widehat{\alpha}^{-}}$		list of variables
	$\vert$	$\overline{\Theta_i}^{\,i}$		concatenate contexts
	$\vert$	(Θ)	S	
$\vec{\alpha},\ \vec{\beta}$	$::=$			ordered positive or negative variables
	$\vert$	$\cdot$		empty list
	$\vert$	$\overrightarrow{\alpha^{+}}$		list of variables
	$\vert$	$\overrightarrow{\alpha^{-}}$		list of variables
	$\vert$	$\vec{\alpha}_1 \backslash \mathit{vars}$		setminus
	$\vert$	Γ		context
	$\vert$	vars		
	$\vert$	$\overline{\vec{\alpha}_i}^{\,i}$		concatenate contexts
	$\vert$	$(\vec{\alpha})$	S	parenthesis
	$\vert$	$[\mu]\vec{\alpha}$	M	apply moving to list
	$\vert$	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P$	M	
	$\vert$	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N$	M	
	$\vert$	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P$	M	
	$\vert$	$\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N$	M	
vars	$::=$			set of variables
	$\vert$	$\varnothing$		empty set
	$\vert$	$\mathbf{fv}\ P$		free variables
	$\vert$	$\mathbf{fv}\ N$		free variables
	$\vert$	$\mathbf{fv}\ P$		free variables
	$\vert$	$\mathbf{fv}\ N$		free variables
	$\vert$	$\mathit{vars}_1 \cap \mathit{vars}_2$		set intersection
	$\vert$	$\mathit{vars}_1 \cup \mathit{vars}_2$		set union
	$\vert$	$\mathit{vars}_1 \backslash \mathit{vars}_2$		set complement
	$\vert$	$\mathbf{mv}\ P$		movable variables
	$\vert$	$\mathbf{mv}\ N$		movable variables
	$\vert$	$\mathbf{uv}\ N$		unification variables
	$\vert$	$\mathbf{uv}\ P$		unification variables
	$\vert$	$\mathbf{fv}\ N$		free variables
	$\vert$	$\mathbf{fv}\ P$		free variables
	$\vert$	(vars)	S	parenthesis
	$\vert$	$\{\vec{\alpha}\}$		ordered list of variables
	$\vert$	$[\mu]\mathit{vars}$	M	apply moving to varset
μ	$::=$			
	$\vert$	$\cdot$		empty moving
	$\vert$	$\widetilde{\alpha}_1^{+} \mapsto \widetilde{\alpha}_2^{+}$		Positive unit substitution
	$\vert$	$\widetilde{\alpha}_1^{-} \mapsto \widetilde{\alpha}_2^{-}$		Positive unit substitution
	$\vert$	$\mu_1 \cup \mu_2$	M	Set-like union of movings
	$\vert$	$\overline{\mu_i}^{\,i}$		concatenate movings
	$\vert$	$\mu\vert_{\mathit{vars}}$	M	restriction on a set
	$\vert$	μ^{-1}	M	inversion

n	$::=$			cohort index
	$\vert$	0		
	$\vert$	$n+1$		
$\widetilde{\alpha}^{+}$	$::=$			positive movable variable
	$\vert$	$\widetilde{\alpha}^{+n}$		
$\widetilde{\alpha}^{-}$	$::=$			negative movable variable
	$\vert$	$\widetilde{\alpha}^{-n}$		
$\overrightarrow{\widetilde{\alpha}^{+}},\ \overrightarrow{\widetilde{\beta}^{+}}$	$::=$			positive movable variable list
	$\vert$	$\cdot$		empty list
	$\vert$	$\widetilde{\alpha}^{+}$		a variable
	$\vert$	$\overrightarrow{\widetilde{\alpha}^{+n}}$		from a non-movable variable
	$\vert$	$\overline{\overrightarrow{\widetilde{\alpha}^{+}}_{i}}^{i}$		concatenate lists
$\overrightarrow{\widetilde{\alpha}^{-}},\ \overrightarrow{\widetilde{\beta}^{-}}$	$::=$			negatiive movable variable list
	$\vert$	$\cdot$		empty list
	$\vert$	$\widetilde{\alpha}^{-}$		a variable
	$\vert$	$\overrightarrow{\widetilde{\alpha}^{-n}}$		from a non-movable variable
	$\vert$	$\overline{\overrightarrow{\widetilde{\alpha}^{-}}_{i}}^{i}$		concatenate lists
$P,\ Q$	$::=$			multi-quantified positive types with movable variables
	$\vert$	α^{+}		
	$\vert$	$\widetilde{\alpha}^{+}$		
	$\vert$	$\downarrow N$		
	$\vert$	$\exists\overrightarrow{\alpha^{-}}.P$		
	$\vert$	$[\sigma]P$	M	
	$\vert$	$[\mu]P$	M	
$N,\ M$	$::=$			multi-quantified negative types with movable variables
	$\vert$	α^{-}		
	$\vert$	$\widetilde{\alpha}^{-}$		
	$\vert$	$\uparrow P$		
	$\vert$	$P \rightarrow N$		
	$\vert$	$\forall\overrightarrow{\alpha^{+}}.N$		
	$\vert$	$[\sigma]N$	M	
	$\vert$	$[\mu]N$	M	
$\widehat{\alpha}^{+}$	$::=$			positive unification variable
	$\vert$	$\widehat{\alpha}^{+}$		
	$\vert$	$\widehat{\alpha}^{+}\{\Delta\}$		
$\widehat{\alpha}^{-}$	$::=$			negative unification variable
	$\vert$	$\widehat{\alpha}^{-}$		
	$\vert$	$\widehat{\alpha}^{-}_{\{N,M\}}$		
	$\vert$	$\widehat{\alpha}^{-}\{\Delta\}$		
$\overrightarrow{\widetilde{\alpha}^{+}},\ \overrightarrow{\widetilde{\beta}^{+}}$	$::=$			positive unification variable list

	$\vert$	$\cdot$		empty list
	$\vert$	$\widehat{\alpha}^+$		a variable
	$\vert$	$\overrightarrow{\widehat{\alpha}^+\{\Delta\}}$		from a normal variable
	$\vert$	$\overrightarrow{\widehat{\alpha}^+}$		from a normal variable, context unspecified
	$\vert$	$\overline{\overrightarrow{\widetilde{\alpha}^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\widehat{\alpha}^-}$, $\overrightarrow{\widehat{\beta}^-}$	::=			negative unification variable list
	$\vert$	$\cdot$		empty list
	$\vert$	$\widehat{\alpha}^-$		a variable
	$\vert$	$\overrightarrow{\widehat{\alpha}^-\{\Delta\}}$		from a normal variable
	$\vert$	$\overrightarrow{\widehat{\alpha}^-}$		from a normal variable, context unspecified
	$\vert$	$\overline{\overrightarrow{\widehat{\alpha}^-}_i}^{\,i}$		concatenate lists
P, Q	::=			a positive algorithmic type (potentially with metavariables)
	$\vert$	α^+		
	$\vert$	$\widetilde{\alpha}^+$		
	$\vert$	$\widehat{\alpha}^+$		
	$\vert$	$\downarrow N$		
	$\vert$	$\exists\overrightarrow{\alpha^-}.P$		
	$\vert$	$[\sigma]P$	M	
	$\vert$	$[\mu]P$	M	
	$\vert$	$\mathbf{nf}\,(P')$	M	
N, M	::=			a negative algorithmic type (potentially with metavariables)
	$\vert$	α^-		
	$\vert$	$\widehat{\alpha}^-$		
	$\vert$	$\uparrow P$		
	$\vert$	$P \to N$		
	$\vert$	$\forall\overrightarrow{\alpha^+}.N$		
	$\vert$	$[\sigma]N$	M	
	$\vert$	$[\mu]N$	M	
	$\vert$	$\mathbf{nf}\,(N')$	M	
terminals	::=			
	$\vert$	$\exists$		
	$\vert$	$\forall$		
	$\vert$	$\uparrow$		
	$\vert$	$\downarrow$		
	$\vert$	$\to$		
	$\vert$	$\leftrightarrow$		
	$\vert$	$\in$		
	$\vert$	$\notin$		
	$\vert$	$\cdot$		
	$\vert$	$\vdash$		
	$\vert$	$\leqslant$		
	$\vert$	$\geqslant$		
	$\vert$	$\simeq$		
	$\vert$	$\cup$		
	$\vert$	$\cap$		
	$\vert$	$\setminus$		

| | $\subseteq$
| | $\mapsto$
| | $\overset{u}{\simeq}$
| | $\overset{a}{\simeq}$
| | $\varnothing$
| | $\vDash$
| | $\dashv$
| | $\neq$
| | $\equiv_n$
| | $\vee$
| | $\Downarrow$
| | $:\geqslant$
| | $:\simeq$

formula	::=		
	|	*judgement*	
	|	$formula_1 \quad .. \quad formula_n$	
	|	$\mu : vars_1 \leftrightarrow vars_2$	
	|	μ **is bijective**	
	|	$\widehat{\sigma}$ **is functional**	
	|	$\widehat{\sigma}_1 \in \widehat{\sigma}_2$	
	|	$vars_1 \subseteq vars_2$	
	|	$vars_1 = vars_2$	
	|	$vars$ **is fresh**	
	|	$\alpha^- \notin vars$	
	|	$\alpha^+ \notin vars$	
	|	$\alpha^- \in vars$	
	|	$\alpha^+ \in vars$	
	|	$\widehat{\alpha}^- \in \Theta$	
	|	$\widehat{\alpha}^+ \in \Theta$	
	|	if any other rule is not applicable	
	|	$N \neq M$	
	|	$P \neq Q$	
A	::=		
	|	$\Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}$	Negative subtyping
	|	$\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}$	Positive supertyping
AU	::=		
	|	$\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)$	
	|	$\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)$	
E1	::=		
	|	$N \simeq_1^D M$	Negative multi-quantified type equivalence
	|	$P \simeq_1^D Q$	Positive multi-quantified type equivalence
D1	::=		
	|	$\Gamma \vdash N \simeq_1^{\leqslant} M$	Negative equivalence on MQ types
	|	$\Gamma \vdash P \simeq_1^{\leqslant} Q$	Positive equivalence on MQ types
	|	$\Gamma \vdash N \leqslant_1 M$	Negative subtyping

	$\mid$ $\Gamma \vdash P \geqslant_1 Q$	Positive supertyping
D0	::=	
	$\mid$ $\Gamma \vdash N \simeq_0^{\leqslant} M$	Negative equivalence
	$\mid$ $\Gamma \vdash P \simeq_0^{\leqslant} Q$	Positive equivalence
	$\mid$ $\Gamma \vdash N \leqslant_0 M$	Negative subtyping
	$\mid$ $\Gamma \vdash P \geqslant_0 Q$	Positive supertyping
LUBF	::=	
	$\mid$ $\mathbf{ord}\ vars\ \mathbf{in}\ P\ ===\ \vec{\alpha}$	
	$\mid$ $\mathbf{ord}\ vars\ \mathbf{in}\ N\ ===\ \vec{\alpha}$	
	$\mid$ $\mathbf{ord}\ vars\ \mathbf{in}\ P\ ===\ \vec{\alpha}$	
	$\mid$ $\mathbf{ord}\ vars\ \mathbf{in}\ N\ ===\ \vec{\alpha}$	
	$\mid$ $\mathbf{nf}\,(N')\ ===\ N$	
	$\mid$ $\mathbf{nf}\,(P')\ ===\ P$	
	$\mid$ $\mathbf{nf}\,(N')\ ===\ N$	
	$\mid$ $\mathbf{nf}\,(P')\ ===\ P$	
	$\mid$ $e_1\ \&\ e_2$	
	$\mid$ $\hat{\sigma}_1\ \&\ \hat{\sigma}_2$	
LUB	::=	
	$\mid$ $\Gamma \models P_1 \vee P_2 = Q$	Least Upper Bound (Least Common Supertype)
	$\mid$ $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$	
Nrm	::=	
	$\mid$ $\mathbf{nf}\,(N) = M$	
	$\mid$ $\mathbf{nf}\,(P) = Q$	
	$\mid$ $\mathbf{nf}\,(N) = M$	
	$\mid$ $\mathbf{nf}\,(P) = Q$	
Order	::=	
	$\mid$ $\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}$	
	$\mid$ $\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}$	
	$\mid$ $\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}$	
	$\mid$ $\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}$	
SM	::=	
	$\mid$ $e_1\ \&\ e_2 = e_3$	Unification Solution Entry Merge
	$\mid$ $\hat{\sigma}_1 \& \hat{\sigma}_2 = \hat{\sigma}_3$	Merge unification solutions
U	::=	
	$\mid$ $\Gamma; \Theta \models N \stackrel{u}{\simeq} M \dashv \hat{\sigma}$	Negative unification
	$\mid$ $\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \dashv \hat{\sigma}$	Positive unification
WF	::=	
	$\mid$ $\Gamma \vdash N$	Negative type well-formedness
	$\mid$ $\Gamma \vdash P$	Positive type well-formedness
	$\mid$ $\Gamma \vdash N$	Negative type well-formedness
	$\mid$ $\Gamma \vdash P$	Positive type well-formedness

$judgement$ ::=
$\mid$ A
$\mid$ AU
$\mid$ $E1$
$\mid$ $D1$
$\mid$ $D0$
$\mid$ LUB
$\mid$ Nrm
$\mid$ $Order$
$\mid$ SM
$\mid$ U
$\mid$ WF

$user\_syntax$ ::=
$\mid$ α
$\mid$ n
$\mid$ α^{+}
$\mid$ α^{-}
$\mid$ σ
$\mid$ e
$\mid$ $\widehat{\sigma}$
$\mid$ P
$\mid$ N
$\mid$ $\overrightarrow{\alpha^{+}}$
$\mid$ $\overrightarrow{\alpha^{-}}$
$\mid$ P
$\mid$ N
$\mid$ $\overrightarrow{P}$
$\mid$ $\overrightarrow{N}$
$\mid$ Γ
$\mid$ Θ
$\mid$ $\overrightarrow{\alpha}$
$\mid$ $vars$
$\mid$ μ
$\mid$ n
$\mid$ $\widetilde{\alpha}^{+}$
$\mid$ $\widetilde{\alpha}^{-}$
$\mid$ $\overrightarrow{\widetilde{\alpha^{+}}}$
$\mid$ $\overrightarrow{\widetilde{\alpha^{-}}}$
$\mid$ P
$\mid$ N
$\mid$ $\widehat{\alpha}^{+}$
$\mid$ $\widehat{\alpha}^{-}$
$\mid$ $\overrightarrow{\widetilde{\alpha^{+}}}$
$\mid$ $\overrightarrow{\widehat{\alpha^{-}}}$
$\mid$ P
$\mid$ N
$\mid$ $terminals$
$\mid$ $formula$

$\boxed{\Gamma;\Theta \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma;\Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot}\ \textsc{ANVar}$$

$$\frac{\Gamma;\cdot \models \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma;\Theta \models {\uparrow}P \leqslant {\uparrow}Q \dashv \widehat{\sigma}}\ \textsc{AShiftU}$$

$$\frac{\Gamma;\Theta \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma;\Theta \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma;\Theta \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\ \textsc{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+};\Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma;\Theta \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}}\ \textsc{AForall}$$

$\boxed{\Gamma;\Theta \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma;\Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \textsc{APVar}$$

$$\frac{\Gamma;\cdot \models \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma;\Theta \models {\downarrow}N \geqslant {\downarrow}M \dashv \widehat{\sigma}}\ \textsc{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-};\Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma;\Theta \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}}\ \textsc{AExists}$$

$$\frac{\mathbf{upgrade}\,\Gamma \vdash \mathbf{nf}\,(P)\,\mathbf{to}\,\Delta = Q}{\Gamma;\Theta \models \widehat{\alpha}^+\{\Delta\} \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)}\ \textsc{APUVar}$$

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\alpha^+, \cdot, \cdot)}\ \textsc{AUPVar}$$

$$\frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models {\downarrow}N_1 \overset{a}{\simeq} {\downarrow}N_2 \dashv ({\downarrow}M, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \textsc{AUPShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\exists\overrightarrow{\alpha^-}.Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \textsc{AUPExists}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\alpha^-, \cdot, \cdot)}\ \textsc{AUNVar}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models {\uparrow}P_1 \overset{a}{\simeq} {\uparrow}P_2 \dashv ({\uparrow}Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \textsc{AUNShift}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}'_1, \widehat{\sigma}'_2)}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (Q \to M, \widehat{\sigma}_1 \cup \widehat{\sigma}'_1, \widehat{\sigma}_2 \cup \widehat{\sigma}'_2)}\ \textsc{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, (\Gamma \vdash \widehat{\alpha}^-_{\{N,M\}} :\approx N), (\Gamma \vdash \widehat{\alpha}^-_{\{N,M\}} :\approx M))}\ \textsc{AUNAU}$$

$\boxed{N \simeq^D_1 M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq^D_1 \alpha^-}\ \textsc{E1NVar}$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad \textsc{E1ShiftU}$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M} \quad \textsc{E1Arrow}$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M} \quad \textsc{E1Forall}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad \textsc{E1PVar}$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad \textsc{E1ShiftD}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \quad \textsc{E1Exists}$$

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \quad \textsc{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \quad \textsc{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad \textsc{D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \quad \textsc{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \quad \textsc{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M} \quad \textsc{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad \textsc{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \quad \textsc{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q} \quad \textsc{D1ExistsL}$$

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \quad \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \text{ D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \text{ D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \text{ D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \text{ D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \text{ D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \text{ D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \text{ D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \text{ D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \text{ D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \text{ D0ExistsR}$$

$\boxed{\textbf{ord } \mathit{vars} \textbf{ in } P}$

$\boxed{\textbf{ord } \mathit{vars} \textbf{ in } N}$

$\boxed{\textbf{ord } \mathit{vars} \textbf{ in } P}$

$\boxed{\textbf{ord } \mathit{vars} \textbf{ in } N}$

$\boxed{\textbf{nf } (N')}$

$\boxed{\textbf{nf } (P')}$

$\boxed{\textbf{nf } (N')}$

$\boxed{\mathbf{nf}\,(P')}$

$\boxed{e_1 \,\&\, e_2}$

$\boxed{\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}$

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+}\ \textsc{LUBVar}$$

$$\frac{\Gamma \vDash \downarrow N \overset{a}{\simeq} \downarrow M \dashv (P, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/(\mathbf{uv}\, P)]P}\ \textsc{LUBShift}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q}\ \textsc{LUBExists}$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\ \textsc{NrmNVar}$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\ \textsc{NrmShiftU}$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\ \textsc{NrmArrow}$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall \overrightarrow{\alpha^+}.N) = \forall \overrightarrow{\alpha^+}'.N'}\ \textsc{NrmForall}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\ \textsc{NrmPVar}$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\ \textsc{NrmShiftD}$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.P) = \exists \overrightarrow{\alpha^-}'.P'}\ \textsc{NrmExists}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\ \textsc{NrmNUVar}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\quad \textsc{NrmPUVar}$$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \overrightarrow{\alpha}}$

$$\frac{\alpha^- \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^- = \alpha^-}\quad \textsc{ONVarIn}$$

$$\frac{\alpha^- \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^- = \cdot}\quad \textsc{ONVarNIn}$$

$$\frac{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \overrightarrow{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \uparrow P = \overrightarrow{\alpha}}\quad \textsc{OShiftU}$$

$$\frac{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \overrightarrow{\alpha}_1 \quad \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \overrightarrow{\alpha}_2}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P \to N = \overrightarrow{\alpha}_1, (\overrightarrow{\alpha}_2 \backslash \{\overrightarrow{\alpha}_1\})}\quad \textsc{OArrow}$$

$$\frac{\mathit{vars} \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \overrightarrow{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \forall \overrightarrow{\alpha^+}.N = \overrightarrow{\alpha}}\quad \textsc{OForall}$$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \overrightarrow{\alpha}}$

$$\frac{\alpha^+ \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^+ = \alpha^+}\quad \textsc{OPVarIn}$$

$$\frac{\alpha^+ \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^+ = \cdot}\quad \textsc{OPVarNIn}$$

$$\frac{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \overrightarrow{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \downarrow N = \overrightarrow{\alpha}}\quad \textsc{OShiftD}$$

$$\frac{\mathit{vars} \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \overrightarrow{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \exists \overrightarrow{\alpha^-}.P = \overrightarrow{\alpha}}\quad \textsc{OExists}$$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \overrightarrow{\alpha}}$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \widehat{\alpha}^- = \cdot}\quad \textsc{ONUVar}$$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \overrightarrow{\alpha}}$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \widehat{\alpha}^+ = \cdot}\quad \textsc{OPUVar}$$

$\boxed{e_1\ \&\ e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{\Gamma \vDash P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1)\ \&\ (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)}\quad \textsc{SMESupSup}$$

$$\frac{\Gamma; \cdot \vDash P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P)\ \&\ (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\quad \textsc{SMEEqSup}$$

$$\frac{\Gamma; \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P)\ \&\ (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)}\quad \textsc{SMESupEq}$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P)\ \&\ (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\quad \textsc{SMEPEqEq}$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N)\ \&\ (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)}\quad \textsc{SMENEqEq}$$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$\boxed{\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Gamma; \Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot} \quad \text{UNVAR}$$

$$\frac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma; \Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}} \quad \text{USHIFTU}$$

$$\frac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2} \quad \text{UARROW}$$

$$\frac{\Gamma, \overrightarrow{\alpha^+}; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma; \Theta \models \forall \overrightarrow{\alpha^+}. N \overset{u}{\simeq} \forall \overrightarrow{\alpha^+}. M \dashv \widehat{\sigma}} \quad \text{UFORALL}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma; \Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)} \quad \text{UNUVAR}$$

$\boxed{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Gamma; \Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot} \quad \text{UPVAR}$$

$$\frac{\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma; \Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}} \quad \text{USHIFTD}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma; \Theta \models \exists \overrightarrow{\alpha^-}. P \overset{u}{\simeq} \exists \overrightarrow{\alpha^-}. Q \dashv \widehat{\sigma}} \quad \text{UEXISTS}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma; \Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)} \quad \text{UPUVAR}$$

$\boxed{\Gamma \vdash N}$ Negative type well-formedness
$\boxed{\Gamma \vdash P}$ Positive type well-formedness
$\boxed{\Gamma \vdash N}$ Negative type well-formedness
$\boxed{\Gamma \vdash P}$ Positive type well-formedness

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Definition rules:         79 good    0 bad
Definition rule clauses: 137 good    0 bad
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