1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q,\ R & ::= & & \text{positive types} \\
& | & \alpha^+ & \\
& | & \downarrow N & \\
& | & \exists \alpha^-.P & \\
\\
N,\ M,\ K & ::= & & \text{negative types} \\
& | & \alpha^- & \\
& | & \uparrow P & \\
& | & \forall \alpha^+.N & \\
& | & P \to N & \\
\\
v,\ w & ::= & & \text{value terms} \\
& | & x & \\
& | & \{c\} & \\
& | & (v : P) & \\
\\
c,\ d & ::= & & \text{computation terms} \\
& | & (c : N) & \\
& | & \lambda x : P.c & \\
& | & \Lambda \alpha^+.c & \\
& | & \mathbf{return}\, v & \\
& | & \mathbf{let}\, x = v; c & \\
& | & \mathbf{let}\, x : P = v(\overrightarrow{v}); c & \\
& | & \mathbf{let}\, x = v(\overrightarrow{v}); c & \\
& | & \mathbf{let}^{\exists}(\alpha^-, x) = v; c &
\end{array}
$$

1.2 Declarative Typing

$\boxed{\Gamma; \Phi \vdash v \colon P}$ Positive type inference

$$\frac{v : P \in \Phi}{\Gamma; \Phi \vdash v \colon P}\ \text{DTVar}$$

$$\frac{\Gamma; \Phi \vdash c \colon N}{\Gamma; \Phi \vdash \{c\} \colon \downarrow N}\ \text{DTThunk}$$

$$\frac{\Gamma; \Phi \vdash v \colon P \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma; \Phi \vdash (v : Q) \colon Q}\ \text{DTPAnnot}$$

$\boxed{\Gamma; \Phi \vdash c \colon N}$ Negative type inference

$$\frac{\Gamma; \Phi, x : P \vdash c \colon N}{\Gamma; \Phi \vdash \lambda x : P.c \colon P \to N}\ \text{DTtLam}$$

$$\frac{\Gamma, \alpha^+; \Phi \vdash c \colon N}{\Gamma; \Phi \vdash \Lambda \alpha^+.c \colon \forall \alpha^+.N}\ \text{DTTLam}$$

$$\frac{\Gamma; \Phi \vdash v \colon P}{\Gamma; \Phi \vdash \mathbf{return}\, v \colon \uparrow P}\ \text{DTReturn}$$

$$\frac{\Gamma; \Phi \vdash v \colon P \quad \Gamma; \Phi, x : P \vdash c \colon N}{\Gamma; \Phi \vdash \mathbf{let}\, x = v; c \colon N}\ \text{DTVarLet}$$

$$\frac{\Gamma; \Phi \vdash v \colon \downarrow M \quad \Gamma; \Phi \vdash M \bullet \overrightarrow{v} \Rrightarrow \uparrow Q \text{ uniquely} \quad \Gamma; \Phi, x : Q \vdash c \colon N}{\Gamma; \Phi \vdash \mathbf{let}\, x = v(\overrightarrow{v}); c \colon N}\ \text{DTAppLet}$$

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi \vdash v \colon \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q \quad \Gamma \vdash \uparrow Q \leqslant_1 \uparrow P \quad \Gamma; \Phi, x : P \vdash c \colon N}{\Gamma; \Phi \vdash \mathbf{let}\, x : P = v(\vec{v}); c \colon N} \text{ DTAppLetAnn}$$

$$\frac{\Gamma; \Phi \vdash v \colon \exists \alpha^-.P \quad \Gamma, \alpha^-; \Phi, x : P \vdash c \colon N \quad \Gamma \vdash N}{\Gamma; \Phi \vdash \mathbf{let}^\exists(\alpha^-, x) = v; c \colon N} \text{ DTUnpack}$$

$$\frac{\Gamma; \Phi \vdash c \colon N \quad \Gamma \vdash N \leqslant_1 M}{\Gamma; \Phi \vdash (c : M) \colon M} \text{ DTNAnnot}$$

$\boxed{\Gamma; \Phi \vdash N \bullet \vec{v} \Rrightarrow M}$ Application type inference

$$\frac{\Gamma \vdash N \simeq_1^\leqslant N'}{\Gamma; \Phi \vdash N \bullet \cdot \Rrightarrow N'} \text{ DTEmptyApp}$$

$$\frac{\Gamma; \Phi \vdash v \colon P \quad \Gamma \vdash Q \geqslant_1 P \quad \Gamma; \Phi \vdash N \bullet \vec{v} \Rrightarrow M}{\Gamma; \Phi \vdash Q \to N \bullet v, \vec{v} \Rrightarrow M} \text{ DTArrowApp}$$

$$\frac{\begin{array}{l}\Gamma \vdash \sigma : \overrightarrow{\alpha^+} \quad \Gamma; \Phi \vdash [\sigma]N \bullet \vec{v} \Rrightarrow M \\ \vec{v} \neq \cdot\end{array}}{\Gamma; \Phi \vdash \forall \overrightarrow{\alpha^+}.N \bullet \vec{v} \Rrightarrow M} \text{ DTForallApp}$$

1.3 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^\leqslant M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^\leqslant M} \text{ D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^\leqslant Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^\leqslant Q} \text{ D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \text{ D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^\leqslant Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \text{ D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \text{ D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \text{ D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \text{ D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \text{ D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^\leqslant M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \text{ D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \text{ D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \text{ D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$$
\begin{array}{llll}
P,\ Q,\ R & ::= & & \text{multi-quantified positive types} \\
& | & \alpha^+ & \\
& | & \downarrow N & \\
& | & \exists\overrightarrow{\alpha^-}.P & P \neq \exists\ldots \\
& | & (P) \quad \mathsf{S} &
\end{array}
$$

$$
\begin{array}{llll}
N,\ M,\ K & ::= & & \text{multi-quantified negative types} \\
& | & \alpha^- & \\
& | & \uparrow P & \\
& | & P \rightarrow N & \\
& | & \forall\overrightarrow{\alpha^+}.N & N \neq \forall\ldots \\
& | & (N) \quad \mathsf{S} &
\end{array}
$$

2.2 Declarative Multiquantified Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \quad (\simeq_1^{\leqslant}{}^-)$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \quad (\simeq_1^{\leqslant}{}^+)$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad (\textsc{Var}^{-\leqslant_1})$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \quad (\uparrow^{\leqslant_1})$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \rightarrow N \leqslant_1 Q \rightarrow M} \quad (\rightarrow^{\leqslant_1})$$

$$\frac{\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M} \quad (\forall^{\leqslant_1})$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad (\textsc{Var}^{+\geqslant_1})$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \quad (\downarrow^{\geqslant_1})$$

$$\frac{\mathbf{fv}\, P \cap \overrightarrow{\beta^-} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q} \quad (\exists^{\geqslant_1})$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1}$ Equivalence of substitutions

$\boxed{\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : vars}$ Equivalence of substitutions

$\boxed{\Theta \vdash \widehat{\sigma}_1 \simeq_1^{\leqslant} \widehat{\sigma}_2 : vars}$ Equivalence of unification substitutions

2.3 Declarative Equivalence

$\boxed{N \simeq_1^{D} M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^{D} \alpha^-} \quad (\textsc{Var}^{-\simeq_1^{D}})$$

$$\frac{P \simeq_1^{D} Q}{\uparrow P \simeq_1^{D} \uparrow Q} \quad (\uparrow^{\simeq_1^{D}})$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M} \quad (\to^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M} \quad (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\mathrm{VAR}^{+\simeq_1^D})$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists\overrightarrow{\alpha^-}.P \simeq_1^D \exists\overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{\alpha^- \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^- = \alpha^-} \quad (\mathrm{VAR}^-_{\in})$$

$$\frac{\alpha^- \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^- = \cdot} \quad (\mathrm{VAR}^-_{\notin})$$

$$\frac{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \uparrow P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)} \quad (\to)$$

$$\frac{\mathit{vars} \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{\alpha^+ \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^+ = \alpha^+} \quad (\mathrm{VAR}^+_{\in})$$

$$\frac{\alpha^+ \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^+ = \cdot} \quad (\mathrm{VAR}^+_{\notin})$$

$$\frac{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \downarrow N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{\mathit{vars} \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \exists\overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad (\exists)$$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \widehat{\alpha}^- = \cdot} \quad (\mathrm{UVAR}^-)$$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \widehat{\alpha}^+ = \cdot} \quad (\mathrm{UVAR}^+)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-} \quad (\textsc{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q} \quad (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M} \quad (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'} \quad (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+} \quad (\textsc{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M} \quad (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'} \quad (\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-} \quad (\textsc{UVar}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+} \quad (\textsc{UVar}^+)$$

3.2 Singularity

$\boxed{e_1\ \mathbf{singular}}$ Subtyping Constraint Entry Is Singular

$$\frac{}{\widehat{\alpha}^+ :\approx P\ \mathbf{singular}} \quad \textsc{SINGPEq}$$

$$\frac{}{\widehat{\alpha}^- :\approx N\ \mathbf{singular}} \quad \textsc{SINGNEq}$$

$$\frac{}{\widehat{\alpha}^+ :\geqslant \exists\overrightarrow{\alpha^-}.\alpha^+\ \mathbf{singular}} \quad \textsc{SINGSupVar}$$

$$\frac{N \simeq_1^D \alpha_i^-}{\widehat{\alpha}^+ :\geqslant \exists\overrightarrow{\alpha^-}.\downarrow N\ \mathbf{singular}} \quad \textsc{SINGSupShift}$$

$\boxed{SC\ \mathbf{singular}}$ Subtyping Constraint Is Singular

3.3 Unification

$\boxed{\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC}$ Negative unification

$$\frac{}{\Gamma; \Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot} \quad (\textsc{Var}^{-\overset{u}{\simeq}})$$

$$\frac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC}{\Gamma; \Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv UC} \quad (\uparrow^{\overset{u}{\simeq}})$$

$$\frac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC_1 \quad \Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC_2}{\Gamma; \Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv UC_1\ \&\ UC_2} \quad (\to^{\overset{u}{\simeq}})$$

$$\frac{\Gamma, \overrightarrow{\alpha^+}; \Theta \models N \overset{u}{\simeq} M \dashv UC}{\Gamma; \Theta \models \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv UC} \quad (\forall^{\overset{u}{\simeq}})$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma; \Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\widehat{\alpha}^- :\approx N)} \quad (\text{UVAR}^{-\overset{u}{\simeq}})$$

$\boxed{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC}$ Positive unification

$$\frac{}{\Gamma; \Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot} \quad (\text{VAR}^{+\overset{u}{\simeq}})$$

$$\frac{\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC}{\Gamma; \Theta \models {\downarrow}N \overset{u}{\simeq} {\downarrow}M \dashv UC} \quad ({\downarrow}^{\overset{u}{\simeq}})$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}; \Theta \models P \overset{u}{\simeq} Q \dashv UC}{\Gamma; \Theta \models \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv UC} \quad (\exists^{\overset{u}{\simeq}})$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma; \Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\widehat{\alpha}^+ :\approx P)} \quad (\text{UVAR}^{+\overset{u}{\simeq}})$$

3.4 Algorithmic Subtyping

$\boxed{\Gamma;\ \Theta \models N \leqslant M \dashv SC}$ Negative subtyping

$$\frac{}{\Gamma;\ \Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot} \quad (\text{VAR}^{-\leqslant})$$

$$\frac{\Gamma; \Theta \models \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv UC}{\Gamma;\ \Theta \models {\uparrow}P \leqslant {\uparrow}Q \dashv UC} \quad ({\uparrow}^{\leqslant})$$

$$\frac{\Gamma;\ \Theta \models P \geqslant Q \dashv SC_1 \quad \Gamma;\ \Theta \models N \leqslant M \dashv SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC}{\Gamma;\ \Theta \models P \to N \leqslant Q \to M \dashv SC} \quad (\to^{\leqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^+};\ \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma, \overrightarrow{\beta^+}\} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv SC}{\Gamma;\ \Theta \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv SC \backslash \overrightarrow{\widehat{\alpha}^+}} \quad (\forall^{\leqslant})$$

$\boxed{\Gamma;\ \Theta \models P \geqslant Q \dashv SC}$ Positive supertyping

$$\frac{}{\Gamma;\ \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad (\text{VAR}^{+\geqslant})$$

$$\frac{\Gamma; \Theta \models \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv UC}{\Gamma;\ \Theta \models {\downarrow}N \geqslant {\downarrow}M \dashv UC} \quad ({\downarrow}^{\geqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^-};\ \Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv SC}{\Gamma;\ \Theta \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv SC \backslash \overrightarrow{\widehat{\alpha}^-}} \quad (\exists^{\geqslant})$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{\Gamma;\ \Theta \models \widehat{\alpha}^+ \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant Q)} \quad (\text{UVAR}^{\geqslant})$$

3.5 Constraint Merge

Unification and subtyping constraints is are by a list of constraint entries. Each entry restricts an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\widehat{\alpha}^+ :\approx P$ or $\widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries)**.** *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.
$\boxed{\Gamma \vdash e_1 \;\&\; e_2 = e_3}$ *Subtyping Constraint Entry Merge*

$$\frac{\Gamma \models P_1 \vee P_2 = Q}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P_1) \;\&\; (\widehat{\alpha}^+ :\geqslant P_2) = (\widehat{\alpha}^+ :\geqslant Q)} \quad (\geqslant \&^+ \geqslant)$$

$$\frac{\Gamma; \cdot \models P \geqslant Q \dashv \cdot}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \;\&\; (\widehat{\alpha}^+ :\geqslant Q) = (\widehat{\alpha}^+ :\approx P)} \quad (\simeq \&^+ \geqslant)$$

$$\frac{\Gamma; \cdot \models Q \geqslant P \dashv \cdot}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P) \;\&\; (\widehat{\alpha}^+ :\approx Q) = (\widehat{\alpha}^+ :\approx Q)} \quad (\geqslant \&^+ \simeq)$$

$$\frac{\mathbf{nf}\,(P) = \mathbf{nf}\,(P')}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \;\&\; (\widehat{\alpha}^+ :\approx P') = (\widehat{\alpha}^+ :\approx P)} \quad (\simeq \&^+ \simeq)$$

$$\frac{\mathbf{nf}\,(N) = \mathbf{nf}\,(N')}{\Gamma \vdash (\widehat{\alpha}^- :\approx N_1) \;\&\; (\widehat{\alpha}^- :\approx N') = (\widehat{\alpha}^- :\approx N)} \quad (\simeq \&^- \simeq)$$

To merge two constraints, we merge each pair of matching entries, and unite the results. `Ilya: add contexts`

Definition 3. $SC_1 \;\&\; SC_2 = \{e_1 \;\&\; e_2 \mid e_1 \in SC_1, e_2 \in SC_2, \textit{s.t. } e_1 \textit{ matches with } e_2\}$
$\cup \{e_1 \mid e_1 \in SC_1,\; \textit{s.t. } \forall e_2 \in SC_2, e_1 \textit{ does not match with } e_2\}$
$\cup \{e_2 \mid e_2 \in SC_2,\; \textit{s.t. } \forall e_1 \in SC_1, e_1 \textit{ does not match with } e_2\}$

3.6 Constraint Satisfaction

$\boxed{\Gamma \vdash P : e}$ Positive type satisfies with the subtyping constraint entry

$$\frac{\Gamma \vdash P \geqslant_1 Q}{\Gamma \vdash P : (\widehat{\alpha}^+ :\geqslant Q)} \quad \text{SATSCESup}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash P : (\widehat{\alpha}^+ :\approx Q)} \quad \text{SATSCEPEq}$$

$\boxed{\Gamma \vdash N : e}$ Negative type satisfies with the subtyping constraint entry

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash N : (\widehat{\alpha}^- :\approx M)} \quad \text{SATSCENEq}$$

3.7 Least Upper Bound

$\boxed{\Gamma \models P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \quad (\text{Var}^{\vee})$$

$$\frac{\Gamma, \cdot \models \mathbf{nf}\,(\downarrow N) \overset{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi] P} \quad (\downarrow^{\vee})$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q}{\Gamma \models \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q} \quad (\exists^{\vee})$$

$\boxed{\mathbf{upgrade}\, \Gamma \vdash P\, \mathbf{to}\, \Delta = Q}$

$$\frac{\Gamma = \Delta, \overrightarrow{\alpha^{\pm}} \quad \overrightarrow{\beta^{\pm}}\ \textbf{is fresh} \quad \overrightarrow{\gamma^{\pm}}\ \textbf{is fresh} \quad \Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \models [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P = Q}{\mathbf{upgrade}\, \Gamma \vdash P\, \mathbf{to}\, \Delta = Q} \quad (\text{Upg})$$

3.8 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \quad (\text{Var}^{+\overset{a}{\simeq}})$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \dashv (\Xi, \downarrow M, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\downarrow^{\overset{a}{\simeq}})$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \exists\overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists\overrightarrow{\alpha^-}.Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\exists^{\overset{a}{\simeq}})$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\cdot, \alpha^-, \cdot, \cdot)} \quad (\text{Var}^{-\overset{a}{\simeq}})$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\uparrow^{\overset{a}{\simeq}})$$

$$\frac{\overrightarrow{\alpha^+} \cap \Gamma = \varnothing \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \forall\overrightarrow{\alpha^+}.N_1 \overset{a}{\simeq} \forall\overrightarrow{\alpha^+}.N_2 \dashv (\Xi, \forall\overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\forall^{\overset{a}{\simeq}})$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)} \quad (\to^{\overset{a}{\simeq}})$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad (\text{AU}^-)$$

3.9 Typing

$\boxed{\Gamma; \Phi \vDash v \colon P}$ Positive type inference

$$\frac{v : P \in \Phi}{\Gamma; \Phi \vDash v \colon P} \quad \text{ATVar}$$

$$\frac{\Gamma; \Phi \vDash c \colon N}{\Gamma; \Phi \vDash \{c\} \colon \downarrow N} \quad \text{ATThunk}$$

$$\frac{\Gamma; \Phi \vDash v \colon P \quad \Gamma; \cdot \vDash Q \geqslant P \dashv \cdot}{\Gamma; \Phi \vDash (v : Q) \colon Q} \quad \text{ATPAnnot}$$

$\boxed{\Gamma; \Phi \vDash c \colon N}$ Negative type inference

$$\frac{\Gamma; \Phi \vDash c \colon N \quad \Gamma; \cdot \vDash N \leqslant M \dashv \cdot}{\Gamma; \Phi \vDash (c : M) \colon M} \quad \text{ATNAnnot}$$

$$\frac{\Gamma; \Phi, x : P \vDash c \colon N}{\Gamma; \Phi \vDash \lambda x : P.c \colon P \to N} \quad \text{ATtLam}$$

$$\frac{\Gamma, \alpha^+; \Phi \vDash c \colon N}{\Gamma; \Phi \vDash \Lambda\alpha^+.c \colon \forall\alpha^+.N} \quad \text{ATTLam}$$

$$\frac{\Gamma; \Phi \vDash v \colon P}{\Gamma; \Phi \vDash \mathbf{return}\, v \colon \uparrow P} \quad \text{ATReturn}$$

$$\frac{\Gamma; \Phi \vDash v \colon P \quad \Gamma; \Phi, x : P \vDash c \colon N}{\Gamma; \Phi \vDash \mathbf{let}\, x = v; c \colon N} \quad \text{ATVarLet}$$

$$\frac{\begin{array}{l}\Gamma \vdash P \quad \Gamma; \Phi \vDash v \colon \downarrow M \quad \Gamma; \Phi; \cdot \vDash M \bullet \overrightarrow{v} \Longrightarrow\!\!\!> \uparrow Q \dashv \Theta; SC_1 \quad \Gamma;\, \Theta \vDash \uparrow Q \leqslant \uparrow P \dashv SC_2 \\ \Theta \vdash SC_1 \& SC_2 = SC \quad \Gamma; \Phi, x : P \vDash c \colon N\end{array}}{\Gamma; \Phi \vDash \mathbf{let}\, x : P = v(\overrightarrow{v}); c \colon N} \quad \text{ATAppLetAnn}$$

$$\dfrac{\begin{array}{l}\Gamma;\Phi \vDash v\colon {\downarrow}M \quad \Gamma;\Phi;\cdot \vDash M \bullet \vec{v} \Rrightarrow {\uparrow}Q \dashv \Theta; SC\\ \text{<<no parses (char 57): uv uQ dom(SC) SC | uv uQ singular u*** : SC >>}\\ \text{<<no parses (char 17): ; , x:[u***]uQ c : iN >>}\end{array}}{\Gamma;\Phi \vDash \mathbf{let}\; x = v(\vec{v});\, c\colon N}\ \textsc{ATAppLet}$$

$$\dfrac{\Gamma;\Phi \vDash v\colon \exists\alpha^-.P \quad \Gamma,\alpha^-;\Phi,x:P \vDash c\colon N \quad \Gamma \vdash N}{\Gamma;\Phi \vDash \mathbf{let}^{\exists}(\alpha^-,x) = v;\, c\colon N}\ \textsc{ATUnpack}$$

$\boxed{\Gamma;\Phi;\Theta_1 \vDash N \bullet \vec{v} \Rrightarrow M \dashv \Theta_2; SC}$ Application type inference

$$\dfrac{}{\Gamma;\Phi;\Theta \vDash N \bullet \cdot \Rrightarrow N \dashv \Theta;\cdot}\ \textsc{ATEmptyApp}$$

$$\dfrac{\begin{array}{l}\Gamma;\Phi \vDash v\colon P \quad \Gamma;\Theta \vDash Q \geqslant P \dashv SC_1 \quad \Gamma;\Phi;\Theta \vDash N \bullet \vec{v} \Rrightarrow M \dashv \Theta'; SC_2\\ \Theta \vdash SC_1 \& SC_2 = SC\end{array}}{\Gamma;\Phi;\Theta \vDash Q \to N \bullet v, \vec{v} \Rrightarrow M \dashv \Theta'; SC}\ \textsc{ATArrowApp}$$

$$\dfrac{\begin{array}{l}\Gamma;\Phi;\Theta,\overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \vec{v} \Rrightarrow M \dashv \Theta'; SC\\ \vec{v} \neq \cdot\end{array}}{\Gamma;\Phi;\Theta \vDash \forall\overrightarrow{\alpha^+}.N \bullet \vec{v} \Rrightarrow M \dashv \Theta'; SC}\ \textsc{ATForallApp}$$

4 Proofs

4.1 Substitution

Lemma 1 (Substitution strengthening)**.** *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

+ *if* $\Gamma_1 \vdash P$ *then* $[\sigma]P = [\sigma|_{\mathbf{fv}\,P}]P$,

− *if* $\Gamma_1 \vdash N$ *then* $[\sigma]N = [\sigma|_{\mathbf{fv}\,N}]N$

Proof. Ilya: todo □

Corollary 1 (Substitution preserves equivalence)**.** *Suppose that $\Gamma \vdash \sigma : \Gamma_1$. Then*

+ *if* $\Gamma_1 \vdash P$, $\Gamma_1 \vdash Q$, *and* $\Gamma_1 \vdash P \simeq_1^{\leqslant} Q$ *then* $\Gamma \vdash [\sigma]P \simeq_1^{\leqslant} [\sigma]Q$

− *if* $\Gamma_1 \vdash N$, $\Gamma_1 \vdash M$, *and* $\Gamma_1 \vdash N \simeq_1^{\leqslant} M$ *then* $\Gamma \vdash [\sigma]N \simeq_1^{\leqslant} [\sigma]M$

Lemma 2. *Suppose that $\Gamma' \subseteq \Gamma$, σ_1 and σ_2 are substitutions of signature $\Gamma \vdash \sigma_i : \Gamma'$. Then*

+ *for a type* $\Gamma \vdash P$, *if* $\Gamma \vdash [\sigma_1]P \simeq_1^{\leqslant} [\sigma_2]P$ *then* $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \mathbf{fv}\,P \cap \Gamma'$;

− *for a type* $\Gamma \vdash N$, *if* $\Gamma \vdash [\sigma_1]N \simeq_1^{\leqslant} [\sigma_2]N$ *then* $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \mathbf{fv}\,N \cap \Gamma'$.

Proof. Let us make an additional assumption that σ_1, σ_2, and the mentioned types are normalized. If they are not, we normalize them first.

Notice that the normalization preserves the set of free variables (lemma 15), well-formedness (corollary 12), and equivalence (lemma 29), and distributes over substitution (lemma 17). This way, the assumed and desired properties are equivalent to their normalized versions.

We prove it by induction on the structure of P and mutually, N. Let us consider the shape of this type.

Case 1. $P = \alpha^+ \in \Gamma'$. Then $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \mathbf{fv}\,P \cap \Gamma'$ means $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \alpha^+$, i.e. $\Gamma \vdash [\sigma_1]\alpha^+ \simeq_1^{\leqslant} [\sigma_2]\alpha^+$, which holds by assumption.

Case 2. $P = \alpha^+ \in \Gamma \backslash \Gamma'$. Then $\mathbf{fv}\,P \cap \Gamma' = \varnothing$, so $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \mathbf{fv}\,P \cap \Gamma'$ holds vacuously.

Case 3. $P = {\downarrow}N$. Then the induction hypothesis is applicable to type N:

1. N is normalized,
2. $\Gamma \vdash N$ by inversion of $\Gamma \vdash {\downarrow}N$,
3. $\Gamma \vdash [\sigma_1]N \simeq_1^{\leqslant} [\sigma_2]N$ holds by inversion of $\Gamma \vdash [\sigma_1]{\downarrow}N \simeq_1^{\leqslant} [\sigma_2]{\downarrow}N$, i.e. $\Gamma \vdash {\downarrow}[\sigma_1]N \simeq_1^{\leqslant} {\downarrow}[\sigma_2]N$.

This way, we obtain $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \mathbf{fv}\,N \cap \Gamma'$, which implies the required equivalence since $\mathbf{fv}\,P \cap \Gamma' = \mathbf{fv}\,{\downarrow}N \cap \Gamma' = \mathbf{fv}\,N \cap \Gamma'$.

Case 4. $P = \exists\overrightarrow{\alpha^-}.Q$ Then the induction hypothesis is applicable to type Q well-formed in context $\Gamma, \overrightarrow{\alpha^-}$:

1. $\Gamma' \subseteq \Gamma, \overrightarrow{\alpha^-}$ since $\Gamma' \subseteq \Gamma$,
2. $\Gamma, \overrightarrow{\alpha^-} \vdash \sigma_i : \Gamma'$ by weakening,
3. Q is normalized,
4. $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.Q$,
5. Notice that $[\sigma_i]\exists\overrightarrow{\alpha^-}.Q$ is normalized, and thus, $[\sigma_1]\exists\overrightarrow{\alpha^-}.Q \simeq_1^D [\sigma_2]\exists\overrightarrow{\alpha^-}.Q$ implies $[\sigma_1]\exists\overrightarrow{\alpha^-}.Q = [\sigma_2]\exists\overrightarrow{\alpha^-}.Q$ (by lemma 29).). This equality means $[\sigma_1]Q = [\sigma_2]Q$, which implies $\Gamma \vdash [\sigma_1]Q \simeq_1^{\leqslant} [\sigma_2]Q$.

Case 5. $N = P \to M$

□

4.2 Declarative Subtyping

Lemma 3 (Free Variable Propagation)**.** *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context* Γ*,*

- $-$ *if* $\Gamma \vdash N \leqslant_1 M$ *then* $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$
- $+$ *if* $\Gamma \vdash P \geqslant_1 Q$ *then* $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$

Proof. Mutual induction on $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash P \geqslant_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
It is self-evident that $\alpha^- \subseteq \alpha^-$.

Case 2. $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq_1^{\leqslant} Q$), we have $\Gamma \vdash P \geqslant_1 Q$. Then by the induction hypothesis, $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$. The desired inclusion inclusion holds, since $\mathbf{fv}(\uparrow P) = \mathbf{fv}(P)$ and $\mathbf{fv}(\uparrow Q) = \mathbf{fv}(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant_1 Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$ and $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$. Then $\mathbf{fv}(P \to N) = \mathbf{fv}(P) \cup \mathbf{fv}(N) \subseteq \mathbf{fv}(Q) \cup \mathbf{fv}(M) = \mathbf{fv}(Q \to M)$.

Case 4. $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$

$$\begin{aligned}
\mathbf{fv}\,\forall\overrightarrow{\alpha^+}.N &\subseteq \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \setminus \overrightarrow{\beta^+} && \text{here } \overrightarrow{\beta^+} \text{ is excluded by the premise } \mathbf{fv}\,N \cap \overrightarrow{\beta^+} = \varnothing\\
&\subseteq \mathbf{fv}\,M \setminus \overrightarrow{\beta^+} && \text{by the induction hypothesis, } \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \subseteq \mathbf{fv}\,M\\
&\subseteq \mathbf{fv}\,\forall\overrightarrow{\beta^+}.M
\end{aligned}$$

Case 5. The positive cases are symmetric.

□

Corollary 2 (Free Variables of mutual subtypes)**.**

$-$ *If* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $\mathbf{fv}\,N = \mathbf{fv}\,M$*,*

$+$ *If* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $\mathbf{fv}\,P = \mathbf{fv}\,Q$

Lemma 4 (Subtypes and supertypes of a variable)**.** *Assuming* $\Gamma \vdash \alpha^-$, $\Gamma \vdash \alpha^+$, $\Gamma \vdash N$*, and* $\Gamma \vdash P$*,*

$+$ *if* $\Gamma \vdash P \geqslant_1 \alpha^+$ *or* $\Gamma \vdash \alpha^+ \geqslant_1 P$ *then* $P = \exists\overrightarrow{\alpha^-}.\alpha^+$ *(for some potentially empty* $\overrightarrow{\alpha^-}$*)*

$-$ *if* $\Gamma \vdash N \leqslant_1 \alpha^-$ *or* $\Gamma \vdash \alpha^- \leqslant_1 N$ *then* $N = \forall\overrightarrow{\alpha^+}.\alpha^-$ *(for some potentially empty* $\overrightarrow{\alpha^+}$*)*

Proof. We prove by induction on the tree inferring $\Gamma \vdash P \geqslant_1 \alpha^+$ or $\Gamma \vdash \alpha^+ \geqslant_1 P$ or or $\Gamma \vdash N \leqslant_1 \alpha^-$ or $\Gamma \vdash \alpha^- \leqslant_1 N$.
Let us consider which of these judgments the tree is inferring.

Case 1. $\Gamma \vdash P \geqslant_1 \alpha^+$
If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\text{Var}^{+\geqslant_1}$), which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists^{\geqslant_1}$). By inverting this rule, $P = \exists\overrightarrow{\alpha^-}.P'$ where P' does not start with $\exists$ and $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P' \geqslant_1 \alpha^+$ for some $\Gamma, \overrightarrow{\beta^-} \vdash N_i$.

By the induction hypothesis, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]P' = \exists\overrightarrow{\beta^-}.\alpha^+$. Notice that P' must be a variable, because P' does not start with $\exists$, nor does it start with $\uparrow$ (otherwise, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]P'$ would also started with $\uparrow$ and would not be equal to $\exists\overrightarrow{\beta^-}.\alpha^+$). Since P' is a *positive* variable, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]P' = P'$, and then $P' = \exists\overrightarrow{\beta^-}.\alpha^+$ means that $P' = \alpha^+$. This way, $P = \exists\overrightarrow{\alpha^-}.P' = \exists\overrightarrow{\alpha^-}.\alpha^+$, as required.

Case 2. $\Gamma \vdash \alpha^+ \geqslant_1 P$
If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\mathrm{Var}^{+\geqslant_1}$), which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists^{\geqslant_1}$). By inverting this rule, $P = \exists\overrightarrow{\beta^-}.Q$ where and $\Gamma, \overrightarrow{\beta^-} \vdash \alpha^+ \geqslant_1 Q$.

By the induction hypothesis, $Q = \exists\overrightarrow{\beta^-}'.\alpha^+$. This way, $P = \exists\overrightarrow{\beta^-}.Q = \exists\overrightarrow{\beta^-}.\exists\overrightarrow{\beta^-}'.\alpha^+$, as required.

Case 3. The negative cases ($\Gamma \vdash N \leqslant_1 \alpha^-$ and $\Gamma \vdash \alpha^- \leqslant_1 N$) are proved analogously.

□

Corollary 3 (Variables have no proper subtypes and supertypes)**.** *Assuming that all mentioned types are well-formed in* Γ*,*

$$\begin{array}{ccccccc}
\Gamma \vdash P \geqslant_1 \alpha^+ & \iff & P = \exists\overrightarrow{\beta^-}.\alpha^+ & \iff & \Gamma \vdash P \simeq_1^{\leqslant} \alpha^+ & \iff & P \simeq_1^{D} \alpha^+ \\
\Gamma \vdash \alpha^+ \geqslant_1 P & \iff & P = \exists\overrightarrow{\beta^-}.\alpha^+ & \iff & \Gamma \vdash P \simeq_1^{\leqslant} \alpha^+ & \iff & P \simeq_1^{D} \alpha^+ \\
\Gamma \vdash N \leqslant_1 \alpha^- & \iff & N = \forall\overrightarrow{\beta^+}.\alpha^- & \iff & \Gamma \vdash N \simeq_1^{\leqslant} \alpha^- & \iff & N \simeq_1^{D} \alpha^- \\
\Gamma \vdash \alpha^- \leqslant_1 N & \iff & N = \forall\overrightarrow{\beta^+}.\alpha^- & \iff & \Gamma \vdash N \simeq_1^{\leqslant} \alpha^- & \iff & N \simeq_1^{D} \alpha^-
\end{array}$$

Proof. Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \simeq_1^{\leqslant} \alpha^+$ and $\exists\overrightarrow{\alpha^-}.\alpha^+ \simeq \alpha^+$ and apply lemma 4. Ilya: fix □

Lemma 5 (Reflexivity of subtyping)**.** *Assuming all the types are well-formed in* Γ*,*

$-$ $\Gamma \vdash N \leqslant_1 N$

$+$ $\Gamma \vdash P \geqslant_1 P$

Proof. Let us prove it by the size of N and mutually, P.

Case 1. $N = \alpha^-$
Then $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$ is inferred immediately by Rule ($\mathrm{Var}^{-\leqslant_1}$).

Case 2. $N = \forall\overrightarrow{\alpha^+}.N'$ where $\overrightarrow{\alpha^+}$ is not empty
First, we rename $\overrightarrow{\alpha^+}$ to fresh $\overrightarrow{\beta^+}$ in $\forall\overrightarrow{\alpha^+}.N'$ to avoid name clashes: $\forall\overrightarrow{\alpha^+}.N' = \forall\overrightarrow{\beta^+}.[\overrightarrow{\alpha^+}/\overrightarrow{\beta^+}]N'$. Then to infer $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant_1 \forall\overrightarrow{\beta^+}.[\overrightarrow{\alpha^+}/\overrightarrow{\beta^+}]N'$ we can apply Rule ($\forall^{\leqslant_1}$), instantiating $\overrightarrow{\alpha^+}$ with $\overrightarrow{\beta^+}$:

- **fv** $N \cap \overrightarrow{\beta^+} = \varnothing$ by choice of $\overrightarrow{\beta^+}$,
- $\Gamma, \overrightarrow{\beta^+} \vdash \beta_i^+$,
- $\Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{\beta^+}/\overrightarrow{\alpha^+}]N' \leqslant_1 [\overrightarrow{\beta^+}/\overrightarrow{\alpha^+}]N'$ by the induction hypothesis, since the size of $[\overrightarrow{\beta^+}/\overrightarrow{\alpha^+}]N'$ is equal to the size of N', which is smaller than the size of $N = \forall\overrightarrow{\alpha^+}.N'$.

Case 3. $N = P \to M$
Then $\Gamma \vdash P \to M \leqslant_1 P \to M$ is inferred by Rule ($\to^{\leqslant_1}$), since $\Gamma \vdash P \geqslant_1 P$ and $\Gamma \vdash M \leqslant_1 M$ hold the induction hypothesis.

Case 4. $N = \uparrow P$
Then $\Gamma \vdash \uparrow P \leqslant_1 \uparrow P$ is inferred by Rule ($\uparrow^{\leqslant_1}$), since $\Gamma \vdash P \geqslant_1 P$ holds by the induction hypothesis.

Case 5. The positive cases are symmetric to the negative ones.

□

Lemma 6 (Substitution preserves subtyipng)**.** *Assuming that all mentioned types are well-formed in* Γ*, and* $\Gamma' \vdash \sigma : \Gamma$*, where* Γ' *is disjoint from* Γ*,*

- $-$ *If* $\Gamma \vdash N \leqslant_1 M$ *then* $\Gamma' \vdash [\sigma]N \leqslant_1 [\sigma]M$
- $+$ *If* $\Gamma \vdash P \geqslant_1 Q$ *then* $\Gamma' \vdash [\sigma]P \geqslant_1 [\sigma]Q$

Proof. We prove it by induction on the size of the derivation of $\Gamma \vdash N \leqslant_1 M$ and mutually, $\Gamma \vdash P \geqslant_1 Q$. Let us consider the last rule used in the derivation:

Case 1. Rule ($\mathrm{Var}^{-\leqslant_1}$). Then by inversion, $N = \alpha^-$ and $M = \alpha^-$. By reflexivity of subtyping (lemma 5), we have $\Gamma' \vdash [\sigma]\alpha^- \leqslant_1 [\sigma]\alpha^-$, i.e. $\Gamma' \vdash [\sigma]N \leqslant_1 [\sigma]M$, as required.

Case 2. Rule ($\forall^{\leqslant_1}$). Then by inversion, $N = \forall\overrightarrow{\alpha^+}.N'$, $M = \forall\overrightarrow{\beta^+}.M'$, where $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is not empty. Moreover, $\Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant_1 M'$ for some $\Gamma, \overrightarrow{\beta^+} \vdash \overrightarrow{P}$, and $\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing$.

Notice that since the derivation of $\Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant_1 M'$ is a subderivation of the derivation of $\Gamma \vdash N \leqslant_1 M$, its size is smaller, and hence, the induction hypothesis applies: $\Gamma', \overrightarrow{\beta^+} \vdash [\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant_1 [\sigma]M'$.

First, let us assume that $\overrightarrow{\alpha^+} \cap \Gamma' = \varnothing$ and $\overrightarrow{\beta^+} \cap \Gamma' = \varnothing$ (otherwise, we rename $\overrightarrow{\alpha^+}$ and $\overrightarrow{\beta^+}$ to fresh $\overrightarrow{\alpha^+}'$ and $\overrightarrow{\beta^+}'$). Then $[\sigma]\forall\overrightarrow{\alpha^+}.N' = \forall\overrightarrow{\alpha^+}.[\sigma]N'$ and $[\sigma]\forall\overrightarrow{\beta^+}.M' = \forall\overrightarrow{\beta^+}.[\sigma]M'$, which means that the required $\Gamma' \vdash [\sigma]\forall\overrightarrow{\alpha^+}.N' \leqslant_1 [\sigma]\forall\overrightarrow{\beta^+}.M'$ is rewritten as $\Gamma' \vdash \forall\overrightarrow{\alpha^+}.[\sigma]N' \leqslant_1 \forall\overrightarrow{\beta^+}.[\sigma]M'$.

To infer it, we apply Rule ($\forall^{\leqslant_1}$), instantiating α_i^+ with $[\sigma]P_i$:

- $\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing$ as noted before, from the inversion;
- $\Gamma', \overrightarrow{\beta^+} \vdash [\sigma]P_i$, by corollary 6 since from the inversion, $\Gamma, \overrightarrow{\beta^+} \vdash P_i$;
- $\Gamma, \overrightarrow{\beta^+} \vdash [[\sigma]\overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N' \leqslant_1 [\sigma]M'$ holds because $[[\sigma]\overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N' = [\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N$ (since $\overrightarrow{\alpha^+} \cap \Gamma = \varnothing$), and $\Gamma', \overrightarrow{\beta^+} \vdash [\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant_1 [\sigma]M'$ holds by the induction hypothesis.

Case 3. Rule ($\rightarrow^{\leqslant_1}$). Then by inversion, $N = P \to N_1$, $M = Q \to M_1$, $\Gamma \vdash P \geqslant_1 Q$, and $\Gamma \vdash N_1 \leqslant_1 M_1$. And by the induction hypothesis, $\Gamma' \vdash [\sigma]P \geqslant_1 [\sigma]Q$ and $\Gamma' \vdash [\sigma]N_1 \leqslant_1 [\sigma]M_1$. Then $\Gamma' \vdash [\sigma]N \leqslant_1 [\sigma]M$, i.e. $\Gamma' \vdash [\sigma]P \to [\sigma]N_1 \leqslant_1 [\sigma]Q \to [\sigma]M_1$, is inferred by Rule ($\rightarrow^{\leqslant_1}$).

Case 4. Rule ($\uparrow^{\leqslant_1}$). Then by inversion, $N = \uparrow P$, $M = \uparrow Q$, and $\Gamma \vdash P \simeq_1^{\leqslant} Q$, which by inversion means that $\Gamma \vdash P \geqslant_1 Q$ and $\Gamma \vdash Q \geqslant_1 P$. Then the induction hypothesis applies, and we have $\Gamma' \vdash [\sigma]P \geqslant_1 [\sigma]Q$ and $\Gamma' \vdash [\sigma]Q \geqslant_1 [\sigma]P$. Then by sequential application of Rule ($\simeq_1^{\leqslant}{}^-$) and Rule ($\uparrow^{\leqslant_1}$) to these judgments, we have $\Gamma' \vdash \uparrow[\sigma]P \leqslant_1 \uparrow[\sigma]Q$, i.e. $\Gamma' \vdash [\sigma]N \leqslant_1 [\sigma]M$, as required.

Case 5. The positive cases are proved symmetrically.

□

Lemma 7 (Strong transitivity of subtyping). *Assuming all the types are well-formed in* Γ,

- $-$ *if* $\Gamma \vdash N \leqslant_1 M_1$, $\Gamma \vdash M_2 \leqslant_1 K$, *and for* $\Gamma' \vdash \sigma : \Gamma$, $[\sigma]M_1 = [\sigma]M_2$ *then* $\Gamma' \vdash [\sigma]N \leqslant_1 [\sigma]K$
- $+$ *if* $\Gamma \vdash P \geqslant_1 Q_1$, $\Gamma \vdash Q_2 \geqslant_1 R$, *and for* $\Gamma' \vdash \sigma : \Gamma$, $[\sigma]Q_1 = [\sigma]Q_2$ *then* $\Gamma' \vdash [\sigma]P \geqslant_1 [\sigma]R$

Proof. We prove it by induction on $\mathsf{depth}(\Gamma \vdash N \leqslant_1 M_1) + \mathsf{depth}(\Gamma \vdash M_2 \leqslant_1 K)$ and mutually, on $\mathsf{depth}(\Gamma \vdash P \geqslant_1 Q_1) + \mathsf{depth}(\Gamma \vdash Q_2 \geqslant_1 R)$.

Case 1. Firs, let us consider the case when the last rule applied to infer $\Gamma \vdash N \leqslant_1 M_1$ is Rule ($\mathrm{Var}^{-\leqslant_1}$). Notice that this case covers the base of the induction: the sum of the depths is minimal when both derivations are inferred by the non-recursive rules (i.e. Rule ($\mathrm{Var}^{-\leqslant_1}$)).

By inverting the rule, $N = \alpha^-$ and $M_1 = \alpha^-$ Then $[\sigma]N = [\sigma]\alpha^- = [\sigma]M_1 = [\sigma]M_2$. And $\Gamma' \vdash [\sigma]M_2 \leqslant_1 [\sigma]K$ by hello

□

Corollary 4 (Transitivity of subtyping). *Assuming the types are well-formed in* Γ,

$-$ *if* $\Gamma \vdash N_1 \leqslant_1 N_2$ *and* $\Gamma \vdash N_2 \leqslant_1 N_3$ *then* $\Gamma \vdash N_1 \leqslant_1 N_3$,

$+$ *if* $\Gamma \vdash P_1 \geqslant_1 P_2$ *and* $\Gamma \vdash P_2 \geqslant_1 P_3$ *then* $\Gamma \vdash P_1 \geqslant_1 P_3$.

Corollary 5 (Transitivity of equivalence). *Assuming the types are well-formed in* Γ,

$-$ *if* $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_2$ *and* $\Gamma \vdash N_2 \simeq_1^{\leqslant} N_3$ *then* $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_3$,

$+$ *if* $\Gamma \vdash P_1 \simeq_1^{\leqslant} P_2$ *and* $\Gamma \vdash P_2 \simeq_1^{\leqslant} P_3$ *then* $\Gamma \vdash P_1 \simeq_1^{\leqslant} P_3$.

4.3 Type well-formedness

Lemma 8 (Well-formedness agrees with substitution)**.** *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then*

+ $\Gamma, \Gamma_1 \vdash P \;\Leftrightarrow\; \Gamma, \Gamma_2 \vdash [\sigma]P$

− $\Gamma, \Gamma_1 \vdash N \;\Leftrightarrow\; \Gamma, \Gamma_2 \vdash [\sigma]N$

Proof. `Ilya: todo` □

Corollary 6. *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then*

+ $\Gamma_1, \Gamma_2 \vdash P \;\Leftrightarrow\; \Gamma_2 \vdash [\sigma]P$

− $\Gamma_1, \Gamma_2 \vdash N \;\Leftrightarrow\; \Gamma_2 \vdash [\sigma]N$

Lemma 9 (Equivalent Contexts)**.** *In the well-formedness judgment, only used variables matter:*

+ *if* $\Gamma_1 \cap \textbf{fv}\,P = \Gamma_2 \cap \textbf{fv}\,P$ *then* $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$,

− *if* $\Gamma_1 \cap \textbf{fv}\,N = \Gamma_2 \cap \textbf{fv}\,N$ *then* $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$.

Proof. By simple mutual induction on P and Q. □

Corollary 7. *Suppose that all the types below are well-formed in* Γ *and* $\Gamma' \subseteq \Gamma$. *Then*

+ $\Gamma \vdash P \simeq^{\leqslant}_1 Q$ *implies* $\Gamma' \vdash P \iff \Gamma' \vdash Q$

− $\Gamma \vdash N \simeq^{\leqslant}_1 M$ *implies* $\Gamma' \vdash N \iff \Gamma' \vdash M$

Proof. From lemma 9 and corollary 2. □

4.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\dfrac{}{\textbf{ord}\ vars\ \textbf{in}\ N \equiv vars \cap \textbf{fv}\ N}$	$\dfrac{N \simeq^D_1 M}{\textbf{ord}\ vars\ \textbf{in}\ N = \textbf{ord}\ vars\ \textbf{in}\ M}$	—
Normalization	$\dfrac{}{N \simeq^D_1 \textbf{nf}\,(N)}$	$\dfrac{N \simeq^D_1 M}{\textbf{nf}\,(N) = \textbf{nf}\,(M)}$	—
Equivalence	$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq^D_1 Q}{\Gamma \vdash P \simeq^{\leqslant}_1 Q}$	$\dfrac{\Gamma \vdash P \simeq^{\leqslant}_1 Q}{P \simeq^D_1 Q}$	—
Uppgrade	$\dfrac{\textbf{upgrade}\,\Gamma \vdash P\,\textbf{to}\,\Delta = Q}{Q \text{ is sound} \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \textbf{upgrade}\,\Gamma \vdash P\,\textbf{to}\,\Delta = Q}$	$\dfrac{\begin{array}{c} Q' \text{ is sound} \\ \textbf{upgrade}\,\Gamma \vdash P\,\textbf{to}\,\Delta = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \models P_1 \vee P_2 = Q}{Q \text{ is sound} \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \models P_1 \vee P_2 = Q}$	$\dfrac{\begin{array}{c} Q' \text{ is sound} \\ \Gamma \models P_1 \vee P_2 = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \Dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\begin{array}{l}(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \\ \text{is sound}\end{array} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \widehat{\tau}_i : \Xi \\ [\widehat{\tau}_i]\, Q = P_i \end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)}{\begin{array}{l} \exists (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \text{ s.t.} \\ \Gamma \models P_1 \overset{a}{\simeq} P_2 \Dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \end{array}}$	$\dfrac{\begin{array}{c} (\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2) \text{ is sound} \\ \Gamma \models P_1 \overset{a}{\simeq} P_2 \Dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \end{array}}{\exists \Gamma; \Xi \vdash \widehat{\tau} : \Xi' \text{ s.t. } [\widehat{\tau}]\, Q' = Q}$
Unification (matching)			—
Subtyping			—

4.5 Variable Ordering

Definition 4 (Collision free bijection)**.** *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is* ***collision free on sets*** *P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 10 (Soundness of variable ordering)**.** *Variable ordering extracts precisely used free variables.*

$-$ $\mathbf{ord}\, vars\, \mathbf{in}\, N \equiv vars \cap \mathbf{fv}\, N$ *(as sets)*

$+$ $\mathbf{ord}\, vars\, \mathbf{in}\, P \equiv vars \cap \mathbf{fv}\, P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\, vars\, \mathbf{in}\, N = \vec{\alpha}$ and $\mathbf{ord}\, vars\, \mathbf{in}\, P = \vec{\alpha}$ $\square$

Corollary 8 (Additivity of ordering)**.** *Variable ordering is additive (in terms of set union) with respect to its first argument.*

$-$ $\mathbf{ord}\, (vars_1 \cup vars_2)\, \mathbf{in}\, N \equiv \mathbf{ord}\, vars_1\, \mathbf{in}\, N \cup \mathbf{ord}\, vars_2\, \mathbf{in}\, N$ *(as sets)*

$+$ $\mathbf{ord}\, (vars_1 \cup vars_2)\, \mathbf{in}\, P \equiv \mathbf{ord}\, vars_1\, \mathbf{in}\, P \cup \mathbf{ord}\, vars_2\, \mathbf{in}\, P$ *(as sets)*

Corollary 9 (Weakening of ordering)**.** *Extending the first argument of the ordering with unused variables does not change the result.*

$-$ $\mathbf{ord}\, (vars \cap \mathbf{fv}\, N)\, \mathbf{in}\, N = \mathbf{ord}\, vars\, \mathbf{in}\, N$

$+$ $\mathbf{ord}\, (vars \cap \mathbf{fv}\, P)\, \mathbf{in}\, P = \mathbf{ord}\, vars\, \mathbf{in}\, P$

Lemma 11 (Distributivity of renaming over variable ordering)**.** *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

$-$ *If μ is collision free on vars and* $\mathbf{fv}\, N$ *then* $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = \mathbf{ord}\, ([\mu]vars)\, \mathbf{in}\, [\mu]N$

$+$ *If μ is collision free on vars and* $\mathbf{fv}\, P$ *then* $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, P) = \mathbf{ord}\, ([\mu]vars)\, \mathbf{in}\, [\mu]P$

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$
Then
$$\begin{aligned}
[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) && \\
&= [\mu]\alpha^- && \text{by Rule } (\mathrm{Var}^+_\in) \\
&= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars) \\
&= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \beta^- && \text{by Rule } (\mathrm{Var}^+_\in)\text{, because } \beta^- \in [\mu]vars \\
&= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- &&
\end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$
Notice that $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. On the other hand, $\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- = \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_\notin)$, because μ is collision free on $vars$ and $\mathbf{fv}\, N$, so $\mathbf{fv}\, N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$
Then $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. To prove that $\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_\notin)$. Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

(i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.
(ii) Since μ is collision free on $vars$ and $\mathbf{fv}\, N$, $\mu(A \cap \mathbf{fv}\, N) \ni \mu(\alpha^-) \notin vars$.

d. $\alpha^- \notin A$ but $\alpha^- \in vars$
$\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- = \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_\notin)$, because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$

$$
\begin{aligned}
[\mu](\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N) &= [\mu](\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \uparrow P) \\
&= [\mu](\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P) && \text{by Rule } (\uparrow) \\
&= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]P && \text{by the induction hypothesis} \\
&= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ \uparrow [\mu]P && \text{by Rule } (\uparrow) \\
&= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]\uparrow P && \text{by the definition of substitution} \\
&= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]N
\end{aligned}
$$

Case 3. $N = P \to M$

$$
\begin{aligned}
[\mu](\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N) &= [\mu](\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P \to M) \\
&= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)) && \text{where } \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ M = \vec{\alpha}_2 \\
&= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \vec{\alpha}_1) \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to case 1. Notice that } \mu \text{ is} \\
& && \text{collision free on } \vec{\alpha}_1 \text{ and } \vec{\alpha}_2 \text{ since } \vec{\alpha}_1 \subseteq \mathit{vars} \text{ and } \vec{\alpha}_2 \subseteq \mathbf{fv}\ N \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]N) &= (\mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]P \to [\mu]M) \\
&= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \vec{\beta}_1)) && \text{where } \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]M = \vec{\beta}_2 \\
& && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)
\end{aligned}
$$

Case 4. $N = \forall \overrightarrow{\alpha^+}.M$

$$
\begin{aligned}
[\mu](\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N) &= [\mu]\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \forall \overrightarrow{\alpha^+}.M \\
&= [\mu]\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ M \\
&= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]M && \text{by the induction hypothesis}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]N) &= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]\forall \overrightarrow{\alpha^+}.M \\
&= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ \forall \overrightarrow{\alpha^+}.[\mu]M \\
&= \mathbf{ord}\ [\mu]\mathit{vars}\ \mathbf{in}\ [\mu]M
\end{aligned}
$$

□

Lemma 12 (Ordering is not affected by independent substitutions). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2, and vars is a set of variables disjoint with both Γ_1 and Γ_2. Then*

– $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ [\sigma]N = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N$

+ $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ [\sigma]P = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P$

Proof. `Ilya: Should be easy` □

Lemma 13 (Completeness of variable ordering). *Variable ordering is invariant under equivalence. For arbitrary vars,*

– *If $N \simeq_1^D M$ then $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ M$ (as lists)*

+ *If $P \simeq_1^D Q$ then $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ Q$ (as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. □

4.6 Normaliztaion

Lemma 14. *Set of free variables is invariant under equivalence.*

– *If $N \simeq_1^D M$ then $\mathbf{fv}\ N \equiv \mathbf{fv}\ M$ (as sets)*

+ *If $P \simeq_1^D Q$ then $\mathbf{fv}\ P \equiv \mathbf{fv}\ Q$ (as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

Lemma 15. *Free variables are not changed by the normalization*

– $\mathbf{fv}\ N \equiv \mathbf{fv}\ \mathbf{nf}\ (N)$

$+$ $\mathbf{fv}\, P \equiv \mathbf{fv}\, \mathbf{nf}\,(P)$

Proof. By straightforward induction on $\mathbf{nf}\,(N) = M$. ◻

Lemma 16 (Soundness of quantifier normalization)**.**

$-$ $N \simeq_1^D \mathbf{nf}\,(N)$

$+$ $P \simeq_1^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var$^-$) and (Var$^+$)
By the corresponding equivalence rules.

Case 2. ($\uparrow$), ($\downarrow$), and ($\rightarrow$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall$), i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 14, $\mathbf{fv}\, N \equiv \mathbf{fv}\, N'$. Then by lemma 10, $\overrightarrow{\alpha^+}' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$, and thus, $\overrightarrow{\alpha^+}' \cap \mathbf{fv}\, N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$.

To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^+}'.N'$, it suffices to provide a bijection $\mu : \overrightarrow{\alpha^+}' \cap \mathbf{fv}\, N' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. ($\exists$) Same as for case 3.

◻

Corollary 10 (Normalization preserves ordering)**.** *For any vars,*

$-$ $\mathbf{ord}\; vars\; \mathbf{in}\; \mathbf{nf}\,(N) = \mathbf{ord}\; vars\; \mathbf{in}\; M$

$+$ $\mathbf{ord}\; vars\; \mathbf{in}\; \mathbf{nf}\,(P) = \mathbf{ord}\; vars\; \mathbf{in}\; Q$

Proof. Immediately from lemmas 13 and 16. ◻

Lemma 17 (Distributivity of normalization over substitution)**.** *Normalization of a term distributes over substitution. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*, i.e.* σ *maps variables from* Γ_1 *into types taking free variables from* Γ_2*.*

$-$ $\mathbf{nf}\,([\sigma]N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$

$+$ $\mathbf{nf}\,([\sigma]P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P)$

where $\mathbf{nf}\,(\sigma)$ *means pointwise normalization:* $[\mathbf{nf}\,(\sigma)]\alpha^- = \mathbf{nf}\,([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.

Case 2. $P = \alpha^+$
Similar to case 1.

Case 3. If the type is formed by $\rightarrow$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \rightarrow M$ then

$$
\begin{aligned}
\mathbf{nf}\,([\sigma]N) &= \mathbf{nf}\,([\sigma](P \rightarrow M)) \\
&= \mathbf{nf}\,([\sigma]P \rightarrow [\sigma]M) && \text{By the congruence of substitution} \\
&= \mathbf{nf}\,([\sigma]P) \rightarrow \mathbf{nf}\,([\sigma]M) && \text{By the congruence of normalization, i.e. Rule } (\rightarrow) \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) \rightarrow [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M) && \text{By the induction hypothesis} \\
&= [\mathbf{nf}\,(\sigma)](\mathbf{nf}\,(P) \rightarrow \mathbf{nf}\,(M)) && \text{By the congruence of substitution} \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P \rightarrow M) && \text{By the congruence of normalization} \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)
\end{aligned}
$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$\begin{aligned}[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) &= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M)\\ &= [\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M) \quad \text{Where } \overrightarrow{\alpha^+}' = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(M) = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M \text{ (the latter is by corollary 10)}\end{aligned}$$

$$\begin{aligned}\mathbf{nf}\,([\sigma]N) &= \mathbf{nf}\,([\sigma]\forall\overrightarrow{\alpha^+}.M)\\ &= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\sigma]M) && \text{Assuming } \overrightarrow{\alpha^+}\cap\Gamma_1=\varnothing \text{ and } \overrightarrow{\alpha^+}\cap\Gamma_2=\varnothing\\ &= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\sigma]M) && \text{Where } \overrightarrow{\beta^+} = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,([\sigma]M) = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M \text{ (the latter is by corollary 10)}\\ &= \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\sigma]M) && \text{By lemma 12, } \overrightarrow{\beta^+} = \overrightarrow{\alpha^+}' \text{ since } \overrightarrow{\alpha^+} \text{ is disjoint with } \Gamma_1 \text{ and } \Gamma_2\\ &= \forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M) && \text{By the induction hypothesis}\end{aligned}$$

To show alpha-equivalence of $[\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$, we can assume that $\overrightarrow{\alpha^+}'\cap\Gamma_1=\varnothing$, and $\overrightarrow{\alpha^+}'\cap\Gamma_2=\varnothing$.

Case 5. $P = \exists\overrightarrow{\alpha^-}.Q$
Same as for case 4.

□

Corollary 11 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

– $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$

+ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Proof. Immediately from lemma 17, after noticing that $\mathbf{nf}\,(\mu) = \mu$. □

Lemma 18 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

– *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

+ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. $(\forall^{\simeq_1^D})$

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\overrightarrow{\beta^+}\cap\mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+}\cap\mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 10 and 15, the domain and the codomain of μ can be written as $\mu : \overrightarrow{\beta^+}' \leftrightarrow \overrightarrow{\alpha^+}'$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by corollary 11, the second—by the induction hypothesis.

(ii)
$$\begin{aligned}[\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}'\\ &= [\mu]\mathbf{ord}\,(\overrightarrow{\beta^+}\cap\mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 15 and corollary 9}\\ &= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+}\cap\mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 11, because } \overrightarrow{\alpha^+}\cap\mathbf{fv}\,N\cap\mathbf{fv}\,\mathbf{nf}\,(M)\subseteq\overrightarrow{\alpha^+}\cap\mathbf{fv}\,M=\varnothing\\ & && \text{and } \overrightarrow{\alpha^+}\cap\mathbf{fv}\,N\cap(\overrightarrow{\beta^+}\cap\mathbf{fv}\,M)\subseteq\overrightarrow{\alpha^+}\cap\mathbf{fv}\,M=\varnothing\\ &= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+}\cap\mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M)=\mathbf{nf}\,(N) \text{ is proved}\\ &= \mathbf{ord}\,(\overrightarrow{\alpha^+}\cap\mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \overrightarrow{\alpha^+}\cap\mathbf{fv}\,N \text{ and } \overrightarrow{\beta^+}\cap\mathbf{fv}\,M\\ &= \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 15 and corollary 9}\\ &= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'\end{aligned}$$

Case 2. ($\exists^{\simeq^D_1}$) Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□

Lemma 19 (Idempotence of normalization)**.** *Normalization is idempotent*

– $\mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$

+ $\mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

Proof. By applying lemma 18 to lemma 16. □

Lemma 20. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized. Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then*

+ $[\sigma]P$ *is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \textit{is normal} \\ P & \textit{is normal} \end{cases}$

– $[\sigma]N$*is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \textit{is normal} \\ N & \textit{is normal} \end{cases}$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$
Then N is always normal, and the normality of $\sigma|_{\alpha^-}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$\begin{aligned}
[\sigma](P \to M) \text{ is normal} &\iff [\sigma]P \to [\sigma]M \text{ is normal} && \text{by the substitution congruence} \\
&\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases} && \text{by congruence of normality } \texttt{Ilya: lemma?} \\
&\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(M)} & \text{is normal} \end{cases} && \text{by the induction hypothesis} \\
&\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P) \cup \mathbf{fv}\,(M)} & \text{is normal} \end{cases} \iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P \to M)} & \text{is normal} \end{cases}
\end{aligned}$$

Case 3. $N = \uparrow P$
By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall \overrightarrow{\alpha^+}.M$

$$\begin{aligned}
[\sigma](\forall \overrightarrow{\alpha^+}.M) \text{ is normal} &\iff (\forall \overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} && \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing \\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+} \end{cases} && \text{by the definition of normalization} \\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} && \text{by lemma 12} \\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} && \text{by the induction hypothesis} \\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(\forall \overrightarrow{\alpha^+}.M)} \text{ is normal} \\ \forall \overrightarrow{\alpha^+}.M \text{ is normal} \end{cases} && \begin{array}{l} \text{since } \mathbf{fv}\,(\forall \overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M); \\ \text{by the definition of normalization} \end{array}
\end{aligned}$$

Case 5. $P = \ldots$
The positive cases are done in the same way as the negative ones.

□

4.7 Equivalence

Lemma 21 (Declarative equivalence is transitive).

+ *if* $P_1 \simeq_1^D P_2$ *and* $P_2 \simeq_1^D P_3$ *then* $P_1 \simeq_1^D P_3$,

− *if* $N_1 \simeq_1^D N_2$ *and* $N_2 \simeq_1^D N_3$ *then* $N_1 \simeq_1^D N_3$.

Proof. `Ilya: should be easy to do by induction since the types are getting smaller` □

Lemma 22 (Algorithmization of declarative equivalence). *Declarative equivalence is equality of normal forms.*

+ $P \simeq_1^D Q \iff \mathbf{nf}(P) = \mathbf{nf}(Q)$,

− $N \simeq_1^D M \iff \mathbf{nf}(N) = \mathbf{nf}(M)$.

Proof.

+ Let us prove both directions separately.
  - ⇒ exactly by lemma 18,
  - ⇐ from lemma 16, we know $P \simeq_1^D \mathbf{nf}(P) = \mathbf{nf}(Q) \simeq_1^D Q$, then by transitivity (lemma 21), $P \simeq_1^D Q$.

− The proof is exactly the same.

□

Lemma 23 (Type well-formedness is invariant under equivalence). *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,

− *if* $N \simeq_1^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. `Ilya: todo` □

Corollary 12 (Normalization preserves well-formedness).

+ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}(P)$,

− $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}(N)$

Proof. Immediately from lemmas 16 and 23. □

Corollary 13 (Normalization preserves well-formedness of substitution).
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}(\sigma) : \Gamma_1$

Lemma 24 (Soundness of equivalence). *Declarative equivalence implies mutual subtyping.*

+ *if* $\Gamma \vdash P$, $\Gamma \vdash Q$, *and* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \simeq_1^{\leqslant} Q$,

− *if* $\Gamma \vdash N$, $\Gamma \vdash M$, *and* $N \simeq_1^D M$ *then* $\Gamma \vdash N \simeq_1^{\leqslant} M$.

Proof. We prove it by mutual induction on $P \simeq_1^D Q$ and $N \simeq_1^D M$.

Case 1. $\alpha^- \simeq_1^D \alpha^-$
Then $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$ by Rule ($\mathrm{Var}^{-\leqslant_1}$), which immediately implies $\Gamma \vdash \alpha^- \simeq_1^{\leqslant} \alpha^-$ by Rule ($\simeq_1^{\leqslant}-$).

Case 2. $\uparrow P \simeq_1^D \uparrow Q$
Then by inversion of Rule ($\uparrow^{\leqslant_1}$), $P \simeq_1^D Q$, and from the induction hypothesis, $\Gamma \vdash P \simeq_1^{\leqslant} Q$, and (by symmetry) $\Gamma \vdash Q \simeq_1^{\leqslant} P$. When Rule ($\uparrow^{\leqslant_1}$) is applied to $\Gamma \vdash P \simeq_1^{\leqslant} Q$, it gives us $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$; when it is applied to $\Gamma \vdash Q \simeq_1^{\leqslant} P$, we obtain $\Gamma \vdash \uparrow Q \leqslant_1 \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \simeq_1^{\leqslant} \uparrow Q$.

Case 3. $P \to N \simeq_1^D Q \to M$
Then by inversion of Rule ($\to^{\leqslant_1}$), $P \simeq_1^D Q$ and $N \simeq_1^D M$. By the induction hypothesis, $\Gamma \vdash P \simeq_1^{\leqslant} Q$ and $\Gamma \vdash N \simeq_1^{\leqslant} M$, which means by inversion: (i) $\Gamma \vdash P \geqslant_1 Q$, (ii) $\Gamma \vdash Q \geqslant_1 P$, (iii) $\Gamma \vdash N \leqslant_1 M$, (iv) $\Gamma \vdash M \leqslant_1 N$. Applying Rule ($\to^{\leqslant_1}$) to (i) and (iii), we obtain $\Gamma \vdash P \to N \leqslant_1 Q \to M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \to M \leqslant_1 P \to N$. Together, it implies $\Gamma \vdash P \to N \simeq_1^{\leqslant} Q \to M$.

Case 4. $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M$

Then by inversion, there exists bijection $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N)$, such that $N \simeq_1^D [\mu]M$. By the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu]M$. From corollary 1 and the fact that μ is bijective, we also have $\Gamma, \overrightarrow{\beta^+} \vdash [\mu^{-1}]N \simeq_1^{\leqslant} M$.

Let us construct a subsitution $\overrightarrow{\alpha^+} \vdash \overrightarrow{P}/\overrightarrow{\beta^+} : \overrightarrow{\beta^+}$ by extending μ with arbitrary positive types on $\overrightarrow{\beta^+} \backslash \mathbf{fv}\, M$.

Notice that $[\mu]M = [\overrightarrow{P}/\overrightarrow{\beta^+}]M$, and therefore, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu]M$ implies $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{P}/\overrightarrow{\beta^+}]M \leqslant_1 N$. Then by Rule $(\forall^{\leqslant_1})$, $\Gamma \vdash \forall\overrightarrow{\beta^+}.M \leqslant_1 \forall\overrightarrow{\alpha^+}.N$.

Analogously, we construct the substitution from μ^{-1}, and use it to instantiate $\overrightarrow{\alpha^+}$ in the application of Rule $(\forall^{\leqslant_1})$ to infer $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$.

This way, $\Gamma \vdash \forall\overrightarrow{\beta^+}.M \leqslant_1 \forall\overrightarrow{\alpha^+}.N$ and $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$ gives us $\Gamma \vdash \forall\overrightarrow{\beta^+}.M \simeq_1^{\leqslant} \forall\overrightarrow{\alpha^+}.N$.

Case 5. For the cases of the positive types, the proofs are symmetric.

□

Corollary 14 (Normalization is sound w.r.t. subtyping-induced equivalence)**.**

+ *if* $\Gamma \vdash P$ *then* $\Gamma \vdash P \simeq_1^{\leqslant} \mathbf{nf}\,(P)$,

− *if* $\Gamma \vdash N$ *then* $\Gamma \vdash N \simeq_1^{\leqslant} \mathbf{nf}\,(N)$.

Proof. Immediately from lemmas 16 and 24 and corollary 12. □

Corollary 15 (Normalization preserves subtyping)**.** *Assuming all the types are well-formed in context* Γ*,*

+ $\Gamma \vdash P \geqslant_1 Q \iff \Gamma \vdash \mathbf{nf}\,(P) \geqslant_1 \mathbf{nf}\,(Q)$,

− $\Gamma \vdash N \leqslant_1 M \iff \Gamma \vdash \mathbf{nf}\,(N) \leqslant_1 \mathbf{nf}\,(M)$.

Proof.

+ $\Rightarrow$ Let us assume $\Gamma \vdash P \geqslant_1 Q$. By corollary 14, $\Gamma \vdash P \simeq_1^{\leqslant} \mathbf{nf}\,(P)$ and $\Gamma \vdash Q \simeq_1^{\leqslant} \mathbf{nf}\,(Q)$, in particular, by inversion, $\Gamma \vdash \mathbf{nf}\,(P) \geqslant_1 P$ and $\Gamma \vdash Q \geqslant_1 \mathbf{nf}\,(Q)$. Then by the transitivity of subtyping (corollary 4), $\Gamma \vdash \mathbf{nf}\,(P) \geqslant_1 \mathbf{nf}\,(Q)$.

  $\Leftarrow$ Let us assume $\Gamma \vdash \mathbf{nf}\,(P) \geqslant_1 \mathbf{nf}\,(Q)$. Also by corollary 14 and inversion, $\Gamma \vdash P \geqslant_1 \mathbf{nf}\,(P)$ and $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant_1 Q$. Then by the transitivity, $\Gamma \vdash P \geqslant_1 Q$.

− The negative case is proved symmetrically.

□

Lemma 25 (Subtyping induced by disjoint substitutions)**.** *If two disjoint substitutions induce subtyping, they are degenerate (so is the subtyping). Suppose that* $\Gamma \vdash \sigma_1 : \Gamma_1$ *and* $\Gamma \vdash \sigma_2 : \Gamma_1$*, where* $\Gamma_i \subseteq \Gamma$ *and* $\Gamma_1 \cap \Gamma_2 = \varnothing$*. Then*

− *assuming* $\Gamma \vdash N$, $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ *implies* $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, N$

+ *assuming* $\Gamma \vdash P$, $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ *implies* $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$

Proof. Proof by induciton on $\Gamma \vdash N$ (and mutually on $\Gamma \vdash P$).

Case 1. $N = \alpha^-$

Then $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\alpha^- \leqslant_1 [\sigma_2]\alpha^-$. Let us consider the following cases:

a. $\alpha^- \notin \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \alpha^-$ holds immediately, since $[\sigma_i]\alpha^- = [\mathsf{id}]\alpha^- = \alpha^-$ and $\Gamma \vdash \alpha^- \simeq_1^{\leqslant} \alpha^-$.

b. $\alpha^- \in \Gamma_1$ and $\alpha^- \in \Gamma_2$
This case is not possible by assumption: $\Gamma_1 \cap \Gamma_2 = \varnothing$.

c. $\alpha^- \in \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then we have $\Gamma \vdash [\sigma_1]\alpha^- \leqslant_1 \alpha^-$, which by corollary 3 means $\Gamma \vdash [\sigma_1]\alpha^- \simeq_1^{\leqslant} \alpha^-$, and hence, $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \alpha^-$.
$\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \alpha^-$ holds since $[\sigma_2]\alpha^- = \alpha^-$, similarly to case 1.*a*.

d. $\alpha^- \notin \Gamma_1$ and $\alpha^- \in \Gamma_2$
Then we have $\Gamma \vdash \alpha^- \leqslant_1 [\sigma_2]\alpha^-$, which by corollary 3 means $\Gamma \vdash \alpha^- \simeq_1^{\leqslant} [\sigma_2]\alpha^-$, and hence, $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \alpha^-$.
$\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \alpha^-$ holds since $[\sigma_1]\alpha^- = \alpha^-$, similarly to case 1.*a*.

Case 2. $N = \forall\overrightarrow{\alpha^+}.M$
Then by inversion, $\Gamma, \overrightarrow{\alpha^+} \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\forall\overrightarrow{\alpha^+}.M \leqslant_1 [\sigma_2]\forall\overrightarrow{\alpha^+}.M$. By the congruence of substitution and by the inversion of Rule ($\forall^{\leqslant_1}$), $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{Q}/\overrightarrow{\alpha^+}][\sigma_1]M \leqslant_1 [\sigma_2]M$, where $\Gamma, \overrightarrow{\alpha^+} \vdash Q_i$. Let us denote the (Kleisli) composition of σ_1 and $\overrightarrow{Q}/\overrightarrow{\alpha^+}$ as σ_1', noting that $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' : \Gamma_1, \overrightarrow{\alpha^+}$, and $\Gamma_1, \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$.

Let us apply the induction hypothesis to M and the substitutions σ_1' and σ_2 with $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_1']M \leqslant_1 [\sigma_2]M$ to obtain:

$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{1}$$
$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{2}$$

Then $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ holds by strengthening of 2: for any $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M \backslash \overrightarrow{\alpha^+}$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$ is strengthened to $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$, because $\mathbf{fv}\, [\sigma_2]\beta^\pm = \mathbf{fv}\, \beta^\pm = \{\beta^\pm\} \subseteq \Gamma$.

To show that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$, let us take an arbitrary $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M \backslash \overrightarrow{\alpha^+}$.

$$\begin{aligned}
\beta^\pm &= [\mathsf{id}]\beta^\pm && \text{by definition of id}\\
&\simeq_1^{\leqslant} [\sigma_1']\beta^\pm && \text{by 1}\\
&= [\overrightarrow{Q}/\overrightarrow{\alpha^+}][\sigma_1]\beta^\pm && \text{by definition of } \sigma_1'\\
&= [\sigma_1]\beta^\pm && \text{because } \overrightarrow{\alpha^+} \cap \mathbf{fv}\, [\sigma_1]\beta^\pm \subseteq \overrightarrow{\alpha^+} \cap \Gamma = \varnothing
\end{aligned}$$

This way, $\Gamma \vdash [\sigma_1]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$ for any $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ and thus, $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$.

Case 3. $N = P \to M$
Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1](P \to M) \leqslant_1 [\sigma_2](P \to M)$, then by congruence of substitution, $\Gamma \vdash [\sigma_1]P \to [\sigma_1]M \leqslant_1 [\sigma_2]P \to [\sigma_2]M$, then by inversion $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ and $\Gamma \vdash [\sigma_1]M \leqslant_1 [\sigma_2]M$.

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ and to $\Gamma \vdash [\sigma_1]M \leqslant_1 [\sigma_2]M$, we obtain (respectively):

$$\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P \tag{3}$$
$$\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{4}$$

Noting that $\mathbf{fv}\, (P \to M) = \mathbf{fv}\, P \cup \mathbf{fv}\, M$, we combine eqs. (3) and (4) to conclude: $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, (P \to M)$.

Case 4. $N = \uparrow P$
Then by inversion, $\Gamma \vdash P$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\uparrow P \leqslant_1 [\sigma_2]\uparrow P$, then by congruence of substitution and by inversion, $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$, we obtain $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$. Since $\mathbf{fv}\uparrow P = \mathbf{fv}\, P$, we can conclude: $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\uparrow P$.

Case 5. The positive cases are proved symmetrically.

□

Corollary 16 (Substitution cannot induce proper subtypes or supertypes)**.** *Assuming all mentioned types are well-formed in* Γ *and* σ *is a substitution* $\Gamma \vdash \sigma : \Gamma$,

$$\begin{aligned}
\Gamma \vdash [\sigma]N \leqslant_1 N &\Rightarrow \Gamma \vdash [\sigma]N \simeq_1^{\leqslant} N \textit{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, N\\
\Gamma \vdash N \leqslant_1 [\sigma]N &\Rightarrow \Gamma \vdash N \simeq_1^{\leqslant} [\sigma]N \textit{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, N\\
\Gamma \vdash [\sigma]P \geqslant_1 P &\Rightarrow \Gamma \vdash [\sigma]P \simeq_1^{\leqslant} P \textit{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P\\
\Gamma \vdash P \geqslant_1 [\sigma]P &\Rightarrow \Gamma \vdash P \simeq_1^{\leqslant} [\sigma]P \textit{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P
\end{aligned}$$

Lemma 26. *Asssuming that the mentioned types (P, Q, N, and M) are well-formed in Γ and that the substitutions (σ_1 and σ_2) have signature $\Gamma \vdash \sigma_i : \Gamma$,*

+ *if* $\Gamma \vdash [\sigma_1]P \geqslant_1 Q$ *and* $\Gamma \vdash [\sigma_2]Q \geqslant_1 P$
then there exists a bijection $\mu : \mathbf{fv}\, P \leftrightarrow \mathbf{fv}\, Q$ *such that* $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, P$ *and* $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, Q$*;*

− *if* $\Gamma \vdash [\sigma_1]N \leqslant_1 M$ *and* $\Gamma \vdash [\sigma_2]N \leqslant_1 M$
then there exists a bijection $\mu : \mathbf{fv}\, N \leftrightarrow \mathbf{fv}\, M$ *such that* $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, N$ *and* $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, M$*.*

Proof.

+ Applying σ_2 to both sides of $\Gamma \vdash [\sigma_1]P \geqslant_1 Q$ (by **??**), we have: $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant_1 [\sigma_2]Q$. Composing it with $\Gamma \vdash [\sigma_2]Q \geqslant_1 P$ (by transitivity **??**), we have $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant_1 P$. Then by corollary 16, $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$.

  By a symmetric argument, we also have: $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, Q$.

  Now, we prove that $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$ and $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, Q$ implies that σ_1 and σ_1 are (equivalent to) mutually inverse bijections.

  To do so, it suffices to prove that

  (i) for any $\alpha^{\pm} \in \mathbf{fv}\, P$ there exists $\beta^{\pm} \in \mathbf{fv}\, Q$ such that $\Gamma \vdash [\sigma_1]\alpha^{\pm} \simeq_1^{\leqslant} \beta^{\pm}$ and $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$; and

  (ii) for any $\beta^{\pm} \in \mathbf{fv}\, Q$ there exists $\alpha^{\pm} \in \mathbf{fv}\, P$ such that $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$ and $\Gamma \vdash [\sigma_1]\alpha^{\pm} \simeq_1^{\leqslant} \beta^{\pm}$.

  Then the these correspondences between $\mathbf{fv}\, P$ and $\mathbf{fv}\, Q$ are mutually inverse functions, since for any $\beta^{\pm}$ there can be at most one $\alpha^{\pm}$ such that $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$ (and vice versa).

  (i) Let us take $\alpha^{\pm} \in \mathbf{fv}\, P$.

    (a) if $\alpha^{\pm}$ is positive ($\alpha^{\pm} = \alpha^{+}$), from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{+} \simeq_1^{\leqslant} \alpha^{+}$, by corollary 3, we have $[\sigma_2][\sigma_1]\alpha^{+} = \exists\overrightarrow{\beta^{-}}.\alpha^{+}$. What shape can $[\sigma_1]\alpha^{+}$ have? It cannot be $\exists\overrightarrow{\alpha^{-}}.{\downarrow}N$ (for potentially empty $\overrightarrow{\alpha^{-}}$), because the outer constructor $\downarrow$ would remain after the substitution σ_2, whereas $\exists\overrightarrow{\beta^{-}}.\alpha^{+}$ does not have $\downarrow$. The only case left is $[\sigma_1]\alpha^{+} = \exists\overrightarrow{\alpha^{-}}.\gamma^{+}$. Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.\gamma^{+} \simeq_1^{\leqslant} \gamma^{+}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{+} \simeq_1^{\leqslant} \gamma^{+}$. Also notice that $[\sigma_2]\exists\overrightarrow{\alpha^{-}}.\gamma^{+} = \exists\overrightarrow{\beta^{-}}.\alpha^{+}$ implies $\Gamma \vdash [\sigma_2]\gamma^{+} \simeq_1^{\leqslant} \alpha^{+}$.

    (b) if $\alpha^{\pm}$ is negative ($\alpha^{\pm} = \alpha^{-}$) from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{-} \simeq_1^{\leqslant} \alpha^{-}$, by corollary 3, we have $[\sigma_2][\sigma_1]\alpha^{-} = \forall\overrightarrow{\beta^{+}}.\alpha^{-}$. What shape can $[\sigma_1]\alpha^{-}$ have? It cannot be $\forall\overrightarrow{\alpha^{+}}.{\uparrow}P$ nor $\forall\overrightarrow{\alpha^{+}}.P \to M$ (for potentially empty $\overrightarrow{\alpha^{+}}$), because the outer constructor ($\to$ or $\uparrow$), remaining after the substitution σ_2, is however absent in the resulting $\forall\overrightarrow{\beta^{+}}.\alpha^{-}$. Hence, the only case left is $[\sigma_1]\alpha^{-} = \forall\overrightarrow{\alpha^{+}}.\gamma^{-}$ Notice that $\Gamma \vdash \gamma^{-} \simeq_1^{\leqslant} \forall\overrightarrow{\alpha^{+}}.\gamma^{-}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{-} \simeq_1^{\leqslant} \gamma^{-}$. Also notice that $[\sigma_2]\forall\overrightarrow{\alpha^{+}}.\gamma^{-} = \forall\overrightarrow{\beta^{+}}.\alpha^{-}$ implies $\Gamma \vdash [\sigma_2]\gamma^{-} \simeq_1^{\leqslant} \alpha^{-}$.

  (ii) The proof is symmetric: We swap P and Q, σ_1 and σ_2, and exploit $\Gamma \vdash [\sigma_1][\sigma_2]\alpha^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$ instead of $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$.

− The proof is symmetric to the positive case.

□

Lemma 27 (Equivalence of polymorphic types)**.**

− *For* $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N$ *and* $\Gamma \vdash \forall\overrightarrow{\beta^{+}}.M$*,*
  if $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N \simeq_1^{\leqslant} \forall\overrightarrow{\beta^{+}}.M$ *then there exists a bijection* $\mu : \overrightarrow{\beta^{+}} \cap \mathbf{fv}\, M \leftrightarrow \overrightarrow{\alpha^{+}} \cap \mathbf{fv}\, N$ *such that* $\Gamma, \overrightarrow{\alpha^{+}} \vdash N \simeq_1^{\leqslant} [\mu]N$*,*

+ *For* $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.P$ *and* $\Gamma \vdash \exists\overrightarrow{\beta^{-}}.Q$*,*
  if $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.P \simeq_1^{\leqslant} \exists\overrightarrow{\beta^{-}}.Q$ *then there exists a bijection* $\mu : \overrightarrow{\beta^{-}} \cap \mathbf{fv}\, Q \leftrightarrow \overrightarrow{\alpha^{-}} \cap \mathbf{fv}\, P$ *such that* $\Gamma, \overrightarrow{\beta^{-}} \vdash P \simeq_1^{\leqslant} [\mu]Q$*.*

Proof.

− First, by α-conversion, we ensure $\overrightarrow{\alpha^{+}} \cap \mathbf{fv}\, M = \varnothing$ and $\overrightarrow{\beta^{+}} \cap \mathbf{fv}\, N = \varnothing$. By inversion, $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N \simeq_1^{\leqslant} \forall\overrightarrow{\beta^{+}}.M$ implies

  1. $\Gamma, \overrightarrow{\beta^{+}} \vdash [\sigma_1]N \leqslant_1 M$ for $\Gamma, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \overrightarrow{\alpha^{+}}$ and
  2. $\Gamma, \overrightarrow{\alpha^{+}} \vdash [\sigma_2]M \leqslant_1 N$ for $\Gamma, \overrightarrow{\alpha^{+}} \vdash \sigma_2 : \overrightarrow{\beta^{+}}$.

  To apply lemma 26, we weaken and rearrange the contexts, and extend the substitutions to act as identity outside of their initial domain:

  1. $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\sigma_1]N \leqslant_1 M$ for $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}}$ and
  2. $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\sigma_2]M \leqslant_1 N$ for $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_2 : \Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}}$.

  Then from lemma 26, there exists a bijection $\mu : \mathbf{fv}\, M \leftrightarrow \mathbf{fv}\, N$ such that $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_2 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, M$ and $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, N$.

  Let us show that if we restrict the domain of μ to $\overrightarrow{\beta^{+}}$, its range will be contained in $\overrightarrow{\alpha^{+}}$. Let us take $\gamma^{+} \in \overrightarrow{\beta^{+}} \cap \mathbf{fv}\, M$ and assume $[\mu]\gamma^{+} \notin \overrightarrow{\alpha^{+}}$. Then since $\Gamma, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \overrightarrow{\alpha^{+}}$, σ_1 acts as identity outside of $\overrightarrow{\alpha^{+}}$, i.e. $[\sigma_1][\mu]\gamma^{+} = [\mu]\gamma^{+}$. Since $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, N$, application of σ_1 is equivalent to application of μ^{-1}, then $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\mu^{-1}][\mu]\gamma^{+} \simeq_1^{\leqslant} [\mu]\gamma^{+}$, i.e.

$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \gamma^+ \simeq_1^{\leqslant} [\mu]\gamma^+$, which means $\gamma^+ \in \mathbf{fv}\,[\mu]\gamma^+ \subseteq \mathbf{fv}\,N$. By assumption, $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv}\,M$, i.e. $\overrightarrow{\beta^+} \cap \mathbf{fv}\,N \neq \varnothing$, hence contradiction.

By **??**, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq_1^{\leqslant} \mu|_{\overrightarrow{\beta^+}} : \mathbf{fv}\,M$ implies $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \simeq_1^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$. By similar reasoning, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \simeq_1^{\leqslant} [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N$.

This way,

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N \leqslant_1 M \tag{5}$$
$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu|_{\overrightarrow{\beta^+}}]M \leqslant_1 N \tag{6}$$

By applying $\mu|_{\overrightarrow{\beta^+}}$ to both sides of 5 (**??**) and contracting $\mu^{-1}|_{\overrightarrow{\alpha^+}} \circ \mu|_{\overrightarrow{\beta^+}} = \mu|^{-1}_{\overrightarrow{\beta^+}} \circ \mu|_{\overrightarrow{\beta^+}} = \mathsf{id}$, we have: $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \leqslant_1 [\mu|_{\overrightarrow{\beta^+}}]M$, which together with 6 means $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \simeq_1^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$, and by strengthening, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$. Symmetrically, $\Gamma, \overrightarrow{\beta^+} \vdash M \simeq_1^{\leqslant} [\mu|^{-1}_{\overrightarrow{\beta^+}}]N$.

- $+$ The proof is symmetric to the proof of the negative case.

□

Lemma 28 (Completeness of equivalence)**.** *Mutual subtyping implies declarative equivalence. Assuming all the types below are well-formed in* Γ*:*

$+$ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $P \simeq_1^{D} Q$*,*

$-$ *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $N \simeq_1^{D} M$*.*

Proof. $-$ Induction on the sum of sizes of N and M. By inversion, $\Gamma \vdash N \simeq_1^{\leqslant} M$ means $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash M \leqslant_1 N$. Let us consider the last rule that forms $\Gamma \vdash N \leqslant_1 M$:

Case 1. Rule $(\mathrm{Var}^{-\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
Then $N \simeq_1^{D} M$ (i.e. $\alpha^- \simeq_1^{D} \alpha^-$) holds immediately by Rule $(\mathrm{Var}^{-\simeq_1^{D}})$.

Case 2. Rule $(\uparrow^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$
Then by inversion, $\Gamma \vdash P \simeq_1^{\leqslant} Q$, and by induction hypothesis, $P \simeq_1^{D} Q$. Then $N \simeq_1^{D} M$ (i.e. $\uparrow P \simeq_1^{D} \uparrow Q$) holds by Rule $(\uparrow^{\simeq_1^{D}})$.

Case 3. Rule $(\rightarrow^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash P \rightarrow N' \leqslant_1 Q \rightarrow M'$
Then by inversion, $\Gamma \vdash P \geqslant_1 Q$ and $\Gamma \vdash N' \leqslant_1 M'$. Notice that $\Gamma \vdash M \leqslant_1 N$ is of the form $\Gamma \vdash Q \rightarrow M' \leqslant_1 P \rightarrow N'$, which by inversion means $\Gamma \vdash Q \geqslant_1 P$ and $\Gamma \vdash M' \leqslant_1 N'$.
This way, $\Gamma \vdash Q \simeq_1^{\leqslant} P$ and $\Gamma \vdash M' \simeq_1^{\leqslant} N'$. Then by induction hypothesis, $Q \simeq_1^{D} P$ and $M' \simeq_1^{D} N'$. Then $N \simeq_1^{D} M$ (i.e. $P \rightarrow N' \simeq_1^{D} Q \rightarrow M'$) holds by Rule $(\rightarrow^{\simeq_1^{D}})$.

Case 4. Rule $(\forall^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant_1 \forall\overrightarrow{\beta^+}.M'$
Then by **??**, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \simeq_1^{\leqslant} \forall\overrightarrow{\beta^+}.M'$ means that there exists a bijection $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv}\,M' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N'$ such that $\Gamma, \overrightarrow{\alpha^+} \vdash [\mu]M' \simeq_1^{\leqslant} N'$.
Notice that the application of bijection μ to M' does not change its size (which is less than the size of M), hence the induction hypothesis applies. This way, $[\mu]M' \simeq_1^{D} N'$ (and by symmetry, $N' \simeq_1^{D} [\mu]M'$) holds by induction. Then we apply Rule $(\forall^{\simeq_1^{D}})$ to get $\forall\overrightarrow{\alpha^+}.N' \simeq_1^{D} \forall\overrightarrow{\beta^+}.M'$, i.e. $N \simeq_1^{D} M$.

$+$ The proof is symmetric to the proof of the negative case.

□

Corollary 17 (Normalization is complete w.r.t. subtyping-induced equivalence)**.** *Assuming all the types below are well-formed in* Γ*:*

$+$ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$*,*

$-$ *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$*.*

Proof. Immediately from lemmas 18 and 28. □

Lemma 29 (Algorithmization of subtyping-induced equivalence)**.** *Mutual subtyping is equality of normal forms. Assuming all the types below are well-formed in* Γ*:*

$+$ $\Gamma \vdash P \simeq_1^{\leqslant} Q \iff \mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$,

– $\Gamma \vdash N \simeq_1^{\leqslant} M \iff \mathbf{nf}\,(N) = \mathbf{nf}\,(M)$.

Proof. Let us prove the positive case, the negative case is symmetric. We prove both directions of $\iff$ separately:

$\Rightarrow$ exactly corollary 17;

$\Leftarrow$ by lemmas 22 and 24.

$\square$

4.8 Unification Constraint Merge

Lemma 30 (Soundness of Unification Constraint Merge)**.** *Suppose that $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$ are normalized unification constraints. If $\Theta \vdash UC_1 \& UC_2 = UC$ is defined then $UC = UC_1 \cup UC_2$.*

Proof.

- $UC_1 \;\&\; UC_2 \subseteq UC_1 \cup UC_2$
  By definition, $UC_1 \;\&\; UC_2$ consists of three parts: entries of UC_1 that do not have matching entries of UC_2, entries of UC_2 that do not have matching entries of UC_1, and the merge of matching entries.

  If e is from the first or the second part, then $e \in UC_1 \cup UC_2$ holds immediately. If e is from the third part, then e is the merge of two matching entries $e_1 \in UC_1$ and $e_2 \in UC_2$. Since UC_1 and UC_2 are normalized unification , e_1 and e_2 have one of the following forms:

  – $\widehat{\alpha}^+ :\approx P_1$ and $\widehat{\alpha}^+ :\approx P_2$, where P_1 and P_2 are normalized, and then since $\Theta(\widehat{\alpha}^+) \vdash e_1 \& e_2 = e$ exists, Rule **??** was applied to infer it. It means that $e = e_1 = e_2$;

  – $\widehat{\alpha}^- :\approx N_1$ and $\widehat{\alpha}^- :\approx N_2$, then symmetrically, $\Theta(\widehat{\alpha}^-) \vdash e_1 \& e_2 = e = e_1 = e_2$

  In both cases, $e \in UC_1 \cup UC_2$.

- $UC_1 \cup UC_2 \subseteq UC_1 \;\&\; UC_2$
  Let us take an arbitrary $e_1 \in UC_1$. Then since UC_1 is a unification constraint, e_1 has one of the following forms:

  – $\widehat{\alpha}^+ :\approx P$ where P is normalized. If $\widehat{\alpha}^+ \notin \mathbf{dom}\,(UC_2)$, then $e_1 \in UC_1 \;\&\; UC_2$. Otherwise, there is a normalized matching $e_2 = (\widehat{\alpha}^+ :\approx P') \in UC_2$ and then since $UC_1 \;\&\; UC_2$ exists, Rule **??** was applied to construct $e_1 \;\&\; e_2 \in UC_1 \;\&\; UC_2$. By inversion of Rule **??**, $e_1 \;\&\; e_2 = e_1$, and $\mathbf{nf}\,(P) = \mathbf{nf}\,(P')$, which since P and P' are normalized, implies that $P = P'$, that is $e_1 = e_2 \in UC_1 \;\&\; UC_2$.

  – $\widehat{\alpha}^- :\approx N$ where N is normalized. Then symmetrically, $e_1 = e_2 \in UC_1 \;\&\; UC_2$.

  Similarly, if we take an arbitrary $e_2 \in UC_2$, then $e_1 = e_2 \in UC_1 \;\&\; UC_2$.

$\square$

Corollary 18. *Suppose that $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$ are normalized unification constraints. If $\Theta \vdash UC_1 \& UC_2 = UC$ is defined then*

1. *$\Theta \vdash UC$ is normalized unification constraint,*

2. *for any substitution $\Theta \vdash \widehat{\sigma}$, $\Theta \vdash \widehat{\sigma} : UC$ implies $\Theta \vdash \widehat{\sigma} : UC_1$ and $\Theta \vdash \widehat{\sigma} : UC_2$.*

Proof. It is clear that since $UC = UC_1 \cup UC_2$ (by lemma 30), and being normalized means that all entries are normalized, UC is a normalized unification constraint. Analogously, $\Theta \vdash UC = UC_1 \cup UC_2$ holds immediately, since $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$.

Let us take an arbitrary substitution $\Theta \vdash \widehat{\sigma}$ and assume that $\Theta \vdash \widehat{\sigma} : UC$. Then $\Theta \vdash \widehat{\sigma} : UC_i$ holds by definition: If $e \in UC_i \subseteq UC_1 \cup UC_2 = UC$. So $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$ holds. $\square$

Lemma 31 (Completeness of Unification Constraint Entry Merge)**.** *For a fixed context Γ, suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$ are matching constraint entries.*

- *for a type P such that $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$, $\Gamma \vdash e_1 \& e_2 = e$ is defined and $\Gamma \vdash P : e$.*

- *for a type N such that $\Gamma \vdash N : e_1$ and $\Gamma \vdash N : e_2$, $\Gamma \vdash e_1 \& e_2 = e$ is defined and $\Gamma \vdash N : e$.*

Proof. The proof repeats the one of lemma 49 and is done by the case analysis on the shape of e_1 and e_2. However, it only needs to consider two cases.

Case 1. e_1 is $\widehat{\alpha}^+ :\approx Q_1$ and e_2 is $\widehat{\alpha}^+ :\approx Q_2$.

Case 2. e_1 is $\widehat{\alpha}^- :\approx N_1$ and e_2 is $\widehat{\alpha}^- :\approx M_2$.

The proof of these cases is based only on lemma 29 and corollary 5, and does not require the properties of the least upper bound or subtyping. □

Lemma 32 (Completeness of Unification Constraint Merge)**.** *Suppose that* $\Theta \vdash UC_1$ *and* $\Theta \vdash UC_2$*. Then for any substitution* $\Theta \vdash \widehat{\sigma}$ *such that* $\Theta \vdash \widehat{\sigma} : UC_1$ *and* $\Theta \vdash \widehat{\sigma} : UC_2$*,*

1. $\Theta \vdash UC_1 \& UC_2 = UC$ *is defined and*
2. $\Theta \vdash \widehat{\sigma} : UC$*.*

Proof. The proof repeats the proof of lemma 50 but uses lemma 31 instead of lemma 49. □

4.9 Unification

Lemma 33 (Soundness of Unification)**.**

+ *For normalized* P *and* Q *such that* $\Gamma; \Theta \vdash P$ *and* $\Gamma \vdash Q$*,*
  if $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC$ *then* $\Theta \vdash UC$ *and for any normalized* $\widehat{\sigma}$ *such that* $\Theta \vdash \widehat{\sigma} : UC$*,* $[\widehat{\sigma}]P = Q$*.*

− *For normalized* N *and* M *such that* $\Gamma; \Theta \vdash N$ *and* $\Gamma \vdash M$*,*
  if $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$ *then* $\Theta \vdash UC$ *and for any normalized* $\widehat{\sigma}$ *such that* $\Theta \vdash \widehat{\sigma} : UC$*,* $[\widehat{\sigma}]N = M$*.*

Proof. We prove by induction on the derivation of $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$ and mutually $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC$. Let us consider the last rule forming this derivation.

Case 1. Rule ($\mathrm{Var}^{-\overset{u}{\simeq}}$), then $N = \alpha^- = M$. The resulting unification constraint is empty: $UC = \cdot$. It satisfies $\Theta \vdash UC$ vacuously, and $[\widehat{\sigma}]\alpha^- = \alpha^-$, that is $[\widehat{\sigma}]N = M$.

Case 2. Rule ($\uparrow^{\overset{u}{\simeq}}$), then $N = \uparrow P$ and $M = \uparrow Q$. The algorithm makes a recursive call to $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC$ returning UC. By induction hypothesis, $\Theta \vdash UC$ and for any $\Theta \vdash \widehat{\sigma} : UC$, $[\widehat{\sigma}]N = [\widehat{\sigma}]\uparrow P = \uparrow[\widehat{\sigma}]P = \uparrow Q = M$, as required.

Case 3. Rule ($\to^{\overset{u}{\simeq}}$), then $N = P \to N'$ and $M = Q \to M'$. The algorithm makes two recursive calls to $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC_1$ and $\Gamma; \Theta \models N' \overset{u}{\simeq} M' \dashv UC_2$ returning $\Theta \vdash UC_1 \& UC_2 = UC$ as the result.

It is clear that P, N', Q, and M' are normalized, and that $\Gamma; \Theta \vdash P$, $\Gamma; \Theta \vdash N'$, $\Gamma \vdash Q$, and $\Gamma \vdash M'$. This way, the induction hypothesis is applicable to both recursive calls.

By applying the induction hypothesis to $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC_1$, we have:

- $\Theta \vdash UC_1$,
- for any $\Theta \vdash \widehat{\sigma}' : UC_1$, $[\widehat{\sigma}']P = Q$.

By applying it to $\Gamma; \Theta \models N' \overset{u}{\simeq} M' \dashv UC_2$, we have:

- $\Theta \vdash UC_2$,
- for any $\Theta \vdash \widehat{\sigma}' : UC_2$, $[\widehat{\sigma}']N' = M'$.

Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : UC$. By the soundness of the constraint merge (lemma 48), $\Theta \vdash UC_1 \& UC_2 = UC$ implies $\Theta \vdash \widehat{\sigma} : UC_1$ and $\Theta \vdash \widehat{\sigma} : UC_2$.

Applying the induction hypothesis to $\Theta \vdash \widehat{\sigma} : UC_1$, we have $[\widehat{\sigma}]P = Q$; applying it to $\Theta \vdash \widehat{\sigma} : UC_2$, we have $[\widehat{\sigma}]N' = M'$. This way, $[\widehat{\sigma}]N = [\widehat{\sigma}]P \to [\widehat{\sigma}]N' = Q \to M' = M$.

Case 4. Rule ($\forall^{\overset{u}{\simeq}}$), then $N = \forall\overrightarrow{\alpha^+}.N'$ and $M = \forall\overrightarrow{\alpha^+}.M'$. The algorithm makes a recursive call to $\Gamma, \overrightarrow{\alpha^+}; \Theta \models N' \overset{u}{\simeq} M' \dashv UC$ returning UC as the result.

The induction hypothesis is applicable: $\Gamma, \overrightarrow{\alpha^+}; \Theta \vdash N'$ and $\Gamma, \overrightarrow{\alpha^+} \vdash M'$ hold by inversion, and N' and M' are normalized, since N and M are. Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : UC$. By the induction hypothesis, $[\widehat{\sigma}]N' = M'$. Then $[\widehat{\sigma}]N = [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N' = \forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N' = \forall\overrightarrow{\alpha^+}.M' = M$.

Case 5. Rule ($\mathrm{UVar}^{-\overset{u}{\simeq}}$), then $N = \widehat{\alpha}^-$, $\widehat{\alpha}^-\{\Delta\} \in \Theta$, and $\Delta \vdash M$. As the result, the algorithm returns $UC = (\widehat{\alpha}^- :\approx M)$.

It is clear that $\widehat{\alpha}^-\{\Delta\} \vdash (\widehat{\alpha}^- :\approx M)$, since $\Delta \vdash M$, meaning that $\Theta \vdash UC$.

Let us take an arbitrary $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : UC$. Since $UC = (\widehat{\alpha}^- :\approx M)$, $\Theta \vdash \widehat{\sigma} : UC$ implies $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]\widehat{\alpha}^- : (\widehat{\alpha}^- :\approx M)$. By inversion of Rule SATSCENEq, it means $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]\widehat{\alpha}^- \simeq_1^{\leqslant} M$. This way, $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]N \simeq_1^{\leqslant} M$. Notice that $\widehat{\sigma}$ and N are normalized, and by **??**, so is $[\widehat{\sigma}]N$. Since both sides of $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]N \simeq_1^{\leqslant} M$ are normalized, by lemma 29, we have $[\widehat{\sigma}]N = M$.

Case 6. The positive cases are proved symmetrically.

$\square$

Lemma 34 (Completeness of Unification)**.**

+ *For normalized P and Q such that $\Gamma; \Theta \vdash P$ and $\Gamma \vdash Q$, for any $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]P = Q$, there exists $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC$, and $\Theta \vdash \hat{\sigma} : UC$.*

− *For normalized N and M such that $\Gamma; \Theta \vdash N$ and $\Gamma \vdash M$, for any $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]N = M$, there exists $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$, and $\Theta \vdash \hat{\sigma} : UC$.*

Proof. We prove it by induction on the structure of P and mutually, N.

Case 1. $N = \hat{\alpha}^-$
$\Gamma; \Theta \vdash \hat{\alpha}^-$ means that $\hat{\alpha}^-\{\Delta\} \in \Theta$ for some Δ.

Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]\hat{\alpha}^- = M$. $\Theta \vdash \hat{\sigma}$ means that $\Delta \vdash M$. This way, Rule (UVar$^{-\overset{u}{\simeq}}$) is applicable to infer $\Gamma; \Theta \models \hat{\alpha}^- \overset{u}{\simeq} M \dashv (\hat{\alpha}^- :\approx M)$. $\Theta \vdash \hat{\sigma} : (\hat{\alpha}^- :\approx M)$ holds by Rule SATSCENEq.

Case 2. $N = \alpha^-$
Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]\alpha^- = M$. The latter means $M = \alpha^-$.

Then $[\hat{\sigma}]\alpha^- = M$ means $M = \alpha^-$. This way, Rule (Var$^{-\overset{u}{\simeq}}$) infers $\Gamma; \Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot$, which is rewritten as $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \cdot$, and $\Theta \vdash \hat{\sigma} : \cdot$ holds trivially.

Case 3. $N = \uparrow P$
Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]\uparrow P = M$. The latter means $\uparrow[\hat{\sigma}]P = M$, i.e. $M = \uparrow Q$ for some Q and $[\hat{\sigma}]P = Q$.

Let us show that the induction hypothesis is applicable to $[\hat{\sigma}]P = Q$. Notice that P is normalized, since $N = \uparrow P$ is normalized, $\Gamma; \Theta \vdash P$ holds by inversion of $\Gamma; \Theta \vdash \uparrow P$, and $\Gamma \vdash Q$ holds by inversion of $\Gamma \vdash \uparrow Q$.

This way, by the induction hypothesis there exists UC such that $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC$, and moreover, $\Theta \vdash \hat{\sigma} : UC$.

Case 4. $N = P \to N'$
Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}](P \to N') = M$. The latter means $[\hat{\sigma}]P \to [\hat{\sigma}]N' = M$, i.e. $M = Q \to M'$ for some Q and M', such that $[\hat{\sigma}]P = Q$ and $[\hat{\sigma}]N' = M'$.

Let us show that the induction hypothesis is applicable to $[\hat{\sigma}]P = Q$ and to $[\hat{\sigma}]N' = M'$:

- P and N' are normalized, since $N = P \to N'$ is normalized
- $\Gamma; \Theta \vdash P$ and $\Gamma; \Theta \vdash N'$ follow from the inversion of $\Gamma; \Theta \vdash P \to N'$,
- $\Gamma \vdash Q$ and $\Gamma \vdash M'$ follow from inversion of $\Gamma \vdash Q \to M'$.

Then by the induction hypothesis, $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv UC_1$ and $\Theta \vdash \hat{\sigma} : UC_1$, $\Gamma; \Theta \models N' \overset{u}{\simeq} M' \dashv UC_2$ and $\Theta \vdash \hat{\sigma} : UC_2$. To apply Rule ($\to^{\overset{u}{\simeq}}$) and infer the required $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$, we need to show that $\Theta \vdash UC_1 \& UC_2 = UC$ is defined and $\Theta \vdash \hat{\sigma} : UC$. It holds by the completeness of the unification constraint merge (lemma 50):

- $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$ holds by the soundness of unification (lemma 33)
- $\Theta \vdash \hat{\sigma} : UC_1$ and $\Theta \vdash \hat{\sigma} : UC_2$ holds as noted above

.

Case 5. $N = \forall\overrightarrow{\alpha^+}.N'$
Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]\forall\overrightarrow{\alpha^+}.N' = M$. The latter means $\forall\overrightarrow{\alpha^+}.[\hat{\sigma}]N' = M$, i.e. $M = \forall\overrightarrow{\alpha^+}.M'$ for some M' such that $[\hat{\sigma}]N' = M'$.

Let us show that the induction hypothesis is applicable to $[\hat{\sigma}]N' = M'$. Notice that N' is normalized, since $N = \forall\overrightarrow{\alpha^+}.N'$ is normalized, $\Gamma, \overrightarrow{\alpha^+}; \Theta \vdash N'$ follows from inversion of $\Gamma; \Theta \vdash \forall\overrightarrow{\alpha^+}.N'$, $\Gamma, \overrightarrow{\alpha^+} \vdash M'$ follows from inversion of $\Gamma \vdash \forall\overrightarrow{\alpha^+}.M'$, and $\Theta \vdash \hat{\sigma}$ by assumption.

This way, by the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+}; \Theta \models N' \overset{u}{\simeq} M' \dashv UC$ exists and moreover, $\Theta \vdash \hat{\sigma} : UC$. Hence, Rule ($\forall^{\overset{u}{\simeq}}$) is applicable to infer $\Gamma; \Theta \models \forall\overrightarrow{\alpha^+}.N' \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M' \dashv UC$, that is $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$.

Case 6. The positive cases are proved symmetrically.

$\square$

4.10 Anti-unification

Observation 1 (Anti-unification algorithm is deterministic)**.**

+ *If* $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ *and* $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$*, then* $\Xi = \Xi'$, $Q = Q'$, $\widehat{\tau}_1 = \widehat{\tau}'_1$*, and* $\widehat{\tau}_2 = \widehat{\tau}'_2$.

− *If* $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ *and* $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi', M', \widehat{\tau}'_1, \widehat{\tau}'_2)$*, then* $\Xi = \Xi'$, $M = M'$, $\widehat{\tau}_1 = \widehat{\tau}'_1$*, and* $\widehat{\tau}_2 = \widehat{\tau}'_2$.

Proof. By trivial induction on $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$. ☐

Observation 2. *Names of the anti-unification variables are uniquely defined by the types they are mapped to by the resulting substitutions.*

+ *Assuming* P_1 *and* P_2 *are normalized, if* $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ *then for any* $\widehat{\beta}^- \in \Xi$, $\widehat{\beta}^- = \widehat{\alpha}^-_{\{[\widehat{\tau}_1]\widehat{\beta}^-, [\widehat{\tau}_2]\widehat{\beta}^-\}}$

− *Assuming* N_1 *and* N_2 *are normalized, if* $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ *then for any* $\widehat{\beta}^- \in \Xi$, $\widehat{\beta}^- = \widehat{\alpha}^-_{\{[\widehat{\tau}_1]\widehat{\beta}^-, [\widehat{\tau}_2]\widehat{\beta}^-\}}$

Proof. By simple induction on $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$. Let us consider tha last rule applied to infer this judgment.

Case 1. Rule ($\mathrm{Var}^{+\overset{a}{\simeq}}$) or Rule ($\mathrm{Var}^{-\overset{a}{\simeq}}$), then $\Xi = \cdot$, and the property holds vacuously.

Case 2. Rule (AU^-) Then $\Xi = \widehat{\alpha}^-_{\{N_1,N_2\}}$, $\widehat{\tau}_1 = \widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_1$, and $\widehat{\tau}_2 = \widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_2$. So the property holds trivially.

Case 3. Rule **??** In this case, $\Xi = \Xi' \cup \Xi''$, $\widehat{\tau}_1 = \widehat{\tau}'_1 \cup \widehat{\tau}''_1$, and $\widehat{\tau}_2 = \widehat{\tau}'_2 \cup \widehat{\tau}''_2$, where the property holds for (Ξ', $\widehat{\tau}'_1$, $\widehat{\tau}'_2$) and (Ξ'', $\widehat{\tau}''_1$, $\widehat{\tau}''_2$) by the induction hypothesis. Then since the union of solutions does not change the types the variables are mapped to, the required property holds for Ξ, $\widehat{\tau}_1$, and $\widehat{\tau}_2$.

Case 4. For the other rules, the resulting Ξ is taken from the recursive call and the required property holds immediately by the induction hypothesis.

☐

Lemma 35 (Soundness of Anti-Unification)**.**

+ *Assuming* P_1 *and* P_2 *are normalized, if* $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ *then*

  1. $\Gamma; \Xi \vdash Q$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ *for* $i \in \{1, 2\}$ *are anti-unification solutions, and*
  3. $[\widehat{\tau}_i]Q = P_i$ *for* $i \in \{1, 2\}$.

− *Assuming* N_1 *and* N_2 *are normalized, if* $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ *then*

  1. $\Gamma; \Xi \vdash M$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ *for* $i \in \{1, 2\}$ *are anti-unification solutions, and*
  3. $[\widehat{\tau}_i]M = N_i$ *for* $i \in \{1, 2\}$.

Proof. We prove it by induction on $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually, $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$. Let us consider the last rule applied to infer this judgement.

Case 1. Rule ($\mathrm{Var}^{-\overset{a}{\simeq}}$), then $N_1 = \alpha^- = N_2$, $\Xi = \cdot$, $M = \alpha^-$, and $\widehat{\tau}_1 = \widehat{\tau}_2 = \cdot$.

1. $\Gamma; \cdot \vdash \alpha^-$ follows from the assumption $\Gamma \vdash \alpha^-$,
2. $\Gamma; \cdot \vdash \cdot : \cdot$ holds trivially, and
3. $[\cdot]\alpha^- = \alpha^-$ holds trivially.

Case 2. Rule ($\uparrow^{\overset{a}{\simeq}}$), then $N_1 = \uparrow P_1$, $N_2 = \uparrow P_2$, and the algorithm makes the recursive call: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, returning $(\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ as the result.

Since $N_1 = \uparrow P_1$ and $N_2 = \uparrow P_2$ are normalized, so are P_1 and P_2, and thus, the induction hypothesis is applicable to $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$:

1. $\Gamma; \Xi \vdash Q$, and hence, $\Gamma; \Xi \vdash \uparrow Q$,

2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ for $i \in \{1, 2\}$, and
3. $[\widehat{\tau}_i] Q = P_i$ for $i \in \{1, 2\}$, and then by the definition of the substitution, $[\widehat{\tau}_i]{\uparrow} Q = {\uparrow} P_i$ for $i \in \{1, 2\}$.

Case 3. Rule $(\rightarrow^{\overset{a}{\simeq}})$, then $N_1 = P_1 \to N_1'$, $N_2 = P_2 \to N_2'$, and the algorithm makes two recursive calls: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and $\Gamma \vDash N_1' \overset{a}{\simeq} N_2' \dashv (\Xi', M, \widehat{\tau}_1', \widehat{\tau}_2')$ and and returns $(\Xi \cup \Xi', Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}_1', \widehat{\tau}_2 \cup \widehat{\tau}_2')$ as the result.

Notice that the induction hypothesis is applicable to $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$: P_1 and P_2 are normalized, since $N_1 = P_1 \to N_1'$ and $N_2 = P_2 \to N_2'$ are normalized. Similarly, the induction hypothesis is applicable to $\Gamma \vDash N_1' \overset{a}{\simeq} N_2' \dashv (\Xi', M, \widehat{\tau}_1', \widehat{\tau}_2')$.

This way, by the induction hypothesis:

1. $\Gamma; \Xi \vdash Q$ and $\Gamma; \Xi' \vdash M$. Then by weakening (**??**), $\Gamma; \Xi \cup \Xi' \vdash Q$ and $\Gamma; \Xi \cup \Xi' \vdash M$, which implies $\Gamma; \Xi \cup \Xi' \vdash Q \to M$;
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$ Then $\Gamma; \cdot \vdash \widehat{\tau}_i \cup \widehat{\tau}_i' : \Xi \cup \Xi'$ are well-defined anti-unification solution. Let us take an arbitrary $\widehat{\beta}^- \in \Xi \cup \Xi'$. If $\widehat{\beta}^- \in \Xi$. then $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ implies that $\widehat{\tau}_i$, and hence, $\widehat{\tau}_i \cup \widehat{\tau}_i'$ contains an entry well-formed in Γ. If $\widehat{\beta}^- \in \Xi'$, the reasoning is symmetric.
   $\widehat{\tau}_i \cup \widehat{\tau}_i'$ is a well-defined anti-unification solution: any anti-unification variable occurs uniquely $\widehat{\tau}_i \cup \widehat{\tau}_i'$, since by observation 2, the name of the variable is in one-to-one correspondence with the pair of types it is mapped to by $\widehat{\tau}_1$ and $\widehat{\tau}_2$, an is in one-to-one correspondence with the pair of types it is mapped to by $\widehat{\tau}_1'$ and $\widehat{\tau}_2'$ i.e. if $\widehat{\beta}^- \in \Xi \cap \Xi'$ then $[\widehat{\tau}_1]\widehat{\beta}^- = [\widehat{\tau}_1']\widehat{\beta}^-$, and $[\widehat{\tau}_2]\widehat{\beta}^- = [\widehat{\tau}_2']\widehat{\beta}^-$.
3. $[\widehat{\tau}_i] Q = P_i$ and $[\widehat{\tau}_i'] M = N_i'$. Since $\widehat{\tau}_i \cup \widehat{\tau}_i'$ restricted to Ξ is $\widehat{\tau}_i$, and $\widehat{\tau}_i \cup \widehat{\tau}_i'$ restricted to Ξ' is $\widehat{\tau}_i'$, we have $[\widehat{\tau}_i \cup \widehat{\tau}_i'] Q = P_i$ and $[\widehat{\tau}_i \cup \widehat{\tau}_i'] M = N_i'$, and thus, $[\widehat{\tau}_i \cup \widehat{\tau}_i'] Q \to M = P_1 \to N_1'$

Case 4. Rule $(\forall^{\overset{a}{\simeq}})$, then $N_1 = \forall \overrightarrow{\alpha^+}. N_1'$, $N_2 = \forall \overrightarrow{\alpha^+}. N_2'$, and the algorithm makes a recursive call: $\Gamma \vDash N_1' \overset{a}{\simeq} N_2' \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ and returns $(\Xi, \forall \overrightarrow{\alpha^+}. M, \widehat{\tau}_1, \widehat{\tau}_2)$ as the result.

Similarly to case 2, we apply the induction hypothesis to $\Gamma \vDash N_1' \overset{a}{\simeq} N_2' \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ to obtain:

1. $\Gamma; \Xi \vdash M$, and hence, $\Gamma; \Xi \vdash \forall \overrightarrow{\alpha^+}. M$;
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ for $i \in \{1, 2\}$, and
3. $[\widehat{\tau}_i] M = N_i'$ for $i \in \{1, 2\}$, and then by the definition of the substitution, $[\widehat{\tau}_i] \forall \overrightarrow{\alpha^+}. M = \forall \overrightarrow{\alpha^+}. N_i'$ for $i \in \{1, 2\}$.

Case 5. Rule (AU^-), which applies when other rules do not, and $\Gamma \vdash N_i$, returning as the result $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2) = (\widehat{\alpha}^-_{\{N_1,N_2\}}, \widehat{\alpha}^-_{\{N_1,N_2\}}, N_1), (\widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_2))$.

1. $\Gamma; \Xi \vdash M$ is rewritten as $\Gamma; \widehat{\alpha}^-_{\{N_1,N_2\}} \vdash \widehat{\alpha}^-_{\{N_1,N_2\}}$, which holds trivially;
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ is rewritten as $\Gamma; \cdot \vdash (\widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_i) : \widehat{\alpha}^-_{\{N_1,N_2\}}$, which holds since $\Gamma \vdash N_i$ by the premise of the rule;
3. $[\widehat{\tau}_i] M = N_i$ is rewritten as $[\widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_i] \widehat{\alpha}^-_{\{N_1,N_2\}} = N_i$, which holds trivially by the definition of substitution.

Case 6. Positive cases are proved symmetrically.

□

Lemma 36 (Completeness of Anti-Unification)**.**

\+ *Assume that P_1 and P_2 are normalized, and there exists $(\Xi', Q', \widehat{\tau}_1', \widehat{\tau}_2')$ such that*

1. $\Gamma; \Xi' \vdash Q'$,
2. $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$ *for* $i \in \{1, 2\}$ *are anti-unification solutions, and*
3. $[\widehat{\tau}_i'] Q' = P_i$ *for* $i \in \{1, 2\}$.

Then the anti-unification algorithm terminates, that is there exists $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$

− *Assume that N_1 and N_2 are normalized, and there exists $(\Xi', M', \widehat{\tau}_1', \widehat{\tau}_2')$ such that*

1. $\Gamma; \Xi' \vdash M'$,
2. $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$ *for* $i \in \{1, 2\}$, *are anti-unification solutions, and*
3. $[\widehat{\tau}_i'] M' = N_i$ *for* $i \in \{1, 2\}$.

Then the anti-unification algorithm succeeds, that is there exists $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$.

Proof. We prove it by the induction on M' and mutually on Q'.

Case 1. $M' = \widehat{\alpha}^-$ Then since $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$, $\Gamma \vdash [\widehat{\tau}'_i]M' = N_i$. This way, Rule ($\text{AU}^-$) is always applicable if other rules are not.

Case 2. $M' = \alpha^-$ Then $\alpha^- = [\widehat{\tau}'_i]\alpha^- = N_i$, which means that Rule ($\text{Var}^{-\overset{a}{\simeq}}$) is applicable.

Case 3. $M' = \uparrow Q'$ Then $\uparrow[\widehat{\tau}'_i]Q' = [\widehat{\tau}'_i]\uparrow Q' = N_i$, that is N_1 and N_2 have form $\uparrow P_1$ and $\uparrow P_2$ respectively.

Moreover, $[\widehat{\tau}'_i]Q' = P_i$, which means that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2. Then by the induction hypothesis, there exists $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, and hence, $\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ by Rule ($\uparrow^{\overset{a}{\simeq}}$).

Case 4. $M' = \forall\overrightarrow{\alpha^+}.M''$ This case is similar to the previous one: we consider $\forall\overrightarrow{\alpha^+}$ as a constructor. Notice that $\forall\overrightarrow{\alpha^+}.[\widehat{\tau}'_i]M'' = [\widehat{\tau}'_i]\forall\overrightarrow{\alpha^+}.M'' = N_i$, that is N_1 and N_2 have form $\forall\overrightarrow{\alpha^+}.N''_1$ and $\forall\overrightarrow{\alpha^+}.N''_2$ respectively.

Moreover, $[\widehat{\tau}'_i]M'' = N''_i$, which means that $(\Xi', M'', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of N''_1 and N''_2. Then by the induction hypothesis, there exists $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \vDash N''_1 \overset{a}{\simeq} N''_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$, and hence, $\Gamma \vDash \forall\overrightarrow{\alpha^+}.N''_1 \overset{a}{\simeq} \forall\overrightarrow{\alpha^+}.N''_2 \dashv (\Xi, \forall\overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)$ by Rule ($\forall^{\overset{a}{\simeq}}$).

Case 5. $M' = Q' \to M''$ Then $[\widehat{\tau}'_i]Q' \to [\widehat{\tau}'_i]M'' = [\widehat{\tau}'_i](Q' \to M'') = N_i$, that is N_1 and N_2 have form $P_1 \to N'_1$ and $P_2 \to N'_2$ respectively.

Moreover, $[\widehat{\tau}'_i]Q' = P_i$ and $[\widehat{\tau}'_i]M'' = N''_i$, which means that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2, and $(\Xi', M'', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of N''_1 and N''_2. Then by the induction hypothesis, $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and $\Gamma \vDash N''_1 \overset{a}{\simeq} N''_2 \dashv (\Xi_2, M, \widehat{\tau}_3, \widehat{\tau}_4)$ succeed. The result of the algorithm is $(\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}_3, \widehat{\tau}_2 \cup \widehat{\tau}_4)$.

Case 6. $Q' = \widehat{\alpha}^+$ This case if not possible, since $\Gamma; \Xi' \vdash Q'$ means $\widehat{\alpha}^+ \in \Xi'$, but Ξ' can only contain negative variables.

Case 7. Other positive cases are proved symmetrically to the corresponding negative ones.

□

Lemma 37 (Initiality of Anti-Unification)**.**

+ *Assume that P_1 and P_2 are normalized, and $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, then $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than any other sound anti-unifier $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$, i.e. if*

  1. $\Gamma; \Xi' \vdash Q'$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ *for* $i \in \{1, 2\}$ *are anti-unification solutions , and*
  3. $[\widehat{\tau}'_i]Q' = P_i$ *for* $i \in \{1, 2\}$

  then there exists $\widehat{\rho}$ such that $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\,Q'})$ and $[\widehat{\rho}]Q' = Q$. Moreover, $[\widehat{\rho}]\widehat{\beta}^-$ can be uniquely determined by $[\widehat{\tau}'_1]\widehat{\beta}^-$, $[\widehat{\tau}'_2]\widehat{\beta}^-$, and Γ.

− *Assume that N_1 and N_2 are normalized, and $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$, then $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than any other sound anti-unifier $(\Xi', M', \widehat{\tau}'_1, \widehat{\tau}'_2)$, i.e. if*

  1. $\Gamma; \Xi' \vdash M'$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ *for* $i \in \{1, 2\}$ *are anti-unification solutions, and*
  3. $[\widehat{\tau}'_i]M' = N_i$ *for* $i \in \{1, 2\}$

  then there exists $\widehat{\rho}$ such that $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\,M'})$ and $[\widehat{\rho}]M' = M$. Moreover, $[\widehat{\rho}]\widehat{\beta}^-$ can be uniquely determined by $[\widehat{\tau}'_1]\widehat{\beta}^-$, $[\widehat{\tau}'_2]\widehat{\beta}^-$, and Γ.

Proof. First, let us assume that M' is a metavariable $\widehat{\alpha}^-$. Then we can take $\widehat{\rho} = \widehat{\alpha}^- \mapsto M$, which satisfies the required properties:

- $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\,M'})$ holds since $\Xi'|_{\mathbf{uv}\,M'} = \widehat{\alpha}^-$ and $\Gamma; \Xi \vdash M$ by the soundness of anti-unification (lemma 35);
- $[\widehat{\rho}]M' = M$ holds by construction
- $[\widehat{\rho}]\widehat{\alpha}^- = M$ is the anti-unifier of $N_1 = [\widehat{\tau}'_1]\widehat{\alpha}^-$ and $N_2 = [\widehat{\tau}'_2]\widehat{\alpha}^-$ in context Γ, and hence, it is uniquely determined by them (observation 1).

Now, we can assume that M' is not a metavariable. We prove by induction on the derivation of $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on the derivation of $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$.

Since M' is not a metavariable, the substitution acting on M' preserves its outer constructor. In other words, $[\widehat{\tau}'_i]M' = N_i$ means that M', N_1 and N_2 have the same outer constructor. Let us consider the algorithmic anti-unification rule corresponding to this constructor, and show that it was successfully applied to anti-unify N_1 and N_2 (or P_1 and P_2).

Case 1. Rule ($\text{Var}^{-\overset{a}{\simeq}}$), i.e. $N_1 = \alpha^- = N_2$. This rule is applicable since it has no premises.

Then $\Xi = \cdot$, $M = \alpha^-$, and $\widehat{\tau}_1 = \widehat{\tau}_2 = \cdot$. Since $[\widehat{\tau}'_i]M' = N_i = \alpha^-$ and M' is not a metavariable, $M' = \alpha^-$. Then we can take $\widehat{\rho} = \cdot$, which satisfies the required properties:

- $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\ M'})$ holds vacuously since $\Xi'|_{\mathbf{uv}\ M'} = \varnothing$;
- $[\widehat{\rho}]M' = M$, that is $[\cdot]\alpha^- = \alpha^-$ holds by substitution properties;
- the unique determination of $[\widehat{\rho}]\widehat{\alpha}^-$ for $\widehat{\alpha}^- \in \Xi'|_{\mathbf{uv}\ M'} = \varnothing$ holds vacuously.

Case 2. Rule ($\uparrow^{\overset{a}{\simeq}}$), i.e. $N_1 = \uparrow P_1$ and $N_2 = \uparrow P_2$.

Then since $[\widehat{\tau}'_i]M' = N_i = \uparrow P_i$ and M' is not a metavariable, $M' = \uparrow Q'$, where $[\widehat{\tau}'_i]Q' = P_i$. Let us show that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2.

1. $\Gamma; \Xi' \vdash Q'$ holds by inversion of $\Gamma; \Xi' \vdash \uparrow Q'$;
2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ holds by assumption;
3. $[\widehat{\tau}'_i]Q' = P_i$ holds by assumption.

This way, by the completeness of anti-unification (lemma 36), the anti-unification algorithm succeeds on P_1 and P_2: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, which means that Rule ($\uparrow^{\overset{a}{\simeq}}$) is applicable to infer $\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$.

Moreover, by the induction hypothesis, $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$, which immediately implies that $(\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than $(\Xi', \uparrow Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ (we keep the same $\widehat{\rho}$).

Case 3. Rule ($\forall^{\overset{a}{\simeq}}$), i.e. $N_1 = \forall\overrightarrow{\alpha^+}.N'_1$ and $N_2 = \forall\overrightarrow{\alpha^+}.N'_2$. The proof is symmetric to the previous case. Notice that the context Γ is not changed in Rule ($\forall^{\overset{a}{\simeq}}$), as it represents the context in which the anti-unification variables must be instantiated, rather than the context forming the types that are being anti-unified.

Case 4. Rule ($\rightarrow^{\overset{a}{\simeq}}$), i.e. $N_1 = P_1 \to N'_1$ and $N_2 = P_2 \to N'_2$.

Then since $[\widehat{\tau}'_i]M' = N_i = P_i \to N'_i$ and M' is not a metavariable, $M' = Q' \to M''$, where $[\widehat{\tau}'_i]Q' = P_i$ and $[\widehat{\tau}'_i]M'' = N''_i$.

Let us show that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2.

1. $\Gamma; \Xi' \vdash Q'$ holds by inversion of $\Gamma; \Xi' \vdash Q' \to M''$;
2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ holds by assumption;
3. $[\widehat{\tau}'_i]Q' = P_i$ holds by assumption.

Similarly, $(\Xi', M'', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of N''_1 and N''_2.

Then by the completeness of anti-unification (lemma 36), the anti-unification algorithm succeeds on P_1 and P_2: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2)$; and on N'_1 and N'_2: $\Gamma \vDash N''_1 \overset{a}{\simeq} N''_2 \dashv (\Xi_2, M''', \widehat{\tau}_3, \widehat{\tau}_4)$. Notice that $\widehat{\tau}_1$ & $\widehat{\tau}_3$ and $\widehat{\tau}_2$ & $\widehat{\tau}_4$ are defined, in other words, for any $\widehat{\beta}^- \in \Xi_1 \cap \Xi_2$, $[\widehat{\tau}_1]\widehat{\beta}^- = [\widehat{\tau}_2]\widehat{\beta}^-$ and $[\widehat{\tau}_3]\widehat{\beta}^- = [\widehat{\tau}_4]\widehat{\beta}^-$, which follows immediately from observation 2. This way, the algorithm proceeds by applying Rule ($\rightarrow^{\overset{a}{\simeq}}$) and returns $(\Xi_1 \cup \Xi_2, Q \to M''', \widehat{\tau}_1 \cup \widehat{\tau}_3, \widehat{\tau}_2 \cup \widehat{\tau}_4)$.

It is left to construct $\widehat{\rho}$ such that $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\ M'})$ and $[\widehat{\rho}]M' = M$. By the induction hypothesis, there exist $\widehat{\rho}_1$ and $\widehat{\rho}_2$ such that $\Gamma; \Xi_1 \vdash \widehat{\rho}_1 : (\Xi'|_{\mathbf{uv}\ Q'})$, $\Gamma; \Xi_2 \vdash \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\ M''})$, $[\widehat{\rho}_1]Q' = Q$, and $[\widehat{\rho}_2]M'' = M'''$.

Let us show that $\widehat{\rho} = \widehat{\rho}_1 \cup \widehat{\rho}_2$ satisfies the required properties:

- $\Gamma; \Xi_1 \cup \Xi_2 \vdash \widehat{\rho}_1 \cup \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\ M'})$ holds since $\Xi'|_{\mathbf{uv}\ M'} = \Xi'|_{\mathbf{uv}\ Q' \to M''} = (\Xi'|_{\mathbf{uv}\ Q'}) \cup (\Xi'|_{\mathbf{uv}\ M''})$, $\Gamma; \Xi_1 \vdash \widehat{\rho}_1 : (\Xi'|_{\mathbf{uv}\ Q'})$ and $\Gamma; \Xi_2 \vdash \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\ M''})$;
- $[\widehat{\rho}]M' = [\widehat{\rho}](Q' \to M'') = [\widehat{\rho}|_{\mathbf{uv}\ Q'}]Q' \to [\widehat{\rho}|_{\mathbf{uv}\ M''}]M'' = [\widehat{\rho}_1]Q' \to [\widehat{\rho}_2]M'' = Q \to M''' = M$;
- Since $[\widehat{\rho}]\widehat{\beta}^-$ is either equal to $[\widehat{\rho}_1]\widehat{\beta}^-$ or $[\widehat{\rho}_2]\widehat{\beta}^-$, it inherits their property that it is uniquely determined by $[\widehat{\tau}'_1]\widehat{\beta}^-$, $[\widehat{\tau}'_2]\widehat{\beta}^-$, and Γ.

Case 5. $P_1 = P_2 = \alpha^+$. This case is symmetric to case 1.

Case 6. $P_1 = \downarrow N_1$ and $P_2 = \downarrow N_2$. This case is symmetric to case 2

Case 7. $P_1 = \exists\overrightarrow{\alpha^-}.P'_1$ and $P_2 = \exists\overrightarrow{\alpha^-}.P'_2$. This case is symmetric to case 3

□

4.11 Upper Bounds

Lemma 38 (Decomposition of the quantifier rule). `Ilya: move somewhere` *Whenever the quantifier rule (Rule $(\exists^{\geqslant_1})$ or Rule $(\forall^{\leqslant_1})$) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.*

$-$ *If* $\Gamma \vdash N \leqslant_1 \forall\overrightarrow{\beta^+}.M$ *then* $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant_1 M$.

$+$ *If* $\Gamma \vdash P \geqslant_1 \exists\overrightarrow{\beta^-}.Q$ *then* $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant_1 Q$.

Lemma 39 (Characterization of the Supertypes). *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \textit{for } \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash {\downarrow}M$	$\left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-},\ M',\ \textit{and } \overrightarrow{N}\ \textit{s.t.} \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Immediately from lemma 4

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$
Then if $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'$, then by lemma 38, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$, and $\mathbf{fv}\, Q \cap \overrightarrow{\beta^-} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule $(\exists^{\geqslant_1})$. This way, $\{Q \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$ s.t. $\mathbf{fv}\,(Q) \cap \overrightarrow{\beta^-} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = {\downarrow}M$
Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\nexists}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists\overrightarrow{\beta^-}.Q'$
Then the last applied rule to infer $\Gamma \vdash Q \geqslant_1 {\downarrow}M$ must be Rule $({\downarrow}^{\geqslant_1})$, which means $Q = {\downarrow}M'$, and by inversion, $\Gamma \vdash M' \simeq_1^{\leqslant} M$, then by lemma 28 and Rule $({\downarrow}^{\simeq_1^D})$, ${\downarrow}M' \simeq_1^D {\downarrow}M$. This way, $Q = {\downarrow}M' \in \{{\downarrow}M' \mid {\downarrow}M' \simeq_1^D {\downarrow}M\} = \mathsf{UB}(\Gamma \vdash {\downarrow}M)|_{\nexists}$.
In the other direction,
$$\begin{aligned} {\downarrow}M' \simeq_1^D {\downarrow}M &\Rightarrow \Gamma \vdash {\downarrow}M' \simeq_1^{\leqslant} {\downarrow}M && \text{by lemma 24, since } \Gamma \vdash {\downarrow}M' \text{ by lemma 23} \\ &\Rightarrow \Gamma \vdash {\downarrow}M' \geqslant_1 {\downarrow}M && \text{by inversion} \end{aligned}$$

b. $Q = \exists\overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)
Then the last rule applied to infer $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q' \geqslant_1 {\downarrow}M$ must be Rule $(\exists^{\geqslant_1})$. Inversion of this rule gives us $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \geqslant_1 {\downarrow}M$ for some $\Gamma \vdash N_i$. Notice that $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.*a*, $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \simeq_1^D {\downarrow}M$, which is only possible if $Q' = {\downarrow}M'$. This way, $Q = \exists\overrightarrow{\beta^-}.{\downarrow}M' \in \mathsf{UB}(\Gamma \vdash {\downarrow}M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).
In the other direction,
$$\begin{aligned} [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \simeq_1^D {\downarrow}M &\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \simeq_1^{\leqslant} {\downarrow}M && \text{by lemma 24, since } \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \text{ by lemma 23} \\ &\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]{\downarrow}M' \geqslant_1 {\downarrow}M && \text{by inversion} \\ &\Rightarrow \Gamma \vdash \exists\overrightarrow{\beta^-}.{\downarrow}M' \geqslant_1 {\downarrow}M && \text{by Rule } (\exists^{\geqslant_1}) \end{aligned}$$

□

Lemma 40 (Characterization of the Normalized Supertypes). *For a normalized positive type* $P = \mathbf{nf}\,(P)$, *let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash {\downarrow}M$	$\left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-},\ M',\ \textit{and } \overrightarrow{N}\ \textit{s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M \end{array} \right\}$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \;\mid\; \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Then from lemma 39, $\{\mathbf{nf}\,(Q) \;\mid\; \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\beta^+) \;\mid\; \text{for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

$$
\begin{aligned}
\mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') &= \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ not using } \overrightarrow{\beta^-} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothesis} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}\,(Q) = \mathbf{fv}\,Q \text{ by lemma 15} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{by lemma 39} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} && \text{by the definition of UB} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} && \text{by lemma 39}
\end{aligned}
$$

Case 3. $P = \downarrow M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 3. *Suppose that* $\nu : A \to A$ *is an idempotent function,* P *is a predicate on* A*,* $F : A \to B$ *is a function. Then*

$$
\begin{aligned}
&\{F(\nu x) \mid x \in A \text{ s.t. } P(\nu x)\} = \\
=&\{F(x) \mid x \in A \text{ s.t. } \nu x = x \text{ and } P(x)\}.
\end{aligned}
$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 4. *For functions* F *and* ν*, and predicates* P *and* Q*,*

$$
\begin{aligned}
&\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } P(x)\} = \\
=&\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } (\exists x' \in A \text{ s.t. } P(x') \text{ and } \nu x' = \nu x)\}.
\end{aligned}
$$

Observation 5. *There exist positive and negative types well-formed in empty context, hence, a type substitution can be extended to an arbitrary domain (if its values on the domain extension are irrelevant). Specifically, Suppose that* $vars_1 \subseteq vars_2$*. Then* $\Gamma \vdash \sigma|_{vars_1} : vars_1$ *implies* $\exists\ \sigma'$ *s.t.* $\Gamma \vdash \sigma' : vars_2$ *and* $\sigma|_{vars_1} = \sigma'|_{vars_1}$*.*

$$
\begin{aligned}
&\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 {\downarrow}M\} = \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash {\downarrow}M)\} && \text{by lemma 39} \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M',\ \text{and } \overrightarrow{N} \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash N_i,\ \text{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by the definition of UB} \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M',\ \text{and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \text{and } [\sigma]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{we reassigned the substitution } \overrightarrow{N}/\overrightarrow{\alpha^-} \text{ as } \sigma \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M',\ \text{and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by lemma 1} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by the definition of normalization} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } \mathbf{nf}\,([\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M') = \mathbf{nf}\,({\downarrow}M) \end{array} \right\} && \begin{array}{l}\text{from lemmas 16 and 18, equivalence of types can be} \\ \text{replaced with the equality of their normal forms}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})]{\downarrow}\mathbf{nf}\,(M') = {\downarrow}\mathbf{nf}\,(M) \end{array} \right\} && \text{by congruence of normalization and lemma 17} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{by lemma 20, } {\downarrow}M' \text{ and } \sigma|_{\mathbf{fv}\,M'} \text{ are already normal,} \\ \text{since the result of the substitution is normal;} \\ M \text{ is normal by assumption}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ (\exists\sigma' \text{ s.t. } \Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')}) \\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{We apply observation 4 (with } \nu\sigma = \sigma|_{\mathbf{fv}\,M'}, \text{ and} \\ P(\sigma) = \Gamma \vdash \sigma : \overrightarrow{\alpha^-})\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv}\,M'} : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{Notice that} \\ \text{¡¡}\exists\sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')})\text{¿¿} \\ \text{is equivalent to } \Gamma \vdash \sigma|_{\mathbf{fv}\,({\downarrow}M')} : \overrightarrow{\alpha^-}{}' \text{ (observation 5)}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{We apply observation 3 to the restriction of } \sigma, \text{ and} \\ \text{remove } \sigma|_{\mathbf{fv}\,M'} = \sigma \text{ as it follows from } \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}'\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}{}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{by lemma 9, since } \Gamma, \overrightarrow{\alpha^-} \cap \mathbf{fv}\,M' = \Gamma, \overrightarrow{\alpha^-}{}' \cap \mathbf{fv}\,M' \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}{}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\overrightarrow{\alpha^-}{}'\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{We apply observation 3 to the ordering of } \overrightarrow{\alpha^-} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ M',\ \text{and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}{}'\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}', \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-}{}' \vdash M',\ \text{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}{}']{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{By reassigning } \sigma \text{ explicitly as } \overrightarrow{N}/\overrightarrow{\alpha^-}{}' \\
&= \mathsf{NFUB}({\downarrow}M) && \text{by definition}
\end{aligned}
$$

hello

□

Observation 6. *Upper bounds of a type do not depend on the context as soon as the type are well-formed in it.*
If $\Gamma_1 \vdash M$ *and* $\Gamma_2 \vdash M$ *then* $\mathsf{UB}(\Gamma_1 \vdash M) = \mathsf{UB}(\Gamma \vdash M)$ *and* $\mathsf{NFUB}(\Gamma_1 \vdash M) = \mathsf{NFUB}(\Gamma \vdash M)$

Proof. We prove both inclusions by induction on $\Gamma_1 \vdash M$. Notice that if $[\sigma]M' \simeq_1^D M$ and $\Gamma_2 \vdash M$ then the types from the range of $\sigma|_{\mathbf{fv}\,M'}$ are well-formed in 2 `Ilya: lemma`. □

Lemma 41 (Soundness of the Least Upper Bound)**.** *For types* $\Gamma \vdash P_1$*, and* $\Gamma \vdash P_2$*, if* $\Gamma \vDash P_1 \vee P_2 = Q$ *then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Proof. Induction on $\Gamma \models P_1 \vee P_2 = Q$.

Case 1. $\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+$
Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule ($\mathrm{Var}^{+\geqslant_1}$).

Case 2. $\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q$
Then by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applies to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q$. Then

(i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,

(ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$,

(iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}\,(Q) \cap \Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} = \mathbf{fv}\,(Q) \cap \Gamma$ (and then apply lemma 9). The inclusion right-to-left is self-evident. To show $\mathbf{fv}\,(Q) \cap \Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \subseteq \mathbf{fv}\,(Q) \cap \Gamma$, we prove that $\mathbf{fv}\,(Q) \subseteq \Gamma$

$\mathbf{fv}\,(Q) \subseteq \mathbf{fv}\,P_1 \cap \mathbf{fv}\,P_2$ by lemma 3

$\subseteq (\Gamma, \overrightarrow{\alpha^-} \backslash \overrightarrow{\beta^-}) \cap (\Gamma, \overrightarrow{\beta^-} \backslash \overrightarrow{\alpha^-})$ since $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_1$, $\mathbf{fv}\,(P_1) \subseteq \Gamma, \overrightarrow{\alpha^-} = \Gamma, \overrightarrow{\alpha^-} \backslash \overrightarrow{\beta^-}$ (the latter is because by the Barendregt's convention, $\Gamma, \overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing$); similarly, $\mathbf{fv}\,(P_2) \subseteq \Gamma, \overrightarrow{\beta^-} \backslash \overrightarrow{\alpha^-}$

$\subseteq \Gamma$

To show $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\alpha^-}.P_1$, we apply Rule ($\exists^{\geqslant_1}$). Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant_1 P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \models {\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$. By the inversion, $\Gamma, \cdot \models \mathbf{nf}\,({\downarrow}N) \overset{a}{\simeq} \mathbf{nf}\,({\downarrow}M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$. Then by the soundness of anti-unification (**??**),

(i) $\Gamma; \Xi \vdash P$, then by **??**,

$$\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]P \tag{7}$$

(ii) $\Gamma; \cdot \vdash \widehat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_2 : \Xi$. Assuming that $\Xi = \widehat{\beta}_1^-, .., \widehat{\beta}_n^-$, the antiunification solutions $\widehat{\tau}_1$ and $\widehat{\tau}_2$ can be put explicitly as $\widehat{\tau}_1 = (\widehat{\beta}_1^- :\approx N_1, .., \widehat{\beta}_n^- :\approx N_n)$, and $\widehat{\tau}_2 = (\widehat{\beta}_1^- :\approx M_1, .., \widehat{\beta}_n^- :\approx M_n)$. Then

$$\widehat{\tau}_1 = (\overrightarrow{N}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \tag{8}$$

$$\widehat{\tau}_2 = (\overrightarrow{M}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \tag{9}$$

(iii) $[\widehat{\tau}_1]Q = P_1$ and $[\widehat{\tau}_2]Q = P_1$, which, by 8 and 9, means

$$[\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P = \mathbf{nf}\,({\downarrow}N) \tag{10}$$

$$[\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P = \mathbf{nf}\,({\downarrow}M) \tag{11}$$

Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$ follows directly from 7.

To show $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}N$, we apply Rule ($\exists^{\geqslant_1}$), instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{N}$. Then $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}N$ follows from 10 and since $\Gamma \vdash \mathbf{nf}\,({\downarrow}N) \geqslant_1 {\downarrow}N$ (by corollary 14).

Analogously, instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{M}$, gives us $\Gamma \vdash [\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}M$ (from 11), and hence, $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}M$. $\square$

Lemma 42 (Completeness and Initiality of the Least Upper Bound). *For types* $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, *and* $\Gamma \vdash Q$ *such that* $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$, *there exists* Q' *s.t.* $\Gamma \models P_1 \vee P_2 = Q'$ *and* $\Gamma \vdash Q \geqslant_1 Q'$.

Proof. Induction on the pair (P_1, P_2). From lemma 40, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$. Let us consider the cases of what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_i$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$, where either $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then $Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_2.Q_2)$

$$\begin{aligned}
&\subseteq \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_2 \vdash Q_2) && \text{from the definition of UB}\\
&= \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_2) && \text{by observation 6, weakening and exchange}\\
&= \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant_1 Q_1\} \cap \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant_1 Q_2\} && \text{by lemma 39,}
\end{aligned}$$

meaning that $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$. Then the next step of the algorithm—the recursive call $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vDash Q_1 \vee Q_2 = Q'$ terminates by the induction hypothesis, and moreover, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$. This way, the result of the algorithm is Q', i.e. $\Gamma \vDash P_1 \vee P_2 = Q'$.

Since both Q and Q' are sound, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$ can be strengthened to $\Gamma \vdash Q \geqslant_1 Q'$ by **??**.

Case 2. $P_1 = \alpha^+$ and $P_2 = \downarrow N$

Then the set of common upper bounds of $\downarrow N$ and α^+ is empty, and thus, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$ gives a contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \downarrow N)$

$$\begin{aligned}
&= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\downarrow M' \mid \cdots\} && \text{by the definition of UB}\\
&= \varnothing && \text{since } \alpha^+ \neq \downarrow M' \text{ for any } M'
\end{aligned}$$

Case 3. $P_1 = \downarrow N$ and $P_2 = \alpha^+$

Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)

Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \beta^+)$

$$\begin{aligned}
&= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\beta^+ \mid \cdots\} && \text{by the definition of UB}\\
&= \varnothing && \text{since } \alpha^+ \neq \beta^+
\end{aligned}$$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Then the algorithm terminates in one step (Rule ($\mathrm{Var}^{\vee}$)) and the result is α^+, i.e. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$.

Since $Q \in \mathsf{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists\overrightarrow{\alpha^-}.\alpha^+$. Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant_1 \alpha^+$ by Rule ($\exists^{\geqslant_1}$): $\overrightarrow{\alpha^-}$ can be instantiated with arbitrary negative types (for example $\forall\beta^+.\uparrow\beta^+$), since the substitution for unused variables does not change the term $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule ($\mathrm{Var}^{+\geqslant_1}$).

Case 6. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$

Then on the next step, the algorithm tries to anti-unify $\mathbf{nf}\,(\downarrow M_1)$ and $\mathbf{nf}\,(\downarrow M_2)$. By **??**, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$\mathbf{nf}\,(Q) \in \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,(\downarrow M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,(\downarrow M_2))$

$$\begin{aligned}
&= \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}\,(M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}\,(M_2))\\
&= \begin{array}{c}\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l}\text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-},\\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1)\end{array} \right\}\\ \cap \\ \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l}\text{for } \overrightarrow{\alpha^-},\, M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-},\\ \Gamma \vdash \overrightarrow{N}_1,\ \Gamma \vdash \overrightarrow{N}_2,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2)\end{array} \right\}\end{array}\\
&= \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l}\text{for } \overrightarrow{\alpha^-},\, M',\, \overrightarrow{N}_1 \text{ and } \overrightarrow{N}_2 \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-},\\ \Gamma \vdash \overrightarrow{N}_1,\ \Gamma \vdash \overrightarrow{N}_2,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1), \text{ and } [\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2)\end{array} \right\}
\end{aligned}$$

The fact that the latter set is non-empty means that there exist $\overrightarrow{\alpha^-}$, M', $\overrightarrow{N}_1$ and $\overrightarrow{N}_2$ such that

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$ (notice that M' is normal)

(ii) $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma \vdash \overrightarrow{N}_1$,

(iii) $[\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1)$ and $[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2)$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative anti-unification variable $\widehat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\widehat{\alpha}^-}$. Let us show that $(\overrightarrow{\widehat{\alpha}^-},\ [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M',\ \vec{N}_1/\overrightarrow{\widehat{\alpha}^-},\ \vec{N}_2/\overrightarrow{\widehat{\alpha}^-})$ is a sound anti-unifier of $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$ in context Γ:

- $\overrightarrow{\widehat{\alpha}^-}$ is negative by construction,
- $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M'$ because $\Gamma, \overrightarrow{\alpha^-} \vdash {\downarrow}M'$ Ilya: lemma!,
- $\Gamma; \cdot \vdash (\vec{N}_1/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \vec{N}_1$ and $\Gamma; \cdot \vdash (\vec{N}_2/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \vec{N}_2$,
- $[\vec{N}_1/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\vec{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_1) = \mathbf{nf}\,({\downarrow}M_1)$.
- $[\vec{N}_2/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\vec{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_2) = \mathbf{nf}\,({\downarrow}M_2)$.

Then by the completeness of the anti-unification (**??**), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it, i.e. $Q' = \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, where $(\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$ is the result of the anti-unification of $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$ in context Γ.

Moreover, **??** also says that the found anti-unification solution is initial, i.e. there exists $\widehat{\tau}$ such that $\Gamma; \Xi \vdash \widehat{\tau} : \overrightarrow{\widehat{\alpha}^-}$ and $[\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}$, (ii) $\widehat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi][\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi]P$. In particular, from the reflexivity of subtyping: $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]{\downarrow}M' \geqslant_1 [\overrightarrow{\beta^-}/\Xi]P$.

It allows us to show $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant_1 Q'$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.{\downarrow}M' \geqslant_1 \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, by applying Rule $(\exists^{\geqslant_1})$, instantiating $\overrightarrow{\alpha^-}$ with respect to σ. Finally, $\Gamma \vdash Q \geqslant_1 Q'$ since $\Gamma \vdash \mathbf{nf}\,(Q) \simeq_1^{\leqslant} Q$, and equivalence implies subtyping by Ilya: lemma.

□

4.12 Upgrade

Let us consider a type P well-formed in Γ. Some of its Γ-supertypes are also well-formed in a smaller context $\Delta \subseteq \Gamma$. The upgrade is the operation that returns the least of such supertypes.

Lemma 43 (Soundness of Upgrade). *Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^\pm}$, if* $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ *then*

1. $\Delta \vdash Q$
2. $\Gamma \vdash Q \geqslant_1 P$

Proof. By inversion, $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ means that for fresh $\overrightarrow{\beta^\pm}$ and $\overrightarrow{\gamma^\pm}$, $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vDash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q$. Then by the soundness of the least upper bound (lemma 41),

1. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q$,
2. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant_1 [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$, and
3. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant_1 [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P$.

$$\begin{aligned}
\mathbf{fv}\,Q &\subseteq \mathbf{fv}\,[\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \cap \mathbf{fv}\,[\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P && \text{Since by lemma 3, } \mathbf{fv}\,Q \subseteq \mathbf{fv}\,[\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \text{ and } \mathbf{fv}\,Q \subseteq \mathbf{fv}\,[\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P\\
&\subseteq ((\mathbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) \cup \overrightarrow{\beta^\pm}) \cap ((\mathbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) \cup \overrightarrow{\gamma^\pm}) &&\\
&= (\mathbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) \cap (\mathbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) && \text{since } \overrightarrow{\beta^\pm} \text{ and } \overrightarrow{\gamma^\pm} \text{ are fresh}\\
&= \mathbf{fv}\,P \backslash \overrightarrow{\alpha^\pm} &&\\
&\subseteq \Gamma \backslash \overrightarrow{\alpha^\pm} && \text{since } P \text{ is well-formed in } \Gamma\\
&\subseteq \Delta &&
\end{aligned}$$

This way, by lemma 9, $\Delta \vdash Q$.

Let us apply $\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}$—the inverse of the substitution $\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}$ to both sides of $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant_1 [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$ and by **??**, get $\Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm} \vdash [\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}]Q \geqslant_1 P$. Notice that $\Delta \vdash Q$ implies that $\mathbf{fv}\,Q \cap \overrightarrow{\beta^\pm} = \varnothing$, then by **??**, $[\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}]Q = Q$, and thus $\Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant_1 P$. By context strengthening, $\Delta, \overrightarrow{\alpha^\pm} \vdash Q \geqslant_1 P$. □

Lemma 44 (Completeness and Initiality of Upgrade)**.** *The upgrade returns the least Γ-supertype of P well-formed in Δ. Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$,*
For any Q' such that

1. *$\Delta \vdash Q'$ and*
2. *$\Gamma \vdash Q' \geqslant_1 P$,*

The result of the upgrade algorithm Q exists ($\mathbf{upgrade}\, \Gamma \vdash P\, \mathbf{to}\, \Delta = Q$) and satisfies $\Delta \vdash Q' \geqslant_1 Q$.

Proof. Let us consider fresh (not intersecting with Γ) $\overrightarrow{\beta^{\pm}}$ and $\overrightarrow{\gamma^{\pm}}$.

If we apply substitution $\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q' \geqslant_1 P$, we have $\Delta, \overrightarrow{\beta^{\pm}} \vdash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]Q' \geqslant_1 [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$, which by **??**, since Q' is well-formed in Δ, simplifies to $\Delta, \overrightarrow{\beta^{\pm}} \vdash Q' \geqslant_1 [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$.

Analogously, if we apply substitution $\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q' \geqslant_1 P$, we have $\Delta, \overrightarrow{\gamma^{\pm}} \vdash Q' \geqslant_1 [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$.

This way, Q' is a common supertype of $[\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$ and $[\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$ in context $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}}$. It means that we can apply the completeness of the least upper bound (lemma 42):

1. there exists Q s.t. $\Gamma \vDash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P = Q$
2. $\Gamma \vdash Q' \geqslant_1 Q$.

The former means that the upgrade algorithm terminates and returns Q. The latter means that since both Q' and Q are well-formed in Δ, by **??**, $\Delta \vdash Q' \geqslant_1 Q$. □

4.13 Positive Subtyping

Lemma 45 (Soundness of the Positive Subtyping)**.** *If $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash Q$, $\Gamma; \Theta \vdash P$, and $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$, then $\Theta \vdash SC$ and for any normalized $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : SC$, $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$.*

Proof. We prove it by induction on $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$. Let us consider the last rule to infer this judgment.

Case 1. Rule ($\mathrm{UVar}^{\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \vDash \widehat{\alpha}^{+} \geqslant P' \dashv (\widehat{\alpha}^{+} :\geqslant Q')$ where $\widehat{\alpha}^{+}\{\Delta\} \in \Theta$ and $\mathbf{upgrade}\, \Gamma \vdash P'\, \mathbf{to}\, \Delta = Q'$.

Notice that $\widehat{\alpha}^{+}\{\Delta\} \in \Theta$ and $\Gamma \vdash^{\supseteq} \Theta$ implies $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$ for some $\overrightarrow{\alpha^{\pm}}$, hence, the soundness of upgrade (lemma 43) is applicable:

1. $\Delta \vdash Q'$ and
2. $\Gamma \vdash Q' \geqslant_1 P$.

Since $\widehat{\alpha}^{+}\{\Delta\} \in \Theta$ and $\Delta \vdash Q'$, it is clear that $\Theta \vdash (\widehat{\alpha}^{+} :\geqslant Q')$.

It is left to show that $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^{+} \geqslant_1 P'$ for any normalized $\widehat{\sigma}$ s.t. $\Theta \vdash \widehat{\sigma} : (\widehat{\alpha}^{+} :\geqslant Q')$. The latter means that $\Theta(\widehat{\alpha}^{+}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{+} \geqslant_1 Q'$, i.e. $\Delta \vdash [\widehat{\sigma}]\widehat{\alpha}^{+} \geqslant_1 Q'$. By weakening the context to Γ and combining this judgment transitively with $\Gamma \vdash Q' \geqslant_1 P$, we have $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^{+} \geqslant_1 P$, as required.

Case 2. Rule ($\mathrm{Var}^{+\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \vDash \alpha^{+} \geqslant \alpha^{+} \dashv \cdot$. Then $\mathbf{uv}\, \alpha^{+} = \varnothing$, and $SC = \cdot$ satisfies $\Theta \vdash \cdot$. Since $\mathbf{uv}\, \alpha^{+} = \varnothing$, application of any substitution $\widehat{\sigma}$ does not change α^{+}, i.e. $[\widehat{\sigma}]\alpha^{+} = \alpha^{+}$. Therefore, $\Gamma \vdash [\widehat{\sigma}]\alpha^{+} \geqslant_1 \alpha^{+}$ holds by Rule ($\mathrm{Var}^{-\leqslant_1}$).

Case 3. Rule ($\downarrow^{\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \vDash \downarrow N \geqslant \downarrow M \dashv SC$.
Then the next step of the algorithm is the unification of $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$, and it returns the resulting unification constraint $UC = SC$ as the result. By the soundness of unification (lemma 33), $\Theta \vdash SC$ and for any normalized $\widehat{\sigma}$, $\Theta \vdash \widehat{\sigma} : SC$ implies $[\widehat{\sigma}]\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$, then we rewrite the left-hand side by lemma 17: $\mathbf{nf}\,([\widehat{\sigma}]N) = \mathbf{nf}\,(M)$ and apply lemma 29: $\Gamma \vdash [\widehat{\sigma}]N \simeq_1^{\leqslant} M$, then by Rule ($\uparrow^{\leqslant_1}$), $\Gamma \vdash \downarrow[\widehat{\sigma}]N \geqslant_1 \downarrow M$.

Case 4. Rule ($\exists^{\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \vDash \exists\overrightarrow{\alpha^{-}}.P' \geqslant \exists\overrightarrow{\beta^{-}}.Q' \dashv SC$ s.t. either $\overrightarrow{\alpha^{-}}$ or $\overrightarrow{\beta^{-}}$ is not empty.
Then the algorithm creates fresh unification variables $\overrightarrow{\widehat{\alpha}^{-}}\{\Gamma, \overrightarrow{\beta^{-}}\}$, substitutes the old $\overrightarrow{\alpha^{-}}$ with them in P', and makes the recursive call: $\Gamma, \overrightarrow{\beta^{-}}; \Theta, \overrightarrow{\widehat{\alpha}^{-}}\{\Gamma, \overrightarrow{\beta^{-}}\} \vDash [\overrightarrow{\widehat{\alpha}^{-}}/\overrightarrow{\alpha^{-}}]P' \geqslant Q' \dashv SC'$, returning as the result $SC = SC' \backslash \overrightarrow{\widehat{\alpha}^{-}}$.

Let us take an arbitrary normalized $\widehat{\sigma}$ s.t. $\Theta \vdash \widehat{\sigma} : SC' \backslash \overrightarrow{\widehat{\alpha}^{-}}$. We wish to show $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.[\widehat{\sigma}]P' \geqslant_1 \exists\overrightarrow{\beta^{-}}.Q'$. To do that, we apply Rule ($\exists^{\geqslant_1}$), and what is left to show is $\Gamma, \overrightarrow{\beta^{-}} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^{-}}][\widehat{\sigma}]P' \geqslant_1 Q'$ for some $\overrightarrow{N}$. If we construct a normalized $\widehat{\sigma}'$ such that $\Theta, \overrightarrow{\widehat{\alpha}^{-}}\{\Gamma, \overrightarrow{\beta^{-}}\} \vdash \widehat{\sigma}' : SC'$ and for some $\overrightarrow{N}$, $[\overrightarrow{N}/\overrightarrow{\alpha^{-}}][\widehat{\sigma}]P' = [\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^{-}}/\overrightarrow{\alpha^{-}}]P'$, we can apply the induction hypothesis to $\Gamma, \overrightarrow{\beta^{-}}; \Theta, \overrightarrow{\widehat{\alpha}^{-}}\{\Gamma, \overrightarrow{\beta^{-}}\} \vDash [\overrightarrow{\widehat{\alpha}^{-}}/\overrightarrow{\alpha^{-}}]P \geqslant Q \dashv SC'$ and infer the required subtyping.

Let us construct such $\widehat{\sigma}'$ by extending $\widehat{\sigma}$ with $\overrightarrow{\widehat{\alpha}^-}$ mapped to the corresponding types in SC':

$$[\widehat{\sigma}']\widehat{\beta}^{\pm} = \begin{cases} [\widehat{\sigma}]\widehat{\beta}^{\pm} & \text{if } \widehat{\beta}^{\pm} \in \mathbf{dom}\,(SC')\backslash\overrightarrow{\widehat{\alpha}^-} \\ \mathbf{nf}\,(N) & \text{if } \widehat{\beta}^{\pm} \in \overrightarrow{\widehat{\alpha}^-} \text{ and } (\widehat{\beta}^{\pm} :\approx N) \in SC' \end{cases}$$

It is easy to see that $\widehat{\sigma}'$ is normalized. Let us show that $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}' : SC'$. Let us take an arbitrary entry e from SC' restricting a variable $\widehat{\beta}^{\pm}$. Suppose $\widehat{\beta}^{\pm} \in \mathbf{dom}\,(SC')\backslash\overrightarrow{\widehat{\alpha}^-}$. Then $(\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\})(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}']\widehat{\beta}^{\pm} : e$ is rewritten as $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\beta}^{\pm} : e$, which holds since $\Theta \vdash \widehat{\sigma} : SC'$. Suppose $\widehat{\beta}^{\pm} = \widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-}$. Then $e = (\widehat{\alpha_i}^- :\approx N)$ for some N, $[\widehat{\sigma}']\widehat{\alpha_i}^- = \mathbf{nf}\,(N)$ by the definition, and $\Gamma, \overrightarrow{\beta^-} \vdash \mathbf{nf}\,(N) : (\widehat{\alpha_i}^- :\approx N)$ by Rule SATSCENEq, since $\Gamma \vdash \mathbf{nf}\,(N) \simeq_1^{\leqslant} N$ by lemma 29.

Finally, let us show that $[\overrightarrow{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]P' = [\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$. For N_i, we take the *normalized* type restricting $\widehat{\alpha_i}^-$ in SC'. Let us take an arbitrary variable from P.

1. If this variable is a unification variable $\widehat{\beta}^{\pm}$, then $[\overrightarrow{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$, since $\Theta \vdash \widehat{\sigma} : SC'\backslash\overrightarrow{\widehat{\alpha}^-}$ and $\mathbf{dom}\,(\Theta) \cap \overrightarrow{\alpha^-} = \varnothing$. Notice that $\widehat{\beta}^{\pm} \in \mathbf{dom}\,(\Theta)$, which is disjoint from $\overrightarrow{\widehat{\alpha}^-}$, that is $\widehat{\beta}^{\pm} \in \mathbf{dom}\,(SC')\backslash\overrightarrow{\widehat{\alpha}^-}$. This way, $[\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\widehat{\beta}^{\pm} = [\widehat{\sigma}']\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$ by the definition of $\widehat{\sigma}'$,
2. If this variable is a regular variable $\beta^{\pm} \notin \overrightarrow{\alpha^-}$, then $[\overrightarrow{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]\beta^{\pm} = \beta^{\pm}$ and $[\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\beta^{\pm} = \beta^{\pm}$.
3. If this variable is a regular variable $\alpha_i^- \in \overrightarrow{\alpha^-}$, then $[\overrightarrow{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]\alpha_i^- = N_i = \mathbf{nf}\,(N_i)$ (the latter equality holds since N_i is normalized) and $[\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\alpha_i^- = [\widehat{\sigma}']\widehat{\alpha_i}^- = \mathbf{nf}\,(N_i)$.

□

Lemma 46 (Completeness of the Positive Subtyping). *Suppose that* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash Q$ *and* $\Gamma; \Theta \vdash P$. *Then for any* $\Theta \vdash \widehat{\sigma}$ *such that* $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$, *there exists* $\Gamma; \Theta \models P \geqslant Q \dashv SC$ *such that* $\Theta \vdash SC$ *and moreover,* $\Theta \vdash \widehat{\sigma} : SC$.

Proof. Let us prove this lemma by induction on $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$. Let us consider the last rule used in the derivation, but first, consider the base case for the substitution $[\widehat{\sigma}]P$:

Case 1. $P = \exists\overrightarrow{\beta^-}.\widehat{\alpha}^+$ (for potentially empty $\overrightarrow{\beta^-}$)

Then by assumption, $\Gamma \vdash \exists\overrightarrow{\beta^-}.[\widehat{\sigma}]\widehat{\alpha}^+ \geqslant_1 Q$ (where $\overrightarrow{\beta^-} \cap \mathbf{fv}\,[\widehat{\sigma}]\widehat{\alpha}^+ = \varnothing$). By **??**, $\Gamma \vdash \exists\overrightarrow{\beta^-}.[\widehat{\sigma}]\widehat{\alpha}^+ \geqslant_1 Q$ means $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}][\widehat{\sigma}]\widehat{\alpha}^+ \geqslant_1 Q$, and hence $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant_1 Q$. By inversion, $\Gamma; \Theta \vdash \exists\overrightarrow{\beta^-}.\widehat{\alpha}^+$ implies $\widehat{\alpha}^+\{\Delta\} \in \Theta$ for some Δ.

In the algorithm trying to infer the subtyping $\Gamma; \Theta \models \exists\overrightarrow{\beta^-}.\widehat{\alpha}^+ \geqslant Q \dashv SC$, after multiple applications of Rule $(\exists^{\geqslant})$ the type $\exists\overrightarrow{\beta^-}.\widehat{\alpha}^+$ is reduced to $\widehat{\alpha}^+$. Next, the algorithm tries to apply Rule $(\mathrm{UVar}^{\geqslant})$ and the resulting restriction is $SC' = (\widehat{\alpha}^+ :\geqslant Q')$ where $\mathbf{upgrade}\,\Gamma \vdash Q\,\mathbf{to}\,\Delta = Q'$.

Why does the upgrade procedure terminates? Because $[\widehat{\sigma}]\widehat{\alpha}^+$ satisfies the pre-conditions of the completeness of the upgrade (lemma 44):

- $\Delta \vdash P'$ because $P' = [\widehat{\sigma}]\widehat{\alpha}^+$ and $\Theta \vdash \widehat{\sigma}$ and $\widehat{\alpha}^+\{\Delta\} \in \Theta$,
- $\Gamma \vdash P' \geqslant_1 Q$ as noted above

Moreover, the completeness of the upgrade also gives us $\Gamma \vdash P' \geqslant_1 Q'$ and further, we strengthen it to $\Delta \vdash P' \geqslant_1 Q'$ (since by the soundness of the upgrade (lemma 43), $\Delta \vdash Q'$). It means that $\Delta \vdash P' : (\widehat{\alpha}^+ :\geqslant Q')$, that is $\Theta \vdash \widehat{\sigma} : (\widehat{\alpha}^+ :\geqslant Q')$, as required.

Case 2. $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$ is derived by Rule $(\mathrm{Var}^{+\geqslant_1})$, i.e. $P = [\widehat{\sigma}]P = \alpha^+ = Q$. Here the first equality holds because P is not a unification variable: it has been covered by case 1. The second equality hold because Rule $(\mathrm{Var}^{+\geqslant_1})$ was applied.

The algorithm applies Rule $(\mathrm{Var}^{+\geqslant})$ and infers $SC = \cdot$, i.e. $\Gamma; \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot$. Then $\Theta \vdash \widehat{\sigma} : \cdot$ holds trivially.

Case 3. $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$ is derived by Rule $(\downarrow^{\geqslant_1})$,

Then $P = \downarrow N$, since the substitution $[\widehat{\sigma}]P$ must preserve the top-level constructor of $P \neq \widehat{\alpha}^+$ (the case $P = \widehat{\alpha}^+$ has been covered by case 1), and $Q = \downarrow M$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]N \simeq_1^{\leqslant} M$.

Since both types start with $\downarrow$, the algorithm tries to apply Rule $(\downarrow^{\geqslant})$: $\Gamma; \Theta \models \downarrow N \geqslant \downarrow M \dashv SC$. The premise of this rule is the unification of $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$: $\Gamma; \Theta \models \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv UC$. And the algorithm returns it as a subtyping constraint $SC = UC$.

To demonstrate that the unification terminates ant $\widehat{\sigma}$ satisfies the resulting constraints, we apply the completeness of the unification algorithm (lemma 34). In order to do that, we need to provide a substitution unifying $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$. Let us show that $\mathbf{nf}\,(\widehat{\sigma})$ is such a substitution.

- $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$ are normalized

- $\Gamma; \Theta \vdash \mathbf{nf}\,(N)$ because $\Gamma; \Theta \vdash N$ (**??**)
- $\Gamma \vdash \mathbf{nf}\,(M)$ because $\Gamma \vdash M$ (corollary 12)
- $\Theta \vdash \mathbf{nf}\,(\widehat{\sigma})$ because $\Theta \vdash \widehat{\sigma}$ (**??**)
- $\Gamma \vdash [\widehat{\sigma}]N \simeq_1^{\leqslant} M \Rightarrow [\widehat{\sigma}]N \simeq_1^{D} M$ by lemma 28
  $\Rightarrow \mathbf{nf}\,([\widehat{\sigma}]N) = \mathbf{nf}\,(M)$ by lemma 18
  $\Rightarrow [\mathbf{nf}\,(\widehat{\sigma})]\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$ by lemma 17

Then by the completeness of the unification, $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv UC$ exists, and $\Theta \vdash \mathbf{nf}\,(\widehat{\sigma}) : UC$. Then by **??**, $\Theta \vdash \widehat{\sigma} : UC$.

Case 4. $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$ is derived by Rule ($\exists^{\geqslant_1}$).

We should only consider the case when the substitution $[\widehat{\sigma}]P$ results in the existential type $\exists\overrightarrow{\alpha^-}.P''$ (for $P'' \neq \exists \ldots$) by congruence, i.e. $P = \exists\overrightarrow{\alpha^-}.P'$ (for $P' \neq \exists \ldots$) and $[\widehat{\sigma}]P' = P''$. This is because the case when $P = \exists\overrightarrow{\beta^-}.\widehat{\alpha}^+$ has been covered (case 1), and thus, the substitution $\widehat{\sigma}$ must preserve all the outer quantifiers of P and does not generate any new ones.

This way, $P = \exists\overrightarrow{\alpha^-}.P'$, $[\widehat{\sigma}]P = \exists\overrightarrow{\alpha^-}.[\widehat{\sigma}]P'$ (assuming $\overrightarrow{\alpha^-}$ does not intersect with the range of $\widehat{\sigma}$) and $Q = \exists\overrightarrow{\beta^-}.Q'$, where either $\overrightarrow{\alpha^-}$ or $\overrightarrow{\beta^-}$ is not empty.

By inversion, $\Gamma \vdash [\sigma][\widehat{\sigma}]P' \geqslant_1 Q'$ for some $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$. Since σ and $\widehat{\sigma}$ have disjoint domains, and the range of one does not intersect with the domain of the other, they commute, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}][\sigma]P' \geqslant_1 Q'$ (notice that the tree inferring this judgement is a proper subtree of the tree inferring $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$).

At the next step, the algorithm creates fresh (disjoint with $\mathbf{uv}\,P'$) unification variables $\overrightarrow{\widehat{\alpha}^-}$, replaces $\overrightarrow{\alpha^-}$ with them in P', and makes the recursive call: $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \models P_0 \geqslant Q' \dashv SC_1$, (where $P_0 = [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$), returning $SC_1 \backslash \overrightarrow{\widehat{\alpha}^-}$ as the result.

To show that the recursive call terminates and that $\Theta \vdash \widehat{\sigma} : SC_1 \backslash \overrightarrow{\widehat{\alpha}^-}$, it suffices to build $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}_0$—an extension of $\widehat{\sigma}$ with $\overrightarrow{\widehat{\alpha}^-}$ such that $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant_1 Q$. Then by the induction hypothesis, $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}_0 : SC_1$, and hence, $\Theta \vdash \widehat{\sigma} : SC_1 \backslash \overrightarrow{\widehat{\alpha}^-}$, as required.

Let us construct such a substitution $\widehat{\sigma}_0$:

$$[\widehat{\sigma}_0]\widehat{\beta}^{\pm} = \begin{cases} [\sigma]\alpha_i^- & \text{if } \widehat{\beta}^{\pm} = \widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-} \\ [\widehat{\sigma}]\widehat{\beta}^{\pm} & \text{if } \widehat{\beta}^{\pm} \in \mathbf{uv}\,(P') \end{cases}$$

It is easy to see $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}_0$:

1. for $\widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-}$, $(\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\})(\widehat{\alpha_i}^-) \vdash [\widehat{\sigma}_0]\widehat{\alpha_i}^-$, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]\alpha_i^-$ holds since $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$,
2. for $\widehat{\beta}^{\pm} \in \mathbf{uv}\,(P') \subseteq \mathbf{dom}\,(\Theta)$, $(\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\})(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}_0]\widehat{\beta}^{\pm}$, i.e. $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\beta}^{\pm}$ holds since $\Theta \vdash \widehat{\sigma}$.

Now, let us show that $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant_1 Q$. To do that, we notice that $[\widehat{\sigma}_0]P_0 = [\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P_0$: let us consider an arbitrary variable appearing freely in P_0:

1. if this variable is a metavariable $\widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-}$, then $[\widehat{\sigma}_0]\widehat{\alpha_i}^- = [\sigma]\alpha_i^-$ and $[\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]\widehat{\alpha_i}^- = [\widehat{\sigma}][\sigma]\alpha_i^- = [\sigma]\alpha_i^-$,
2. if this variable is a metavariable $\widehat{\beta}^{\pm} \in \mathbf{uv}\,(P_0) \backslash \overrightarrow{\widehat{\alpha}^-} = \mathbf{uv}\,(P')$, then $[\widehat{\sigma}_0]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$ and $[\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]\widehat{\beta}^{\pm} = [\widehat{\sigma}][\sigma]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$,
3. if this variable is a regular variable from $\mathbf{fv}\,(P_0)$, both substitutions do not change it: $\widehat{\sigma}_0$, $\widehat{\sigma}$ and $\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}$ act on metavariables, and σ is defined on $\overrightarrow{\alpha^-}$, however, $\overrightarrow{\alpha^-} \cap \mathbf{fv}\,(P_0) = \varnothing$.

This way, $[\widehat{\sigma}_0]P_0 = [\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P_0 = [\widehat{\sigma}][\sigma]P'$, and thus, $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant_1 Q'$.

□

4.14 Subtyping Constraint Merge

Lemma 47 (Soundness of Constraint Entry Merge). *For a fixed context* Γ*, suppose that* $\Gamma \vdash e_1$ *and* $\Gamma \vdash e_2$*. If* $\Gamma \vdash e_1 \,\&\, e_2 = e$ *is defined then*

1. $\Gamma \vdash e$

2. For any $\Gamma \vdash P$*,* $\Gamma \vdash P : e$ *implies* $\Gamma \vdash P : e_1$ *and* $\Gamma \vdash P : e_2$

Proof. Let us consider the rule forming $\Gamma \vdash e_1 \,\&\, e_2 = e$.

Case 1. Rule ($\simeq \&^+ \simeq$), i.e. $\Gamma \vdash e_1 \,\&\, e_2 = e$ has form $\Gamma \vdash (\widehat{\alpha}^+ :\approx Q) \,\&\, (\widehat{\alpha}^+ :\approx Q') = (\widehat{\alpha}^+ :\approx Q)$ and $\mathbf{nf}\,(Q) = \mathbf{nf}\,(Q')$. The latter implies $\Gamma \vdash Q \simeq_1^{\leqslant} Q'$ by lemma 29. Then

1. $\Gamma \vdash e$, i.e. $\Gamma \vdash \widehat{\alpha}^+ :\approx Q$ holds by assumption;
2. by inversion, $\Gamma \vdash P : (\widehat{\alpha}^+ :\approx Q)$ means $\Theta \vdash \quad P \simeq_1^D Q$, and by transitivity of equivalence (corollary 5), $\Theta \vdash \quad P \simeq_1^D Q'$. Thus, $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$ hold by Rule SATSCEPEq.

Case 2. Rule ($\simeq \&^- \simeq$) the negative case is proved in exactly the same way as the positive one.

Case 3. Rule ($\geqslant \&^+ \geqslant$) Then e_1 is $\widehat{\alpha}^+ :\geqslant Q_1$, e_2 is $\widehat{\alpha}^+ :\geqslant Q_2$, and $e_1 \,\&\, e_2 = e$ is $\widehat{\alpha}^+ :\geqslant Q$ where Q is the least upper bound of Q_1 and Q_2. Then by lemma 41,

- $\Gamma \vdash Q$,
- $\Gamma \vdash Q \geqslant_1 Q_1$,
- $\Gamma \vdash Q \geqslant_1 Q_2$.

Let us show the required properties.

- $\Gamma \vdash e$ holds from $\Gamma \vdash Q$,
- Assuming $\Gamma \vdash P : e$, by inversion, we have $\Gamma \vdash P \geqslant_1 Q$. Combining it transitively with $\Gamma \vdash Q \geqslant_1 Q_1$, we have $\Gamma \vdash P \geqslant_1 Q_1$. Analogously, $\Gamma \vdash P \geqslant_1 Q_2$. Then $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$ hold by Rule SATSCESup.

Case 4. Rule ($\geqslant \&^+ \simeq$) Then e_1 is $\widehat{\alpha}^+ :\geqslant Q_1$, e_2 is $\widehat{\alpha}^+ :\approx Q_2$, where $\Gamma; \cdot \vDash Q_2 \geqslant Q_1 \dashv \cdot$, and the resulting $e_1 \,\&\, e_2 = e$ is equal to e_2, that is $\widehat{\alpha}^+ :\approx Q_2$.

Let us show the required properties.

- By assumption, $\Gamma \vdash Q$, and hence $\Gamma \vdash e$.
- Since $\mathbf{uv}\,(Q_2) = \varnothing$, $\Gamma; \cdot \vDash Q_2 \geqslant Q_1 \dashv \cdot$ implies $\Gamma \vdash Q_2 \geqslant_1 Q_1$ by the soundness of positive subtyping (lemma 45). Then let us take an arbitrary $\Gamma \vdash P$ such that $\Gamma \vdash P : e$. Since $e_2 = e$, $\Gamma \vdash P : e_2$ holds immediately.

  By inversion, $\Gamma \vdash P : (\widehat{\alpha}^+ :\approx Q_2)$ means $\Gamma \vdash P \simeq_1^{\leqslant} Q_2$, and then by transitivity of subtyping (lemma 7), $\Gamma \vdash P \geqslant_1 Q_1$. Then $\Gamma \vdash P : e_1$ holds by Rule SATSCESup.

Case 5. Rule ($\simeq \&^+ \geqslant$) Thee proof is analogous to the previous case.

$\square$

Lemma 48 (Soundness of Constraint Merge)**.** *Suppose that $\Theta \vdash SC_1$ and $\Theta \vdash SC_2$ and $\Theta \vdash SC_1 \& SC_2 = SC$ is defined. Then*

1. *$\Theta \vdash SC$,*
2. *for any substitution $\Theta \vdash \widehat{\sigma}$, $\Theta \vdash \widehat{\sigma} : SC$ implies $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$.*

Proof. By definition, $SC_1 \,\&\, SC_2 = SC$ consists of three parts: entries of SC_1 that do not have matching entries of SC, entries of SC_2 that do not have matching entries of SC_1, and the merge of matching entries.

Let us show $\Theta \vdash SC$. First, let us assume that an entry $e \in SC$ belongs to the first group, i.e. $e \in SC_1$. Let us denote the variable e as $\widehat{\alpha}^{\pm}$. Then $\Theta(\widehat{\alpha}^{\pm}) \vdash e$ holds since $\Theta \vdash SC_1 \ni e$. Analogously, if e belongs to the second group, then $\Theta(\widehat{\alpha}^{\pm}) \vdash e$ holds since $\Theta \vdash SC_2 \ni e$. Finally, if e belongs to the third group, then e is a merge of two entries $\Theta(\widehat{\alpha}^{\pm}) \vdash e_1$ and $\Theta(\widehat{\alpha}^{\pm}) \vdash e_2$. Then $\Theta(\widehat{\alpha}^{\pm}) \vdash e$ holds by lemma 47.

Let us show the second property. We take an arbitrary $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma}$ and $\Theta \vdash \widehat{\sigma} : SC$. To prove $\Theta \vdash \widehat{\sigma} : SC_1$, we need to show that for any $e_1 \in SC_1$, restricting $\widehat{\alpha}^{\pm}$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$ holds.

Let us assume that $\widehat{\alpha}^{\pm} \notin \mathbf{dom}\,(SC_2)$. It means that $SC \ni e_1$, and then since $\Theta \vdash \widehat{\sigma} : SC$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$.

Otherwise, SC_2 contains an entry e_2 restricting $\widehat{\alpha}^{\pm}$, and $SC \ni e$ where $\Theta(\widehat{\alpha}^{\pm}) \vdash e_1 \,\&\, e_2 = e$. Then since $\Theta \vdash \widehat{\sigma} : SC$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$, and by lemma 47, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$.

The proof of $\Theta \vdash \widehat{\sigma} : SC_2$ is symmetric. $\square$

Lemma 49 (Completeness of Constraint Entry Merge)**.** *For a fixed context Γ, suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$ are matching constraint entries.*

- *for a type P such that $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$, $\Gamma \vdash e_1 \,\&\, e_2 = e$ is defined and $\Gamma \vdash P : e$.*
- *for a type N such that $\Gamma \vdash N : e_1$ and $\Gamma \vdash N : e_2$, $\Gamma \vdash e_1 \,\&\, e_2 = e$ is defined and $\Gamma \vdash N : e$.*

Proof. Let us consider the shape of e_1 and e_2.

Case 1. e_1 is $\widehat{\alpha}^+ :\approx Q_1$ and e_2 is $\widehat{\alpha}^+ :\approx Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \simeq_1^{\leqslant} Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \simeq_1^{\leqslant} Q_2$. Then by transitivity of equivalence (corollary 5), $\Gamma \vdash Q_1 \simeq_1^{\leqslant} Q_2$, which means $\mathbf{nf}\,(Q_1) = \mathbf{nf}\,(Q_2)$ by lemma 29. Hence, Rule ($\simeq \&^+ \simeq$) applies to infer $\Gamma \vdash e_1 \;\&\; e_2 = e_2$, and $\Gamma \vdash P : e_2$ holds by assumption.

Case 2. e_1 is $\widehat{\alpha}^+ :\approx Q_1$ and e_2 is $\widehat{\alpha}^+ :\geqslant Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \simeq_1^{\leqslant} Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \geqslant_1 Q_2$. Then by transitivity of subtyping, $\Gamma \vdash Q_1 \geqslant_1 Q_2$, which means $\Gamma; \cdot \vDash Q_1 \geqslant Q_2 \dashv \cdot$ by lemma 46. This way, Rule ($\simeq \&^+ \geqslant$) applies to infer $\Gamma \vdash e_1 \;\&\; e_2 = e_1$, and $\Gamma \vdash P : e_1$ holds by assumption.

Case 3. e_1 is $\widehat{\alpha}^+ :\geqslant Q_1$ and e_2 is $\widehat{\alpha}^+ :\geqslant Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \geqslant_1 Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \geqslant_1 Q_2$. By the completeness of the least upper bound (lemma 42), $\Gamma \vDash Q_1 \vee Q_2 = Q$, and $\Gamma \vdash P \geqslant_1 Q$. This way, Rule ($\geqslant \&^+ \geqslant$) applies to infer $\Gamma \vdash e_1 \;\&\; e_2 = (\widehat{\alpha}^+ :\geqslant Q)$, and $\Gamma \vdash P : (\widehat{\alpha}^+ :\geqslant Q)$ holds by Rule SATSCESup.

Case 4. The negative cases are proved symmetrically.

□

Lemma 50 (Completeness of Constraint Merge)**.** *Suppose that $\Theta \vdash SC_1$ and $\Theta \vdash SC_2$. Then for any substitution $\Theta \vdash \widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$,*

1. *$\Theta \vdash SC_1 \& SC_2 = SC$ is defined,*
2. *$\Theta \vdash SC$, and*
3. *$\Theta \vdash \widehat{\sigma} : SC$.*

Proof. By definition, $SC_1 \;\&\; SC_2$ is a union of

1. entries of SC_1, which do not have matching entries in SC_2,
2. entries of SC_2, which do not have matching entries in SC_1, and
3. the merge of matching entries.

This way, to show that $\Theta \vdash SC_1 \& SC_2 = SC$ is defined, we need to demonstrate that each of these components is defined. To prove $\Theta \vdash \widehat{\sigma} : SC$ we need to show that for every entry e of SC restricting a variable $\widehat{\alpha}^{\pm}$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$ holds.

It is clear that the first two components of this union exist, and that if e is an entry restricting $\widehat{\alpha}^{\pm}$ $\mathbf{dom}\,(SC_1) \backslash \mathbf{dom}\,(SC_2)$ then then $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$ because $\Theta \vdash \widehat{\sigma} : SC_1$ and $e \in SC_1$. Analogously, if e is an entry from the second component of the union, the property holds for it. Let us show that the third component exists, and each of its entries satisfies the property. Let us take two entries $e_1 \in SC_1$ and $e_2 \in SC_2$ restricting the same variable $\widehat{\alpha}^{\pm}$. $\Theta \vdash \widehat{\sigma} : SC_1$ means that $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$ means $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_2$. Then by lemma 49, $\Theta(\widehat{\alpha}^{\pm}) \vdash e_1 \;\&\; e_2 = e$ is defined and $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$. □

Lemma 51 (Substitution existence)**.** *If $\Theta \vdash SC$ then there exists $\Theta \vdash \widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : SC$.*

Proof. □

4.15 Constraint Satisfaction

Lemma 52. *Suppose that $\Theta \vdash SC$ then there exist s $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : SC$.*

Proof. Let us define $\widehat{\sigma}$ on $\mathbf{dom}\,(SC)$ in the following way:

$$[\widehat{\sigma}]\widehat{\alpha}^{\pm} = \begin{cases} P & \text{if } (\widehat{\alpha}^{\pm} :\approx P) \in SC \\ P & \text{if } (\widehat{\alpha}^{\pm} :\geqslant P) \in SC \\ N & \text{if } (\widehat{\alpha}^{\pm} :\approx N) \in SC \end{cases}$$

Then $\Theta \vdash \widehat{\sigma} : SC$ follows immediately from the reflexivity of equivalence and subtyping (lemma 5) and the corresponding rules Rule SATSCEPEq, Rule SATSCENEq, and Rule SATSCESup. □

Lemma 53. *Suppose $\Theta \vdash SC$, and SC* **singular***. Then there exist a unique (up-to equivalence) $\Theta \vdash \widehat{\sigma} : SC$.*

Proof. □

4.16 Negative Subtyping

Lemma 54 (Soundness of Negative Subtyping). *If $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash M$, $\Gamma; \Theta \vdash N$ and $\Gamma; \Theta \models N \leqslant M \dashv SC$, then $\Theta \vdash SC$ and for any normalized $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : SC$, $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$.*

Proof. We prove it by induction on $\Gamma; \Theta \models N \leqslant M \dashv SC$.

Suppose that $\widehat{\sigma}$ is normalized and $\Theta \vdash \widehat{\sigma} : SC$, Let us consider the last rule to infer this judgment.

Case 1. Rule $(\to^{\leqslant})$. Then $\Gamma; \Theta \models N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \models P \to N' \leqslant Q \to M' \dashv SC$
On the next step, the the algorithm makes two recursive calls: $\Gamma; \Theta \models P \geqslant Q \dashv SC_1$ and $\Gamma; \Theta \models N' \leqslant M' \dashv SC_2$ and returns $\Theta \vdash SC_1 \& SC_2 = SC$ as the result.

By the soundness of constraint merge (lemma 48), $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$. Then by the soundness of positive subtyping (lemma 45), $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$; and by the induction hypothesis, $\Gamma \vdash [\widehat{\sigma}]N' \leqslant_1 M'$. This way, by Rule $(\to^{\leqslant_1})$, $\Gamma \vdash [\widehat{\sigma}](P \to N') \leqslant_1 Q \to M'$.

Case 2. Rule $(\mathrm{Var}^{-\leqslant})$, and then $\Gamma; \Theta \models N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot$
This case is symmetric to case 2 of lemma 45.

Case 3. Rule $(\uparrow^{\leqslant})$, and then $\Gamma; \Theta \models N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \models {\uparrow}P \leqslant {\uparrow}Q \dashv SC$
This case is symmetric to case 3 of lemma 45.

Case 4. Rule $(\forall^{\leqslant})$, and then $\Gamma; \Theta \models N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \models \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.M' \dashv SC$ s.t. either $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is not empty
This case is symmetric to case 4 of lemma 45.

□

Lemma 55 (Completeness of the Negative Subtyping). *Suppose that $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash M$, $\Gamma; \Theta \vdash N$, and N does not contain negative unification variables ($\widehat{\alpha}^- \notin \mathbf{uv}\, N$). Then for any $\Theta \vdash \widehat{\sigma}$ such that $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$, there exists $\Gamma; \Theta \models N \leqslant M \dashv SC$, such that $\Theta \vdash SC$ and moreover, $\Theta \vdash \widehat{\sigma} : SC$.*

Proof. We prove it by induction on $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$. Let us consider the last rule used in the derivation of $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$.

Case 1. $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$ is derived by Rule $(\uparrow^{\leqslant_1})$
Then $N = {\uparrow}P$, since the substitution $[\widehat{\sigma}]N$ must preserve the top-level constructor of $N \neq \widehat{\alpha}^-$ (since by assumption, $\widehat{\alpha}^- \notin \mathbf{uv}\, N$), and $Q = {\downarrow}M$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]N \simeq_1^{\leqslant} M$. The rest of the proof is symmetric to case 3 of lemma 46: notice that the algorithm does not make a recursive call, and the difference in the induction statement for the positive and the negative case here does not matter.

Case 2. $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$ is derived by Rule $(\to^{\leqslant_1})$, i.e. $[\widehat{\sigma}]N = [\widehat{\sigma}]P \to [\widehat{\sigma}]N'$ and $M = Q \to M'$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$ and $\Gamma \vdash [\widehat{\sigma}]N' \leqslant_1 M'$.

The algorithm makes two recursive calls: $\Gamma; \Theta \models P \geqslant Q \dashv SC_1$ and $\Gamma; \Theta \models N' \leqslant M' \dashv SC_2$, and then returns $\Theta \vdash SC_1 \& SC_2 = SC$ as the result. Let us show that these recursive calls are successful and the returning constraints are fulfilled by $\widehat{\sigma}$.

Notice that from the inversion of $\Gamma \vdash M$, we have: $\Gamma \vdash Q$ and $\Gamma \vdash M'$; from the inversion of $\Gamma; \Theta \vdash N$, we have: $\Gamma; \Theta \vdash P$ and $\Gamma; \Theta \vdash N'$; and since N does not contain negative unification variables, N' does not contain negative unification variables either.

This way, we can apply the induction hypothesis to $\Gamma \vdash [\widehat{\sigma}]N' \leqslant_1 M'$ to obtain $\Gamma; \Theta \models N' \leqslant M' \dashv SC_2$ such that $\Theta \vdash SC_2$ and $\Theta \vdash \widehat{\sigma} : SC_2$. Also, we can apply the completeness of the positive subtyping (lemma 46) to $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$ to obtain $\Gamma; \Theta \models P \geqslant Q \dashv SC_1$ such that $\Theta \vdash SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_1$.

Finally, we need to show that the merge of SC_1 and SC_2 is successful and satisfies the required properties. To do so, we apply the completeness of subtyping constraint merge (lemma 50). This way, $\Theta \vdash SC_1 \& SC_2 = SC$ is defined, $\Theta \vdash SC$, and $\Theta \vdash \widehat{\sigma} : SC$, as required.

Case 3. $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$ is derived by Rule $(\forall^{\leqslant_1})$. Since N does not contain negative unification variables, N must be of the form $\forall\overrightarrow{\alpha^+}.N'$, such that $[\widehat{\sigma}]N = \forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N'$ and $[\widehat{\sigma}]N' \neq \forall\ldots$ (assuming $\overrightarrow{\alpha^+}$ does not intersect with the range of $\widehat{\sigma}$). Also, $M = \forall\overrightarrow{\beta^+}.M'$ and either $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is non-empty.

The rest of the proof is symmetric to **??** of lemma 46. To apply the induction hypothesis, we need to show additionally that there are no negative unification variables in $N_0 = [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N'$. This is because $\mathbf{uv}\, N_0 \subseteq \mathbf{uv}\, N \cup \overrightarrow{\widehat{\alpha}^+}$, and N is free of negative unification variables by assumption.

Case 4. $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$ is derived by Rule ($\text{Var}^{-\leqslant_1}$).
Then $N = [\widehat{\sigma}]N = \alpha^- = M$. Here the first equality holds because N is not a unification variable: by assumption, N is free of negative unification variables. The second and the third equations hold because Rule ($\text{Var}^{-\leqslant_1}$) was applied.

The rest of the proof is symmetric to case 2 of lemma 46.

□

4.17 Singularity

Lemma 56 (Soundness of Entry Singularity)**.** *Suppose* e **singular***. Then*

+ *for any* P_1 *and* P_2 *well-formed in* Γ*, if* $\Gamma \vdash P_1 : e$ *and* $\Gamma \vdash P_2 : e$ *then* $\Gamma \vdash P_1 \simeq_1^{\leqslant} P_2$*;*

− *for any* N_1 *and* N_2 *well-formed in* Γ*, if* $\Gamma \vdash N_1 : e$ *and* $\Gamma \vdash N_2 : e$ *then* $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_2$*.*

Proof. Let us consider how e **singular** is formed.

Case 1. Rule SINGNEq, that is $e = \widehat{\alpha}^- :\approx M$. Then $\Gamma \vdash N_i : e$ means $\Gamma \vdash N_i \simeq_1^{\leqslant} M$ (by inversion of Rule SATSCENEq). Then by symmetry and transitivity of equivalence (lemma 5 and corollary 5), $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_2$.

Case 2. Rule SINGPEq. This case is symmetric to the previous one.

Case 3. Rule SINGSupVar, that is $e = \widehat{\alpha}^+ :\geqslant \exists\overrightarrow{\alpha^-}.\beta^+$. Then $\Gamma \vdash P_i : e$ means $\Gamma \vdash P_i \geqslant_1 \exists\overrightarrow{\alpha^-}.\beta^+$.

Let us show that it implies $\Gamma \vdash P_i \simeq_1^{\leqslant} \exists\overrightarrow{\alpha^-}.\beta^+$. By applying lemma 39 once, we have $\Gamma, \overrightarrow{\alpha^-} \vdash P_i \geqslant_1 \beta^+$. By applying it again, we notice that $\Gamma, \overrightarrow{\alpha^-} \vdash P_i \geqslant_1 \beta^+$ implies $P_i = \exists\overrightarrow{\alpha^-}'.\beta^+$. Finally, it is easy to see that $\Gamma \vdash \exists\overrightarrow{\alpha^-}'.\beta^+ \simeq_1^{\leqslant} \exists\overrightarrow{\alpha^-}.\beta^+$

This way, by symmetry and transitivity of equivalence (lemma 5 and corollary 5), $\Gamma \vdash P_1 \simeq_1^{\leqslant} \exists\overrightarrow{\alpha^-}.\beta^+$ and $\Gamma \vdash P_2 \simeq_1^{\leqslant} \exists\overrightarrow{\alpha^-}.\beta^+$ implies $\Gamma \vdash P_1 \simeq_1^{\leqslant} P_2$.

Case 4. Rule SINGSupShift, that is $e = \widehat{\alpha}^+ :\geqslant \exists\overrightarrow{\beta^-}.{\downarrow}N$, where $N \simeq_1^D \beta_j^-$. Then $\Gamma \vdash P_i : e$ means $\Gamma \vdash P_i \geqslant_1 \exists\overrightarrow{\beta^-}.{\downarrow}N$

Let us show that it implies $\Gamma \vdash P_i \simeq_1^{\leqslant} \exists\overrightarrow{\beta^-}.{\downarrow}N$.

$[h]\Gamma \vdash P_i \geqslant_1 \exists\overrightarrow{\beta^-}.{\downarrow}N$	$\Rightarrow \Gamma \vdash \mathbf{nf}\,(P_i) \geqslant_1 \exists\overrightarrow{\beta^-}'.{\downarrow}\mathbf{nf}\,(N)$ where $\mathbf{ord}\,\overrightarrow{\beta^-}\,\mathbf{in}\,N' = \overrightarrow{\beta^-}'$	by corollary 15
	$\Rightarrow \Gamma \vdash \mathbf{nf}\,(P_i) \geqslant_1 \exists\overrightarrow{\beta^-}'.{\downarrow}\mathbf{nf}\,(\beta_j^-)$	by lemma 18
	$\Rightarrow \Gamma \vdash \mathbf{nf}\,(P_i) \geqslant_1 \exists\overrightarrow{\beta^-}'.{\downarrow}\beta_n^-$	by definition of normalization
	$\Rightarrow \Gamma \vdash \mathbf{nf}\,(P_i) \geqslant_1 \exists\beta_j^-.{\downarrow}\beta_j^-$	since $\mathbf{ord}\,\overrightarrow{\beta^-}\,\mathbf{in}\,\mathbf{nf}\,(N') = \beta_j^-$
	$\Rightarrow \Gamma, \beta_j^- \vdash \mathbf{nf}\,(P) \geqslant_1 {\downarrow}\beta_j^-$ and $\beta_j^- \notin \mathbf{fv}\,(\mathbf{nf}\,(P_i))$	by lemma 40

By lemma 40, the last subtyping means that $\mathbf{nf}\,(P_i) = \exists\overrightarrow{\alpha^-}.{\downarrow}N'$, such that

1. $\Gamma, \beta_j^-, \overrightarrow{\alpha^-} \vdash N'$
2. $\mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,N' = \overrightarrow{\alpha^-}$
3. for some substitution $\Gamma, \beta_j^- \vdash \sigma : \overrightarrow{\alpha^-}$, $[\sigma]N' = \beta_j^-$.

Since $\beta_j^- \notin \mathbf{fv}\,(\mathbf{nf}\,(P_i))$, the latter means that $N' = \alpha^-$, and then $\mathbf{nf}\,(P_i) = \exists\alpha^-.{\downarrow}\alpha^-$ for some α^-. Finally, notice that all the types of shape $\exists\alpha^-.{\downarrow}\alpha^-$ are equal, and hence, $\mathbf{nf}\,(P_1) = \mathbf{nf}\,(P_2)$, which implies $\Gamma \vdash P_1 \simeq_1^{\leqslant} P_2$ by lemma 29.

□

Lemma 57 (Completeness of Entry Singularity)**.** − *Suppose that for any* N_1 *and* N_2 *well-formed in* Γ *such that* $\Gamma \vdash N_1 : e$ *and* $\Gamma \vdash N_2 : e$*,* $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_2$ *holds. Then* e **singular** *holds.*

+ *Suppose that for any* P_1 *and* P_2 *well-formed in* Γ *such that* $\Gamma \vdash P_1 : e$ *and* $\Gamma \vdash P_2 : e$*,* $\Gamma \vdash P_1 \simeq_1^{\leqslant} P_2$ *holds. Then* e **singular** *holds.*

Proof. − Since N_i are negative, by inversion of $\Gamma \vdash N_1 : e$, e has shape $\widehat{\alpha}^- :\approx M$. Then e **singular** by Rule SINGNEq.

+ Let us consider the shape of e:

  Case 1. $e = (\widehat{\alpha}^+ :\approx P)$ then e is singular by Rule SINGPEq;

  Case 2. $e = (\widehat{\alpha}^+ :\geqslant P)$. Let us consider the shape of the positive type P:

a. $P = \exists\overrightarrow{\beta^-}.\beta^+$ (for potentially empty $\overrightarrow{\beta^-}$) then e is singular by Rule SINGSupVar;

b. $P = \exists\overrightarrow{\beta^-}.{\downarrow}N$ (for potentially empty $\overrightarrow{\beta^-}$). Notice since $\Gamma \vdash P \geqslant_1 P$, $\Gamma \vdash P : e$ holds.
Notice that $\Gamma \vdash \exists\gamma^-.{\downarrow}\gamma^- \geqslant_1 \exists\overrightarrow{\beta^-}.{\downarrow}N$ (by Rule $(\exists^{\geqslant_1})$, with substitution N/γ^-), and thus, $\Gamma \vdash \exists\gamma^-.{\downarrow}\gamma^- : e$ by Rule SATSCESup.
This way, by assumption, $\Gamma \vdash \exists\gamma^-.{\downarrow}\gamma^- \simeq_1^{\leqslant} \exists\overrightarrow{\beta^-}.{\downarrow}N$, which implies $\mathbf{nf}\,(\exists\overrightarrow{\beta^-}.{\downarrow}N) = \exists\gamma^-.{\downarrow}\gamma^-$ (by lemma 29), which by definition of normalization means $\exists\overrightarrow{\beta^-}'.{\downarrow}\mathbf{nf}\,(N) = \exists\gamma^-.{\downarrow}\gamma^-$, where $\mathbf{ord}\,\overrightarrow{\beta^-}\,\mathbf{in}\,N' = \overrightarrow{\beta^-}'$. This way, $\overrightarrow{\beta^-}'$ is a variable β^-, and $\mathbf{nf}\,(N) = \beta^-$. Notice that $\beta^- \in \overrightarrow{\beta^-}' \subseteq \overrightarrow{\beta^-}$ by lemma 10.
This way, $\Gamma \vdash N \simeq_1^{\leqslant} \beta^-$ for $\beta^- \in \overrightarrow{\beta^-}$ (by lemma 29), and hence, e is singular by Rule SINGSupShift.

□

Lemma 58 (Soundness of Singularity)**.** *Suppose $\Theta \vdash SC$, and SC* **singular***. Then for any $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ such that $\Theta \vdash \widehat{\sigma}_1 : SC$ and $\Theta \vdash \widehat{\sigma}_2 : SC$, $\Theta \vdash \widehat{\sigma}_1 \simeq_1^{\leqslant} \widehat{\sigma}_2 : \mathbf{dom}\,(SC)$.*

Proof. Suppose that $\Theta \vdash \widehat{\sigma} : SC$. It means that for any $e \in SC$ restricting $\widehat{\alpha}^{\pm}$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$ holds. SC **singular** means e **singular**, and hence, by lemma 57, $[\widehat{\sigma}]\widehat{\alpha}^{\pm}$ is unique up-to equivalence.
Since the uniqueness holds for every variable from $\mathbf{dom}\,(SC)$, $\widehat{\sigma}$ is defined uniquely on this set. □

Lemma 59 (Completeness of Singularity)**.**

4.18 Typing

Lemma 60 (Soundness of typing)**.**

+ *If $\Gamma; \Phi \models v : P$ then $\Gamma \vdash P$ and $\Gamma; \Phi \vdash v : P$*

− *If $\Gamma; \Phi \models c : N$ then $\Gamma \vdash N$ and $\Gamma; \Phi \vdash c : N$*

- *For $\Gamma \vdash^{\supseteq} \Theta$ and $\Gamma; \Theta \vdash N$, if $\Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rrightarrow M \dashv \Theta'; SC$ then*
  1. *$\Gamma \vdash^{\supseteq} \Theta'$*
  2. *$\Theta \subseteq \Theta'$*
  3. *$\Gamma; \Theta' \vdash M$*
  4. *$\Theta' \vdash SC$*
  5. *for any $\Theta' \vdash \widehat{\sigma} : SC$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]M$*

Proof. We prove it by induction on the typing derivation. Let us consider the last rule used to infer the derivation.

Case 1. Rule ATVar

Case 2. Rule ATThunk

Case 3. Rule ATPAnnot

Case 4. Rule ATNAnnot

Case 5. Rule ATtLam

Case 6. Rule ATTLam

Case 7. Rule ATReturn

Case 8. Rule ATVarLet

Case 9. Rule ATAppLetAnn By inversion, we have:

1. c is $\mathbf{let}\, x : P = v(\vec{v}); c'$
2. $\Gamma \vdash P$
3. $\Gamma; \Phi \models v : {\downarrow}M$
4. $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rrightarrow {\uparrow}Q \dashv \Theta; SC_1$
5. $\Gamma; \Theta \models {\uparrow}Q \leqslant {\uparrow}P \dashv SC_2$
6. $\Theta \vdash SC_1 \& SC_2 = SC$
7. $\Gamma; \Phi, x : P \models c' : N$

By the soundness of constraint merge (lemma 48), we have $\Theta \vdash SC$. Let us take $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : SC$ (it exists by lemma 51). Notice that by the soundness of constraint merge, $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$.

By the induction hypothesis applied to $\Gamma; \Phi \models v \colon \downarrow M$, we have $\Gamma; \Phi \vdash v \colon \downarrow M$ and $\Gamma \vdash \downarrow M$ (and hence,$\Gamma; \Theta \vdash M$).

By the induction hypothesis applied to $I\Gamma; \Phi, x : P \models c' \colon N$, we have $\Gamma; \Phi, x : P \vdash c' \colon N$ and $\Gamma \vdash N$.

By the induction hypothesis applied to $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Longrightarrow \uparrow Q \dashv \Theta; SC_1$, we have:

1. $\Gamma \vdash^{\supseteq} \Theta$,
2. $\Gamma; \Theta \vdash \uparrow Q$,
3. $\Theta' \vdash SC_1$,
4. for any $\Theta' \vdash \widehat{\sigma} : SC_1$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}] M \bullet \vec{v} \Longrightarrow [\widehat{\sigma}] \uparrow Q$. In particular, it holds for the $\widehat{\sigma}$ chosen above.

By the soundness of negative subtyping (**??**) applied to $\Gamma;\ \Theta \models \uparrow Q \leqslant \uparrow P \dashv SC$, we have $\Gamma \vdash \uparrow [\widehat{\sigma}] Q \leqslant_1 \uparrow P$.

To infer $\Gamma; \Phi \vdash \mathbf{let}\, x : P = v(\vec{v}); c' \colon N$, we apply the corresponding declarative rule Rule DTAppLetAnn, where Q is $[\widehat{\sigma}] Q$. Notice that all the premises were already shown to hold above:

1. $\Gamma \vdash P$ and $\Gamma; \Phi \vdash v \colon \downarrow M$ from the inversion,
2. $\Gamma; \Phi \vdash M \bullet \vec{v} \Longrightarrow \uparrow [\widehat{\sigma}] Q$ holds since $[\widehat{\sigma}] \uparrow Q = \uparrow [\widehat{\sigma}] Q$,
3. $\Gamma \vdash \uparrow [\widehat{\sigma}] Q \leqslant_1 \uparrow P$ by the soundness of negative subtyping,
4. $\Gamma; \Phi, x : P \vdash c' \colon N$ from the the induction hypothesis.

Case 10. Rule ATAppLet By the inversion, we have:

1. c is $\mathbf{let}\, x = v(\vec{v}); c'$
2. $\Gamma; \Phi \models v \colon \downarrow M$
3. $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Longrightarrow \uparrow Q \dashv \Theta; SC$
4. $\mathbf{uv}\ Q \subseteq \mathbf{dom}\,(SC)$
5. $\mathbf{SC}|_{\mathbf{uv}\,(\,Q\,)}$ **singular**
6. $\Gamma; \Phi, x : Q \models c' \colon N$

By the induction hypothesis applied to $\Gamma; \Phi \models v \colon \downarrow M$, we have $\Gamma; \Phi \vdash v \colon \downarrow M$ and $\Gamma \vdash \downarrow M$ (and thus, $\Gamma; \Theta \vdash M$).

By the induction hypothesis applied to $\Gamma; \Phi, x : Q \models c' \colon N$, we have $\Gamma \vdash N$ and $\Gamma; \Phi, x : Q \vdash c' \colon N$.

By the induction hypothesis applied to $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Longrightarrow \uparrow Q \dashv \Theta; SC$, we have:

1. $\Gamma \vdash^{\supseteq} \Theta$
2. $\Gamma; \Theta \vdash \uparrow Q$
3. $\Theta \vdash SC$
4. for any $\Theta \vdash \widehat{\sigma} : SC$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}] M \bullet \vec{v} \Longrightarrow [\widehat{\sigma}] \uparrow Q$, which, since M is ground means $\Gamma; \Phi \vdash M \bullet \vec{v} \Longrightarrow \uparrow [\widehat{\sigma}] Q$

To infer $\Gamma; \Phi \vdash \mathbf{let}\, x = v(\vec{v}); c' \colon N$, we apply the corresponding declarative rule Rule DTAppLet. Let us show that the premises hold:

- $\Gamma; \Phi \vdash v \colon \downarrow M$ holds by the induction hypothesis;
- $\Gamma; \Phi, x : Q \vdash c' \colon N$ also holds by the induction hypothesis, as noted above;
- Let us take an arbitrary substitution $\widehat{\sigma}$ satisfying $\Theta \vdash \widehat{\sigma} : SC$ (it exists by lemma 52). Then $\Gamma; \Phi \vdash M \bullet \vec{v} \Longrightarrow \uparrow [\widehat{\sigma}] Q$ holds, as noted above;
- To show the uniqueness of $\uparrow [\widehat{\sigma}] Q$, we assume that for some other type K holds $\Gamma; \Phi \vdash M \bullet \vec{v} \Longrightarrow K$ that is $\Gamma; \Phi \vdash [\cdot] M \bullet \vec{v} \Longrightarrow K$. Then by the completeness of typing (lemma 62), there exist N', Θ', and SC' such that
  1. $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Longrightarrow N' \dashv \Theta'; SC'$ and
  2. there exists a substitution $\Theta' \vdash \widehat{\sigma}' : SC'$ such that $\Gamma \vdash [\widehat{\sigma}'] N' \simeq_1^{\leqslant} K$.

  By the determinicity of the typing algorithm (**??**) $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Longrightarrow N' \dashv \Theta'; SC'$, means that SC' is SC, Θ' is Θ, and N' is $\uparrow Q$. This way, $\Gamma \vdash [\widehat{\sigma}'] \uparrow Q \simeq_1^{\leqslant} K$ for a substitution $\Theta \vdash \widehat{\sigma}' : SC$.

  It is left to show that $\Gamma \vdash [\widehat{\sigma}'] \uparrow Q \simeq_1^{\leqslant} [\widehat{\sigma}] \uparrow Q$, then by transitivity of the equivalence, we will have $\Gamma \vdash [\widehat{\sigma}] \uparrow Q \simeq_1^{\leqslant} K$. Since $\Theta \vdash \widehat{\sigma} : \mathbf{SC}|_{\mathbf{uv}\,(\,Q\,)}$ and $\Theta \vdash \widehat{\sigma}' : \mathbf{SC}|_{\mathbf{uv}\,(\,Q\,)}$, and $\mathbf{SC}|_{\mathbf{uv}\,(\,Q\,)}$ **singular**, we have $\Theta \vdash \widehat{\sigma} \simeq_1^{\leqslant} \widehat{\sigma}' : \mathbf{dom}\,(\mathbf{SC}|_{\mathbf{uv}\,(\,Q\,)})$. Then since $\mathbf{uv}\,(\,Q\,) \subseteq \mathbf{dom}\,(SC)$, we have $\mathbf{dom}\,(\mathbf{SC}|_{\mathbf{uv}\,(\,Q\,)}) = \mathbf{uv}\,(\,Q\,)$. This way, $\Theta \vdash \widehat{\sigma} \simeq_1^{\leqslant} \widehat{\sigma}' : \mathbf{uv}\,(\,Q\,)$, which implies $\Gamma \vdash [\widehat{\sigma}'] \uparrow Q \simeq_1^{\leqslant} [\widehat{\sigma}] \uparrow Q$.

Case 11. Rule ATUnpack By the inversion, we have:

1. c is $\mathbf{let}^{\exists}(\alpha^-, x) = v; c'$
2. $\Gamma; \Phi \models v \colon \exists\alpha^-.P$
3. $\Gamma, \alpha^-; \Phi, x : P \models c' \colon N$
4. $\Gamma \vdash N$

By the induction hypothesis applied to $\Gamma; \Phi \models v \colon \exists\alpha^-.P$, we have $\Gamma; \Phi \vdash v \colon \exists\alpha^-.P$. By the induction hypothesis applied to $\Gamma, \alpha^-; \Phi, x : P \models c' \colon N$, we have $\Gamma, \alpha^-; \Phi, x : P \vdash c' \colon N$.

To show $\Gamma; \Phi \vdash \mathbf{let}^{\exists}(\alpha^-, x) = v; c' \colon N$, we apply the corresponding declarative rule Rule DTUnpack. Let us show that the premises hold:

1. $\Gamma; \Phi \vdash v \colon \exists\alpha^-.P$ holds by the induction hypothesis, as noted above,
2. $\Gamma, \alpha^-; \Phi, x : P \vdash c' \colon N$ also holds by the induction hypothesis,
3. $\Gamma \vdash N$ holds by the inversion, as noted above.

Case 12. Rule ATEmptyApp Then by assumption:

- $\Gamma \vdash^{\supseteq} \Theta$,
- $\Gamma; \Theta \vdash N$,
- $\Gamma; \Phi; \Theta \models N \bullet \cdot \Longrightarrow N \dashv \Theta; \cdot$, which by inversion means that $N \neq \forall\overrightarrow{\alpha^+}.M$.

Let us show the required properties:

1. $\Gamma \vdash^{\supseteq} \Theta$ holds by assumption,
2. $\Theta \subseteq \Theta$ holds trivially,
3. $\Gamma; \Theta \vdash N$ holds by assumption,
4. $\Theta \vdash \cdot$ holds trivially,
5. for any $\Theta \vdash \widehat{\sigma} : \cdot$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \cdot \Longrightarrow [\widehat{\sigma}]N$. To show $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \cdot \Longrightarrow [\widehat{\sigma}]N$, we apply the corresponding declarative rule Rule DTEmptyApp.

Case 13. Rule ATArrowApp
By assumption:

1. $\Gamma \vdash^{\supseteq} \Theta$,
2. $\Gamma; \Theta \vdash Q \to N$,
3. $\Theta \vdash SC_1 \& SC_2 = SC$,
4. $\Gamma; \Phi; \Theta \models Q \to N \bullet v, \vec{v} \Longrightarrow M \dashv \Theta'; SC$, and by inversion:
    (a) $\Gamma; \Phi \models v \colon P$, and by the induction hypothesis applied to this judgment, $\Gamma; \Phi \vdash v \colon P$,
    (b) $\Gamma;\, \Theta \models Q \geqslant P \dashv SC_1$, and by the soundness of subtyping: $\Theta \vdash SC$ and for any $\Theta \vdash \widehat{\sigma} : SC_1$, we have $\Gamma \vdash [\widehat{\sigma}]Q \geqslant_1 P$,
    (c) $\Gamma; \Phi; \Theta \models N \bullet \vec{v} \Longrightarrow M \dashv \Theta'; SC_2$, and by the induction hypothesis applied to this judgment,
        i. $\Gamma \vdash^{\supseteq} \Theta'$,
        ii. $\Theta \subseteq \Theta'$,
        iii. $\Gamma; \Theta' \vdash M$,
        iv. $\Theta' \vdash SC_2$,
        v. for any $\Theta' \vdash \widehat{\sigma} : SC_2$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \vec{v} \Longrightarrow [\widehat{\sigma}]M$.

Let us show the required properties:

1. $\Gamma \vdash^{\supseteq} \Theta'$ is shown above,
2. $\Theta \subseteq \Theta'$ is shown above,
3. $\Gamma; \Theta' \vdash M$ is shown above,
4. $\Theta' \vdash SC$ holds: $\Theta \vdash SC_1$ implies $\Theta' \vdash SC_1$, then we apply the soundness of constraint merge (lemma 48) to $\Theta' \vdash SC_1 \;\&\; SC_2$,
    (a) $\Theta' \vdash SC_1$,
    (b) for any $\Theta' \vdash \widehat{\sigma} : SC$, $\Theta' \vdash \widehat{\sigma} : SC_i$ holds;

5. suppose that $\Theta' \vdash \hat{\sigma} : SC$. Then to show $\Gamma; \Phi \vdash [\hat{\sigma}](Q \rightarrow N) \bullet v, \vec{v} \Rightarrow\!\!\!> [\hat{\sigma}]M$, that is $\Gamma; \Phi \vdash [\hat{\sigma}]Q \rightarrow [\hat{\sigma}]N \bullet v, \vec{v} \Rightarrow\!\!\!> [\hat{\sigma}]M$, we apply the corresponding declarative rule Rule DTArrowApp. Let us show the required premises:
   (a) $\Gamma; \Phi \vdash v : P$ holds as shown above,
   (b) $\Gamma \vdash [\hat{\sigma}]Q \geqslant_1 P$ holds by the soundness of subtyping as noted above, since $\Theta' \vdash \hat{\sigma} : SC$ implies $\Theta \vdash \hat{\sigma} : SC_1$.
   (c) $\Gamma; \Phi \vdash [\hat{\sigma}]N \bullet \vec{v} \Rightarrow\!\!\!> [\hat{\sigma}]M$ holds by the induction hypothesis as shown above, since $\Theta' \vdash \hat{\sigma} : SC$ implies $\Theta' \vdash \hat{\sigma} : SC_2$.

Case 14. Rule ATForallApp
By assumption:

1. $\Gamma \vdash^{\supseteq} \Theta$,
2. $\Gamma; \Theta \vdash \forall \overrightarrow{\alpha^+}. N$,
3. $\Gamma; \Phi; \Theta \models \forall \overrightarrow{\alpha^+}. N \bullet \vec{v} \Rightarrow\!\!\!> M \dashv \Theta'; SC$, which by inversion means $\vec{v} \neq \cdot$ and $\Gamma; \Phi; \Theta, \overrightarrow{\hat{\alpha}^+}\{\Gamma\} \models [\overrightarrow{\hat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \vec{v} \Rightarrow\!\!\!> M \dashv \Theta'; SC$. It is easy to see that the induction hypothesis is applicable to the latter judgment: $\Gamma \vdash^{\supseteq} \Theta, \overrightarrow{\hat{\alpha}^+}\{\Gamma\}$ is implied by $\Gamma \vdash^{\supseteq} \Theta$, and $\Gamma; \Theta, \overrightarrow{\hat{\alpha}^+}\{\Gamma\} \vdash [\overrightarrow{\hat{\alpha}^+}/\overrightarrow{\alpha^+}]N$ is holds since $\Gamma; \Theta \vdash \forall \overrightarrow{\alpha^+}. N$. Let us apply the inductive hypothesis to the latter judgment to obtain:
   (a) $\Gamma \vdash^{\supseteq} \Theta'$,
   (b) $\Theta, \overrightarrow{\hat{\alpha}^+}\{\Gamma\} \subseteq \Theta'$,
   (c) $\Gamma; \Theta' \vdash M$,
   (d) $\Theta' \vdash SC$,
   (e) for any $\Theta' \vdash \hat{\sigma} : SC$, we have $\Gamma; \Phi \vdash [\hat{\sigma}][\overrightarrow{\hat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \vec{v} \Rightarrow\!\!\!> [\hat{\sigma}]M$.

Let us show the required properties:

1. $\Gamma \vdash^{\supseteq} \Theta'$ is shown above,
2. $\Theta \subseteq \Theta'$ since $\Theta, \overrightarrow{\hat{\alpha}^+}\{\Gamma\} \subseteq \Theta'$,
3. $\Gamma; \Theta' \vdash M$ is shown above,
4. $\Theta' \vdash SC$ is shown above,
5. let us assume $\Theta' \vdash \hat{\sigma} : SC$ Then to show $\Gamma; \Phi \vdash [\hat{\sigma}]\forall \overrightarrow{\alpha^+}. N \bullet \vec{v} \Rightarrow\!\!\!> [\hat{\sigma}]M$, we apply the corresponding declarative rule Rule DTForallApp with substitution $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}$ defined in the following way: $[\sigma]\alpha_i^+ = [\hat{\sigma}]\widehat{\alpha_i}^+$.
   Let us show that its premises hold:
   (a) $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}$, i.e. $\Gamma \vdash [\sigma]\alpha_i^+$ holds since $\Theta' \vdash \hat{\sigma}$ and $\Gamma \vdash^{\supseteq} \Theta'$;
   (b) $\Gamma; \Phi \vdash [\sigma][\hat{\sigma}]N \bullet \vec{v} \Rightarrow\!\!\!> [\hat{\sigma}]M$ holds by rewriting $\Gamma; \Phi \vdash [\hat{\sigma}][\overrightarrow{\hat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \vec{v} \Rightarrow\!\!\!> [\hat{\sigma}]M$ using equality $[\hat{\sigma}][\overrightarrow{\hat{\alpha}^+}/\overrightarrow{\alpha^+}]N = [\sigma][\hat{\sigma}]N$:
      i. for $\alpha_i^+ \in \overrightarrow{\alpha^+}$, $[\hat{\sigma}][\overrightarrow{\hat{\alpha}^+}/\overrightarrow{\alpha^+}]\alpha_i^+ = [\hat{\sigma}]\widehat{\alpha_i}^+ = [\sigma]\alpha_i^+ = [\sigma][\hat{\sigma}]\alpha_i^+$,
      ii. for $\hat{\beta}^\pm \in \mathbf{dom}(\hat{\sigma})$, $[\hat{\sigma}][\overrightarrow{\hat{\alpha}^+}/\overrightarrow{\alpha^+}]\hat{\beta}^\pm = [\hat{\sigma}]\hat{\beta}^\pm = [\sigma][\hat{\sigma}]\hat{\beta}^\pm$, where the latter equality holds since $\overrightarrow{\alpha^+} \cap \Gamma = \varnothing$.
   (c) $\vec{v} \neq \cdot$ holds by assumption

□

Lemma 61. *If* $\Gamma; \Phi \vdash N_1 \bullet \vec{v} \Rightarrow\!\!\!> M$ *and* $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_2$ *then* $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Rightarrow\!\!\!> M$.

Proof. Induction on $\Gamma; \Phi \vdash N_1 \bullet \vec{v} \Rightarrow\!\!\!> M$. Let us consider the last rule used in the derivation:

Case 1. Rule DTEmptyApp Then $\vec{v} = \cdot$ and by inversion of $\Gamma; \Phi \vdash N_1 \bullet \cdot \Rightarrow\!\!\!> M$, $\Gamma \vdash N_1 \simeq_1^{\leqslant} M$. By transitivity corollary 5, $\Gamma \vdash N_2 \simeq_1^{\leqslant} M$, and then Rule DTEmptyApp is applicable to infer $\Gamma; \Phi \vdash N_2 \bullet \cdot \Rightarrow\!\!\!> M$.

Case 2. Rule DTArrowApp Then we are proving that $\Gamma; \Phi \vdash (Q_1 \rightarrow N_1) \bullet v, \vec{v} \Rightarrow\!\!\!> M$ and $\Gamma \vdash (Q_1 \rightarrow N_1) \simeq_1^{\leqslant} (Q_2 \rightarrow N_2)$ imply $\Gamma; \Phi \vdash (Q_2 \rightarrow N_2) \bullet v, \vec{v} \Rightarrow\!\!\!> M$.

By inversion, $\Gamma \vdash (Q_1 \rightarrow N_1) \simeq_1^{\leqslant} (Q_2 \rightarrow N_2)$ means $\Gamma \vdash Q_1 \simeq_1^{\leqslant} Q_2$ and $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_2$.

By inversion of $\Gamma; \Phi \vdash (Q_1 \rightarrow N_1) \bullet v, \vec{v} \Rightarrow\!\!\!> M$:

1. $\Gamma; \Phi \vdash v : P$
2. $\Gamma \vdash Q_1 \geqslant_1 P$, and then by transitivity corollary 4, $\Gamma \vdash Q_2 \geqslant_1 P$;
3. $\Gamma; \Phi \vdash N_1 \bullet \vec{v} \Rightarrow\!\!\!> M$, and then by induction hypothesis, $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Rightarrow\!\!\!> M$.

Since we have $\Gamma; \Phi \vdash v : P$, $\Gamma \vdash Q_2 \geqslant_1 P$ and $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Rightarrow\!\!\!> M$, we can apply Rule DTArrowApp to infer $\Gamma; \Phi \vdash (Q_2 \rightarrow N_2) \bullet v, \vec{v} \Rightarrow\!\!\!> M$.

Case 3. Rule DTArrowApp Then we are proving that $\Gamma; \Phi \vdash \forall \overrightarrow{\alpha^+}_1 . N_1' \bullet \vec{v} \Rrightarrow M$ and $\Gamma \vdash \forall \overrightarrow{\alpha^+}_1 . N_1' \simeq_1^{\leqslant} N_2$ imply $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Rrightarrow M$.

Let us write N_2 as $\forall \overrightarrow{\alpha^+}_2 . N_2'$ with potentially empty $\overrightarrow{\alpha^+}_2$.

By lemma 28, $\Gamma \vdash \forall \overrightarrow{\alpha^+}_1 . N_1' \simeq_1^{\leqslant} \forall \overrightarrow{\alpha^+}_2 . N_2'$ means $\forall \overrightarrow{\alpha^+}_1 . N_1' \simeq_1^{D} \forall \overrightarrow{\alpha^+}_2 . N_2'$, and by inversion:

1. $\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, N_1 = \varnothing$,
2. there exists a bijection $\mu : (\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, N_2') \leftrightarrow (\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\, N_1')$ such that $N_1' \simeq_1^{D} [\mu] N_2'$.

By inversion of $\Gamma; \Phi \vdash \forall \overrightarrow{\alpha^+}_1 . N_1' \bullet \vec{v} \Rrightarrow M$:

1. $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}_1$
2. $\Gamma; \Phi \vdash [\sigma] N_1' \bullet \vec{v} \Rrightarrow M$
3. $\vec{v} \neq \cdot$

Let us construct $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^+}_2$ in the following way:

$$\begin{cases} [\sigma_0]\alpha^+ = [\sigma][\mu]\alpha^+ & \text{if } \alpha^+ \in \overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, N_2' \\ [\sigma_0]\alpha^+ = \exists \beta^- . \downarrow \beta^- & \text{otherwise (the type does not matter here)} \end{cases}$$

Then to infer $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Rrightarrow M$, we apply Rule DTArrowApp with σ_0. Let us show the required premises:

1. $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^+}_2$ by construction;
2. $\vec{v} \neq \cdot$ as noted above;
3. To show $\Gamma; \Phi \vdash [\sigma_0] N_2' \bullet \vec{v} \Rrightarrow M$, Notice that $[\sigma_0] N_2' = [\sigma][\mu] N_2'$ and since $[\mu] N_2' \simeq_1^{D} N_1'$, $[\sigma][\mu] N_2' \simeq_1^{D} [\sigma] N_1'$. This way, by lemma 24, $\Gamma \vdash [\sigma] N_1' \simeq_1^{\leqslant} [\sigma_0] N_2'$. Then the required judgement holds by the induction hypothesis applied to $\Gamma; \Phi \vdash [\sigma] N_1' \bullet \vec{v} \Rrightarrow M$.

□

Lemma 62 (Completeness of Typing)**.**

\+ *If $\Gamma; \Phi \vdash v \colon P$ then there exists P' such that $\Gamma; \Phi \vDash v \colon P'$ and $\Gamma \vdash P' \simeq_1^{\leqslant} P$;*

− *If $\Gamma; \Phi \vdash c \colon N$ then there exists N' such that $\Gamma; \Phi \vDash c \colon N'$ and $\Gamma \vdash N' \simeq_1^{\leqslant} N$;*

- *Suppose that $\Gamma; \Phi \vdash [\widehat{\sigma}] N \bullet \vec{v} \Rrightarrow M$ holds for some $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma; \Theta \vdash N$ (free from negative metavariables, that is $\widehat{\alpha}^- \notin \mathbf{uv}\, N$), $\Theta \vdash \widehat{\sigma}$, and $\Gamma \vdash M$. Then there exist M', Θ', and SC such that*
  1. *$\Gamma; \Phi; \Theta \vDash N \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC$ and*
  2. *for any $\Theta \vdash \widehat{\sigma}$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\widehat{\sigma}] N \bullet \vec{v} \Rrightarrow M$, there exists $\widehat{\sigma}'$ such that*
     (a) *$\Theta' \vdash \widehat{\sigma}' : SC$,*
     (b) *$\Theta \vdash \widehat{\sigma}' \simeq_1^{\leqslant} \widehat{\sigma} : \mathbf{dom}\,(\Theta)$, and*
     (c) *$\Gamma \vdash [\widehat{\sigma}'] M' \simeq_1^{\leqslant} M$.*

Proof. By induction on the typing derivation. Let us consider the last rule applied to infer the derivation.

Case 1. Rule DTAppLet
Then by assumption, c is $\mathbf{let}\, x = v(\vec{v}); c'$. Then by inversion of $\Gamma; \Phi \vdash \mathbf{let}\, x = v(\vec{v}); c' \colon N$:

- $\Gamma; \Phi \vdash v \colon \downarrow M$, which by the induction hypothesis means that there exists M_2 such that <<no parses (char 17): ; v :
and $\Gamma \vdash \downarrow M_2 \simeq_1^{\leqslant} \downarrow M$;
-
$$\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q$$
uniquely.*Thenby*??, *since*$\Gamma \vdash \downarrow M_2 \simeq_1^{\leqslant} \downarrow M$ By the induction hypothesis, $\Gamma; \Phi \vdash [\cdot] M \bullet \vec{v} \Rrightarrow \uparrow Q$ means that there exist M', Θ', and SC such that (considering M is ground):
  1. $\Gamma; \Phi; \cdot \vDash M \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC$ and
  2. for any $\Gamma \vdash M_1$ such that $\Gamma; \Phi \vdash [\cdot] M \bullet \vec{v} \Rrightarrow M_1$, and in particular, for $\Gamma \vdash \uparrow Q$, there exists $\widehat{\sigma}'$ such that
     (a) $\Theta' \vdash \widehat{\sigma}' : SC$, and
     (b) $\Gamma \vdash [\widehat{\sigma}'] M' \simeq_1^{\leqslant} \uparrow Q$.

- $\Gamma; \Phi, x : Q \vdash c' \colon N$, which by the induction hypothesis means that there exists N_2 such that $\Gamma; \Phi, x : Q \vDash c' \colon N_2$ and $\Gamma \vdash N_2 \simeq_1^{\leqslant} N$.

Let us show that N' from 1 satisfies the required properties. $\Gamma \vdash N' \simeq_1^{\leqslant} N$ holds as noted above. To infer $\Gamma; \Phi \vdash \mathbf{let}\, x = v(\vec{v}); c' \colon N$, let us apply the corresponding algorithmic rule (Rule ATAppLet) with the type of v being

Case 2. Rule DTForallApp
Since N cannot be a metavariable, if $[\widehat{\sigma}]N$ starts from $\forall$, so does N. This way, $N = \forall\overrightarrow{\alpha^+}.N_1$. Then by assumption:

1. $\Gamma; \Theta \vdash \forall\overrightarrow{\alpha^+}.N_1$ is free from negative metavariables, and then $\Gamma, \overrightarrow{\alpha^+}; \Theta \vdash N_1$ is free from negative metavariables;
2. $\Theta \vdash \widehat{\sigma}$;
3. $\Gamma \vdash M$;
4. $\Gamma; \Phi \vdash [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N_1 \bullet \vec{v} \Rightarrow\!\!\!> M$, that is $\Gamma; \Phi \vdash (\forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N_1) \bullet \vec{v} \Rightarrow\!\!\!> M$. Then by inversion there exists σ such that
   (a) $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}$;
   (b) $\vec{v} \neq \cdot$; and
   (c) $\Gamma; \Phi \vdash [\sigma][\widehat{\sigma}]N_1 \bullet \vec{v} \Rightarrow\!\!\!> M$. Notice that σ and $\widehat{\sigma}$ commute because the codomain of σ does not contain metavariables (and thus, does not intersect with the domain of $\widehat{\sigma}$), and the codomain of $\widehat{\sigma}$ is Γ and does not intersect with $\overrightarrow{\alpha^+}$—the domain of σ.
   Let us construct $N_0 = [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N_1$ and $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}_0$ defined as

$$\begin{cases} [\widehat{\sigma}_0]\widehat{\alpha_i}^+ = [\sigma]\alpha_i^+ & \text{for } \widehat{\alpha_i}^+ \in \overrightarrow{\widehat{\alpha}^+} \\ [\widehat{\sigma}_0]\widehat{\beta}^\pm = [\widehat{\sigma}]\widehat{\beta}^\pm & \text{for } \widehat{\beta}^\pm \in \mathbf{dom}\,(\Theta) \end{cases}$$

   Then it is easy to see that $[\widehat{\sigma}_0][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N_1 = [\sigma][\widehat{\sigma}]N_1$ because this substitution compositions coincide on $\overrightarrow{\alpha^+} \cup \mathbf{dom}\,(\Theta)$, their domain. In other words, $[\widehat{\sigma}_0]N_0 = [\sigma][\widehat{\sigma}]N_1$.
   Then let us apply the induction hypothesis to $\Gamma; \Phi \vdash [\widehat{\sigma}_0]N_0 \bullet \vec{v} \Rightarrow\!\!\!> M$ and obtain M', Θ', and SC such that
   - $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash N_0 \bullet \vec{v} \Rightarrow\!\!\!> M' \dashv \Theta'; SC$ and
   - for any $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}_0$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\widehat{\sigma}_0]N_0 \bullet \vec{v} \Rightarrow\!\!\!> M$, there exists $\widehat{\sigma}'_0$ such that
     i. $\Theta' \vdash \widehat{\sigma}'_0 : SC$,
     ii. $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}'_0 \simeq_1^{\leqslant} \widehat{\sigma}_0 : \mathbf{dom}\,(\Theta) \cup \overrightarrow{\widehat{\alpha}^+}$, and
     iii. $\Gamma \vdash [\widehat{\sigma}'_0]M' \simeq_1^{\leqslant} M$.

Let us take M', Θ', and SC from the induction hypothesis (4c) and show they satisfy the required properties.

1. to infer $\Gamma; \Phi; \Theta \vDash \forall\overrightarrow{\alpha^+}.N_1 \bullet \vec{v} \Rightarrow\!\!\!> M' \dashv \Theta'; SC$ we apply the corresponding algorithmic rule Rule ATForallApp, not that the required premises hold, as noted above:
   (a) $\vec{v} \neq \cdot$, and
   (b) $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N_1 \bullet \vec{v} \Rightarrow\!\!\!> M' \dashv \Theta'; SC$ can be rewritten as $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash N_0 \bullet \vec{v} \Rightarrow\!\!\!> M' \dashv \Theta'; SC$.
2. Let us take and arbitrary $\Theta \vdash \widehat{\sigma}$ and $\Gamma \vdash M$ and assume $\Gamma; \Phi \vdash [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N_1 \bullet \vec{v} \Rightarrow\!\!\!> M$. Then the same reasoning as in 4c applies. In particular, we construct $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}_0$ as an extension of $\widehat{\sigma}$ and obtain $\Gamma; \Phi \vdash [\widehat{\sigma}_0]N_0 \bullet \vec{v} \Rightarrow\!\!\!> M$.
   It means, we can apply the property inferred from the induction hypothesis (4c) to obtain $\widehat{\sigma}'_0$ such that
   (a) $\Theta' \vdash \widehat{\sigma}'_0 : SC$,
   (b) $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}'_0 \simeq_1^{\leqslant} \widehat{\sigma}_0 : \mathbf{dom}\,(\Theta) \cup \overrightarrow{\widehat{\alpha}^+}$, and
   (c) $\Gamma \vdash [\widehat{\sigma}'_0]M' \simeq_1^{\leqslant} M$.
   Let us show that the obtained $\widehat{\sigma}'_0$ satisfies the required properties.
   (a) $\Theta' \vdash \widehat{\sigma}'_0 : SC$ holds as shown,
   (b) $\Gamma \vdash [\widehat{\sigma}'_0]M' \simeq_1^{\leqslant} M$ holds as shown,
   (c) $\Theta \vdash \widehat{\sigma}'_0 \simeq_1^{\leqslant} \widehat{\sigma} : \mathbf{dom}\,(\Theta)$, holds. Let us take an arbitrary $\widehat{\beta}^\pm \in \mathbf{dom}\,(\Theta) \subseteq \mathbf{dom}\,(\Theta) \cup \overrightarrow{\widehat{\alpha}^+}$. Then since $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}'_0 \simeq_1^{\leqslant} \widehat{\sigma}_0 : \mathbf{dom}\,(\Theta) \cup \overrightarrow{\widehat{\alpha}^+}$, we have $[\widehat{\sigma}'_0]\widehat{\beta}^\pm = [\widehat{\sigma}_0]\widehat{\beta}^\pm$ and by definition of $\widehat{\sigma}_0$, $[\widehat{\sigma}_0]\widehat{\beta}^\pm = [\widehat{\sigma}]\widehat{\beta}^\pm$.

Case 3. Rule DTArrowApp
Since N cannot be a metavariable, if the shape of $[\widehat{\sigma}]N$ is an arrow, so is the shape of N. This way, $N = Q \to N_1$. Then by assumption:

1. $\Gamma; \Theta \vdash Q \to N_1$ is free from negative metavariables;

2. $\Theta \vdash \hat{\sigma}$;
3. $\Gamma \vdash M$;
4. $\Gamma; \Phi \vdash [\hat{\sigma}](Q \to N_1) \bullet v, \vec{v} \Rrightarrow M$, that is $\Gamma; \Phi \vdash ([\hat{\sigma}]Q \to [\hat{\sigma}]N_1) \bullet v, \vec{v} \Rrightarrow M$, and by inversion:
   (a) $\Gamma; \Phi \vdash v : P$, and by the induction hypothsis, $\Gamma; \Phi \vDash v : P'$ for some P' such that $\Gamma \vdash P' \simeq_1^{\leqslant} P$;
   (b) $\Gamma \vdash [\hat{\sigma}]Q \geqslant_1 P$, which by transitivity (lemma 7) means $\Gamma \vdash [\hat{\sigma}]Q \geqslant_1 P'$, and then by completeness of subtyping (lemma 46), $\Gamma; \Theta \vDash Q \geqslant P' \dashv SC_1$, for some $\Theta \vdash SC_1$, and moreover, $\Theta \vdash \hat{\sigma} : SC_1$;
   (c) $\Gamma; \Phi \vdash [\hat{\sigma}]N_1 \bullet \vec{v} \Rrightarrow M$. Notice that the induction hypothesis is applicable to this case: $\Gamma; \Theta \vdash N_1$ is free from negative metavariables because so is $Q \to N_1$. This way, there exist M', Θ', and SC_2 such that
      i. $\Gamma; \Phi; \Theta \vDash N_1 \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC_2$ and then by the soundness of typing (i.e. the induction hypothesis),
         A. $\Theta \subseteq \Theta'$
         B. $\Gamma; \Theta' \vdash M'$
      ii. for any $\Theta \vdash \hat{\sigma}$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\hat{\sigma}]N_1 \bullet \vec{v} \Rrightarrow M$, there exists $\hat{\sigma}'$ such that
         A. $\Theta' \vdash \hat{\sigma}' : SC_2$,
         B. $\Theta \vdash \hat{\sigma}' \simeq_1^{\leqslant} \hat{\sigma} : \mathbf{dom}(\Theta)$, and
         C. $\Gamma \vdash [\hat{\sigma}']M' \simeq_1^{\leqslant} M$.

Let us take $\Theta \vdash \hat{\sigma}$ and M and construct $\Theta' \vdash \hat{\sigma}'$ by the induction hypothesis (4(c)ii). Then $\Theta' \vdash \hat{\sigma}' : SC_2$ and $\Theta' \vdash \hat{\sigma}' : SC_1$ holds and since $\Theta \vdash \hat{\sigma} : SC_1$ and $\Theta \vdash \hat{\sigma}' \simeq_1^{\leqslant} \hat{\sigma} : \mathbf{dom}(\Theta)$. Then by the completeness of constraint merge (lemma 50), $\Theta' \vdash SC_1 \& SC_2 = SC$ exists, $\Theta' \vdash SC$, and $\Theta' \vdash \hat{\sigma} : SC$.

To show the required properties, we take M' and Θ' from the induction hypothesis (4(c)ii), and SC defined above. Then

1. $\Gamma; \Phi; \Theta \vDash Q \to N_1 \bullet v, \vec{v} \Rrightarrow M' \dashv \Theta'; SC$ is inferred by Rule ATArrowApp:
   (a) $\Gamma; \Phi \vDash v : P'$ as noted above,
   (b) $\Gamma; \Theta \vDash Q \geqslant P' \dashv SC_1$ as noted above,
   (c) $\Gamma; \Phi; \Theta \vDash N_1 \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC_2$ as noted above;
2. let us take an arbitrary $\Theta \vdash \hat{\sigma}_0$ and $\Gamma \vdash M_0$ such that $\Gamma; \Phi \vdash [\hat{\sigma}_0](Q \to N_1) \bullet v, \vec{v} \Rrightarrow M_0$. Then by inversion of $\Gamma; \Phi \vdash [\hat{\sigma}_0]Q \to [\hat{\sigma}_0]N_1 \bullet v, \vec{v} \Rrightarrow M_0$, we have the same properties as in 4. In particular,
   - $\Gamma; \Phi \vdash [\hat{\sigma}_0]N_1 \bullet \vec{v} \Rrightarrow M_0$. Then by 4(c)ii, there exists $\hat{\sigma}'$ such that
     (a) $\Theta' \vdash \hat{\sigma}' : SC_2$,
     (b) $\Theta \vdash \hat{\sigma}' \simeq_1^{\leqslant} \hat{\sigma}_0 : \mathbf{dom}(\Theta)$, and
     (c) $\Gamma \vdash [\hat{\sigma}']M' \simeq_1^{\leqslant} M_0$.
   - $\Gamma \vdash [\hat{\sigma}_0]Q \geqslant_1 P'$ and by the completeness of subtyping (lemma 46), $\Theta \vdash \hat{\sigma}_0 : SC_1$.

   This way,
   - $\Theta \vdash \hat{\sigma}' \simeq_1^{\leqslant} \hat{\sigma}_0 : \mathbf{dom}(\Theta)$ holds as noted above;
   - $\Theta' \vdash \hat{\sigma}' : SC_1$ holds because $\Theta \vdash \hat{\sigma}_0 : SC_1$ and $\Theta \vdash \hat{\sigma}' \simeq_1^{\leqslant} \hat{\sigma}_0 : \mathbf{dom}(\Theta)$, and $\Theta' \vdash \hat{\sigma}' : SC_1$ together with $\Theta' \vdash \hat{\sigma}' : SC_2$ implies $\Theta' \vdash \hat{\sigma}' : SC$ by the completeness of constraint merge (lemma 50); and
   - $\Gamma \vdash [\hat{\sigma}']M' \simeq_1^{\leqslant} M_0$ holds as noted above.

Case 4. Rule DTEmptyApp
By assumption:

1. $\Gamma; \Theta \vdash N$,
2. $\Theta \vdash \hat{\sigma}$,
3. $\Gamma; \Phi \vdash [\hat{\sigma}]N \bullet \cdot \Rrightarrow [\hat{\sigma}]N$.

Then we can apply the corresponding algorithmic rule Rule ATEmptyApp to infer $\Gamma; \Phi; \Theta \vDash N \bullet \cdot \Rrightarrow N \dashv \Theta; \cdot$. Let us show the required properties. Let us take an arbitrary $\Theta \vdash \hat{\sigma}_1$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\hat{\sigma}_1]N \bullet \cdot \Rrightarrow M$. Then we can take $\hat{\sigma}' = \hat{\sigma}_1$:

1. $\Theta \vdash \hat{\sigma}' : \cdot$ holds vacuously,
2. $\Theta \vdash \hat{\sigma}' \simeq_1^{\leqslant} \hat{\sigma}_1 : \mathbf{dom}(\Theta)$ holds by reflexivity of equivalence,
3. $\Gamma \vdash [\hat{\sigma}']N \simeq_1^{\leqslant} M$ or equivalently, $\Gamma \vdash [\hat{\sigma}]N \simeq_1^{\leqslant} M$ holds because $\Gamma; \Phi \vdash [\hat{\sigma}_1]N \bullet \cdot \Rrightarrow M$ can only be inferred by Rule DTEmptyApp, meaning $[\hat{\sigma}_1]N = M$.

Case 5. Rule DTVar

Case 6. Rule DTThunk

Case 7. Rule DTPAnnot

Case 8. Rule DTtLam

Case 9. Rule DTTLam

Case 10. Rule DTReturn

Case 11. Rule DTVarLet

Case 12. Rule DTAppLetAnn

Case 13. Rule DTUnpack

Case 14. Rule DTNAnnot

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