1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
& | & \alpha^+ & \\
& | & \downarrow N & \\
& | & \exists \alpha^-.P & \\
\end{array}
$$

$$
\begin{array}{llll}
N,\ M & ::= & & \text{negative types} \\
& | & \alpha^- & \\
& | & \uparrow P & \\
& | & \forall \alpha^+.N & \\
& | & P \to N & \\
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq^{\leqslant}_0 M}$ Negative equivalence

$$
\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq^{\leqslant}_0 M} \quad \text{D0NDEF}
$$

$\boxed{\Gamma \vdash P \simeq^{\leqslant}_0 Q}$ Positive equivalence

$$
\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq^{\leqslant}_0 Q} \quad \text{D0PDEF}
$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$
\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \quad \text{D0NVAR}
$$

$$
\frac{\Gamma \vdash P \simeq^{\leqslant}_0 Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \quad \text{D0SHIFTU}
$$

$$
\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \quad \text{D0FORALLL}
$$

$$
\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \quad \text{D0FORALLR}
$$

$$
\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \quad \text{D0ARROW}
$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$
\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \quad \text{D0PVAR}
$$

$$
\frac{\Gamma \vdash N \simeq^{\leqslant}_0 M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \quad \text{D0SHIFTD}
$$

$$
\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \quad \text{D0EXISTSL}
$$

$$
\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \quad \text{D0EXISTSR}
$$

2 Multi-Quantified System

2.1 Grammar

$$
\begin{array}{lll}
P,\ Q & ::= & \text{multi-quantified positive types} \\
& | \quad \alpha^+ & \\
& | \quad {\downarrow}N & \\
& | \quad \exists\overrightarrow{\alpha^-}.P & \qquad P \neq \exists\ldots \\
& | \quad (P) \quad \mathsf{S} &
\end{array}
$$

$$
\begin{array}{lll}
N,\ M & ::= & \text{multi-quantified negative types} \\
& | \quad \alpha^- & \\
& | \quad {\uparrow}P & \\
& | \quad P \to N & \\
& | \quad \forall\overrightarrow{\alpha^+}.N & \qquad N \neq \forall\ldots \\
& | \quad (N) \quad \mathsf{S} &
\end{array}
$$

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^\leqslant M}$ Negative equivalence on MQ types

$$
\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^\leqslant M} \quad (\simeq_1^\leqslant{}^-)
$$

$\boxed{\Gamma \vdash P \simeq_1^\leqslant Q}$ Positive equivalence on MQ types

$$
\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^\leqslant Q} \quad (\simeq_1^\leqslant{}^+)
$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$
\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad (\textsc{Var}^{-\leqslant_1})
$$

$$
\frac{\Gamma \vdash P \simeq_1^\leqslant Q}{\Gamma \vdash {\uparrow}P \leqslant_1 {\uparrow}Q} \quad ({\uparrow}^{\leqslant_1})
$$

$$
\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \quad (\to^{\leqslant_1})
$$

$$
\frac{\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M} \quad (\forall^{\leqslant_1})
$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$
\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad (\textsc{Var}^{+\geqslant_1})
$$

$$
\frac{\Gamma \vdash N \simeq_1^\leqslant M}{\Gamma \vdash {\downarrow}N \geqslant_1 {\downarrow}M} \quad ({\downarrow}^{\geqslant_1})
$$

$$
\frac{\mathbf{fv}\, P \cap \overrightarrow{\beta^-} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q} \quad (\exists^{\geqslant_1})
$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^\leqslant \sigma_2 : \Gamma_1}$ Equivalence of substitutions

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$
\frac{}{\alpha^- \simeq_1^D \alpha^-} \quad (\textsc{Var}^{-\simeq_1^D})
$$

$$
\frac{P \simeq_1^D Q}{{\uparrow}P \simeq_1^D {\uparrow}Q} \quad ({\uparrow}^{\simeq_1^D})
$$

$$
\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M} \quad (\to^{\simeq_1^D})
$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu] M}{\overrightarrow{\forall \alpha^+}.N \simeq_1^D \overrightarrow{\forall \beta^+}.M} \quad (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\textsc{Var}^{+\simeq_1^D})$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu] Q}{\overrightarrow{\exists \alpha^-}.P \simeq_1^D \overrightarrow{\exists \beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3   Algorithm

3.1   Normalization

3.1.1   Ordering

$\boxed{\mathbf{ord}\, vars\, \mathbf{in}\, N = \overrightarrow{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^- = \alpha^-} \quad (\textsc{Var}_\in^-)$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^- = \cdot} \quad (\textsc{Var}_{\notin}^-)$$

$$\frac{\mathbf{ord}\, vars\, \mathbf{in}\, P = \overrightarrow{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in}\, \uparrow P = \overrightarrow{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\, vars\, \mathbf{in}\, P = \overrightarrow{\alpha}_1 \quad \mathbf{ord}\, vars\, \mathbf{in}\, N = \overrightarrow{\alpha}_2}{\mathbf{ord}\, vars\, \mathbf{in}\, P \rightarrow N = \overrightarrow{\alpha}_1, (\overrightarrow{\alpha}_2 \backslash \overrightarrow{\alpha}_1)} \quad (\rightarrow)$$

$$\frac{vars \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\, vars\, \mathbf{in}\, N = \overrightarrow{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in}\, \overrightarrow{\forall \alpha^+}.N = \overrightarrow{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\, vars\, \mathbf{in}\, P = \overrightarrow{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^+ = \alpha^+} \quad (\textsc{Var}_\in^+)$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^+ = \cdot} \quad (\textsc{Var}_{\notin}^+)$$

$$\frac{\mathbf{ord}\, vars\, \mathbf{in}\, N = \overrightarrow{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in}\, \downarrow N = \overrightarrow{\alpha}} \quad (\downarrow)$$

$$\frac{vars \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\, vars\, \mathbf{in}\, P = \overrightarrow{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in}\, \overrightarrow{\exists \alpha^-}.P = \overrightarrow{\alpha}} \quad (\exists)$$

$\boxed{\mathbf{ord}\, vars\, \mathbf{in}\, N = \overrightarrow{\alpha}}$

$$\frac{}{\mathbf{ord}\, vars\, \mathbf{in}\, \widehat{\alpha}^- = \cdot} \quad (\textsc{UVar}^-)$$

$\boxed{\mathbf{ord}\, vars\, \mathbf{in}\, P = \overrightarrow{\alpha}}$

$$\frac{}{\mathbf{ord}\, vars\, \mathbf{in}\, \widehat{\alpha}^+ = \cdot} \quad (\textsc{UVar}^+)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\ \ (\text{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\ \ (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\ \ (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ N' = \overrightarrow{\alpha^{+\prime}}}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.N'}\ \ (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\ \ (\text{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\ \ (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\ \overrightarrow{\alpha^-}\ \mathbf{in}\ P' = \overrightarrow{\alpha^{-\prime}}}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^{-\prime}}.P'}\ \ (\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\ \ (\text{UVar}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\ \ (\text{UVar}^+)$$

3.2 Unification

$\boxed{\Gamma;\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Gamma;\Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\ \ \text{UNVar}$$

$$\frac{\Gamma;\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma;\Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\ \ \text{UShiftU}$$

$$\frac{\Gamma;\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Gamma;\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Gamma;\Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2}\ \ \text{UArrow}$$

$$\frac{\Gamma,\overrightarrow{\alpha^+};\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma;\Theta \models \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\ \ \text{UForall}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma;\Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\widehat{\alpha}^- :\approx N)}\ \ \text{UNUVar}$$

$\boxed{\Gamma;\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Gamma;\Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\ \ \text{UPVar}$$

$$\frac{\Gamma;\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma;\Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\ \ \text{UShiftD}$$

$$\frac{\Gamma,\overrightarrow{\alpha^-};\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma;\Theta \models \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\ \ \text{UExists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma;\Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\widehat{\alpha}^+ :\approx P)}\ \ \text{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma; \Theta \vDash \alpha^- \leqslant \alpha^- \dashv \cdot} \quad (\text{Var}^{-\leqslant})$$

$$\frac{\Gamma; \Theta \vDash \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad (\uparrow^{\leqslant})$$

$$\frac{\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \vDash P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2} \quad (\to^{\leqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vDash [\overrightarrow{\widehat{\alpha}^+/\overrightarrow{\alpha^+}}] N \leqslant M \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^+}} \quad (\forall^{\leqslant})$$

$\boxed{\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma; \Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad (\text{Var}^{+\geqslant})$$

$$\frac{\Gamma; \Theta \vDash \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \quad (\downarrow^{\geqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vDash [\overrightarrow{\widehat{\alpha}^-/\overrightarrow{\alpha^-}}] P \geqslant Q \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \exists \overrightarrow{\alpha^-}.P \geqslant \exists \overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}} \quad (\exists^{\geqslant})$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \mathbf{upgrade}\,\Gamma \vdash P \,\mathbf{to}\,\Delta = Q}{\Gamma; \Theta \vDash \widehat{\alpha}^+ \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant Q)} \quad (\text{UVar}^{\geqslant})$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restricts an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\widehat{\alpha}^+ :\approx P$ or $\widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries). *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.
$\boxed{\Theta \vdash e_1 \mathbin{\&} e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\widehat{\alpha}^+\{\Gamma\} \in \Theta \quad \Gamma \vDash P_1 \vee P_2 = Q}{\Theta \vdash (\widehat{\alpha}^+ :\geqslant P_1) \mathbin{\&} (\widehat{\alpha}^+ :\geqslant P_2) = (\widehat{\alpha}^+ :\geqslant Q)} \quad (\geqslant \mathbin{\&} \geqslant)$$

$$\frac{\widehat{\alpha}^+\{\Gamma\} \in \Theta \quad \Gamma; \cdot \vDash P \geqslant Q \dashv \widehat{\sigma}'}{\Theta \vdash (\widehat{\alpha}^+ :\approx P) \mathbin{\&} (\widehat{\alpha}^+ :\geqslant Q) = (\widehat{\alpha}^+ :\approx P)} \quad (\simeq \mathbin{\&} \geqslant)$$

$$\frac{\widehat{\alpha}^+\{\Gamma\} \in \Theta \quad \Gamma; \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'}{\Theta \vdash (\widehat{\alpha}^+ :\geqslant P) \mathbin{\&} (\widehat{\alpha}^+ :\approx Q) = (\widehat{\alpha}^+ :\approx Q)} \quad (\geqslant \mathbin{\&} \simeq)$$

$$\frac{}{\Theta \vdash (\widehat{\alpha}^+ :\approx P) \mathbin{\&} (\widehat{\alpha}^+ :\approx P) = (\widehat{\alpha}^+ :\approx P)} \quad (\simeq \mathbin{\&} \simeq^+)$$

$$\frac{}{\Theta \vdash (\widehat{\alpha}^- :\approx N) \mathbin{\&} (\widehat{\alpha}^- :\approx N) = (\widehat{\alpha}^- :\approx N)} \quad (\simeq \mathbin{\&} \simeq^-)$$

Notice that in case of equivalence, the assigned types must be equal (i.e. alpha-equivalent) to be merged. This is because the unification algorithm assumes that every type is normalized, and hence, equivalence is alpha-equivalence (corollaries 14 and 16).

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3. $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2 = \{e_1 \mathbin{\&} e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, s.t.\ e_1\ matches\ with\ e_2\}$
$\qquad\qquad \cup \{e_1 \mid e_1 \in \widehat{\sigma}_1,\ s.t.\ \forall e_2 \in \widehat{\sigma}_2, e_1\ does\ not\ match\ with\ e_2\}$
$\qquad\qquad \cup \{e_2 \mid e_2 \in \widehat{\sigma}_2,\ s.t.\ \forall e_1 \in \widehat{\sigma}_1, e_1\ does\ not\ match\ with\ e_2\}$

3.5 Least Upper Bound

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+} \quad (\text{Var}^\vee)$$

$$\frac{\Gamma, \cdot \vDash \mathbf{nf}\,(\downarrow N) \stackrel{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \hat\tau_1, \hat\tau_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\, P} \quad (\downarrow^\vee)$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q} \quad (\exists^\vee)$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

$$\frac{\Gamma = \Delta, \overrightarrow{\alpha^\pm} \quad \overrightarrow{\beta^\pm}\,\mathbf{is\,fresh} \quad \overrightarrow{\gamma^\pm}\,\mathbf{is\,fresh}}{\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vDash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q}{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q} \quad (\text{Upg})$$

3.6 Antiunification

$\boxed{\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \stackrel{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \quad \text{AUPVar}$$

$$\frac{\Gamma \vDash N_1 \stackrel{a}{\simeq} N_2 \dashv (\Xi, M, \hat\tau_1, \hat\tau_2)}{\Gamma \vDash \downarrow N_1 \stackrel{a}{\simeq} \downarrow N_2 \dashv (\Xi, \downarrow M, \hat\tau_1, \hat\tau_2)} \quad \text{AUPShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \stackrel{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists \overrightarrow{\alpha^-}.Q, \hat\tau_1, \hat\tau_2)} \quad \text{AUPExists}$$

$\boxed{\Gamma \vDash N_1 \stackrel{a}{\simeq} N_2 \dashv (\Xi, M, \hat\tau_1, \hat\tau_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \stackrel{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)} \quad \text{AUNVar}$$

$$\frac{\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2)}{\Gamma \vDash \uparrow P_1 \stackrel{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \hat\tau_1, \hat\tau_2)} \quad \text{AUNShift}$$

$$\frac{\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi_1, Q, \hat\tau_1, \hat\tau_2) \quad \Gamma \vDash N_1 \stackrel{a}{\simeq} N_2 \dashv (\Xi_2, M, \hat\tau_1', \hat\tau_2')}{\Gamma \vDash P_1 \to N_1 \stackrel{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \hat\tau_1 \cup \hat\tau_1', \hat\tau_2 \cup \hat\tau_2')} \quad \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \stackrel{a}{\simeq} M \dashv (\hat\alpha^-_{\{N,M\}}, \hat\alpha^-_{\{N,M\}}, (\hat\alpha^-_{\{N,M\}} :\approx N), (\hat\alpha^-_{\{N,M\}} :\approx M))} \quad \text{AUNAU}$$

4 Proofs

4.1 Declarative Subtyping

Lemma 1 (Free Variable Propagation). *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context* Γ,

- $-$ *if* $\Gamma \vdash N \leqslant_1 M$ *then* $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$

- $+$ *if* $\Gamma \vdash P \geqslant_1 Q$ *then* $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$

Proof. Mutual induction on $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash P \geqslant_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
It is self-evident that $\alpha^- \subseteq \alpha^-$.

Case 2. $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq^\leqslant_1 Q$), we have $\Gamma \vdash P \geqslant_1 Q$. Then by the induction hypothesis, $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$. The desired inclusion inclusion holds, since $\mathbf{fv}\,(\uparrow P) = \mathbf{fv}\,(P)$ and $\mathbf{fv}\,(\uparrow Q) = \mathbf{fv}\,(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant_1 Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$ and $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$.
Then $\mathbf{fv}(P \to N) = \mathbf{fv}(P) \cup \mathbf{fv}(N) \subseteq \mathbf{fv}(Q) \cup \mathbf{fv}(M) = \mathbf{fv}(Q \to M)$.

Case 4. $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$

$\mathbf{fv}\,\forall\overrightarrow{\alpha^+}.N \subseteq \mathbf{fv}\,([\overrightarrow{P/\alpha^+}]N) \setminus \overrightarrow{\beta^+}$ here $\overrightarrow{\beta^+}$ is excluded by the premise $\mathbf{fv}\,N \cap \overrightarrow{\beta^+} = \varnothing$

$\subseteq \mathbf{fv}\,M \setminus \overrightarrow{\beta^+}$ by the induction hypothesis, $\mathbf{fv}\,([\overrightarrow{P/\alpha^+}]N) \subseteq \mathbf{fv}\,M$

$\subseteq \mathbf{fv}\,\forall\overrightarrow{\beta^+}.M$

Case 5. The positive cases are symmetric.

$\square$

Corollary 1 (Free Variables of mutual subtypes)**.**

$-$ If $\Gamma \vdash N \simeq_1^\leqslant M$ then $\mathbf{fv}\,N = \mathbf{fv}\,M$,

$+$ If $\Gamma \vdash P \simeq_1^\leqslant Q$ then $\mathbf{fv}\,P = \mathbf{fv}\,Q$

Lemma 2 (Subtypes and supertypes of a variable)**.** *Assuming* $\Gamma \vdash \alpha^-$, $\Gamma \vdash \alpha^+$, $\Gamma \vdash N$, *and* $\Gamma \vdash P$,

$+$ *if* $\Gamma \vdash P \geqslant_1 \alpha^+$ *or* $\Gamma \vdash \alpha^+ \geqslant_1 P$ *then* $P = \exists\overrightarrow{\alpha^-}.\alpha^+$ *(for some potentially empty* $\overrightarrow{\alpha^-}$*)*

$-$ *if* $\Gamma \vdash N \leqslant_1 \alpha^-$ *or* $\Gamma \vdash \alpha^- \leqslant_1 N$ *then* $N = \forall\overrightarrow{\alpha^+}.\alpha^-$ *(for some potentially empty* $\overrightarrow{\alpha^+}$*)*

Proof. We prove by induction on the tree inferring $\Gamma \vdash P \geqslant_1 \alpha^+$ or $\Gamma \vdash \alpha^+ \geqslant_1 P$ or or $\Gamma \vdash N \leqslant_1 \alpha^-$ or $\Gamma \vdash \alpha^- \leqslant_1 N$.
Let us consider which of these judgments the tree is inferring.

Case 1. $\Gamma \vdash P \geqslant_1 \alpha^+$

If the size of the inference tree is 1 then the only rule that can infer it is Rule $(\text{Var}^{+\geqslant_1})$, which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule $(\exists^{\geqslant_1})$. By inverting this rule, $P = \exists\overrightarrow{\alpha^-}.P'$ where P' does not start with $\exists$ and $\Gamma \vdash [\overrightarrow{N/\alpha^-}]P' \geqslant_1 \alpha^+$ for some $\Gamma, \overrightarrow{\beta^-} \vdash N_i$.

By the induction hypothesis, $[\overrightarrow{N/\alpha^-}]P' = \exists\overrightarrow{\beta^-}.\alpha^+$. Notice that P' must be a variable, because P' does not start with $\exists$, nor does it start with $\uparrow$ (otherwise, $[\overrightarrow{N/\alpha^-}]P'$ would also started with $\uparrow$ and would not be equal to $\exists\overrightarrow{\beta^-}.\alpha^+$). Since P' is a *positive* variable, $[\overrightarrow{N/\alpha^-}]P' = P'$, and then $P' = \exists\overrightarrow{\beta^-}.\alpha^+$ means that $P' = \alpha^+$. This way, $P = \exists\overrightarrow{\alpha^-}.P' = \exists\overrightarrow{\alpha^-}.\alpha^+$, as required.

Case 2. $\Gamma \vdash \alpha^+ \geqslant_1 P$

If the size of the inference tree is 1 then the only rule that can infer it is Rule $(\text{Var}^{+\geqslant_1})$, which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule $(\exists^{\geqslant_1})$. By inverting this rule, $P = \exists\overrightarrow{\beta^-}.Q$ where and $\Gamma, \overrightarrow{\beta^-} \vdash \alpha^+ \geqslant_1 Q$.

By the induction hypothesis, $Q = \exists\overrightarrow{\beta^-}'.\alpha^+$. This way, $P = \exists\overrightarrow{\beta^-}.Q = \exists\overrightarrow{\beta^-}.\exists\overrightarrow{\beta^-}'.\alpha^+$, as required.

Case 3. The negative cases ($\Gamma \vdash N \leqslant_1 \alpha^-$ and $\Gamma \vdash \alpha^- \leqslant_1 N$) are proved analogously.

$\square$

Corollary 2 (Variables have no proper subtypes and supertypes)**.** *Assuming that all mentioned types are well-formed in* Γ,

$$\Gamma \vdash P \geqslant_1 \alpha^+ \iff P = \exists\overrightarrow{\beta^-}.\alpha^+ \iff \Gamma \vdash P \simeq_1^\leqslant \alpha^+ \iff P \simeq_1^D \alpha^+$$

$$\Gamma \vdash \alpha^+ \geqslant_1 P \iff P = \exists\overrightarrow{\beta^-}.\alpha^+ \iff \Gamma \vdash P \simeq_1^\leqslant \alpha^+ \iff P \simeq_1^D \alpha^+$$

$$\Gamma \vdash N \leqslant_1 \alpha^- \iff N = \forall\overrightarrow{\beta^+}.\alpha^- \iff \Gamma \vdash N \simeq_1^\leqslant \alpha^- \iff N \simeq_1^D \alpha^-$$

$$\Gamma \vdash \alpha^- \leqslant_1 N \iff N = \forall\overrightarrow{\beta^+}.\alpha^- \iff \Gamma \vdash N \simeq_1^\leqslant \alpha^- \iff N \simeq_1^D \alpha^-$$

Proof. Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \simeq_1^\leqslant \alpha^+$ and $\exists\overrightarrow{\alpha^-}.\alpha^+ \simeq \alpha^+$ and apply lemma 2. Ilya: fix $\square$

Corollary 3 (Transitivity of subtyping)**.** *Assuming the types are well-formed in* Γ,

$-$ *if* $\Gamma \vdash N_1 \leqslant_1 N_2$ *and* $\Gamma \vdash N_2 \leqslant_1 N_3$ *then* $\Gamma \vdash N_1 \leqslant_1 N_3$,

$+$ *if* $\Gamma \vdash P_1 \geqslant_1 P_2$ *and* $\Gamma \vdash P_2 \geqslant_1 P_3$ *then* $\Gamma \vdash P_1 \geqslant_1 P_3$.

Corollary 4 (Transitivity of equivalence)**.** *Assuming the types are well-formed in* Γ,

$-$ *if* $\Gamma \vdash N_1 \simeq_1^\leqslant N_2$ *and* $\Gamma \vdash N_2 \simeq_1^\leqslant N_3$ *then* $\Gamma \vdash N_1 \simeq_1^\leqslant N_3$,

$+$ *if* $\Gamma \vdash P_1 \simeq_1^\leqslant P_2$ *and* $\Gamma \vdash P_2 \simeq_1^\leqslant P_3$ *then* $\Gamma \vdash P_1 \simeq_1^\leqslant P_3$.

4.2 Substitution

Lemma 3 (Substitution strengthening)**.** *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

+ *if $\Gamma_1 \vdash P$ then $[\sigma]P = [\sigma|_{\mathbf{fv}\,P}]P$,*

− *if $\Gamma_1 \vdash N$ then $[\sigma]N = [\sigma|_{\mathbf{fv}\,N}]N$*

Proof. `Ilya: todo` $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Lemma 4 (Substitution preserves subtyping)**.** *Suppose that $\Gamma \vdash \sigma : \Gamma_1$. Then*

+ *if $\Gamma_1 \vdash P$, $\Gamma_1 \vdash Q$, and $\Gamma_1 \vdash P \geqslant_1 Q$ then $\Gamma \vdash [\sigma]P \geqslant_1 [\sigma]Q$*

− *if $\Gamma_1 \vdash N$, $\Gamma_1 \vdash M$, and $\Gamma_1 \vdash N \leqslant_1 M$ then $\Gamma \vdash [\sigma]N \leqslant_1 [\sigma]M$*

Proof. `Ilya: todo` $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Corollary 5 (Substitution preserves equivalence)**.** *Suppose that $\Gamma \vdash \sigma : \Gamma_1$. Then*

+ *if $\Gamma_1 \vdash P$, $\Gamma_1 \vdash Q$, and $\Gamma_1 \vdash P \simeq_1^{\leqslant} Q$ then $\Gamma \vdash [\sigma]P \simeq_1^{\leqslant} [\sigma]Q$*

− *if $\Gamma_1 \vdash N$, $\Gamma_1 \vdash M$, and $\Gamma_1 \vdash N \simeq_1^{\leqslant} M$ then $\Gamma \vdash [\sigma]N \simeq_1^{\leqslant} [\sigma]M$*

4.3 Type well-formedness

Lemma 5 (Well-formedness agrees with substitution)**.** *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

+ $\Gamma, \Gamma_1 \vdash P \Leftrightarrow \Gamma, \Gamma_2 \vdash [\sigma]P$

− $\Gamma, \Gamma_1 \vdash N \Leftrightarrow \Gamma, \Gamma_2 \vdash [\sigma]N$

Proof. `Ilya: todo` $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Corollary 6. *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

+ $\Gamma_1, \Gamma_2 \vdash P \Leftrightarrow \Gamma_2 \vdash [\sigma]P$

− $\Gamma_1, \Gamma_2 \vdash N \Leftrightarrow \Gamma_2 \vdash [\sigma]N$

Lemma 6 (Equivalent Contexts)**.** *In the well-formedness judgment, only used variables matter:*

+ *if $\Gamma_1 \cap \mathbf{fv}\,P = \Gamma_2 \cap \mathbf{fv}\,P$ then $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$,*

− *if $\Gamma_1 \cap \mathbf{fv}\,N = \Gamma_2 \cap \mathbf{fv}\,N$ then $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$.*

Proof. By simple mutual induction on P and Q. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Corollary 7. *Suppose that all the types below are well-formed in Γ and $\Gamma' \subseteq \Gamma$. Then*

+ $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *implies* $\Gamma' \vdash P \iff \Gamma' \vdash Q$

− $\Gamma \vdash N \simeq_1^{\leqslant} M$ *implies* $\Gamma' \vdash N \iff \Gamma' \vdash M$

Proof. From lemma 6 and corollary 1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

4.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\dfrac{}{\mathbf{ord}\ vars\ \mathbf{in}\ N \equiv vars \cap \mathbf{fv}\ N}$	$\dfrac{N \simeq^D_1 M}{\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ M}$	—
Normalization	$\dfrac{}{N \simeq^D_1 \mathbf{nf}\,(N)}$	$\dfrac{N \simeq^D_1 M}{\mathbf{nf}\,(N) = \mathbf{nf}\,(M)}$	—
Equivalence	$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq^D_1 Q}{\Gamma \vdash P \simeq^{\leqslant}_1 Q}$	$\dfrac{\Gamma \vdash P \simeq^{\leqslant}_1 Q}{P \simeq^D_1 Q}$	—
Uppgrade	$\dfrac{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{Q \text{ is sound } \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$	$Q' \text{ is sound}$ $\dfrac{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \models P_1 \vee P_2 = Q}{Q \text{ is sound } \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \models P_1 \vee P_2 = Q}$	$Q' \text{ is sound}$ $\dfrac{\Gamma \models P_1 \vee P_2 = Q}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \text{ is sound} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \widehat{\tau}_i : \Xi \\ [\widehat{\tau}_i]\,Q = P_i \end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)}{\exists(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2) \text{ s.t.}}$ $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$	$(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2) \text{ is sound}$ $\dfrac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\exists \Gamma; \Xi \vdash \widehat{\tau} : \Xi' \text{ s.t. } [\widehat{\tau}]\,Q' = Q}$
Unification (matching)	$\dfrac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\widehat{\sigma} \text{ is sound } \begin{cases} \Gamma \vdash \widehat{\sigma} : \Theta \\ [\widehat{\sigma}]\,P = Q \end{cases}}$	$\dfrac{\exists \text{ sound } \widehat{\sigma}'}{\exists \widehat{\sigma} \text{ s.t. } \Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$	—
Subtyping	$\dfrac{\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}}{\widehat{\sigma} \text{ is sound } \begin{cases} \Gamma \vdash \widehat{\sigma} : \Theta \\ \Gamma \vdash [\widehat{\sigma}]\,N \leqslant_1 M \end{cases}}$	$\dfrac{\exists \text{ sound } \widehat{\sigma}'}{\exists \widehat{\sigma} \text{ s.t. } \Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}}$	—

4.5 Variable Ordering

Definition 4 (Collision free bijection)**.** *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is **collision free on sets** P and Q if and only if*

1. *$\mu(P \cap A) \cap Q = \varnothing$*

2. *$\mu(Q \cap A) \cap P = \varnothing$*

Lemma 7 (Soundness of variable ordering)**.** *Variable ordering extracts precisely used free variables.*

- **ord** $vars\,\mathbf{in}\ N \equiv vars \cap \mathbf{fv}\ N$ *(as sets)*

+ **ord** $vars\,\mathbf{in}\ P \equiv vars \cap \mathbf{fv}\ P$ *(as sets)*

Proof. Straightforward mutual induction on **ord** $vars\,\mathbf{in}\ N = \vec{\alpha}$ and **ord** $vars\,\mathbf{in}\ P = \vec{\alpha}$ □

Corollary 8 (Additivity of ordering)**.** *Variable ordering is additive (in terms of set union) with respect to its first argument.*

- **ord** $(vars_1 \cup vars_2)\,\mathbf{in}\ N \equiv \mathbf{ord}\ vars_1\,\mathbf{in}\ N \cup \mathbf{ord}\ vars_2\,\mathbf{in}\ N$ *(as sets)*

+ **ord** $(vars_1 \cup vars_2)\,\mathbf{in}\ P \equiv \mathbf{ord}\ vars_1\,\mathbf{in}\ P \cup \mathbf{ord}\ vars_2\,\mathbf{in}\ P$ *(as sets)*

Corollary 9 (Weakening of ordering)**.** *Extending the first argument of the ordering with unused variables does not change the result.*

- **ord** $(vars \cap \mathbf{fv}\ N)\,\mathbf{in}\ N = \mathbf{ord}\ vars\,\mathbf{in}\ N$

$+\ \mathbf{ord}\,(vars \cap \mathbf{fv}\,P)\,\mathbf{in}\,P = \mathbf{ord}\,vars\,\mathbf{in}\,P$

Lemma 8 (Distributivity of renaming over variable ordering). *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

$-\ $ *If μ is collision free on vars and $\mathbf{fv}\,N$ then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]N$*

$+\ $ *If μ is collision free on vars and $\mathbf{fv}\,P$ then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]P$*

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^{-}$
let us consider four cases:

    a. $\alpha^{-} \in A$ and $\alpha^{-} \in vars$
        Then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^{-})$

$$
\begin{aligned}
&= [\mu]\alpha^{-} && \text{by Rule } (\mathrm{Var}^{+}_{\in})\\
&= \beta^{-} && \text{for some } \beta^{-} \in B \text{ (notice that } \beta^{-} \in [\mu]vars)\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\beta^{-} && \text{by Rule } (\mathrm{Var}^{+}_{\in}), \text{ because } \beta^{-} \in [\mu]vars\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^{-}
\end{aligned}
$$

    b. $\alpha^{-} \notin A$ and $\alpha^{-} \notin vars$
        Notice that $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^{-}) = \cdot$ by Rule $(\mathrm{Var}^{+}_{\notin})$. On the other hand, $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^{-} = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^{-} = \cdot$ The latter equality is from Rule $(\mathrm{Var}^{+}_{\notin})$, because μ is collision free on vars and $\mathbf{fv}\,N$, so $\mathbf{fv}\,N \ni \alpha^{-} \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

    c. $\alpha^{-} \in A$ but $\alpha^{-} \notin vars$
        Then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^{-}) = \cdot$ by Rule $(\mathrm{Var}^{+}_{\notin})$. To prove that $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^{-} = \cdot$, we apply Rule $(\mathrm{Var}^{+}_{\notin})$. Let us show that $[\mu]\alpha^{-} \notin [\mu]vars$. Since $[\mu]\alpha^{-} = \mu(\alpha^{-})$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^{-}) \notin \mu(A \cap vars) \cup vars$.

      (i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^{-}$, then $x = \alpha^{-}$ by bijectivity of μ, which contradicts with $\alpha^{-} \notin vars$. This way, $\mu(\alpha^{-}) \notin \mu(A \cap vars)$.

      (ii) Since μ is collision free on vars and $\mathbf{fv}\,N$, $\mu(A \cap \mathbf{fv}\,N) \ni \mu(\alpha^{-}) \notin vars$.

    d. $\alpha^{-} \notin A$ but $\alpha^{-} \in vars$
        $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^{-} = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^{-} = \alpha^{-}$. The latter is by Rule $(\mathrm{Var}^{+}_{\in})$, because $\alpha^{-} = [\mu]\alpha^{-} \in [\mu]vars$ since $\alpha^{-} \in vars$. On the other hand, $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^{-}) = [\mu]\alpha^{-} = \alpha^{-}$.

Case 2. $N = {\uparrow}P$
$[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,{\uparrow}P)$

$$
\begin{aligned}
&= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) && \text{by Rule } ({\uparrow})\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P && \text{by the induction hypothesis}\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,{\uparrow}[\mu]P && \text{by Rule } ({\uparrow})\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]{\uparrow}P && \text{by the definition of substitution}\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N
\end{aligned}
$$

Case 3. $N = P \to M$
$[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P \to M)$

$$
\begin{aligned}
&= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)) && \text{where } \mathbf{ord}\,vars\,\mathbf{in}\,P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\,vars\,\mathbf{in}\,M = \vec{\alpha}_2\\
&= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \vec{\alpha}_1)\\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1) && \text{by induction on } \vec{\alpha}_2; \text{ the inductive step is similar to case 1. Notice that } \mu \text{ is}\\
&&& \text{collision free on } \vec{\alpha}_1 \text{ and } \vec{\alpha}_2 \text{ since } \vec{\alpha}_1 \subseteq vars \text{ and } \vec{\alpha}_2 \subseteq \mathbf{fv}\,N\\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)
\end{aligned}
$$

$(\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N) = (\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P \to [\mu]M)$

$$
\begin{aligned}
&= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \vec{\beta}_1)) && \text{where } \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M = \vec{\beta}_2\\
&&& \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2,\\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)
\end{aligned}
$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu]\mathbf{ord}\,vars\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.M$$
$$= [\mu]\mathbf{ord}\,vars\,\mathbf{in}\,M$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M \quad \text{by the induction hypothesis}$$
$$(\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N) = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\forall\overrightarrow{\alpha^+}.M$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.[\mu]M$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M$$

$\square$

Lemma 9 (Ordering is not affected by independent substitutions). *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$, *i.e.* σ *maps variables from* Γ_1 *into types taking free variables from* Γ_2, *and vars is a set of variables disjoint with both* Γ_1 *and* Γ_2. *Then*

– $\mathbf{ord}\,vars\,\mathbf{in}\,[\sigma]N = \mathbf{ord}\,vars\,\mathbf{in}\,N$

+ $\mathbf{ord}\,vars\,\mathbf{in}\,[\sigma]P = \mathbf{ord}\,vars\,\mathbf{in}\,P$

Proof. `Ilya: Should be easy` $\square$

Lemma 10 (Completeness of variable ordering). *Variable ordering is invariant under equivalence. For arbitrary vars,*

– *If* $N \simeq_1^D M$ *then* $\mathbf{ord}\,vars\,\mathbf{in}\,N = \mathbf{ord}\,vars\,\mathbf{in}\,M$ *(as lists)*

+ *If* $P \simeq_1^D Q$ *then* $\mathbf{ord}\,vars\,\mathbf{in}\,P = \mathbf{ord}\,vars\,\mathbf{in}\,Q$ *(as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. $\square$

4.6   Normaliztaion

Lemma 11. *Set of free variables is invariant under equivalence.*

– *If* $N \simeq_1^D M$ *then* $\mathbf{fv}\,N \equiv \mathbf{fv}\,M$ *(as sets)*

+ *If* $P \simeq_1^D Q$ *then* $\mathbf{fv}\,P \equiv \mathbf{fv}\,Q$ *(as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ $\square$

Lemma 12. *Free variables are not changed by the normalization*

– $\mathbf{fv}\,N \equiv \mathbf{fv}\,\mathbf{nf}\,(N)$

+ $\mathbf{fv}\,P \equiv \mathbf{fv}\,\mathbf{nf}\,(P)$

Proof. By straightforward induction on $\mathbf{nf}\,(N) = M$. $\square$

Lemma 13 (Soundness of quantifier normalization).

– $N \simeq_1^D \mathbf{nf}\,(N)$

+ $P \simeq_1^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var$^-$) and (Var$^+$)
By the corresponding equivalence rules.

Case 2. ($\uparrow$), ($\downarrow$), and ($\rightarrow$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall$), i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 11, $\mathbf{fv}\,N \equiv \mathbf{fv}\,N'$. Then by lemma 7, $\overrightarrow{\alpha^{+\prime}} \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$, and thus, $\overrightarrow{\alpha^{+\prime}} \cap \mathbf{fv}\,N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$.
To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^{+\prime}}.N'$, it suffices to provide a bijection $\mu : \overrightarrow{\alpha^{+\prime}} \cap \mathbf{fv}\,N' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. ($\exists$) Same as for case 3.

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$\square$

Corollary 10 (Normalization preserves ordering)**.** *For any vars,*

- $\mathbf{ord}\ vars\,\mathbf{in}\,\mathbf{nf}\ (\,N\,) = \mathbf{ord}\ vars\,\mathbf{in}\ M$

+ $\mathbf{ord}\ vars\,\mathbf{in}\,\mathbf{nf}\ (\,P\,) = \mathbf{ord}\ vars\,\mathbf{in}\ Q$

Proof. Immediately from lemmas 10 and 13. $\square$

Lemma 14 (Distributivity of normalization over substitution)**.** *Normalization of a term distributes over substitution. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$, *i.e.* σ *maps variables from* Γ_1 *into types taking free variables from* Γ_2. *Then*

- $\mathbf{nf}\ ([\sigma]N) = [\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (N)$

+ $\mathbf{nf}\ ([\sigma]P) = [\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (P)$

where $\mathbf{nf}\ (\sigma)$ *means pointwise normalization:* $[\mathbf{nf}\ (\sigma)]\alpha^- = \mathbf{nf}\ ([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

    Case 1. $N = \alpha^-$
      $\mathbf{nf}\ ([\sigma]N) = \mathbf{nf}\ ([\sigma]\alpha^-) = [\mathbf{nf}\ (\sigma)]\alpha^-$.
    $[\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (N) = [\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (\alpha^-) = [\mathbf{nf}\ (\sigma)]\alpha^-$.

    Case 2. $P = \alpha^+$
    Similar to case 1.

    Case 3. If the type is formed by $\to$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \to M$ then
    $\mathbf{nf}\ ([\sigma]N) = \mathbf{nf}\ ([\sigma](P \to M))$

$$
\begin{aligned}
&= \mathbf{nf}\ ([\sigma]P \to [\sigma]M) &&\text{By the congruence of substitution}\\
&= \mathbf{nf}\ ([\sigma]P) \to \mathbf{nf}\ ([\sigma]M) &&\text{By the congruence of normalization, i.e. Rule } (\to)\\
&= [\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (P) \to [\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (M) &&\text{By the induction hypothesis}\\
&= [\mathbf{nf}\ (\sigma)](\mathbf{nf}\ (P) \to \mathbf{nf}\ (M)) &&\text{By the congruence of substitution}\\
&= [\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (P \to M) &&\text{By the congruence of normalization}\\
&= [\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (N)
\end{aligned}
$$

    Case 4. $N = \forall\overrightarrow{\alpha^+}.M$
    $[\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (N) = [\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (\forall\overrightarrow{\alpha^+}.M)$

$$
= [\mathbf{nf}\ (\sigma)]\forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\ (M) \quad \text{Where } \overrightarrow{\alpha^{+\prime}} = \mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ \mathbf{nf}\ (M) = \mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ M \text{ (the latter is by corollary 10)}
$$

    $\mathbf{nf}\ ([\sigma]N) = \mathbf{nf}\ ([\sigma]\forall\overrightarrow{\alpha^+}.M)$

$$
\begin{aligned}
&= \mathbf{nf}\ (\forall\overrightarrow{\alpha^+}.[\sigma]M) &&\text{Assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing\\
&= \forall\overrightarrow{\beta^+}.\mathbf{nf}\ ([\sigma]M) &&\text{Where } \overrightarrow{\beta^+} = \mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ \mathbf{nf}\ ([\sigma]M) = \mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ [\sigma]M \text{ (the latter is by corollary 10)}\\
&= \forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\ ([\sigma]M) &&\text{By lemma 9, } \overrightarrow{\beta^+} = \overrightarrow{\alpha^{+\prime}} \text{ since } \overrightarrow{\alpha^+} \text{ is disjoint with } \Gamma_1 \text{ and } \Gamma_2\\
&= \forall\overrightarrow{\alpha^{+\prime}}.[\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (M) &&\text{By the induction hypothesis}
\end{aligned}
$$

    To show alpha-equivalence of $[\mathbf{nf}\ (\sigma)]\forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\ (M)$ and $\forall\overrightarrow{\alpha^{+\prime}}.[\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (M)$, we can assume that $\overrightarrow{\alpha^{+\prime}} \cap \Gamma_1 = \varnothing$, and $\overrightarrow{\alpha^{+\prime}} \cap \Gamma_2 = \varnothing$.

    Case 5. $P = \exists\overrightarrow{\alpha^-}.Q$
    Same as for case 4.

$\square$

Corollary 11 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that* μ *is a bijection between two sets of variables* $\mu : A \leftrightarrow B$. *Then*

- $\mathbf{nf}\ ([\mu]N) = [\mu]\mathbf{nf}\ (N)$

+ $\mathbf{nf}\ ([\mu]P) = [\mu]\mathbf{nf}\ (P)$

Proof. Immediately from lemma 14, after noticing that $\mathbf{nf}(\mu) = \mu$. □

Lemma 15 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

    − If $N \simeq_1^D M$ then $\mathbf{nf}(N) = \mathbf{nf}(M)$

    + If $P \simeq_1^D Q$ then $\mathbf{nf}(P) = \mathbf{nf}(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

    Case 1. $(\forall^{\simeq_1^P})$

      From the definition of the normalization,

- $\mathbf{nf}(\forall \overrightarrow{\alpha^+}.N) = \forall \overrightarrow{\alpha^{+\prime}}.\mathbf{nf}(N)$ where $\overrightarrow{\alpha^{+\prime}}$ is $\mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}(N)$
- $\mathbf{nf}(\forall \overrightarrow{\beta^+}.M) = \forall \overrightarrow{\beta^{+\prime}}.\mathbf{nf}(M)$ where $\overrightarrow{\beta^{+\prime}}$ is $\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}(M)$

Let us take $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 7 and 12, the domain and the codomain of μ can be written as $\mu : \overrightarrow{\beta^{+\prime}} \leftrightarrow \overrightarrow{\alpha^{+\prime}}$.

To show the alpha-equivalence of $\forall \overrightarrow{\alpha^{+\prime}}.\mathbf{nf}(N)$ and $\forall \overrightarrow{\beta^{+\prime}}.\mathbf{nf}(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}(M) = \mathbf{nf}(N)$ and (ii) $[\mu]\overrightarrow{\beta^{+\prime}} = \overrightarrow{\alpha^{+\prime}}$.

    (i) $[\mu]\mathbf{nf}(M) = \mathbf{nf}([\mu]M) = \mathbf{nf}(N)$. The first equality holds by corollary 11, the second—by the induction hypothesis.

$$
\begin{aligned}
\text{(ii)}\quad [\mu]\overrightarrow{\beta^{+\prime}} &= [\mu]\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}(M) && \text{by the definition of } \overrightarrow{\beta^{+\prime}} \\
&= [\mu]\mathbf{ord}\,(\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}(M) && \text{from lemma 12 and corollary 9} \\
&= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}(M) && \text{by lemma 8, because } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}(M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \\
& && \text{and } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \\
&= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}(N) && \text{since } [\mu]\mathbf{nf}(M) = \mathbf{nf}(N) \text{ is proved} \\
&= \mathbf{ord}\,(\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}(N) && \text{because } \mu \text{ is a bijection between } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \text{ and } \overrightarrow{\beta^+} \cap \mathbf{fv}\,M \\
&= \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}(N) && \text{from lemma 12 and corollary 9} \\
&= \overrightarrow{\alpha^{+\prime}} && \text{by the definition of } \overrightarrow{\alpha^{+\prime}}
\end{aligned}
$$

    Case 2. $(\exists^{\simeq_1^D})$ Same as for case 1.

    Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□

Lemma 16 (Idempotence of normalization)**.** *Normalization is idempotent*

    − $\mathbf{nf}(\mathbf{nf}(N)) = \mathbf{nf}(N)$

    + $\mathbf{nf}(\mathbf{nf}(P)) = \mathbf{nf}(P)$

Proof. By applying lemma 15 to lemma 13. □

Lemma 17. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized.*
    Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then

$$
+\ [\sigma]P \text{ is normal} \iff \begin{cases} \sigma|_{\mathbf{fv}(P)} & \text{is normal} \\ P & \text{is normal} \end{cases}
$$

$$
-\ [\sigma]N \text{ is normal} \iff \begin{cases} \sigma|_{\mathbf{fv}(N)} & \text{is normal} \\ N & \text{is normal} \end{cases}
$$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$

Then N is always normal, and the normality of $\sigma|_{\alpha^-}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$[\sigma](P \to M) \text{ is normal} \iff [\sigma]P \to [\sigma]M \text{ is normal} \qquad \text{by the substitution congruence}$$

$$\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases} \qquad \text{by congruence of normality \texttt{\color{red}Ilya: lemma?}}$$

$$\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}(M)} & \text{is normal} \end{cases} \qquad \text{by the induction hypothesis}$$

$$\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}(P) \cup \mathbf{fv}(M)} & \text{is normal} \end{cases} \iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}(P \to M)} & \text{is normal} \end{cases}$$

Case 3. $N = {\uparrow}P$

By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$[\sigma](\forall\overrightarrow{\alpha^+}.M) \text{ is normal} \iff (\forall\overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} \qquad \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$$

$$\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by the definition of normalization}$$

$$\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by lemma 9}$$

$$\iff \begin{cases} \sigma|_{\mathbf{fv}(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by the induction hypothesis}$$

$$\iff \begin{cases} \sigma|_{\mathbf{fv}(\forall\overrightarrow{\alpha^+}.M)} \text{ is normal} \\ \forall\overrightarrow{\alpha^+}.M \text{ is normal} \end{cases} \qquad \begin{array}{l} \text{since } \mathbf{fv}(\forall\overrightarrow{\alpha^+}.M) = \mathbf{fv}(M); \\ \text{by the definition of normalization} \end{array}$$

Case 5. $P = \ldots$

The positive cases are done in the same way as the negative ones.

$\square$

4.7 Equivalence

Lemma 18 (Type well-formedness is invariant under equivalence)**.** *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,

− *if* $N \simeq_1^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. {\color{red}\texttt{Ilya: todo}} $\square$

Corollary 12 (Normalization preserves well-formedness)**.**

+ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}(P)$,

− $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}(N)$

Proof. Immediately from lemmas 13 and 18. $\square$

Corollary 13 (Normalization preserves well-formedness of substitution)**.**
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}(\sigma) : \Gamma_1$

Lemma 19 (Soundness of equivalence)**.** *Declarative equivalence implies mutual subtyping.*

+ *if* $\Gamma \vdash P$, $\Gamma \vdash Q$, *and* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \simeq_1^\leqslant Q$,

− *if* $\Gamma \vdash N$, $\Gamma \vdash M$, *and* $N \simeq_1^D M$ *then* $\Gamma \vdash N \simeq_1^\leqslant M$.

Proof. We prove it by mutual induction on $P \simeq_1^D Q$ and $N \simeq_1^D M$.

Case 1. $\alpha^- \simeq_1^D \alpha^-$
Then $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$ by Rule (Var$^{-\leqslant_1}$), which immediately implies $\Gamma \vdash \alpha^- \simeq_1^\leqslant \alpha^-$ by Rule ($\simeq_1^\leqslant {}^-$).

Case 2. $\uparrow P \simeq_1^D \uparrow Q$
Then by inversion of Rule ($\uparrow^{\leqslant_1}$), $P \simeq_1^D Q$, and from the induction hypothesis, $\Gamma \vdash P \simeq_1^\leqslant Q$, and (by symmetry) $\Gamma \vdash Q \simeq_1^\leqslant P$.
When Rule ($\uparrow^{\leqslant_1}$) is applied to $\Gamma \vdash P \simeq_1^\leqslant Q$, it gives us $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$; when it is applied to $\Gamma \vdash Q \simeq_1^\leqslant P$, we obtain $\Gamma \vdash \uparrow Q \leqslant_1 \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \simeq_1^\leqslant \uparrow Q$.

Case 3. $P \to N \simeq_1^D Q \to M$
Then by inversion of Rule ($\to^{\leqslant_1}$), $P \simeq_1^D Q$ and $N \simeq_1^D M$. By the induction hypothesis, $\Gamma \vdash P \simeq_1^\leqslant Q$ and $\Gamma \vdash N \simeq_1^\leqslant M$, which means by inversion: (i) $\Gamma \vdash P \geqslant_1 Q$, (ii) $\Gamma \vdash Q \geqslant_1 P$, (iii) $\Gamma \vdash N \leqslant_1 M$, (iv) $\Gamma \vdash M \leqslant_1 N$. Applying Rule ($\to^{\leqslant_1}$) to (i) and (iii), we obtain $\Gamma \vdash P \to N \leqslant_1 Q \to M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \to M \leqslant_1 P \to N$. Together, it implies $\Gamma \vdash P \to N \simeq_1^\leqslant Q \to M$.

Case 4. $\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M$
Then by inversion, there exists bijection $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N)$, such that $N \simeq_1^D [\mu]M$. By the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^\leqslant [\mu]M$. From corollary 5 and the fact that μ is bijective, we also have $\Gamma, \overrightarrow{\beta^+} \vdash [\mu^{-1}]N \simeq_1^\leqslant M$.

Let us construct a subsitution $\overrightarrow{\alpha^+} \vdash \overrightarrow{P}/\overrightarrow{\beta^+} : \overrightarrow{\beta^+}$ by extending μ with arbitrary positive types on $\overrightarrow{\beta^+} \backslash \mathbf{fv}\, M$.

Notice that $[\mu]M = [\overrightarrow{P}/\overrightarrow{\beta^+}]M$, and therefore, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^\leqslant [\mu]M$ implies $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{P}/\overrightarrow{\beta^+}]M \leqslant_1 N$. Then by Rule ($\forall^{\leqslant_1}$), $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \leqslant_1 \forall \overrightarrow{\alpha^+}.N$.

Analogously, we construct the substitution from μ^{-1}, and use it to instantiate $\overrightarrow{\alpha^+}$ in the application of Rule ($\forall^{\leqslant_1}$) to infer $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M$.

This way, $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \leqslant_1 \forall \overrightarrow{\alpha^+}.N$ and $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M$ gives us $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \simeq_1^\leqslant \forall \overrightarrow{\alpha^+}.N$.

Case 5. For the cases of the positive types, the proofs are symmetric.

$\square$

Corollary 14 (Normalization is sound w.r.t. subtyping-induced equivalence)**.**

+ *if* $\Gamma \vdash P$ *then* $\Gamma \vdash P \simeq_1^\leqslant \mathbf{nf}\,(P)$,

− *if* $\Gamma \vdash N$ *then* $\Gamma \vdash N \simeq_1^\leqslant \mathbf{nf}\,(N)$.

Proof. Immediately from lemmas 13 and 19 and corollary 12. $\square$

Lemma 20 (Subtyping induced by disjoint substitutions)**.** *If two disjoint substitutions induce subtyping, they are degenerate (so is the subtyping). Suppose that $\Gamma \vdash \sigma_1 : \Gamma_1$ and $\Gamma \vdash \sigma_2 : \Gamma_1$, where $\Gamma_i \subseteq \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$. Then*

− *assuming* $\Gamma \vdash N$, $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ *implies* $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\, N$

+ *assuming* $\Gamma \vdash P$, $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ *implies* $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\, P$

Proof. Proof by induciton on $\Gamma \vdash N$ (and mutually on $\Gamma \vdash P$).

Case 1. $N = \alpha^-$
Then $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\alpha^- \leqslant_1 [\sigma_2]\alpha^-$. Let us consider the following cases:

    a. $\alpha^- \notin \Gamma_1$ and $\alpha^- \notin \Gamma_2$
    Then $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \alpha^-$ holds immediately, since $[\sigma_i]\alpha^- = [\mathsf{id}]\alpha^- = \alpha^-$ and $\Gamma \vdash \alpha^- \simeq_1^\leqslant \alpha^-$.

    b. $\alpha^- \in \Gamma_1$ and $\alpha^- \in \Gamma_2$
    This case is not possible by assumption: $\Gamma_1 \cap \Gamma_2 = \varnothing$.

    c. $\alpha^- \in \Gamma_1$ and $\alpha^- \notin \Gamma_2$
    Then we have $\Gamma \vdash [\sigma_1]\alpha^- \leqslant_1 \alpha^-$, which by corollary 2 means $\Gamma \vdash [\sigma_1]\alpha^- \simeq_1^\leqslant \alpha^-$, and hence, $\Gamma \vdash \sigma_1 \simeq_1^\leqslant \mathsf{id} : \alpha^-$.
    $\Gamma \vdash \sigma_2 \simeq_1^\leqslant \mathsf{id} : \alpha^-$ holds since $[\sigma_2]\alpha^- = \alpha^-$, similarly to case 1.*a*.

    d. $\alpha^- \notin \Gamma_1$ and $\alpha^- \in \Gamma_2$
    Then we have $\Gamma \vdash \alpha^- \leqslant_1 [\sigma_2]\alpha^-$, which by corollary 2 means $\Gamma \vdash \alpha^- \simeq_1^\leqslant [\sigma_2]\alpha^-$, and hence, $\Gamma \vdash \sigma_2 \simeq_1^\leqslant \mathsf{id} : \alpha^-$.
    $\Gamma \vdash \sigma_1 \simeq_1^\leqslant \mathsf{id} : \alpha^-$ holds since $[\sigma_1]\alpha^- = \alpha^-$, similarly to case 1.*a*.

Case 2. $N = \forall\overrightarrow{\alpha^+}.M$

Then by inversion, $\Gamma, \overrightarrow{\alpha^+} \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\forall\overrightarrow{\alpha^+}.M \leqslant_1 [\sigma_2]\forall\overrightarrow{\alpha^+}.M$. By the congruence of substitution and by the inversion of Rule $(\forall^{\leqslant_1})$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{Q/\alpha^+}][\sigma_1]M \leqslant_1 [\sigma_2]M$, where $\Gamma, \overrightarrow{\alpha^+} \vdash Q_i$. Let us denote the (Kleisli) composition of σ_1 and $\overrightarrow{Q/\alpha^+}$ as σ_1', noting that $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' : \Gamma_1, \overrightarrow{\alpha^+}$, and $\Gamma_1, \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$.

Let us apply the induction hypothesis to M and the substitutions σ_1' and σ_2 with $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_1']M \leqslant_1 [\sigma_2]M$ to obtain:

$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, M \tag{1}$$

$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, M \tag{2}$$

Then $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ holds by strengthening of 2: for any $\beta^\pm \in \mathbf{fv}\, \overrightarrow{\alpha^+}.M = \mathbf{fv}\, M\backslash\overrightarrow{\alpha^+}$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$ is strengthened to $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$, because $\mathbf{fv}\, [\sigma_2]\beta^\pm = \mathbf{fv}\, \beta^\pm = \{\beta^\pm\} \subseteq \Gamma$.

To show that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$, let us take an arbitrary $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M\backslash\overrightarrow{\alpha^+}$.

$$\begin{aligned}
\beta^\pm &= [\text{id}]\beta^\pm &&\text{by definition of id} \\
&\simeq_1^{\leqslant} [\sigma_1']\beta^\pm &&\text{by 1} \\
&= [\overrightarrow{Q/\alpha^+}][\sigma_1]\beta^\pm &&\text{by definition of } \sigma_1' \\
&= [\sigma_1]\beta^\pm &&\text{because } \overrightarrow{\alpha^+} \cap \mathbf{fv}\, [\sigma_1]\beta^\pm \subseteq \overrightarrow{\alpha^+} \cap \Gamma = \varnothing
\end{aligned}$$

This way, $\Gamma \vdash [\sigma_1]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$ for any $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ and thus, $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$.

Case 3. $N = P \to M$

Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1](P \to M) \leqslant_1 [\sigma_2](P \to M)$, then by congruence of substitution, $\Gamma \vdash [\sigma_1]P \to [\sigma_1]M \leqslant_1 [\sigma_2]P \to [\sigma_2]M$, then by inversion $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ and $\Gamma \vdash [\sigma_1]M \leqslant_1 [\sigma_2]M$.

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ and to $\Gamma \vdash [\sigma_1]M \leqslant_1 [\sigma_2]M$, we obtain (respectively):

$$\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, P \tag{3}$$

$$\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, M \tag{4}$$

Noting that $\mathbf{fv}\, (P \to M) = \mathbf{fv}\, P \cup \mathbf{fv}\, M$, we combine eqs. (3) and (4) to conclude: $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, (P \to M)$.

Case 4. $N = \uparrow P$

Then by inversion, $\Gamma \vdash P$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\uparrow P \leqslant_1 [\sigma_2]\uparrow P$, then by congruence of substitution and by inversion, $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$, we obtain $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, P$. Since $\mathbf{fv}\, \uparrow P = \mathbf{fv}\, P$, we can conclude: $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, \uparrow P$.

Case 5. The positive cases are proved symmetrically.

$\square$

Corollary 15 (Substitution cannot induce proper subtypes or supertypes)**.** *Assuming all mentioned types are well-formed in Γ and σ is a substitution $\Gamma \vdash \sigma : \Gamma$,*

$$\Gamma \vdash [\sigma]N \leqslant_1 N \;\Rightarrow\; \Gamma \vdash [\sigma]N \simeq_1^{\leqslant} N \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, N$$

$$\Gamma \vdash N \leqslant_1 [\sigma]N \;\Rightarrow\; \Gamma \vdash N \simeq_1^{\leqslant} [\sigma]N \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, N$$

$$\Gamma \vdash [\sigma]P \geqslant_1 P \;\Rightarrow\; \Gamma \vdash [\sigma]P \simeq_1^{\leqslant} P \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, P$$

$$\Gamma \vdash P \geqslant_1 [\sigma]P \;\Rightarrow\; \Gamma \vdash P \simeq_1^{\leqslant} [\sigma]P \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \text{id} : \mathbf{fv}\, P$$

Lemma 21. *Assuming that the mentioned types (P, Q, N, and M) are well-formed in Γ and that the substitutions (σ_1 and σ_2) have signature $\Gamma \vdash \sigma_i : \Gamma$,*

$+$ *if $\Gamma \vdash [\sigma_1]P \geqslant_1 Q$ and $\Gamma \vdash [\sigma_2]Q \geqslant_1 P$*
then there exists a bijection $\mu : \mathbf{fv}\, P \leftrightarrow \mathbf{fv}\, Q$ such that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, P$ and $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, Q$;

$-$ *if $\Gamma \vdash [\sigma_1]N \leqslant_1 M$ and $\Gamma \vdash [\sigma_2]N \leqslant_1 M$*
then there exists a bijection $\mu : \mathbf{fv}\, N \leftrightarrow \mathbf{fv}\, M$ such that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, N$ and $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, M$.

Proof.

+ Applying σ_2 to both sides of $\Gamma \vdash [\sigma_1]P \geqslant_1 Q$ (by **??**), we have: $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant_1 [\sigma_2]Q$. Composing it with $\Gamma \vdash [\sigma_2]Q \geqslant_1 P$ (by transitivity **??**), we have $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant_1 P$. Then by corollary 15, $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\,P$.

  By a symmetric argument, we also have: $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\,Q$.

  Now, we prove that $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\,P$ and $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\,Q$ implies that σ_1 and σ_1 are (equivalent to) mutually inverse bijections.

  To do so, it suffices to prove that

  (i) for any $\alpha^{\pm} \in \mathbf{fv}\,P$ there exists $\beta^{\pm} \in \mathbf{fv}\,Q$ such that $\Gamma \vdash [\sigma_1]\alpha^{\pm} \simeq_1^{\leqslant} \beta^{\pm}$ and $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$; and

  (ii) for any $\beta^{\pm} \in \mathbf{fv}\,Q$ there exists $\alpha^{\pm} \in \mathbf{fv}\,P$ such that $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$ and $\Gamma \vdash [\sigma_1]\alpha^{\pm} \simeq_1^{\leqslant} \beta^{\pm}$.

  Then the these correspondences between $\mathbf{fv}\,P$ and $\mathbf{fv}\,Q$ are mutually inverse functions, since for any $\beta^{\pm}$ there can be at most one $\alpha^{\pm}$ such that $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$ (and vice versa).

  (i) Let us take $\alpha^{\pm} \in \mathbf{fv}\,P$.

  (a) if $\alpha^{\pm}$ is positive ($\alpha^{\pm} = \alpha^{+}$), from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{+} \simeq_1^{\leqslant} \alpha^{+}$, by corollary 2, we have $[\sigma_2][\sigma_1]\alpha^{+} = \exists \overrightarrow{\beta^{-}}.\alpha^{+}$.
  What shape can $[\sigma_1]\alpha^{+}$ have? It cannot be $\exists \overrightarrow{\alpha^{-}}.{\downarrow}N$ (for potentially empty $\overrightarrow{\alpha^{-}}$), because the outer constructor $\downarrow$ would remain after the substitution σ_2, whereas $\exists \overrightarrow{\beta^{-}}.\alpha^{+}$ does not have $\downarrow$. The only case left is $[\sigma_1]\alpha^{+} = \exists \overrightarrow{\alpha^{-}}.\gamma^{+}$.
  Notice that $\Gamma \vdash \exists \overrightarrow{\alpha^{-}}.\gamma^{+} \simeq_1^{\leqslant} \gamma^{+}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{+} \simeq_1^{\leqslant} \gamma^{+}$. Also notice that $[\sigma_2]\exists \overrightarrow{\alpha^{-}}.\gamma^{+} = \exists \overrightarrow{\beta^{-}}.\alpha^{+}$ implies $\Gamma \vdash [\sigma_2]\gamma^{+} \simeq_1^{\leqslant} \alpha^{+}$.

  (b) if $\alpha^{\pm}$ is negative ($\alpha^{\pm} = \alpha^{-}$) from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{-} \simeq_1^{\leqslant} \alpha^{-}$, by corollary 2, we have $[\sigma_2][\sigma_1]\alpha^{-} = \forall \overrightarrow{\beta^{+}}.\alpha^{-}$.
  What shape can $[\sigma_1]\alpha^{-}$ have? It cannot be $\forall \overrightarrow{\alpha^{+}}.{\uparrow}P$ nor $\forall \overrightarrow{\alpha^{+}}.P \to M$ (for potentially empty $\overrightarrow{\alpha^{+}}$), because the outer constructor ($\to$ or $\uparrow$), remaining after the substitution σ_2, is however absent in the resulting $\forall \overrightarrow{\beta^{+}}.\alpha^{-}$. Hence, the only case left is $[\sigma_1]\alpha^{-} = \forall \overrightarrow{\alpha^{+}}.\gamma^{-}$ Notice that $\Gamma \vdash \gamma^{-} \simeq_1^{\leqslant} \forall \overrightarrow{\alpha^{+}}.\gamma^{-}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{-} \simeq_1^{\leqslant} \gamma^{-}$. Also notice that $[\sigma_2]\forall \overrightarrow{\alpha^{+}}.\gamma^{-} = \forall \overrightarrow{\beta^{+}}.\alpha^{-}$ implies $\Gamma \vdash [\sigma_2]\gamma^{-} \simeq_1^{\leqslant} \alpha^{-}$.

  (ii) The proof is symmetric: We swap P and Q, σ_1 and σ_2, and exploit $\Gamma \vdash [\sigma_1][\sigma_2]\alpha^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$ instead of $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$.

− The proof is symmetric to the positive case.

$\square$

Lemma 22 (Equivalence of polymorphic types).

− *For $\Gamma \vdash \forall \overrightarrow{\alpha^{+}}.N$ and $\Gamma \vdash \forall \overrightarrow{\beta^{+}}.M$,*
  if $\Gamma \vdash \forall \overrightarrow{\alpha^{+}}.N \simeq_1^{\leqslant} \forall \overrightarrow{\beta^{+}}.M$ then there exists a bijection $\mu : \overrightarrow{\beta^{+}} \cap \mathbf{fv}\,M \leftrightarrow \overrightarrow{\alpha^{+}} \cap \mathbf{fv}\,N$ such that $\Gamma, \overrightarrow{\alpha^{+}} \vdash N \simeq_1^{\leqslant} [\mu]N$,

+ *For $\Gamma \vdash \exists \overrightarrow{\alpha^{-}}.P$ and $\Gamma \vdash \exists \overrightarrow{\beta^{-}}.Q$,*
  if $\Gamma \vdash \exists \overrightarrow{\alpha^{-}}.P \simeq_1^{\leqslant} \exists \overrightarrow{\beta^{-}}.Q$ then there exists a bijection $\mu : \overrightarrow{\beta^{-}} \cap \mathbf{fv}\,Q \leftrightarrow \overrightarrow{\alpha^{-}} \cap \mathbf{fv}\,P$ such that $\Gamma, \overrightarrow{\beta^{-}} \vdash P \simeq_1^{\leqslant} [\mu]Q$.

Proof.

− First, by α-conversion, we ensure $\overrightarrow{\alpha^{+}} \cap \mathbf{fv}\,M = \varnothing$ and $\overrightarrow{\beta^{+}} \cap \mathbf{fv}\,N = \varnothing$. By inversion, $\Gamma \vdash \forall \overrightarrow{\alpha^{+}}.N \simeq_1^{\leqslant} \forall \overrightarrow{\beta^{+}}.M$ implies

  1. $\Gamma, \overrightarrow{\beta^{+}} \vdash [\sigma_1]N \leqslant_1 M$ for $\Gamma, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \overrightarrow{\alpha^{+}}$ and
  2. $\Gamma, \overrightarrow{\alpha^{+}} \vdash [\sigma_2]M \leqslant_1 N$ for $\Gamma, \overrightarrow{\alpha^{+}} \vdash \sigma_2 : \overrightarrow{\beta^{+}}$.

  To apply lemma 21, we weaken and rearrange the contexts, and extend the substitutions to act as identity outside of their initial domain:

  1. $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\sigma_1]N \leqslant_1 M$ for $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}}$ and
  2. $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\sigma_2]M \leqslant_1 N$ for $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_2 : \Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}}$.

  Then from lemma 21, there exists a bijection $\mu : \mathbf{fv}\,M \leftrightarrow \mathbf{fv}\,N$ such that $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_2 \simeq_1^{\leqslant} \mu : \mathbf{fv}\,M$ and $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\,N$.

  Let us show that if we restrict the domain of μ to $\overrightarrow{\beta^{+}}$, its range will be contained in $\overrightarrow{\alpha^{+}}$. Let us take $\gamma^{+} \in \overrightarrow{\beta^{+}} \cap \mathbf{fv}\,M$ and assume $[\mu]\gamma^{+} \notin \overrightarrow{\alpha^{+}}$. Then since $\Gamma, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \overrightarrow{\alpha^{+}}$, σ_1 acts as identity outside of $\overrightarrow{\alpha^{+}}$, i.e. $[\sigma_1][\mu]\gamma^{+} = [\mu]\gamma^{+}$. Since $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\,N$, application of σ_1 is equivalent to application of μ^{-1}, then $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\mu^{-1}][\mu]\gamma^{+} \simeq_1^{\leqslant} [\mu]\gamma^{+}$, i.e.

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$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \gamma^+ \simeq^{\leqslant}_1 [\mu]\gamma^+$, which means $\gamma^+ \in \mathbf{fv}\,[\mu]\gamma^+ \subseteq \mathbf{fv}\,N$. By assumption, $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv}\,M$, i.e. $\overrightarrow{\beta^+} \cap \mathbf{fv}\,N \neq \varnothing$, hence contradiction.

By **??**, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq^{\leqslant}_1 \mu|_{\overrightarrow{\beta^+}} : \mathbf{fv}\,M$ implies $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \simeq^{\leqslant}_1 [\mu|_{\overrightarrow{\beta^+}}]M$. By similar reasoning, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \simeq^{\leqslant}_1 [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N$.

This way,

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N \leqslant_1 M \tag{5}$$

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu|_{\overrightarrow{\beta^+}}]M \leqslant_1 N \tag{6}$$

By applying $\mu|_{\overrightarrow{\beta^+}}$ to both sides of 5 (**??**) and contracting $\mu^{-1}|_{\overrightarrow{\alpha^+}} \circ \mu|_{\overrightarrow{\beta^+}} = \mu|^{-1}_{\overrightarrow{\beta^+}} \circ \mu|_{\overrightarrow{\beta^+}} = \mathsf{id}$, we have: $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \leqslant_1 [\mu|_{\overrightarrow{\beta^+}}]M$, which together with 6 means $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \simeq^{\leqslant}_1 [\mu|_{\overrightarrow{\beta^+}}]M$, and by strengthening, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq^{\leqslant}_1 [\mu|_{\overrightarrow{\beta^+}}]M$. Symmetrically, $\Gamma, \overrightarrow{\beta^+} \vdash M \simeq^{\leqslant}_1 [\mu|^{-1}_{\overrightarrow{\beta^+}}]N$.

- $+$ The proof is symmetric to the proof of the negative case.

$\square$

Lemma 23 (Completeness of equivalence). *Mutual subtyping implies declarative equivalence. Assuming all the types below are well-formed in Γ:*

$+$ *if $\Gamma \vdash P \simeq^{\leqslant}_1 Q$ then $P \simeq^D_1 Q$,*

$-$ *if $\Gamma \vdash N \simeq^{\leqslant}_1 M$ then $N \simeq^D_1 M$.*

Proof. $-$ Induction on the sum of sizes of N and M. By inversion, $\Gamma \vdash N \simeq^{\leqslant}_1 M$ means $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash M \leqslant_1 N$. Let us consider the last rule that forms $\Gamma \vdash N \leqslant_1 M$:

Case 1. Rule $(\mathrm{Var}^{-\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
Then $N \simeq^D_1 M$ (i.e. $\alpha^- \simeq^D_1 \alpha^-$) holds immediately by Rule $(\mathrm{Var}^{-\simeq^D_1})$.

Case 2. Rule $(\uparrow^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \,\uparrow P \leqslant_1 \,\uparrow Q$
Then by inversion, $\Gamma \vdash P \simeq^{\leqslant}_1 Q$, and by induction hypothesis, $P \simeq^D_1 Q$. Then $N \simeq^D_1 M$ (i.e. $\uparrow P \simeq^D_1 \,\uparrow Q$) holds by Rule $(\uparrow^{\simeq^D_1})$.

Case 3. Rule $(\rightarrow^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash P \rightarrow N' \leqslant_1 Q \rightarrow M'$
Then by inversion, $\Gamma \vdash P \geqslant_1 Q$ and $\Gamma \vdash N' \leqslant_1 M'$. Notice that $\Gamma \vdash M \leqslant_1 N$ is of the form $\Gamma \vdash Q \rightarrow M' \leqslant_1 P \rightarrow N'$, which by inversion means $\Gamma \vdash Q \geqslant_1 P$ and $\Gamma \vdash M' \leqslant_1 N'$.
This way, $\Gamma \vdash Q \simeq^{\leqslant}_1 P$ and $\Gamma \vdash M' \simeq^{\leqslant}_1 N'$. Then by induction hypothesis, $Q \simeq^D_1 P$ and $M' \simeq^D_1 N'$. Then $N \simeq^D_1 M$ (i.e. $P \rightarrow N' \simeq^D_1 Q \rightarrow M'$) holds by Rule $(\rightarrow^{\simeq^D_1})$.

Case 4. Rule $(\forall^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant_1 \forall\overrightarrow{\beta^+}.M'$
Then by **??**, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \simeq^{\leqslant}_1 \forall\overrightarrow{\beta^+}.M'$ means that there exists a bijection $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv}\,M' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N'$ such that $\Gamma, \overrightarrow{\alpha^+} \vdash [\mu]M' \simeq^{\leqslant}_1 N'$.
Notice that the application of bijection μ to M' does not change its size (which is less than the size of M), hence the induction hypothesis applies. This way, $[\mu]M' \simeq^D_1 N'$ (and by symmetry, $N' \simeq^D_1 [\mu]M'$) holds by induction. Then we apply Rule $(\forall^{\simeq^D_1})$ to get $\forall\overrightarrow{\alpha^+}.N' \simeq^D_1 \forall\overrightarrow{\beta^+}.M'$, i.e. $N \simeq^D_1 M$.

$+$ The proof is symmetric to the proof of the negative case.

$\square$

Corollary 16 (Normalization is complete w.r.t. subtyping-induced equivalence). *Assuming all the types below are well-formed in Γ:*

$+$ *if $\Gamma \vdash P \simeq^{\leqslant}_1 Q$ then $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$,*

$-$ *if $\Gamma \vdash N \simeq^{\leqslant}_1 M$ then $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$.*

Proof. Immediately from lemmas 15 and 23.

$\square$

4.8 Upgrade

Let us consider a type P well-formed in Γ. Some of its Γ-supertypes are also well-formed in a smaller context $\Delta \subseteq \Gamma$. The upgrade is the operation that returns the least of such supertypes.

Lemma 24 (Soundness of Upgrade). *Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$,*
if $\mathbf{upgrade}\,\Gamma \vdash P \,\mathbf{to}\, \Delta = Q$ then

1. $\Delta \vdash Q$

2. $\Gamma \vdash Q \geqslant_1 P$

Proof. By inversion, $\mathbf{upgrade}\,\Gamma \vdash P \,\mathbf{to}\, \Delta = Q$ means that for fresh $\overrightarrow{\beta^{\pm}}$ and $\overrightarrow{\gamma^{\pm}}$, $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vDash [\overrightarrow{\beta^{\pm}/\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}/\alpha^{\pm}}]P = Q$. Then by the soundness of the least upper bound (lemma 29),

1. $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q$,

2. $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 [\overrightarrow{\beta^{\pm}/\alpha^{\pm}}]P$, and

3. $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 [\overrightarrow{\gamma^{\pm}/\alpha^{\pm}}]P$.

$$\begin{aligned}
\mathbf{fv}\,Q &\subseteq \mathbf{fv}\,[\overrightarrow{\beta^{\pm}/\alpha^{\pm}}]P \cap \mathbf{fv}\,[\overrightarrow{\gamma^{\pm}/\alpha^{\pm}}]P && \text{Since by lemma 1, } \mathbf{fv}\,Q \subseteq \mathbf{fv}\,[\overrightarrow{\beta^{\pm}/\alpha^{\pm}}]P \text{ and } \mathbf{fv}\,Q \subseteq \mathbf{fv}\,[\overrightarrow{\gamma^{\pm}/\alpha^{\pm}}]P \\
&\subseteq ((\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) \cup \overrightarrow{\beta^{\pm}}) \cap ((\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) \cup \overrightarrow{\gamma^{\pm}}) \\
&= (\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) \cap (\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) && \text{since } \overrightarrow{\beta^{\pm}} \text{ and } \overrightarrow{\gamma^{\pm}} \text{ are fresh} \\
&= \mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}} \\
&\subseteq \Gamma \backslash \overrightarrow{\alpha^{\pm}} && \text{since } P \text{ is well-formed in } \Gamma \\
&\subseteq \Delta
\end{aligned}$$

This way, by lemma 6, $\Delta \vdash Q$.

Let us apply $\overrightarrow{\alpha^{\pm}/\beta^{\pm}}$—the inverse of the substitution $\overrightarrow{\beta^{\pm}/\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 [\overrightarrow{\beta^{\pm}/\alpha^{\pm}}]P$ and by **??**, get $\Delta, \overrightarrow{\alpha^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash [\overrightarrow{\alpha^{\pm}/\beta^{\pm}}]Q \geqslant_1 P$. Notice that $\Delta \vdash Q$ implies that $\mathbf{fv}\,Q \cap \overrightarrow{\beta^{\pm}} = \varnothing$, then by **??**, $[\overrightarrow{\alpha^{\pm}/\beta^{\pm}}]Q = Q$, and thus $\Delta, \overrightarrow{\alpha^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 P$. By context strengthening, $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q \geqslant_1 P$. $\qquad\square$

Lemma 25 (Completeness and Initiality of Upgrade). *The upgrade returns the least Γ-supertype of P well-formed in Δ. Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$,*
For any Q' such that

1. $\Delta \vdash Q'$ *and*

2. $\Gamma \vdash Q' \geqslant_1 P$,

The result of the upgrade algorithm Q exists ($\mathbf{upgrade}\,\Gamma \vdash P \,\mathbf{to}\, \Delta = Q$) and satisfies $\Delta \vdash Q' \geqslant_1 Q$.

Proof. Let us consider fresh (not intersecting with Γ) $\overrightarrow{\beta^{\pm}}$ and $\overrightarrow{\gamma^{\pm}}$.

If we apply substitution $\overrightarrow{\beta^{\pm}/\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q' \geqslant_1 P$, we have $\Delta, \overrightarrow{\beta^{\pm}} \vdash [\overrightarrow{\beta^{\pm}/\alpha^{\pm}}]Q' \geqslant_1 [\overrightarrow{\beta^{\pm}/\alpha^{\pm}}]P$, which by **??**, since Q' is well-formed in Δ, simplifies to $\Delta, \overrightarrow{\beta^{\pm}} \vdash Q' \geqslant_1 [\overrightarrow{\beta^{\pm}/\alpha^{\pm}}]P$.

Analogously, if we apply substitution $\overrightarrow{\gamma^{\pm}/\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q' \geqslant_1 P$, we have $\Delta, \overrightarrow{\gamma^{\pm}} \vdash Q' \geqslant_1 [\overrightarrow{\gamma^{\pm}/\alpha^{\pm}}]P$.

This way, Q' is a common supertype of $[\overrightarrow{\beta^{\pm}/\alpha^{\pm}}]P$ and $[\overrightarrow{\gamma^{\pm}/\alpha^{\pm}}]P$ in context $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}}$. It means that we can apply the completeness of the least upper bound (lemma 30):

1. there exists Q s.t. $\Gamma \vDash [\overrightarrow{\beta^{\pm}/\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}/\alpha^{\pm}}]P = Q$

2. $\Gamma \vdash Q' \geqslant_1 Q$.

The former means that the upgrade algorithm terminates and returns Q. The latter means that since both Q' and Q are well-formed in Δ, by **??**, $\Delta \vdash Q' \geqslant_1 Q$. $\qquad\square$

4.9 Upper Bounds

Lemma 26 (Decomposition of the quantifier rule). *`Ilya: move somewhere` Whenever the quantifier rule (Rule ($\exists^{\geqslant_1}$) or Rule ($\forall^{\leqslant_1}$)) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.*

- *If $\Gamma \vdash N \leqslant_1 \forall \overrightarrow{\beta^+}.M$ then $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant_1 M$.*

+ *If $\Gamma \vdash P \geqslant_1 \exists \overrightarrow{\beta^-}.Q$ then $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant_1 Q$.*

Lemma 27 (Characterization of the Supertypes). *Let us define the set of upper bounds of a positive type $\mathsf{UB}(P)$ in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists \overrightarrow{\alpha^-}.\beta^+ \mid for\ \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists \overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)\ not\ using\ \overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{\exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} for\ \overrightarrow{\alpha^-},\ M',\ and\ \overrightarrow{N}\ s.t. \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ and\ [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array}\right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Immediately from lemma 2

Case 2. $P = \exists \overrightarrow{\beta^-}.P'$
Then if $\Gamma \vdash Q \geqslant_1 \exists \overrightarrow{\beta^-}.P'$, then by lemma 26, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$, and $\mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule ($\exists^{\geqslant_1}$). This way, $\{Q \mid \Gamma \vdash Q \geqslant_1 \exists \overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$ s.t. $\mathbf{fv}\,(Q) \cap \overrightarrow{\beta^-} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists \overrightarrow{\beta^-}.P')$.

Case 3. $P = \downarrow M$
Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\sharp}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists \overrightarrow{\beta^-}.Q'$
Then the last applied rule to infer $\Gamma \vdash Q \geqslant_1 \downarrow M$ must be Rule ($\downarrow^{\geqslant_1}$), which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq_1^{\leqslant} M$, then by lemma 23 and Rule ($\downarrow^{\simeq_1^D}$), $\downarrow M' \simeq_1^D \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq_1^D \downarrow M\} = \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\sharp}$.
In the other direction, $\downarrow M' \simeq_1^D \downarrow M \Rightarrow \Gamma \vdash \downarrow M' \simeq_1^{\leqslant} \downarrow M$ by lemma 19, since $\Gamma \vdash \downarrow M'$ by lemma 18
$$\Rightarrow \Gamma \vdash \downarrow M' \geqslant_1 \downarrow M \quad \text{by inversion}$$

b. $Q = \exists \overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)
Then the last rule applied to infer $\Gamma \vdash \exists \overrightarrow{\beta^-}.Q' \geqslant_1 \downarrow M$ must be Rule ($\exists^{\geqslant_1}$). Inversion of this rule gives us $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \geqslant_1 \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.a, $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \simeq_1^D \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists \overrightarrow{\beta^-}.\downarrow M' \in \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).
In the other direction, $[\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^D \downarrow M \Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^{\leqslant} \downarrow M$ by lemma 19, since $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M'$ by lemma 18
$$\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \geqslant_1 \downarrow M \quad \text{by inversion}$$
$$\Rightarrow \Gamma \vdash \exists \overrightarrow{\beta^-}.\downarrow M' \geqslant_1 \downarrow M \qquad \text{by Rule } (\exists^{\geqslant_1})$$

$\square$

Lemma 28 (Characterization of the Normalized Supertypes). *For a normalized positive type $P = \mathbf{nf}\,(P)$, let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists \overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)\ not\ using\ \overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{\exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} for\ \overrightarrow{\alpha^-},\ M',\ and\ \overrightarrow{N}\ s.t.\ \mathbf{ord}\ \overrightarrow{\alpha^-}\ \mathbf{in}\ M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ and\ [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \end{array}\right\}$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 P\}.$

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
   Then from lemma 27, $\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\beta^+) \mid \text{ for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$
   $\mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') = \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$

$$
\begin{aligned}
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothesis} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}\,(Q) = \mathbf{fv}\,Q \text{ by lemma 12} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{by lemma 27} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} && \text{by the definition of UB} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} && \text{by lemma 27}
\end{aligned}
$$

Case 3. $P = {\downarrow}M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 1. *Suppose that $\nu : A \to A$ is an idempotent function, P is a predicate on A, $F : A \to B$ is a function. Then*

$$
\begin{aligned}
&\{F(\nu x) \mid x \in A \ s.t. \ P(\nu x)\} = \\
=&\{F(x) \mid x \in A \ s.t. \ \nu x = x \ and \ P(x)\}.
\end{aligned}
$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 2. *For functions F and ν, and predicates P and Q,*

$$
\begin{aligned}
&\{F(\nu x) \mid x \in A \ s.t. \ Q(\nu x) \ and \ P(x)\} = \\
=&\{F(\nu x) \mid x \in A \ s.t. \ Q(\nu x) \ and \ (\exists x' \in A \ s.t. \ P(x') \ and \ \nu x' = \nu x)\}.
\end{aligned}
$$

Observation 3. *There exist positive and negative types well-formed in empty context, hence, a type substitution can be extended to an arbitrary domain (if its values on the domain extension are irrelevant). Specifically, Suppose that $vars_1 \subseteq vars_2$. Then $\Gamma \vdash \sigma|_{vars_1} : vars_1$ implies $\exists\, \sigma'$ s.t. $\Gamma \vdash \sigma' : vars_2$ and $\sigma|_{vars_1} = \sigma'|_{vars_1}$.*

$$\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \downarrow M\} =$$

$$= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \downarrow M)\} \qquad\qquad \text{by lemma 27}$$

$$= \left\{ \mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\text{ and } \overrightarrow{N} \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash N_i,\text{ and } [\overrightarrow{N/\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \qquad \text{by the definition of } \mathsf{UB}$$

$$= \left\{ \mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\text{ and } [\sigma]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \qquad \text{we reassigned the substitution } \overrightarrow{N/\alpha^-} \text{ as } \sigma$$

$$= \left\{ \mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\text{ and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \qquad \text{by lemma 3}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}'.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\, \mathbf{ord}\,\overrightarrow{\alpha^-}\text{ in }M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \qquad \text{by the definition of normalization}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}'.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\, \mathbf{ord}\,\overrightarrow{\alpha^-}\text{ in }M' = \overrightarrow{\alpha^-}' \\ \text{and } \mathbf{nf}\,([\sigma|_{\mathbf{fv}\,M'}]\downarrow M') = \mathbf{nf}\,(\downarrow M) \end{array} \right\} \qquad \begin{array}{l}\text{from lemmas 13 and 15, equivalence of types can be}\\ \text{replaced with the equality of their normal forms}\end{array}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}'.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\, \mathbf{ord}\,\overrightarrow{\alpha^-}\text{ in }M' = \overrightarrow{\alpha^-}' \\ \text{and } [\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})]\downarrow\mathbf{nf}\,(M') = \downarrow\mathbf{nf}\,(M) \end{array} \right\} \qquad \text{by congruence of normalization and lemma 14}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\, \mathbf{ord}\,\overrightarrow{\alpha^-}\text{ in }M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} \qquad \begin{array}{l}\text{by lemma 17, } \downarrow M' \text{ and } \sigma|_{\mathbf{fv}\,M'} \text{ are already normal,}\\ \text{since the result of the substitution is normal;}\\ M \text{ is normal by assumption}\end{array}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ (\exists \sigma' \text{ s.t. } \Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,(\downarrow M')} = \sigma'|_{\mathbf{fv}\,(\downarrow M')}) \\ \mathbf{ord}\,\overrightarrow{\alpha^-}\text{ in }M' = \overrightarrow{\alpha^-}' \text{ and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} \qquad \begin{array}{l}\text{We apply observation 2 (with } \nu\sigma = \sigma|_{\mathbf{fv}\,M'},\text{ and}\\ P(\sigma) = \Gamma \vdash \sigma : \overrightarrow{\alpha^-})\end{array}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv}\,M'} : \overrightarrow{\alpha^-}',\, \mathbf{ord}\,\overrightarrow{\alpha^-}\text{ in }M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} \qquad \begin{array}{l}\text{Notice that}\\ \text{``}\exists \sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,(\downarrow M')} = \sigma'|_{\mathbf{fv}\,(\downarrow M')})\text{''}\\ \text{is equivalent to } \Gamma \vdash \sigma|_{\mathbf{fv}\,(\downarrow M')} : \overrightarrow{\alpha^-}' \text{ (observation 3)}\end{array}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\, \mathbf{ord}\,\overrightarrow{\alpha^-}\text{ in }M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} \qquad \begin{array}{l}\text{We apply observation 1 to the restriction of } \sigma,\text{ and}\\ \text{remove } \sigma|_{\mathbf{fv}\,M'} = \sigma \text{ as it follows from } \Gamma \vdash \sigma : \overrightarrow{\alpha^-}'\end{array}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\, \mathbf{ord}\,\overrightarrow{\alpha^-}\text{ in }M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} \qquad \text{by lemma 6, since } \Gamma, \overrightarrow{\alpha^-} \cap \mathbf{fv}\,M' = \Gamma, \overrightarrow{\alpha^-}' \cap \mathbf{fv}\,M'$$

$$= \left\{ \exists \overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}',\, \mathbf{ord}\,\overrightarrow{\alpha^-}'\text{ in }M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} \qquad \text{We apply observation 1 to the ordering of } \overrightarrow{\alpha^-}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, M',\text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}'\text{ in }M' = \overrightarrow{\alpha^-}', \\ \Gamma \vdash N_i,\, \Gamma, \overrightarrow{\alpha^-}' \vdash M',\text{ and } [\overrightarrow{N/\alpha^-}']\downarrow M' = \downarrow M \end{array} \right\} \qquad \text{By reassigning } \sigma \text{ explicitly as } \overrightarrow{N/\alpha^-}'$$

$$= \mathsf{NFUB}(\downarrow M) \qquad\qquad \text{by definition}$$

$$\square$$

Observation 4. *Upper bounds of a type do not depend on the context as soon as the type are well-formed in it.*
If $\Gamma_1 \vdash M$ and $\Gamma_2 \vdash M$ then $\mathsf{UB}(\Gamma_1 \vdash M) = \mathsf{UB}(\Gamma \vdash M)$ and $\mathsf{NFUB}(\Gamma_1 \vdash M) = \mathsf{NFUB}(\Gamma \vdash M)$

Proof. We prove both inclusions by induction on $\Gamma_1 \vdash M$. Notice that if $[\sigma]M' \simeq_1^D M$ and $\Gamma_2 \vdash M$ then the types from the range of $\sigma|_{\mathbf{fv}\,M'}$ are well-formed in 2 `Ilya:   lemma`. $\square$

Lemma 29 (Soundness of the Least Upper Bound). *For types $\Gamma \vdash P_1$, and $\Gamma \vdash P_2$, if $\Gamma \models P_1 \vee P_2 = Q$ then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Proof. Induction on $\Gamma \models P_1 \vee P_2 = Q$.

Case 1. $\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+$
Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule (Var$^{+\geqslant_1}$).

Case 2. $\Gamma \models \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q$
Then by inversion of $\Gamma \vdash \exists \overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applies to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q$. Then

  (i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,

  (ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$,

  (iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}(Q) \cap \Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} = \mathbf{fv}(Q) \cap \Gamma$ (and then apply lemma 6). The inclusion right-to-left is self-evident. To show $\mathbf{fv}(Q) \cap \Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \subseteq \mathbf{fv}(Q) \cap \Gamma$, we prove that $\mathbf{fv}(Q) \subseteq \Gamma$

$\mathbf{fv}(Q) \subseteq \mathbf{fv}\, P_1 \cap \mathbf{fv}\, P_2$ by lemma 1

                     since $\Gamma \vdash \exists \overrightarrow{\alpha^-}.P_1,$ $\mathbf{fv}(P_1) \subseteq \Gamma, \overrightarrow{\alpha^-} = \Gamma, \overrightarrow{\alpha^-} \backslash \overrightarrow{\beta^-}$

$\subseteq (\Gamma, \overrightarrow{\alpha^-} \backslash \overrightarrow{\beta^-}) \cap (\Gamma, \overrightarrow{\beta^-} \backslash \overrightarrow{\alpha^-})$ (the latter is because by the Barendregt's convention,

                     $\Gamma, \overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing$); similarly, $\mathbf{fv}(P_2) \subseteq \Gamma, \overrightarrow{\beta^-} \backslash \overrightarrow{\alpha^-}$

$\subseteq \Gamma$

To show $\Gamma \vdash Q \geqslant_1 \exists \overrightarrow{\alpha^-}.P_1$, we apply Rule ($\exists^{\geqslant_1}$). Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant_1 P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant_1 \exists \overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \models {\downarrow}N \vee {\downarrow}M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P$. By the inversion, $\Gamma, \cdot \models \mathbf{nf}({\downarrow}N) \overset{a}{\simeq} \mathbf{nf}({\downarrow}M) \dashv (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)$. Then by the soundness of anti-unification (**??**),

  (i) $\Gamma; \Xi \vdash P$, then by **??**,

$$\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]\,P \tag{7}$$

  (ii) $\Gamma; \cdot \vdash \hat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \hat{\tau}_2 : \Xi$. Assuming that $\Xi = \widehat{\beta}_1^-, .., \widehat{\beta}_n^-$, the antiunification solutions $\hat{\tau}_1$ and $\hat{\tau}_2$ can be put explicitly as $\hat{\tau}_1 = (\widehat{\beta}_1^- :\approx N_1, .., \widehat{\beta}_n^- :\approx N_n)$, and $\hat{\tau}_2 = (\widehat{\beta}_1^- :\approx M_1, .., \widehat{\beta}_n^- :\approx M_n)$. Then

$$\hat{\tau}_1 = (\overrightarrow{N/\alpha^-}) \circ (\overrightarrow{\alpha^-/\Xi}) \tag{8}$$

$$\hat{\tau}_2 = (\overrightarrow{M/\alpha^-}) \circ (\overrightarrow{\alpha^-/\Xi}) \tag{9}$$

(iii) $[\hat{\tau}_1]\,Q = P_1$ and $[\hat{\tau}_2]\,Q = P_1$, which, by 8 and 9, means

$$[\overrightarrow{N/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P = \mathbf{nf}({\downarrow}N) \tag{10}$$

$$[\overrightarrow{M/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P = \mathbf{nf}({\downarrow}M) \tag{11}$$

Then $\Gamma \vdash \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-/\Xi}]\,P$ follows directly from 7.

To show $\Gamma \vdash \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 {\downarrow}N$, we apply Rule ($\exists^{\geqslant_1}$), instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{N}$. Then $\Gamma \vdash [\overrightarrow{N/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 {\downarrow}N$ follows from 10 and since $\Gamma \vdash \mathbf{nf}({\downarrow}N) \geqslant_1 {\downarrow}N$ (by corollary 14).

Analogously, instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{M}$, gives us $\Gamma \vdash [\overrightarrow{M/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 {\downarrow}M$ (from 11), and hence, $\Gamma \vdash \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 {\downarrow}M$.

$\square$

Lemma 30 (Completeness and Initiality of the Least Upper Bound). *For types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q$ such that $\Gamma \vdash Q \geqslant_1 P_1$ and $\Gamma \vdash Q \geqslant_1 P_2$, there exists Q' s.t. $\Gamma \models P_1 \vee P_2 = Q'$ and $\Gamma \vdash Q \geqslant_1 Q'$.*

Proof. Induction on the pair (P_1, P_2). From lemma 28, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$. Let us consider the cases of what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_i$).

Case 1. $P_1 = \exists \overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists \overrightarrow{\beta^-}_2.Q_2$, where either $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then $Q \in \mathsf{UB}(\Gamma \vdash \exists \overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{UB}(\Gamma \vdash \exists \overrightarrow{\beta^-}_2.Q_2)$

$\subseteq \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_2 \vdash Q_2)$ from the definition of UB

$= \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_2)$ by observation 4, weakening and exchange

$= \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant_1 Q_1\} \cap \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant_1 Q_2\}$ by lemma 27,

meaning that $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$. Then the next step of the algorithm—the recursive call $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \models Q_1 \vee Q_2 = Q'$ terminates by the induction hypothesis, and moreover, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$. This way, the result of the algorithm is Q', i.e. $\Gamma \models P_1 \vee P_2 = Q'$.

Since both Q and Q' are sound, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$ can be strengthened to $\Gamma \vdash Q \geqslant_1 Q'$ by **??**.

Case 2. $P_1 = \alpha^+$ and $P_2 = \downarrow N$

Then the set of common upper bounds of $\downarrow N$ and α^+ is empty, and thus, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$ gives a contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \downarrow N)$

$= \{\exists \overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists \overrightarrow{\beta^-}.\downarrow M' \mid \cdots\}$ by the definition of UB

$= \varnothing$ since $\alpha^+ \neq \downarrow M'$ for any M'

Case 3. $P_1 = \downarrow N$ and $P_2 = \alpha^+$

Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)

Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \beta^+)$

$= \{\exists \overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists \overrightarrow{\beta^-}.\beta^+ \mid \cdots\}$ by the definition of UB

$= \varnothing$ since $\alpha^+ \neq \beta^+$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Then the algorithm terminates in one step (Rule $(\mathrm{Var}^\vee)$) and the result is α^+, i.e. $\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+$.

Since $Q \in \mathsf{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists \overrightarrow{\alpha^-}.\alpha^+$. Then $\Gamma \vdash \exists \overrightarrow{\alpha^-}.\alpha^+ \geqslant_1 \alpha^+$ by Rule $(\exists^{\geqslant_1})$: $\overrightarrow{\alpha^-}$ can be instantiated with arbitrary negative types (for example $\forall \beta^+.\uparrow \beta^+$), since the substitution for unused variables does not change the term $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule $(\mathrm{Var}^{+\geqslant_1})$.

Case 6. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$

Then on the next step, the algorithm tries to anti-unify $\mathbf{nf}\,(\downarrow M_1)$ and $\mathbf{nf}\,(\downarrow M_2)$. By **??**, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$\mathbf{nf}\,(Q) \in \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,(\downarrow M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,(\downarrow M_2))$

$= \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}\,(M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}\,(M_2))$

$$= \left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \text{and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\, \Gamma, \overrightarrow{\alpha^-} \vdash M',\, \text{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1) \end{array} \right\}$$

$$\cap$$

$$\left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \text{and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \overrightarrow{N}_1,\, \Gamma \vdash \overrightarrow{N}_2,\, \Gamma, \overrightarrow{\alpha^-} \vdash M',\, \text{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2) \end{array} \right\}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \overrightarrow{N}_1 \text{ and } \overrightarrow{N}_2 \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \overrightarrow{N}_1,\, \Gamma \vdash \overrightarrow{N}_2,\, \Gamma, \overrightarrow{\alpha^-} \vdash M',\, [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1),\, \text{and } [\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2) \end{array} \right\}$$

The fact that the latter set is non-empty means that there exist $\overrightarrow{\alpha^-}, M', \overrightarrow{N}_1$ and $\overrightarrow{N}_2$ such that

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$ (notice that M' is normal)

(ii) $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma \vdash \overrightarrow{N}_1$,

(iii) $[\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1)$ and $[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2)$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative anti-unification variable $\widehat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\widehat{\alpha}^-}$. Let us show that $(\overrightarrow{\widehat{\alpha}^-},\ [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M',\ \overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-},\ \overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-})$ is a sound anti-unifier of $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$ in context Γ:

- $\overrightarrow{\widehat{\alpha}^-}$ is negative by construction,
- $\Gamma;\overrightarrow{\widehat{\alpha}^-} \vdash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M'$ because $\Gamma,\overrightarrow{\alpha^-} \vdash {\downarrow}M'$ `Ilya: lemma!`,
- $\Gamma;\cdot \vdash (\overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma;\cdot \vdash (\overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N}_2$,
- $[\overrightarrow{N}_1/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_1) = \mathbf{nf}\,({\downarrow}M_1)$.
- $[\overrightarrow{N}_2/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_2) = \mathbf{nf}\,({\downarrow}M_2)$.

Then by the completeness of the anti-unification (**??**), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it, i.e. $Q' = \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]\,P$, where $(\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$ is the result of the anti-unification of $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$ in context Γ.

Moreover, **??** also says that the found solution is initial, i.e. there exists $\widehat{\tau}$ such that $\Gamma;\Xi \vdash \widehat{\tau} : \overrightarrow{\widehat{\alpha}^-}$ and $[\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}$, (ii) $\widehat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma,\overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi][\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi]\,P$. In particular, from the reflexivity of subtyping: $\Gamma,\overrightarrow{\beta^-} \vdash [\sigma]{\downarrow}M' \geqslant_1 [\overrightarrow{\beta^-}/\Xi]\,P$.

It allows us to show $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant_1 Q'$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.{\downarrow}M' \geqslant_1 \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]\,P$, by applying Rule $(\exists^{\geqslant_1})$, instantiating $\overrightarrow{\alpha^-}$ with respect to σ. Finally, $\Gamma \vdash Q \geqslant_1 Q'$ since $\Gamma \vdash \mathbf{nf}\,(Q) \simeq_1^{\leqslant} Q$, and equivalence implies subtyping by `Ilya: lemma`.

$\square$

4.10 Unification

Lemma 31 (Soundness of Unification)**.**

- $+$ *For normalized P and Q such that $\Gamma;\Theta \vdash P$ and $\Gamma \vdash Q$,*
  if $\Gamma;\Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}$ then $\widehat{\sigma} : \Theta|_{\mathbf{uv}\ P}$, $\widehat{\sigma}$ is equivalence-only, and $[\widehat{\sigma}]\,P = Q$.

- $-$ *For normalized N and M such that $\Gamma;\Theta \vdash N$ and $\Gamma \vdash M$,*
  if $\Gamma;\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}$ then $\widehat{\sigma} : \Theta|_{\mathbf{uv}\ N}$, $\widehat{\sigma}$ is equivalence-only, and $[\widehat{\sigma}]\,N = M$.

Lemma 32 (Completeness of Unification)**.**

- $+$ *For normalized P and Q such that $\Gamma;\Theta \vdash P$ and $\Gamma \vdash Q$,*
  assume there exists $\widehat{\sigma} : \Theta|_{\mathbf{uv}\ P}$ such that $[\widehat{\sigma}]\,P = Q$. Then $\Gamma;\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}$.

- $-$ *For normalized N and M such that $\Gamma;\Theta \vdash N$ and $\Gamma \vdash M$,*
  assume there exists $\widehat{\sigma} : \Theta|_{\mathbf{uv}\ N}$ such that $[\widehat{\sigma}]\,N = M$. Then $\Gamma;\Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}$.

4.11 Anti-unification

Lemma 33 (Soundness of Anti-Unification)**.**

- $+$

- $-$

Lemma 34 (Completeness of Anti-Unification)**.**

- $+$

- $-$

4.12 Solution Merge

Lemma 35 (Soundness of Solution Merge)**.** *Suppose that $\widehat{\sigma}_1 : \Theta_1$ and $\widehat{\sigma}_2 : \Theta_2$ and $\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2$ is defined. Then*

1. $\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 : \Theta_1 \cup \Theta_2$,

2. $\Theta_1 \vdash \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_1$, and

3. $\Theta_2 \vdash \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_2$.

Lemma 36 (Completeness and Initiality of Solution Merge)**.** *Suppose that $\widehat{\sigma}_1 : \Theta_1$ and $\widehat{\sigma}_2 : \Theta_2$ and there exists $\widehat{\sigma}$ such that $\Theta_1 \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}$ and $\Theta_2 \vdash \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}$. Then $\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2$ is defined and $\Theta_1 \cup \Theta_2 \vdash \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}$.*

Lemma 37 (Solution Weakening is Monotonous)**.** *if $\widehat{\sigma}_1 : \Theta$ and $\widehat{\sigma}_2 : \Theta$ and $\widehat{\sigma}_1 \subseteq \widehat{\sigma}_2$ then $\Theta \vdash \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_1$.*

Lemma 38 (Solution Weakening is transitive)**.** *Assuming $\widehat{\sigma}_1 : \Theta$, $\widehat{\sigma}_2 : \Theta$ and $\widehat{\sigma}_3 : \Theta$, if $\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2$ and $\Theta \vdash \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_3$ then $\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_3$.*

4.13 Subtyping Algorithm

Theorem 1 (Soundness of Subtyping Algorithm)**.**

− *If $\Gamma \vdash^{\sqsupseteq} \Theta$, $\Gamma \vdash M$, and $\Gamma; \Theta \vdash N$ then*
  $\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}$ *implies* $\widehat{\sigma} : \Theta|_{\mathbf{uv}\ N}$ *and for any $\widehat{\sigma}'$ such that $\Theta \vdash \widehat{\sigma}' \Rightarrow \widehat{\sigma}$, $\Gamma \vdash [\widehat{\sigma}'] N \leqslant_1 M$*

+ *If $\Gamma \vdash^{\sqsupseteq} \Theta$, $\Gamma \vdash Q$, and $\Gamma; \Theta \vdash P$ then*
  $\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}$ *implies* $\widehat{\sigma} : \Theta|_{\mathbf{uv}\ P}$ *and for any $\widehat{\sigma}'$ such that $\Theta \vdash \widehat{\sigma}' \Rightarrow \widehat{\sigma}$, $\Gamma \vdash [\widehat{\sigma}'] P \geqslant_1 Q$.*

Proof. We prove it by induction on $\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}$ (and mutually, on $\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}$). Let us consider the last rule to infer this judgment.

Case 1. $\Gamma; \Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot$
Then $\mathbf{uv}\, \alpha^- = \varnothing$, and $\widehat{\sigma} = \cdot : \cdot$ satisfies $\widehat{\sigma} : \Theta|_\varnothing$. Since $\mathbf{uv}\, \alpha^- = \varnothing$, application of any unification solution $\widehat{\sigma}'$ does not change α^-, i.e. $[\widehat{\sigma}'] \alpha^- = \alpha^-$. Therefore, $\Gamma \vdash [\widehat{\sigma}'] \alpha^- \leqslant_1 \alpha^-$ holds by Rule $(\mathrm{Var}^{-\leqslant_1})$.

Case 2. $\Gamma; \Theta \models {\uparrow}P \leqslant {\uparrow}Q \dashv \widehat{\sigma}$
Then the next step of the algorithm is the unification of $\mathbf{nf}\,(P)$ and $\mathbf{nf}\,(Q)$. By the soundness of the unification algorithm (lemma 31), it returns an equivalence-only solution $\widehat{\sigma}$ such that $\widehat{\sigma} : \mathbf{uv}\ P$. By **??**, since $\widehat{\sigma}$ is equivalence-only and $\Gamma \vdash^{\sqsupseteq} \Theta$, $\Theta \vdash \widehat{\sigma}' \Rightarrow \widehat{\sigma}$ means $\Gamma \vdash \widehat{\sigma}' \simeq_1^{\leqslant} \widehat{\sigma} : \mathbf{uv}\ P$ as substitutions.

$[\widehat{\sigma}]\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$ implies $\Gamma \vdash [\widehat{\sigma}]\mathbf{nf}\,(P) \simeq_1^{\leqslant} \mathbf{nf}\,(Q)$, and then $\Gamma \vdash [\widehat{\sigma}']\mathbf{nf}\,(P) \simeq_1^{\leqslant} \mathbf{nf}\,(Q)$. Ilya: add lemmas

Let us rewrite the left-hand side and the right-hand side of $\Gamma \vdash [\widehat{\sigma}']\mathbf{nf}\,(P) \simeq_1^{\leqslant} \mathbf{nf}\,(Q)$ by transitivity of equivalence (corollary 4). By corollaries 5 and 14, $\Gamma \vdash [\widehat{\sigma}']\mathbf{nf}\,(P) \simeq_1^{\leqslant} [\widehat{\sigma}']P$. By corollary 14, $\Gamma \vdash \mathbf{nf}\,(Q) \simeq_1^{\leqslant} Q$. This way, we have $\Gamma \vdash [\widehat{\sigma}']P \simeq_1^{\leqslant} Q$. Then by Rule $({\uparrow}^{\leqslant_1})$ and congruence of substitution, $\Gamma \vdash [\widehat{\sigma}']{\uparrow}P \leqslant_1 {\uparrow}Q$.

Case 3. $\Gamma; \Theta \models P \to N' \leqslant Q \to M' \dashv \widehat{\sigma}$
The next step of the algorithm is two recursive calls: $\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}_1$ and $\Gamma; \Theta \models N' \leqslant M' \dashv \widehat{\sigma}_2$. By the induction hypothesis,

1. $\widehat{\sigma}_1 : \Theta|_{\mathbf{uv}\ P}$ and $\Gamma \vdash [\widehat{\sigma}_1']P \geqslant_1 Q$ for any $\widehat{\sigma}_1'$ s.t. $\Theta \vdash \widehat{\sigma}_1' \Rightarrow \widehat{\sigma}_1$
2. $\widehat{\sigma}_2 : \Theta|_{\mathbf{uv}\ N'}$ and $\Gamma \vdash [\widehat{\sigma}_2']N' \leqslant_1 M'$ for any $\widehat{\sigma}_2'$ s.t. $\Theta \vdash \widehat{\sigma}_2' \Rightarrow \widehat{\sigma}_2$

Then the algorithm merges two unification solutions $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$. By lemma 35, since $\mathbf{uv}\ P \cup \mathbf{uv}\ N' = \mathbf{uv}\,(P \to N')$, we have $\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 : \Theta|_{\mathbf{uv}\,(P \to N')}$, and also $\Theta \vdash \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_1$ and $\Theta \vdash \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_2$. By the transitivity of solution weakening (lemma 38), $\Theta \vdash \widehat{\sigma}' \Rightarrow \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2$ implies $\Theta \vdash \widehat{\sigma}' \Rightarrow \widehat{\sigma}_1$ and $\Theta \vdash \widehat{\sigma}' \Rightarrow \widehat{\sigma}_2$.

The application of the induction hypothesis gives us $\Gamma \vdash [\widehat{\sigma}']P \geqslant_1 Q$ and $\Gamma \vdash [\widehat{\sigma}']N' \leqslant_1 M'$. Finally, by Rule $(\to^{\leqslant_1})$, $\Gamma \vdash [\widehat{\sigma}'](P \to N') \leqslant_1 Q \to M'$.

Case 4. $\Gamma; \Theta \models \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.M' \dashv \widehat{\sigma}$ s.t. either $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is not empty
Then the algorithm creates fresh unification variables $\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}$, substitutes the old $\overrightarrow{\alpha^+}$ with them in N', and makes the recursive call: $\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}'$, returning as the resulting solution $\widehat{\sigma} = \widehat{\sigma}' \backslash \overrightarrow{\widehat{\alpha}^+}$.

Notice that $\Gamma, \overrightarrow{\beta^+} \vdash^{\sqsupseteq} \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}}$, $\Gamma, \overrightarrow{\beta^+} \vdash M'$, and $\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vdash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N$, so the induction hypothesis is applicable, that is $\widehat{\sigma}' : \Theta|_{\mathbf{uv}\ [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N'}$ and $\Gamma, \overrightarrow{\beta^+} \vdash [\widehat{\sigma}_2'][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N' \leqslant_1 M'$ for any $\widehat{\sigma}_2'$ s.t. $\Theta \vdash \widehat{\sigma}_2' \Rightarrow \widehat{\sigma}'$.

Since the domain of $\hat{\sigma}'$ is $\mathbf{uv}\,[\overrightarrow{\widehat{\alpha^+}/\alpha^+}]\,N'$, the domain of $\hat{\sigma} = \hat{\sigma}'\backslash\overrightarrow{\widehat{\alpha^+}}$ is $\mathbf{uv}\,[\overrightarrow{\widehat{\alpha^+}/\alpha^+}]\,N\backslash\overrightarrow{\widehat{\alpha^+}} = \mathbf{uv}\,N' = \mathbf{uv}\,\forall\overrightarrow{\alpha^+}.N'$, this way, $\hat{\sigma} : \Theta|_{\mathbf{uv}\,\forall\overrightarrow{\alpha^+}.N'}$, as required.

It is left to show that $\Gamma \vdash [\hat{\sigma}_2]\forall\overrightarrow{\alpha^+}.N' \leqslant_1 \forall\overrightarrow{\beta^+}.M'$ for any $\hat{\sigma}_2$ s.t. $\Theta \vdash \hat{\sigma}_2 \Rightarrow \hat{\sigma}$. Let us consider an arbitrary such $\hat{\sigma}_2$. Let us construct $\hat{\sigma}_2'$, extending $\hat{\sigma}_2$ to the domain $\mathbf{uv}\,[\overrightarrow{\widehat{\alpha^+}/\alpha^+}]\,N'$ with the values of $\hat{\sigma}'$, i.e. $\hat{\sigma}_2' : \Theta|_{\mathbf{uv}\,[\overrightarrow{\widehat{\alpha^+}/\alpha^+}]\,N'}$, and

$$\hat{\sigma}_2'(\widehat{\beta^\pm}) = \begin{cases} \hat{\sigma}_2(\widehat{\beta^\pm}) & \text{if } \widehat{\beta^\pm} \in \mathbf{uv}\,N' \\ \hat{\sigma}'(\widehat{\beta^\pm}) & \text{if } \widehat{\beta^\pm} \in \overrightarrow{\widehat{\alpha^+}} \end{cases}$$

, where the application of the unification solution to a variable returns the corresponding *unification entry*. Notice that $\hat{\sigma}_2'|_{\mathbf{uv}\,N'} = \hat{\sigma}_2$. It is easy to see that $\Theta \vdash \hat{\sigma}_2' \Rightarrow \hat{\sigma}'$:

1. if $\widehat{\beta^\pm} \in \mathbf{uv}\,N'$ then $\hat{\sigma}_2'(\widehat{\beta^\pm}) = \hat{\sigma}_2(\widehat{\beta^\pm}) \Rightarrow \hat{\sigma}(\widehat{\beta^\pm}) = \hat{\sigma}'(\widehat{\beta^\pm})$;

2. it $\widehat{\beta^\pm} \in \overrightarrow{\widehat{\alpha^+}}$ then $\hat{\sigma}_2'(\widehat{\beta^\pm}) = \hat{\sigma}'(\widehat{\beta^\pm}) \Rightarrow \hat{\sigma}'(\widehat{\beta^\pm})$,

which means that the induction hypothesis can be applied to $\hat{\sigma}_2'$, i.e. $\Gamma, \overrightarrow{\beta^+} \vdash [\hat{\sigma}_2'][\overrightarrow{\widehat{\alpha^+}/\alpha^+}]\,N' \leqslant_1 M'$.
Notice that $[\hat{\sigma}_2'][\overrightarrow{\widehat{\alpha^+}/\alpha^+}]\,N' = [\hat{\sigma}_2'|_{\overrightarrow{\widehat{\alpha^+}}} \circ \overrightarrow{\widehat{\alpha^+}/\alpha^+}][\hat{\sigma}_2'|_{\mathbf{uv}\,N'}]\,N'$ by substitution properties Ilya: todo

$$= [\hat{\sigma}_2'|_{\overrightarrow{\widehat{\alpha^+}}} \circ \overrightarrow{\widehat{\alpha^+}/\alpha^+}][\hat{\sigma}_2]\,N' \qquad \text{since } \hat{\sigma}_2'|_{\mathbf{uv}\,N'} = \hat{\sigma}_2.$$

Also notice that the domain of $\hat{\sigma}_2'|_{\overrightarrow{\widehat{\alpha^+}}} \circ \overrightarrow{\widehat{\alpha^+}/\alpha^+}$ is $\overrightarrow{\alpha^+}$, and the range is $\Gamma, \overrightarrow{\beta^+}$, i.e. $\Gamma, \overrightarrow{\beta^+} \vdash \hat{\sigma}_2'|_{\overrightarrow{\widehat{\alpha^+}}} \circ \overrightarrow{\widehat{\alpha^+}/\alpha^+} : \overrightarrow{\alpha^+}$, which means that we can apply Rule $(\forall^{\leqslant_1})$ to $\Gamma, \overrightarrow{\beta^+} \vdash [\hat{\sigma}_2'|_{\overrightarrow{\widehat{\alpha^+}}} \circ \overrightarrow{\widehat{\alpha^+}/\alpha^+}][\hat{\sigma}_2]\,N' \leqslant_1 M'$ to get $\Gamma \vdash [\hat{\sigma}_2]\forall\overrightarrow{\alpha^+}.N' \leqslant_1 \forall\overrightarrow{\beta^+}.M'$, as required.

Case 5. $\Gamma; \Theta \vDash \widehat{\alpha^+} \geqslant P' \dashv (\widehat{\alpha^+} :\geqslant Q')$ where $\widehat{\alpha^+}\{\Delta\} \in \Theta$ and $\mathbf{upgrade}\,\Gamma \vdash P'\,\mathbf{to}\,\Delta = Q'$

Notice that $\widehat{\alpha^+}\{\Delta\} \in \Theta$ and $\Gamma \vdash^{\sqsupseteq} \Theta$ implies $\Gamma = \Delta, \overrightarrow{\alpha^\pm}$ for some $\overrightarrow{\alpha^\pm}$, hence, the soundness of upgrade (lemma 24) is applicable:

1. $\Delta \vdash Q'$ and
2. $\Gamma \vdash Q \geqslant_1 P$.

Since $\widehat{\alpha^+}\{\Delta\} \in \Theta|_{\mathbf{uv}\,\widehat{\alpha^+}}$ and $\Delta \vdash Q'$, it is clear that $(\widehat{\alpha^+} :\geqslant Q') : \Theta|_{\mathbf{uv}\,\widehat{\alpha^+}}$.

It is left to show that $\Gamma \vdash [\hat{\sigma}']\widehat{\alpha^+} \geqslant_1 P'$ for any $\hat{\sigma}'$ s.t. $\Theta \vdash \hat{\sigma}' \Rightarrow (\widehat{\alpha^+} :\geqslant Q')$. The latter weakening statement means that either $\hat{\sigma}' \ni \widehat{\alpha^+} :\geqslant Q''$ or $\hat{\sigma}' \ni (\widehat{\alpha^+} :\approx Q'')$ for $\Delta \vdash Q'' \geqslant_1 Q'$. In any case, $\Delta \vdash [\hat{\sigma}']\widehat{\alpha^+} \geqslant_1 Q$. By weakening the context to Γ and combining this judgment transitivity (**??**) with $\Gamma \vdash Q \geqslant_1 P$, we have $\Gamma \vdash [\hat{\sigma}']\widehat{\alpha^+} \geqslant_1 P$, as required.

Case 6. For the other positive cases, the proof is symmetric to the corresponding negative cases.

$\square$

Notice that the induction statement in following theorem is not symmetric for the positive and negative cases.

Theorem 2 (Completeness and Initiality of the Subtyping Algorithm)**.**

$-$ *Suppose that $\Gamma \vdash^{\sqsupseteq} \Theta$, $\Gamma \vdash M$, $\Gamma; \Theta \vdash N$, N does not contain negative unification variables ($\widehat{\alpha^-} \notin \mathbf{uv}\,N$) and there exists $\hat{\sigma} : \Theta|_{\mathbf{uv}\,N}$ such that $\Gamma \vdash [\hat{\sigma}]\,N \leqslant_1 M$. Then there exists $\hat{\sigma}'$ such that $\Gamma; \Theta \vDash N \leqslant M \dashv \hat{\sigma}'$ and $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'$.*

$+$ *Suppose that $\Gamma \vdash^{\sqsupseteq} \Theta$, $\Gamma \vdash Q$ and $\Gamma; \Theta \vdash P$ and there exists $\hat{\sigma} : \Theta|_{\mathbf{uv}\,P}$ such that $\Gamma \vdash [\hat{\sigma}]\,P \geqslant_1 Q$. Then there exists $\hat{\sigma}'$ such that $\Gamma; \Theta \vDash P \geqslant Q \dashv \hat{\sigma}'$ and $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'$.*

Proof. We prove it by induction on $\Gamma \vdash [\hat{\sigma}]\,N \leqslant_1 M$ (and mutually, on $\Gamma \vdash [\hat{\sigma}]\,P \geqslant_1 Q$). Let us consider the last rule used in the derivation of $\Gamma \vdash [\hat{\sigma}]\,N \leqslant_1 M$ or $\Gamma \vdash [\hat{\sigma}]\,P \geqslant_1 Q$, but first consider the base case for the substitution $[\hat{\sigma}]\,P$:

Case 1. $P = \exists\overrightarrow{\beta^-}.\widehat{\alpha^+}$ (for potentially empty $\overrightarrow{\beta^+}$)

Then by assumption, there exists $\hat{\sigma} : \Theta|_{\mathbf{uv}\,P}$ such that $\Gamma \vdash \exists\overrightarrow{\beta^-}.[\hat{\sigma}]\widehat{\alpha^+} \geqslant_1 Q$. $\hat{\sigma} : \Theta|_{\mathbf{uv}\,\widehat{\alpha^+}}$ means that $\hat{\sigma}$ is either $(\widehat{\alpha^+} :\approx P')$ or $(\widehat{\alpha^+} :\geqslant P')$, where $\Delta \vdash P'$ and $\widehat{\alpha^+}\{\Delta\} \in \Theta$. $\Gamma \vdash \exists\overrightarrow{\beta^-}.[\hat{\sigma}]\widehat{\alpha^+} \geqslant_1 Q$ means that $\Gamma \vdash P' \geqslant_1 Q$ because multiple inversions of Rule $(\exists^{\geqslant_1})$ gives us $\Gamma \vdash [\hat{\sigma}]\widehat{\alpha^+} \geqslant_1 Q$ since $\overrightarrow{\beta^-} \cap \mathbf{fv}\,[\hat{\sigma}]\widehat{\alpha^+} = \varnothing$.

In the algorithm, after multiple applications of Rule $(\exists^{\geqslant})$ the type $\exists\overrightarrow{\beta^-}.\widehat{\alpha^+}$ is reduced to $\widehat{\alpha^+}$. Next, the algorithm tries to apply Rule $(\mathrm{UVar}^{\geqslant})$ and the resulting solution is $\hat{\sigma}' = (\widehat{\alpha^+} :\geqslant Q')$ where $\mathbf{upgrade}\,\Gamma \vdash Q\,\mathbf{to}\,\Delta = Q'$.

Why does the upgrade procedure terminates? Because P' satisfies the pre-conditions of the completeness of the upgrade (lemma 25) :

- $\Delta \vdash P'$ because $P' = [\hat{\sigma}]\widehat{\alpha^+}$, $\hat{\sigma} : \Theta|_{\mathbf{uv}\,P}$ and $\widehat{\alpha^+}\{\Delta\} \in \Theta|_{\mathbf{uv}\,P}$,

- $\Gamma \vdash P' \geqslant_1 Q$ as noted before

Moreover, completeness of the upgrade also gives us $\Gamma \vdash P' \geqslant_1 Q'$ and further, we strengthen it to $\Delta \vdash P' \geqslant_1 Q'$ (since by the soundness of the upgrade (lemma 24), $\Delta \vdash Q'$).

It means that $\Theta \vdash (\widehat{\alpha}^+ :\approx P') \Rightarrow (\widehat{\alpha}^+ :\geqslant Q')$ and $\Theta \vdash (\widehat{\alpha}^+ :\geqslant P') \Rightarrow (\widehat{\alpha}^+ :\geqslant Q')$, which means that in any case, $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}'$.

Case 2. $\Gamma \vdash [\widehat{\sigma}] N \leqslant_1 M$ is derived by Rule $(\mathrm{Var}^{-\leqslant_1})$, i.e. $[\widehat{\sigma}] N = M = \alpha^- = N$ (the latter equality holds because $N \neq \widehat{\alpha}^-$).

Notice that $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, \alpha^-} = \widehat{\sigma} : \Theta|_\varnothing = \cdot$.

The algorithm applies Rule $(\mathrm{Var}^{-\leqslant})$ and infers $\widehat{\sigma}' = \cdot$, i.e. $\Gamma; \Theta \vDash \alpha^- \leqslant \alpha^- \dashv \cdot$.

Since $\Theta \vdash \cdot \Rightarrow \cdot$, we have $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}'$.

Case 3. $\Gamma \vdash [\widehat{\sigma}] N \leqslant_1 M$ is derived by Rule $(\uparrow^{\leqslant_1})$, i.e. $[\widehat{\sigma}] N = \uparrow[\widehat{\sigma}] P$ (since the substitution $[\widehat{\sigma}] N$ must preserve the top-level constructor of $N \neq \widehat{\alpha}^-$), and $M = \uparrow Q$, and by inversion, $\Gamma \vdash [\widehat{\sigma}] P \simeq_1^{\leqslant} Q$.

Since both types start with $\uparrow$, the algorithm tries to apply Rule $(\uparrow^{\leqslant})$: $\Gamma; \Theta \vDash \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}'$. The premise of this rule is the unification of $\mathbf{nf}\,(P)$ and $\mathbf{nf}\,(Q)$: $\Gamma; \Theta \vDash \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}'$. Let us show that the unification successfully terminates and returns $\widehat{\sigma}' = \mathbf{nf}\,(\widehat{\sigma})$.

To demonstrate that the unification terminates, we apply the completeness of the unification algorithm (lemma 32). In order to do that, we need to provide a unifier of $\mathbf{nf}\,(P)$ and $\mathbf{nf}\,(Q)$. Thankfully, $\mathbf{nf}\,(\widehat{\sigma})$ does it.

- $\mathbf{nf}\,(P)$ and $\mathbf{nf}\,(Q)$ are normalized
- $\Gamma; \Theta \vdash \mathbf{nf}\,(P)$ because $\Gamma; \Theta \vdash P$ (**??**)
- $\Gamma \vdash \mathbf{nf}\,(Q)$ because $\Gamma \vdash Q$ (corollary 12)
- $\mathbf{nf}\,(\widehat{\sigma}) : \Theta|_{\mathbf{uv}\,\mathbf{nf}\,(P)}$ because $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, P}$ (**??**)
- $\Gamma \vdash [\widehat{\sigma}] P \simeq_1^{\leqslant} Q \Rightarrow [\widehat{\sigma}] P \simeq_1^{D} Q$ by lemma 23
  $\Rightarrow \mathbf{nf}\,([\widehat{\sigma}] P) = \mathbf{nf}\,(Q)$ by lemma 15
  $\Rightarrow [\mathbf{nf}\,(\widehat{\sigma})]\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$ by lemma 14

Then by the completeness of the unification, $\Gamma; \Theta \vDash \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \mathbf{nf}\,(\widehat{\sigma})$. This way, the subtyping algorithm terminates and the resulting solution is $\mathbf{nf}\,(\widehat{\sigma})$.

It is left to note that $\Theta \vdash \widehat{\sigma} \Rightarrow \mathbf{nf}\,(\widehat{\sigma})$, by **??**, since $\Theta \vdash \widehat{\sigma} \simeq \mathbf{nf}\,(\widehat{\sigma})$

Case 4. $\Gamma \vdash [\widehat{\sigma}] N \leqslant_1 M$ is derived by Rule $(\rightarrow^{\leqslant_1})$, i.e. $[\widehat{\sigma}] N = [\widehat{\sigma}] P \rightarrow [\widehat{\sigma}] N'$ and $M = Q \rightarrow M'$, and by inversion, $\Gamma \vdash [\widehat{\sigma}] P \geqslant_1 Q$ and $\Gamma \vdash [\widehat{\sigma}] N' \leqslant_1 M'$.

Let us consider restrictions of $\widehat{\sigma}$ to the set of unification variables in P and N': $\widehat{\sigma}|\mathbf{uv}\, P : \Theta|_{\mathbf{uv}\, P}$ and $\widehat{\sigma}|\mathbf{uv}\, N' : \Theta|_{\mathbf{uv}\, N'}$. Notice that $[\widehat{\sigma}] P = [\widehat{\sigma}|\mathbf{uv}\, P] P$ and $[\widehat{\sigma}] N' = [\widehat{\sigma}|\mathbf{uv}\, N'] N'$.

Let us apply the induction hypothesis to $\Gamma \vdash [\widehat{\sigma}|\mathbf{uv}\, P] P \geqslant_1 Q$ and $\Gamma \vdash [\widehat{\sigma}|\mathbf{uv}\, N'] N' \leqslant_1 M'$ (notice that since $\mathbf{uv}\, N' \subseteq \mathbf{uv}\, N$, there are no negative unification variables in N') to obtain $\widehat{\sigma}_1'$ and $\widehat{\sigma}_2'$ such that

1. $\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}_1'$ and $\Theta \vdash \widehat{\sigma}|\mathbf{uv}\, P \Rightarrow \widehat{\sigma}_1'$
2. $\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}_2'$ and $\Theta \vdash \widehat{\sigma}|\mathbf{uv}\, N' \Rightarrow \widehat{\sigma}_2'$

This way, the algorithm applies Rule $(\rightarrow^{\leqslant})$ and terminates returning $\widehat{\sigma}_1'\, \&\, \widehat{\sigma}_2'$ as the result.

It is left to show that $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}_1'\, \&\, \widehat{\sigma}_2'$. Since $\widehat{\sigma}|\mathbf{uv}\, P \subseteq \widehat{\sigma}$, by lemma 37, $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}|\mathbf{uv}\, P$, and then by lemma 38, $\Theta \vdash \widehat{\sigma}|\mathbf{uv}\, P \Rightarrow \widehat{\sigma}_1'$. Analogously, $\Theta \vdash \widehat{\sigma}|\mathbf{uv}\, N' \Rightarrow \widehat{\sigma}_2'$. Then since by lemma 36, $\widehat{\sigma}_1'\, \&\, \widehat{\sigma}_2'$ is the weakest restriction implying both $\widehat{\sigma}_1'$ and $\widehat{\sigma}_2'$, we have $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}_1'\, \&\, \widehat{\sigma}_2'$, as required.

Case 5. $\Gamma \vdash [\widehat{\sigma}] N \leqslant_1 M$ is derived by Rule $(\forall^{\leqslant_1})$, i.e. $[\widehat{\sigma}] N = \forall \overrightarrow{\alpha^+}.[\widehat{\sigma}] N'$ (assuming $\overrightarrow{\alpha^+}$ does not intersect with the range of $\widehat{\sigma}$) and $M = \forall \overrightarrow{\beta^+}.M'$, where either $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is non-empty.

By inversion, $\Gamma, \overrightarrow{\beta^+} \vdash [\sigma][\widehat{\sigma}] N' \leqslant_1 M'$ for $\Gamma, \overrightarrow{\beta^+} \vdash \sigma : \overrightarrow{\alpha^+}$. Since σ and $\widehat{\sigma}$ have disjoint domains, and the range of one does not intersect with the domain of the other, they commute, i.e. $\Gamma, \overrightarrow{\beta^+} \vdash [\widehat{\sigma}][\sigma] N' \leqslant_1 M'$ (notice that the tree inferring this judgement is a proper subtree of the tree inferring $\Gamma \vdash [\widehat{\sigma}] N \leqslant_1 M$).

On the next step, the algorithm creates fresh (disjoint with $\mathbf{uv}\, N'$) unification variables $\overrightarrow{\widehat{\alpha}^+}$, replaces $\overrightarrow{\alpha^+}$ with them in $[\widehat{\sigma}] N'$, and makes the recursive call: $\Gamma, \overrightarrow{\beta^+}; \Theta, \widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\} \vDash N_0 \leqslant M \dashv \widehat{\sigma}_1$, (where $N_0 = [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}] N'$), returning $\widehat{\sigma}_1 \backslash \overrightarrow{\widehat{\alpha}^+}$ as the result.

Notice that $[\hat{\sigma}][\sigma][\overrightarrow{\alpha^+/\widehat{\alpha}^+}]N_0 = [\hat{\sigma}][\sigma]N'$, and thus, $\Gamma, \overrightarrow{\beta^+} \vdash [\hat{\sigma}][\sigma][\overrightarrow{\alpha^+/\widehat{\alpha}^+}]N_0 \leqslant_1 M'$. Let us combine $\hat{\sigma}$ and $\sigma \circ \overrightarrow{\alpha^+/\widehat{\alpha}^+}$ into one $\hat{\sigma}_0$ an unification solution for N_0:

$$\hat{\sigma}_0(\hat{\beta}^\pm) = \begin{cases} \hat{\beta}^\pm :\approx [\sigma]\alpha_i^+ & \text{if } \hat{\beta}^\pm = \widehat{\alpha_i}^+ \in \overrightarrow{\widehat{\alpha}^+} \\ \hat{\sigma}(\hat{\beta}^\pm) & \text{if } \hat{\beta}^\pm \in \mathbf{uv}\ N' \end{cases}$$

Let us show that the induction hypothesis is applicable to $\Gamma, \overrightarrow{\beta^+} \vdash [\hat{\sigma}_0]N_0 \leqslant_1 M'$.

- $\hat{\sigma}_0 : \Theta|_{\mathbf{uv}\ N'}, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}}$. Notice that for every $\widehat{\alpha_i}^+$, the type corresponding to the entry $\hat{\sigma}_0(\widehat{\alpha_i}^+)$ is well-formed in $\Gamma, \overrightarrow{\beta^+}$ since $\Gamma, \overrightarrow{\beta^+} \vdash \sigma : \overrightarrow{\alpha^+}$; and for every $\hat{\beta}^\pm \in \mathbf{uv}\ N'$, the type corresponding to the entry $\hat{\sigma}_0(\hat{\beta}^\pm)$ is well-formed in context $\Theta(\hat{\beta}^\pm)$ since $\hat{\sigma} : \Theta|_{\mathbf{uv}\ N'}$.
- $\Gamma, \overrightarrow{\beta^+} \vdash [\hat{\sigma}_0]N_0 \leqslant_1 M'$. This is because $[\hat{\sigma}_0]N_0 = [\hat{\sigma}][\sigma][\overrightarrow{\alpha^+/\widehat{\alpha}^+}]N_0 = [\hat{\sigma}][\sigma]N'$.
- $\Gamma, \overrightarrow{\beta^+} \vdash^\supseteq \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}}$ since $\Gamma, \overrightarrow{\beta^+} \vdash^\supseteq \Theta$ and $\Gamma, \overrightarrow{\beta^+} \subseteq \Gamma, \overrightarrow{\beta^+}$.
- $\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vdash N_0$
- There are no negative unification variables in N_0 because $\mathbf{uv}\ N_0 \subseteq \mathbf{uv}\ N \cup \overrightarrow{\widehat{\alpha}^+}$.

This way, we apply the induction hypothesis to $\Gamma, \overrightarrow{\beta^+} \vdash [\hat{\sigma}_0]N_0 \leqslant_1 M'$ and infer that there exists $\hat{\sigma}_0' : (\Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}})|_{\mathbf{uv}\ N_0}$ such that $\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vDash N_0 \leqslant M \dashv \hat{\sigma}_0'$, and $\Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vdash \hat{\sigma}_0' \Rightarrow \hat{\sigma}_0$.

This way, the algorithm terminates with the result $\hat{\sigma}_0' \backslash \overrightarrow{\widehat{\alpha}^+}$ and it is left to show $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}_0' \backslash \overrightarrow{\widehat{\alpha}^+}$. Notice that $\hat{\sigma}_0' \backslash \overrightarrow{\widehat{\alpha}^+} : \Theta|_{\mathbf{uv}\ N'}$. This way, to show $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}_0' \backslash \overrightarrow{\widehat{\alpha}^+}$. it suffices to consider an unification variable $\hat{\beta}^\pm \mathbf{uv}\ N'$ and show that $\Gamma, \overrightarrow{\beta^+} \vdash \hat{\sigma}(\hat{\beta}^\pm) \Rightarrow \hat{\sigma}_0'(\hat{\beta}^\pm)$. It holds by the reflexivity of weakening (**??**) since $\hat{\sigma}_0'(\hat{\beta}^\pm) = \hat{\sigma}(\hat{\beta}^\pm)$.

Case 6. The positive case when $\Gamma \vdash [\hat{\sigma}]P \geqslant_1 Q$ is derived by Rule $(\downarrow^{\geqslant_1})$ is symmetric to the corresponding negative case 3. Notice that the algorithm does not make a recursive call in this case but rather invoke the unification. This way, the proof does not apply the induction hypothesis, and hence, the absence of negative unification variables in a negative type ($\widehat{\alpha}^- \notin N$ for $\downarrow N = P$) is not required.

Case 7. The positive case when $\Gamma \vdash [\hat{\sigma}]P \geqslant_1 Q$ is derived by Rule $(\exists^{\geqslant_1})$ is symmetric to the corresponding negative case 5. Notice that we only consider the case when the substitution $[\hat{\sigma}]P$ results in the existential type $\exists\overrightarrow{\alpha^-}.P'$ (for $P' \neq \exists \dots$) by congruence, i.e. $P = \exists\overrightarrow{\alpha^-}.P''$ (for $P'' \neq \exists \dots$) and $[\hat{\sigma}]P'' = P'$. This is because the case when $P = \exists\overrightarrow{\beta^-}.\widehat{\alpha}^+$ has been covered (case 1), and thus, the substitution $\hat{\sigma}$ must preserve all the outer quantifiers of P and does not generate any new ones.

Case 8. The positive case when $\Gamma \vdash [\hat{\sigma}]P \geqslant_1 Q$ is derived by Rule $(\mathrm{Var}^{+\geqslant_1})$ is symmetric to the corresponding negative case 2. Notice that $[\hat{\sigma}]P = \beta^+$ implies $P = \beta^+$ because the case when $P = \widehat{\alpha}^+$ has been covered (case 1).

$\square$