1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
 & | & \alpha^+ & \\
 & | & \downarrow N & \\
 & | & \exists \alpha^-.P & \\
 & & & \\
N,\ M & ::= & & \text{negative types} \\
 & | & \alpha^- & \\
 & | & \uparrow P & \\
 & | & \forall \alpha^+.N & \\
 & | & P \to N &
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M}\ \text{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q}\ \text{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-}\ \text{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q}\ \text{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M}\ \text{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M}\ \text{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M}\ \text{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+}\ \text{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M}\ \text{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q}\ \text{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q}\ \text{D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{multi-quantified positive types} \\
 & \mid & \alpha^+ & \\
 & \mid & {\downarrow}N & \\
 & \mid & \exists\overrightarrow{\alpha^-}.P & P \neq \exists\ldots \\
 & \mid & (P) & \mathsf{S} \\
\end{array}
$$

$$
\begin{array}{llll}
N,\ M & ::= & & \text{multi-quantified negative types} \\
 & \mid & \alpha^- & \\
 & \mid & {\uparrow}P & \\
 & \mid & P \to N & \\
 & \mid & \forall\overrightarrow{\alpha^+}.N & N \neq \forall\ldots \\
 & \mid & (N) & \mathsf{S} \\
\end{array}
$$

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \quad ({\simeq_1^{\leqslant}}^-)$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \quad ({\simeq_1^{\leqslant}}^+)$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad (\mathrm{VAR}^{-\leqslant_1})$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash {\uparrow}P \leqslant_1 {\uparrow}Q} \quad ({\uparrow}^{\leqslant_1})$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \quad ({\to}^{\leqslant_1})$$

$$\frac{\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M} \quad (\forall^{\leqslant_1})$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad (\mathrm{VAR}^{+\geqslant_1})$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash {\downarrow}N \geqslant_1 {\downarrow}M} \quad ({\downarrow}^{\geqslant_1})$$

$$\frac{\mathbf{fv}\, P \cap \overrightarrow{\beta^-} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q} \quad (\exists^{\geqslant_1})$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1}$ Equivalence of substitutions

2.3 Declarative Equivalence

$\boxed{N \simeq_1^{D} M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^{D} \alpha^-} \quad (\mathrm{VAR}^{-\simeq_1^{D}})$$

$$\frac{P \simeq_1^{D} Q}{{\uparrow}P \simeq_1^{D} {\uparrow}Q} \quad ({\uparrow}^{\simeq_1^{D}})$$

$$\frac{P \simeq_1^{D} Q \quad N \simeq_1^{D} M}{P \to N \simeq_1^{D} Q \to M} \quad ({\to}^{\simeq_1^{D}})$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\textsc{Var}^{+\simeq_1^D})$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \alpha^-} \quad (\textsc{Var}^-_{\in})$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \cdot} \quad (\textsc{Var}^-_{\notin})$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \uparrow P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ vars\ \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)} \quad (\to)$$

$$\frac{vars \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \alpha^+} \quad (\textsc{Var}^+_{\in})$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \cdot} \quad (\textsc{Var}^+_{\notin})$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \downarrow N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{vars \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \exists \overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad (\exists)$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^- = \cdot} \quad (\textsc{UVar}^-)$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^+ = \cdot} \quad (\textsc{UVar}^+)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\quad(\textsc{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\quad(\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\quad(\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'}\quad(\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\quad(\textsc{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\quad(\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'}\quad(\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\quad(\textsc{UVar}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\quad(\textsc{UVar}^+)$$

3.2 Unification

$\boxed{\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Gamma; \Theta \models \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\quad\textsc{UNVar}$$

$$\frac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma; \Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\quad\textsc{UShiftU}$$

$$\frac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2}\quad\textsc{UArrow}$$

$$\frac{\Gamma, \overrightarrow{\alpha^+}; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma; \Theta \models \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\quad\textsc{UForall}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma; \Theta \models \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)}\quad\textsc{UNUVar}$$

$\boxed{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Gamma; \Theta \models \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\quad\textsc{UPVar}$$

$$\frac{\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma; \Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\quad\textsc{UShiftD}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma; \Theta \models \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\quad\textsc{UExists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma; \Theta \models \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)}\quad\textsc{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma;\,\Theta \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma;\,\Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot}\ (\textsc{Var}^{-\leqslant})$$

$$\frac{\Gamma;\Theta \models \mathbf{nf}\,(P) \stackrel{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma;\,\Theta \models {\uparrow}P \leqslant {\uparrow}Q \dashv \widehat{\sigma}}\ ({\uparrow}^{\leqslant})$$

$$\frac{\Gamma;\,\Theta \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma;\,\Theta \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma;\,\Theta \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\ ({\to}^{\leqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^+};\, \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma;\,\Theta \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}}\ (\forall^{\leqslant})$$

$\boxed{\Gamma;\,\Theta \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma;\,\Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ (\textsc{Var}^{+\geqslant})$$

$$\frac{\Gamma;\Theta \models \mathbf{nf}\,(N) \stackrel{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma;\,\Theta \models {\downarrow}N \geqslant {\downarrow}M \dashv \widehat{\sigma}}\ ({\downarrow}^{\geqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^-};\, \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma;\,\Theta \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}}\ (\exists^{\geqslant})$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{\Gamma;\,\Theta \models \widehat{\alpha}^+ \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)}\ (\textsc{UVar}^{\geqslant})$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restricts an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\Delta \vdash \widehat{\alpha}^+ :\approx P$ or $\Delta \vdash \widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\Delta \vdash \widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries)**.** *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.
$\boxed{e_1 \,\&\, e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\Gamma \models P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)}\ ({\geqslant} \,\&\, {\geqslant})$$

$$\frac{\Gamma;\,\cdot \models P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\ ({\simeq} \,\&\, {\geqslant})$$

$$\frac{\Gamma;\,\cdot \models Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)}\ ({\geqslant} \,\&\, {\simeq})$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \,\&\, (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)}\ ({\simeq} \,\&\, {\simeq}^+)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N) \,\&\, (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)}\ ({\simeq} \,\&\, {\simeq}^-)$$

Notice that in case of equivalence, the assigned types must be equal (i.e. alpha-equivalent) to be merged. This is because the unification algorithm assumes that every type is normalized, and hence, equivalence is alpha-equivalence (corollaries 14 and 16).

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3. $\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 = \{e_1 \,\&\, e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, \textit{s.t. } e_1 \textit{ matches with } e_2\}$
$\cup \{e_1 \mid e_1 \in \widehat{\sigma}_1,\ \textit{s.t. } \forall e_2 \in \widehat{\sigma}_2, e_1 \textit{ does not match with } e_2\}$
$\cup \{e_2 \mid e_2 \in \widehat{\sigma}_2,\ \textit{s.t. } \forall e_1 \in \widehat{\sigma}_1, e_1 \textit{ does not match with } e_2\}$

3.5 Least Upper Bound

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+} \quad (\text{VAR}^{\vee})$$

$$\frac{\Gamma, \cdot \vDash \mathbf{nf}\,(\downarrow N) \overset{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi] P} \quad (\downarrow^{\vee})$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q} \quad (\exists^{\vee})$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

$$\frac{\Gamma = \Delta, \overrightarrow{\alpha^{\pm}} \quad \overrightarrow{\beta^{\pm}}\ \mathbf{is\ fresh} \quad \overrightarrow{\gamma^{\pm}}\ \mathbf{is\ fresh} \quad \Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vDash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P = Q}{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q} \quad (\text{UPG})$$

3.6 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \quad \text{AUPVAR}$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \dashv (\Xi, \downarrow M, \widehat{\tau}_1, \widehat{\tau}_2)} \quad \text{AUPSHIFT}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists \overrightarrow{\alpha^-}.Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad \text{AUPEXISTS}$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)} \quad \text{AUNVAR}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad \text{AUNSHIFT}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)} \quad \text{AUNARROW}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad \text{AUNAU}$$

4 Proofs

4.1 Declarative Subtyping

Lemma 1 (Free Variable Propagation). *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context* Γ,

- $-$ *if* $\Gamma \vdash N \leqslant_1 M$ *then* $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$
- $+$ *if* $\Gamma \vdash P \geqslant_1 Q$ *then* $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$

Proof. Mutual induction on $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash P \geqslant_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
It is self-evident that $\alpha^- \subseteq \alpha^-$.

Case 2. $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq_1^{\leqslant} Q$), we have $\Gamma \vdash P \geqslant_1 Q$. Then by the induction hypothesis, $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$. The desired inclusion inclusion holds, since $\mathbf{fv}\,(\uparrow P) = \mathbf{fv}\,(P)$ and $\mathbf{fv}\,(\uparrow Q) = \mathbf{fv}\,(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant_1 Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$ and $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$. Then $\mathbf{fv}\,(P \to N) = \mathbf{fv}\,(P) \cup \mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(Q) \cup \mathbf{fv}\,(M) = \mathbf{fv}\,(Q \to M)$.

Case 4. $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$

$$\begin{aligned}
\mathbf{fv}\,\forall\overrightarrow{\alpha^+}.N &\subseteq \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \setminus \overrightarrow{\beta^+} && \text{here } \overrightarrow{\beta^+} \text{ is excluded by the premise } \mathbf{fv}\,N \cap \overrightarrow{\beta^+} = \varnothing\\
&\subseteq \mathbf{fv}\,M \setminus \overrightarrow{\beta^+} && \text{by the induction hypothesis, } \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^+}]N) \subseteq \mathbf{fv}\,M\\
&\subseteq \mathbf{fv}\,\forall\overrightarrow{\beta^+}.M
\end{aligned}$$

Case 5. The positive cases are symmetric.

$\square$

Corollary 1 (Free Variables of mutual subtypes)**.**

− *If* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $\mathbf{fv}\,N = \mathbf{fv}\,M$*,*

+ *If* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $\mathbf{fv}\,P = \mathbf{fv}\,Q$

Lemma 2 (Subtypes and supertypes of a variable)**.** *Assuming* $\Gamma \vdash \alpha^-$, $\Gamma \vdash \alpha^+$, $\Gamma \vdash N$, *and* $\Gamma \vdash P$*,*

+ *if* $\Gamma \vdash P \geqslant_1 \alpha^+$ *or* $\Gamma \vdash \alpha^+ \geqslant_1 P$ *then* $P = \exists\overrightarrow{\alpha^-}.\alpha^+$ *(for some potentially empty* $\overrightarrow{\alpha^-}$*)*

− *if* $\Gamma \vdash N \leqslant_1 \alpha^-$ *or* $\Gamma \vdash \alpha^- \leqslant_1 N$ *then* $N = \forall\overrightarrow{\alpha^+}.\alpha^-$ *(for some potentially empty* $\overrightarrow{\alpha^+}$*)*

Proof. We prove by induction on the tree inferring $\Gamma \vdash P \geqslant_1 \alpha^+$ or $\Gamma \vdash \alpha^+ \geqslant_1 P$ or or $\Gamma \vdash N \leqslant_1 \alpha^-$ or $\Gamma \vdash \alpha^- \leqslant_1 N$.
Let us consider which of these judgments the tree is inferring.

Case 1. $\Gamma \vdash P \geqslant_1 \alpha^+$

If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\mathrm{Var}^{+\geqslant_1}$), which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists^{\geqslant_1}$). By inverting this rule, $P = \exists\overrightarrow{\alpha^-}.P'$ where P' does not start with $\exists$ and $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P' \geqslant_1 \alpha^+$ for some $\Gamma, \overrightarrow{\beta^-} \vdash N_i$.

By the induction hypothesis, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]P' = \exists\overrightarrow{\beta^-}.\alpha^+$. Notice that P' must be a variable, because P' does not start with $\exists$, nor does it start with $\uparrow$ (otherwise, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]P'$ would also started with $\uparrow$ and would not be equal to $\exists\overrightarrow{\beta^-}.\alpha^+$). Since P' is a *positive* variable, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]P' = P'$, and then $P' = \exists\overrightarrow{\beta^-}.\alpha^+$ means that $P' = \alpha^+$. This way, $P = \exists\overrightarrow{\alpha^-}.P' = \exists\overrightarrow{\alpha^-}.\alpha^+$, as required.

Case 2. $\Gamma \vdash \alpha^+ \geqslant_1 P$

If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\mathrm{Var}^{+\geqslant_1}$), which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists^{\geqslant_1}$). By inverting this rule, $P = \exists\overrightarrow{\beta^-}.Q$ where and $\Gamma, \overrightarrow{\beta^-} \vdash \alpha^+ \geqslant_1 Q$.

By the induction hypothesis, $Q = \exists\overrightarrow{\beta^-}'.\alpha^+$. This way, $P = \exists\overrightarrow{\beta^-}.Q = \exists\overrightarrow{\beta^-}.\exists\overrightarrow{\beta^-}'.\alpha^+$, as required.

Case 3. The negative cases ($\Gamma \vdash N \leqslant_1 \alpha^-$ and $\Gamma \vdash \alpha^- \leqslant_1 N$) are proved analogously.

$\square$

Corollary 2 (Variables have no proper subtypes and supertypes)**.** *Assuming that all mentioned types are well-formed in* Γ*,*

$$\begin{array}{lllllll}
\Gamma \vdash P \geqslant_1 \alpha^+ & \iff & P = \exists\overrightarrow{\beta^-}.\alpha^+ & \iff & \Gamma \vdash P \simeq_1^{\leqslant} \alpha^+ & \iff & P \simeq_1^{D} \alpha^+\\
\Gamma \vdash \alpha^+ \geqslant_1 P & \iff & P = \exists\overrightarrow{\beta^-}.\alpha^+ & \iff & \Gamma \vdash P \simeq_1^{\leqslant} \alpha^+ & \iff & P \simeq_1^{D} \alpha^+\\
\Gamma \vdash N \leqslant_1 \alpha^- & \iff & N = \forall\overrightarrow{\beta^+}.\alpha^- & \iff & \Gamma \vdash N \simeq_1^{\leqslant} \alpha^- & \iff & N \simeq_1^{D} \alpha^-\\
\Gamma \vdash \alpha^- \leqslant_1 N & \iff & N = \forall\overrightarrow{\beta^+}.\alpha^- & \iff & \Gamma \vdash N \simeq_1^{\leqslant} \alpha^- & \iff & N \simeq_1^{D} \alpha^-
\end{array}$$

Proof. Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \simeq_1^{\leqslant} \alpha^+$ and $\exists\overrightarrow{\alpha^-}.\alpha^+ \simeq \alpha^+$ and apply lemma 2. `Ilya: fix` $\square$

Corollary 3 (Transitivity of subtyping)**.** *Assuming the types are well-formed in* Γ*,*

− *if* $\Gamma \vdash N_1 \leqslant_1 N_2$ *and* $\Gamma \vdash N_2 \leqslant_1 N_3$ *then* $\Gamma \vdash N_1 \leqslant_1 N_3$*,*

+ *if* $\Gamma \vdash P_1 \geqslant_1 P_2$ *and* $\Gamma \vdash P_2 \geqslant_1 P_3$ *then* $\Gamma \vdash P_1 \geqslant_1 P_3$*.*

Corollary 4 (Transitivity of equivalence)**.** *Assuming the types are well-formed in* Γ*,*

− *if* $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_2$ *and* $\Gamma \vdash N_2 \simeq_1^{\leqslant} N_3$ *then* $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_3$*,*

+ *if* $\Gamma \vdash P_1 \simeq_1^{\leqslant} P_2$ *and* $\Gamma \vdash P_2 \simeq_1^{\leqslant} P_3$ *then* $\Gamma \vdash P_1 \simeq_1^{\leqslant} P_3$*.*

4.2 Substitution

Lemma 3 (Substitution strengthening). *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

$+$ *if* $\Gamma_1 \vdash P$ *then* $[\sigma]P = [\sigma|_{\mathbf{fv}\, P}]P$*,*

$-$ *if* $\Gamma_1 \vdash N$ *then* $[\sigma]N = [\sigma|_{\mathbf{fv}\, N}]N$

Proof. `Ilya: todo` □

Lemma 4 (Substitution preserves subtyping). *Suppose that* $\Gamma \vdash \sigma : \Gamma_1$*. Then*

$+$ *if* $\Gamma_1 \vdash P$, $\Gamma_1 \vdash Q$, *and* $\Gamma_1 \vdash P \geqslant_1 Q$ *then* $\Gamma \vdash [\sigma]P \geqslant_1 [\sigma]Q$

$-$ *if* $\Gamma_1 \vdash N$, $\Gamma_1 \vdash M$, *and* $\Gamma_1 \vdash N \leqslant_1 M$ *then* $\Gamma \vdash [\sigma]N \leqslant_1 [\sigma]M$

Proof. `Ilya: todo` □

Corollary 5 (Substitution preserves equivalence). *Suppose that* $\Gamma \vdash \sigma : \Gamma_1$*. Then*

$+$ *if* $\Gamma_1 \vdash P$, $\Gamma_1 \vdash Q$, *and* $\Gamma_1 \vdash P \simeq_1^{\leqslant} Q$ *then* $\Gamma \vdash [\sigma]P \simeq_1^{\leqslant} [\sigma]Q$

$-$ *if* $\Gamma_1 \vdash N$, $\Gamma_1 \vdash M$, *and* $\Gamma_1 \vdash N \simeq_1^{\leqslant} M$ *then* $\Gamma \vdash [\sigma]N \simeq_1^{\leqslant} [\sigma]M$

4.3 Type well-formedness

Lemma 5 (Well-formedness agrees with substitution). *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

$+$ $\Gamma, \Gamma_1 \vdash P \;\Leftrightarrow\; \Gamma, \Gamma_2 \vdash [\sigma]P$

$-$ $\Gamma, \Gamma_1 \vdash N \;\Leftrightarrow\; \Gamma, \Gamma_2 \vdash [\sigma]N$

Proof. `Ilya: todo` □

Corollary 6. *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

$+$ $\Gamma_1, \Gamma_2 \vdash P \;\Leftrightarrow\; \Gamma_2 \vdash [\sigma]P$

$-$ $\Gamma_1, \Gamma_2 \vdash N \;\Leftrightarrow\; \Gamma_2 \vdash [\sigma]N$

Lemma 6 (Equivalent Contexts). *In the well-formedness judgment, only used variables matter:*

$+$ *if* $\Gamma_1 \cap \mathbf{fv}\, P = \Gamma_2 \cap \mathbf{fv}\, P$ *then* $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$,

$-$ *if* $\Gamma_1 \cap \mathbf{fv}\, N = \Gamma_2 \cap \mathbf{fv}\, N$ *then* $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$.

Proof. By simple mutual induction on P and Q. □

Corollary 7. *Suppose that all the types below are well-formed in* Γ *and* $\Gamma' \subseteq \Gamma$*. Then*

$+$ $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *implies* $\Gamma' \vdash P \iff \Gamma' \vdash Q$

$-$ $\Gamma \vdash N \simeq_1^{\leqslant} M$ *implies* $\Gamma' \vdash N \iff \Gamma' \vdash M$

Proof. From lemma 6 and corollary 1. □

4.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\dfrac{}{\mathbf{ord}\ vars\ \mathbf{in}\ N \equiv vars \cap \mathbf{fv}\ N}$	$\dfrac{N \simeq_1^D M}{\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ M}$	—
Normalization	$\dfrac{}{N \simeq_1^D \mathbf{nf}\,(N)}$	$\dfrac{N \simeq_1^D M}{\mathbf{nf}\,(N) = \mathbf{nf}\,(M)}$	—
Equivalence	$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq_1^D Q}{\Gamma \vdash P \simeq_1^{\leqslant} Q}$	$\dfrac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{P \simeq_1^D Q}$	—
Uppgrade	$\dfrac{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{Q \text{ is sound} \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$	$\dfrac{\begin{array}{l} Q' \text{ is sound} \\ \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \vDash P_1 \vee P_2 = Q}{Q \text{ is sound} \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \vDash P_1 \vee P_2 = Q}$	$\dfrac{\begin{array}{l} Q' \text{ is sound} \\ \Gamma \vDash P_1 \vee P_2 = Q \end{array}}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\begin{array}{l}(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \\ \text{is sound}\end{array} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat{\tau}_i : \Xi \\ [\hat{\tau}_i]\, Q = P_i \end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \hat{\tau}_1', \hat{\tau}_2')}{\begin{array}{l} \exists (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \text{ s.t.} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \end{array}}$	$\dfrac{\begin{array}{l} (\Xi', Q', \hat{\tau}_1', \hat{\tau}_2') \text{ is sound} \\ \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \end{array}}{\exists \Gamma; \Xi \vdash \hat{\tau} : \Xi' \text{ s.t. } [\hat{\tau}]\, Q' = Q}$
Unification (matching)	$\dfrac{\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}}{\hat{\sigma} \text{ is sound} \begin{cases} \Gamma \vdash \hat{\sigma} : \Theta \\ [\hat{\sigma}]\, P = Q \end{cases}}$	$\dfrac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}}$	—
Subtyping	$\dfrac{\Gamma;\ \Theta \vDash N \leqslant M \dashv \hat{\sigma}}{\hat{\sigma} \text{ is sound} \begin{cases} \Gamma \vdash \hat{\sigma} : \Theta \\ \Gamma \vdash [\hat{\sigma}]\, N \leqslant_1 M \end{cases}}$	$\dfrac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Gamma;\ \Theta \vDash N \leqslant M \dashv \hat{\sigma}}$	—

4.5 Variable Ordering

Definition 4 (Collision free bijection)**.** *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is* ***collision free on sets*** *P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 7 (Soundness of variable ordering)**.** *Variable ordering extracts precisely used free variables.*

− $\mathbf{ord}\ vars\ \mathbf{in}\ N \equiv vars \cap \mathbf{fv}\ N$ *(as sets)*

+ $\mathbf{ord}\ vars\ \mathbf{in}\ P \equiv vars \cap \mathbf{fv}\ P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}$ and $\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}$ □

Corollary 8 (Additivity of ordering)**.** *Variable ordering is additive (in terms of set union) with respect to its first argument.*

− $\mathbf{ord}\,(vars_1 \cup vars_2)\ \mathbf{in}\ N \equiv \mathbf{ord}\ vars_1\ \mathbf{in}\ N \cup \mathbf{ord}\ vars_2\ \mathbf{in}\ N$ *(as sets)*

+ $\mathbf{ord}\,(vars_1 \cup vars_2)\ \mathbf{in}\ P \equiv \mathbf{ord}\ vars_1\ \mathbf{in}\ P \cup \mathbf{ord}\ vars_2\ \mathbf{in}\ P$ *(as sets)*

Corollary 9 (Weakening of ordering)**.** *Extending the first argument of the ordering with unused variables does not change the result.*

− $\mathbf{ord}\,(vars \cap \mathbf{fv}\ N)\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ N$

+ $\mathbf{ord}\,(\mathit{vars} \cap \mathbf{fv}\,P)\,\mathbf{in}\,P = \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P$

Lemma 8 (Distributivity of renaming over variable ordering)**.** *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

– *If μ is collision free on vars and $\mathbf{fv}\,N$ then* $[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) = \mathbf{ord}\,([\mu]\mathit{vars})\,\mathbf{in}\,[\mu]N$

+ *If μ is collision free on vars and $\mathbf{fv}\,P$ then* $[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P) = \mathbf{ord}\,([\mu]\mathit{vars})\,\mathbf{in}\,[\mu]P$

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in \mathit{vars}$
Then
$$\begin{aligned}[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^-)\\ &= [\mu]\alpha^- && \text{by Rule } (\mathrm{Var}^+_\in)\\ &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]\mathit{vars})\\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,\beta^- && \text{by Rule } (\mathrm{Var}^+_\in)\text{, because } \beta^- \in [\mu]\mathit{vars}\\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\alpha^-\end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin \mathit{vars}$
Notice that $[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. On the other hand, $\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,\alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_\notin)$, because μ is collision free on *vars* and $\mathbf{fv}\,N$, so $\mathbf{fv}\,N \ni \alpha^- \notin \mu(A \cap \mathit{vars}) \cup \mathit{vars} \supseteq [\mu]\mathit{vars}$.

c. $\alpha^- \in A$ but $\alpha^- \notin \mathit{vars}$
Then $[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. To prove that $\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_\notin)$. Let us show that $[\mu]\alpha^- \notin [\mu]\mathit{vars}$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]\mathit{vars} \subseteq \mu(A \cap \mathit{vars}) \cup \mathit{vars}$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap \mathit{vars}) \cup \mathit{vars}$.

(i) If there is an element $x \in A \cap \mathit{vars}$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin \mathit{vars}$. This way, $\mu(\alpha^-) \notin \mu(A \cap \mathit{vars})$.

(ii) Since μ is collision free on *vars* and $\mathbf{fv}\,N$, $\mu(A \cap \mathbf{fv}\,N) \ni \mu(\alpha^-) \notin \mathit{vars}$.

d. $\alpha^- \notin A$ but $\alpha^- \in \mathit{vars}$
$\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_\notin)$, because $\alpha^- = [\mu]\alpha^- \in [\mu]\mathit{vars}$ since $\alpha^- \in \mathit{vars}$. On the other hand, $[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$
$$\begin{aligned}[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\uparrow P)\\ &= [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P) && \text{by Rule } (\uparrow)\\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]P && \text{by the induction hypothesis}\\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,\uparrow[\mu]P && \text{by Rule } (\uparrow)\\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]\uparrow P && \text{by the definition of substitution}\\ &= \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]N\end{aligned}$$

Case 3. $N = P \to M$
$$\begin{aligned}[\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P \to M)\\ &= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)) && \text{where } \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,M = \vec{\alpha}_2\\ &= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \vec{\alpha}_1)\\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to case 1. Notice that } \mu \text{ is}\\ & && \text{collision free on } \vec{\alpha}_1 \text{ and } \vec{\alpha}_2 \text{ since } \vec{\alpha}_1 \subseteq \mathit{vars} \text{ and } \vec{\alpha}_2 \subseteq \mathbf{fv}\,N\\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)\end{aligned}$$
$$\begin{aligned}(\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]N) &= (\mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]P \to [\mu]M)\\ &= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \vec{\beta}_1)) && \text{where } \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\,[\mu]\mathit{vars}\,\mathbf{in}\,[\mu]M = \vec{\beta}_2\\ & && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2,\\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)\end{aligned}$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$\begin{aligned}[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) &= [\mu]\mathbf{ord}\ vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.M \\ &= [\mu]\mathbf{ord}\ vars\ \mathbf{in}\ M \\ &= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]M \quad \text{by the induction hypothesis}\end{aligned}$$

$$\begin{aligned}(\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]N) &= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]\forall\overrightarrow{\alpha^+}.M \\ &= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.[\mu]M \\ &= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]M\end{aligned}$$

□

Lemma 9 (Ordering is not affected by independent substitutions)**.** *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2, and vars is a set of variables disjoint with both Γ_1 and Γ_2. Then*

− $\mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]N = \mathbf{ord}\ vars\ \mathbf{in}\ N$

+ $\mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]P = \mathbf{ord}\ vars\ \mathbf{in}\ P$

Proof. `Ilya: Should be easy` □

Lemma 10 (Completeness of variable ordering)**.** *Variable ordering is invariant under equivalence. For arbitrary vars,*

− *If $N \simeq_1^D M$ then $\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ M$ (as lists)*

+ *If $P \simeq_1^D Q$ then $\mathbf{ord}\ vars\ \mathbf{in}\ P = \mathbf{ord}\ vars\ \mathbf{in}\ Q$ (as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. □

4.6 Normaliztaion

Lemma 11. *Set of free variables is invariant under equivalence.*

− *If $N \simeq_1^D M$ then $\mathbf{fv}\ N \equiv \mathbf{fv}\ M$ (as sets)*

+ *If $P \simeq_1^D Q$ then $\mathbf{fv}\ P \equiv \mathbf{fv}\ Q$ (as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

Lemma 12. *Free variables are not changed by the normalization*

− $\mathbf{fv}\ N \equiv \mathbf{fv}\ \mathbf{nf}\ (N)$

+ $\mathbf{fv}\ P \equiv \mathbf{fv}\ \mathbf{nf}\ (P)$

Proof. By straightforward induction on $\mathbf{nf}\ (N) = M$. □

Lemma 13 (Soundness of quantifier normalization)**.**

− $N \simeq_1^D \mathbf{nf}\ (N)$

+ $P \simeq_1^D \mathbf{nf}\ (P)$

Proof. Mutual induction on $\mathbf{nf}\ (N) = M$ and $\mathbf{nf}\ (P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var^-) and (Var^+)
By the corresponding equivalence rules.

Case 2. ($\uparrow$), ($\downarrow$), and ($\rightarrow$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall$), i.e. $\mathbf{nf}\ (\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 11, $\mathbf{fv}\ N \equiv \mathbf{fv}\ N'$. Then by lemma 7, $\overrightarrow{\alpha^+}' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\ N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\ N$, and thus, $\overrightarrow{\alpha^+}' \cap \mathbf{fv}\ N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\ N$.

To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^+}'.N'$, it suffices to provide a bijection $\mu : \overrightarrow{\alpha^+}' \cap \mathbf{fv}\ N' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\ N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. ($\exists$) Same as for case 3.

□

Corollary 10 (Normalization preserves ordering). *For any vars,*

- $-$ $\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\mathbf{nf}\,(N) = \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,M$
- $+$ $\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\mathbf{nf}\,(P) = \mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,Q$

Proof. Immediately from lemmas 10 and 13. □

Lemma 14 (Distributivity of normalization over substitution). *Normalization of a term distributes over substitution. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2. Then*

- $-$ $\mathbf{nf}\,([\sigma]N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$
- $+$ $\mathbf{nf}\,([\sigma]P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P)$

where $\mathbf{nf}\,(\sigma)$ means pointwise normalization: $[\mathbf{nf}\,(\sigma)]\alpha^- = \mathbf{nf}\,([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.

Case 2. $P = \alpha^+$
Similar to case 1.

Case 3. If the type is formed by $\rightarrow$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \rightarrow M$ then

$$
\begin{aligned}
\mathbf{nf}\,([\sigma]N) &= \mathbf{nf}\,([\sigma](P \rightarrow M)) \\
&= \mathbf{nf}\,([\sigma]P \rightarrow [\sigma]M) && \text{By the congruence of substitution} \\
&= \mathbf{nf}\,([\sigma]P) \rightarrow \mathbf{nf}\,([\sigma]M) && \text{By the congruence of normalization, i.e. Rule } (\rightarrow) \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) \rightarrow [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M) && \text{By the induction hypothesis} \\
&= [\mathbf{nf}\,(\sigma)](\mathbf{nf}\,(P) \rightarrow \mathbf{nf}\,(M)) && \text{By the congruence of substitution} \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P \rightarrow M) && \text{By the congruence of normalization} \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)
\end{aligned}
$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$
\begin{aligned}
[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) &= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M) \\
&= [\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M) && \text{Where } \overrightarrow{\alpha^+}' = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(M) = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M \text{ (the latter is by corollary 10)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{nf}\,([\sigma]N) &= \mathbf{nf}\,([\sigma]\forall\overrightarrow{\alpha^+}.M) \\
&= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\sigma]M) && \text{Assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing \\
&= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\sigma]M) && \text{Where } \overrightarrow{\beta^+} = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,([\sigma]M) = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M \text{ (the latter is by corollary 10)} \\
&= \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,([\sigma]M) && \text{By lemma 9, } \overrightarrow{\beta^+} = \overrightarrow{\alpha^+}' \text{ since } \overrightarrow{\alpha^+} \text{ is disjoint with } \Gamma_1 \text{ and } \Gamma_2 \\
&= \forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M) && \text{By the induction hypothesis}
\end{aligned}
$$

To show alpha-equivalence of $[\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^+}'.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$, we can assume that $\overrightarrow{\alpha^+}' \cap \Gamma_1 = \varnothing$, and $\overrightarrow{\alpha^+}' \cap \Gamma_2 = \varnothing$.

Case 5. $P = \exists\overrightarrow{\alpha^-}.Q$
Same as for case 4.

□

Corollary 11 (Commutativity of normalization and renaming). *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

- $-$ $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$
- $+$ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Proof. Immediately from lemma 14, after noticing that $\mathbf{nf}\,(\mu) = \mu$. $\square$

Lemma 15 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

$-$ *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

$+$ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. $(\forall^{\simeq_1^D})$

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 7 and 12, the domain and the codomain of μ can be written as $\mu : \overrightarrow{\beta^+}' \leftrightarrow \overrightarrow{\alpha^+}'$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by corollary 11, the second—by the induction hypothesis.

(ii)
$$
\begin{aligned}
[\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}' \\
&= [\mu]\mathbf{ord}\,(\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 12 and corollary 9} \\
&= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 8, because } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \\
& && \text{and } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \\
&= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved} \\
&= \mathbf{ord}\,(\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \text{ and } \overrightarrow{\beta^+} \cap \mathbf{fv}\,M \\
&= \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 12 and corollary 9} \\
&= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'
\end{aligned}
$$

Case 2. $(\exists^{\simeq_1^D})$ Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

$\square$

Lemma 16 (Idempotence of normalization)**.** *Normalization is idempotent*

$-$ $\mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$

$+$ $\mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

Proof. By applying lemma 15 to lemma 13. $\square$

Lemma 17. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized. Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then*

$$+\ [\sigma]P \textit{ is normal} \iff \begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \textit{is normal} \\ P & \textit{is normal} \end{cases}$$

$$-\ [\sigma]N \textit{is normal} \iff \begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \textit{is normal} \\ N & \textit{is normal} \end{cases}$$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$

Then N is always normal, and the normality of $\sigma|_{\alpha^-}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$\begin{aligned}
[\sigma](P \to M) \text{ is normal} &\iff [\sigma]P \to [\sigma]M \text{ is normal} && \text{by the substitution congruence}\\
&\iff \begin{cases} [\sigma]P & \text{is normal}\\ [\sigma]M & \text{is normal}\end{cases} && \text{by congruence of normality } \texttt{Ilya: lemma?}\\
&\iff \begin{cases} P & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(P)} & \text{is normal}\\ M & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(M)} & \text{is normal}\end{cases} && \text{by the induction hypothesis}\\
&\iff \begin{cases} P \to M & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(P)\cup\mathbf{fv}\,(M)} & \text{is normal}\end{cases} \iff && \begin{cases} P \to M & \text{is normal}\\ \sigma|_{\mathbf{fv}\,(P\to M)} & \text{is normal}\end{cases}
\end{aligned}$$

Case 3. $N = \uparrow P$

By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$\begin{aligned}
[\sigma](\forall\overrightarrow{\alpha^+}.M) \text{ is normal} &\iff (\forall\overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} && \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing\\
&\iff \begin{cases} [\sigma]M \text{ is normal}\\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+}\end{cases} && \text{by the definition of normalization}\\
&\iff \begin{cases} [\sigma]M \text{ is normal}\\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+}\end{cases} && \text{by lemma 9}\\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(M)} \text{ is normal}\\ M \text{ is normal}\\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+}\end{cases} && \text{by the induction hypothesis}\\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M)} \text{ is normal}\\ \forall\overrightarrow{\alpha^+}.M \text{ is normal}\end{cases} && \begin{array}{l}\text{since } \mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M);\\ \text{by the definition of normalization}\end{array}
\end{aligned}$$

Case 5. $P = \dots$

The positive cases are done in the same way as the negative ones.

□

4.7 Equivalence

Lemma 18 (Type well-formedness is invariant under equivalence)**.** *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,

− *if* $N \simeq_1^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. `Ilya: todo` □

Corollary 12 (Normalization preserves well-formedness)**.**

+ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}\,(P)$,

− $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}\,(N)$

Proof. Immediately from lemmas 13 and 18. □

Corollary 13 (Normalization preserves well-formedness of substitution)**.**
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}\,(\sigma) : \Gamma_1$

Lemma 19 (Soundness of equivalence)**.** *Declarative equivalence implies mutual subtyping.*

+ *if* $\Gamma \vdash P$, $\Gamma \vdash Q$, *and* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \simeq_1^{\leqslant} Q$,

− *if* $\Gamma \vdash N$, $\Gamma \vdash M$, *and* $N \simeq_1^D M$ *then* $\Gamma \vdash N \simeq_1^{\leqslant} M$.

Proof. We prove it by mutual induction on $P \simeq_1^D Q$ and $N \simeq_1^D M$.

Case 1. $\alpha^- \simeq_1^D \alpha^-$
Then $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$ by Rule ($\mathrm{Var}^{-\leqslant_1}$), which immediately implies $\Gamma \vdash \alpha^- \simeq_1^\leqslant \alpha^-$ by Rule ($\simeq_1^\leqslant -$).

Case 2. $\uparrow P \simeq_1^D \uparrow Q$
Then by inversion of Rule ($\uparrow^{\leqslant_1}$), $P \simeq_1^D Q$, and from the induction hypothesis, $\Gamma \vdash P \simeq_1^\leqslant Q$, and (by symmetry) $\Gamma \vdash Q \simeq_1^\leqslant P$.

When Rule ($\uparrow^{\leqslant_1}$) is applied to $\Gamma \vdash P \simeq_1^\leqslant Q$, it gives us $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$; when it is applied to $\Gamma \vdash Q \simeq_1^\leqslant P$, we obtain $\Gamma \vdash \uparrow Q \leqslant_1 \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \simeq_1^\leqslant \uparrow Q$.

Case 3. $P \to N \simeq_1^D Q \to M$
Then by inversion of Rule ($\to^{\leqslant_1}$), $P \simeq_1^D Q$ and $N \simeq_1^D M$. By the induction hypothesis, $\Gamma \vdash P \simeq_1^\leqslant Q$ and $\Gamma \vdash N \simeq_1^\leqslant M$, which means by inversion: (i) $\Gamma \vdash P \geqslant_1 Q$, (ii) $\Gamma \vdash Q \geqslant_1 P$, (iii) $\Gamma \vdash N \leqslant_1 M$, (iv) $\Gamma \vdash M \leqslant_1 N$. Applying Rule ($\to^{\leqslant_1}$) to (i) and (iii), we obtain $\Gamma \vdash P \to N \leqslant_1 Q \to M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \to M \leqslant_1 P \to N$. Together, it implies $\Gamma \vdash P \to N \simeq_1^\leqslant Q \to M$.

Case 4. $\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M$
Then by inversion, there exists bijection $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N)$, such that $N \simeq_1^D [\mu]M$. By the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^\leqslant [\mu]M$. From corollary 5 and the fact that μ is bijective, we also have $\Gamma, \overrightarrow{\beta^+} \vdash [\mu^{-1}]N \simeq_1^\leqslant M$.

Let us construct a subsitution $\overrightarrow{\alpha^+} \vdash \overrightarrow{P}/\overrightarrow{\beta^+} : \overrightarrow{\beta^+}$ by extending μ with arbitrary positive types on $\overrightarrow{\beta^+} \backslash \mathbf{fv}\, M$.

Notice that $[\mu]M = [\overrightarrow{P}/\overrightarrow{\beta^+}]M$, and therefore, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^\leqslant [\mu]M$ implies $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{P}/\overrightarrow{\beta^+}]M \leqslant_1 N$. Then by Rule ($\forall^{\leqslant_1}$), $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \leqslant_1 \forall \overrightarrow{\alpha^+}.N$.

Analogously, we construct the substitution from μ^{-1}, and use it to instantiate $\overrightarrow{\alpha^+}$ in the application of Rule ($\forall^{\leqslant_1}$) to infer $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M$.

This way, $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \leqslant_1 \forall \overrightarrow{\alpha^+}.N$ and $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M$ gives us $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \simeq_1^\leqslant \forall \overrightarrow{\alpha^+}.N$.

Case 5. For the cases of the positive types, the proofs are symmetric.

$\square$

Corollary 14 (Normalization is sound w.r.t. subtyping-induced equivalence)**.**

+ *if* $\Gamma \vdash P$ *then* $\Gamma \vdash P \simeq_1^\leqslant \mathbf{nf}\,(P)$,

− *if* $\Gamma \vdash N$ *then* $\Gamma \vdash N \simeq_1^\leqslant \mathbf{nf}\,(N)$.

Proof. Immediately from lemmas 13 and 19 and corollary 12. $\square$

Lemma 20 (Subtyping induced by disjoint substitutions)**.** *If two disjoint substitutions induce subtyping, they are degenerate (so is the subtyping). Suppose that* $\Gamma \vdash \sigma_1 : \Gamma_1$ *and* $\Gamma \vdash \sigma_2 : \Gamma_1$*, where* $\Gamma_i \subseteq \Gamma$ *and* $\Gamma_1 \cap \Gamma_2 = \varnothing$*. Then*

− *assuming* $\Gamma \vdash N$, $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ *implies* $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\, N$

+ *assuming* $\Gamma \vdash P$, $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ *implies* $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\, P$

Proof. Proof by induciton on $\Gamma \vdash N$ (and mutually on $\Gamma \vdash P$).

Case 1. $N = \alpha^-$
Then $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\alpha^- \leqslant_1 [\sigma_2]\alpha^-$. Let us consider the following cases:

a. $\alpha^- \notin \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \alpha^-$ holds immediately, since $[\sigma_i]\alpha^- = [\mathsf{id}]\alpha^- = \alpha^-$ and $\Gamma \vdash \alpha^- \simeq_1^\leqslant \alpha^-$.

b. $\alpha^- \in \Gamma_1$ and $\alpha^- \in \Gamma_2$
This case is not possible by assumption: $\Gamma_1 \cap \Gamma_2 = \varnothing$.

c. $\alpha^- \in \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then we have $\Gamma \vdash [\sigma_1]\alpha^- \leqslant_1 \alpha^-$, which by corollary 2 means $\Gamma \vdash [\sigma_1]\alpha^- \simeq_1^\leqslant \alpha^-$, and hence, $\Gamma \vdash \sigma_1 \simeq_1^\leqslant \mathsf{id} : \alpha^-$.
$\Gamma \vdash \sigma_2 \simeq_1^\leqslant \mathsf{id} : \alpha^-$ holds since $[\sigma_2]\alpha^- = \alpha^-$, similarly to case 1.*a*.

d. $\alpha^- \notin \Gamma_1$ and $\alpha^- \in \Gamma_2$
Then we have $\Gamma \vdash \alpha^- \leqslant_1 [\sigma_2]\alpha^-$, which by corollary 2 means $\Gamma \vdash \alpha^- \simeq_1^\leqslant [\sigma_2]\alpha^-$, and hence, $\Gamma \vdash \sigma_2 \simeq_1^\leqslant \mathsf{id} : \alpha^-$.
$\Gamma \vdash \sigma_1 \simeq_1^\leqslant \mathsf{id} : \alpha^-$ holds since $[\sigma_1]\alpha^- = \alpha^-$, similarly to case 1.*a*.

Case 2. $N = \forall\overrightarrow{\alpha^+}.M$
Then by inversion, $\Gamma, \overrightarrow{\alpha^+} \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\forall\overrightarrow{\alpha^+}.M \leqslant_1 [\sigma_2]\forall\overrightarrow{\alpha^+}.M$. By the congruence of substitution and by the inversion of Rule ($\forall^{\leqslant_1}$), $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{Q}/\overrightarrow{\alpha^+}][\sigma_1]M \leqslant_1 [\sigma_2]M$, where $\Gamma, \overrightarrow{\alpha^+} \vdash Q_i$. Let us denote the (Kleisli) composition of σ_1 and $\overrightarrow{Q}/\overrightarrow{\alpha^+}$ as σ_1', noting that $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' : \Gamma_1, \overrightarrow{\alpha^+}$, and $\Gamma_1, \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$.

Let us apply the induction hypothesis to M and the substitutions σ_1' and σ_2 with $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_1']M \leqslant_1 [\sigma_2]M$ to obtain:

$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{1}$$
$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{2}$$

Then $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ holds by strengthening of 2: for any $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M \backslash \overrightarrow{\alpha^+}$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$ is strengthened to $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$, because $\mathbf{fv}\, [\sigma_2]\beta^\pm = \mathbf{fv}\, \beta^\pm = \{\beta^\pm\} \subseteq \Gamma$.

To show that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$, let us take an arbitrary $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M \backslash \overrightarrow{\alpha^+}$.

$$\begin{aligned}
\beta^\pm &= [\mathsf{id}]\beta^\pm && \text{by definition of id}\\
&\simeq_1^{\leqslant} [\sigma_1']\beta^\pm && \text{by 1}\\
&= [\overrightarrow{Q}/\overrightarrow{\alpha^+}][\sigma_1]\beta^\pm && \text{by definition of } \sigma_1'\\
&= [\sigma_1]\beta^\pm && \text{because } \overrightarrow{\alpha^+} \cap \mathbf{fv}\, [\sigma_1]\beta^\pm \subseteq \overrightarrow{\alpha^+} \cap \Gamma = \varnothing
\end{aligned}$$

This way, $\Gamma \vdash [\sigma_1]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$ for any $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ and thus, $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$.

Case 3. $N = P \to M$
Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1](P \to M) \leqslant_1 [\sigma_2](P \to M)$, then by congruence of substitution, $\Gamma \vdash [\sigma_1]P \to [\sigma_1]M \leqslant_1 [\sigma_2]P \to [\sigma_2]M$, then by inversion $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ and $\Gamma \vdash [\sigma_1]M \leqslant_1 [\sigma_2]M$.

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ and to $\Gamma \vdash [\sigma_1]M \leqslant_1 [\sigma_2]M$, we obtain (respectively):

$$\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P \tag{3}$$
$$\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{4}$$

Noting that $\mathbf{fv}\, (P \to M) = \mathbf{fv}\, P \cup \mathbf{fv}\, M$, we combine eqs. (3) and (4) to conclude: $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, (P \to M)$.

Case 4. $N = \uparrow P$
Then by inversion, $\Gamma \vdash P$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\uparrow P \leqslant_1 [\sigma_2]\uparrow P$, then by congruence of substitution and by inversion, $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$, we obtain $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$. Since $\mathbf{fv}\, \uparrow P = \mathbf{fv}\, P$, we can conclude: $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \uparrow P$.

Case 5. The positive cases are proved symmetrically.

$\square$

Corollary 15 (Substitution cannot induce proper subtypes or supertypes)**.** *Assuming all mentioned types are well-formed in Γ and σ is a substitution $\Gamma \vdash \sigma : \Gamma$,*

$$\begin{aligned}
\Gamma \vdash [\sigma]N \leqslant_1 N &\Rightarrow \Gamma \vdash [\sigma]N \simeq_1^{\leqslant} N \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, N\\
\Gamma \vdash N \leqslant_1 [\sigma]N &\Rightarrow \Gamma \vdash N \simeq_1^{\leqslant} [\sigma]N \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, N\\
\Gamma \vdash [\sigma]P \geqslant_1 P &\Rightarrow \Gamma \vdash [\sigma]P \simeq_1^{\leqslant} P \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P\\
\Gamma \vdash P \geqslant_1 [\sigma]P &\Rightarrow \Gamma \vdash P \simeq_1^{\leqslant} [\sigma]P \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P
\end{aligned}$$

Lemma 21. *Asssuming that the mentioned types (P, Q, N, and M) are well-formed in Γ and that the substitutions (σ_1 and σ_2) have signature $\Gamma \vdash \sigma_i : \Gamma$,*

+ *if $\Gamma \vdash [\sigma_1]P \geqslant_1 Q$ and $\Gamma \vdash [\sigma_2]Q \geqslant_1 P$*
  then there exists a bijection $\mu : \mathbf{fv}\, P \leftrightarrow \mathbf{fv}\, Q$ such that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, P$ and $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, Q$;

− *if $\Gamma \vdash [\sigma_1]N \leqslant_1 M$ and $\Gamma \vdash [\sigma_2]N \leqslant_1 M$*
  then there exists a bijection $\mu : \mathbf{fv}\, N \leftrightarrow \mathbf{fv}\, M$ such that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, N$ and $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, M$.

Proof.

+ Applying σ_2 to both sides of $\Gamma \vdash [\sigma_1]P \geqslant_1 Q$ (by **??**), we have: $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant_1 [\sigma_2]Q$. Composing it with $\Gamma \vdash [\sigma_2]Q \geqslant_1 P$ (by transitivity **??**), we have $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant_1 P$. Then by corollary 15, $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$.

  By a symmetric argument, we also have: $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, Q$.

  Now, we prove that $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$ and $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, Q$ implies that σ_1 and σ_1 are (equivalent to) mutually inverse bijections.

  To do so, it suffices to prove that

  (i) for any $\alpha^\pm \in \mathbf{fv}\, P$ there exists $\beta^\pm \in \mathbf{fv}\, Q$ such that $\Gamma \vdash [\sigma_1]\alpha^\pm \simeq_1^{\leqslant} \beta^\pm$ and $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \alpha^\pm$; and

  (ii) for any $\beta^\pm \in \mathbf{fv}\, Q$ there exists $\alpha^\pm \in \mathbf{fv}\, P$ such that $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \alpha^\pm$ and $\Gamma \vdash [\sigma_1]\alpha^\pm \simeq_1^{\leqslant} \beta^\pm$.

  Then the these correspondences between $\mathbf{fv}\, P$ and $\mathbf{fv}\, Q$ are mutually inverse functions, since for any $\beta^\pm$ there can be at most one $\alpha^\pm$ such that $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \alpha^\pm$ (and vice versa).

  (i) Let us take $\alpha^\pm \in \mathbf{fv}\, P$.

    (a) if $\alpha^\pm$ is positive ($\alpha^\pm = \alpha^+$), from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^+ \simeq_1^{\leqslant} \alpha^+$, by corollary 2, we have $[\sigma_2][\sigma_1]\alpha^+ = \exists\overrightarrow{\beta^-}.\alpha^+$.
    What shape can $[\sigma_1]\alpha^+$ have? It cannot be $\exists\overrightarrow{\alpha^-}.{\downarrow}N$ (for potentially empty $\overrightarrow{\alpha^-}$), because the outer constructor $\downarrow$ would remain after the substitution σ_2, whereas $\exists\overrightarrow{\beta^-}.\alpha^+$ does not have $\downarrow$. The only case left is $[\sigma_1]\alpha^+ = \exists\overrightarrow{\alpha^-}.\gamma^+$.
    Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\gamma^+ \simeq_1^{\leqslant} \gamma^+$, meaning that $\Gamma \vdash [\sigma_1]\alpha^+ \simeq_1^{\leqslant} \gamma^+$. Also notice that $[\sigma_2]\exists\overrightarrow{\alpha^-}.\gamma^+ = \exists\overrightarrow{\beta^-}.\alpha^+$ implies $\Gamma \vdash [\sigma_2]\gamma^+ \simeq_1^{\leqslant} \alpha^+$.

    (b) if $\alpha^\pm$ is negative ($\alpha^\pm = \alpha^-$) from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^- \simeq_1^{\leqslant} \alpha^-$, by corollary 2, we have $[\sigma_2][\sigma_1]\alpha^- = \forall\overrightarrow{\beta^+}.\alpha^-$.
    What shape can $[\sigma_1]\alpha^-$ have? It cannot be $\forall\overrightarrow{\alpha^+}.{\uparrow}P$ nor $\forall\overrightarrow{\alpha^+}.P \to M$ (for potentially empty $\overrightarrow{\alpha^+}$), because the outer constructor ($\to$ or $\uparrow$), remaining after the substitution σ_2, is however absent in the resulting $\forall\overrightarrow{\beta^+}.\alpha^-$. Hence, the only case left is $[\sigma_1]\alpha^- = \forall\overrightarrow{\alpha^+}.\gamma^-$ Notice that $\Gamma \vdash \gamma^- \simeq_1^{\leqslant} \forall\overrightarrow{\alpha^+}.\gamma^-$, meaning that $\Gamma \vdash [\sigma_1]\alpha^- \simeq_1^{\leqslant} \gamma^-$. Also notice that $[\sigma_2]\forall\overrightarrow{\alpha^+}.\gamma^- = \forall\overrightarrow{\beta^+}.\alpha^-$ implies $\Gamma \vdash [\sigma_2]\gamma^- \simeq_1^{\leqslant} \alpha^-$.

  (ii) The proof is symmetric: We swap P and Q, σ_1 and σ_2, and exploit $\Gamma \vdash [\sigma_1][\sigma_2]\alpha^\pm \simeq_1^{\leqslant} \alpha^\pm$ instead of $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^\pm \simeq_1^{\leqslant} \alpha^\pm$.

− The proof is symmetric to the positive case.

□

Lemma 22 (Equivalence of polymorphic types)**.**

− *For* $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N$ *and* $\Gamma \vdash \forall\overrightarrow{\beta^+}.M$,
  if $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \simeq_1^{\leqslant} \forall\overrightarrow{\beta^+}.M$ *then there exists a bijection* $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv}\, M \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$ *such that* $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu]N$,

+ *For* $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P$ *and* $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$,
  if $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \simeq_1^{\leqslant} \exists\overrightarrow{\beta^-}.Q$ *then there exists a bijection* $\mu : \overrightarrow{\beta^-} \cap \mathbf{fv}\, Q \leftrightarrow \overrightarrow{\alpha^-} \cap \mathbf{fv}\, P$ *such that* $\Gamma, \overrightarrow{\beta^-} \vdash P \simeq_1^{\leqslant} [\mu]Q$.

Proof.

− First, by α-conversion, we ensure $\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing$ and $\overrightarrow{\beta^+} \cap \mathbf{fv}\, N = \varnothing$. By inversion, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \simeq_1^{\leqslant} \forall\overrightarrow{\beta^+}.M$ implies

  1. $\Gamma, \overrightarrow{\beta^+} \vdash [\sigma_1]N \leqslant_1 M$ for $\Gamma, \overrightarrow{\beta^+} \vdash \sigma_1 : \overrightarrow{\alpha^+}$ and
  2. $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]M \leqslant_1 N$ for $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 : \overrightarrow{\beta^+}$.

  To apply lemma 21, we weaken and rearrange the contexts, and extend the substitutions to act as identity outside of their initial domain:

  1. $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \leqslant_1 M$ for $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 : \Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$ and
  2. $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \leqslant_1 N$ for $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 : \Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$.

  Then from lemma 21, there exists a bijection $\mu : \mathbf{fv}\, M \leftrightarrow \mathbf{fv}\, N$ such that $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, M$ and $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, N$.

  Let us show that if we restrict the domain of μ to $\overrightarrow{\beta^+}$, its range will be contained in $\overrightarrow{\alpha^+}$. Let us take $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv}\, M$ and assume $[\mu]\gamma^+ \notin \overrightarrow{\alpha^+}$. Then since $\Gamma, \overrightarrow{\beta^+} \vdash \sigma_1 : \overrightarrow{\alpha^+}$, σ_1 acts as identity outside of $\overrightarrow{\alpha^+}$, i.e. $[\sigma_1][\mu]\gamma^+ = [\mu]\gamma^+$. Since $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, N$, application of σ_1 is equivalent to application of μ^{-1}, then $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}][\mu]\gamma^+ \simeq_1^{\leqslant} [\mu]\gamma^+$, i.e.

$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \gamma^+ \simeq_1^{\leqslant} [\mu]\gamma^+$, which means $\gamma^+ \in \mathbf{fv}\,[\mu]\gamma^+ \subseteq \mathbf{fv}\,N$. By assumption, $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv}\,M$, i.e. $\overrightarrow{\beta^+} \cap \mathbf{fv}\,N \neq \varnothing$, hence contradiction.

By **??**, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq_1^{\leqslant} \mu|_{\overrightarrow{\beta^+}} : \mathbf{fv}\,M$ implies $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \simeq_1^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$. By similar reasoning, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \simeq_1^{\leqslant} [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N$.

This way,

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N \leqslant_1 M \tag{5}$$

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu|_{\overrightarrow{\beta^+}}]M \leqslant_1 N \tag{6}$$

By applying $\mu|_{\overrightarrow{\beta^+}}$ to both sides of 5 (**??**) and contracting $\mu^{-1}|_{\overrightarrow{\alpha^+}} \circ \mu|_{\overrightarrow{\beta^+}} = \mu|^{-1}_{\overrightarrow{\beta^+}} \circ \mu|_{\overrightarrow{\beta^+}} = \mathsf{id}$, we have: $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \leqslant_1 [\mu|_{\overrightarrow{\beta^+}}]M$, which together with 6 means $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \simeq_1^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$, and by strengthening, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$. Symmetrically, $\Gamma, \overrightarrow{\beta^+} \vdash M \simeq_1^{\leqslant} [\mu|^{-1}_{\overrightarrow{\beta^+}}]N$.

- + The proof is symmetric to the proof of the negative case.

□

Lemma 23 (Completeness of equivalence)**.** *Mutual subtyping implies declarative equivalence. Assuming all the types below are well-formed in* Γ*:*

+ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $P \simeq_1^{D} Q$*,*

− *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $N \simeq_1^{D} M$*.*

Proof. − Induction on the sum of sizes of N and M. By inversion, $\Gamma \vdash N \simeq_1^{\leqslant} M$ means $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash M \leqslant_1 N$. Let us consider the last rule that forms $\Gamma \vdash N \leqslant_1 M$:

Case 1. Rule $(\mathrm{Var}^{-\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
Then $N \simeq_1^{D} M$ (i.e. $\alpha^- \simeq_1^{D} \alpha^-$) holds immediately by Rule $(\mathrm{Var}^{-\simeq_1^{D}})$.

Case 2. Rule $(\uparrow^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$
Then by inversion, $\Gamma \vdash P \simeq_1^{\leqslant} Q$, and by induction hypothesis, $P \simeq_1^{D} Q$. Then $N \simeq_1^{D} M$ (i.e. $\uparrow P \simeq_1^{D} \uparrow Q$) holds by Rule $(\uparrow^{\simeq_1^{D}})$.

Case 3. Rule $(\rightarrow^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash P \rightarrow N' \leqslant_1 Q \rightarrow M'$
Then by inversion, $\Gamma \vdash P \geqslant_1 Q$ and $\Gamma \vdash N' \leqslant_1 M'$. Notice that $\Gamma \vdash M \leqslant_1 N$ is of the form $\Gamma \vdash Q \rightarrow M' \leqslant_1 P \rightarrow N'$, which by inversion means $\Gamma \vdash Q \geqslant_1 P$ and $\Gamma \vdash M' \leqslant_1 N'$.
This way, $\Gamma \vdash Q \simeq_1^{\leqslant} P$ and $\Gamma \vdash M' \simeq_1^{\leqslant} N'$. Then by induction hypothesis, $Q \simeq_1^{D} P$ and $M' \simeq_1^{D} N'$. Then $N \simeq_1^{D} M$ (i.e. $P \rightarrow N' \simeq_1^{D} Q \rightarrow M'$) holds by Rule $(\rightarrow^{\simeq_1^{D}})$.

Case 4. Rule $(\forall^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N' \leqslant_1 \forall \overrightarrow{\beta^+}.M'$
Then by **??**, $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N' \simeq_1^{\leqslant} \forall \overrightarrow{\beta^+}.M'$ means that there exists a bijection $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv}\,M' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N'$ such that $\Gamma, \overrightarrow{\alpha^+} \vdash [\mu]M' \simeq_1^{\leqslant} N'$.
Notice that the application of bijection μ to M' does not change its size (which is less than the size of M), hence the induction hypothesis applies. This way, $[\mu]M' \simeq_1^{D} N'$ (and by symmetry, $N' \simeq_1^{D} [\mu]M'$) holds by induction. Then we apply Rule $(\forall^{\simeq_1^{D}})$ to get $\forall \overrightarrow{\alpha^+}.N' \simeq_1^{D} \forall \overrightarrow{\beta^+}.M'$, i.e. $N \simeq_1^{D} M$.

+ The proof is symmetric to the proof of the negative case.

□

Corollary 16 (Normalization is complete w.r.t. subtyping-induced equivalence)**.** *Assuming all the types below are well-formed in* Γ*:*

+ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$*,*

− *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$*.*

Proof. Immediately from lemmas 15 and 23. □

4.8 Upgrade

Let us consider a type P well-formed in Γ. Some of its Γ-supertypes are also well-formed in a smaller context $\Delta \subseteq \Gamma$. The upgrade is the operation that returns the least of such supertypes.

Lemma 24 (Soundness of Upgrade)**.** *Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^\pm}$,*
if $\textbf{upgrade}\,\Gamma \vdash P\,\textbf{to}\,\Delta = Q$ *then*

1. $\Delta \vdash Q$
2. $\Gamma \vdash Q \geqslant_1 P$

Proof. By inversion, $\textbf{upgrade}\,\Gamma \vdash P\,\textbf{to}\,\Delta = Q$ means that for fresh $\overrightarrow{\beta^\pm}$ and $\overrightarrow{\gamma^\pm}$, $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vDash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q$. Then by the soundness of the least upper bound (lemma 29),

1. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q$,
2. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant_1 [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$, and
3. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant_1 [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P$.

$$\begin{aligned}
\textbf{fv}\,Q &\subseteq \textbf{fv}\,[\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \cap \textbf{fv}\,[\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P && \text{Since by lemma 1, } \textbf{fv}\,Q \subseteq \textbf{fv}\,[\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \text{ and } \textbf{fv}\,Q \subseteq \textbf{fv}\,[\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P \\
&\subseteq ((\textbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) \cup \overrightarrow{\beta^\pm}) \cap ((\textbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) \cup \overrightarrow{\gamma^\pm}) \\
&= (\textbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) \cap (\textbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) && \text{since } \overrightarrow{\beta^\pm} \text{ and } \overrightarrow{\gamma^\pm} \text{ are fresh} \\
&= \textbf{fv}\,P \backslash \overrightarrow{\alpha^\pm} \\
&\subseteq \Gamma \backslash \overrightarrow{\alpha^\pm} && \text{since } P \text{ is well-formed in } \Gamma \\
&\subseteq \Delta
\end{aligned}$$

This way, by lemma 6, $\Delta \vdash Q$.

Let us apply $\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}$—the inverse of the substitution $\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}$ to both sides of $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant_1 [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$ and by **??**, get $\Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm} \vdash [\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}]Q \geqslant_1 P$. Notice that $\Delta \vdash Q$ implies that $\textbf{fv}\,Q \cap \overrightarrow{\beta^\pm} = \varnothing$, then by **??**, $[\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}]Q = Q$, and thus $\Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant_1 P$. By context strengthening, $\Delta, \overrightarrow{\alpha^\pm} \vdash Q \geqslant_1 P$. ☐

Lemma 25 (Completeness and Initiality of Upgrade)**.** *The upgrade returns the least Γ-supertype of P well-formed in Δ. Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^\pm}$,*
For any Q' such that

1. $\Delta \vdash Q'$ *and*
2. $\Gamma \vdash Q' \geqslant_1 P$,

The result of the upgrade algorithm Q exists ($\textbf{upgrade}\,\Gamma \vdash P\,\textbf{to}\,\Delta = Q$) *and satisfies* $\Delta \vdash Q' \geqslant_1 Q$.

Proof. Let us consider fresh (not intersecting with Γ) $\overrightarrow{\beta^\pm}$ and $\overrightarrow{\gamma^\pm}$.

If we apply substitution $\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}$ to both sides of $\Delta, \overrightarrow{\alpha^\pm} \vdash Q' \geqslant_1 P$, we have $\Delta, \overrightarrow{\beta^\pm} \vdash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]Q' \geqslant_1 [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$, which by **??**, since Q' is well-formed in Δ, simplifies to $\Delta, \overrightarrow{\beta^\pm} \vdash Q' \geqslant_1 [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$.

Analogously, if we apply substitution $\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}$ to both sides of $\Delta, \overrightarrow{\alpha^\pm} \vdash Q' \geqslant_1 P$, we have $\Delta, \overrightarrow{\gamma^\pm} \vdash Q' \geqslant_1 [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P$.

This way, Q' is a common supertype of $[\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$ and $[\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P$ in context $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm}$. It means that we can apply the completeness of the least upper bound (lemma 30):

1. there exists Q s.t. $\Gamma \vDash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q$
2. $\Gamma \vdash Q' \geqslant_1 Q$.

The former means that the upgrade algorithm terminates and returns Q. The latter means that since both Q' and Q are well-formed in Δ, by **??**, $\Delta \vdash Q' \geqslant_1 Q$. ☐

4.9 Upper Bounds

Lemma 26 (Decomposition of the quantifier rule). `Ilya: move somewhere` *Whenever the quantifier rule (Rule ($\exists^{\geqslant_1}$) or Rule ($\forall^{\leqslant_1}$)) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.*

$-$ *If* $\Gamma \vdash N \leqslant_1 \forall\overrightarrow{\beta^+}.M$ *then* $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant_1 M$.

$+$ *If* $\Gamma \vdash P \geqslant_1 \exists\overrightarrow{\beta^-}.Q$ *then* $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant_1 Q$.

Lemma 27 (Characterization of the Supertypes). *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \textit{for } \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle\vert\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-}, M', \textit{and } \overrightarrow{N} \textit{ s.t.} \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Immediately from lemma 2

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$
Then if $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'$, then by lemma 26, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$, and $\mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule ($\exists^{\geqslant_1}$). This way, $\{Q \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,(Q) \cap \overrightarrow{\beta^-} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = \downarrow M$
Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\nexists}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists\overrightarrow{\beta^-}.Q'$
Then the last applied rule to infer $\Gamma \vdash Q \geqslant_1 \downarrow M$ must be Rule ($\downarrow^{\geqslant_1}$), which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq_1^{\leqslant} M$, then by lemma 23 and Rule ($\downarrow^{\simeq_1^D}$), $\downarrow M' \simeq_1^D \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq_1^D \downarrow M\} = \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\nexists}$.
In the other direction,
$$\begin{aligned} \downarrow M' \simeq_1^D \downarrow M &\Rightarrow \Gamma \vdash \downarrow M' \simeq_1^{\leqslant} \downarrow M && \text{by lemma 19, since } \Gamma \vdash \downarrow M' \text{ by lemma 18} \\ &\Rightarrow \Gamma \vdash \downarrow M' \geqslant_1 \downarrow M && \text{by inversion} \end{aligned}$$

b. $Q = \exists\overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)
Then the last rule applied to infer $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q' \geqslant_1 \downarrow M$ must be Rule ($\exists^{\geqslant_1}$). Inversion of this rule gives us $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \geqslant_1 \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.*a*, $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \simeq_1^D \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists\overrightarrow{\beta^-}.\downarrow M' \in \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).
In the other direction,
$$\begin{aligned} [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^D \downarrow M &\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^{\leqslant} \downarrow M && \text{by lemma 19, since } \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \text{ by lemma 18} \\ &\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \geqslant_1 \downarrow M && \text{by inversion} \\ &\Rightarrow \Gamma \vdash \exists\overrightarrow{\beta^-}.\downarrow M' \geqslant_1 \downarrow M && \text{by Rule } (\exists^{\geqslant_1}) \end{aligned}$$

□

Lemma 28 (Characterization of the Normalized Supertypes). *For a normalized positive type* $P = \mathbf{nf}\,(P)$*, let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle\vert\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-}, M', \textit{and } \overrightarrow{N} \textit{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \end{array} \right\}$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \ \mid\ \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Then from lemma 27, $\{\mathbf{nf}\,(Q) \ \mid\ \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\beta^+) \ \mid\ \text{for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

$$
\begin{aligned}
\mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') &= \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ not using } \overrightarrow{\beta^-} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothesis} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}\,(Q) = \mathbf{fv}\,Q \text{ by lemma 12} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{by lemma 27} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} && \text{by the definition of } \mathsf{UB} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} && \text{by lemma 27}
\end{aligned}
$$

Case 3. $P = \downarrow M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 1. *Suppose that $\nu : A \to A$ is an idempotent function, P is a predicate on A, $F : A \to B$ is a function. Then*

$$
\begin{aligned}
&\{F(\nu x) \mid x \in A \ s.t.\ P(\nu x)\} = \\
=&\{F(x) \mid x \in A \ s.t.\ \nu x = x \ and\ P(x)\}.
\end{aligned}
$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 2. *For functions F and ν, and predicates P and Q,*

$$
\begin{aligned}
&\{F(\nu x) \mid x \in A \ s.t.\ Q(\nu x) \ and\ P(x)\} = \\
=&\{F(\nu x) \mid x \in A \ s.t.\ Q(\nu x) \ and\ (\exists x' \in A \ s.t.\ P(x') \ and\ \nu x' = \nu x)\}.
\end{aligned}
$$

Observation 3. *There exist positive and negative types well-formed in empty context, hence, a type substitution can be extended to an arbitrary domain (if its values on the domain extension are irrelevant). Specifically, Suppose that $vars_1 \subseteq vars_2$. Then $\Gamma \vdash \sigma|_{vars_1} : vars_1$ implies $\exists\ \sigma'$ s.t. $\Gamma \vdash \sigma' : vars_2$ and $\sigma|_{vars_1} = \sigma'|_{vars_1}$.*

$$
\begin{aligned}
&\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 {\downarrow}M\} = \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash {\downarrow}M)\} && \text{by lemma 27} \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \overrightarrow{N} \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash N_i, \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by the definition of } \mathsf{UB} \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{we reassigned the substitution } \overrightarrow{N}/\overrightarrow{\alpha^-} \text{ as } \sigma \\
&= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by lemma 3} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} && \text{by the definition of normalization} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } \mathbf{nf}\,([\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M') = \mathbf{nf}\,({\downarrow}M) \end{array} \right\} && \begin{array}{l}\text{from lemmas 13 and 15, equivalence of types can be} \\ \text{replaced with the equality of their normal forms}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})]{\downarrow}\mathbf{nf}\,(M') = {\downarrow}\mathbf{nf}\,(M) \end{array} \right\} && \text{by congruence of normalization and lemma 14} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{by lemma 17, } {\downarrow}M' \text{ and } \sigma|_{\mathbf{fv}\,M'} \text{ are already normal,} \\ \text{since the result of the substitution is normal;} \\ M \text{ is normal by assumption}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ (\exists\sigma' \text{ s.t. } \Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')}) \\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{We apply observation 2 (with } \nu\sigma = \sigma|_{\mathbf{fv}\,M'}\text{, and} \\ P(\sigma) = \Gamma \vdash \sigma : \overrightarrow{\alpha^-})\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv}\,M'} : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{Notice that} \\ \text{“}\exists\sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')})\text{”} \\ \text{is equivalent to } \Gamma \vdash \sigma|_{\mathbf{fv}\,({\downarrow}M')} : \overrightarrow{\alpha^-}{}' \text{ (observation 3)}\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \begin{array}{l}\text{We apply observation 1 to the restriction of } \sigma\text{, and} \\ \text{remove } \sigma|_{\mathbf{fv}\,M'} = \sigma \text{ as it follows from } \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}'\end{array} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ \overrightarrow{\alpha^-},\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}{}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{by lemma 6, since } \Gamma, \overrightarrow{\alpha^-} \cap \mathbf{fv}\,M' = \Gamma, \overrightarrow{\alpha^-}{}' \cap \mathbf{fv}\,M' \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ M',\ \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}{}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}{}',\ \mathbf{ord}\,\overrightarrow{\alpha^-}{}'\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}' \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{We apply observation 1 to the ordering of } \overrightarrow{\alpha^-} \\
&= \left\{ \exists\overrightarrow{\alpha^-}{}'.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}{}',\ M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}{}'\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}{}', \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-}{}' \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}{}']{\downarrow}M' = {\downarrow}M \end{array} \right\} && \text{By reassigning } \sigma \text{ explicitly as } \overrightarrow{N}/\overrightarrow{\alpha^-}{}' \\
&= \mathsf{NFUB}({\downarrow}M) && \text{by definition}
\end{aligned}
$$

□

Observation 4. *Upper bounds of a type do not depend on the context as soon as the type are well-formed in it.*
If $\Gamma_1 \vdash M$ *and* $\Gamma_2 \vdash M$ *then* $\mathsf{UB}(\Gamma_1 \vdash M) = \mathsf{UB}(\Gamma \vdash M)$ *and* $\mathsf{NFUB}(\Gamma_1 \vdash M) = \mathsf{NFUB}(\Gamma \vdash M)$

Proof. We prove both inclusions by induction on $\Gamma_1 \vdash M$. Notice that if $[\sigma]M' \simeq_1^D M$ and $\Gamma_2 \vdash M$ then the types from the range of $\sigma|_{\mathbf{fv}\,M'}$ are well-formed in 2 `Ilya: lemma`. □

Lemma 29 (Soundness of the Least Upper Bound)**.** *For types* $\Gamma \vdash P_1$*, and* $\Gamma \vdash P_2$*, if* $\Gamma \models P_1 \vee P_2 = Q$ *then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Proof. Induction on $\Gamma \vDash P_1 \vee P_2 = Q$.

Case 1. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$
Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule ($\text{Var}^{+\geqslant_1}$).

Case 2. $\Gamma \vDash \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q$
Then by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applies to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q$. Then

(i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,

(ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$,

(iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}\,(Q) \cap \Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} = \mathbf{fv}\,(Q) \cap \Gamma$ (and then apply lemma 6). The inclusion right-to-left is self-evident. To show $\mathbf{fv}\,(Q) \cap \Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \subseteq \mathbf{fv}\,(Q) \cap \Gamma$, we prove that $\mathbf{fv}\,(Q) \subseteq \Gamma$

$\mathbf{fv}\,(Q) \subseteq \mathbf{fv}\,P_1 \cap \mathbf{fv}\,P_2$ — by lemma 1

$\subseteq (\Gamma, \overrightarrow{\alpha^-} \backslash \overrightarrow{\beta^-}) \cap (\Gamma, \overrightarrow{\beta^-} \backslash \overrightarrow{\alpha^-})$ — since $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_1$, $\mathbf{fv}\,(P_1) \subseteq \Gamma, \overrightarrow{\alpha^-} = \Gamma, \overrightarrow{\alpha^-} \backslash \overrightarrow{\beta^-}$ (the latter is because by the Barendregt's convention, $\Gamma, \overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing$); similarly, $\mathbf{fv}\,(P_2) \subseteq \Gamma, \overrightarrow{\beta^-} \backslash \overrightarrow{\alpha^-}$

$\subseteq \Gamma$

To show $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\alpha^-}.P_1$, we apply Rule ($\exists^{\geqslant_1}$). Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant_1 P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \vDash {\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$. By the inversion, $\Gamma, \cdot \vDash \mathbf{nf}\,({\downarrow}N) \stackrel{a}{\simeq} \mathbf{nf}\,({\downarrow}M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$. Then by the soundness of anti-unification (??),

(i) $\Gamma; \Xi \vdash P$, then by ??,

$$\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]P \tag{7}$$

(ii) $\Gamma; \cdot \vdash \widehat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_2 : \Xi$. Assuming that $\Xi = \widehat{\beta}_1^-, .., \widehat{\beta}_n^-$, the antiunification solutions $\widehat{\tau}_1$ and $\widehat{\tau}_2$ can be put explicitly as $\widehat{\tau}_1 = (\widehat{\beta}_1^- :\approx N_1, .., \widehat{\beta}_n^- :\approx N_n)$, and $\widehat{\tau}_2 = (\widehat{\beta}_1^- :\approx M_1, .., \widehat{\beta}_n^- :\approx M_n)$. Then

$$\widehat{\tau}_1 = (\overrightarrow{N}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \tag{8}$$

$$\widehat{\tau}_2 = (\overrightarrow{M}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \tag{9}$$

(iii) $[\widehat{\tau}_1]Q = P_1$ and $[\widehat{\tau}_2]Q = P_1$, which, by 8 and 9, means

$$[\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P = \mathbf{nf}\,({\downarrow}N) \tag{10}$$

$$[\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P = \mathbf{nf}\,({\downarrow}M) \tag{11}$$

Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$ follows directly from 7.

To show $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}N$, we apply Rule ($\exists^{\geqslant_1}$), instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{N}$. Then $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}N$ follows from 10 and since $\Gamma \vdash \mathbf{nf}\,({\downarrow}N) \geqslant_1 {\downarrow}N$ (by corollary 14).

Analogously, instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{M}$, gives us $\Gamma \vdash [\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}M$ (from 11), and hence, $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant_1 {\downarrow}M$.

□

Lemma 30 (Completeness and Initiality of the Least Upper Bound)**.** *For types* $\Gamma \vdash P_1$*,* $\Gamma \vdash P_2$*, and* $\Gamma \vdash Q$ *such that* $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$*, there exists* Q' *s.t.* $\Gamma \vDash P_1 \vee P_2 = Q'$ *and* $\Gamma \vdash Q \geqslant_1 Q'$*.*

Proof. Induction on the pair (P_1, P_2). From lemma 28, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$. Let us consider the cases of what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_i$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$, where either $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then $Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_2.Q_2)$

$$\begin{aligned}
&\subseteq \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_2 \vdash Q_2) && \text{from the definition of UB}\\
&= \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_2) && \text{by observation 4, weakening and exchange}\\
&= \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant_1 Q_1\} \cap \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant_1 Q_2\} && \text{by lemma 27,}
\end{aligned}$$

meaning that $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$. Then the next step of the algorithm—the recursive call $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vDash Q_1 \vee Q_2 = Q'$ terminates by the induction hypothesis, and moreover, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$. This way, the result of the algorithm is Q', i.e. $\Gamma \vDash P_1 \vee P_2 = Q'$.

Since both Q and Q' are sound, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$ can be strengthened to $\Gamma \vdash Q \geqslant_1 Q'$ by **??**.

Case 2. $P_1 = \alpha^+$ and $P_2 = \downarrow N$

Then the set of common upper bounds of $\downarrow N$ and α^+ is empty, and thus, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$ gives a contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \downarrow N)$

$$\begin{aligned}
&= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\downarrow M' \mid \cdots\} && \text{by the definition of UB}\\
&= \varnothing && \text{since } \alpha^+ \neq \downarrow M' \text{ for any } M'
\end{aligned}$$

Case 3. $P_1 = \downarrow N$ and $P_2 = \alpha^+$

Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)

Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \beta^+)$

$$\begin{aligned}
&= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\beta^+ \mid \cdots\} && \text{by the definition of UB}\\
&= \varnothing && \text{since } \alpha^+ \neq \beta^+
\end{aligned}$$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Then the algorithm terminates in one step (Rule (Var$^\vee$)) and the result is α^+, i.e. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$.

Since $Q \in \mathsf{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists\overrightarrow{\alpha^-}.\alpha^+$. Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant_1 \alpha^+$ by Rule ($\exists^{\geqslant_1}$): $\overrightarrow{\alpha^-}$ can be instantiated with arbitrary negative types (for example $\forall\beta^+.\uparrow\beta^+$), since the substitution for unused variables does not change the term $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule (Var$^{+\geqslant_1}$).

Case 6. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$

Then on the next step, the algorithm tries to anti-unify $\mathbf{nf}(\downarrow M_1)$ and $\mathbf{nf}(\downarrow M_2)$. By **??**, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$\mathbf{nf}(Q) \in \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}(\downarrow M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}(\downarrow M_2))$

$$\begin{aligned}
&= \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}(M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}(M_2))\\
&= \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \text{and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i, \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}(M_1) \end{array} \right\}\\
&\qquad \cap\\
&\quad\; \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \text{and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \overrightarrow{N}_1, \Gamma \vdash \overrightarrow{N}_2, \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}(M_2) \end{array} \right\}\\
&= \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \overrightarrow{N}_1 \text{ and } \overrightarrow{N}_2 \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \overrightarrow{N}_1, \Gamma \vdash \overrightarrow{N}_2, \Gamma, \overrightarrow{\alpha^-} \vdash M', [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}(M_1), \text{and } [\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}(M_2) \end{array} \right\}
\end{aligned}$$

The fact that the latter set is non-empty means that there exist $\overrightarrow{\alpha^-}, M', \overrightarrow{N}_1$ and $\overrightarrow{N}_2$ such that

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$ (notice that M' is normal)

(ii) $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma \vdash \overrightarrow{N}_1$,

(iii) $[\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}(M_1)$ and $[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}(M_2)$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative anti-unification variable $\widehat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\widehat{\alpha}^-}$. Let us show that $(\overrightarrow{\widehat{\alpha}^-}, [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M', \overrightarrow{N_1}/\overrightarrow{\widehat{\alpha}^-}, \overrightarrow{N_2}/\overrightarrow{\widehat{\alpha}^-})$ is a sound anti-unifier of $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$ in context Γ:

- $\overrightarrow{\widehat{\alpha}^-}$ is negative by construction,
- $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M'$ because $\Gamma, \overrightarrow{\alpha^-} \vdash {\downarrow}M'$ Ilya: lemma!,
- $\Gamma; \cdot \vdash (\overrightarrow{N_1}/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N_1}$ and $\Gamma; \cdot \vdash (\overrightarrow{N_2}/\overrightarrow{\widehat{\alpha}^-}) : \overrightarrow{\widehat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N_2}$,
- $[\overrightarrow{N_1}/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{N_1}/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_1) = \mathbf{nf}\,({\downarrow}M_1)$.
- $[\overrightarrow{N_2}/\overrightarrow{\widehat{\alpha}^-}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{N_2}/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_2) = \mathbf{nf}\,({\downarrow}M_2)$.

Then by the completeness of the anti-unification (**??**), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it, i.e. $Q' = \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, where $(\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$ is the result of the anti-unification of $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$ in context Γ.

Moreover, **??** also says that the found solution is initial, i.e. there exists $\widehat{\tau}$ such that $\Gamma; \Xi \vdash \widehat{\tau} : \overrightarrow{\widehat{\alpha}^-}$ and $[\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}$, (ii) $\widehat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi][\widehat{\tau}][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi]P$. In particular, from the reflexivity of subtyping: $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]{\downarrow}M' \geqslant_1 [\overrightarrow{\beta^-}/\Xi]P$.

It allows us to show $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant_1 Q'$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.{\downarrow}M' \geqslant_1 \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, by applying Rule $(\exists^{\geqslant_1})$, instantiating $\overrightarrow{\alpha^-}$ with respect to σ. Finally, $\Gamma \vdash Q \geqslant_1 Q'$ since $\Gamma \vdash \mathbf{nf}\,(Q) \simeq_1^{\leqslant} Q$, and equivalence implies subtyping by Ilya: lemma.

□

4.10 Unification

Lemma 31 (Soundness of Unification)**.**

\+ *For normalized P and Q such that $\Gamma; \Theta \vdash P$ and $\Gamma \vdash Q$,*
if $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}$ then $\widehat{\sigma} : \Theta \cap \mathbf{uv}\,P$, $\widehat{\sigma}$ is equivalence-only, and $[\widehat{\sigma}]P = Q$.

− *For normalized N and M such that $\Gamma; \Theta \vdash N$ and $\Gamma \vdash M$,*
if $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}$ then $\widehat{\sigma} : \Theta \cap \mathbf{uv}\,N$, $\widehat{\sigma}$ is equivalence-only, and $[\widehat{\sigma}]N = M$.

Lemma 32 (Completeness of Unification)**.**

\+ *For normalized P and Q such that $\Gamma; \Theta \vdash P$ and $\Gamma \vdash Q$,*
assume there exists $\widehat{\sigma}' : \Theta \cap \mathbf{uv}\,P$ such that $[\widehat{\sigma}']P = Q$. Then there exists $\widehat{\sigma}$ such that $\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}$.

− *For normalized N and M such that $\Gamma; \Theta \vdash N$ and $\Gamma \vdash M$,*
assume there exists $\widehat{\sigma}' : \Theta \cap \mathbf{uv}\,N$ such that $[\widehat{\sigma}']N = M$. Then there exists $\widehat{\sigma}$ such that $\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}$.

4.11 Anti-unification

Lemma 33 (Soundness of Anti-Unification)**.**

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Lemma 34 (Completeness of Anti-Unification)**.**

\+ *haha!!*

−

4.12 Solution Merge

Hello!

4.13 Subtyping Algorithm

Theorem 1 (Soundness of Subtyping Algorithm)**.**

– *If* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash M$, *and* $\Gamma; \Theta \vdash N$ *then*
$\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}$ *implies* $\widehat{\sigma} : \Theta \cap \mathbf{uv}\, N$ *and for any* $\widehat{\sigma}'$ *such that* $\widehat{\sigma}' \Rightarrow \widehat{\sigma}$, $\Gamma \vdash [\widehat{\sigma}']N \leqslant_1 M$

+ *If* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash Q$, *and* $\Gamma; \Theta \vdash P$ *then*
$\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}$ *implies* $\widehat{\sigma} : \Theta \cap \mathbf{uv}\, P$ *and for any* $\widehat{\sigma}'$ *such that* $\widehat{\sigma}' \Rightarrow \widehat{\sigma}$, $\Gamma \vdash [\widehat{\sigma}']P \geqslant_1 Q$.

Proof. We prove it by induction on $\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}$ (and mutually, on $\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}$). Let us consider the last rule to infer this judgment.

Case 1. $\Gamma; \Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot$
Then $\mathbf{uv}\, \alpha^- = \varnothing$, and $\widehat{\sigma} = \cdot : \cdot$ satisfies $\widehat{\sigma} : \Theta \cap \varnothing$. Since $\mathbf{uv}\, \alpha^- = \varnothing$, any unification solution $\widehat{\sigma}'$ does not change α^-, i.e. $[\widehat{\sigma}']\alpha^- = \alpha^-$. Therefore, $\Gamma \vdash [\widehat{\sigma}']\alpha^- \leqslant_1 \alpha^-$ holds by Rule ($\mathrm{Var}^{-\leqslant_1}$).

Case 2. $\Gamma; \Theta \models \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}$
Then the next step of the algorithm is the unification of $\mathbf{nf}\,(P)$ and $\mathbf{nf}\,(Q)$. By the soundness of the unification algorithm (lemma 31), it returns an equivalence-only solution $\widehat{\sigma}$ such that $\widehat{\sigma} : \mathbf{uv}\, P$. By **??**, since $\widehat{\sigma}$ is equivalence-only and $\Gamma \vdash^{\supseteq} \Theta$, $\widehat{\sigma}' \Rightarrow \widehat{\sigma}$ means $\Gamma \vdash \widehat{\sigma}' \simeq_1^{\leqslant} \widehat{\sigma} : \mathbf{uv}\, P$ as substitutions.

$[\widehat{\sigma}]\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$ implies $\Gamma \vdash [\widehat{\sigma}]\mathbf{nf}\,(P) \simeq_1^{\leqslant} \mathbf{nf}\,(Q)$, and then $\Gamma \vdash [\widehat{\sigma}']\mathbf{nf}\,(P) \simeq_1^{\leqslant} \mathbf{nf}\,(Q)$. `Ilya: add lemmas`

Let us rewrite the left-hand side and the right-hand side of $\Gamma \vdash [\widehat{\sigma}']\mathbf{nf}\,(P) \simeq_1^{\leqslant} \mathbf{nf}\,(Q)$ by transitivity of equivalence (corollary 4). By corollaries 5 and 14, $\Gamma \vdash [\widehat{\sigma}']\mathbf{nf}\,(P) \simeq_1^{\leqslant} [\widehat{\sigma}']P$. By corollary 14, $\Gamma \vdash \mathbf{nf}\,(Q) \simeq_1^{\leqslant} Q$. This way, we have $\Gamma \vdash [\widehat{\sigma}']P \simeq_1^{\leqslant} Q$. Then by Rule ($\uparrow^{\leqslant_1}$) and congruence of substitution, $\Gamma \vdash [\widehat{\sigma}']\uparrow P \leqslant_1 \uparrow Q$.

Case 3. $\Gamma; \Theta \models P \to N' \leqslant Q \to M' \dashv \widehat{\sigma}$
The next step of the algorithm is two recursive calls: $\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}_1$ and $\Gamma; \Theta \models N' \leqslant M' \dashv \widehat{\sigma}_2$. By the induction hypothesis,

1. $\widehat{\sigma}_1 : \Theta \cap \mathbf{uv}\, P$ and $\Gamma \vdash [\widehat{\sigma}_1']P \geqslant_1 Q$ for any $\widehat{\sigma}_1' \Rightarrow \widehat{\sigma}_1$
2. $\widehat{\sigma}_2 : \Theta \cap \mathbf{uv}\, N'$ and $\Gamma \vdash [\widehat{\sigma}_2']N' \leqslant_1 M'$ for any $\widehat{\sigma}_2' \Rightarrow \widehat{\sigma}_2$

Then the algorithm merges two unification solutions $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$. By **??**, since $\mathbf{uv}\, P \cup \mathbf{uv}\, N' = \mathbf{uv}\,(P \to N')$, we have $\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 : \Theta \cap \mathbf{uv}\,(P \to N')$, and also $\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_1$ and $\widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_2$. By the transitivity of solution weakening (**??**), $\widehat{\sigma}' \Rightarrow \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2$ implies $\widehat{\sigma}' \Rightarrow \widehat{\sigma}_1$ and $\widehat{\sigma}' \Rightarrow \widehat{\sigma}_2$.

The application of the induction hypothesis gives us $\Gamma \vdash [\widehat{\sigma}']P \geqslant_1 Q$ and $\Gamma \vdash [\widehat{\sigma}']N' \leqslant_1 M'$. Finally, by Rule ($\to^{\leqslant_1}$), $\Gamma \vdash [\widehat{\sigma}'](P \to N') \leqslant_1 Q \to M'$.

Case 4. $\Gamma; \Theta \models \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.M' \dashv \widehat{\sigma}$ s.t. either $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is not empty
Then the algorithm creates fresh unification variables $\overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}}$, substitutes the old $\overrightarrow{\alpha^+}$ with them in N', and makes the recursive call: $\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}'$, returning as the resulting solution $\widehat{\sigma} = \widehat{\sigma}' \backslash \overrightarrow{\widehat{\alpha}^+}$.

Notice that $\Gamma, \overrightarrow{\beta^+} \vdash^{\supseteq} \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}}$, $\Gamma, \overrightarrow{\beta^+} \vdash M'$, and $\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vdash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N$, so the induction hypothesis is applicable, that is $\widehat{\sigma}' : \Theta \cap \mathbf{uv}\, [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N'$ and $\Gamma, \overrightarrow{\beta^+} \vdash [\widehat{\sigma}_2'][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N' \leqslant_1 M'$ for any $\widehat{\sigma}_2' \Rightarrow \widehat{\sigma}'$.

Since the domain of $\widehat{\sigma}'$ is $\mathbf{uv}\, [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N'$, the domain of $\widehat{\sigma} = \widehat{\sigma}' \backslash \overrightarrow{\widehat{\alpha}^+}$ is $\mathbf{uv}\, [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \backslash \overrightarrow{\widehat{\alpha}^+} = \mathbf{uv}\, N' = \mathbf{uv}\, \forall\overrightarrow{\alpha^+}.N'$, this way, $\widehat{\sigma} : \Theta \cap \mathbf{uv}\, \forall\overrightarrow{\alpha^+}.N'$, as required.

It is left to show that $\Gamma \vdash [\widehat{\sigma}_2]\forall\overrightarrow{\alpha^+}.N' \leqslant_1 \forall\overrightarrow{\beta^+}.M'$ for any $\widehat{\sigma}_2 \Rightarrow \widehat{\sigma}$. Let us consider an arbitrary $\widehat{\sigma}_2 \Rightarrow \widehat{\sigma}$. Let us construct $\widehat{\sigma}_2'$, extending $\widehat{\sigma}_2$ to the domain $\mathbf{uv}\, [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N'$ with the values of $\widehat{\sigma}'$, i.e. $\widehat{\sigma}_2' : \Theta \cap \mathbf{uv}\, [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N'$, and

$$\widehat{\sigma}_2'(\widehat{\beta}^{\pm}) = \begin{cases} \widehat{\sigma}_2(\widehat{\beta}^{\pm}) & \text{if } \widehat{\beta}^{\pm} \in \mathbf{uv}\, N' \\ \widehat{\sigma}'(\widehat{\beta}^{\pm}) & \text{if } \widehat{\beta}^{\pm} \in \overrightarrow{\widehat{\alpha}^+} \end{cases}$$

, where the application of the unification solution to a variable returns the corresponding *unification entry*. Notice that $\widehat{\sigma}_2'|_{\mathbf{uv}\, N'} = \widehat{\sigma}_2$. It is easy to see that $\widehat{\sigma}_2' \Rightarrow \widehat{\sigma}'$:

1. if $\widehat{\beta}^{\pm} \in \mathbf{uv}\, N'$ then $\widehat{\sigma}_2'(\widehat{\beta}^{\pm}) = \widehat{\sigma}_2(\widehat{\beta}^{\pm}) \Rightarrow \widehat{\sigma}(\widehat{\beta}^{\pm}) = \widehat{\sigma}'(\widehat{\beta}^{\pm})$;
2. it $\widehat{\beta}^{\pm} \in \overrightarrow{\widehat{\alpha}^+}$ then $\widehat{\sigma}_2'(\widehat{\beta}^{\pm}) = \widehat{\sigma}'(\widehat{\beta}^{\pm}) \Rightarrow \widehat{\sigma}'(\widehat{\beta}^{\pm})$,

which means that the induction hypothesis can be applied to $\widehat{\sigma}'_2$, i.e. $\Gamma, \overrightarrow{\beta^+} \vdash [\widehat{\sigma}'_2][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N' \leqslant_1 M'$.

Notice that $[\widehat{\sigma}'_2][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N' = [\widehat{\sigma}'_2|_{\overrightarrow{\widehat{\alpha}^+}} \circ \overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}][\widehat{\sigma}'_2|_{\mathbf{uv}\, N'}]N'$ by substitution properties Ilya: todo

$= [\widehat{\sigma}'_2|_{\overrightarrow{\widehat{\alpha}^+}} \circ \overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}][\widehat{\sigma}_2]N'$ Since $\widehat{\sigma}'_2|_{\mathbf{uv}\, N'} = \widehat{\sigma}_2$.

Also notice that the domain of $\widehat{\sigma}'_2|_{\overrightarrow{\widehat{\alpha}^+}} \circ \overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}$ is $\overrightarrow{\alpha^+}$, and the range is $\Gamma, \overrightarrow{\beta^+}$, i.e. $\Gamma, \overrightarrow{\beta^+} \vdash \widehat{\sigma}'_2|_{\overrightarrow{\widehat{\alpha}^+}} \circ \overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+} : \overrightarrow{\alpha^+}$, which means that we can apply Rule ($\forall^{\leqslant_1}$) to $\Gamma, \overrightarrow{\beta^+} \vdash [\widehat{\sigma}'_2|_{\overrightarrow{\widehat{\alpha}^+}} \circ \overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}][\widehat{\sigma}_2]N' \leqslant_1 M'$ to get $\Gamma \vdash [\widehat{\sigma}_2]\forall\overrightarrow{\alpha^+}.N' \leqslant_1 \forall\overrightarrow{\beta^+}.M'$, as required.

Case 5. $\Gamma; \Theta \vDash \widehat{\alpha}^+ \geqslant P' \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q')$ where $\widehat{\alpha}^+\{\Delta\} \in \Theta$ and **upgrade** $\Gamma \vdash P'$ **to** $\Delta = Q'$

Notice that $\widehat{\alpha}^+\{\Delta\} \in \Theta$ and $\Gamma \vdash^{\supseteq} \Theta$ implies $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$ for some $\overrightarrow{\alpha^{\pm}}$, hence, the soundness of upgrade (lemma 24) is applicable:

1. $\Delta \vdash Q'$ and
2. $\Gamma \vdash Q \geqslant_1 P$.

Since $\widehat{\alpha}^+\{\Delta\} \in \Theta$ and $\Delta \vdash Q'$, it is clear that $(\Delta \vdash \widehat{\alpha}^+ :\geqslant Q') : \Theta \cap \mathbf{uv}\,\widehat{\alpha}^+$.

It is left to show that $\Gamma \vdash [\widehat{\sigma}']\widehat{\alpha}^+ \geqslant_1 P'$ for any $\widehat{\sigma}' \Rightarrow \Delta \vdash \widehat{\alpha}^+ :\geqslant Q'$. The latter weakening statement means that either $\widehat{\sigma}' \ni \Delta \vdash \widehat{\alpha}^+ :\geqslant Q''$ or $\widehat{\sigma}' \ni (\Delta \vdash \widehat{\alpha}^+ :\approx Q'')$ for $\Delta \vdash Q'' \geqslant_1 Q'$. In any case, $\Delta \vdash [\widehat{\sigma}']\widehat{\alpha}^+ \geqslant_1 Q$. By weakening the context to Γ and combining this judgment transitivity (**??**) with $\Gamma \vdash Q \geqslant_1 P$, we have $\Gamma \vdash [\widehat{\sigma}']\widehat{\alpha}^+ \geqslant_1 P$, as required.

Case 6. For the other positive cases, the proof is symmetric to the corresponding negative cases.

□

Notice that the induction statement in following theorem is not symmetric for the positive and negative cases.

Theorem 2 (Completeness and Initiality of the Subtyping Algorithm)**.**

− *Suppose that* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash M$ *and* $\Gamma; \Theta \vdash N \neq \widehat{\alpha}^-$ *and there exists* $\widehat{\sigma} : \Theta \cap \mathbf{uv}\, N$ *such that* $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$*. Then there exists* $\widehat{\sigma}'$ *such that* $\Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}'$ *and* $\widehat{\sigma} \Rightarrow \widehat{\sigma}'$*.*

\+ *Suppose that* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash Q$ *and* $\Gamma; \Theta \vdash P$ *and there exists* $\widehat{\sigma} : \Theta \cap \mathbf{uv}\, P$ *such that* $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$*. Then there exists* $\widehat{\sigma}'$ *such that* $\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}'$ *and* $\widehat{\sigma} \Rightarrow \widehat{\sigma}'$*.*

Proof. We prove it by induction on $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$ (and mutually, on $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$). Let us consider the last rule used in the derivation of $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$ or $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$, but first consider the base case for the substitution $[\widehat{\sigma}]P$:

Case 1. $P = \widehat{\alpha}^+$ is a unification variable

Then by assumption, there exists $\widehat{\sigma} : \Theta \cap \mathbf{uv}\, P$ such that $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant_1 Q$. It means that $\widehat{\sigma}$ sends $\widehat{\alpha}^+$ to P' such that $\Gamma \vdash P' \geqslant_1 Q$. This way, $\widehat{\sigma}$ is either $(\Delta \vdash \widehat{\alpha}^+ :\approx P')$ or $(\Delta \vdash \widehat{\alpha}^+ :\geqslant P')$, where $\widehat{\alpha}^+\{\Delta\} \in \Theta$.

In this case, the algorithm tries to apply Rule ($\mathrm{UVar}^{\geqslant}$) and the resulting solution is $\widehat{\sigma}' = (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q')$ where **upgrade** $\Gamma \vdash Q$ **to** $\Delta = Q'$.

Why does the upgrade procedure terminates? Because P' satisfies the pre-conditions of the completeness of the upgrade (lemma 25) :

- $\Delta \vdash P'$ because $P' = [\widehat{\sigma}]\widehat{\alpha}^+$, $\widehat{\sigma} : \Theta \cap \mathbf{uv}\, P$ and $\widehat{\alpha}^+\{\Delta\} \in \Theta \cap \mathbf{uv}\, P$,
- $\Gamma \vdash P' \geqslant_1 Q$ as noted before

Moreover, completeness of the upgrade also gives us $\Gamma \vdash P' \geqslant_1 Q'$ and further, we strengthen it to $\Delta \vdash P' \geqslant_1 Q'$ (since by the soundness of the upgrade (lemma 24), $\Delta \vdash Q'$).

It means that $(\Delta \vdash \widehat{\alpha}^+ :\approx P') \Rightarrow (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q')$ and $(\Delta \vdash \widehat{\alpha}^+ :\geqslant P') \Rightarrow (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q')$, which means that in any case, $\widehat{\sigma} \Rightarrow \widehat{\sigma}'$.

□