

$\alpha, \beta, \alpha, \beta, \gamma, \delta$	type variables
n, m, i, j	index variables
x, y, z	term variables

UC	$::=$ \cdot e $UC \backslash vars$ $UC vars$ $UC_1 \cup UC_2$ \overline{UC}_i^i (UC) S $UC' vars$ M $UC_1 \ \& \ UC_2$ M $UC_1 \cup UC_2$ M $ SC $ M	unification constraint
SC	$::=$ \cdot e $SC \backslash vars$ $SC vars$ $SC_1 \cup SC_2$ UC \overline{SC}_i^i (SC) S $SC' vars$ M $SC_1 \ \& \ SC_2$ M	subtyping constraint
$\hat{\sigma}$	$::=$ \cdot $P/\hat{\alpha}^+$ $N/\hat{\alpha}^-$ $\vec{P}/\vec{\hat{\alpha}}^+$ $\vec{N}/\vec{\hat{\alpha}}^-$ $(\hat{\sigma})$ S $\hat{\sigma}_1 \circ \hat{\sigma}_2$ $\overline{\hat{\sigma}}_i^i$ $\mathbf{nf}(\hat{\sigma}')$ M $\hat{\sigma}' vars$ M	unification substitution
$\hat{\tau}, \hat{\rho}$	$::=$ \cdot $\hat{\alpha}^- \mapsto N$ $\hat{\alpha}^- \mapsto \textcolor{gray}{N}$ $\vec{\alpha}^- / \vec{\hat{\alpha}}^-$ $\vec{N} / \vec{\hat{\alpha}}^-$ $\hat{\tau}_1 \cup \hat{\tau}_2$ $\overline{\hat{\tau}}_i^i$ $(\hat{\tau})$ S $\hat{\tau}' vars$ M $\hat{\tau}_1 \ \& \ \hat{\tau}_2$ M	anti-unification substitution
$\vec{\alpha}^+, \vec{\beta}^+, \vec{\gamma}^+, \vec{\delta}^+$	$::=$	positive variable list

		\cdot	empty list
		α^+	a variable
		$\overrightarrow{\alpha^+}$	a variable
		$\overrightarrow{\overrightarrow{\alpha^+}}^i$	concatenate lists
$\overrightarrow{\alpha^-}, \overrightarrow{\beta^-}, \overrightarrow{\gamma^-}, \overrightarrow{\delta^-}$::=		negative variables
		\cdot	empty list
		α^-	a variable
		$\overrightarrow{\alpha^-}$	variables
		$\overrightarrow{\overrightarrow{\alpha^-}}^i$	concatenate lists
$\overrightarrow{\alpha^\pm}, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm}, \overrightarrow{\delta^\pm}$::=		positive or negative variable list
		\cdot	empty list
		α^\pm	a variable
		$\overrightarrow{\alpha^\pm}$	variables
		$\overrightarrow{\overrightarrow{\alpha^\pm}}^i$	concatenate lists
P, Q, R	::=		positive declarative types
		α^+	
		$\downarrow N$	
		$\exists \alpha^-. P$	
		$[\sigma]P$	M
		$[\hat{\tau}]P$	M
		$[\hat{\sigma}]P$	M
		$[\mu]P$	M
		(P)	S
		$P_1 \vee P_2$	M
		$\mathbf{nf}(P')$	M
N, M, K	::=		negative declarative types
		α^-	
		$\uparrow P$	
		$P \rightarrow N$	
		$\forall \alpha^+. N$	
		$[\sigma]N$	M
		$[\hat{\tau}]N$	M
		$[\mu]N$	M
		$[\hat{\sigma}]N$	M
		(N)	S
		$\mathbf{nf}(N')$	M
\vec{P}, \vec{Q}	::=		list of positive types
		\cdot	empty list
		P	a singel type
		$[\sigma]\vec{P}$	M
		$\vec{\vec{P}}^i$	concatenate lists
		(\vec{P})	S
		$\mathbf{nf}(\vec{P}')$	M

\vec{N}, \vec{M}	$::=$		list of negative types
	\cdot		empty list
	N		a singel type
	$[\sigma]\vec{N}$	M	
	\vec{N}_i^i		concatenate lists
	(\vec{N})	S	
	$\mathbf{nf}(\vec{N}')$	M	
Δ, Γ	$::=$		declarative type context
	\cdot		empty context
	$\vec{\alpha}^+$		list of variables
	$\vec{\alpha}^-$		list of variables
	$\{\alpha^\pm\}$		list of variables
	$\vec{\Gamma}_i^i$		concatenate contexts
	(Γ)	S	
	$\Gamma, \vec{\alpha}^+$	M	append a list of variables
	$\Gamma, \vec{\alpha}^-$	M	append a list of variables
	Γ, α^\pm	M	append a list of variables
	$\Theta(\hat{\alpha}^+)$	M	
	$\Theta(\hat{\alpha}^-)$	M	
	$\Gamma_1 \cup \Gamma_2$		
	$\Gamma_1 \cap vars$		
	$\Gamma_1 \cup \Gamma_2$	M	
	$\mathbf{fv} N$	M	
	$\mathbf{fv} P$	M	
	$\mathbf{fv} P$	M	
	$\mathbf{fv} N$	M	
Θ	$::=$		algorithmic variable context
	\cdot		empty context
	$\vec{\alpha}\{\Delta\}$		from an ordered list of variables
	$\hat{\alpha}^+\{\Delta\}$		from a variable to a list
	$\vec{\Theta}_i^i$		concatenate contexts
	(Θ)	S	
	$\Theta _{vars}$		leave only those variables that are in the set
	$\Theta_1 \cup \Theta_2$		
Ξ	$::=$		anti-unification type variable context
	\cdot		empty context
	$\vec{\hat{\alpha}}^+$		list of positive variables
	$\vec{\hat{\alpha}}^-$		list of negative variables
	$\Xi, \vec{\hat{\alpha}}^+$	M	append a list of variables
	$\Xi, \vec{\hat{\alpha}}^-$	M	append a list of variables
	$\vec{\Xi}_i^i$		concatenate contexts
	(Ξ)	S	
	$\Xi_1 \cup \Xi_2$		
	$\Xi_1 \cap vars$		
	$\Xi' _{vars}$	M	
	$\mathbf{dom}(UC)$	M	
	$\mathbf{dom}(SC)$	M	

		dom ($\hat{\sigma}$)	M	
		dom ($\hat{\tau}$)	M	
		dom (Θ)	M	
		uv N	M	
		uv P	M	
$\vec{\alpha}, \vec{\beta}$::=			ordered positive or negative variables
		\cdot		empty list
		$\vec{\alpha}^+$		list of variables
		$\vec{\alpha}^-$		list of variables
		$\vec{\alpha}^\pm$		list of variables
		$\vec{\hat{\alpha}}^+$		list of variables
		$\vec{\hat{\alpha}}^-$		list of variables
		$\vec{\alpha}_1 \setminus vars$		setminus
		$vars$		
		$\vec{\alpha}_i^i$		concatenate contexts
		$(\vec{\alpha})$	S	parenthesis
		$[\mu]\vec{\alpha}$	M	apply moving to list
		$[\vec{\mu}]\vec{\alpha}$	M	apply umoving to list
		ord vars in P	M	
		ord vars in N	M	
		ord vars in P	M	
		ord vars in N	M	
$vars$::=			set of variables
		\emptyset		empty set
		$vars_1 \cap vars_2$		set intersection
		$vars_1 \cup vars_2$		set union
		$vars_1 \setminus vars_2$		set complement
		$(vars)$	S	parenthesis
		$\vec{\alpha}$		ordered list of variables
		$[\mu]vars$	M	apply moving to varset
		Ξ		algorithmic type context
		Γ		declarative type context
μ	::=			
		\cdot		empty moving
		$pma1 \mapsto pma2$		Positive unit substitution
		$nma1 \mapsto nma2$		Positive unit substitution
		$\mu_1 \cup \mu_2$	M	Set-like union of movings
		$\mu_1 \circ \mu_2$	M	Composition
		$\vec{\mu}_i^i$		concatenate movings
		$\mu _{vars}$	M	restriction on a set
		μ^{-1}	M	inversion
		nf (μ')	M	
$\vec{\mu}$::=			
		\cdot		empty moving
		$\vec{\hat{\alpha}}^+ / \vec{\alpha}^+$		
		$\vec{\hat{\alpha}}^- / \vec{\alpha}^-$		

$\hat{\alpha}^\pm$	$::=$ $\hat{\alpha}^\pm$	positive/negative unification variable
$\hat{\alpha}^+$	$::=$ $\hat{\alpha}^+$ $\hat{\alpha}^+\{\Delta\}$ $\hat{\alpha}^\pm$	positive unification variable
$\hat{\alpha}^-, \hat{\beta}^-$	$::=$ $\hat{\alpha}^-$ $\hat{\alpha}^-_{\{N,M\}}$ $\hat{\alpha}^-\{\Delta\}$ $\hat{\alpha}^\pm$	negative unification variable
$\overrightarrow{\hat{\alpha}^+}, \overrightarrow{\hat{\beta}^+}$	$::=$ \cdot $\hat{\alpha}^+$ $\overrightarrow{\hat{\alpha}^+}$ $\overrightarrow{\hat{\alpha}^+}_i$	positive unification variable list empty list a variable from a normal variable, context unspecified concatenate lists
$\overrightarrow{\hat{\alpha}^-}, \overrightarrow{\hat{\beta}^-}$	$::=$ \cdot $\hat{\alpha}^-$ Ξ $\overrightarrow{\hat{\alpha}^-\{\Delta\}}$ $\overrightarrow{\hat{\alpha}^-}$ $\overrightarrow{\hat{\alpha}^-}_i$	negative unification variable list empty list a variable from an antiunification context from a normal variable from a normal variable, context unspecified concatenate lists
P, Q	$::=$ $\hat{\alpha}^+$ α^+ $\downarrow N$ $\exists \alpha^-. P$ $[\sigma] P$ M $[\hat{\tau}] P$ M $[\mu] P$ M $[\hat{\sigma}] P$ M $[\vec{\mu}] P$ M (P) S $\mathbf{nf}(P')$ M	a positive algorithmic type
N, M	$::=$ $\hat{\alpha}^-$ α^- $\uparrow P$ $P \rightarrow N$ $\forall \alpha^+. N$ $[\sigma] N$ M $[\hat{\tau}] N$ M	a negative algorithmic type

	$[\mu]N$	M
	$[\hat{\sigma}]N$	M
	$[\vec{\mu}]N$	M
	(N)	S
	$\mathbf{nf}(N')$	M

$auSol$	$::=$	
		$(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$
		$(\Xi, N, \hat{\tau}_1, \hat{\tau}_2)$

$terminals$	$::=$	
		\exists
		\forall
		\uparrow
		\downarrow
		\mapsto
		\leftrightarrow
		\in
		\notin
		\cdot
		\perp
		\preceq
		\succeq
		\wr
		\subset
		\supset
		\diagdown
		\sqcup
		\mapsto
		\wr^u
		\wr^a
		\emptyset
		\circ
		\Rightarrow
		Π
		\equiv
		\neq
		\equiv_n
		\prec
		\Downarrow
		$\colon\geq$
		$\colon\wr$
		Λ
		λ
		\mathbf{let}^\exists
		\bullet
		$\Rightarrow\Rightarrow$
		$\Leftarrow\Leftarrow$

\vec{v}	$::=$	list of arguments
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	\cdot v \overrightarrow{v}_i^i	concatenate
v, w	$::=$ x $\{c\}$ $(v : P)$ (v)	value terms variable thunk annotated value M
c, d	$::=$ $(c : N)$ $\lambda x : P. c$ $\Lambda \alpha^+. c$ $\mathbf{return} \ v$ $\mathbf{let} \ x = v; c$ $\mathbf{let} \ x : P = v(\overrightarrow{v}); c$ $\mathbf{let} \ x = v(\overrightarrow{v}); c$ $\mathbf{let}^{\exists}(\overrightarrow{\alpha}^-, x) = v; c$	computation terms annotated computation annotated abstraction annotated type abstraction computation embedding a value standard let-binding annotated applicative let unannotated applicative let unpack
$vctx, \Phi$	$::=$ \cdot $x : P$ $\overrightarrow{\Phi}_i^i$	variable context concatenate contexts
$formula$	$::=$ $judgement$ $judgement \text{ unique}$ $formula_1 \ .. \ formula_n$ $\mu : vars_1 \leftrightarrow vars_2$ $\mu \text{ is bijective}$ $x : P \in \Phi$ $UC_1 \subseteq UC_2$ $UC_1 = UC_2$ $SC_1 \subseteq SC_2$ $e \in SC$ $e \in UC$ $vars_1 \subseteq vars_2$ $vars_1 \subseteq vars_2 \subseteq vars_3$ $vars_1 = vars_2$ $vars \text{ are fresh}$ $\alpha^- \notin vars$ $\alpha^+ \notin vars$ $\alpha^- \in vars$ $\alpha^+ \in vars$ $\hat{\alpha}^+ \in vars$ $\hat{\alpha}^- \in vars$ $\hat{\alpha}^- \in \Theta$ $\hat{\alpha}^+ \in \Theta$ $\hat{\alpha}^- \notin vars$	

	$\hat{\alpha}^+ \notin vars$ $\hat{\alpha}^- \notin \Theta$ $\hat{\alpha}^+ \notin \Theta$ $\hat{\alpha}^- \in \Xi$ $\hat{\alpha}^- \notin \Xi$ $\hat{\alpha}^+ \in \Xi$ $\hat{\alpha}^+ \notin \Xi$ if other rules are not applicable $\vec{\alpha}_1 = \vec{\alpha}_2$ $e_1 = e_2$ $e_1 = e_2$ $\hat{\sigma}_1 = \hat{\sigma}_2$ $\boxed{N} = \boxed{M}$ $\Theta \subseteq \Theta'$ $\vec{v}_1 = \vec{v}_2$ $\mathbf{N} \neq \mathbf{M}$ $\mathbf{P} \neq \mathbf{Q}$ $N \neq M$ $P \neq Q$ $\boxed{P} \neq \boxed{Q}$ $\boxed{N} \neq \boxed{M}$ $\vec{v}_1 \neq \vec{v}_2$ $\vec{\alpha}_1^+ \neq \vec{\alpha}_2^+$ $ \vec{\alpha}^- + \vec{\beta}^- > 0$ $ \vec{\alpha}^+ + \vec{\beta}^+ > 0$	
A	$::=$ $\Gamma; \Theta \models \boxed{N} \leq M \Rightarrow SC$ $\Gamma; \Theta \models \boxed{P} \geq Q \Rightarrow SC$	Negative subtyping Positive supertyping
AT	$::=$ $\Gamma; \Phi \models v : P$ $\Gamma; \Phi \models c : N$ $\Gamma; \Phi; \Theta_1 \models \boxed{N} \bullet \vec{v} \Rightarrow \boxed{M} \Rightarrow \Theta_2; SC$	Positive typing Negative typing Application typing
AU	$::=$ $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \boxed{Q}, \hat{\tau}_1, \hat{\tau}_2)$ $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \boxed{M}, \hat{\tau}_1, \hat{\tau}_2)$	
SCM	$::=$ $\Gamma \vdash e_1 \& e_2 = e_3$ $\Theta \vdash SC_1 \& SC_2 = SC_3$	Subtyping Constraint Entry Merge Merge of subtyping constraints
UCM	$::=$ $\Gamma \vdash e_1 \& e_2 = e_3$ $\Theta \vdash UC_1 \& UC_2 = UC_3$	Merge of unification constraints
$SATSCE$	$::=$ $\Gamma \vdash P : e$	Positive constraint entry satisfaction

	$\Gamma \vdash N : e$	Negative constraint entry satisfaction
<i>SING</i>	$::=$ e_1 singular with P e_1 singular with N SC singular with \hat{o}	Subtyping Constraint Is Singular
<i>E1</i>	$::=$ $N \simeq^D M$ $P \simeq^D Q$ $\boxed{P} \simeq^D \boxed{Q}$ $\boxed{N} \simeq^D \boxed{M}$	Negative type equivalence Positive type equivalence Positive unification type equivalence Positive unification type equivalence
<i>D1</i>	$::=$ $\Gamma \vdash N \simeq^{\leq} M$ $\Gamma \vdash P \simeq^{\leq} Q$ $\Gamma \vdash N \leq M$ $\Gamma \vdash P \geq Q$	Negative subtyping-induced equivalence Positive subtyping-induced equivalence Negative subtyping Positive supertyping
<i>D1S</i>	$::=$ $\Gamma_2 \vdash \sigma_1 \simeq^{\leq} \sigma_2 : \Gamma_1$ $\Gamma \vdash \sigma_1 \simeq^{\leq} \sigma_2 : vars$ $\Theta \vdash \hat{\sigma}_1 \simeq^{\leq} \hat{\sigma}_2 : vars$ $\Gamma \vdash \hat{\sigma}_1 \simeq^{\leq} \hat{\sigma}_2 : vars$	Equivalence of substitutions Equivalence of substitutions Equivalence of unification substitutions Equivalence of unification substitutions
<i>D1C</i>	$::=$ $\Gamma \vdash \Phi_1 \simeq^{\leq} \Phi_2$	Equivalence of contexts
<i>DT</i>	$::=$ $\Gamma; \Phi \vdash v : P$ $\Gamma; \Phi \vdash c : N$ $\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M$	Positive typing Negative typing Application typing
<i>EQ</i>	$::=$ $N = M$ $P = Q$ $\boxed{P} = \boxed{Q}$	Negative type equality (alpha-equivalence) Positive type equality (alpha-equivalence)
<i>LUBF</i>	$::=$ $P_1 \vee P_2 === Q$ ord vars in $\boxed{P} === \vec{\alpha}$ ord vars in $\boxed{N} === \vec{\alpha}$ ord vars in $P === \vec{\alpha}$ ord vars in $N === \vec{\alpha}$ nf $(N') === N$ nf $(P') === P$ nf $(\boxed{N'}) === \boxed{N}$ nf $(\boxed{P'}) === \boxed{P}$ nf $(\vec{N'}) === \vec{N}$	

	$\text{nf}(\vec{P}') === \vec{P}$ $\text{nf}(\sigma') === \sigma$ $\text{nf}(\hat{\sigma}') === \hat{\sigma}$ $\text{nf}(\mu') === \mu$ $\sigma' _{vars}$ $\hat{\sigma}' _{vars}$ $\hat{\tau}' _{vars}$ $\Xi' _{vars}$ $SC' _{vars}$ $UC' _{vars}$ $e_1 \ \& \ e_2$ $e_1 \ \& \ e_2$ $UC_1 \ \& \ UC_2$ $UC_1 \cup UC_2$ $\Gamma_1 \cup \Gamma_2$ $SC_1 \ \& \ SC_2$ $\hat{\tau}_1 \ \& \ \hat{\tau}_2$ $\text{dom}(UC) === \Xi$ $\text{dom}(SC) === \Xi$ $\text{dom}(\hat{\sigma}) === \Xi$ $\text{dom}(\hat{\tau}) === \Xi$ $\text{dom}(\Theta) === \Xi$ $ SC === UC$ $\text{fv } N === \Gamma$ $\text{fv } P === \Gamma$ $\text{fv } P === \Gamma$ $\text{fv } N === \Gamma$ $\text{uv } N === \Xi$ $\text{uv } P === \Xi$	
<i>LUB</i>	$::=$ $\Gamma \models P_1 \vee P_2 = Q$ $\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q$	Least Upper Bound
<i>Nrm</i>	$::=$ $\text{nf}(N) = M$ $\text{nf}(P) = Q$ $\text{nf}(N) = M$ $\text{nf}(P) = Q$	
<i>Order</i>	$::=$ $\text{ord vars in } N = \vec{\alpha}$ $\text{ord vars in } P = \vec{\alpha}$ $\text{ord vars in } N = \vec{\alpha}$ $\text{ord vars in } P = \vec{\alpha}$	variable ordering in a negative type
<i>U</i>	$::=$ $\Gamma; \Theta \models N \stackrel{u}{\simeq} M = UC$ $\Gamma; \Theta \models P \stackrel{u}{\simeq} Q = UC$	Negative unification Positive unification

WFT	$::=$		
		$\Gamma \vdash N$	Negative type well-formedness
		$\Gamma \vdash P$	Positive type well-formedness
$WFAT$	$::=$		
		$\Gamma; \Xi \vdash N$	Negative algorithmic type well-formedness
		$\Gamma; \Xi \vdash P$	Positive algorithmic type well-formedness
$WFALL$	$::=$		
		$\Gamma \vdash \vec{N}$	Negative type list well-formedness
		$\Gamma \vdash \vec{P}$	Positive type list well-formedness
		$\Gamma; \Xi_2 \vdash \hat{\tau} : \Xi_1$	Antiunification substitution well-formedness
		$\Gamma \vdash^\exists \Theta$	Unification context well-formedness
		$\Gamma_1 \vdash \sigma : \Gamma_2$	Substitution signature
		$\Theta \vdash \hat{\sigma} : \Xi$	Unification substitution signature
		$\Gamma \vdash \hat{\sigma} : \Xi$	Unification substitution general signature
		$\Theta \vdash \hat{\sigma} : UC$	Unification substitution satisfies unification constraint
		$\Theta \vdash \hat{\sigma} : SC$	Unification substitution satisfies subtyping constraint
		$\Gamma \vdash e$	Unification constraint entry well-formedness
		$\Gamma \vdash e$	Subtyping constraint entry well-formedness
		$\Gamma \vdash P : e$	Positive type satisfies unification constraint
		$\Gamma \vdash N : e$	Negative type satisfies unification constraint
		$\Gamma \vdash P : e$	Positive type satisfies subtyping constraint
		$\Gamma \vdash N : e$	Negative type satisfies subtyping constraint
		$\Theta \vdash UC : \Xi$	Unification constraint well-formedness with specified domain
		$\Theta \vdash SC : \Xi$	Subtyping constraint well-formedness with specified domain
		$\Theta \vdash UC$	Unification constraint well-formedness
		$\Theta \vdash SC$	Subtyping constraint well-formedness
		$\Gamma \vdash \vec{v}$	Argument List well-formedness
		$\Gamma \vdash \Phi$	Context well-formedness
		$\Gamma \vdash v$	Value well-formedness
		$\Gamma \vdash c$	Computation well-formedness
$judgement$	$::=$		
		A	
		AT	
		AU	
		SCM	
		UCM	
		$SATSCE$	
		$SING$	
		$E1$	
		$D1$	
		$D1S$	
		$D1C$	
		DT	
		EQ	
		LUB	
		Nrm	
		$Order$	

		U
		WFT
		$WFAT$
		$WFALL$
$user_syntax$	$::=$	
		α
		n
		x
		n
		α^+
		α^-
		α^\pm
		σ
		e
		e
		UC
		SC
		$\hat{\sigma}$
		$\hat{\tau}$
		$\overrightarrow{\alpha^+}$
		$\overrightarrow{\alpha^-}$
		$\overrightarrow{\alpha^\pm}$
		P
		N
		\vec{P}
		\vec{N}
		Γ
		Θ
		Ξ
		$\vec{\alpha}$
		$vars$
		μ
		$\vec{\mu}$
		$\hat{\alpha}^\pm$
		$\hat{\alpha}^+$
		$\hat{\alpha}^-$
		$\overrightarrow{\hat{\alpha}^+}$
		$\overrightarrow{\hat{\alpha}^-}$
		P
		N
		$auSol$
		$terminals$
		\vec{v}
		v
		c
		$vctx$
		$formula$

$\boxed{\Gamma; \Theta \models N \leqslant M \Rightarrow SC}$ Negative subtyping

$$\begin{array}{c}
\frac{}{\Gamma; \Theta \models \alpha^- \leq \alpha^- \Rightarrow} \text{ANVAR} \\
\frac{\Gamma; \Theta \models \mathbf{nf}(P) \stackrel{u}{\simeq} \mathbf{nf}(Q) \Rightarrow UC}{\Gamma; \Theta \models \uparrow P \leq \uparrow Q \Rightarrow UC} \text{ASHIFTU} \\
\frac{\begin{array}{c} \overrightarrow{\hat{\alpha}^+} \text{ are fresh} \\ \text{<<multiple parses>>} \end{array}}{\Gamma; \Theta \models \forall \alpha^+. N \leq \forall \beta^+. M \Rightarrow SC \setminus \overrightarrow{\hat{\alpha}^+}} \text{AFORALL} \\
\frac{\begin{array}{c} \Gamma; \Theta \models P \geq Q \Rightarrow SC_1 \quad \Gamma; \Theta \models N \leq M \Rightarrow SC_2 \\ \Theta \vdash SC_1 \& SC_2 = SC \end{array}}{\Gamma; \Theta \models P \rightarrow N \leq Q \rightarrow M \Rightarrow SC} \text{AARROW} \\
\boxed{\Gamma; \Theta \models P \geq Q \Rightarrow SC} \quad \text{Positive supertyping}
\end{array}$$

$$\begin{array}{c}
\frac{}{\Gamma; \Theta \models \alpha^+ \geq \alpha^+ \Rightarrow} \text{APVAR} \\
\frac{\Gamma; \Theta \models \mathbf{nf}(N) \stackrel{u}{\simeq} \mathbf{nf}(M) \Rightarrow UC}{\Gamma; \Theta \models \downarrow N \geq \downarrow M \Rightarrow UC} \text{ASHIFTD} \\
\frac{\begin{array}{c} \overrightarrow{\hat{\alpha}^-} \text{ are fresh} \\ \Gamma, \beta^-; \Theta, \hat{\alpha}^- \{ \Gamma, \beta^- \} \models [\hat{\alpha}^- / \alpha^-] P \geq Q \Rightarrow SC \end{array}}{\Gamma; \Theta \models \exists \alpha^-. P \geq \exists \beta^-. Q \Rightarrow SC \setminus \overrightarrow{\hat{\alpha}^-}} \text{AEXISTS} \\
\frac{\text{upgrade } \Gamma \vdash P \text{ to } \Theta(\hat{\alpha}^+) = Q}{\Gamma; \Theta \models \hat{\alpha}^+ \geq P \Rightarrow (\hat{\alpha}^+ : \geq Q)} \text{APUVAR} \\
\boxed{\Gamma; \Phi \models v : P} \quad \text{Positive typing}
\end{array}$$

$$\begin{array}{c}
\frac{x : P \in \Phi}{\Gamma; \Phi \models x : \mathbf{nf}(P)} \text{ATVAR} \\
\frac{\Gamma; \Phi \models c : N}{\Gamma; \Phi \models \{c\} : \downarrow N} \text{ATT HUNK} \\
\frac{\Gamma \vdash Q \quad \Gamma; \Phi \models v : P \quad \Gamma; \cdot \models Q \geq P \Rightarrow \cdot}{\Gamma; \Phi \models (v : Q) : \mathbf{nf}(Q)} \text{ATPANNOT} \\
\boxed{\Gamma; \Phi \models c : N} \quad \text{Negative typing}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash M \quad \Gamma; \Phi \models c : N \quad \Gamma; \cdot \models N \leq M \Rightarrow \cdot}{\Gamma; \Phi \models (c : M) : \mathbf{nf}(M)} \text{ATNANNOT} \\
\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \lambda x : P. c : \mathbf{nf}(P \rightarrow N)} \text{ATTLAM} \\
\frac{\Gamma, \alpha^+; \Phi \models c : N}{\Gamma; \Phi \models \Lambda \alpha^+. c : \mathbf{nf}(\forall \alpha^+. N)} \text{ATTLAM} \\
\frac{\Gamma; \Phi \models v : P}{\Gamma; \Phi \models \mathbf{return} v : \uparrow P} \text{ATRETURN} \\
\frac{\Gamma; \Phi \models v : P \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \mathbf{let} x = v; c : N} \text{ATVARLET}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash P \quad \Gamma; \Phi \models v : \downarrow M \quad \Gamma; \Phi; \cdot \models \overline{M} \bullet \vec{v} \Rightarrow \uparrow Q \models \Theta; SC_1 \quad \Gamma; \Theta \models \uparrow Q \leq \uparrow P \models SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \text{let } x : P = v(\vec{v}); c : N} \text{ ATAPPLETANN} \\
\\
\frac{\Gamma; \Phi \models v : \downarrow M \quad \Gamma; \Phi; \cdot \models \overline{M} \bullet \vec{v} \Rightarrow \uparrow Q \models \Theta; SC \quad \llcorner \text{multiple parses} \gg \quad \Gamma; \Phi, x : [\hat{\sigma}] Q \models c : N}{\Gamma; \Phi \models \text{let } x = v(\vec{v}); c : N} \text{ ATAPPLET} \\
\\
\frac{\Gamma; \Phi \models v : \exists \alpha^-. \overline{P} \quad \Gamma, \alpha^-; \Phi, x : P \models c : N \quad \Gamma \vdash N}{\Gamma; \Phi \models \text{let}^3(\alpha^-, x) = v; c : N} \text{ ATUNPACK} \\
\\
\boxed{\Gamma; \Phi; \Theta_1 \models \overline{N} \bullet \vec{v} \Rightarrow \overline{M} \models \Theta_2; SC} \quad \text{Application typing} \\
\\
\frac{}{\Gamma; \Phi; \Theta \models \overline{N} \bullet \cdot \Rightarrow \text{nf}(N) \models \Theta; \cdot} \text{ AEMPTYAPP} \\
\\
\frac{\Gamma; \Phi \models v : P \quad \Gamma; \Theta \models \overline{Q} \geq P \models SC_1 \quad \Gamma; \Phi; \Theta \models \overline{N} \bullet \vec{v} \Rightarrow \overline{M} \models \Theta'; SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC}{\Gamma; \Phi; \Theta \models \overline{Q} \rightarrow \overline{N} \bullet v, \vec{v} \Rightarrow \overline{M} \models \Theta'; SC} \text{ ATARROWAPP} \\
\\
\frac{\llcorner \text{multiple parses} \gg \quad \hat{\alpha}^+ \text{ are fresh} \quad \vec{v} \neq \cdot \quad \alpha^+ \neq \cdot}{\llcorner \text{multiple parses} \gg} \text{ ATFORALLAPP} \\
\\
\boxed{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)} \\
\\
\frac{}{\Gamma \models \alpha^+ \stackrel{a}{\simeq} \alpha^+ \models (\cdot, \alpha^+, \cdot, \cdot)} \text{ AUPVAR} \\
\\
\frac{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \models (\Xi, \overline{M}, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N_1 \stackrel{a}{\simeq} \downarrow N_2 \models (\Xi, \downarrow \overline{M}, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUSHIFTD} \\
\\
\frac{\llcorner \text{multiple parses} \gg}{\Gamma \models \exists \alpha^-. \overline{P}_1 \stackrel{a}{\simeq} \exists \alpha^-. \overline{P}_2 \models (\Xi, \exists \alpha^-. \overline{Q}, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUEXISTS} \\
\\
\boxed{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \models (\Xi, \overline{M}, \hat{\tau}_1, \hat{\tau}_2)} \\
\\
\frac{}{\Gamma \models \alpha^- \stackrel{a}{\simeq} \alpha^- \models (\cdot, \alpha^-, \cdot, \cdot)} \text{ AUNVAR} \\
\\
\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (\Xi, \overline{Q}, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \uparrow P_1 \stackrel{a}{\simeq} \uparrow P_2 \models (\Xi, \uparrow \overline{Q}, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUSHIFTU} \\
\\
\frac{\llcorner \text{multiple parses} \gg}{\Gamma \models \forall \alpha^+. \overline{N}_1 \stackrel{a}{\simeq} \forall \alpha^+. \overline{N}_2 \models (\Xi, \forall \alpha^+. \overline{M}, \hat{\tau}_1, \hat{\tau}_2)} \text{ AUFORALL} \\
\\
\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (\Xi_1, \overline{Q}, \hat{\tau}_1, \hat{\tau}_2) \quad \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \models (\Xi_2, \overline{M}, \hat{\tau}'_1, \hat{\tau}'_2)}{\Gamma \models P_1 \rightarrow N_1 \stackrel{a}{\simeq} P_2 \rightarrow N_2 \models (\Xi_1 \cup \Xi_2, \overline{Q} \rightarrow \overline{M}, \hat{\tau}_1 \cup \hat{\tau}'_1, \hat{\tau}_2 \cup \hat{\tau}'_2)} \text{ AUARROW} \\
\\
\frac{\text{if other rules are not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \stackrel{a}{\simeq} M \models (\hat{\alpha}^-_{\{N,M\}}, \hat{\alpha}^-_{\{N,M\}}, (\hat{\alpha}^-_{\{N,M\}} \mapsto N), (\hat{\alpha}^-_{\{N,M\}} \mapsto M))} \text{ AUAU}
\end{array}$$

$\boxed{\Gamma \vdash e_1 \& e_2 = e_3}$ Subtyping Constraint Entry Merge

$$\frac{\Gamma \models P_1 \vee P_2 = Q}{\Gamma \vdash (\hat{\alpha}^+ : \geq P_1) \& (\hat{\alpha}^+ : \geq P_2) = (\hat{\alpha}^+ : \geq Q)} \quad \text{SCMESUPSUP}$$

$$\frac{\Gamma; \cdot \models \textcolor{gray}{P} \geq Q = \cdot}{\Gamma \vdash (\hat{\alpha}^+ : \simeq P) \& (\hat{\alpha}^+ : \geq Q) = (\hat{\alpha}^+ : \simeq P)} \quad \text{SCMEEQSUP}$$

$$\frac{\Gamma; \cdot \models \textcolor{gray}{Q} \geq P = \cdot}{\Gamma \vdash (\hat{\alpha}^+ : \geq P) \& (\hat{\alpha}^+ : \simeq Q) = (\hat{\alpha}^+ : \simeq Q)} \quad \text{SCMESUPEQ}$$

$$\frac{\textcolor{red}{\langle\langle \text{multiple parses} \rangle\rangle}}{\Gamma \vdash (\hat{\alpha}^+ : \simeq P) \& (\hat{\alpha}^+ : \simeq P') = (\hat{\alpha}^+ : \simeq P)} \quad \text{SCMEPEQEQ}$$

$$\frac{\textcolor{red}{\langle\langle \text{multiple parses} \rangle\rangle}}{\Gamma \vdash (\hat{\alpha}^- : \simeq N) \& (\hat{\alpha}^- : \simeq N') = (\hat{\alpha}^- : \simeq N)} \quad \text{SCMENEQEQ}$$

$\boxed{\Theta \vdash SC_1 \& SC_2 = SC_3}$ Merge of subtyping constraints

$\boxed{\Gamma \vdash e_1 \& e_2 = e_3}$

$$\frac{\textcolor{red}{\langle\langle \text{multiple parses} \rangle\rangle}}{\Gamma \vdash (\hat{\alpha}^+ : \simeq P) \& (\hat{\alpha}^+ : \simeq P') = (\hat{\alpha}^+ : \simeq P)} \quad \text{UCMEPEQEQ}$$

$$\frac{\textcolor{red}{\langle\langle \text{multiple parses} \rangle\rangle}}{\Gamma \vdash (\hat{\alpha}^- : \simeq N) \& (\hat{\alpha}^- : \simeq N') = (\hat{\alpha}^- : \simeq N)} \quad \text{UCMENEQEQ}$$

$\boxed{\Theta \vdash UC_1 \& UC_2 = UC_3}$ Merge of unification constraints

$\boxed{\Gamma \vdash P : e}$ Positive constraint entry satisfaction

$$\frac{\Gamma \vdash P \geq Q}{\Gamma \vdash P : (\hat{\alpha}^+ : \geq Q)} \quad \text{SATSCESUP}$$

$$\frac{\textcolor{red}{\langle\langle \text{multiple parses} \rangle\rangle}}{\Gamma \vdash P : (\hat{\alpha}^+ : \simeq Q)} \quad \text{SATSCEPEQ}$$

$\boxed{\Gamma \vdash N : e}$ Negative constraint entry satisfaction

$$\frac{\textcolor{red}{\langle\langle \text{multiple parses} \rangle\rangle}}{\Gamma \vdash N : (\hat{\alpha}^- : \simeq M)} \quad \text{SATSCENEQ}$$

$\boxed{e_1 \text{ singular with } P}$

$$\overline{\hat{\alpha}^+ : \simeq P \text{ singular with nf } (P)} \quad \text{SINGPEQ}$$

$$\frac{}{\hat{\alpha}^+ : \geq \overrightarrow{\exists \alpha^- . \alpha^+} \text{ singular with } \alpha^+} \quad \text{SINGSUPVAR}$$

$$\frac{\textcolor{red}{\langle\langle \text{multiple parses} \rangle\rangle}}{\hat{\alpha}^+ : \geq \overrightarrow{\exists \alpha^- . \downarrow N} \text{ singular with } \exists \alpha^- . \downarrow \alpha^-} \quad \text{SINGSUPSHIFT}$$

$\boxed{e_1 \text{ singular with } N}$

$$\overline{\hat{\alpha}^- : \simeq N \text{ singular with nf } (N)} \quad \text{SINGNEQ}$$

$\boxed{SC \text{ singular with } \hat{o}}$ Subtyping Constraint Is Singular

$\boxed{N \simeq^D M}$ Negative type equivalence

$$\overline{\alpha^- \simeq^D \alpha^-} \quad \text{E1NVAR}$$

$$\begin{array}{c}
\frac{P \simeq^D Q}{\uparrow P \simeq^D \uparrow Q} \text{ E1SHIFTU} \\
\frac{P \simeq^D Q \quad N \simeq^D M}{P \rightarrow N \simeq^D Q \rightarrow M} \text{ E1ARROW} \\
\frac{\mu : (\vec{\beta}^+ \cap \mathbf{fv} M) \leftrightarrow (\vec{\alpha}^+ \cap \mathbf{fv} N) \quad \langle\langle \text{multiple parses} \rangle\rangle}{\forall \vec{\alpha}^+. N \simeq^D \forall \vec{\beta}^+. M} \text{ E1FORALL}
\end{array}$$

$\boxed{P \simeq^D Q}$ Positive type equivalence

$$\begin{array}{c}
\frac{}{\alpha^+ \simeq^D \alpha^+} \text{ E1PVAR} \\
\frac{N \simeq^D M}{\downarrow N \simeq^D \downarrow M} \text{ E1SHIFTD} \\
\frac{\mu : (\vec{\beta}^- \cap \mathbf{fv} Q) \leftrightarrow (\vec{\alpha}^- \cap \mathbf{fv} P) \quad \langle\langle \text{multiple parses} \rangle\rangle}{\exists \vec{\alpha}^-. P \simeq^D \exists \vec{\beta}^-. Q} \text{ E1EXISTS}
\end{array}$$

$\boxed{P \simeq^D Q}$ Positive unification type equivalence

$\boxed{N \simeq^D M}$ Positive unification type equivalence

$\boxed{\Gamma \vdash N \simeq^{\leq} M}$ Negative subtyping-induced equivalence

$$\frac{\Gamma \vdash N \leq M \quad \Gamma \vdash M \leq N}{\Gamma \vdash N \simeq^{\leq} M} \text{ D1NDEF}$$

$\boxed{\Gamma \vdash P \simeq^{\leq} Q}$ Positive subtyping-induced equivalence

$$\frac{\Gamma \vdash P \geq Q \quad \Gamma \vdash Q \geq P}{\Gamma \vdash P \simeq^{\leq} Q} \text{ D1PDEF}$$

$\boxed{\Gamma \vdash N \leq M}$ Negative subtyping

$$\begin{array}{c}
\frac{}{\Gamma \vdash \alpha^- \leq \alpha^-} \text{ D1NVAR} \\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\Gamma \vdash \uparrow P \leq \uparrow Q} \text{ D1SHIFTU} \\
\frac{\Gamma \vdash P \geq Q \quad \Gamma \vdash N \leq M}{\Gamma \vdash P \rightarrow N \leq Q \rightarrow M} \text{ D1ARROW} \\
\frac{\Gamma, \vec{\beta}^+ \vdash \sigma : \vec{\alpha}^+ \quad \Gamma, \vec{\beta}^+ \vdash [\sigma]N \leq M}{\Gamma \vdash \forall \vec{\alpha}^+. N \leq \forall \vec{\beta}^+. M} \text{ D1FORALL}
\end{array}$$

$\boxed{\Gamma \vdash P \geq Q}$ Positive supertyping

$$\begin{array}{c}
\frac{}{\Gamma \vdash \alpha^+ \geq \alpha^+} \text{ D1PVAR} \\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\Gamma \vdash \downarrow N \geq \downarrow M} \text{ D1SHIFTD} \\
\frac{\Gamma, \vec{\beta}^- \vdash \sigma : \vec{\alpha}^- \quad \Gamma, \vec{\beta}^- \vdash [\sigma]P \geq Q}{\Gamma \vdash \exists \vec{\alpha}^-. P \geq \exists \vec{\beta}^-. Q} \text{ D1EXISTS}
\end{array}$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq^{\leq} \sigma_2 : \Gamma_1}$	Equivalence of substitutions
$\boxed{\Gamma \vdash \sigma_1 \simeq^{\leq} \sigma_2 : vars}$	Equivalence of substitutions
$\boxed{\Theta \vdash \hat{\sigma}_1 \simeq^{\leq} \hat{\sigma}_2 : vars}$	Equivalence of unification substitutions
$\boxed{\Gamma \vdash \hat{\sigma}_1 \simeq^{\leq} \hat{\sigma}_2 : vars}$	Equivalence of unification substitutions
$\boxed{\Gamma \vdash \Phi_1 \simeq^{\leq} \Phi_2}$	Equivalence of contexts
$\boxed{\Gamma; \Phi \vdash v : P}$	Positive typing

$$\frac{x : P \in \Phi}{\Gamma; \Phi \vdash x : P} \text{DTV}_{\text{VAR}}$$

$$\frac{\Gamma; \Phi \vdash c : N}{\Gamma; \Phi \vdash \{c\} : \downarrow N} \text{DTT}_{\text{HUNK}}$$

$$\frac{\Gamma \vdash Q \quad \Gamma; \Phi \vdash v : P \quad \Gamma \vdash Q \geq P}{\Gamma; \Phi \vdash (v : Q) : Q} \text{DTP}_{\text{ANNOT}}$$

$$\frac{\text{<<multiple parses>>}}{\Gamma; \Phi \vdash v : P'} \text{DTPE}_{\text{EQUIV}}$$

$$\boxed{\Gamma; \Phi \vdash c : N} \quad \text{Negative typing}$$

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \lambda x : P. c : P \rightarrow N} \text{DTT}_{\text{LAM}}$$

$$\frac{\Gamma, \alpha^+; \Phi \vdash c : N}{\Gamma; \Phi \vdash \Lambda \alpha^+. c : \forall \alpha^+. N} \text{DTT}_{\text{LAM}}$$

$$\frac{\Gamma; \Phi \vdash v : P}{\Gamma; \Phi \vdash \mathbf{return} v : \uparrow P} \text{DTR}_{\text{RETURN}}$$

$$\frac{\Gamma; \Phi \vdash v : P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x = v; c : N} \text{DTV}_{\text{VARLET}}$$

$$\frac{\Gamma; \Phi \vdash v : \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q \text{ unique} \quad \Gamma; \Phi, x : Q \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x = v(\vec{v}); c : N} \text{DTAPP}_{\text{LET}}$$

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi \vdash v : \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q \quad \Gamma \vdash \uparrow Q \leq \uparrow P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x : P = v(\vec{v}); c : N} \text{DTAPP}_{\text{LETANN}}$$

$$\frac{\text{<<multiple parses>>} \quad \Gamma, \vec{\alpha}^-; \Phi, x : P \vdash c : N \quad \Gamma \vdash N}{\Gamma; \Phi \vdash \mathbf{let}^{\exists}(\vec{\alpha}^-, x) = v; c : N} \text{DTUN}_{\text{PACK}}$$

$$\frac{\Gamma \vdash M \quad \Gamma; \Phi \vdash c : N \quad \Gamma \vdash N \leq M}{\Gamma; \Phi \vdash (c : M) : M} \text{DTN}_{\text{ANNOT}}$$

$$\frac{\text{<<multiple parses>>}}{\Gamma; \Phi \vdash c : N'} \text{DTNE}_{\text{EQUIV}}$$

$$\boxed{\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M} \quad \text{Application typing}$$

$$\frac{\text{<<multiple parses>>}}{\Gamma; \Phi \vdash N \bullet \cdot \Rightarrow N'} \text{DTE}_{\text{EMPTYAPP}}$$

$$\frac{\Gamma; \Phi \vdash v : P \quad \Gamma \vdash Q \geq P \quad \Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M}{\Gamma; \Phi \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M} \text{DTARROW}_{\text{APP}}$$

$$\frac{\vec{v} \neq \cdot \quad \vec{\alpha}^+ \neq \cdot \quad \Gamma \vdash \sigma : \vec{\alpha}^+ \quad \Gamma; \Phi \vdash [\sigma]N \bullet \vec{v} \Rightarrow M}{\Gamma; \Phi \vdash \forall \vec{\alpha}^+. N \bullet \vec{v} \Rightarrow M} \text{DTForallApp}$$

$\boxed{N = M}$ Negative type equality (alpha-equivalence)

$\boxed{P = Q}$ Positive type equality (alpha-equivalence)

$\boxed{P = Q}$

$\boxed{P_1 \vee P_2}$

$\boxed{\text{ord vars in } P}$

$\boxed{\text{ord vars in } N}$

$\boxed{\text{ord vars in } P}$

$\boxed{\text{ord vars in } N}$

$\boxed{\text{nf}(N')}$

$\boxed{\text{nf}(P')}$

$\boxed{\text{nf}(N')}$

$\boxed{\text{nf}(P')}$

$\boxed{\text{nf}(\vec{N}')}$

$\boxed{\text{nf}(\vec{P}')}$

$\boxed{\text{nf}(\sigma')}$

$\boxed{\text{nf}(\hat{\sigma}')}$

$$\mathbf{nf}(\mu')$$

$$\sigma'|_{vars}$$

$$\hat{\sigma}'|_{vars}$$

$$\hat{\tau}'|_{vars}$$

$$\Xi'|_{vars}$$

$$SC'|_{vars}$$

$$UC'|_{vars}$$

$$e_1 \ \& \ e_2$$

$$e_1 \ \& \ e_2$$

$$UC_1 \ \& \ UC_2$$

$$UC_1 \cup UC_2$$

$$\Gamma_1 \cup \Gamma_2$$

$$SC_1 \ \& \ SC_2$$

$$\hat{\tau}_1 \ \& \ \hat{\tau}_2$$

$$\mathbf{dom}(UC)$$

$\text{dom}(SC)$ $\text{dom}(\hat{\sigma})$ $\text{dom}(\hat{\tau})$ $\text{dom}(\Theta)$ $|SC|$ $\text{fv } N$ $\text{fv } P$ $\text{fv } P$ $\text{fv } N$ $\text{uv } N$ $\text{uv } P$ $\Gamma \models P_1 \vee P_2 = Q$ Least Upper Bound

$$\begin{array}{c}
\overline{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVAR} \\
\frac{\Gamma \models \mathbf{nf}(\downarrow N) \stackrel{a}{\simeq} \mathbf{nf}(\downarrow M) \Rightarrow (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-} . [\overrightarrow{\alpha^-} / \Xi] P} \quad \text{LUBSHIFT} \\
\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q}{\Gamma \models \exists \overrightarrow{\alpha^-} . P_1 \vee \exists \overrightarrow{\beta^-} . P_2 = Q} \quad \text{LUBEXISTS}
\end{array}$$

 $\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q$

$$\frac{\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \models [\overrightarrow{\beta^\pm} / \overrightarrow{\alpha^\pm}] P \vee [\overrightarrow{\gamma^\pm} / \overrightarrow{\alpha^\pm}] P = Q}{\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q} \quad \text{LUBUPGRADE}$$

$$\boxed{\mathbf{nf}(N) = M}$$

$$\begin{array}{c} \overline{\mathbf{nf}(\alpha^-) = \alpha^-} \quad \text{NRMNVAR} \\ \frac{\text{\textcolor{red}{<<multiple parses>>}}}{\mathbf{nf}(\uparrow P) = \uparrow Q} \quad \text{NRMSHIFTU} \\ \frac{\text{\textcolor{red}{<<multiple parses>>}}}{\mathbf{nf}(P \rightarrow N) = Q \rightarrow M} \quad \text{NRMARROW} \\ \frac{\text{\textcolor{red}{<<multiple parses>>}}}{\mathbf{nf}(\overrightarrow{\forall \alpha^+}.N) = \overrightarrow{\forall \alpha^{+'}}.N'} \quad \text{NRMFORALL} \end{array}$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\begin{array}{c} \overline{\mathbf{nf}(\alpha^+) = \alpha^+} \quad \text{NRMPVAR} \\ \frac{\text{\textcolor{red}{<<multiple parses>>}}}{\mathbf{nf}(\downarrow N) = \downarrow M} \quad \text{NRMSHIFTD} \\ \frac{\text{\textcolor{red}{<<multiple parses>>}}}{\mathbf{nf}(\overrightarrow{\exists \alpha^-}.P) = \overrightarrow{\exists \alpha^{-'}}.P'} \quad \text{NRME EXISTS} \end{array}$$

$$\boxed{\mathbf{nf}(N) = M}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^-) = \hat{\alpha}^-} \quad \text{NRMNUVAR}$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^+) = \hat{\alpha}^+} \quad \text{NRMPUVAR}$$

$$\boxed{\mathbf{ord vars in } N = \vec{\alpha}} \quad \text{variable ordering in a negative type}$$

$$\begin{array}{c} \frac{\alpha^- \in \text{vars}}{\mathbf{ord vars in } \alpha^- = \alpha^-} \quad \text{ONVARIN} \\ \frac{\alpha^- \notin \text{vars}}{\text{\textcolor{red}{<<multiple parses>>}} \quad \text{ONVARNIN}} \\ \frac{\mathbf{ord vars in } P = \vec{\alpha}}{\mathbf{ord vars in } \uparrow P = \vec{\alpha}} \quad \text{OSHIFTU} \\ \frac{\mathbf{ord vars in } P = \vec{\alpha}_1 \quad \mathbf{ord vars in } N = \vec{\alpha}_2}{\mathbf{ord vars in } P \rightarrow N = \vec{\alpha}_1, (\vec{\alpha}_2 \setminus \vec{\alpha}_1)} \quad \text{OARROW} \\ \frac{\text{\textcolor{red}{<<multiple parses>>}}}{\mathbf{ord vars in } \overrightarrow{\forall \alpha^+}.N = \vec{\alpha}} \quad \text{OFORALL} \end{array}$$

$$\boxed{\mathbf{ord vars in } P = \vec{\alpha}}$$

variable ordering in a positive type

$$\begin{array}{c} \frac{\alpha^+ \in \text{vars}}{\mathbf{ord vars in } \alpha^+ = \alpha^+} \quad \text{OPVARIN} \\ \frac{\alpha^+ \notin \text{vars}}{\text{\textcolor{red}{<<multiple parses>>}} \quad \text{OPVARNIN}} \end{array}$$

$$\begin{array}{c}
\frac{\text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \downarrow N = \vec{\alpha}} \quad \text{OSHIFTD} \\
\frac{\text{<<multiple parses>>}}{\text{ord vars in } \exists \alpha^{\rightarrow}. P = \vec{\alpha}} \quad \text{OEXISTS} \\
\boxed{\text{ord vars in } N = \vec{\alpha}} \\
\frac{}{\text{<<multiple parses>>}} \quad \text{ONUVar} \\
\boxed{\text{ord vars in } P = \vec{\alpha}} \\
\frac{}{\text{<<multiple parses>>}} \quad \text{OPUVar} \\
\boxed{\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC} \quad \text{Negative unification} \\
\frac{}{\Gamma; \Theta \models \alpha^- \stackrel{u}{\simeq} \alpha^- \Rightarrow \cdot} \quad \text{UNVar} \\
\frac{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC}{\Gamma; \Theta \models \uparrow P \stackrel{u}{\simeq} \uparrow Q \Rightarrow UC} \quad \text{USHIFTU} \\
\frac{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC_1 \quad \Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC_2}{\Gamma; \Theta \models P \rightarrow N \stackrel{u}{\simeq} Q \rightarrow M \Rightarrow UC_1 \ \& \ UC_2} \quad \text{UArrow} \\
\frac{\Gamma, \vec{\alpha}^+; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC}{\Gamma; \Theta \models \forall \alpha^+. N \stackrel{u}{\simeq} \forall \alpha^+. M \Rightarrow UC} \quad \text{Uforall} \\
\frac{\Theta(\hat{\alpha}^-) \vdash N}{\Gamma; \Theta \models \hat{\alpha}^- \stackrel{u}{\simeq} N \Rightarrow (\hat{\alpha}^- : \simeq N)} \quad \text{UNUVar} \\
\boxed{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC} \quad \text{Positive unification} \\
\frac{}{\Gamma; \Theta \models \alpha^+ \stackrel{u}{\simeq} \alpha^+ \Rightarrow \cdot} \quad \text{UPVar} \\
\frac{\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC}{\Gamma; \Theta \models \downarrow N \stackrel{u}{\simeq} \downarrow M \Rightarrow UC} \quad \text{USHIFTD} \\
\frac{\Gamma, \vec{\alpha}^-; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC}{\Gamma; \Theta \models \exists \alpha^-. P \stackrel{u}{\simeq} \exists \alpha^-. Q \Rightarrow UC} \quad \text{UEXISTS} \\
\frac{\Theta(\hat{\alpha}^+) \vdash P}{\Gamma; \Theta \models \hat{\alpha}^+ \stackrel{u}{\simeq} P \Rightarrow (\hat{\alpha}^+ : \simeq P)} \quad \text{UPUVar} \\
\boxed{\Gamma \vdash N} \quad \text{Negative type well-formedness} \\
\frac{\alpha^- \in \Gamma}{\Gamma \vdash \alpha^-} \quad \text{WFTNVar} \\
\frac{\Gamma \vdash P}{\Gamma \vdash \uparrow P} \quad \text{WFTSHIFTU} \\
\frac{\Gamma \vdash P \quad \Gamma \vdash N}{\Gamma \vdash P \rightarrow N} \quad \text{WFTArrow}
\end{array}$$

$$\frac{\Gamma, \vec{\alpha}^+ \vdash N}{\Gamma \vdash \forall \vec{\alpha}^+. N} \quad \text{WFT}_{\text{FORALL}}$$

$\boxed{\Gamma \vdash P}$ Positive type well-formedness

$$\frac{\alpha^+ \in \Gamma}{\Gamma \vdash \alpha^+} \quad \text{WFTP}_{\text{VAR}}$$

$$\frac{\Gamma \vdash N}{\Gamma \vdash \downarrow N} \quad \text{WFT}_{\text{SHIFTD}}$$

$$\frac{\Gamma, \vec{\alpha}^- \vdash P}{\Gamma \vdash \exists \vec{\alpha}^-. P} \quad \text{WFT}_{\text{EXISTS}}$$

$\boxed{\Gamma; \Xi \vdash N}$ Negative algorithmic type well-formedness

$$\frac{\alpha^- \in \Gamma}{\Gamma; \Xi \vdash \alpha^-} \quad \text{WFATN}_{\text{VAR}}$$

$$\frac{\hat{\alpha}^- \in \Xi}{\Gamma; \Xi \vdash \hat{\alpha}^-} \quad \text{WFATNU}_{\text{VAR}}$$

$$\frac{\Gamma; \Xi \vdash P}{\Gamma; \Xi \vdash \uparrow P} \quad \text{WFAT}_{\text{SHIFTU}}$$

$$\frac{\Gamma; \Xi \vdash P \quad \Gamma; \Xi \vdash N}{\Gamma; \Xi \vdash P \rightarrow N} \quad \text{WFATA}_{\text{ARROW}}$$

$$\frac{\Gamma, \vec{\alpha}^+; \Xi \vdash N}{\Gamma; \Xi \vdash \forall \vec{\alpha}^+. N} \quad \text{WFAT}_{\text{FORALL}}$$

$\boxed{\Gamma; \Xi \vdash P}$ Positive algorithmic type well-formedness

$$\frac{\alpha^+ \in \Gamma}{\Gamma; \Xi \vdash \alpha^+} \quad \text{WFATP}_{\text{VAR}}$$

$$\frac{\hat{\alpha}^+ \in \Xi}{\Gamma; \Xi \vdash \hat{\alpha}^+} \quad \text{WFATPU}_{\text{VAR}}$$

$$\frac{\Gamma; \Xi \vdash N}{\Gamma; \Xi \vdash \downarrow N} \quad \text{WFAT}_{\text{SHIFTD}}$$

$$\frac{\Gamma, \vec{\alpha}^-; \Xi \vdash P}{\Gamma; \Xi \vdash \exists \vec{\alpha}^-. P} \quad \text{WFATE}_{\text{EXISTS}}$$

$\boxed{\Gamma \vdash \vec{N}}$ Negative type list well-formedness

$\boxed{\Gamma \vdash \vec{P}}$ Positive type list well-formedness

$\boxed{\Gamma; \Xi_2 \vdash \hat{\tau} : \Xi_1}$ Antiunification substitution well-formedness

$\boxed{\Gamma \vdash^= \Theta}$ Unification context well-formedness

$\boxed{\Gamma_1 \vdash \sigma : \Gamma_2}$ Substitution signature

$\boxed{\Theta \vdash \hat{\sigma} : \Xi}$ Unification substitution signature

$\boxed{\Gamma \vdash \hat{\sigma} : \Xi}$ Unification substitution general signature

$\boxed{\Theta \vdash \hat{\sigma} : UC}$ Unification substitution satisfies unification constraint

$\boxed{\Theta \vdash \hat{\sigma} : SC}$ Unification substitution satisfies subtyping constraint

$\boxed{\Gamma \vdash e}$ Unification constraint entry well-formedness

$\boxed{\Gamma \vdash e}$ Subtyping constraint entry well-formedness

$\boxed{\Gamma \vdash P : e}$ Positive type satisfies unification constraint

$\boxed{\Gamma \vdash N : e}$	Negative type satisfies unification constraint
$\boxed{\Gamma \vdash P : e}$	Positive type satisfies subtyping constraint
$\boxed{\Gamma \vdash N : e}$	Negative type satisfies subtyping constraint
$\boxed{\Theta \vdash UC : \Xi}$	Unification constraint well-formedness with specified domain
$\boxed{\Theta \vdash SC : \Xi}$	Subtyping constraint well-formedness with specified domain
$\boxed{\Theta \vdash UC}$	Unification constraint well-formedness
$\boxed{\Theta \vdash SC}$	Subtyping constraint well-formedness
$\boxed{\Gamma \vdash \vec{v}}$	Argument List well-formedness
$\boxed{\Gamma \vdash \Phi}$	Context well-formedness
$\boxed{\Gamma \vdash v}$	Value well-formedness

$$\frac{}{\Gamma \vdash x} \text{WFALLVAR}$$

$\boxed{\Gamma \vdash c}$ Computation well-formedness

$$\frac{\Gamma \vdash v \quad \Gamma \vdash c \quad \Gamma \vdash \vec{v}}{\Gamma \vdash \mathbf{let} \, x = v(\vec{v}); c} \text{WFALLAPPLET}$$

Definition rules: 95 good 32 bad
Definition rule clauses: 217 good 33 bad