1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
 & | & \alpha^+ & \\
 & | & \downarrow N & \\
 & | & \exists \alpha^-.P &
\end{array}
$$

$$
\begin{array}{llll}
N,\ M & ::= & & \text{negative types} \\
 & | & \alpha^- & \\
 & | & \uparrow P & \\
 & | & \forall \alpha^+.N & \\
 & | & P \to N &
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M}\ \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q}\ \textsc{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-}\ \textsc{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q}\ \textsc{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M}\ \textsc{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M}\ \textsc{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M}\ \textsc{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+}\ \textsc{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M}\ \textsc{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q}\ \textsc{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q}\ \textsc{D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$$
\begin{array}{lll}
P,\ Q & ::= & \text{multi-quantified positive types} \\
& \mid & \alpha^+ \\
& \mid & \downarrow N \\
& \mid & \exists \overrightarrow{\alpha^-}.P \qquad P \neq \exists \ldots \\
& \mid & (P) \qquad \mathsf{S}
\end{array}
$$

$$
\begin{array}{lll}
N,\ M & ::= & \text{multi-quantified negative types} \\
& \mid & \alpha^- \\
& \mid & \uparrow P \\
& \mid & P \to N \\
& \mid & \forall \overrightarrow{\alpha^+}.N \qquad N \neq \forall \ldots \\
& \mid & (N) \qquad \mathsf{S}
\end{array}
$$

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \quad (\simeq_1^{\leqslant\ -})$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \quad (\simeq_1^{\leqslant\ +})$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad (\textsc{Var}^{-\leqslant_1})$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \quad (\uparrow^{\leqslant_1})$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \quad (\to^{\leqslant_1})$$

$$\frac{\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\vec{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\leqslant_1})$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad (\textsc{Var}^{+\geqslant_1})$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \quad (\downarrow^{\geqslant_1})$$

$$\frac{\mathbf{fv}\, P \cap \overrightarrow{\beta^-} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\vec{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\geqslant_1})$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1}$ Equivalence of substitutions

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-} \quad (\textsc{Var}^{-\simeq_1^D})$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad (\uparrow^{\simeq_1^D})$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M} \quad (\to^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M} \quad (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\textsc{Var}^{+\simeq_1^D})$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists\overrightarrow{\alpha^-}.P \simeq_1^D \exists\overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \alpha^-} \quad (\textsc{Var}_\in^-)$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \cdot} \quad (\textsc{Var}_\notin^-)$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \uparrow P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ vars\ \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)} \quad (\to)$$

$$\frac{vars \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \alpha^+} \quad (\textsc{Var}_\in^+)$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \cdot} \quad (\textsc{Var}_\notin^+)$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \downarrow N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{vars \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \exists\overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad (\exists)$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^- = \cdot} \quad (\textsc{UVar}^-)$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^+ = \cdot} \quad (\textsc{UVar}^+)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\quad (\textsc{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\quad (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\quad (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,N' = \overrightarrow{\alpha^{+\prime}}}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.N'}\quad (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\quad (\textsc{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\quad (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,P' = \overrightarrow{\alpha^{-\prime}}}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^{-\prime}}.P'}\quad (\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\quad (\textsc{UVar}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\quad (\textsc{UVar}^+)$$

3.2 Unification

$\boxed{\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Gamma; \Theta \vDash \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\quad (\textsc{Var}^{-\overset{u}{\simeq}})$$

$$\frac{\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\quad (\uparrow^{\overset{u}{\simeq}})$$

$$\frac{\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \vDash P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2}\quad (\to^{\overset{u}{\simeq}})$$

$$\frac{\Gamma, \overrightarrow{\alpha^+}; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\quad (\forall^{\overset{u}{\simeq}})$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma; \Theta \vDash \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\widehat{\alpha}^- :\approx N)}\quad (\textsc{UVar}^{-\overset{u}{\simeq}})$$

$\boxed{\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Gamma; \Theta \vDash \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\quad (\textsc{Var}^{+\overset{u}{\simeq}})$$

$$\frac{\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\quad (\downarrow^{\overset{u}{\simeq}})$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}; \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\quad (\exists^{\overset{u}{\simeq}})$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma; \Theta \vDash \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\widehat{\alpha}^+ :\approx P)}\quad (\textsc{UVar}^{+\overset{u}{\simeq}})$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma; \Theta \vDash \alpha^- \leqslant \alpha^- \dashv \cdot} \quad (\textsc{Var}^{-\leqslant})$$

$$\frac{\Gamma; \Theta \vDash \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad (\uparrow^{\leqslant})$$

$$\frac{\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \vDash P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2} \quad (\to^{\leqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^+}} \quad (\forall^{\leqslant})$$

$\boxed{\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma; \Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad (\textsc{Var}^{+\geqslant})$$

$$\frac{\Gamma; \Theta \vDash \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \quad (\downarrow^{\geqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vDash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \exists \overrightarrow{\alpha^-}.P \geqslant \exists \overrightarrow{\beta^-}.Q \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^-}} \quad (\exists^{\geqslant})$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{\Gamma; \Theta \vDash \widehat{\alpha}^+ \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant Q)} \quad (\textsc{UVar}^{\geqslant})$$

3.4 Unification Solution Weakening

Unification solution is represented by a list of unification solution entries. Each entry restricts an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\widehat{\alpha}^+ :\approx P$ or $\widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries)**.** *We call two entries matching if they are restricting the same unification variable.*

Unification solutions are preordered by the weakening relation. First, let us define the weakening on the unification solution entries.

Definition 2.
$\boxed{\Gamma \vdash e_1 \Rightarrow e_2}$ *Weakening of unification solution entries*

$$\frac{\Gamma \vdash P_1 \geqslant_1 P_2}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P_1) \Rightarrow (\widehat{\alpha}^+ :\geqslant P_2)} \quad (\geqslant\Rightarrow^+\geqslant)$$

$$\frac{\Gamma \vdash P_1 \geqslant_1 P_2}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P_1) \Rightarrow (\widehat{\alpha}^+ :\geqslant P_2)} \quad (\simeq\Rightarrow^+\geqslant)$$

$$\frac{\Gamma \vdash P_1 \simeq_1^{\leqslant} P_2}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P_1) \Rightarrow (\widehat{\alpha}^+ :\approx P_2)} \quad (\simeq\Rightarrow^+\simeq)$$

$$\frac{\Gamma \vdash N_1 \simeq_1^{\leqslant} N_2}{\Gamma \vdash (\widehat{\alpha}^- :\approx N_1) \Rightarrow (\widehat{\alpha}^- :\approx N_2)} \quad (\simeq\Rightarrow^-\simeq)$$

Notice that $\Gamma \vdash e_1 \Rightarrow e_2$ means that e_1 and e_2 are matching. And matching is a equivalence relation on the set of unification solutions.

Next, we lift the weakening relation to unification solutions. Informally, $\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2$ means that for any entry of $\widehat{\sigma}_2$, there is a matching entry in $\widehat{\sigma}_1$ that is stronger than it.

Definition 3. *Assuming $\widehat{\sigma}_2 : \Theta$ and $\widehat{\sigma}_1 : \Theta' \supseteq \Theta$, we say $\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2$ iff $\forall e_2 \in \widehat{\sigma}_2$, $\exists e_1 \in \widehat{\sigma}_1$ s.t.*

1. *e_1 and e_2 are matching, i.e. restricting the same unification variable $\widehat{\alpha}^{\pm}$;*
2. *$\Theta(\widehat{\alpha}^{\pm}) \vdash e_1 \Rightarrow e_2$ (where $\Theta(\widehat{\alpha}^{\pm})$ is the context corresponding to $\widehat{\alpha}^{\pm}$ in Θ).*

3.5 Unification Solution Merge

Two matching entries can be merged in the following way:

Definition 4.
$\boxed{\Gamma \vdash e_1 \;\&\; e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\Gamma \vDash P_1 \vee P_2 = Q}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P_1) \;\&\; (\widehat{\alpha}^+ :\geqslant P_2) = (\widehat{\alpha}^+ :\geqslant Q)} \quad (\geqslant \&^+ \geqslant)$$

$$\frac{\Gamma; \cdot \vDash P \geqslant Q \dashv \widehat{\sigma}'}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \;\&\; (\widehat{\alpha}^+ :\geqslant Q) = (\widehat{\alpha}^+ :\approx P)} \quad (\simeq \&^+ \geqslant)$$

$$\frac{\Gamma; \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P) \;\&\; (\widehat{\alpha}^+ :\approx Q) = (\widehat{\alpha}^+ :\approx Q)} \quad (\geqslant \&^+ \simeq)$$

$$\frac{\mathbf{nf}\,(P) = \mathbf{nf}\,(P')}{\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \;\&\; (\widehat{\alpha}^+ :\approx P') = (\widehat{\alpha}^+ :\approx P)} \quad (\simeq \&^+ \simeq)$$

$$\frac{\mathbf{nf}\,(N) = \mathbf{nf}\,(N')}{\Gamma \vdash (\widehat{\alpha}^- :\approx N_1) \;\&\; (\widehat{\alpha}^- :\approx N') = (\widehat{\alpha}^- :\approx N)} \quad (\simeq \&^- \simeq)$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 5. $\widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2 = \{e_1 \;\&\; e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, \textit{s.t. } e_1 \textit{ matches with } e_2\}$
$\cup \{e_1 \mid e_1 \in \widehat{\sigma}_1, \textit{ s.t. } \forall e_2 \in \widehat{\sigma}_2, e_1 \textit{ does not match with } e_2\}$
$\cup \{e_2 \mid e_2 \in \widehat{\sigma}_2, \textit{ s.t. } \forall e_1 \in \widehat{\sigma}_1, e_1 \textit{ does not match with } e_2\}$

3.6 Least Upper Bound

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+} \quad (\textsc{Var}^\vee)$$

$$\frac{\Gamma, \cdot \vDash \mathbf{nf}\,(\downarrow N) \overset{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P} \quad (\downarrow^\vee)$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q} \quad (\exists^\vee)$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

$$\frac{\Gamma = \Delta, \overrightarrow{\alpha^\pm} \quad \overrightarrow{\beta^\pm}\,\mathbf{is\,fresh} \quad \overrightarrow{\gamma^\pm}\,\mathbf{is\,fresh} \quad \Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vDash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q}{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q} \quad (\textsc{Upg})$$

3.7 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \quad (\textsc{Var}^{+\overset{a}{\simeq}})$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \dashv (\Xi, \downarrow M, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\downarrow^{\overset{a}{\simeq}})$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists \overrightarrow{\alpha^-}.Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\exists^{\overset{a}{\simeq}})$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\cdot, \alpha^-, \cdot, \cdot)} \quad (\textsc{Var}^{-\overset{a}{\simeq}})$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\cong} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \uparrow P_1 \overset{a}{\cong} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\uparrow^{\overset{a}{\cong}})$$

$$\frac{\overrightarrow{\alpha^+} \cap \Gamma = \varnothing \quad \Gamma \vDash N_1 \overset{a}{\cong} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \forall \overrightarrow{\alpha^+}.N_1 \overset{a}{\cong} \forall \overrightarrow{\alpha^+}.N_2 \dashv (\Xi, \forall \overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\forall^{\overset{a}{\cong}})$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\cong} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \vDash N_1 \overset{a}{\cong} N_2 \dashv (\Xi_2, M, \widehat{\tau}_1', \widehat{\tau}_2')}{\Gamma \vDash P_1 \to N_1 \overset{a}{\cong} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \,\&\, \widehat{\tau}_1', \widehat{\tau}_2 \,\&\, \widehat{\tau}_2')} \quad (\to^{\overset{a}{\cong}})$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\cong} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad (\mathrm{AU}^-)$$

4 Proofs

4.1 Declarative Subtyping

Lemma 1 (Free Variable Propagation)**.** *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context* Γ*,*

- $-$ *if* $\Gamma \vdash N \leqslant_1 M$ *then* $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$
- $+$ *if* $\Gamma \vdash P \geqslant_1 Q$ *then* $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$

Proof. Mutual induction on $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash P \geqslant_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
It is self-evident that $\alpha^- \subseteq \alpha^-$.

Case 2. $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq_1^{\leqslant} Q$), we have $\Gamma \vdash P \geqslant_1 Q$. Then by the induction hypothesis, $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$. The desired inclusion inclusion holds, since $\mathbf{fv}\,(\uparrow P) = \mathbf{fv}\,(P)$ and $\mathbf{fv}\,(\uparrow Q) = \mathbf{fv}\,(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant_1 Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$ and $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$. Then $\mathbf{fv}\,(P \to N) = \mathbf{fv}\,(P) \cup \mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(Q) \cup \mathbf{fv}\,(M) = \mathbf{fv}\,(Q \to M)$.

Case 4. $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M$

$$\begin{aligned}
\mathbf{fv}\,\forall \overrightarrow{\alpha^+}.N &\subseteq \mathbf{fv}\,([\vec{P}/\overrightarrow{\alpha^+}]N) \setminus \overrightarrow{\beta^+} && \text{here } \overrightarrow{\beta^+} \text{ is excluded by the premise } \mathbf{fv}\,N \cap \overrightarrow{\beta^+} = \varnothing \\
&\subseteq \mathbf{fv}\,M \setminus \overrightarrow{\beta^+} && \text{by the induction hypothesis, } \mathbf{fv}\,([\vec{P}/\overrightarrow{\alpha^+}]N) \subseteq \mathbf{fv}\,M \\
&\subseteq \mathbf{fv}\,\forall \overrightarrow{\beta^+}.M
\end{aligned}$$

Case 5. The positive cases are symmetric.

□

Corollary 1 (Free Variables of mutual subtypes)**.**

- $-$ *If* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $\mathbf{fv}\,N = \mathbf{fv}\,M$*,*
- $+$ *If* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $\mathbf{fv}\,P = \mathbf{fv}\,Q$

Lemma 2 (Subtypes and supertypes of a variable)**.** *Assuming* $\Gamma \vdash \alpha^-$*,* $\Gamma \vdash \alpha^+$*,* $\Gamma \vdash N$*, and* $\Gamma \vdash P$*,*

- $+$ *if* $\Gamma \vdash P \geqslant_1 \alpha^+$ *or* $\Gamma \vdash \alpha^+ \geqslant_1 P$ *then* $P = \exists \overrightarrow{\alpha^-}.\alpha^+$ *(for some potentially empty* $\overrightarrow{\alpha^-}$*)*
- $-$ *if* $\Gamma \vdash N \leqslant_1 \alpha^-$ *or* $\Gamma \vdash \alpha^- \leqslant_1 N$ *then* $N = \forall \overrightarrow{\alpha^+}.\alpha^-$ *(for some potentially empty* $\overrightarrow{\alpha^+}$*)*

Proof. We prove by induction on the tree inferring $\Gamma \vdash P \geqslant_1 \alpha^+$ or $\Gamma \vdash \alpha^+ \geqslant_1 P$ or or $\Gamma \vdash N \leqslant_1 \alpha^-$ or $\Gamma \vdash \alpha^- \leqslant_1 N$.
Let us consider which of these judgments the tree is inferring.

Case 1. $\Gamma \vdash P \geqslant_1 \alpha^+$
If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\mathrm{Var}^{+\geqslant_1}$), which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists^{\geqslant_1}$). By inverting this rule, $P = \exists \overrightarrow{\alpha^-}.P'$ where P' does not start with $\exists$ and $\Gamma \vdash [\vec{N}/\overrightarrow{\alpha^-}]P' \geqslant_1 \alpha^+$ for some $\Gamma, \overrightarrow{\beta^-} \vdash N_i$.

By the induction hypothesis, $[\vec{N}/\overrightarrow{\alpha^-}]P' = \exists \overrightarrow{\beta^-}.\alpha^+$. Notice that P' must be a variable, because P' does not start with $\exists$, nor does it start with $\uparrow$ (otherwise, $[\vec{N}/\overrightarrow{\alpha^-}]P'$ would also started with $\uparrow$ and would not be equal to $\exists \overrightarrow{\beta^-}.\alpha^+$). Since P' is a *positive* variable, $[\vec{N}/\overrightarrow{\alpha^-}]P' = P'$, and then $P' = \exists \overrightarrow{\beta^-}.\alpha^+$ means that $P' = \alpha^+$. This way, $P = \exists \overrightarrow{\alpha^-}.P' = \exists \overrightarrow{\alpha^-}.\alpha^+$, as required.

Case 2. $\Gamma \vdash \alpha^+ \geqslant_1 P$

If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\mathrm{Var}^{+\geqslant_1}$), which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists^{\geqslant_1}$). By inverting this rule, $P = \exists\overrightarrow{\beta^-}.Q$ where and $\Gamma, \overrightarrow{\beta^-} \vdash \alpha^+ \geqslant_1 Q$.

By the induction hypothesis, $Q = \exists\overrightarrow{\beta^-}'.\alpha^+$. This way, $P = \exists\overrightarrow{\beta^-}.Q = \exists\overrightarrow{\beta^-}.\exists\overrightarrow{\beta^-}'.\alpha^+$, as required.

Case 3. The negative cases ($\Gamma \vdash N \leqslant_1 \alpha^-$ and $\Gamma \vdash \alpha^- \leqslant_1 N$) are proved analogously.

□

Corollary 2 (Variables have no proper subtypes and supertypes)**.** *Assuming that all mentioned types are well-formed in* Γ*,*

$$\begin{array}{lllll}
\Gamma \vdash P \geqslant_1 \alpha^+ & \iff & P = \exists\overrightarrow{\beta^-}.\alpha^+ & \iff & \Gamma \vdash P \simeq_1^{\leqslant} \alpha^+ \iff P \simeq_1^{D} \alpha^+ \\
\Gamma \vdash \alpha^+ \geqslant_1 P & \iff & P = \exists\overrightarrow{\beta^-}.\alpha^+ & \iff & \Gamma \vdash P \simeq_1^{\leqslant} \alpha^+ \iff P \simeq_1^{D} \alpha^+ \\
\Gamma \vdash N \leqslant_1 \alpha^- & \iff & N = \forall\overrightarrow{\beta^+}.\alpha^- & \iff & \Gamma \vdash N \simeq_1^{\leqslant} \alpha^- \iff N \simeq_1^{D} \alpha^- \\
\Gamma \vdash \alpha^- \leqslant_1 N & \iff & N = \forall\overrightarrow{\beta^+}.\alpha^- & \iff & \Gamma \vdash N \simeq_1^{\leqslant} \alpha^- \iff N \simeq_1^{D} \alpha^-
\end{array}$$

Proof. Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \simeq_1^{\leqslant} \alpha^+$ and $\exists\overrightarrow{\alpha^-}.\alpha^+ \simeq \alpha^+$ and apply lemma 2. `Ilya: fix` □

Corollary 3 (Transitivity of subtyping)**.** *Assuming the types are well-formed in* Γ*,*

- *if* $\Gamma \vdash N_1 \leqslant_1 N_2$ *and* $\Gamma \vdash N_2 \leqslant_1 N_3$ *then* $\Gamma \vdash N_1 \leqslant_1 N_3$*,*
- *if* $\Gamma \vdash P_1 \geqslant_1 P_2$ *and* $\Gamma \vdash P_2 \geqslant_1 P_3$ *then* $\Gamma \vdash P_1 \geqslant_1 P_3$*.*

Corollary 4 (Transitivity of equivalence)**.** *Assuming the types are well-formed in* Γ*,*

- *if* $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_2$ *and* $\Gamma \vdash N_2 \simeq_1^{\leqslant} N_3$ *then* $\Gamma \vdash N_1 \simeq_1^{\leqslant} N_3$*,*
- *if* $\Gamma \vdash P_1 \simeq_1^{\leqslant} P_2$ *and* $\Gamma \vdash P_2 \simeq_1^{\leqslant} P_3$ *then* $\Gamma \vdash P_1 \simeq_1^{\leqslant} P_3$*.*

4.2 Substitution

Lemma 3 (Substitution strengthening)**.** *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

- *if* $\Gamma_1 \vdash P$ *then* $[\sigma]P = [\sigma|_{\mathbf{fv}\,P}]P$*,*
- *if* $\Gamma_1 \vdash N$ *then* $[\sigma]N = [\sigma|_{\mathbf{fv}\,N}]N$

Proof. `Ilya: todo` □

Lemma 4 (Substitution preserves subtyping)**.** *Suppose that* $\Gamma \vdash \sigma : \Gamma_1$*. Then*

- *if* $\Gamma_1 \vdash P$*,* $\Gamma_1 \vdash Q$*, and* $\Gamma_1 \vdash P \geqslant_1 Q$ *then* $\Gamma \vdash [\sigma]P \geqslant_1 [\sigma]Q$
- *if* $\Gamma_1 \vdash N$*,* $\Gamma_1 \vdash M$*, and* $\Gamma_1 \vdash N \leqslant_1 M$ *then* $\Gamma \vdash [\sigma]N \leqslant_1 [\sigma]M$

Proof. `Ilya: todo` □

Corollary 5 (Substitution preserves equivalence)**.** *Suppose that* $\Gamma \vdash \sigma : \Gamma_1$*. Then*

- *if* $\Gamma_1 \vdash P$*,* $\Gamma_1 \vdash Q$*, and* $\Gamma_1 \vdash P \simeq_1^{\leqslant} Q$ *then* $\Gamma \vdash [\sigma]P \simeq_1^{\leqslant} [\sigma]Q$
- *if* $\Gamma_1 \vdash N$*,* $\Gamma_1 \vdash M$*, and* $\Gamma_1 \vdash N \simeq_1^{\leqslant} M$ *then* $\Gamma \vdash [\sigma]N \simeq_1^{\leqslant} [\sigma]M$

4.3 Type well-formedness

Lemma 5 (Well-formedness agrees with substitution)**.** *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

+ $\Gamma, \Gamma_1 \vdash P \;\Leftrightarrow\; \Gamma, \Gamma_2 \vdash [\sigma]P$
− $\Gamma, \Gamma_1 \vdash N \;\Leftrightarrow\; \Gamma, \Gamma_2 \vdash [\sigma]N$

Proof. Ilya: todo □

Corollary 6. *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*. Then*

+ $\Gamma_1, \Gamma_2 \vdash P \;\Leftrightarrow\; \Gamma_2 \vdash [\sigma]P$
− $\Gamma_1, \Gamma_2 \vdash N \;\Leftrightarrow\; \Gamma_2 \vdash [\sigma]N$

Lemma 6 (Equivalent Contexts)**.** *In the well-formedness judgment, only used variables matter:*

+ *if* $\Gamma_1 \cap \mathbf{fv}\, P = \Gamma_2 \cap \mathbf{fv}\, P$ *then* $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$*,*
− *if* $\Gamma_1 \cap \mathbf{fv}\, N = \Gamma_2 \cap \mathbf{fv}\, N$ *then* $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$*.*

Proof. By simple mutual induction on P and Q. □

Corollary 7. *Suppose that all the types below are well-formed in* Γ *and* $\Gamma' \subseteq \Gamma$*. Then*

+ $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *implies* $\Gamma' \vdash P \iff \Gamma' \vdash Q$
− $\Gamma \vdash N \simeq_1^{\leqslant} M$ *implies* $\Gamma' \vdash N \iff \Gamma' \vdash M$

Proof. From lemma 6 and corollary 1. □

4.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\dfrac{}{\mathbf{ord}\; vars \;\mathbf{in}\; N \equiv vars \cap \mathbf{fv}\, N}$	$\dfrac{N \simeq_1^D M}{\mathbf{ord}\; vars \;\mathbf{in}\; N = \mathbf{ord}\; vars \;\mathbf{in}\; M}$	—
Normalization	$\dfrac{}{N \simeq_1^D \mathbf{nf}\,(N)}$	$\dfrac{N \simeq_1^D M}{\mathbf{nf}\,(N) = \mathbf{nf}\,(M)}$	—
Equivalence	$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq_1^D Q}{\Gamma \vdash P \simeq_1^{\leqslant} Q}$	$\dfrac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{P \simeq_1^D Q}$	—
Uppgrade	$\dfrac{\mathbf{upgrade}\, \Gamma \vdash P \,\mathbf{to}\, \Delta = Q}{Q \text{ is sound} \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\, \Gamma \vdash P \,\mathbf{to}\, \Delta = Q}$	$\dfrac{Q' \text{ is sound} \quad \mathbf{upgrade}\, \Gamma \vdash P \,\mathbf{to}\, \Delta = Q}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \vDash P_1 \vee P_2 = Q}{Q \text{ is sound} \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \vDash P_1 \vee P_2 = Q}$	$\dfrac{Q' \text{ is sound} \quad \Gamma \vDash P_1 \vee P_2 = Q}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2)}{(\Xi, Q, \hat\tau_1, \hat\tau_2) \text{ is sound} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat\tau_i : \Xi \\ [\hat\tau_i]\, Q = P_i \end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \hat\tau_1', \hat\tau_2')}{\exists (\Xi, Q, \hat\tau_1, \hat\tau_2) \text{ s.t. } \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2)}$	$\dfrac{(\Xi', Q', \hat\tau_1', \hat\tau_2') \text{ is sound} \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2)}{\exists \Gamma; \Xi \vdash \hat\tau : \Xi' \text{ s.t. } [\hat\tau]\, Q' = Q}$
Unification (matching)	$\dfrac{\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \hat\sigma}{\hat\sigma \text{ is sound} \begin{cases} \Gamma \vdash \hat\sigma : \Theta \\ [\hat\sigma]\, P = Q \end{cases}}$	$\dfrac{\exists \text{ sound } \hat\sigma'}{\exists \hat\sigma \text{ s.t. } \Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \hat\sigma}$	—
Subtyping	$\dfrac{\Gamma; \Theta \vDash N \leqslant M \dashv \hat\sigma}{\hat\sigma \text{ is sound} \begin{cases} \Gamma \vdash \hat\sigma : \Theta \\ \Gamma \vdash [\hat\sigma]\, N \leqslant_1 M \end{cases}}$	$\dfrac{\exists \text{ sound } \hat\sigma'}{\exists \hat\sigma \text{ s.t. } \Gamma; \Theta \vDash N \leqslant M \dashv \hat\sigma}$	—

4.5 Variable Ordering

Definition 6 (Collision free bijection)**.** *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is* ***collision free on sets*** *P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \varnothing$
2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 7 (Soundness of variable ordering)**.** *Variable ordering extracts precisely used free variables.*

- $\mathbf{ord}\, vars\, \mathbf{in}\, N \equiv vars \cap \mathbf{fv}\, N$ *(as sets)*
- $\mathbf{ord}\, vars\, \mathbf{in}\, P \equiv vars \cap \mathbf{fv}\, P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\, vars\, \mathbf{in}\, N = \vec{\alpha}$ and $\mathbf{ord}\, vars\, \mathbf{in}\, P = \vec{\alpha}$ □

Corollary 8 (Additivity of ordering)**.** *Variable ordering is additive (in terms of set union) with respect to its first argument.*

- $\mathbf{ord}\,(vars_1 \cup vars_2)\,\mathbf{in}\, N \equiv \mathbf{ord}\, vars_1\, \mathbf{in}\, N \cup \mathbf{ord}\, vars_2\, \mathbf{in}\, N$ *(as sets)*
- $\mathbf{ord}\,(vars_1 \cup vars_2)\,\mathbf{in}\, P \equiv \mathbf{ord}\, vars_1\, \mathbf{in}\, P \cup \mathbf{ord}\, vars_2\, \mathbf{in}\, P$ *(as sets)*

Corollary 9 (Weakening of ordering)**.** *Extending the first argument of the ordering with unused variables does not change the result.*

- $\mathbf{ord}\,(vars \cap \mathbf{fv}\, N)\,\mathbf{in}\, N = \mathbf{ord}\, vars\, \mathbf{in}\, N$
- $\mathbf{ord}\,(vars \cap \mathbf{fv}\, P)\,\mathbf{in}\, P = \mathbf{ord}\, vars\, \mathbf{in}\, P$

Lemma 8 (Distributivity of renaming over variable ordering)**.** *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

- *If μ is collision free on vars and* $\mathbf{fv}\, N$ *then* $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]N$
- *If μ is collision free on vars and* $\mathbf{fv}\, P$ *then* $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, P) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]P$

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$
Then
$$\begin{aligned}
[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) \\
&= [\mu]\alpha^- && \text{by Rule } (\mathrm{Var}^+_\in) \\
&= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars) \\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\beta^- && \text{by Rule } (\mathrm{Var}^+_\in), \text{ because } \beta^- \in [\mu]vars \\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^-
\end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$
Notice that $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. On the other hand, $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_\notin)$, because μ is collision free on *vars* and $\mathbf{fv}\, N$, so $\mathbf{fv}\, N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$
Then $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_\notin)$. To prove that $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_\notin)$. Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

(i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.

(ii) Since μ is collision free on *vars* and $\mathbf{fv}\, N$, $\mu(A \cap \mathbf{fv}\, N) \ni \mu(\alpha^-) \notin vars$.

d. $\alpha^- \notin A$ but $\alpha^- \in vars$
$\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_\notin)$, because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$

$$\begin{aligned}
[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, vars\, \mathbf{in} \uparrow P) \\
&= [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, P) && \text{by Rule } (\uparrow) \\
&= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]P && \text{by the induction hypothesis} \\
&= \mathbf{ord}\, [\mu]vars\, \mathbf{in} \uparrow [\mu]P && \text{by Rule } (\uparrow) \\
&= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\uparrow P && \text{by the definition of substitution} \\
&= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]N
\end{aligned}$$

Case 3. $N = P \to M$

$$\begin{aligned}
[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, P \to M) \\
&= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)) && \text{where } \mathbf{ord}\, vars\, \mathbf{in}\, P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\, vars\, \mathbf{in}\, M = \vec{\alpha}_2 \\
&= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \vec{\alpha}_1) \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to case 1. Notice that } \mu \text{ is} \\
& && \text{collision free on } \vec{\alpha}_1 \text{ and } \vec{\alpha}_2 \text{ since } \vec{\alpha}_1 \subseteq vars \text{ and } \vec{\alpha}_2 \subseteq \mathbf{fv}\, N \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1) \\
(\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]N) &= (\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]P \to [\mu]M) \\
&= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \vec{\beta}_1)) && \text{where } \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]M = \vec{\beta}_2 \\
& && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)
\end{aligned}$$

Case 4. $N = \forall \overrightarrow{\alpha^+}.M$

$$\begin{aligned}
[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu]\mathbf{ord}\, vars\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}.M \\
&= [\mu]\mathbf{ord}\, vars\, \mathbf{in}\, M \\
&= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]M && \text{by the induction hypothesis} \\
(\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]N) &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\forall \overrightarrow{\alpha^+}.M \\
&= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}.[\mu]M \\
&= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]M
\end{aligned}$$

□

Lemma 9 (Ordering is not affected by independent substitutions)**.** *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*, i.e.* σ *maps variables from* Γ_1 *into types taking free variables from* Γ_2*, and vars is a set of variables disjoint with both* Γ_1 *and* Γ_2*. Then*

– $\mathbf{ord}\, vars\, \mathbf{in}\, [\sigma]N = \mathbf{ord}\, vars\, \mathbf{in}\, N$

\+ $\mathbf{ord}\, vars\, \mathbf{in}\, [\sigma]P = \mathbf{ord}\, vars\, \mathbf{in}\, P$

Proof. `Ilya: Should be easy` □

Lemma 10 (Completeness of variable ordering)**.** *Variable ordering is invariant under equivalence. For arbitrary vars,*

– *If* $N \simeq_1^D M$ *then* $\mathbf{ord}\, vars\, \mathbf{in}\, N = \mathbf{ord}\, vars\, \mathbf{in}\, M$ *(as lists)*

\+ *If* $P \simeq_1^D Q$ *then* $\mathbf{ord}\, vars\, \mathbf{in}\, P = \mathbf{ord}\, vars\, \mathbf{in}\, Q$ *(as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. □

4.6 Normaliztaion

Lemma 11. *Set of free variables is invariant under equivalence.*

– *If* $N \simeq_1^D M$ *then* $\mathbf{fv}\, N \equiv \mathbf{fv}\, M$ *(as sets)*

\+ *If* $P \simeq_1^D Q$ *then* $\mathbf{fv}\, P \equiv \mathbf{fv}\, Q$ *(as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ □

Lemma 12. *Free variables are not changed by the normalization*

– $\mathbf{fv}\, N \equiv \mathbf{fv}\, \mathbf{nf}\, (N)$

+ $\mathbf{fv}\, P \equiv \mathbf{fv}\, \mathbf{nf}\, (P)$

Proof. By straightforward induction on $\mathbf{nf}\, (N) = M$. □

Lemma 13 (Soundness of quantifier normalization)**.**

− $N \simeq_1^D \mathbf{nf}\, (N)$

+ $P \simeq_1^D \mathbf{nf}\, (P)$

Proof. Mutual induction on $\mathbf{nf}\, (N) = M$ and $\mathbf{nf}\, (P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var$^-$) and (Var$^+$)
By the corresponding equivalence rules.

Case 2. (↑), (↓), and (→)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. (∀), i.e. $\mathbf{nf}\, (\forall \overrightarrow{\alpha^+}.N) = \forall \overrightarrow{\alpha^{+\prime}}.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 11, $\mathbf{fv}\, N \equiv \mathbf{fv}\, N'$. Then by lemma 7, $\overrightarrow{\alpha^{+\prime}} \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$, and thus, $\overrightarrow{\alpha^{+\prime}} \cap \mathbf{fv}\, N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$.
To prove $\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\alpha^{+\prime}}.N'$, it suffices to provide a bijection $\mu : \overrightarrow{\alpha^{+\prime}} \cap \mathbf{fv}\, N' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. (∃) Same as for case 3.

□

Corollary 10 (Normalization preserves ordering)**.** *For any vars,*

− $\mathbf{ord}\, vars\, \mathbf{in}\, \mathbf{nf}\, (N) = \mathbf{ord}\, vars\, \mathbf{in}\, M$

+ $\mathbf{ord}\, vars\, \mathbf{in}\, \mathbf{nf}\, (P) = \mathbf{ord}\, vars\, \mathbf{in}\, Q$

Proof. Immediately from lemmas 10 and 13. □

Lemma 14 (Distributivity of normalization over substitution)**.** *Normalization of a term distributes over substitution. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$*, i.e.* σ *maps variables from* Γ_1 *into types taking free variables from* Γ_2*. Then*

− $\mathbf{nf}\, ([\sigma]N) = [\mathbf{nf}\, (\sigma)]\mathbf{nf}\, (N)$

+ $\mathbf{nf}\, ([\sigma]P) = [\mathbf{nf}\, (\sigma)]\mathbf{nf}\, (P)$

where $\mathbf{nf}\, (\sigma)$ *means pointwise normalization:* $[\mathbf{nf}\, (\sigma)]\alpha^- = \mathbf{nf}\, ([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\, ([\sigma]N) = \mathbf{nf}\, ([\sigma]\alpha^-) = [\mathbf{nf}\, (\sigma)]\alpha^-$.
$[\mathbf{nf}\, (\sigma)]\mathbf{nf}\, (N) = [\mathbf{nf}\, (\sigma)]\mathbf{nf}\, (\alpha^-) = [\mathbf{nf}\, (\sigma)]\alpha^-$.

Case 2. $P = \alpha^+$
Similar to case 1.

Case 3. If the type is formed by →, ↑, or ↓, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \to M$ then

$$
\begin{aligned}
\mathbf{nf}\, ([\sigma]N) &= \mathbf{nf}\, ([\sigma](P \to M)) \\
&= \mathbf{nf}\, ([\sigma]P \to [\sigma]M) && \text{By the congruence of substitution} \\
&= \mathbf{nf}\, ([\sigma]P) \to \mathbf{nf}\, ([\sigma]M) && \text{By the congruence of normalization, i.e. Rule } (\to) \\
&= [\mathbf{nf}\, (\sigma)]\mathbf{nf}\, (P) \to [\mathbf{nf}\, (\sigma)]\mathbf{nf}\, (M) && \text{By the induction hypothesis} \\
&= [\mathbf{nf}\, (\sigma)](\mathbf{nf}\, (P) \to \mathbf{nf}\, (M)) && \text{By the congruence of substitution} \\
&= [\mathbf{nf}\, (\sigma)]\mathbf{nf}\, (P \to M) && \text{By the congruence of normalization} \\
&= [\mathbf{nf}\, (\sigma)]\mathbf{nf}\, (N)
\end{aligned}
$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M)$

$= [\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,(M)$ Where $\overrightarrow{\alpha^{+\prime}} = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(M) = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M$ (the latter is by corollary 10)

$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\forall\overrightarrow{\alpha^+}.M)$

$= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\sigma]M)$ Assuming $\overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing$ and $\overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$

$= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\sigma]M)$ Where $\overrightarrow{\beta^+} = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,([\sigma]M) = \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M$ (the latter is by corollary 10)

$= \forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,([\sigma]M)$ By lemma 9, $\overrightarrow{\beta^+} = \overrightarrow{\alpha^{+\prime}}$ since $\overrightarrow{\alpha^+}$ is disjoint with Γ_1 and Γ_2

$= \forall\overrightarrow{\alpha^{+\prime}}.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$ By the induction hypothesis

To show alpha-equivalence of $[\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^{+\prime}}.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$, we can assume that $\overrightarrow{\alpha^{+\prime}} \cap \Gamma_1 = \varnothing$, and $\overrightarrow{\alpha^{+\prime}} \cap \Gamma_2 = \varnothing$.

Case 5. $P = \exists\overrightarrow{\alpha^-}.Q$

Same as for case 4.

□

Corollary 11 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

– $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$

\+ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Proof. Immediately from lemma 14, after noticing that $\mathbf{nf}\,(\mu) = \mu$. □

Lemma 15 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

– *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

\+ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. $(\forall^{\simeq_1^D})$

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^{+\prime}}$ is $\mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^{+\prime}}.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^{+\prime}}$ is $\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 7 and 12, the domain and the codomain of μ can be written as $\mu : \overrightarrow{\beta^{+\prime}} \leftrightarrow \overrightarrow{\alpha^{+\prime}}$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^{+\prime}}.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^{+\prime}} = \overrightarrow{\alpha^{+\prime}}$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by corollary 11, the second—by the induction hypothesis.

(ii)
$[\mu]\overrightarrow{\beta^{+\prime}} = [\mu]\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M)$ by the definition of $\overrightarrow{\beta^{+\prime}}$

$= [\mu]\mathbf{ord}\,(\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M)$ from lemma 12 and corollary 9

$= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M)$ by lemma 8, because $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing$ and $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing$

$= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N)$ since $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ is proved

$= \mathbf{ord}\,(\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N)$ because μ is a bijection between $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$ and $\overrightarrow{\beta^+} \cap \mathbf{fv}\,M$

$= \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N)$ from lemma 12 and corollary 9

$= \overrightarrow{\alpha^{+\prime}}$ by the definition of $\overrightarrow{\alpha^{+\prime}}$

Case 2. ($\exists^{\simeq^D_1}$) Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□

Lemma 16 (Idempotence of normalization)**.** *Normalization is idempotent*

– $\mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$

\+ $\mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

Proof. By applying lemma 15 to lemma 13. □

Lemma 17. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized. Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then*

\+ $[\sigma]P$ *is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \text{is normal} \\ P & \text{is normal} \end{cases}$

– $[\sigma]N$*is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \text{is normal} \\ N & \text{is normal} \end{cases}$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$
Then N is always normal, and the normality of $\sigma|_{\alpha^-}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$
\begin{aligned}
[\sigma](P \to M) \text{ is normal} &\iff [\sigma]P \to [\sigma]M \text{ is normal} && \text{by the substitution congruence} \\
&\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases} && \text{by congruence of normality } \texttt{Ilya: lemma?} \\
&\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(M)} & \text{is normal} \end{cases} && \text{by the induction hypothesis} \\
&\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P)\cup\mathbf{fv}\,(M)} & \text{is normal} \end{cases} \iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P\to M)} & \text{is normal} \end{cases}
\end{aligned}
$$

Case 3. $N = \uparrow P$
By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$
\begin{aligned}
[\sigma](\forall\overrightarrow{\alpha^+}.M) \text{ is normal} &\iff (\forall\overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} && \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing \\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+} \end{cases} && \text{by the definition of normalization} \\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} && \text{by lemma 9} \\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} && \text{by the induction hypothesis} \\
&\iff \begin{cases} \sigma|_{\mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M)} \text{ is normal} \\ \forall\overrightarrow{\alpha^+}.M \text{ is normal} \end{cases} && \text{since } \mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M); \\ & && \text{by the definition of normalization}
\end{aligned}
$$

Case 5. $P = \ldots$
The positive cases are done in the same way as the negative ones.

□

4.7 Equivalence

Lemma 18 (Declarative equivalence is transitive)**.**

+ *if* $P_1 \simeq_1^D P_2$ *and* $P_2 \simeq_1^D P_3$ *then* $P_1 \simeq_1^D P_3$,
− *if* $N_1 \simeq_1^D N_2$ *and* $N_2 \simeq_1^D N_3$ *then* $N_1 \simeq_1^D N_3$.

Proof. `Ilya: should be easy to do by induction since the types are getting smaller` □

Lemma 19 (Algorithmization of declarative equivalence)**.** *Declarative equivalence is equality of normal forms.*

+ $P \simeq_1^D Q \iff \mathbf{nf}(P) = \mathbf{nf}(Q)$,
− $N \simeq_1^D M \iff \mathbf{nf}(N) = \mathbf{nf}(M)$.

Proof.

+ Let us prove both directions separately.
  ⇒ exactly by lemma 15,
  ⇐ from lemma 13, we know $P \simeq_1^D \mathbf{nf}(P) = \mathbf{nf}(Q) \simeq_1^D Q$, then by transitivity (lemma 18), $P \simeq_1^D Q$.
− The proof is exactly the same.

□

Lemma 20 (Type well-formedness is invariant under equivalence)**.** *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,
− *if* $N \simeq_1^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. `Ilya: todo` □

Corollary 12 (Normalization preserves well-formedness)**.**

+ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}(P)$,
− $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}(N)$

Proof. Immediately from lemmas 13 and 20. □

Corollary 13 (Normalization preserves well-formedness of substitution)**.**
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}(\sigma) : \Gamma_1$

Lemma 21 (Soundness of equivalence)**.** *Declarative equivalence implies mutual subtyping.*

+ *if* $\Gamma \vdash P$, $\Gamma \vdash Q$, *and* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \simeq_1^\leqslant Q$,
− *if* $\Gamma \vdash N$, $\Gamma \vdash M$, *and* $N \simeq_1^D M$ *then* $\Gamma \vdash N \simeq_1^\leqslant M$.

Proof. We prove it by mutual induction on $P \simeq_1^D Q$ and $N \simeq_1^D M$.

Case 1. $\alpha^- \simeq_1^D \alpha^-$
Then $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$ by Rule ($\mathrm{Var}^{-\leqslant_1}$), which immediately implies $\Gamma \vdash \alpha^- \simeq_1^\leqslant \alpha^-$ by Rule ($\simeq_1^\leqslant {}^-$).

Case 2. $\uparrow P \simeq_1^D \uparrow Q$
Then by inversion of Rule ($\uparrow^{\leqslant_1}$), $P \simeq_1^D Q$, and from the induction hypothesis, $\Gamma \vdash P \simeq_1^\leqslant Q$, and (by symmetry) $\Gamma \vdash Q \simeq_1^\leqslant P$.
When Rule ($\uparrow^{\leqslant_1}$) is applied to $\Gamma \vdash P \simeq_1^\leqslant Q$, it gives us $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$; when it is applied to $\Gamma \vdash Q \simeq_1^\leqslant P$, we obtain $\Gamma \vdash \uparrow Q \leqslant_1 \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \simeq_1^\leqslant \uparrow Q$.

Case 3. $P \to N \simeq_1^D Q \to M$
Then by inversion of Rule ($\to^{\leqslant_1}$), $P \simeq_1^D Q$ and $N \simeq_1^D M$. By the induction hypothesis, $\Gamma \vdash P \simeq_1^\leqslant Q$ and $\Gamma \vdash N \simeq_1^\leqslant M$, which means by inversion: (i) $\Gamma \vdash P \geqslant_1 Q$, (ii) $\Gamma \vdash Q \geqslant_1 P$, (iii) $\Gamma \vdash N \leqslant_1 M$, (iv) $\Gamma \vdash M \leqslant_1 N$. Applying Rule ($\to^{\leqslant_1}$) to (i) and (iii), we obtain $\Gamma \vdash P \to N \leqslant_1 Q \to M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \to M \leqslant_1 P \to N$. Together, it implies $\Gamma \vdash P \to N \simeq_1^\leqslant Q \to M$.

Case 4. $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M$

Then by inversion, there exists bijection $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N)$, such that $N \simeq_1^D [\mu]M$. By the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu]M$. From corollary 5 and the fact that μ is bijective, we also have $\Gamma, \overrightarrow{\beta^+} \vdash [\mu^{-1}]N \simeq_1^{\leqslant} M$.

Let us construct a subsitution $\overrightarrow{\alpha^+} \vdash \vec{P}/\overrightarrow{\beta^+} : \overrightarrow{\beta^+}$ by extending μ with arbitrary positive types on $\overrightarrow{\beta^+} \backslash \mathbf{fv}\, M$.

Notice that $[\mu]M = [\vec{P}/\overrightarrow{\beta^+}]M$, and therefore, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\leqslant} [\mu]M$ implies $\Gamma, \overrightarrow{\alpha^+} \vdash [\vec{P}/\overrightarrow{\beta^+}]M \leqslant_1 N$. Then by Rule $(\forall^{\leqslant_1})$, $\Gamma \vdash \forall\overrightarrow{\beta^+}.M \leqslant_1 \forall\overrightarrow{\alpha^+}.N$.

Analogously, we construct the substitution from μ^{-1}, and use it to instantiate $\overrightarrow{\alpha^+}$ in the application of Rule $(\forall^{\leqslant_1})$ to infer $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$.

This way, $\Gamma \vdash \forall\overrightarrow{\beta^+}.M \leqslant_1 \forall\overrightarrow{\alpha^+}.N$ and $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$ gives us $\Gamma \vdash \forall\overrightarrow{\beta^+}.M \simeq_1^{\leqslant} \forall\overrightarrow{\alpha^+}.N$.

Case 5. For the cases of the positive types, the proofs are symmetric.

□

Corollary 14 (Normalization is sound w.r.t. subtyping-induced equivalence)**.**

+ *if* $\Gamma \vdash P$ *then* $\Gamma \vdash P \simeq_1^{\leqslant} \mathbf{nf}\,(P)$,
− *if* $\Gamma \vdash N$ *then* $\Gamma \vdash N \simeq_1^{\leqslant} \mathbf{nf}\,(N)$.

Proof. Immediately from lemmas 13 and 21 and corollary 12. □

Lemma 22 (Subtyping induced by disjoint substitutions)**.** *If two disjoint substitutions induce subtyping, they are degenerate (so is the subtyping). Suppose that* $\Gamma \vdash \sigma_1 : \Gamma_1$ *and* $\Gamma \vdash \sigma_2 : \Gamma_1$*, where* $\Gamma_i \subseteq \Gamma$ *and* $\Gamma_1 \cap \Gamma_2 = \varnothing$*. Then*

− *assuming* $\Gamma \vdash N$*,* $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ *implies* $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, N$
+ *assuming* $\Gamma \vdash P$*,* $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ *implies* $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$

Proof. Proof by induciton on $\Gamma \vdash N$ (and mutually on $\Gamma \vdash P$).

Case 1. $N = \alpha^-$

Then $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\alpha^- \leqslant_1 [\sigma_2]\alpha^-$. Let us consider the following cases:

a. $\alpha^- \notin \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \alpha^-$ holds immediately, since $[\sigma_i]\alpha^- = [\mathsf{id}]\alpha^- = \alpha^-$ and $\Gamma \vdash \alpha^- \simeq_1^{\leqslant} \alpha^-$.

b. $\alpha^- \in \Gamma_1$ and $\alpha^- \in \Gamma_2$
This case is not possible by assumption: $\Gamma_1 \cap \Gamma_2 = \varnothing$.

c. $\alpha^- \in \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then we have $\Gamma \vdash [\sigma_1]\alpha^- \leqslant_1 \alpha^-$, which by corollary 2 means $\Gamma \vdash [\sigma_1]\alpha^- \simeq_1^{\leqslant} \alpha^-$, and hence, $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \alpha^-$.
$\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \alpha^-$ holds since $[\sigma_2]\alpha^- = \alpha^-$, similarly to case 1.*a*.

d. $\alpha^- \notin \Gamma_1$ and $\alpha^- \in \Gamma_2$
Then we have $\Gamma \vdash \alpha^- \leqslant_1 [\sigma_2]\alpha^-$, which by corollary 2 means $\Gamma \vdash \alpha^- \simeq_1^{\leqslant} [\sigma_2]\alpha^-$, and hence, $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \alpha^-$.
$\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \alpha^-$ holds since $[\sigma_1]\alpha^- = \alpha^-$, similarly to case 1.*a*.

Case 2. $N = \forall\overrightarrow{\alpha^+}.M$

Then by inversion, $\Gamma, \overrightarrow{\alpha^+} \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\forall\overrightarrow{\alpha^+}.M \leqslant_1 [\sigma_2]\forall\overrightarrow{\alpha^+}.M$. By the congruence of substitution and by the inversion of Rule $(\forall^{\leqslant_1})$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\vec{Q}/\overrightarrow{\alpha^+}][\sigma_1]M \leqslant_1 [\sigma_2]M$, where $\Gamma, \overrightarrow{\alpha^+} \vdash Q_i$. Let us denote the (Kleisli) composition of σ_1 and $\vec{Q}/\overrightarrow{\alpha^+}$ as σ_1', noting that $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' : \Gamma_1, \overrightarrow{\alpha^+}$, and $\Gamma_1, \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$.

Let us apply the induction hypothesis to M and the substitutions σ_1' and σ_2 with $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_1']M \leqslant_1 [\sigma_2]M$ to obtain:

$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{1}$$
$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{2}$$

Then $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ holds by strengthening of 2: for any $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M \backslash \overrightarrow{\alpha^+}$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$ is strengthened to $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$, because $\mathbf{fv}\, [\sigma_2]\beta^\pm = \mathbf{fv}\, \beta^\pm = \{\beta^\pm\} \subseteq \Gamma$.

To show that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$, let us take an arbitrary $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M \backslash \overrightarrow{\alpha^+}$.

$\beta^\pm = [\mathsf{id}]\beta^\pm$ by definition of id

$\simeq_1^\leqslant [\sigma_1']\beta^\pm$ by 1

$= [\vec{Q}/\overrightarrow{\alpha^+}][\sigma_1]\beta^\pm$ by definition of σ_1'

$= [\sigma_1]\beta^\pm$ because $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,[\sigma_1]\beta^\pm \subseteq \overrightarrow{\alpha^+} \cap \Gamma = \varnothing$

This way, $\Gamma \vdash [\sigma_1]\beta^\pm \simeq_1^\leqslant \beta^\pm$ for any $\beta^\pm \in \mathbf{fv}\,\forall\overrightarrow{\alpha^+}.M$ and thus, $\Gamma \vdash \sigma_1 \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,\forall\overrightarrow{\alpha^+}.M$.

Case 3. $N = P \to M$

Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1](P \to M) \leqslant_1 [\sigma_2](P \to M)$, then by congruence of substitution, $\Gamma \vdash [\sigma_1]P \to [\sigma_1]M \leqslant_1 [\sigma_2]P \to [\sigma_2]M$, then by inversion $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ and $\Gamma \vdash [\sigma_1]M \leqslant_1 [\sigma_2]M$.

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ and to $\Gamma \vdash [\sigma_1]M \leqslant_1 [\sigma_2]M$, we obtain (respectively):

$$\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,P \tag{3}$$
$$\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,M \tag{4}$$

Noting that $\mathbf{fv}\,(P \to M) = \mathbf{fv}\,P \cup \mathbf{fv}\,M$, we combine eqs. (3) and (4) to conclude: $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,(P \to M)$.

Case 4. $N = \uparrow P$

Then by inversion, $\Gamma \vdash P$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\uparrow P \leqslant_1 [\sigma_2]\uparrow P$, then by congruence of substitution and by inversion, $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$, we obtain $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,P$. Since $\mathbf{fv}\uparrow P = \mathbf{fv}\,P$, we can conclude: $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\uparrow P$.

Case 5. The positive cases are proved symmetrically.

□

Corollary 15 (Substitution cannot induce proper subtypes or supertypes)**.** *Assuming all mentioned types are well-formed in* Γ *and* σ *is a substitution* $\Gamma \vdash \sigma : \Gamma$*,*

$$\begin{array}{rcl}
\Gamma \vdash [\sigma]N \leqslant_1 N & \Rightarrow & \Gamma \vdash [\sigma]N \simeq_1^\leqslant N \text{ and } \Gamma \vdash \sigma \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,N \\
\Gamma \vdash N \leqslant_1 [\sigma]N & \Rightarrow & \Gamma \vdash N \simeq_1^\leqslant [\sigma]N \text{ and } \Gamma \vdash \sigma \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,N \\
\Gamma \vdash [\sigma]P \geqslant_1 P & \Rightarrow & \Gamma \vdash [\sigma]P \simeq_1^\leqslant P \text{ and } \Gamma \vdash \sigma \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,P \\
\Gamma \vdash P \geqslant_1 [\sigma]P & \Rightarrow & \Gamma \vdash P \simeq_1^\leqslant [\sigma]P \text{ and } \Gamma \vdash \sigma \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,P
\end{array}$$

Lemma 23. *Asssuming that the mentioned types (*P*,* Q*,* N*, and* M*) are well-formed in* Γ *and that the substitutions (*σ_1 *and* σ_2*) have signature* $\Gamma \vdash \sigma_i : \Gamma$*,*

+ *if* $\Gamma \vdash [\sigma_1]P \geqslant_1 Q$ *and* $\Gamma \vdash [\sigma_2]Q \geqslant_1 P$
  then there exists a bijection $\mu : \mathbf{fv}\,P \leftrightarrow \mathbf{fv}\,Q$ *such that* $\Gamma \vdash \sigma_1 \simeq_1^\leqslant \mu : \mathbf{fv}\,P$ *and* $\Gamma \vdash \sigma_2 \simeq_1^\leqslant \mu^{-1} : \mathbf{fv}\,Q$*;*
- *if* $\Gamma \vdash [\sigma_1]N \leqslant_1 M$ *and* $\Gamma \vdash [\sigma_2]N \leqslant_1 M$
  then there exists a bijection $\mu : \mathbf{fv}\,N \leftrightarrow \mathbf{fv}\,M$ *such that* $\Gamma \vdash \sigma_1 \simeq_1^\leqslant \mu : \mathbf{fv}\,N$ *and* $\Gamma \vdash \sigma_2 \simeq_1^\leqslant \mu^{-1} : \mathbf{fv}\,M$*.*

Proof.

+ Applying σ_2 to both sides of $\Gamma \vdash [\sigma_1]P \geqslant_1 Q$ (by **??**), we have: $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant_1 [\sigma_2]Q$. Composing it with $\Gamma \vdash [\sigma_2]Q \geqslant_1 P$ (by transitivity **??**), we have $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant_1 P$. Then by corollary 15, $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,P$.

  By a symmetric argument, we also have: $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,Q$.

  Now, we prove that $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,P$ and $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\,Q$ implies that σ_1 and σ_1 are (equivalent to) mutually inverse bijections.

  To do so, it suffices to prove that

  (i) for any $\alpha^\pm \in \mathbf{fv}\,P$ there exists $\beta^\pm \in \mathbf{fv}\,Q$ such that $\Gamma \vdash [\sigma_1]\alpha^\pm \simeq_1^\leqslant \beta^\pm$ and $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^\leqslant \alpha^\pm$; and

  (ii) for any $\beta^\pm \in \mathbf{fv}\,Q$ there exists $\alpha^\pm \in \mathbf{fv}\,P$ such that $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^\leqslant \alpha^\pm$ and $\Gamma \vdash [\sigma_1]\alpha^\pm \simeq_1^\leqslant \beta^\pm$.

  Then the these correspondences between $\mathbf{fv}\,P$ and $\mathbf{fv}\,Q$ are mutually inverse functions, since for any $\beta^\pm$ there can be at most one $\alpha^\pm$ such that $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^\leqslant \alpha^\pm$ (and vice versa).

(i) Let us take $\alpha^{\pm} \in \mathbf{fv}\, P$.

(a) if $\alpha^{\pm}$ is positive ($\alpha^{\pm} = \alpha^{+}$), from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{+} \simeq_1^{\leqslant} \alpha^{+}$, by corollary 2, we have $[\sigma_2][\sigma_1]\alpha^{+} = \exists\overrightarrow{\beta^{-}}.\alpha^{+}$.
What shape can $[\sigma_1]\alpha^{+}$ have? It cannot be $\exists\overrightarrow{\alpha^{-}}.{\downarrow}N$ (for potentially empty $\overrightarrow{\alpha^{-}}$), because the outer constructor $\downarrow$ would remain after the substitution σ_2, whereas $\exists\overrightarrow{\beta^{-}}.\alpha^{+}$ does not have $\downarrow$. The only case left is $[\sigma_1]\alpha^{+} = \exists\overrightarrow{\alpha^{-}}.\gamma^{+}$.
Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.\gamma^{+} \simeq_1^{\leqslant} \gamma^{+}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{+} \simeq_1^{\leqslant} \gamma^{+}$. Also notice that $[\sigma_2]\exists\overrightarrow{\alpha^{-}}.\gamma^{+} = \exists\overrightarrow{\beta^{-}}.\alpha^{+}$ implies $\Gamma \vdash [\sigma_2]\gamma^{+} \simeq_1^{\leqslant} \alpha^{+}$.

(b) if $\alpha^{\pm}$ is negative ($\alpha^{\pm} = \alpha^{-}$) from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{-} \simeq_1^{\leqslant} \alpha^{-}$, by corollary 2, we have $[\sigma_2][\sigma_1]\alpha^{-} = \forall\overrightarrow{\beta^{+}}.\alpha^{-}$.
What shape can $[\sigma_1]\alpha^{-}$ have? It cannot be $\forall\overrightarrow{\alpha^{+}}.{\uparrow}P$ nor $\forall\overrightarrow{\alpha^{+}}.P \to M$ (for potentially empty $\overrightarrow{\alpha^{+}}$), because the outer constructor ($\to$ or $\uparrow$), remaining after the substitution σ_2, is however absent in the resulting $\forall\overrightarrow{\beta^{+}}.\alpha^{-}$. Hence, the only case left is $[\sigma_1]\alpha^{-} = \forall\overrightarrow{\alpha^{+}}.\gamma^{-}$ Notice that $\Gamma \vdash \gamma^{-} \simeq_1^{\leqslant} \forall\overrightarrow{\alpha^{+}}.\gamma^{-}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{-} \simeq_1^{\leqslant} \gamma^{-}$. Also notice that $[\sigma_2]\forall\overrightarrow{\alpha^{+}}.\gamma^{-} = \forall\overrightarrow{\beta^{+}}.\alpha^{-}$ implies $\Gamma \vdash [\sigma_2]\gamma^{-} \simeq_1^{\leqslant} \alpha^{-}$.

(ii) The proof is symmetric: We swap P and Q, σ_1 and σ_2, and exploit $\Gamma \vdash [\sigma_1][\sigma_2]\alpha^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$ instead of $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$.

– The proof is symmetric to the positive case.

□

Lemma 24 (Equivalence of polymorphic types)**.**

– *For* $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N$ *and* $\Gamma \vdash \forall\overrightarrow{\beta^{+}}.M$,
if $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N \simeq_1^{\leqslant} \forall\overrightarrow{\beta^{+}}.M$ *then there exists a bijection* $\mu : \overrightarrow{\beta^{+}} \cap \mathbf{fv}\, M \leftrightarrow \overrightarrow{\alpha^{+}} \cap \mathbf{fv}\, N$ *such that* $\Gamma, \overrightarrow{\alpha^{+}} \vdash N \simeq_1^{\leqslant} [\mu]N$,

\+ *For* $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.P$ *and* $\Gamma \vdash \exists\overrightarrow{\beta^{-}}.Q$,
if $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.P \simeq_1^{\leqslant} \exists\overrightarrow{\beta^{-}}.Q$ *then there exists a bijection* $\mu : \overrightarrow{\beta^{-}} \cap \mathbf{fv}\, Q \leftrightarrow \overrightarrow{\alpha^{-}} \cap \mathbf{fv}\, P$ *such that* $\Gamma, \overrightarrow{\beta^{-}} \vdash P \simeq_1^{\leqslant} [\mu]Q$.

Proof.

– First, by α-conversion, we ensure $\overrightarrow{\alpha^{+}} \cap \mathbf{fv}\, M = \varnothing$ and $\overrightarrow{\beta^{+}} \cap \mathbf{fv}\, N = \varnothing$. By inversion, $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N \simeq_1^{\leqslant} \forall\overrightarrow{\beta^{+}}.M$ implies

1. $\Gamma, \overrightarrow{\beta^{+}} \vdash [\sigma_1]N \leqslant_1 M$ for $\Gamma, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \overrightarrow{\alpha^{+}}$ and
2. $\Gamma, \overrightarrow{\alpha^{+}} \vdash [\sigma_2]M \leqslant_1 N$ for $\Gamma, \overrightarrow{\alpha^{+}} \vdash \sigma_2 : \overrightarrow{\beta^{+}}$.

To apply lemma 23, we weaken and rearrange the contexts, and extend the substitutions to act as identity outside of their initial domain:

1. $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\sigma_1]N \leqslant_1 M$ for $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}}$ and
2. $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\sigma_2]M \leqslant_1 N$ for $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_2 : \Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}}$.

Then from lemma 23, there exists a bijection $\mu : \mathbf{fv}\, M \leftrightarrow \mathbf{fv}\, N$ such that $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_2 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, M$ and $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, N$.

Let us show that if we restrict the domain of μ to $\overrightarrow{\beta^{+}}$, its range will be contained in $\overrightarrow{\alpha^{+}}$. Let us take $\gamma^{+} \in \overrightarrow{\beta^{+}} \cap \mathbf{fv}\, M$ and assume $[\mu]\gamma^{+} \notin \overrightarrow{\alpha^{+}}$. Then since $\Gamma, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \overrightarrow{\alpha^{+}}$, σ_1 acts as identity outside of $\overrightarrow{\alpha^{+}}$, i.e. $[\sigma_1][\mu]\gamma^{+} = [\mu]\gamma^{+}$. Since $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, N$, application of σ_1 is equivalent to application of μ^{-1}, then $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\mu^{-1}][\mu]\gamma^{+} \simeq_1^{\leqslant} [\mu]\gamma^{+}$, i.e. $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \gamma^{+} \simeq_1^{\leqslant} [\mu]\gamma^{+}$, which means $\gamma^{+} \in \mathbf{fv}\, [\mu]\gamma^{+} \subseteq \mathbf{fv}\, N$. By assumption, $\gamma^{+} \in \overrightarrow{\beta^{+}} \cap \mathbf{fv}\, M$, i.e. $\overrightarrow{\beta^{+}} \cap \mathbf{fv}\, N \neq \varnothing$, hence contradiction.
By **??**, $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_2 \simeq_1^{\leqslant} \mu|_{\overrightarrow{\beta^{+}}} : \mathbf{fv}\, M$ implies $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\sigma_2]M \simeq_1^{\leqslant} [\mu|_{\overrightarrow{\beta^{+}}}]M$. By similar reasoning, $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\sigma_1]N \simeq_1^{\leqslant} [\mu^{-1}|_{\overrightarrow{\alpha^{+}}}]N$.
This way,

$$\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\mu^{-1}|_{\overrightarrow{\alpha^{+}}}]N \leqslant_1 M \tag{5}$$

$$\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\mu|_{\overrightarrow{\beta^{+}}}]M \leqslant_1 N \tag{6}$$

By applying $\mu|_{\overrightarrow{\beta^{+}}}$ to both sides of 5 (**??**) and contracting $\mu^{-1}|_{\overrightarrow{\alpha^{+}}} \circ \mu|_{\overrightarrow{\beta^{+}}} = \mu|^{-1}_{\overrightarrow{\beta^{+}}} \circ \mu|_{\overrightarrow{\beta^{+}}} = \mathsf{id}$, we have: $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash N \leqslant_1 [\mu|_{\overrightarrow{\beta^{+}}}]M$, which together with 6 means $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash N \simeq_1^{\leqslant} [\mu|_{\overrightarrow{\beta^{+}}}]M$, and by strengthening, $\Gamma, \overrightarrow{\alpha^{+}} \vdash N \simeq_1^{\leqslant} [\mu|_{\overrightarrow{\beta^{+}}}]M$. Symmetrically, $\Gamma, \overrightarrow{\beta^{+}} \vdash M \simeq_1^{\leqslant} [\mu|^{-1}_{\overrightarrow{\beta^{+}}}]N$.

- \+ The proof is symmetric to the proof of the negative case.

□

Lemma 25 (Completeness of equivalence)**.** *Mutual subtyping implies declarative equivalence. Assuming all the types below are well-formed in* Γ*:*

+ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $P \simeq_1^D Q$,
− *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $N \simeq_1^D M$.

Proof. − Induction on the sum of sizes of N and M. By inversion, $\Gamma \vdash N \simeq_1^{\leqslant} M$ means $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash M \leqslant_1 N$. Let us consider the last rule that forms $\Gamma \vdash N \leqslant_1 M$:

Case 1. Rule $(\mathrm{Var}^{-\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
Then $N \simeq_1^D M$ (i.e. $\alpha^- \simeq_1^D \alpha^-$) holds immediately by Rule $(\mathrm{Var}^{-\simeq_1^D})$.

Case 2. Rule $(\uparrow^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$
Then by inversion, $\Gamma \vdash P \simeq_1^{\leqslant} Q$, and by induction hypothesis, $P \simeq_1^D Q$. Then $N \simeq_1^D M$ (i.e. $\uparrow P \simeq_1^D \uparrow Q$) holds by Rule $(\uparrow^{\simeq_1^D})$.

Case 3. Rule $(\to^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash P \to N' \leqslant_1 Q \to M'$
Then by inversion, $\Gamma \vdash P \geqslant_1 Q$ and $\Gamma \vdash N' \leqslant_1 M'$. Notice that $\Gamma \vdash M \leqslant_1 N$ is of the form $\Gamma \vdash Q \to M' \leqslant_1 P \to N'$, which by inversion means $\Gamma \vdash Q \geqslant_1 P$ and $\Gamma \vdash M' \leqslant_1 N'$.
This way, $\Gamma \vdash Q \simeq_1^{\leqslant} P$ and $\Gamma \vdash M' \simeq_1^{\leqslant} N'$. Then by induction hypothesis, $Q \simeq_1^D P$ and $M' \simeq_1^D N'$. Then $N \simeq_1^D M$ (i.e. $P \to N' \simeq_1^D Q \to M'$) holds by Rule $(\to^{\simeq_1^D})$.

Case 4. Rule $(\forall^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant_1 \forall\overrightarrow{\beta^+}.M'$
Then by **??**, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \simeq_1^{\leqslant} \forall\overrightarrow{\beta^+}.M'$ means that there exists a bijection $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv}\, M' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N'$ such that $\Gamma, \overrightarrow{\alpha^+} \vdash [\mu]M' \simeq_1^{\leqslant} N'$.
Notice that the application of bijection μ to M' does not change its size (which is less than the size of M), hence the induction hypothesis applies. This way, $[\mu]M' \simeq_1^D N'$ (and by symmetry, $N' \simeq_1^D [\mu]M'$) holds by induction. Then we apply Rule $(\forall^{\simeq_1^D})$ to get $\forall\overrightarrow{\alpha^+}.N' \simeq_1^D \forall\overrightarrow{\beta^+}.M'$, i.e. $N \simeq_1^D M$.

\+ The proof is symmetric to the proof of the negative case.

□

Corollary 16 (Normalization is complete w.r.t. subtyping-induced equivalence)**.** *Assuming all the types below are well-formed in* Γ*:*

+ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$,
− *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$.

Proof. Immediately from lemmas 15 and 25. □

Lemma 26 (Algorithmization of subtyping-induced equivalence)**.** *Mutual subtyping is equality of normal forms. Assuming all the types below are well-formed in* Γ*:*

+ $\Gamma \vdash P \simeq_1^{\leqslant} Q \iff \mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$,
− $\Gamma \vdash N \simeq_1^{\leqslant} M \iff \mathbf{nf}\,(N) = \mathbf{nf}\,(M)$.

Proof. Let us prove the positive case, the negative case is symmetric. We prove both directions of $\iff$ separately:

$\Rightarrow$ exactly corollary 16;

$\Leftarrow$ by lemmas 19 and 21.

□

4.8 Unification Solution Merge

Lemma 27 (Soundness of Merge of Unification Solutions)**.** *Suppose that $\widehat{\sigma}_1 : \Theta|_{vars_1}$ and $\widehat{\sigma}_2 : \Theta|_{vars_2}$ are normalized unification solutions (i.e. $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ can only have equivalence-shaped restrictions, which types are normalized), If $\widehat{\sigma}_1$ & $\widehat{\sigma}_2$ is defined then $\widehat{\sigma}_1$ & $\widehat{\sigma}_2 = \widehat{\sigma}_1\widehat{\sigma}_2$, where*

Proof.

- $\widehat{\sigma}_1$ & $\widehat{\sigma}_2 \subseteq \widehat{\sigma}_1 \cup \widehat{\sigma}_2$
  By definition, $\widehat{\sigma}_1$ & $\widehat{\sigma}_2$ consists of three parts: entries of $\widehat{\sigma}_1$ that do not have matching entries of $\widehat{\sigma}_2$, entries of $\widehat{\sigma}_2$ that do not have matching entries of $\widehat{\sigma}_1$, and the merge of matching entries.

  If e is from the first or the second part, then $e \in \widehat{\sigma}_1 \cup \widehat{\sigma}_2$.

  If e is from the third part, then e is the merge of two matching entries $e_1 \in \widehat{\sigma}_1$ and $e_2 \in \widehat{\sigma}_2$. Since $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are normalized unification solutions, e_1 and e_2 have one of the following forms:

  - $\widehat{\alpha_1}^+ :\approx P_1$ and $\widehat{\alpha_2}^+ :\approx P_2$, where P_1 and P_2 are normalized, and then since e_1 & e_2 exists, Rule ($\simeq$ &$^+$ $\simeq$) was applied to infer it. It means that $e = (e_1 \;\&\; e_2) = e_1 = e_2$;
  - $\widehat{\alpha_1}^- :\approx N_1$ and $\widehat{\alpha_2}^- :\approx N_2$, then symmetrically, $e = (e_1 \;\&\; e_2) = e_1 = e_2$

  In both cases, $e \in \widehat{\sigma}_1 \cup \widehat{\sigma}_2$.

- $\widehat{\sigma}_1 \cup \widehat{\sigma}_2 \subseteq \widehat{\sigma}_1$ & $\widehat{\sigma}_2$
  Let us take an arbitrary $e \in \widehat{\sigma}_1$. Then since $\widehat{\sigma}_1$ is a unification solution, e has one of the following forms:

  - $\widehat{\alpha}^+ :\approx P$ where P is normalized. If $\widehat{\alpha}^+ \notin \mathbf{dom}\,(\widehat{\sigma}_2)$, then $e \in \widehat{\sigma}_1$ & $\widehat{\sigma}_2$. Otherwise there is a normalized $e = (\widehat{\alpha}^+ :\approx P') \in \widehat{\sigma}_2$ and then since $\widehat{\sigma}_1$ & $\widehat{\sigma}_2$ exists, Rule ($\simeq$ &$^+$ $\simeq$) was applied to construct e & $e' \in \widehat{\sigma}_1$ & $\widehat{\sigma}_2$. By inversion of Rule ($\simeq$ &$^+$ $\simeq$), e & $e' = e$, and $\mathbf{nf}\,(P) = \mathbf{nf}\,(P')$, which since P and P' are normalized, implies that $P = P'$, that is $e = e'$. This way, $e' = e \in \widehat{\sigma}_1$ & $\widehat{\sigma}_2$.
  - $\widehat{\alpha}^- :\approx N$ where N is normalized. Then symmetrically, $e \in \widehat{\sigma}_1$ & $\widehat{\sigma}_2$.

  Similarly, if we take an arbitrary $e' \in \widehat{\sigma}_2$, then $e' \in \widehat{\sigma}_1$ & $\widehat{\sigma}_2$. In fact, this is the case where normalization is important, since Rule ($\simeq$ &$^+$ $\simeq$) returns the left-hand operand of &, but as noted above, $\mathbf{nf}\,(P) = \mathbf{nf}\,(P')$ implies $P = P'$ for normalized types.

□

Corollary 17. *Suppose that $\widehat{\sigma}_1 : \Theta|_{vars_1}$ and $\widehat{\sigma}_2 : \Theta|_{vars_2}$ are normalized unification solutions If $\widehat{\sigma}_1$ & $\widehat{\sigma}_2$ is defined then*

1. *$\widehat{\sigma}_1$ & $\widehat{\sigma}_2 : \Theta|_{vars_1 \cup vars_2}$,*
2. *$\widehat{\sigma}_1$ & $\widehat{\sigma}_2$ is a normalized unification solution,*
3. *$\widehat{\sigma}_1$ & $\widehat{\sigma}_2|_{vars_i} = \widehat{\sigma}_i$ for $i = 1, 2$,*

Proof. The first two properties follow immediately from lemma 27.

To prove the third property, first notice that $\widehat{\sigma}_1$ & $\widehat{\sigma}_2|_{vars_i} \supseteq \widehat{\sigma}_i$ follows immediately from $\widehat{\sigma}_1$ & $\widehat{\sigma}_2 = \widehat{\sigma}_1 \cup \widehat{\sigma}_2$. For the other inclusion $\widehat{\sigma}_1$ & $\widehat{\sigma}_2|_{vars_1} \subseteq \widehat{\sigma}_1$, let us take an arbitrary $e \in \widehat{\sigma}_1$ & $\widehat{\sigma}_2|_{vars_1}$. Then e is either from $\widehat{\sigma}_1$ (if it does not have a matching entry in $\widehat{\sigma}_2$) or it is a result of merge of two matching entries, but since $\widehat{\sigma}_1$ & $\widehat{\sigma}_2$ exist and $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are normalized unification solutions, the matching entries of $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are equal, and they are equal to their merge. In both cases, $e \in \widehat{\sigma}_1$. Similarly, $\widehat{\sigma}_1$ & $\widehat{\sigma}_2|_{vars_2} \subseteq \widehat{\sigma}_2$. □

Lemma 28 (Completeness of Unification Solution Merge)**.** *Suppose that $\widehat{\sigma}$ is a unification solution, and $vars_1$, $vars_2$ are sets of variables. Then $\widehat{\sigma}|_{vars_1}$ & $\widehat{\sigma}|_{vars_2}$ is defined and equal to $\widehat{\sigma}|_{vars_1 \cup vars_2}$.*

Proof. $\widehat{\sigma}|_{vars_1}$ & $\widehat{\sigma}|_{vars_2}$ is defined as the union of three parts: entries of $\widehat{\sigma}|_{vars_1}$ that do not have matching entries of $\widehat{\sigma}|_{vars_2}$, entries of $\widehat{\sigma}|_{vars_2}$ that do not have matching entries of $\widehat{\sigma}|_{vars_1}$, and the merge of matching entries. The first two parts are defined. The merge of matching entries is defined by Rule ($\simeq$ &$^+$ $\simeq$), since the matching entries must be equal if they both belong to $\widehat{\sigma}$.

It remains to show that $\widehat{\sigma}|_{vars_1}$ & $\widehat{\sigma}|_{vars_2} = \widehat{\sigma}|_{vars_1 \cup vars_2}$. It is easy to see that the three parts comprising $\widehat{\sigma}|_{vars_1}$ & $\widehat{\sigma}|_{vars_2}$ correspond to the three parts comprising $\widehat{\sigma}|_{vars_1 \cup vars_2} = \widehat{\sigma}|_{(vars_1 \setminus vars_2)} \cup \widehat{\sigma}|_{(vars_2 \setminus vars_1)} \cup \widehat{\sigma}|_{vars_1 \cap vars_2}$. □

4.9 Upgrade

Let us consider a type P well-formed in Γ. Some of its Γ-supertypes are also well-formed in a smaller context $\Delta \subseteq \Gamma$. The upgrade is the operation that returns the least of such supertypes.

Lemma 29 (Soundness of Upgrade)**.** *Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$,*
if $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ *then*

1. $\Delta \vdash Q$
2. $\Gamma \vdash Q \geqslant_1 P$

Proof. By inversion, $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ means that for fresh $\overrightarrow{\beta^{\pm}}$ and $\overrightarrow{\gamma^{\pm}}$, $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vDash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P = Q$. Then by the soundness of the least upper bound (lemma 34),

1. $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q$,
2. $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$, and
3. $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$.

$$
\begin{aligned}
\mathbf{fv}\,Q &\subseteq \mathbf{fv}\,[\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \cap \mathbf{fv}\,[\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P && \text{Since by lemma 1, } \mathbf{fv}\,Q \subseteq \mathbf{fv}\,[\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \text{ and } \mathbf{fv}\,Q \subseteq \mathbf{fv}\,[\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \\
&\subseteq ((\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) \cup \overrightarrow{\beta^{\pm}}) \cap ((\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) \cup \overrightarrow{\gamma^{\pm}}) \\
&= (\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) \cap (\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) && \text{since } \overrightarrow{\beta^{\pm}} \text{ and } \overrightarrow{\gamma^{\pm}} \text{ are fresh} \\
&= \mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}} \\
&\subseteq \Gamma \backslash \overrightarrow{\alpha^{\pm}} && \text{since } P \text{ is well-formed in } \Gamma \\
&\subseteq \Delta
\end{aligned}
$$

This way, by lemma 6, $\Delta \vdash Q$.

Let us apply $\overrightarrow{\alpha^{\pm}}/\overrightarrow{\beta^{\pm}}$—the inverse of the substitution $\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$ and by **??**, get $\Delta, \overrightarrow{\alpha^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash [\overrightarrow{\alpha^{\pm}}/\overrightarrow{\beta^{\pm}}]Q \geqslant_1 P$. Notice that $\Delta \vdash Q$ implies that $\mathbf{fv}\,Q \cap \overrightarrow{\beta^{\pm}} = \varnothing$, then by **??**, $[\overrightarrow{\alpha^{\pm}}/\overrightarrow{\beta^{\pm}}]Q = Q$, and thus $\Delta, \overrightarrow{\alpha^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 P$. By context strengthening, $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q \geqslant_1 P$. □

Lemma 30 (Completeness and Initiality of Upgrade)**.** *The upgrade returns the least Γ-supertype of P well-formed in Δ. Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$,*
For any Q' such that

1. $\Delta \vdash Q'$ *and*
2. $\Gamma \vdash Q' \geqslant_1 P$,

The result of the upgrade algorithm Q exists ($\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$*) and satisfies $\Delta \vdash Q' \geqslant_1 Q$.*

Proof. Let us consider fresh (not intersecting with Γ) $\overrightarrow{\beta^{\pm}}$ and $\overrightarrow{\gamma^{\pm}}$.

If we apply substitution $\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q' \geqslant_1 P$, we have $\Delta, \overrightarrow{\beta^{\pm}} \vdash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]Q' \geqslant_1 [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$, which by **??**, since Q' is well-formed in Δ, simplifies to $\Delta, \overrightarrow{\beta^{\pm}} \vdash Q' \geqslant_1 [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$.

Analogously, if we apply substitution $\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q' \geqslant_1 P$, we have $\Delta, \overrightarrow{\gamma^{\pm}} \vdash Q' \geqslant_1 [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$.

This way, Q' is a common supertype of $[\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$ and $[\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$ in context $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}}$. It means that we can apply the completeness of the least upper bound (lemma 35):

1. there exists Q s.t. $\Gamma \vDash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P = Q$
2. $\Gamma \vdash Q' \geqslant_1 Q$.

The former means that the upgrade algorithm terminates and returns Q. The latter means that since both Q' and Q are well-formed in Δ, by **??**, $\Delta \vdash Q' \geqslant_1 Q$. □

4.10 Upper Bounds

Lemma 31 (Decomposition of the quantifier rule). `Ilya: move somewhere` *Whenever the quantifier rule (Rule $(\exists^{\geqslant_1})$ or Rule $(\forall^{\leqslant_1})$) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.*

− *If* $\Gamma \vdash N \leqslant_1 \forall\overrightarrow{\beta^+}.M$ *then* $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant_1 M$.

+ *If* $\Gamma \vdash P \geqslant_1 \exists\overrightarrow{\beta^-}.Q$ *then* $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant_1 Q$.

Lemma 32 (Characterization of the Supertypes). *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \textit{for } \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-}, M', \textit{and } \overrightarrow{N} \textit{ s.t.} \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Immediately from lemma 2

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$
Then if $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'$, then by lemma 31, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$, and $\mathbf{fv}\, Q \cap \overrightarrow{\beta^-} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule $(\exists^{\geqslant_1})$. This way, $\{Q \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,(Q) \cap \overrightarrow{\beta^-} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = \downarrow M$
Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\nexists}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists\overrightarrow{\beta^-}.Q'$
Then the last applied rule to infer $\Gamma \vdash Q \geqslant_1 \downarrow M$ must be Rule $(\downarrow^{\geqslant_1})$, which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq_1^{\leqslant} M$, then by lemma 25 and Rule $(\downarrow^{\simeq_1^D})$, $\downarrow M' \simeq_1^D \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq_1^D \downarrow M\} = \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\nexists}$.
In the other direction, $\downarrow M' \simeq_1^D \downarrow M \Rightarrow \Gamma \vdash \downarrow M' \simeq_1^{\leqslant} \downarrow M$ by lemma 21, since $\Gamma \vdash \downarrow M'$ by lemma 20
$\Rightarrow \Gamma \vdash \downarrow M' \geqslant_1 \downarrow M$ by inversion

b. $Q = \exists\overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)
Then the last rule applied to infer $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q' \geqslant_1 \downarrow M$ must be Rule $(\exists^{\geqslant_1})$. Inversion of this rule gives us $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \geqslant_1 \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.*a*, $[\overrightarrow{N}/\overrightarrow{\beta^-}]Q' \simeq_1^D \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists\overrightarrow{\beta^-}.\downarrow M' \in \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).
In the other direction, $[\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^D \downarrow M \Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^{\leqslant} \downarrow M$ by lemma 21, since $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M'$ by lemma 20
$\Rightarrow \Gamma \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]\downarrow M' \geqslant_1 \downarrow M$ by inversion
$\Rightarrow \Gamma \vdash \exists\overrightarrow{\beta^-}.\downarrow M' \geqslant_1 \downarrow M$ by Rule $(\exists^{\geqslant_1})$

□

Lemma 33 (Characterization of the Normalized Supertypes). *For a normalized positive type* $P = \mathbf{nf}\,(P)$, *let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-}, M', \textit{and } \overrightarrow{N} \textit{ s.t. } \mathbf{ord}\, \overrightarrow{\alpha^-}\, \mathbf{in}\, M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \textit{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \end{array} \right\}$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Then from lemma 32, $\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\beta^+) \mid \text{for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

$$\begin{aligned}
\mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') &= \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ not using } \overrightarrow{\beta^-} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothesis} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}\,(Q) = \mathbf{fv}\,Q \text{ by lemma 12} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{by lemma 32} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} && \text{by the definition of } \mathsf{UB} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} && \text{by lemma 32}
\end{aligned}$$

Case 3. $P = \downarrow M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 1. *Suppose that $\nu : A \to A$ is an idempotent function, P is a predicate on A, $F : A \to B$ is a function. Then*

$$\begin{aligned}
&\{F(\nu x) \mid x \in A \text{ s.t. } P(\nu x)\} = \\
=&\{F(x) \mid x \in A \text{ s.t. } \nu x = x \text{ and } P(x)\}.
\end{aligned}$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 2. *For functions F and ν, and predicates P and Q,*

$$\begin{aligned}
&\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } P(x)\} = \\
=&\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } (\exists x' \in A \text{ s.t. } P(x') \text{ and } \nu x' = \nu x)\}.
\end{aligned}$$

Observation 3. *There exist positive and negative types well-formed in empty context, hence, a type substitution can be extended to an arbitrary domain (if its values on the domain extension are irrelevant). Specifically, Suppose that $vars_1 \subseteq vars_2$. Then $\Gamma \vdash \sigma|_{vars_1} : vars_1$ implies $\exists\ \sigma'$ s.t. $\Gamma \vdash \sigma' : vars_2$ and $\sigma|_{vars_1} = \sigma'|_{vars_1}$.*

$$
\begin{aligned}
&\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \downarrow M\} = \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \downarrow M)\} && \text{by lemma 32}\\
&= \left\{ \mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \vec{N} \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash N_i, \text{ and } [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} && \text{by the definition of } \mathsf{UB}\\
&= \left\{ \mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} && \text{we reassigned the substitution } \vec{N}/\overrightarrow{\alpha^-} \text{ as } \sigma\\
&= \left\{ \mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} && \text{by lemma 3}\\
&= \left\{ \exists \overrightarrow{\alpha^{-\prime}}.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-},\ M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} && \text{by the definition of normalization}\\
&= \left\{ \exists \overrightarrow{\alpha^{-\prime}}.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-},\ M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } \mathbf{nf}\,([\sigma|_{\mathbf{fv}\,M'}]\downarrow M') = \mathbf{nf}\,(\downarrow M) \end{array} \right\} && \text{from lemmas 13 and 15, equivalence of types can be replaced with the equality of their normal forms}\\
&= \left\{ \exists \overrightarrow{\alpha^{-\prime}}.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-},\ M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})]\downarrow \mathbf{nf}\,(M') = \downarrow \mathbf{nf}\,(M) \end{array} \right\} && \text{by congruence of normalization and lemma 14}\\
&= \left\{ \exists \overrightarrow{\alpha^{-\prime}}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-},\ M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} && \begin{array}{l}\text{by lemma 17, } \downarrow M' \text{ and } \sigma|_{\mathbf{fv}\,M'} \text{ are already normal,} \\ \text{since the result of the substitution is normal;} \\ M \text{ is normal by assumption}\end{array}\\
&= \left\{ \exists \overrightarrow{\alpha^{-\prime}}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-},\ M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ (\exists \sigma' \text{ s.t. } \Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,(\downarrow M')} = \sigma'|_{\mathbf{fv}\,(\downarrow M')}) \\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^{-\prime}} \text{ and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} && \begin{array}{l}\text{We apply observation 2 (with } \nu\sigma = \sigma|_{\mathbf{fv}\,M'}, \text{ and} \\ P(\sigma) = \Gamma \vdash \sigma : \overrightarrow{\alpha^-})\end{array}\\
&= \left\{ \exists \overrightarrow{\alpha^{-\prime}}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-},\ M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv}\,M'} : \overrightarrow{\alpha^-},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} && \begin{array}{l}\text{Notice that} \\ \text{“}\exists\sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,(\downarrow M')} = \sigma'|_{\mathbf{fv}\,(\downarrow M')})\text{”} \\ \text{is equivalent to } \Gamma \vdash \sigma|_{\mathbf{fv}\,(\downarrow M')} : \overrightarrow{\alpha^{-\prime}} \text{ (observation 3)}\end{array}\\
&= \left\{ \exists \overrightarrow{\alpha^{-\prime}}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-},\ M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^{-\prime}},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} && \begin{array}{l}\text{We apply observation 1 to the restriction of } \sigma, \text{ and} \\ \text{remove } \sigma|_{\mathbf{fv}\,M'} = \sigma \text{ as it follows from } \Gamma \vdash \sigma : \overrightarrow{\alpha^{-\prime}}\end{array}\\
&= \left\{ \exists \overrightarrow{\alpha^{-\prime}}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-},\ M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^{-\prime}} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^{-\prime}},\ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} && \text{by lemma 6, since } \Gamma, \overrightarrow{\alpha^-} \cap \mathbf{fv}\,M' = \Gamma, \overrightarrow{\alpha^{-\prime}} \cap \mathbf{fv}\,M'\\
&= \left\{ \exists \overrightarrow{\alpha^{-\prime}}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}},\ M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^{-\prime}} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^{-\prime}},\ \mathbf{ord}\,\overrightarrow{\alpha^{-\prime}}\,\mathbf{in}\,M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} && \text{We apply observation 1 to the ordering of } \overrightarrow{\alpha^-}\\
&= \left\{ \exists \overrightarrow{\alpha^{-\prime}}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}},\ M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^{-\prime}}\,\mathbf{in}\,M' = \overrightarrow{\alpha^{-\prime}}, \\ \Gamma \vdash N_i, \Gamma, \overrightarrow{\alpha^{-\prime}} \vdash M', \text{ and } [\vec{N}/\overrightarrow{\alpha^{-\prime}}]\downarrow M' = \downarrow M \end{array} \right\} && \text{By reassigning } \sigma \text{ explicitly as } \vec{N}/\overrightarrow{\alpha^{-\prime}}\\
&= \mathsf{NFUB}(\downarrow M) && \text{by definition}
\end{aligned}
$$

□

Observation 4. *Upper bounds of a type do not depend on the context as soon as the type are well-formed in it.*
If $\Gamma_1 \vdash M$ *and* $\Gamma_2 \vdash M$ *then* $\mathsf{UB}(\Gamma_1 \vdash M) = \mathsf{UB}(\Gamma \vdash M)$ *and* $\mathsf{NFUB}(\Gamma_1 \vdash M) = \mathsf{NFUB}(\Gamma \vdash M)$

Proof. We prove both inclusions by induction on $\Gamma_1 \vdash M$. Notice that if $[\sigma]M' \simeq_1^D M$ and $\Gamma_2 \vdash M$ then the types from the range of $\sigma|_{\mathbf{fv}\,M'}$ are well-formed in 2 Ilya: lemma. □

Lemma 34 (Soundness of the Least Upper Bound)**.** *For types* $\Gamma \vdash P_1$, *and* $\Gamma \vdash P_2$, *if* $\Gamma \vDash P_1 \vee P_2 = Q$ *then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Proof. Induction on $\Gamma \vDash P_1 \vee P_2 = Q$.

Case 1. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$
Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule ($\mathrm{Var}^{+\geqslant_1}$).

Case 2. $\Gamma \vDash \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q$
Then by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applies to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q$. Then

(i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,
(ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$,
(iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}\,(Q) \cap \Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} = \mathbf{fv}\,(Q) \cap \Gamma$ (and then apply lemma 6). The inclusion right-to-left is self-evident. To show $\mathbf{fv}\,(Q) \cap \Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \subseteq \mathbf{fv}\,(Q) \cap \Gamma$, we prove that $\mathbf{fv}\,(Q) \subseteq \Gamma$

$$
\begin{aligned}
\mathbf{fv}\,(Q) &\subseteq \mathbf{fv}\,P_1 \cap \mathbf{fv}\,P_2 && \text{by lemma 1}\\
&\subseteq (\Gamma, \overrightarrow{\alpha^-} \backslash \overrightarrow{\beta^-}) \cap (\Gamma, \overrightarrow{\beta^-} \backslash \overrightarrow{\alpha^-}) && \text{since } \Gamma \vdash \exists\overrightarrow{\alpha^-}.P_1,\ \mathbf{fv}\,(P_1) \subseteq \Gamma, \overrightarrow{\alpha^-} = \Gamma, \overrightarrow{\alpha^-} \backslash \overrightarrow{\beta^-}\\
& && \text{(the latter is because by the Barendregt's convention,}\\
& && \Gamma, \overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing\text{); similarly, } \mathbf{fv}\,(P_2) \subseteq \Gamma, \overrightarrow{\beta^-} \backslash \overrightarrow{\alpha^-}\\
&\subseteq \Gamma
\end{aligned}
$$

To show $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\alpha^-}.P_1$, we apply Rule ($\exists^{\geqslant_1}$). Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant_1 P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \vDash \downarrow N \vee \downarrow M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P$. By the inversion, $\Gamma, \cdot \vDash \mathbf{nf}\,(\downarrow N) \stackrel{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$. Then by the soundness of anti-unification (**??**),

(i) $\Gamma; \Xi \vdash P$, then by **??**,

$$\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]\,P \tag{7}$$

(ii) $\Gamma; \cdot \vdash \widehat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_2 : \Xi$. Assuming that $\Xi = \widehat{\beta}_1^-, .., \widehat{\beta}_n^-$, the antiunification solutions $\widehat{\tau}_1$ and $\widehat{\tau}_2$ can be put explicitly as $\widehat{\tau}_1 = (\widehat{\beta}_1^- :\approx N_1, .., \widehat{\beta}_n^- :\approx N_n)$, and $\widehat{\tau}_2 = (\widehat{\beta}_1^- :\approx M_1, .., \widehat{\beta}_n^- :\approx M_n)$. Then

$$\widehat{\tau}_1 = (\overrightarrow{N}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \tag{8}$$

$$\widehat{\tau}_2 = (\overrightarrow{M}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \tag{9}$$

(iii) $[\widehat{\tau}_1]\,Q = P_1$ and $[\widehat{\tau}_2]\,Q = P_1$, which, by 8 and 9, means

$$[\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]\,P = \mathbf{nf}\,(\downarrow N) \tag{10}$$

$$[\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]\,P = \mathbf{nf}\,(\downarrow M) \tag{11}$$

Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P$ follows directly from 7.

To show $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P \geqslant_1 \downarrow N$, we apply Rule ($\exists^{\geqslant_1}$), instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{N}$. Then $\Gamma \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]\,P \geqslant_1 \downarrow N$ follows from 10 and since $\Gamma \vdash \mathbf{nf}\,(\downarrow N) \geqslant_1 \downarrow N$ (by corollary 14).

Analogously, instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{M}$, gives us $\Gamma \vdash [\overrightarrow{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]\,P \geqslant_1 \downarrow M$ (from 11), and hence, $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P \geqslant_1 \downarrow M$.

□

Lemma 35 (Completeness and Initiality of the Least Upper Bound)**.** *For types* $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, *and* $\Gamma \vdash Q$ *such that* $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$, *there exists* Q' *s.t.* $\Gamma \vDash P_1 \vee P_2 = Q'$ *and* $\Gamma \vdash Q \geqslant_1 Q'$.

Proof. Induction on the pair (P_1, P_2). From lemma 33, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$. Let us consider the cases of what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_i$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$, where either $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then $Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_2.Q_2)$

$\subseteq \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_2 \vdash Q_2)$ from the definition of UB

$= \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_2)$ by observation 4, weakening and exchange

$= \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant_1 Q_1\} \cap \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant_1 Q_2\}$ by lemma 32,

meaning that $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$. Then the next step of the algorithm—the recursive call $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vDash Q_1 \vee Q_2 = Q'$ terminates by the induction hypothesis, and moreover, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$. This way, the result of the algorithm is Q', i.e. $\Gamma \vDash P_1 \vee P_2 = Q'$.

Since both Q and Q' are sound, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$ can be strengthened to $\Gamma \vdash Q \geqslant_1 Q'$ by **??**.

Case 2. $P_1 = \alpha^+$ and $P_2 = {\downarrow}N$

Then the set of common upper bounds of ${\downarrow}N$ and α^+ is empty, and thus, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$ gives a contradiction:
$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash {\downarrow}N)$

$= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.{\downarrow}M' \mid \cdots\}$ by the definition of UB

$= \varnothing$ since $\alpha^+ \neq {\downarrow}M'$ for any M'

Case 3. $P_1 = {\downarrow}N$ and $P_2 = \alpha^+$
Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)
Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \beta^+)$

$= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\beta^+ \mid \cdots\}$ by the definition of UB

$= \varnothing$ since $\alpha^+ \neq \beta^+$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$
Then the algorithm terminates in one step (Rule ($\mathrm{Var}^\vee$)) and the result is α^+, i.e. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$.

Since $Q \in \mathsf{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists\overrightarrow{\alpha^-}.\alpha^+$. Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant_1 \alpha^+$ by Rule ($\exists^{\geqslant_1}$): $\overrightarrow{\alpha^-}$ can be instantiated with arbitrary negative types (for example $\forall\beta^+.{\uparrow}\beta^+$), since the substitution for unused variables does not change the term $[\vec{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule ($\mathrm{Var}^{+\geqslant_1}$).

Case 6. $P_1 = {\downarrow}M_1$ and $P_2 = {\downarrow}M_2$
Then on the next step, the algorithm tries to anti-unify $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$. By **??**, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$\mathbf{nf}\,(Q) \in \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,({\downarrow}M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,({\downarrow}M_2))$

$= \mathsf{NFUB}(\Gamma \vdash {\downarrow}\mathbf{nf}\,(M_1)) \cap \mathsf{NFUB}(\Gamma \vdash {\downarrow}\mathbf{nf}\,(M_2))$

$$= \left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\vec{N}/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_1) \end{array} \right\}$$

$$= \cap$$

$$\left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \vec{N}_1,\ \Gamma \vdash \vec{N}_2,\ \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\vec{N}/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_2) \end{array} \right\}$$

$$= \left\{ \exists\overrightarrow{\alpha^-}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\ M',\ \vec{N}_1 \text{ and } \vec{N}_2 \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \vec{N}_1,\ \Gamma \vdash \vec{N}_2,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ [\vec{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_1), \text{ and } [\vec{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_2) \end{array} \right\}$$

The fact that the latter set is non-empty means that there exist $\overrightarrow{\alpha^-}$, M', $\vec{N}_1$ and $\vec{N}_2$ such that

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$ (notice that M' is normal)

(ii) $\Gamma \vdash \vec{N}_1$ and $\Gamma \vdash \vec{N}_1$,

(iii) $[\vec{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_1)$ and $[\vec{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_2)$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative anti-unification variable $\hat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\widehat{\alpha^-}}$. Let us show that $(\overrightarrow{\widehat{\alpha^-}},\ [\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]{\downarrow}M',\ \vec{N}_1/\overrightarrow{\widehat{\alpha^-}},\ \vec{N}_2/\overrightarrow{\widehat{\alpha^-}})$ is a sound anti-unifier of $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$ in context Γ:

- $\overrightarrow{\widehat{\alpha^-}}$ is negative by construction,
- $\Gamma;\overrightarrow{\widehat{\alpha^-}} \vdash [\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]{\downarrow}M'$ because $\Gamma,\overrightarrow{\alpha^-} \vdash {\downarrow}M'$ Ilya: lemma!,
- $\Gamma;\cdot \vdash (\vec{N}_1/\overrightarrow{\widehat{\alpha^-}}) : \overrightarrow{\widehat{\alpha^-}}$ because $\Gamma \vdash \vec{N}_1$ and $\Gamma;\cdot \vdash (\vec{N}_2/\overrightarrow{\widehat{\alpha^-}}) : \overrightarrow{\widehat{\alpha^-}}$ because $\Gamma \vdash \vec{N}_2$,
- $[\vec{N}_1/\overrightarrow{\widehat{\alpha^-}}][\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\vec{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_1) = \mathbf{nf}\,({\downarrow}M_1)$.
- $[\vec{N}_2/\overrightarrow{\widehat{\alpha^-}}][\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\vec{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_2) = \mathbf{nf}\,({\downarrow}M_2)$.

Then by the completeness of the anti-unification (**??**), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it, i.e. $Q' = \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]\,P$, where $(\Xi, P, \hat{\tau}_1, \hat{\tau}_2)$ is the result of the anti-unification of $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$ in context Γ.

Moreover, **??** also says that the found solution is initial, i.e. there exists $\hat{\tau}$ such that $\Gamma;\Xi \vdash \hat{\tau} : \overrightarrow{\widehat{\alpha^-}}$ and $[\hat{\tau}][\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]{\downarrow}M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}$, (ii) $\hat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma,\overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi][\hat{\tau}][\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi]\,P$. In particular, from the reflexivity of subtyping: $\Gamma,\overrightarrow{\beta^-} \vdash [\sigma]{\downarrow}M' \geqslant_1 [\overrightarrow{\beta^-}/\Xi]\,P$.

It allows us to show $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant_1 Q'$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.{\downarrow}M' \geqslant_1 \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]\,P$, by applying Rule $(\exists^{\geqslant_1})$, instantiating $\overrightarrow{\alpha^-}$ with respect to σ. Finally, $\Gamma \vdash Q \geqslant_1 Q'$ since $\Gamma \vdash \mathbf{nf}\,(Q) \simeq_1^{\leqslant} Q$, and equivalence implies subtyping by Ilya: lemma.

□

4.11 Unification

Lemma 36 (Soundness of Unification)**.**

+ *For normalized* P *and* Q *such that* $\Gamma;\Theta \vdash P$ *and* $\Gamma \vdash Q$*,*
  if $\Gamma;\Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}$ *then* $\hat{\sigma} : \Theta|_{\mathbf{uv}\,P}$*,* $\hat{\sigma}$ *is equivalence-only, and* $[\hat{\sigma}]\,P = Q$*.*
- *For normalized* N *and* M *such that* $\Gamma;\Theta \vdash N$ *and* $\Gamma \vdash M$*,*
  if $\Gamma;\Theta \vDash N \overset{u}{\simeq} M \dashv \hat{\sigma}$ *then* $\hat{\sigma} : \Theta|_{\mathbf{uv}\,N}$*,* $\hat{\sigma}$ *is equivalence-only, and* $[\hat{\sigma}]\,N = M$*.*

Proof. We prove by induction on the derivation of $\Gamma;\Theta \vDash N \overset{u}{\simeq} M \dashv \hat{\sigma}$ and mutually $\Gamma;\Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}$. Let us consider the last rule forming this derivation.

Case 1. Rule $(\mathrm{Var}^{-\overset{u}{\simeq}})$, then $N = \alpha^- = M$. The resulting unification solution is empty: $\hat{\sigma} = \cdot$. Since $\mathbf{uv}\,N = \varnothing$, it satisfies $\hat{\sigma} : \Theta|_{\mathbf{uv}\,P}$. Vacuously, it is equivalence-only, and $[\hat{\sigma}]\alpha^- = \alpha^-$, that is $[\hat{\sigma}]\,N = M$.

Case 2. Rule $(\uparrow^{\overset{u}{\simeq}})$, then $N = \uparrow P$ and $M = \uparrow Q$. The algorithm makes a recursive call to $\Gamma;\Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}$ returning $\hat{\sigma}'$. By induction hypothesis, $\hat{\sigma}' : \Theta|_{\mathbf{uv}\,P}$, and since $\mathbf{uv}\,N = \mathbf{uv}\uparrow P = \mathbf{uv}\,P$, $\hat{\sigma}' : \Theta|_{\mathbf{uv}\,N}$. Finally, $[\hat{\sigma}]\,N = [\hat{\sigma}]\uparrow P = \uparrow[\hat{\sigma}]\,P = \uparrow Q = M$.

Case 3. Rule $(\to^{\overset{u}{\simeq}})$, then $N = P \to N'$ and $M = Q \to M'$. The algorithm makes two recursive calls to $\Gamma;\Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}_1$ and $\Gamma;\Theta \vDash N' \overset{u}{\simeq} M' \dashv \hat{\sigma}_2$ returning $\hat{\sigma}_1$ & $\hat{\sigma}_2$ as the result.

It is clear that P, N', Q, and M' are normalized, and that $\Gamma;\Theta \vdash P$, $\Gamma;\Theta \vdash N'$, $\Gamma \vdash Q$, and $\Gamma \vdash M'$. This way, the induction hypothesis is applicable to both recursive calls. By applying the induction hypothesis to $\Gamma;\Theta \vDash P \overset{u}{\simeq} Q \dashv \hat{\sigma}_1$, we have:

- $\hat{\sigma}_1 : \Theta|_{\mathbf{uv}\,P}$,
- $\hat{\sigma}_1$ is equivalence-only, and
- $[\hat{\sigma}_1]\,P = Q$.

By applying it to $\Gamma;\Theta \vDash N' \overset{u}{\simeq} M' \dashv \hat{\sigma}_2$, we have:

- $\hat{\sigma}_2 : \Theta|_{\mathbf{uv}\,N'}$,
- $\hat{\sigma}_2$ is equivalence-only, and
- $[\hat{\sigma}_2]\,N' = M'$.

By the soundness of the unification solution merge (**??**), we have:

- $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2 : \Theta|_{\mathbf{uv}\, P \cup \mathbf{uv}\, N'}$, and hence, $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2 : \Theta|_{\mathbf{uv}\, N}$;
- $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2$ is equivalence-only;
- $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2$ is normalized;
- $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2|_{\mathbf{uv}\, P} = \widehat{\sigma}_1$ and hence, $[\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2]P = [\widehat{\sigma}_1]P = Q$;
- $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2|_{\mathbf{uv}\, N'} = \widehat{\sigma}_2$ and hence, $[\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2]N' = [\widehat{\sigma}_2]N' = M'$.

This way, $[\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2]N = [\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2]P \to [\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2]N' = Q \to M' = M$.

Case 4. Rule $(\forall^{\overset{u}{\simeq}})$, then $N = \forall\overrightarrow{\alpha^+}.N'$ and $M = \forall\overrightarrow{\alpha^+}.M'$. The algorithm makes a recursive call to $\Gamma, \overrightarrow{\alpha^+}; \Theta \vDash N' \overset{u}{\simeq} M' \dashv \widehat{\sigma}$ returning $\widehat{\sigma}$ as the result.

The induction hypothesis is applicable: $\Gamma, \overrightarrow{\alpha^+}; \Theta \vdash N'$ and $\Gamma, \overrightarrow{\alpha^+} \vdash M'$ hold by inversion, and N' and M' are normalized, since N and M are.

By the induction hypothesis, $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, N'}$, then, since $\mathbf{uv}\, N = \mathbf{uv}\, \forall\overrightarrow{\alpha^+}.N' = \mathbf{uv}\, N'$, we have $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, N}$. Finally, $[\widehat{\sigma}]N = [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N' = \forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N' = \forall\overrightarrow{\alpha^+}.M' = M$.

Case 5. Rule $(\mathrm{UVar}^{-\overset{u}{\simeq}})$, then $N = \widehat{\alpha}^-$, $\widehat{\alpha}^-\{\Delta\} \in \Theta$, and $\Delta \vdash M$. As the result, the algorithm returns $\widehat{\sigma} = (\widehat{\alpha}^- :\approx M)$.

It is clear that $(\widehat{\alpha}^- :\approx M) : (\widehat{\alpha}^-\{\Delta\})$, since $\Delta \vdash M$. Notice that since $\Theta|_{\mathbf{uv}\, N} = \Theta|_{\widehat{\alpha}^-} = (\widehat{\alpha}^-\{\Delta\})$, we can rewrite $(\widehat{\alpha}^- :\approx M) : (\widehat{\alpha}^-\{\Delta\})$ as $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, N}$, as required.

It is also clear that $\widehat{\sigma} = (\widehat{\alpha}^- :\approx M)$ is equivalence-only, and that $[\widehat{\sigma}]N = [\widehat{\alpha}^- :\approx M]\widehat{\alpha}^- = M$, as required.

Case 6. The positive cases are proved symmetrically.

□

Lemma 37 (Completeness of Unification)**.**

+ *For normalized P and Q such that $\Gamma; \Theta \vdash P$ and $\Gamma \vdash Q$,*
assume there exists a normalized unification solution $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, P}$ such that $[\widehat{\sigma}]P = Q$. Then $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}$.
− *For normalized N and M such that $\Gamma; \Theta \vdash N$ and $\Gamma \vdash M$,*
assume there exists a normalized unification solution $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, N}$ such that $[\widehat{\sigma}]N = M$. Then $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}$.

Proof. We prove it by induction on the structure of P and mutually, N.

Case 1. $N = \widehat{\alpha}^-$
Since $\mathbf{uv}\, N = \widehat{\alpha}^-$, $\widehat{\sigma} : \Theta|_{\widehat{\alpha}^-}$ means that $\widehat{\alpha}^-\{\Delta\} \in \Theta$, and since $\widehat{\sigma}$ is equivalence only, $\widehat{\sigma} = (\widehat{\alpha}^- :\approx M')$ for some $\Delta \vdash M'$. $[\widehat{\sigma}]N = M$ means $M' = M$. This way, Rule $(\mathrm{UVar}^{-\overset{u}{\simeq}})$ is applicable to infer $\Gamma; \Theta \vDash \widehat{\alpha}^- \overset{u}{\simeq} M \dashv (\widehat{\alpha}^- :\approx M)$.

Case 2. $N = \alpha^-$
Then $\widehat{\sigma} : \Theta|_{\alpha^-}$ means that $\widehat{\sigma} = \cdot$, and $[\widehat{\sigma}]\alpha^- = M$ means $M = \alpha^-$. This way, Rule $(\mathrm{Var}^{-\overset{u}{\simeq}})$ infers $\Gamma; \Theta \vDash \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot$, which is rewritten as $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}$.

Case 3. $N = \uparrow P$
Notice that $[\widehat{\sigma}]N = M$ means $\uparrow[\widehat{\sigma}]P = M$, i.e. $M = \uparrow Q$ for some Q and $[\widehat{\sigma}]P = Q$.

Let us show that the induction hypothesis is applicable to $[\widehat{\sigma}]P = Q$. Notice that P is normalized, since $N = \uparrow P$ is normalized, $\Gamma; \Theta \vdash P$ follows from inversion of $\Gamma; \Theta \vdash \uparrow P$, and $\Gamma \vdash Q$ follows from inversion of $\Gamma \vdash \uparrow Q$, $\widehat{\sigma}$ is equivalence-only by assumption, and $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, P}$, since $\mathbf{uv}\, P = \mathbf{uv} \uparrow P = \mathbf{uv}\, N$.

Then by the induction hypothesis, $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}$. Hence, Rule $(\uparrow^{\overset{u}{\simeq}})$ is applicable to infer $\Gamma; \Theta \vDash \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}$, that is $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}$.

Case 4. $N = P \to N'$
Notice that $[\widehat{\sigma}]N = M$ means $[\widehat{\sigma}]P \to [\widehat{\sigma}]N' = M$, i.e. $M = Q \to M'$ for some Q and M', such that $[\widehat{\sigma}]P = Q$ and $[\widehat{\sigma}]N' = M'$. Then $[\widehat{\sigma}|_{\mathbf{uv}\, P}]P = Q$ and $[\widehat{\sigma}|_{\mathbf{uv}\, N'}]N' = M'$.

Let us show that the induction hypothesis is applicable to $[\widehat{\sigma}|_{\mathbf{uv}\, P}]P = Q$ and $[\widehat{\sigma}|_{\mathbf{uv}\, N'}]N' = M'$. Notice that P and N' are normalized, since $N = P \to N'$ is normalized, $\Gamma; \Theta \vdash P$ and $\Gamma; \Theta \vdash N'$ follow from inversion of $\Gamma; \Theta \vdash P \to N'$, $\Gamma \vdash Q$ and $\Gamma \vdash M'$ follow from inversion of $\Gamma \vdash Q \to M'$, $\widehat{\sigma}|_{\mathbf{uv}\, P}$ and $\widehat{\sigma}|_{\mathbf{uv}\, N'}$ are equivalence-only since $\widehat{\sigma}$ is equivalence-only, $\widehat{\sigma}|_{\mathbf{uv}\, P} : \Theta|_{\mathbf{uv}\, P}$ and $\widehat{\sigma}|_{\mathbf{uv}\, N'} : \Theta|_{\mathbf{uv}\, N'}$, since $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, P \cup \mathbf{uv}\, N'}$ (**??**).

Then by the induction hypothesis, $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}|_{\mathbf{uv}\, P}$ and $\Gamma; \Theta \vDash N' \overset{u}{\simeq} M' \dashv \widehat{\sigma}|_{\mathbf{uv}\, N'}$. To apply Rule $(\to\overset{u}{\simeq})$ and infer the required $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}$, we need to show that $\widehat{\sigma}|_{\mathbf{uv}\, P}$ & $\widehat{\sigma}|_{\mathbf{uv}\, N'}$ is defined and equal to $\widehat{\sigma}$. It holds by the completeness of the unification solution merge (lemma 28):

- $\widehat{\sigma}|_{\mathbf{uv}\, P} : \Theta|_{\mathbf{uv}\, P}$,
- $\widehat{\sigma}|_{\mathbf{uv}\, N'} : \Theta|_{\mathbf{uv}\, N'}$,
- $\widehat{\sigma}|_{\mathbf{uv}\, P}$ and $\widehat{\sigma}|_{\mathbf{uv}\, N'}$ are normalized unification solution since so is $\widehat{\sigma}$,
- $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, P \cup \mathbf{uv}\, N'}$ since $\mathbf{uv}\, P \cup \mathbf{uv}\, N' = \mathbf{uv}\, N$

Case 5. $N = \forall \overrightarrow{\alpha^+}. N'$

Notice that $[\widehat{\sigma}]N = M$ means $\forall \overrightarrow{\alpha^+}.[\widehat{\sigma}]N' = M$, i.e. $M = \forall \overrightarrow{\alpha^+}.M'$ for some M' such that $[\widehat{\sigma}]N' = M'$.

Let us show that the induction hypothesis is applicable to $[\widehat{\sigma}]N' = M'$. Notice that N' is normalized, since $N = \forall \overrightarrow{\alpha^+}.N'$ is normalized, $\Gamma, \overrightarrow{\alpha^+}; \Theta \vdash N'$ follows from inversion of $\Gamma; \Theta \vdash \forall \overrightarrow{\alpha^+}.N'$, $\Gamma, \overrightarrow{\alpha^+} \vdash M'$ follows from inversion of $\Gamma \vdash \forall \overrightarrow{\alpha^+}.M'$, $\widehat{\sigma}$ is equivalence-only by assumption, and $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, N'}$ since $\mathbf{uv}\, N' = \mathbf{uv}\, \forall \overrightarrow{\alpha^+}.N' = \mathbf{uv}\, N$.

By the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+}; \Theta \vDash N' \overset{u}{\simeq} M' \dashv \widehat{\sigma}$. Hence, Rule $(\forall\overset{u}{\simeq})$ is applicable to infer $\Gamma; \Theta \vDash \forall \overrightarrow{\alpha^+}.N' \overset{u}{\simeq} \forall \overrightarrow{\alpha^+}.M' \dashv \widehat{\sigma}$, that is $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}$.

Case 6. The positive cases are proved symmetrically.

□

4.12 Anti-unification

Lemma 38 (Soundness of Anti-Unification)**.**

+ *Assuming P_1 and P_2 are normalized, if $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ then*
  1. $\Gamma; \Xi \vdash Q$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ *for* $i \in \{1, 2\}$, *and*
  3. $[\widehat{\tau}_i]Q = P_i$ *for* $i \in \{1, 2\}$.
- *Assuming N_1 and N_2 are normalized, if $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ then*
  1. $\Gamma; \Xi \vdash M$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ *for* $i \in \{1, 2\}$, *and*
  3. $[\widehat{\tau}_i]M = N_i$ *for* $i \in \{1, 2\}$.

Proof. We prove it by induction on $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually, $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$. Let us consider the last rule applied to infer this judgement.

Case 1. Rule $(\mathrm{Var}^-\overset{a}{\simeq})$, then $N_1 = \alpha^- = N_2$, $\Xi = \cdot$, $M = \alpha^-$, and $\widehat{\tau}_1 = \widehat{\tau}_2 = \cdot$.

1. $\Gamma; \cdot \vdash \alpha^-$ follows from the assumption $\Gamma \vdash \alpha^-$,
2. $\Gamma; \cdot \vdash \cdot : \cdot$ holds trivially, and
3. $[\cdot]\alpha^- = \alpha^-$ holds trivially.

Case 2. Rule $(\uparrow\overset{a}{\simeq})$, then $N_1 = \uparrow P_1$, $N_2 = \uparrow P_2$, and the algorithm makes the recursive call: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, returning $(\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ as the result.

Since $N_1 = \uparrow P_1$ and $N_2 = \uparrow P_2$ are normalized, so are P_1 and P_2, and thus, the induction hypothesis is applicable to $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$:

1. $\Gamma; \Xi \vdash Q$, and hence, $\Gamma; \Xi \vdash \uparrow Q$,
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ for $i \in \{1, 2\}$, and
3. $[\widehat{\tau}_i]Q = P_i$ for $i \in \{1, 2\}$, and then by the definition of the substitution, $[\widehat{\tau}_i]\uparrow Q = \uparrow P_i$ for $i \in \{1, 2\}$.

Case 3. Rule $(\to\overset{a}{\cong})$, then $N_1 = P_1 \to N_1'$, $N_2 = P_2 \to N_2'$, and the algorithm makes two recursive calls: $\Gamma \vDash P_1 \overset{a}{\cong} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and $\Gamma \vDash N_1' \overset{a}{\cong} N_2' \dashv (\Xi', M, \widehat{\tau}_1', \widehat{\tau}_2')$ and and returns $(\Xi \cup \Xi', Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}_1', \widehat{\tau}_2 \cup \widehat{\tau}_2')$ as the result.

Notice, that the induction hypothesis is applicable to $\Gamma \vDash P_1 \overset{a}{\cong} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$: P_1 and P_2 are normalized, since $N_1 = P_1 \to N_1'$ and $N_2 = P_2 \to N_2'$ are normalized. Similarly, the induction hypothesis is applicable to $\Gamma \vDash N_1' \overset{a}{\cong} N_2' \dashv (\Xi', M, \widehat{\tau}_1', \widehat{\tau}_2')$.

This way, by the induction hypothesis:

1. $\Gamma; \Xi \vdash Q$ and $\Gamma; \Xi' \vdash M$. Then by weakening (**??**), $\Gamma; \Xi \cup \Xi' \vdash Q$ and $\Gamma; \Xi \cup \Xi' \vdash M$, which implies $\Gamma; \Xi \cup \Xi' \vdash Q \to M$;
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$, and hence, $\Gamma; \cdot \vdash \widehat{\tau}_i \cup \widehat{\tau}_i' : \Xi \cup \Xi'$ (**??**);
3. $[\widehat{\tau}_i] Q = P_i$ and $[\widehat{\tau}_i'] M = N_i'$. Since $\widehat{\tau}_i \cup \widehat{\tau}_i'$ restricted to Ξ is $\widehat{\tau}_i$, we have $[\widehat{\tau}_i \cup \widehat{\tau}_i'] Q = P_i$ and $[\widehat{\tau}_i \cup \widehat{\tau}_i'] M = N_i'$, and thus, $[\widehat{\tau}_i \cup \widehat{\tau}_i'] Q \to M = P_1 \to N_1'$

Case 4. Rule $(\forall\overset{a}{\cong})$, then $N_1 = \forall\overrightarrow{\alpha^+}.N_1'$, $N_2 = \forall\overrightarrow{\alpha^+}.N_2'$, and the algorithm makes a recursive call: $\Gamma \vDash N_1' \overset{a}{\cong} N_2' \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ and returns $(\Xi, \forall\overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)$ as the result.

Similarly to case 2, we apply the induction hypothesis to $\Gamma \vDash N_1' \overset{a}{\cong} N_2' \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ to obtain:

1. $\Gamma; \Xi \vdash M$, and hence, $\Gamma; \Xi \vdash \forall\overrightarrow{\alpha^+}.M$;
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ for $i \in \{1, 2\}$, and
3. $[\widehat{\tau}_i] M = N_i'$ for $i \in \{1, 2\}$, and then by the definition of the substitution, $[\widehat{\tau}_i]\forall\overrightarrow{\alpha^+}.M = \forall\overrightarrow{\alpha^+}.N_i'$ for $i \in \{1, 2\}$.

Case 5. Rule (AU$^-$), which applies when other rules do not, and $\Gamma \vdash N_i$, returning as the result $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2) = (\widehat{\alpha}^-_{\{N_1,N_2\}}, \widehat{\alpha}^-_{\{N_1,N_2\}}, N_1), (\widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_2))$.

1. $\Gamma; \Xi \vdash M$ is rewritten as $\Gamma; \widehat{\alpha}^-_{\{N_1,N_2\}} \vdash \widehat{\alpha}^-_{\{N_1,N_2\}}$, which holds trivially;
2. $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ is rewritten as $\Gamma; \cdot \vdash (\widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_i) : \widehat{\alpha}^-_{\{N_1,N_2\}}$, which holds since $\Gamma \vdash N_i$ by the premise of the rule;
3. $[\widehat{\tau}_i] M = N_i$ is rewritten as $[\widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_i]\widehat{\alpha}^-_{\{N_1,N_2\}} = N_i$, which holds trivially by the definition of substitution.

Case 6. Positive cases are proved symmetrically.

□

Observation 5. *Names of the anti-unification variables are uniquely defined by the types they are mapped to by the resulting substitutions.*

\+ *Assuming P_1 and P_2 are normalized, if $\Gamma \vDash P_1 \overset{a}{\cong} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ then for any $\widehat{\beta}^- \in \Xi$, $\widehat{\beta}^- = \widehat{\alpha}^-_{\{[\widehat{\tau}_1]\widehat{\beta}^-, [\widehat{\tau}_2]\widehat{\beta}^-\}}$*

\− *Assuming N_1 and N_2 are normalized, if $\Gamma \vDash N_1 \overset{a}{\cong} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ then for any $\widehat{\beta}^- \in \Xi$, $\widehat{\beta}^- = \widehat{\alpha}^-_{\{[\widehat{\tau}_1]\widehat{\beta}^-, [\widehat{\tau}_2]\widehat{\beta}^-\}}$*

Proof. By simple induction on $\Gamma \vDash P_1 \overset{a}{\cong} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on $\Gamma \vDash N_1 \overset{a}{\cong} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$. Let us consider tha last rule applied to infer this judgment.

Case 1. Rule (Var$^+\overset{a}{\cong}$) or Rule (Var$^-\overset{a}{\cong}$), then $\Xi = \cdot$, and the property holds vacuously.

Case 2. Rule (AU$^-$) Then $\Xi = \widehat{\alpha}^-_{\{N_1,N_2\}}$, $\widehat{\tau}_1 = \widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_1$, and $\widehat{\tau}_2 = \widehat{\alpha}^-_{\{N_1,N_2\}} :\approx N_2$. So the property holds trivially.

Case 3. Rule **??** In this case, $\Xi = \Xi' \cup \Xi''$, $\widehat{\tau}_1 = \widehat{\tau}_1'$ & $\widehat{\tau}_1''$, and $\widehat{\tau}_2 = \widehat{\tau}_2'$ & $\widehat{\tau}_2''$, where the property holds for $(\Xi', \widehat{\tau}_1', \widehat{\tau}_2')$ and $(\Xi'', \widehat{\tau}_1'', \widehat{\tau}_2'')$ by the induction hypothesis. Then since the merge of solutions does not change the types the variables are mapped to, the required property holds for Ξ, $\widehat{\tau}_1$, and $\widehat{\tau}_2$.

Case 4. For the other rules, the resulting Ξ is taken from the recursive call and the required property holds immediately by the induction hypothesis.

□

Observation 6 (Anti-unification algorithm is deterministic)**.**

\+ *If $\Gamma \vDash P_1 \overset{a}{\cong} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and $\Gamma \vDash P_1 \overset{a}{\cong} P_2 \dashv (\Xi', Q', \widehat{\tau}_1', \widehat{\tau}_2')$, then $\Xi = \Xi'$, $Q = Q'$, $\widehat{\tau}_1 = \widehat{\tau}_1'$, and $\widehat{\tau}_2 = \widehat{\tau}_2'$.*

\− *If $\Gamma \vDash N_1 \overset{a}{\cong} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ and $\Gamma \vDash N_1 \overset{a}{\cong} N_2 \dashv (\Xi', M', \widehat{\tau}_1', \widehat{\tau}_2')$, then $\Xi = \Xi'$, $M = M'$, $\widehat{\tau}_1 = \widehat{\tau}_1'$, and $\widehat{\tau}_2 = \widehat{\tau}_2'$.*

Proof. By trivial induction on $\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on $\Gamma \vDash N_1 \stackrel{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$. □

Lemma 39 (Completeness of Anti-Unification)**.**

+ *Assume that P_1 and P_2 are normalized, and there exists $(\Xi', Q', \widehat{\tau}_1', \widehat{\tau}_2')$ such that*

  1. $\Gamma; \Xi' \vdash Q'$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$ *for* $i \in \{1, 2\}$, *and*
  3. $[\widehat{\tau}_i'] Q' = P_i$ *for* $i \in \{1, 2\}$.

  Then the anti-unification algorithm terminates, that is there exists $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$

− *Assume that N_1 and N_2 are normalized, and there exists $(\Xi', M', \widehat{\tau}_1', \widehat{\tau}_2')$ such that*

  1. $\Gamma; \Xi' \vdash M'$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$ *for* $i \in \{1, 2\}$, *and*
  3. $[\widehat{\tau}_i'] M' = N_i$ *for* $i \in \{1, 2\}$.

  Then the anti-unification algorithm succeeds, that is there exists $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \vDash N_1 \stackrel{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$.

Proof. We prove it by the induction on M' and mutually on Q'.

Case 1. $M' = \widehat{\alpha}^-$ Then since $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$, $\Gamma \vdash [\widehat{\tau}_i'] M' = N_i$. This way, Rule ($\mathrm{AU}^-$) is always applicable if other rules are not.

Case 2. $M' = \alpha^-$ Then $\alpha^- = [\widehat{\tau}_i'] \alpha^- = N_i$, which means that Rule ($\mathrm{Var}^{-\stackrel{a}{\simeq}}$) is applicable.

Case 3. $M' = \uparrow Q'$ Then $\uparrow [\widehat{\tau}_i'] Q' = [\widehat{\tau}_i'] \uparrow Q' = N_i$, that is N_1 and N_2 have form $\uparrow P_1$ and $\uparrow P_2$ respectively.
Moreover, $[\widehat{\tau}_i'] Q' = P_i$, which means that $(\Xi', Q', \widehat{\tau}_1', \widehat{\tau}_2')$ is an anti-unifier of P_1 and P_2. Then by the induction hypothesis, there exists $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, and hence, $\Gamma \vDash \uparrow P_1 \stackrel{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ by Rule ($\uparrow^{\stackrel{a}{\simeq}}$).

Case 4. $M' = \forall \overrightarrow{\alpha^+}. M''$ This case is similar to the previous one: we consider $\forall \overrightarrow{\alpha^+}$ as a constructor. Notice that $\forall \overrightarrow{\alpha^+}. [\widehat{\tau}_i'] M'' = [\widehat{\tau}_i'] \forall \overrightarrow{\alpha^+}. M'' = N_i$, that is N_1 and N_2 have form $\forall \overrightarrow{\alpha^+}. N_1''$ and $\forall \overrightarrow{\alpha^+}. N_2''$ respectively.
Moreover, $[\widehat{\tau}_i'] M'' = N_i''$, which means that $(\Xi', M'', \widehat{\tau}_1', \widehat{\tau}_2')$ is an anti-unifier of N_1'' and N_2''. Then by the induction hypothesis, there exists $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \vDash N_1'' \stackrel{a}{\simeq} N_2'' \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$, and hence, $\Gamma \vDash \forall \overrightarrow{\alpha^+}. N_1'' \stackrel{a}{\simeq} \forall \overrightarrow{\alpha^+}. N_2'' \dashv (\Xi, \forall \overrightarrow{\alpha^+}. M, \widehat{\tau}_1, \widehat{\tau}_2)$ by Rule ($\forall^{\stackrel{a}{\simeq}}$).

Case 5. $M' = Q' \to M''$ Then $[\widehat{\tau}_i'] Q' \to [\widehat{\tau}_i'] M'' = [\widehat{\tau}_i'] (Q' \to M'') = N_i$, that is N_1 and N_2 have form $P_1 \to N_1'$ and $P_2 \to N_2'$ respectively.
Moreover, $[\widehat{\tau}_i'] Q' = P_i$ and $[\widehat{\tau}_i'] M'' = N_i''$, which means that $(\Xi', Q', \widehat{\tau}_1', \widehat{\tau}_2')$ is an anti-unifier of P_1 and P_2, and $(\Xi', M'', \widehat{\tau}_1', \widehat{\tau}_2')$ is an anti-unifier of N_1'' and N_2''. Then by the induction hypothesis, $\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and $\Gamma \vDash N_1'' \stackrel{a}{\simeq} N_2'' \dashv (\Xi_2, M, \widehat{\tau}_3, \widehat{\tau}_4)$ succeed. The result of the algorithm is $(\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \,\&\, \widehat{\tau}_3, \widehat{\tau}_2 \,\&\, \widehat{\tau}_4)$, and it is left to show that $\widehat{\tau}_1 \,\&\, \widehat{\tau}_3$ and $\widehat{\tau}_2 \,\&\, \widehat{\tau}_4$ are defined.
From observation 5, we know that if $\widehat{\beta}^- \in \mathbf{dom}(\widehat{\tau}_1) \cap \mathbf{dom}(\widehat{\tau}_3)$ then $[\widehat{\tau}_1] \widehat{\beta}^- = [\widehat{\tau}_3] \widehat{\beta}^-$, and if $\widehat{\beta}^- \in \mathbf{dom}(\widehat{\tau}_2) \cap \mathbf{dom}(\widehat{\tau}_4)$ then $[\widehat{\tau}_2] \widehat{\beta}^- = [\widehat{\tau}_4] \widehat{\beta}^-$. This way, $\widehat{\tau}_1 \,\&\, \widehat{\tau}_3$ and $\widehat{\tau}_2 \,\&\, \widehat{\tau}_4$ are defined.

Case 6. $Q' = \widehat{\alpha}^+$ This case if not possible, since $\Gamma; \Xi' \vdash Q'$ means $\widehat{\alpha}^+ \in \Xi'$, but Ξ' can only contain negative variables.

Case 7. Other positive cases are proved symmetrically to the corresponding negative ones.

□

Lemma 40 (Initiality of Anti-Unification)**.**

+ *Assume that P_1 and P_2 are normalized, and $\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, then $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than any other sound anti-unifier $(\Xi', Q', \widehat{\tau}_1', \widehat{\tau}_2')$, i.e. if*

  1. $\Gamma; \Xi' \vdash Q'$,
  2. $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$ *for* $i \in \{1, 2\}$, *and*
  3. $[\widehat{\tau}_i'] Q' = P_i$ *for* $i \in \{1, 2\}$

  then there exists $\widehat{\rho}$ such that $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\, Q'})$ and $[\widehat{\rho}] Q' = Q$. Moreover, $[\widehat{\rho}] \widehat{\beta}^-$ can be uniquely determined by $[\widehat{\tau}_1'] \widehat{\beta}^-$, $[\widehat{\tau}_2'] \widehat{\beta}^-$, and Γ.

– *Assume that* N_1 *and* N_2 *are normalized, and* $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$*, then* $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ *is more specific than any other sound anti-unifier* $(\Xi', M', \widehat{\tau}'_1, \widehat{\tau}'_2)$*, i.e. if*

1. $\Gamma; \Xi' \vdash M'$*,*
2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ *for* $i \in \{1, 2\}$*, and*
3. $[\widehat{\tau}'_i] M' = N_i$ *for* $i \in \{1, 2\}$

then there exists $\widehat{\rho}$ *such that* $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\, M'})$ *and* $[\widehat{\rho}] M' = M$*. Moreover,* $[\widehat{\rho}]\widehat{\beta}^-$ *can be uniquely determined by* $[\widehat{\tau}'_1]\widehat{\beta}^-$*,* $[\widehat{\tau}'_2]\widehat{\beta}^-$*, and* Γ*.*

Proof. First, let us assume that M' is a metavariable $\widehat{\alpha}^-$. Then we can take $\widehat{\rho} = \widehat{\alpha}^- \mapsto M$, which satisfies the required properties:

- $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\, M'})$ holds since $\Xi'|_{\mathbf{uv}\, M'} = \widehat{\alpha}^-$ and $\Gamma; \Xi \vdash M$ by the soundness of anti-unification (lemma 38);
- $[\widehat{\rho}] M' = M$ holds by construction
- $[\widehat{\rho}]\widehat{\alpha}^- = M$ is the anti-unifier of $N_1 = [\widehat{\tau}'_1]\widehat{\alpha}^-$ and $N_2 = [\widehat{\tau}'_2]\widehat{\alpha}^-$ in context Γ, and hence, it is uniquely determined by them (observation 6).

Now, we can assume that M' is not a metavariable. We prove by induction on the derivation of $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on the derivation of $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$.

Since M' is not a metavariable, the substitution acting on M' preserves its outer constructor. In other words, $[\widehat{\tau}'_i] M' = N_i$ means that M', N_1 and N_2 have the same outer constructor. Let us consider the algorithmic anti-unification rule corresponding to this constructor, and show that it was successfully applied to anti-unify N_1 and N_2 (or P_1 and P_2).

Case 1. Rule $(\mathrm{Var}^{-\overset{a}{\simeq}})$, i.e. $N_1 = \alpha^- = N_2$. This rule is applicable since it has no premises.

Then $\Xi = \cdot$, $M = \alpha^-$, and $\widehat{\tau}_1 = \widehat{\tau}_2 = \cdot$. Since $[\widehat{\tau}'_i] M' = N_i = \alpha^-$ and M' is not a metavariable, $M' = \alpha^-$. Then we can take $\widehat{\rho} = \cdot$, which satisfies the required properties:

- $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\, M'})$ holds vacuously since $\Xi'|_{\mathbf{uv}\, M'} = \varnothing$;
- $[\widehat{\rho}] M' = M$, that is $[\cdot]\alpha^- = \alpha^-$ holds by substitution properties;
- the unique determination of $[\widehat{\rho}]\widehat{\alpha}^-$ for $\widehat{\alpha}^- \in \Xi'|_{\mathbf{uv}\, M'} = \varnothing$ holds vacuously.

Case 2. Rule $(\uparrow^{\overset{a}{\simeq}})$, i.e. $N_1 = \uparrow P_1$ and $N_2 = \uparrow P_2$.

Then since $[\widehat{\tau}'_i] M' = N_i = \uparrow P_i$ and M' is not a metavariable, $M' = \uparrow Q'$, where $[\widehat{\tau}'_i] Q' = P_i$. Let us show that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2.

1. $\Gamma; \Xi' \vdash Q'$ holds by inversion of $\Gamma; \Xi' \vdash \uparrow Q'$;
2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ holds by assumption;
3. $[\widehat{\tau}'_i] Q' = P_i$ holds by assumption.

This way, by the completeness of anti-unification (lemma 39), the anti-unification algorithm succeeds on P_1 and P_2: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, which means that Rule $(\uparrow^{\overset{a}{\simeq}})$ is applicable to infer $\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$.

Moreover, by the induction hypothesis, $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$, which immediately implies that $(\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than $(\Xi', \uparrow Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ (we keep the same $\widehat{\rho}$).

Case 3. Rule $(\forall^{\overset{a}{\simeq}})$, i.e. $N_1 = \forall \overrightarrow{\alpha^+}.N'_1$ and $N_2 = \forall \overrightarrow{\alpha^+}.N'_2$. The proof is symmetric to the previous case. Notice that the context Γ is not changed in Rule $(\forall^{\overset{a}{\simeq}})$, as it represents the context in which the anti-unification variables must be instantiated, rather than the context forming the types that are being anti-unified.

Case 4. Rule $(\to^{\overset{a}{\simeq}})$, i.e. $N_1 = P_1 \to N'_1$ and $N_2 = P_2 \to N'_2$.

Then since $[\widehat{\tau}'_i] M' = N_i = P_i \to N'_i$ and M' is not a metavariable, $M' = Q' \to M''$, where $[\widehat{\tau}'_i] Q' = P_i$ and $[\widehat{\tau}'_i] M'' = N''_i$.

Let us show that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2.

1. $\Gamma; \Xi' \vdash Q'$ holds by inversion of $\Gamma; \Xi' \vdash Q' \to M''$;
2. $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ holds by assumption;
3. $[\widehat{\tau}'_i] Q' = P_i$ holds by assumption.

Similarly, $(\Xi', M'', \widehat{\tau}_1', \widehat{\tau}_2')$ is an anti-unifier of N_1'' and N_2''.

Then by the completeness of anti-unification (lemma 39), the anti-unification algorithm succeeds on P_1 and P_2: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2)$; and on N_1' and N_2': $\Gamma \vDash N_1'' \overset{a}{\simeq} N_2'' \dashv (\Xi_2, M''', \widehat{\tau}_3, \widehat{\tau}_4)$. Notice that $\widehat{\tau}_1$ & $\widehat{\tau}_3$ and $\widehat{\tau}_2$ & $\widehat{\tau}_4$ are defined, in other words, for any $\widehat{\beta}^- \in \Xi_1 \cap \Xi_2$, $[\widehat{\tau}_1]\widehat{\beta}^- = [\widehat{\tau}_2]\widehat{\beta}^-$ and $[\widehat{\tau}_3]\widehat{\beta}^- = [\widehat{\tau}_4]\widehat{\beta}^-$, which follows immediately from observation 5. This way, the algorithm proceeds by applying Rule $(\rightarrow^{\overset{a}{\simeq}})$ and returns $(\Xi_1 \cup \Xi_2, Q \to M''', \widehat{\tau}_1 \,\&\, \widehat{\tau}_3, \widehat{\tau}_2 \,\&\, \widehat{\tau}_4)$.

It is left to construct $\widehat{\rho}$ such that $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\, M'})$ and $[\widehat{\rho}]M' = M$. By the induction hypothesis, there exist $\widehat{\rho}_1$ and $\widehat{\rho}_2$ such that $\Gamma; \Xi_1 \vdash \widehat{\rho}_1 : (\Xi'|_{\mathbf{uv}\, Q'})$, $\Gamma; \Xi_2 \vdash \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\, M''})$, $[\widehat{\rho}_1]Q' = Q$, and $[\widehat{\rho}_2]M'' = M'''$.

Let us show that $\widehat{\rho} = \widehat{\rho}_1$ & $\widehat{\rho}_2$ is defined, in other words, for any $\widehat{\beta}^- \in (\Xi'|_{\mathbf{uv}\, Q'}) \cap (\Xi'|_{\mathbf{uv}\, M''})$, $[\widehat{\rho}_1]\widehat{\beta}^- = [\widehat{\rho}_2]\widehat{\beta}^-$.It holds because by the induction hypothesis, $[\widehat{\rho}_i]\widehat{\beta}^-$ is uniquely determined by $[\widehat{\tau}_1']\widehat{\beta}^-$, $[\widehat{\tau}_2']\widehat{\beta}^-$, and Γ, none of which depends on i.

Let us show that $\widehat{\rho} = \widehat{\rho}_1$ & $\widehat{\rho}_2$ satisfies the required properties:

- $\Gamma; \Xi_1 \cup \Xi_2 \vdash \widehat{\rho}_1$ & $\widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\, M'})$ holds since $\Xi'|_{\mathbf{uv}\, M'} = \Xi'|_{\mathbf{uv}\, Q' \to M''} = (\Xi'|_{\mathbf{uv}\, Q'}) \cup (\Xi'|_{\mathbf{uv}\, M''})$, $\Gamma; \Xi_1 \vdash \widehat{\rho}_1 : (\Xi'|_{\mathbf{uv}\, Q'})$ and $\Gamma; \Xi_2 \vdash \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\, M''})$ Ilya: add a lemma?;
- $[\widehat{\rho}]M' = [\widehat{\rho}](Q' \to M'') = [\widehat{\rho}|_{\mathbf{uv}\, Q'}]Q' \to [\widehat{\rho}|_{\mathbf{uv}\, M''}]M'' = [\widehat{\rho}_1]Q' \to [\widehat{\rho}_2]M'' = Q \to M''' = M$;
- Since $[\widehat{\rho}]\widehat{\beta}^-$ is either equal to $[\widehat{\rho}_1]\widehat{\beta}^-$ or $[\widehat{\rho}_2]\widehat{\beta}^-$, it inherits their property that it is uniquely determined by $[\widehat{\tau}_1']\widehat{\beta}^-$, $[\widehat{\tau}_2']\widehat{\beta}^-$, and Γ.

Case 5. $P_1 = P_2 = \alpha^+$. This case is symmetric to case 1.

Case 6. $P_1 = \downarrow N_1$ and $P_2 = \downarrow N_2$. This case is symmetric to case 2

Case 7. $P_1 = \exists \overrightarrow{\alpha^-}.P_1'$ and $P_2 = \exists \overrightarrow{\alpha^-}.P_2'$. This case is symmetric to case 3

□

4.13 Subtyping Solution Merge

Lemma 41. *Given a fixed context Γ, weakening forms a preorder on the set of entries well-formed in Γ.*

Proof.

- Reflexivity: $\Gamma \vdash e \simeq e$ then $\Gamma \vdash e \Rightarrow e$. Let us consider the shape of e. In all cases, there is a rule inferring $\Gamma \vdash e \Rightarrow e$, since $\Gamma \vdash P \simeq_1^{\leqslant} Q$ implies $\Gamma \vdash P \geqslant_1 Q$ by inversion.
- Transitivity: $\Gamma \vdash e_1 \Rightarrow e_2$ and $\Gamma \vdash e_2 \Rightarrow e_3$ implies $\Gamma \vdash e_1 \Rightarrow e_3$. It follows immediately by considering the rules inferring $\Gamma \vdash e_1 \Rightarrow e_2$ and $\Gamma \vdash e_2 \Rightarrow e_3$, and applying the transitivity of subtyping (corollary 3).

□

Lemma 42 (Solution Weakening forms a preorder)**.** *Let us consider a set of pairs $(\Theta, \widehat{\sigma})$ such that $\widehat{\sigma} : \Theta$. Then the relation defined as $(\Theta_1, \widehat{\sigma}_1) \Rightarrow (\Theta_2, \widehat{\sigma}_2)$ iff $\Theta_1 \supseteq \Theta_2$ and $\Theta_2 \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2$ forms a preorder.*

Proof.

- Reflexivity: $(\Theta, \widehat{\sigma}) \Rightarrow (\Theta, \widehat{\sigma})$.
  It is clear that $\Theta \supseteq \Theta$. $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}$ holds since for any e in $\widehat{\sigma}$ restricting $\widehat{\alpha}^{\pm}$, $\Theta(\widehat{\alpha}^{\pm}) \vdash e \Rightarrow e$ by lemma 41
- Transitivity: If $(\Theta_1, \widehat{\sigma}_1) \Rightarrow (\Theta_2, \widehat{\sigma}_2)$ and $(\Theta_2, \widehat{\sigma}_2) \Rightarrow (\Theta_3, \widehat{\sigma}_3)$ then $(\Theta_1, e_1) \Rightarrow (\Theta_3, e_3)$.
  It is clear that since $\Theta_1 \subseteq \Theta_2$ and $\Theta_2 \subseteq \Theta_3$ then $\Theta_1 \subseteq \Theta_3$. Let us consider $e_3 \in \widehat{\sigma}_3$. Then there exists $e_2 \in \widehat{\sigma}_2$ such that $\Theta_3 \vdash e_2 \Rightarrow e_3$. Also there exists $e_1 \in \widehat{\sigma}_1$ such that $\Theta_2 \vdash e_1 \Rightarrow e_2$, and by weakening, $\Theta_3 \vdash e_1 \Rightarrow e_2$. Then by transitivity of solution entry weakening (lemma 41), $\Theta_3 \vdash e_1 \Rightarrow e_3$.

□

Corollary 18 (Solution Weakening is transitive)**.** *If $\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2$ and $\Theta \vdash \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_3$ then $\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_3$.*

Lemma 43 (Solution Weakening is Monotonous)**.** *if $\widehat{\sigma}_1 : \Theta_1$, $\widehat{\sigma}_2 : \Theta_2$, and $\widehat{\sigma}_1 \subseteq \widehat{\sigma}_2$ then $\Theta_1 \vdash \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_1$.*

Proof. We need to show that for any $e \in \widehat{\sigma}_1$, there exists $e' \in \widehat{\sigma}_2$ such that $\Theta_1 \vdash e' \Rightarrow e$. Let us consider an arbitrary $e \in \widehat{\sigma}_1 \subseteq \widehat{\sigma}_2$ restricting $\widehat{\alpha}^{\pm}$. Notice that $\Theta_1(\widehat{\alpha}^{\pm}) \vdash e$ since $e \in \widehat{\sigma}_1 : \Theta_1$. By lemma 41, $\Theta_1 \vdash e \Rightarrow e$. □

Lemma 44 (Soundness of Solution Entry Merge)**.** *For a fixed context Γ, suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$. If e_1 & e_2 is defined then*

1. $\Gamma \vdash e_1 \mathbin{\&} e_2$
2. $\Gamma \vdash e_1 \mathbin{\&} e_2 \Rightarrow e_1$
3. $\Gamma \vdash e_1 \mathbin{\&} e_2 \Rightarrow e_2$

Proof. Let us consider the rule forming $\Gamma \vdash e_1 \mathbin{\&} e_2 = e$.

Case 1. Rule $(\simeq \&^+ \simeq)$, i.e. $\Gamma \vdash e_1 \mathbin{\&} e_2 = e$ has form $\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \mathbin{\&} (\widehat{\alpha}^+ :\approx P') = (\widehat{\alpha}^+ :\approx P)$ and $\mathbf{nf}(P) = \mathbf{nf}(P')$. Then

1. $\Gamma \vdash e$, i.e. $\Gamma \vdash \widehat{\alpha}^+ :\approx P$ holds by assumption;
2. $\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \Rightarrow (\widehat{\alpha}^+ :\approx P)$ holds by reflexivity (lemma 41);
3. $\Gamma \vdash (\widehat{\alpha}^+ :\approx P) \Rightarrow (\widehat{\alpha}^+ :\approx P')$ holds because $\mathbf{nf}(P) = \mathbf{nf}(P')$ implies $\Gamma \vdash P \simeq_1^{\leqslant} P'$ (by lemma 26), and thus, Rule $(\simeq \Rightarrow^+ \simeq)$ applies.

Case 2. Rule $(\simeq \&^- \simeq)$ the negative case is proved in exactly the same way as the positive one.

Case 3. Rule $(\geqslant \&^+ \geqslant)$ Then e_1 is $\widehat{\alpha}^+ :\geqslant P_1$, e_2 is $\widehat{\alpha}^+ :\geqslant P_2$, and $e_1 \mathbin{\&} e_2$ is $\widehat{\alpha}^+ :\geqslant Q$ where Q is the least upper bound of P_1 and P_2. Then by lemma 34,

- $\Gamma \vdash Q$, and hence $\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q$, that is $\Gamma \vdash e_1 \mathbin{\&} e_2$,
- $\Gamma \vdash Q \geqslant_1 P_1$, and hence $\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q \Rightarrow \widehat{\alpha}^+ :\geqslant P_1$, that is $\Gamma \vdash e_1 \mathbin{\&} e_2 \Rightarrow e_1$,
- $\Gamma \vdash Q \geqslant_1 Q_1$, and hence $\Gamma \vdash e_1 \mathbin{\&} e_2 \Rightarrow e_2$.

Case 4. Rule $(\geqslant \&^+ \simeq)$ Then e_1 is $\widehat{\alpha}^+ :\geqslant P$, e_2 is $\widehat{\alpha}^+ :\approx Q$, where $\Gamma; \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'$, and the resulting $e_1 \mathbin{\&} e_2$ is equal to e_2, that is $\widehat{\alpha}^+ :\approx Q$.

- By assumption, $\Gamma \vdash Q$, and hence $\Gamma \vdash \widehat{\alpha}^+ :\approx Q$, that is $\Gamma \vdash e_1 \mathbin{\&} e_2$.
- Since $\mathbf{uv}(Q) = \varnothing$, $\Gamma; \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'$ implies $\Gamma \vdash Q \geqslant_1 P$ by the soundness of positive subtyping (lemma 48), which means $\Gamma \vdash \widehat{\alpha}^+ :\approx Q \Rightarrow \widehat{\alpha}^+ :\geqslant P$, that is $\Gamma \vdash e_1 \mathbin{\&} e_2 \Rightarrow e_1$.
- $\Gamma \vdash \widehat{\alpha}^+ :\approx Q \Rightarrow \widehat{\alpha}^+ :\approx Q$ by reflexivity (lemma 41), and hence $\Gamma \vdash e_1 \mathbin{\&} e_2 \Rightarrow e_2$.

Case 5. Rule $(\simeq \&^+ \geqslant)$ Thee proof is analogous to the previous case.

□

Lemma 45 (Soundness of Solution Merge)**.** *Suppose that* $\widehat{\sigma}_1 : \Theta_1$ *and* $\widehat{\sigma}_2 : \Theta_2$ *and* $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2$ *is defined. Then*

1. $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2 : \Theta_1 \cup \Theta_2$,
2. $\Theta_1 \vdash \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_1$, *and*
3. $\Theta_2 \vdash \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_2$.

Proof. Let us prove the properties separately:

1. $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2 : \Theta|_{vars_1 \cup vars_2}$. It suffices to prove the following two properties:
   - The set of variables of the entries of $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2$ coincides with $vars_1 \cup vars_2$
     By definition, $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2$ consists of three parts: entries of $\widehat{\sigma}_1$ that do not have matching entries of $\widehat{\sigma}_2$, entries of $\widehat{\sigma}_2$ that do not have matching entries of $\widehat{\sigma}_1$, and the merge of matching entries. It means that $\mathbf{dom}(\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2) = \mathbf{dom}(\widehat{\sigma}_1) \backslash \mathbf{dom}(\widehat{\sigma}_2) \cup \mathbf{dom}(\widehat{\sigma}_2) \backslash \mathbf{dom}(\widehat{\sigma}_1) \cup \mathbf{dom}(\widehat{\sigma}_1) \cap \mathbf{dom}(\widehat{\sigma}_2) = \mathbf{dom}(\widehat{\sigma}_1) \cup \mathbf{dom}(\widehat{\sigma}_2)$, which, since $\widehat{\sigma}_i : \Theta|_{vars_i}$, is equal to $vars_1 \cup vars_2$.
   - Each entry of $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2$ restricting $\widehat{\alpha}^{\pm}$ is well-formed in the corresponding context $\Theta(\widehat{\alpha}^{\pm})$.
     Let us consider an arbitrary entry e of $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2$ restricting $\widehat{\alpha}^{\pm}$. Then there are three cases:
     (a) e the entry is from $\widehat{\sigma}_1$ and does not have a matching entry in $\widehat{\sigma}_2$, i.e. $\widehat{\alpha}^{\pm} \in \mathbf{dom}(\widehat{\sigma}_1) \backslash \mathbf{dom}(\widehat{\sigma}_2)$. Then e is well-formed in $\Theta|_{vars_1}$ by assumption.
     (b) e the entry is from $\widehat{\sigma}_2$ and does not have a matching entry in $\widehat{\sigma}_1$. This case is symmetric.
     (c) e is the merge of two matching entries $e_1 \in \widehat{\sigma}_1$ and $e_2 \in \widehat{\sigma}_2$ restricting $\widehat{\alpha}^{\pm}$. Since $\widehat{\sigma}_1 : \Theta|_{vars_1}$ and $\widehat{\sigma}_2 : \Theta|_{vars_2}$, $\widehat{\alpha}^{\pm} \in \mathbf{dom}(\Theta|_{vars_1}) \cap \mathbf{dom}(\Theta|_{vars_2})$, i.e. there is an entry $\widehat{\alpha}^{\pm}\{\Gamma\} \in \Theta$, and e_1 and e_2 are well-formed in Γ. Then by lemma 44, $\Gamma \vdash e_1 \mathbin{\&} e_2$, where $\Gamma = \Theta(\widehat{\alpha}^{\pm})$.
2. $\Theta|_{vars_1} \vdash \widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2 \Rightarrow \widehat{\sigma}_1$ We need to show that for every entry e from $\widehat{\sigma}_1$, there is an entry e' from $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2$ such that $\Theta|_{vars_1} \vdash e' \Rightarrow e$. Let us consider an arbitrary e from $\widehat{\sigma}_1$ restricting $\widehat{\alpha}^{\pm}$. Then there are two cases:
   - e does not have a matching entry in $\widehat{\sigma}_2$. Then e is also in $\widehat{\sigma}_1 \mathbin{\&} \widehat{\sigma}_2$ and $\Theta|_{vars_1} \vdash e \Rightarrow e$ by reflexivity (lemma 41).

- e has a matching entry e' in $\hat{\sigma}_2$. Then e & e' is in $\hat{\sigma}_1$ & $\hat{\sigma}_2$ and $\Theta|_{vars_1} \vdash e \,\&\, e' \Rightarrow e$ by lemma 44.

3. $\Theta|_{vars_2} \vdash \hat{\sigma}_1 \,\&\, \hat{\sigma}_2 \Rightarrow \hat{\sigma}_2$ is proved analogously.

□

Lemma 46 (Completeness of Solution Entry Merge)**.** *For a fixed context Γ, suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$. Suppose there exists an entry $\Gamma \vdash e$ such that $\Gamma \vdash e \Rightarrow e_1$ and $\Gamma \vdash e \Rightarrow e_2$. Then e_1 & e_2 is defined and $\Gamma \vdash e \Rightarrow e_1$ & e_2.*

Proof. Let us consider the shape of e.

Case 1. e is $\hat{\alpha}^- :\approx M$ Then since e_1, e_2, and e restrict the same variable, they have the same polarity. So e_1 is $\hat{\alpha}^- :\approx N_1$, e_2 is $\hat{\alpha}^- :\approx N_2$, Then by inversion of Rule ($\simeq\Rightarrow^-\simeq$),

- $\Gamma \vdash M \simeq_1^{\leqslant} N_1$, and hence $\mathbf{nf}\,(M) = \mathbf{nf}\,(N_1)$ by lemma 26,
- $\Gamma \vdash M \simeq_1^{\leqslant} N_2$, and hence $\mathbf{nf}\,(M) = \mathbf{nf}\,(N_2)$ by lemma 26,

which implies $\mathbf{nf}\,(N_1) = \mathbf{nf}\,(N_2)$. It means that the merge of e_1 and e_2 is defined by Rule ($\simeq \&^- \simeq$): $\Gamma \vdash (\hat{\alpha}^- :\approx N_1) \,\&\, (\hat{\alpha}^- :\approx N_2) = (\hat{\alpha}^- :\approx N_1)$.

To show that $\Gamma \vdash (\hat{\alpha}^- :\approx M) \Rightarrow (\hat{\alpha}^- :\approx N_1)$, we apply Rule ($\simeq\Rightarrow^-\simeq$): $\Gamma \vdash N_1 \simeq_1^{\leqslant} M$ is obtained from $\Gamma \vdash M \simeq_1^{\leqslant} N_1$ by symmetry of equivalence.

Case 2. e is $\hat{\alpha}^+ :\geqslant P$. Then the only rule inferring $\Gamma \vdash e \Rightarrow e_i$ is Rule ($\geqslant\Rightarrow^+\geqslant$). This way, e_1 is $\hat{\alpha}^+ :\geqslant P_1$ and e_2 is $\hat{\alpha}^+ :\geqslant P_2$, and P is a common supertype of P_1 and P_2, i.e. $\Gamma \vdash P \geqslant_1 P_1$ and $\Gamma \vdash P \geqslant_1 P_2$. Then by lemma 35, $P_1 \vee P_2$ is defined and $\Gamma \vdash P \geqslant_1 P_1 \vee P_2$. Then e_1 & e_2, that is $\hat{\alpha}^+ :\geqslant P_1$ & $\hat{\alpha}^+ :\geqslant P_1$, is defined as $\hat{\alpha}^+ :\geqslant P_1 \vee P_2$ by Rule ($\geqslant \&^+ \geqslant$). Moreover, $\Gamma \vdash e \Rightarrow e_1$ & e_2, that is $\Gamma \vdash \hat{\alpha}^+ :\geqslant P \Rightarrow \hat{\alpha}^+ :\geqslant P_1 \vee P_2$, holds by Rule ($\geqslant\Rightarrow^+\geqslant$).

Case 3. e is $\hat{\alpha}^+ :\approx P$. Then let us consider the rules inferring $\Gamma \vdash e \Rightarrow e_1$ and $\Gamma \vdash e \Rightarrow e_2$ respectively:

a. Rule ($\simeq\Rightarrow^+\simeq$) and Rule ($\simeq\Rightarrow^+\simeq$). Then e_1 is $\hat{\alpha}^+ :\approx P_1$, e_2 is $\hat{\alpha}^+ :\approx P_2$, where $\Gamma \vdash P \simeq_1^{\leqslant} P_1$ and $\Gamma \vdash P \simeq_1^{\leqslant} P_2$. Then e_1 & e_2 is defined as $\hat{\alpha}^+ :\approx P_1$ by Rule ($\simeq \&^+ \simeq$), and $\Gamma \vdash \hat{\alpha}^+ :\approx P \Rightarrow \hat{\alpha}^+ :\approx P_1$ holds by Rule ($\simeq\Rightarrow^+\simeq$), since $\Gamma \vdash P \simeq_1^{\leqslant} P_1$.

b. Rule ($\simeq\Rightarrow^+\simeq$) and Rule ($\simeq\Rightarrow^+\geqslant$). Then e_1 is $\hat{\alpha}^+ :\approx P_1$, e_2 is $\hat{\alpha}^+ :\geqslant P_2$, where $\Gamma \vdash P \simeq_1^{\leqslant} P_1$ and $\Gamma \vdash P \geqslant_1 P_2$. Then by transitivity of subtyping (corollary 3), $\Gamma \vdash P_1 \geqslant_1 P_2$. The completeness of positive subtyping algorithm (lemma 50) implies that $\Gamma; \cdot \vDash P_1 \geqslant P_2 \dashv \hat{\sigma}'$. Then e_1 & e_2 is defined as $\hat{\alpha}^+ :\approx P_1$ by Rule ($\simeq \&^+ \simeq$). Moreover, $\Gamma \vdash \hat{\alpha}^+ :\approx P \Rightarrow \hat{\alpha}^+ :\approx P_1$ holds by Rule ($\simeq\Rightarrow^+\simeq$), as required.

c. Rule ($\simeq\Rightarrow^+\geqslant$) and Rule ($\simeq\Rightarrow^+\simeq$). This case is symmetric to the previous one.

d. Rule ($\simeq\Rightarrow^+\geqslant$) and Rule ($\simeq\Rightarrow^+\geqslant$). Then e_1 is $\hat{\alpha}^+ :\geqslant P_1$, e_2 is $\hat{\alpha}^+ :\geqslant P_2$, where $\Gamma \vdash P \geqslant_1 P_1$ and $\Gamma \vdash P \geqslant_1 P_2$. And similarly to case 2, e_1 & e_2 is defined as $\hat{\alpha}^+ :\geqslant P_1 \vee P_2$. Moreover, $\Gamma \vdash \hat{\alpha}^+ :\approx P \Rightarrow \hat{\alpha}^+ :\geqslant P_1 \vee P_2$ holds by Rule ($\simeq\Rightarrow^+\geqslant$), since $\Gamma \vdash P \geqslant_1 P_1 \vee P_2$ by completeness of the least upper bound algorithm (lemma 35).

□

Lemma 47 (Completeness of Solution Merge)**.** *Suppose that $\hat{\sigma}_1 : \Theta|_{vars_1}$ and $\hat{\sigma}_2 : \Theta|_{vars_2}$ and there exists $\hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}_1$ and $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}_2$. Then $\hat{\sigma}_1$ & $\hat{\sigma}_2$ is defined and $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}_1$ & $\hat{\sigma}_2$.*

Proof. By definition, $\hat{\sigma}_1$ & $\hat{\sigma}_2$ is a union of entries of $\hat{\sigma}_1$, which do not have matching entries in $\hat{\sigma}_2$, entries of $\hat{\sigma}_2$, which do not have matching entries in $\hat{\sigma}_1$, and the merge of matching entries. It is clear that the first two components of this union exist. Let us show that the third component exists, i.e. that for every two entries $e_1 \in \hat{\sigma}_1$ and $e_2 \in \hat{\sigma}_2$ restricting the same variable $\hat{\alpha}^{\pm}$, e_1 & e_2 is defined.

$\Theta|_{vars_1} \vdash \hat{\sigma} \Rightarrow \hat{\sigma}_1$ means that there exists $e \in \hat{\sigma}$ restricting $\hat{\alpha}^{\pm}$ such that $\Theta(\hat{\alpha}^{\pm}) \vdash e \Rightarrow e_1$. Similarly, since there can be only one entry in e restricting the same variable$\hat{\alpha}^{\pm}$, $\Theta_2 \vdash \hat{\sigma} \Rightarrow \hat{\sigma}_2$ means $\Theta(\hat{\alpha}^{\pm}) \vdash e \Rightarrow e_2$. Then by lemma 46, e_1 & e_2 is defined and $\Theta(\hat{\alpha}^{\pm}) \vdash e \Rightarrow e_1$ & e_2.

Let us show: $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}_1$ & $\hat{\sigma}_2$, i.e. for every $e' \in \hat{\sigma}_1$ & $\hat{\sigma}_2$ restricting $\hat{\alpha}^{\pm}$, there exists $e \in \hat{\sigma}$ such that $\Theta(\hat{\alpha}^{\pm}) \vdash e \Rightarrow e'$. Let us consider three cases:

Case 1. e' is from $\hat{\sigma}_1$ and does not have a matching entry in $\hat{\sigma}_2$. Then $\Theta|_{vars_1} \vdash \hat{\sigma} \Rightarrow \hat{\sigma}_1$ means that there exists $e \in \hat{\sigma}$ such that $\Theta(\hat{\alpha}^{\pm}) \vdash e \Rightarrow e'$, as required.

Case 2. e' is from $\hat{\sigma}_2$ and does not have a matching entry in $\hat{\sigma}_1$. This case is symmetric to the previous one.

Case 3. e' is the merge of two matching entries $e_1 \in \hat{\sigma}_1$ and $e_2 \in \hat{\sigma}_2$ restricting $\hat{\alpha}^{\pm}$. Then as noted previously, $\Theta(\hat{\alpha}^{\pm}) \vdash e \Rightarrow e_1$ & $e_2 = e'$.

□

4.14 Subtyping Algorithm

Lemma 48 (Soundness of Positive Subtyping)**.** *If* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash Q$, *and* $\Gamma; \Theta \vdash P$ *then* $\Gamma; \Theta \vDash P \geqslant Q \dashv \hat{\sigma}$ *implies* $\hat{\sigma} : \Theta|_{\mathbf{uv}\, P}$ *and for any* $\hat{\sigma}'$ *such that* $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}$, $\Gamma \vdash [\hat{\sigma}']P \geqslant_1 Q$.

Proof. We prove it by induction on $\Gamma; \Theta \vDash P \geqslant Q \dashv \hat{\sigma}$. Let us consider the last rule to infer this judgment.

Case 1. Rule (UVar$^{\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv \hat{\sigma}$ has shape $\Gamma; \Theta \vDash \hat{\alpha}^+ \geqslant P' \dashv (\hat{\alpha}^+ :\geqslant Q')$ where $\hat{\alpha}^+\{\Delta\} \in \Theta$ and $\mathbf{upgrade}\, \Gamma \vdash P' \,\mathbf{to}\, \Delta = Q'$.

Notice that $\hat{\alpha}^+\{\Delta\} \in \Theta$ and $\Gamma \vdash^{\supseteq} \Theta$ implies $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$ for some $\overrightarrow{\alpha^{\pm}}$, hence, the soundness of upgrade (lemma 29) is applicable:

1. $\Delta \vdash Q'$ and
2. $\Gamma \vdash Q \geqslant_1 P$.

Since $\hat{\alpha}^+\{\Delta\} \in \Theta|_{\mathbf{uv}\,\hat{\alpha}^+}$ and $\Delta \vdash Q'$, it is clear that $(\hat{\alpha}^+ :\geqslant Q') : \Theta|_{\mathbf{uv}\,\hat{\alpha}^+}$.

It is left to show that $\Gamma \vdash [\hat{\sigma}']\hat{\alpha}^+ \geqslant_1 P'$ for any $\hat{\sigma}'$ s.t. $\Theta \vdash \hat{\sigma}' \Rightarrow (\hat{\alpha}^+ :\geqslant Q')$. The latter weakening statement means that either $\hat{\sigma}' \ni \hat{\alpha}^+ :\geqslant Q''$ or $\hat{\sigma}' \ni (\hat{\alpha}^+ :\approx Q'')$ for $\Delta \vdash Q'' \geqslant_1 Q'$. In any case, $\Delta \vdash [\hat{\sigma}']\hat{\alpha}^+ \geqslant_1 Q$. By weakening the context to Γ and combining this judgment transitively (**??**) with $\Gamma \vdash Q \geqslant_1 P$, we have $\Gamma \vdash [\hat{\sigma}']\hat{\alpha}^+ \geqslant_1 P$, as required.

Case 2. Rule (Var$^{+\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv \hat{\sigma}$ has shape $\Gamma; \Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot$. Then $\mathbf{uv}\, \alpha^+ = \varnothing$, and $\hat{\sigma} = \cdot : \cdot$ satisfies $\hat{\sigma} : \Theta|_{\varnothing}$. Since $\mathbf{uv}\, \alpha^+ = \varnothing$, application of any unification solution $\hat{\sigma}'$ does not change α^+, i.e. $[\hat{\sigma}']\alpha^+ = \alpha^+$. Therefore, $\Gamma \vdash [\hat{\sigma}']\alpha^+ \geqslant_1 \alpha^+$ holds by Rule (Var$^{-\leqslant_1}$).

Case 3. Rule ($\downarrow^{\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv \hat{\sigma}$ has shape $\Gamma; \Theta \vDash \downarrow N \geqslant \downarrow M \dashv \hat{\sigma}$.
Then the next step of the algorithm is the unification of $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$. By the soundness of the unification algorithm (lemma 36), it returns an equivalence-only solution $\hat{\sigma}$ such that $\hat{\sigma} : \mathbf{uv}\, N$. By **??**, since $\hat{\sigma}$ is equivalence-only and $\Gamma \vdash^{\supseteq} \Theta$, $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}$ means $\Gamma \vdash \hat{\sigma}' \simeq_1^{\leqslant} \hat{\sigma} : \mathbf{uv}\, N$ as substitutions. $[\hat{\sigma}]\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$ implies $\Gamma \vdash [\hat{\sigma}]\mathbf{nf}\,(N) \simeq_1^{\leqslant} \mathbf{nf}\,(M)$, and then $\Gamma \vdash [\hat{\sigma}']\mathbf{nf}\,(N) \simeq_1^{\leqslant} \mathbf{nf}\,(M)$. Ilya: add lemmas

Let us rewrite the left-hand side and the right-hand side of $\Gamma \vdash [\hat{\sigma}']\mathbf{nf}\,(N) \simeq_1^{\leqslant} \mathbf{nf}\,(M)$ by transitivity of equivalence (corollary 4). By corollaries 5 and 14, $\Gamma \vdash [\hat{\sigma}']\mathbf{nf}\,(N) \simeq_1^{\leqslant} [\hat{\sigma}']N$. By corollary 14, $\Gamma \vdash \mathbf{nf}\,(M) \simeq_1^{\leqslant} M$. This way, we have $\Gamma \vdash [\hat{\sigma}']N \simeq_1^{\leqslant} M$. Then by Rule ($\uparrow^{\leqslant_1}$) and congruence of substitution, $\Gamma \vdash [\hat{\sigma}']\downarrow N \geqslant_1 \downarrow M$.

Case 4. Rule ($\exists^{\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv \hat{\sigma}$ has shape $\Gamma; \Theta \vDash \exists\overrightarrow{\alpha^-}.P' \geqslant \exists\overrightarrow{\beta^-}.Q' \dashv \hat{\sigma}$ s.t. either $\overrightarrow{\alpha^-}$ or $\overrightarrow{\beta^-}$ is not empty.
Then the algorithm creates fresh unification variables $\overrightarrow{\hat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}}$, substitutes the old $\overrightarrow{\alpha^-}$ with them in P', and makes the recursive call: $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\hat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vDash [\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \hat{\sigma}'$, returning as the result $\hat{\sigma} = \hat{\sigma}'\backslash\overrightarrow{\hat{\alpha}^-}$.

Notice that $\Gamma, \overrightarrow{\beta^-} \vdash^{\supseteq} \Theta, \overrightarrow{\hat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}}$, $\Gamma, \overrightarrow{\beta^-} \vdash Q'$, and $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\hat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vdash [\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$, so the induction hypothesis is applicable, that is $\hat{\sigma}' : \Theta, \overrightarrow{\hat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}}|_{\mathbf{uv}\,[\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P'}$ and $\Gamma, \overrightarrow{\beta^-} \vdash [\hat{\sigma}_2'][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P' \geqslant_1 Q'$ for any $\hat{\sigma}_2'$ s.t. $\Theta, \overrightarrow{\hat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vdash \hat{\sigma}_2' \Rightarrow \hat{\sigma}'$.

Since the domain of $\hat{\sigma}'$ is $\mathbf{uv}\,[\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$, the domain of $\hat{\sigma} = \hat{\sigma}'\backslash\overrightarrow{\hat{\alpha}^-}$ is $\mathbf{uv}\,[\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P'\backslash\overrightarrow{\hat{\alpha}^-} = \mathbf{uv}\,\exists\overrightarrow{\alpha^-}.P'$, this way, $\hat{\sigma} : \Theta|_{\mathbf{uv}\,\exists\overrightarrow{\alpha^-}.P'}$, as required.

It is left to show that $\Gamma \vdash [\hat{\sigma}_2]\exists\overrightarrow{\alpha^-}.P' \geqslant_1 \exists\overrightarrow{\beta^-}.Q'$ for any $\hat{\sigma}_2$ s.t. $\Theta \vdash \hat{\sigma}_2 \Rightarrow \hat{\sigma}$. Let us consider an arbitrary such $\hat{\sigma}_2$. Let us construct $\hat{\sigma}_2'$, extending $\hat{\sigma}_2$ to the domain $\mathbf{uv}\,[\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$ with the values of $\hat{\sigma}'$, i.e. $\hat{\sigma}_2' : \Theta|_{\mathbf{uv}\,[\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P'}$, and

$$\hat{\sigma}_2'(\hat{\beta}^{\pm}) = \begin{cases} \hat{\sigma}_2(\hat{\beta}^{\pm}) & \text{if } \hat{\beta}^{\pm} \in \mathbf{uv}\, P' \\ \hat{\sigma}'(\hat{\beta}^{\pm}) & \text{if } \hat{\beta}^{\pm} \in \overrightarrow{\hat{\alpha}^-} \end{cases}$$

, where the application of the unification solution to a variable returns the corresponding *unification entry*. Notice that $\hat{\sigma}_2'|_{\mathbf{uv}\,P'} = \hat{\sigma}_2$. It is easy to see that $\Theta \vdash \hat{\sigma}_2' \Rightarrow \hat{\sigma}'$:

1. if $\hat{\beta}^{\pm} \in \mathbf{uv}\, P'$ then $\hat{\sigma}_2'(\hat{\beta}^{\pm}) = \hat{\sigma}_2(\hat{\beta}^{\pm}) \Rightarrow \hat{\sigma}(\hat{\beta}^{\pm}) = \hat{\sigma}'(\hat{\beta}^{\pm})$;
2. it $\hat{\beta}^{\pm} \in \overrightarrow{\hat{\alpha}^-}$ then $\hat{\sigma}_2'(\hat{\beta}^{\pm}) = \hat{\sigma}'(\hat{\beta}^{\pm}) \Rightarrow \hat{\sigma}'(\hat{\beta}^{\pm})$,

which means that the induction hypothesis can be applied to $\hat{\sigma}_2'$, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash [\hat{\sigma}_2'][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P' \geqslant_1 Q'$.
Notice that $[\hat{\sigma}_2'][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]P' = [\hat{\sigma}_2'|_{\overrightarrow{\hat{\alpha}^-}} \circ \overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}][\hat{\sigma}_2'|_{\mathbf{uv}\,P'}]P'$ by substitution properties Ilya: todo

$$= [\hat{\sigma}_2'|_{\overrightarrow{\hat{\alpha}^-}} \circ \overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}][\hat{\sigma}_2]P' \qquad \text{since } \hat{\sigma}_2'|_{\mathbf{uv}\,P'} = \hat{\sigma}_2.$$

Also notice that the domain of $\hat{\sigma}_2'|_{\overrightarrow{\hat{\alpha}^-}} \circ \overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}$ is $\overrightarrow{\alpha^-}$, and the range is $\Gamma, \overrightarrow{\beta^-}$, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash \hat{\sigma}_2'|_{\overrightarrow{\hat{\alpha}^-}} \circ \overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-} : \overrightarrow{\alpha^-}$, which means that we can apply Rule ($\forall^{\leqslant_1}$) to $\Gamma, \overrightarrow{\beta^-} \vdash [\hat{\sigma}_2'|_{\overrightarrow{\hat{\alpha}^-}} \circ \overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}][\hat{\sigma}_2]N' \leqslant_1 M'$ to obtain $\Gamma \vdash [\hat{\sigma}_2]\exists\overrightarrow{\alpha^-}.P' \geqslant_1 \exists\overrightarrow{\beta^-}.Q'$, as required.

□

Lemma 49 (Soundness of Negative Subtyping)**.** *If* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash M$, *and* $\Gamma; \Theta \vdash N$ *then* $\Gamma; \Theta \vDash N \leqslant M \dashv \hat{\sigma}$ *implies* $\hat{\sigma} : \Theta|_{\mathbf{uv}\, N}$ *and for any* $\hat{\sigma}'$ *such that* $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}$, $\Gamma \vdash [\hat{\sigma}']N \leqslant_1 M$

Proof. We prove it by induction on $\Gamma; \Theta \vDash N \leqslant M \dashv \hat{\sigma}$. Let us consider the last rule to infer this judgment.

Case 1. Rule $(\to^{\leqslant})$, and then $\Gamma; \Theta \vDash N \leqslant M \dashv \hat{\sigma}$ has shape $\Gamma; \Theta \vDash P \to N' \leqslant Q \to M' \dashv \hat{\sigma}$
On the next step, the the algorithm makes two recursive calls: $\Gamma; \Theta \vDash P \geqslant Q \dashv \hat{\sigma}_1$ and $\Gamma; \Theta \vDash N' \leqslant M' \dashv \hat{\sigma}_2$. By the soundness of positive subtyping (lemma 48) and induction hypothesis, respectively

1. $\hat{\sigma}_1 : \Theta|_{\mathbf{uv}\, P}$ and $\Gamma \vdash [\hat{\sigma}_1']P \geqslant_1 Q$ for any $\hat{\sigma}_1'$ s.t. $\Theta \vdash \hat{\sigma}_1' \Rightarrow \hat{\sigma}_1$
2. $\hat{\sigma}_2 : \Theta|_{\mathbf{uv}\, N'}$ and $\Gamma \vdash [\hat{\sigma}_2']N' \leqslant_1 M'$ for any $\hat{\sigma}_2'$ s.t. $\Theta \vdash \hat{\sigma}_2' \Rightarrow \hat{\sigma}_2$

Then the algorithm merges two unification solutions $\hat{\sigma}_1$ and $\hat{\sigma}_2$. By lemma 45, since $\mathbf{uv}\, P \cup \mathbf{uv}\, N' = \mathbf{uv}\,(P \to N')$, we have $\hat{\sigma}_1 \,\&\, \hat{\sigma}_2 : \Theta|_{\mathbf{uv}\,(P \to N')}$, and also $\Theta \vdash \hat{\sigma}_1 \,\&\, \hat{\sigma}_2 \Rightarrow \hat{\sigma}_1$ and $\Theta \vdash \hat{\sigma}_1 \,\&\, \hat{\sigma}_2 \Rightarrow \hat{\sigma}_2$. By the transitivity of solution weakening (corollary 18), $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}_1 \,\&\, \hat{\sigma}_2$ implies $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}_1$ and $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}_2$.

The application of the induction hypothesis gives us $\Gamma \vdash [\hat{\sigma}']P \geqslant_1 Q$ and $\Gamma \vdash [\hat{\sigma}']N' \leqslant_1 M'$. Finally, by Rule $(\to^{\leqslant_1})$, $\Gamma \vdash [\hat{\sigma}'](P \to N') \leqslant_1 Q \to M'$.

Case 2. Rule $(\mathrm{Var}^{-\leqslant})$, and then $\Gamma; \Theta \vDash N \leqslant M \dashv \hat{\sigma}$ has shape $\Gamma; \Theta \vDash \alpha^- \leqslant \alpha^- \dashv \cdot$
This case is symmetric to case 2 of lemma 48.

Case 3. Rule $(\uparrow^{\leqslant})$, and then $\Gamma; \Theta \vDash N \leqslant M \dashv \hat{\sigma}$ has shape $\Gamma; \Theta \vDash \uparrow P \leqslant \uparrow Q \dashv \hat{\sigma}$
This case is symmetric to case 3 of lemma 48.

Case 4. Rule $(\forall^{\leqslant})$, and then $\Gamma; \Theta \vDash N \leqslant M \dashv \hat{\sigma}$ has shape $\Gamma; \Theta \vDash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.M' \dashv \hat{\sigma}$ s.t. either $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is not empty
This case is symmetric to case 4 of lemma 48.

□

Theorem 1 (Soundness of Subtyping Algorithm)**.**

Proof. □

Lemma 50 (Completeness of the Positive Subtyping)**.** *Suppose that* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash Q$ *and* $\Gamma; \Theta \vdash P$ *and there exists* $\hat{\sigma} : \Theta|_{\mathbf{uv}\, P}$ *such that* $\Gamma \vdash [\hat{\sigma}]P \geqslant_1 Q$. *Then there exists* $\hat{\sigma}'$ *such that* $\Gamma; \Theta \vDash P \geqslant Q \dashv \hat{\sigma}'$ *and* $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'$.

Proof. We prove it by induction on $\Gamma \vdash [\hat{\sigma}]P \geqslant_1 Q$. Let us consider the last rule used in the derivation of $\Gamma \vdash [\hat{\sigma}]P \geqslant_1 Q$, but first consider the base case for the substitution $[\hat{\sigma}]P$:

Case 1. $P = \exists\overrightarrow{\beta^-}.\hat{\alpha}^+$ (for potentially empty $\overrightarrow{\beta^+}$)
Then by assumption, there exists $\hat{\sigma} : \Theta|_{\mathbf{uv}\, P}$ such that $\Gamma \vdash \exists\overrightarrow{\beta^-}.[\hat{\sigma}]\hat{\alpha}^+ \geqslant_1 Q$. $\hat{\sigma} : \Theta|_{\mathbf{uv}\, \hat{\alpha}^+}$ means that $\hat{\sigma}$ is either $(\hat{\alpha}^+ :\approx P')$ or $(\hat{\alpha}^+ :\geqslant P')$, where $\Delta \vdash P'$ and $\hat{\alpha}^+\{\Delta\} \in \Theta$. $\Gamma \vdash \exists\overrightarrow{\beta^-}.[\hat{\sigma}]\hat{\alpha}^+ \geqslant_1 Q$ means that $\Gamma \vdash P' \geqslant_1 Q$ because multiple inversions of Rule $(\exists^{\geqslant_1})$ gives us $\Gamma \vdash [\hat{\sigma}]\hat{\alpha}^+ \geqslant_1 Q$ since $\overrightarrow{\beta^-} \cap \mathbf{fv}\,[\hat{\sigma}]\hat{\alpha}^+ = \varnothing$.

In the algorithm, after multiple applications of Rule $(\exists^{\geqslant})$ the type $\exists\overrightarrow{\beta^-}.\hat{\alpha}^+$ is reduced to $\hat{\alpha}^+$. Next, the algorithm tries to apply Rule $(\mathrm{UVar}^{\geqslant})$ and the resulting solution is $\hat{\sigma}' = (\hat{\alpha}^+ :\geqslant Q')$ where $\mathbf{upgrade}\,\Gamma \vdash Q\,\mathbf{to}\,\Delta = Q'$.

Why does the upgrade procedure terminates? Because P' satisfies the pre-conditions of the completeness of the upgrade (lemma 30) :

- $\Delta \vdash P'$ because $P' = [\hat{\sigma}]\hat{\alpha}^+$, $\hat{\sigma} : \Theta|_{\mathbf{uv}\, P}$ and $\hat{\alpha}^+\{\Delta\} \in \Theta|_{\mathbf{uv}\, P}$,
- $\Gamma \vdash P' \geqslant_1 Q$ as noted before

Moreover, completeness of the upgrade also gives us $\Gamma \vdash P' \geqslant_1 Q'$ and further, we strengthen it to $\Delta \vdash P' \geqslant_1 Q'$ (since by the soundness of the upgrade (lemma 29), $\Delta \vdash Q'$).

It means that $\Theta \vdash (\hat{\alpha}^+ :\approx P') \Rightarrow (\hat{\alpha}^+ :\geqslant Q')$ and $\Theta \vdash (\hat{\alpha}^+ :\geqslant P') \Rightarrow (\hat{\alpha}^+ :\geqslant Q')$, which means that in any case, $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'$.

Case 2. $\Gamma \vdash [\hat{\sigma}]P \geqslant_1 Q$ is derived by Rule $(\mathrm{Var}^{+\geqslant_1})$, i.e. $P = [\hat{\sigma}]P = \alpha^+ = Q$. Here the first equality holds because P is not a unification variable: this case has been covered by case 1. The second equality hold because Rule $(\mathrm{Var}^{+\geqslant_1})$ was applied.

Notice that $\hat{\sigma} : \Theta|_{\mathbf{uv}\, \alpha^+} = \hat{\sigma} : \Theta|_{\varnothing} = \cdot$.

The algorithm applies Rule $(\mathrm{Var}^{+\geqslant})$ and infers $\hat{\sigma}' = \cdot$, i.e. $\Gamma; \Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot$.

Since $\Theta \vdash \cdot \Rightarrow \cdot$, we have $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'$.

Case 3. $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$ is derived by Rule $(\downarrow^{\geqslant_1})$,

Then $P = \downarrow N$, since the substitution $[\widehat{\sigma}]P$ must preserve the top-level constructor of $P \neq \widehat{\alpha}^+$ (the case $P = \widehat{\alpha}^+$ has been covered by case 1), and $Q = \downarrow M$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]N \simeq_1^{\leqslant} M$.

Since both types start with $\downarrow$, the algorithm tries to apply Rule $(\downarrow^{\geqslant})$: $\Gamma; \Theta \vDash \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}'$. The premise of this rule is the unification of $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$: $\Gamma; \Theta \vDash \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}'$. Let us show that the unification successfully terminates and returns $\widehat{\sigma}' = \mathbf{nf}\,(\widehat{\sigma})$.

To demonstrate that the unification terminates, we apply the completeness of the unification algorithm (lemma 37). In order to do that, we need to provide a unifier of $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$. Thankfully, $\mathbf{nf}\,(\widehat{\sigma})$ does it.

- $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$ are normalized
- $\Gamma; \Theta \vdash \mathbf{nf}\,(N)$ because $\Gamma; \Theta \vdash N$ (**??**)
- $\Gamma \vdash \mathbf{nf}\,(M)$ because $\Gamma \vdash M$ (corollary 12)
- $\mathbf{nf}\,(\widehat{\sigma}) : \Theta|_{\mathbf{uv}\,\mathbf{nf}\,(N)}$ because $\widehat{\sigma} : \Theta|_{\mathbf{uv}\,N}$ (**??**)
- $\begin{aligned} \Gamma \vdash [\widehat{\sigma}]N \simeq_1^{\leqslant} M &\Rightarrow [\widehat{\sigma}]N \simeq_1^{D} M && \text{by lemma 25} \\ &\Rightarrow \mathbf{nf}\,([\widehat{\sigma}]N) = \mathbf{nf}\,(M) && \text{by lemma 15} \\ &\Rightarrow [\mathbf{nf}\,(\widehat{\sigma})]\mathbf{nf}\,(N) = \mathbf{nf}\,(M) && \text{by lemma 14} \end{aligned}$

Then by the completeness of the unification, $\Gamma; \Theta \vDash \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \mathbf{nf}\,(\widehat{\sigma})$. This way, the subtyping algorithm terminates and the resulting solution is $\mathbf{nf}\,(\widehat{\sigma})$.

It is left to note that $\Theta \vdash \widehat{\sigma} \Rightarrow \mathbf{nf}\,(\widehat{\sigma})$, by **??**, since $\Theta \vdash \widehat{\sigma} \simeq \mathbf{nf}\,(\widehat{\sigma})$

Case 4. $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$ is derived by Rule $(\exists^{\geqslant_1})$.

We should only consider the case when the substitution $[\widehat{\sigma}]P$ results in the existential type $\exists\overrightarrow{\alpha^-}.P''$ (for $P'' \neq \exists\ldots$) by congruence, i.e. $P = \exists\overrightarrow{\alpha^-}.P'$ (for $P' \neq \exists\ldots$) and $[\widehat{\sigma}]P' = P''$. This is because the case when $P = \exists\overrightarrow{\beta^-}.\widehat{\alpha}^+$ has been covered (case 1), and thus, the substitution $\widehat{\sigma}$ must preserve all the outer quantifiers of P and does not generate any new ones.

This way, $P = \exists\overrightarrow{\alpha^-}.P'$, $[\widehat{\sigma}]P = \exists\overrightarrow{\alpha^-}.[\widehat{\sigma}]P'$ (assuming $\overrightarrow{\alpha^-}$ does not intersect with the range of $\widehat{\sigma}$) and $Q = \exists\overrightarrow{\beta^-}.Q'$, where either $\overrightarrow{\alpha^-}$ or $\overrightarrow{\beta^-}$ is not empty.

By inversion, $\Gamma \vdash [\sigma][\widehat{\sigma}]P' \geqslant_1 Q'$ for some $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$. Since σ and $\widehat{\sigma}$ have disjoint domains, and the range of one does not intersect with the domain of the other, they commute, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}][\sigma]P' \geqslant_1 Q'$ (notice that the tree inferring this judgement is a proper subtree of the tree inferring $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$).

At the next step, the algorithm creates fresh (disjoint with $\mathbf{uv}\,P'$) unification variables $\overrightarrow{\widehat{\alpha}^-}$, replaces $\overrightarrow{\alpha^-}$ with them in P', and makes the recursive call: $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vDash P_0 \geqslant Q \dashv \widehat{\sigma}_1$, (where $P_0 = [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$), returning $\widehat{\sigma}_1 \backslash \overrightarrow{\widehat{\alpha}^-}$ as the result.

Notice that $[\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P_0 = [\widehat{\sigma}][\sigma]P'$, and thus, $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P_0 \geqslant_1 Q'$. Let us combine $\widehat{\sigma}$ and $\sigma \circ \overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}$ into one $\widehat{\sigma}_0$—a unification solution for P_0:

$$\widehat{\sigma}_0(\widehat{\beta}^{\pm}) = \begin{cases} \widehat{\beta}^{\pm} :\approx [\sigma]\alpha_i^- & \text{if } \widehat{\beta}^{\pm} = \widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-} \\ \widehat{\sigma}(\widehat{\beta}^{\pm}) & \text{if } \widehat{\beta}^{\pm} \in \mathbf{uv}\,P' \end{cases}$$

Let us show that the induction hypothesis is applicable to $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant_1 Q'$, with the meta-context $\Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}}$

- $\widehat{\sigma}_0 : (\Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}})|_{\mathbf{uv}\,P_0}$. Notice that for every $\widehat{\alpha_i}^- \in \overrightarrow{\widehat{\alpha}^-}$, the type corresponding to the entry $\widehat{\sigma}_0(\widehat{\alpha_i}^-)$ is well-formed in $\Gamma, \overrightarrow{\beta^-}$ since $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$; and for every $\widehat{\beta}^{\pm} \in \mathbf{uv}\,P'$, the type corresponding to the entry $\widehat{\sigma}_0(\widehat{\beta}^{\pm})$ is well-formed in context $\Theta(\widehat{\beta}^{\pm})$ since $\widehat{\sigma} : \Theta|_{\mathbf{uv}\,P'}$.
- $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant_1 Q'$. This is because $[\widehat{\sigma}_0]P_0 = [\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P_0 = [\widehat{\sigma}][\sigma]P'$.
- $\Gamma, \overrightarrow{\beta^-} \vdash^{\supseteq} \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}}$ since $\Gamma, \overrightarrow{\beta^-} \vdash^{\supseteq} \Theta$ and $\Gamma, \overrightarrow{\beta^-} \subseteq \Gamma, \overrightarrow{\beta^-}$.
- $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vdash P_0$

This way, we apply the induction hypothesis to $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant_1 Q'$ and infer that there exists $\widehat{\sigma}_0' : (\Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}})|_{\mathbf{uv}\,P_0}$ such that $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vDash P_0 \geqslant Q \dashv \widehat{\sigma}_0'$, and $\Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vdash \widehat{\sigma}_0' \Rightarrow \widehat{\sigma}_0$.

This way, the algorithm terminates with the result $\widehat{\sigma}_0' \backslash \overrightarrow{\widehat{\alpha}^-}$ and it is left to show $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}_0' \backslash \overrightarrow{\widehat{\alpha}^-}$. Notice that $\widehat{\sigma}_0' \backslash \overrightarrow{\widehat{\alpha}^-} : \Theta|_{\mathbf{uv}\,P'}$. To show $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}_0' \backslash \overrightarrow{\widehat{\alpha}^-}$, it suffices to consider a unification variable $\widehat{\beta}^{\pm} \in \mathbf{uv}\,P'$ and show that $\Gamma \vdash \widehat{\sigma}(\widehat{\beta}^{\pm}) \Rightarrow \widehat{\sigma}_0'(\widehat{\beta}^{\pm})$. It holds by the reflexivity of weakening (lemma 41) since $\widehat{\sigma}_0'(\widehat{\beta}^{\pm}) = \widehat{\sigma}(\widehat{\beta}^{\pm})$.

Case 5. The positive case when $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$ is derived by Rule ($\mathrm{Var}^{+\geqslant_1}$) is symmetric to the corresponding negative **??**. Notice that $[\widehat{\sigma}]P = \beta^+$ implies $P = \beta^+$ because the case when $P = \widehat{\alpha}^+$ has been covered (**??**).

□

Lemma 51 (Completeness of the Negative Subtyping)**.** *Suppose that* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash M$, $\Gamma; \Theta \vdash N$, N *does not contain negative unification variables* ($\widehat{\alpha}^- \notin \mathbf{uv}\, N$) *and there exists* $\widehat{\sigma} : \Theta|_{\mathbf{uv}\, N}$ *such that* $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$. *Then there exists* $\widehat{\sigma}'$ *such that* $\Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}'$ *and* $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}'$.

Proof. We prove it by induction on $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$. Let us consider the last rule used in the derivation of $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$.

Case 1. $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$ is derived by Rule ($\uparrow^{\leqslant_1}$)
Then $N = \uparrow P$, since the substitution $[\widehat{\sigma}]N$ must preserve the top-level constructor of $N \neq \widehat{\alpha}^-$ (since by assumption, $\widehat{\alpha}^- \notin \mathbf{uv}\, N$), and $Q = \downarrow M$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]N \simeq_1^{\leqslant} M$. The rest of the proof is symmetric to case 3 of lemma 50: notice that the algorithm does not make a recursive call, and the difference in the induction statement for the positive and the negative case here does not matter.

Case 2. $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$ is derived by Rule ($\to^{\leqslant_1}$), i.e. $[\widehat{\sigma}]N = [\widehat{\sigma}]P \to [\widehat{\sigma}]N'$ and $M = Q \to M'$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]P \geqslant_1 Q$ and $\Gamma \vdash [\widehat{\sigma}]N' \leqslant_1 M'$.

Let us consider restrictions of $\widehat{\sigma}$ to the set of unification variables in P and N': $\widehat{\sigma}|_{\mathbf{uv}\, P} : \Theta|_{\mathbf{uv}\, P}$ and $\widehat{\sigma}|_{\mathbf{uv}\, N'} : \Theta|_{\mathbf{uv}\, N'}$. Notice that $[\widehat{\sigma}]P = [\widehat{\sigma}|_{\mathbf{uv}\, P}]P$ and $[\widehat{\sigma}]N' = [\widehat{\sigma}|_{\mathbf{uv}\, N'}]N'$.

Let us apply the induction hypothesis to $\Gamma \vdash [\widehat{\sigma}|_{\mathbf{uv}\, P}]P \geqslant_1 Q$ and $\Gamma \vdash [\widehat{\sigma}|_{\mathbf{uv}\, N'}]N' \leqslant_1 M'$ (notice that since $\mathbf{uv}\, N' \subseteq \mathbf{uv}\, N$, there are no negative unification variables in N') to obtain $\widehat{\sigma}'_1$ and $\widehat{\sigma}'_2$ such that

1. $\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}'_1$ and $\Theta \vdash \widehat{\sigma}|_{\mathbf{uv}\, P} \Rightarrow \widehat{\sigma}'_1$
2. $\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}'_2$ and $\Theta \vdash \widehat{\sigma}|_{\mathbf{uv}\, N'} \Rightarrow \widehat{\sigma}'_2$

This way, the algorithm applies Rule ($\to^{\leqslant}$) and terminates returning $\widehat{\sigma}'_1$ & $\widehat{\sigma}'_2$ as the result.

It is left to show that $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}'_1$ & $\widehat{\sigma}'_2$. Since $\widehat{\sigma}|_{\mathbf{uv}\, P} \subseteq \widehat{\sigma}$, by lemma 43, $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}|_{\mathbf{uv}\, P}$, and then by transitivity (corollary 18), $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}'_1$. Analogously, $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}'_2$. Then since by lemma 47, $\widehat{\sigma}'_1$ & $\widehat{\sigma}'_2$ is the weakest restriction implying both $\widehat{\sigma}'_1$ and $\widehat{\sigma}'_2$, we have $\Theta \vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}'_1$ & $\widehat{\sigma}'_2$, as required.

Case 3. $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$ is derived by Rule ($\forall^{\leqslant_1}$). Since N does not contain negative unification variables, N must be of the form $\forall\overrightarrow{\alpha^+}.N'$, such that $[\widehat{\sigma}]N = \forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N'$ and $[\widehat{\sigma}]N' \neq \forall\ldots$ (assuming $\overrightarrow{\alpha^+}$ does not intersect with the range of $\widehat{\sigma}$). Also, $M = \forall\overrightarrow{\beta^+}.M'$ and either $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is non-empty.

The rest of the proof is symmetric to **??** of lemma 50. To apply the induction hypothesis, we need to show additionally that there are no negative unification variables in $N_0 = [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N'$. This is because $\mathbf{uv}\, N_0 \subseteq \mathbf{uv}\, N \cup \overrightarrow{\widehat{\alpha}^+}$, and N is free of negative unification variables by assumption.

Case 4. $\Gamma \vdash [\widehat{\sigma}]N \leqslant_1 M$ is derived by Rule ($\mathrm{Var}^{-\leqslant_1}$).
Then $N = [\widehat{\sigma}]N = \alpha^- = M$. Here the first equality holds because N is not a unification variable: by assumption, N is free of negative unification variables. The second and the third equations hold because Rule ($\mathrm{Var}^{-\leqslant_1}$) was applied.

The rest of the proof is symmetric to case 2 of lemma 50.

□