

1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$P, Q ::=$ positive types

- α^+
- $\downarrow N$
- $\exists \alpha^-. P$

$N, M ::=$ negative types

- α^-
- $\uparrow P$
- $\forall \alpha^+. N$
- $P \rightarrow N$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leq} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leq_0 M \quad \Gamma \vdash M \leq_0 N}{\Gamma \vdash N \simeq_0^{\leq} M} \text{ D0NDEF}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leq} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geq_0 Q \quad \Gamma \vdash Q \geq_0 P}{\Gamma \vdash P \simeq_0^{\leq} Q} \text{ D0PDEF}$$

$\boxed{\Gamma \vdash N \leq_0 M}$ Negative subtyping

$$\begin{array}{c} \overline{\Gamma \vdash \alpha^- \leq_0 \alpha^-} \quad \text{D0NVAR} \\ \frac{\Gamma \vdash P \simeq_0^{\leq} Q}{\Gamma \vdash \uparrow P \leq_0 \uparrow Q} \quad \text{D0SHIFTU} \\ \frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leq_0 M \quad M \neq \forall \beta^+. M'}{\Gamma \vdash \forall \alpha^+. N \leq_0 M} \quad \text{D0FORALLL} \\ \frac{\Gamma, \alpha^+ \vdash N \leq_0 M}{\Gamma \vdash N \leq_0 \forall \alpha^+. M} \quad \text{D0FORALLR} \\ \frac{\Gamma \vdash P \geq_0 Q \quad \Gamma \vdash N \leq_0 M}{\Gamma \vdash P \rightarrow N \leq_0 Q \rightarrow M} \quad \text{D0ARROW} \end{array}$$

$\boxed{\Gamma \vdash P \geq_0 Q}$ Positive supertyping

$$\begin{array}{c} \overline{\Gamma \vdash \alpha^+ \geq_0 \alpha^+} \quad \text{D0PVAR} \\ \frac{\Gamma \vdash N \simeq_0^{\leq} M}{\Gamma \vdash \downarrow N \geq_0 \downarrow M} \quad \text{D0SHIFTD} \\ \frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geq_0 Q \quad Q \neq \exists \alpha^-. Q'}{\Gamma \vdash \exists \alpha^-. P \geq_0 Q} \quad \text{D0EXISTSL} \\ \frac{\Gamma, \alpha^- \vdash P \geq_0 Q}{\Gamma \vdash P \geq_0 \exists \alpha^-. Q} \quad \text{D0EXISTSR} \end{array}$$

2 Multi-Quantified System

2.1 Grammar

P, Q	$::=$	multi-quantified positive types
	α^+	
	$\downarrow N$	
	$\exists \overrightarrow{\alpha^-}.P$	$P \neq \exists \dots$
	(P)	S
N, M	$::=$	multi-quantified negative types
	α^-	
	$\uparrow P$	
	$P \rightarrow N$	
	$\forall \overrightarrow{\alpha^+}.N$	$N \neq \forall \dots$
	(N)	S

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leq} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leq_1 M \quad \Gamma \vdash M \leq_1 N}{\Gamma \vdash N \simeq_1^{\leq} M} \text{ D1NDEF}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leq} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash Q \geq_1 P}{\Gamma \vdash P \simeq_1^{\leq} Q} \text{ D1PDEF}$$

$\boxed{\Gamma \vdash N \leq_1 M}$ Negative subtyping

$$\begin{array}{c} \overline{\Gamma \vdash \alpha^- \leq_1 \alpha^-} \quad \text{D1NVAR} \\ \frac{\Gamma \vdash P \simeq_1^{\leq} Q}{\Gamma \vdash \uparrow P \leq_1 \uparrow Q} \quad \text{D1SHIFTU} \\ \frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash N \leq_1 M}{\Gamma \vdash P \rightarrow N \leq_1 Q \rightarrow M} \quad \text{D1ARROW} \\ \frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leq_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leq_1 \forall \overrightarrow{\beta^+}.M} \quad \text{D1FORALL} \end{array}$$

$\boxed{\Gamma \vdash P \geq_1 Q}$ Positive supertyping

$$\begin{array}{c} \overline{\Gamma \vdash \alpha^+ \geq_1 \alpha^+} \quad \text{D1PVAR} \\ \frac{\Gamma \vdash N \simeq_1^{\leq} M}{\Gamma \vdash \downarrow N \geq_1 \downarrow M} \quad \text{D1SHIFTD} \\ \frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geq_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geq_1 \exists \overrightarrow{\beta^-}.Q} \quad \text{D1EXISTSL} \end{array}$$

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\begin{array}{c} \overline{\alpha^- \simeq_1^D \alpha^-} \quad (\text{VAR}^{-\simeq_1^D}) \\ \frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad (\uparrow \simeq_1^D) \\ \frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \rightarrow N \simeq_1^D Q \rightarrow M} \quad (\rightarrow \simeq_1^D) \end{array}$$

$$\frac{\{\vec{\alpha}^+\} \cap \mathbf{fv} M = \emptyset \quad \mu : (\{\vec{\beta}^+\} \cap \mathbf{fv} M) \leftrightarrow (\{\vec{\alpha}^+\} \cap \mathbf{fv} N) \quad N \simeq_1^D [\mu]M}{\forall \vec{\alpha}^+. N \simeq_1^D \forall \vec{\beta}^+. M} \quad (\forall \simeq_1^D)$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{\overline{\alpha^+ \simeq_1^D \alpha^+} \quad (\text{VAR}^+) \quad \frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow \simeq_1^D)}{\frac{\{\vec{\alpha}^-\} \cap \mathbf{fv} Q = \emptyset \quad \mu : (\{\vec{\beta}^-\} \cap \mathbf{fv} Q) \leftrightarrow (\{\vec{\alpha}^-\} \cap \mathbf{fv} P) \quad P \simeq_1^D [\mu]Q}{\exists \vec{\alpha}^-. P \simeq_1^D \exists \vec{\beta}^-. Q}} \quad (\exists \simeq_1^D)$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\text{ord vars in } N = \vec{\alpha}}$

$$\frac{\alpha^- \in \text{vars}}{\text{ord vars in } \alpha^- = \alpha^-} \quad (\text{VAR}_{\in}^-)$$

$$\frac{\alpha^- \notin \text{vars}}{\text{ord vars in } \alpha^- = .} \quad (\text{VAR}_{\notin}^-)$$

$$\frac{\text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \uparrow P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\text{ord vars in } P = \vec{\alpha}_1 \quad \text{ord vars in } N = \vec{\alpha}_2}{\text{ord vars in } P \rightarrow N = \vec{\alpha}_1, (\vec{\alpha}_2 \setminus \{\vec{\alpha}_1\})} \quad (\rightarrow)$$

$$\frac{\text{vars} \cap \{\vec{\alpha}^+\} = \emptyset \quad \text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \forall \vec{\alpha}^+. N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\text{ord vars in } P = \vec{\alpha}}$

$$\frac{\alpha^+ \in \text{vars}}{\text{ord vars in } \alpha^+ = \alpha^+} \quad (\text{VAR}_{\in}^+)$$

$$\frac{\alpha^+ \notin \text{vars}}{\text{ord vars in } \alpha^+ = .} \quad (\text{VAR}_{\notin}^+)$$

$$\frac{\text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \downarrow N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{\text{vars} \cap \{\vec{\alpha}^-\} = \emptyset \quad \text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \exists \vec{\alpha}^-. P = \vec{\alpha}} \quad (\exists)$$

$\boxed{\text{ord vars in } N = \vec{\alpha}}$

$$\frac{}{\text{ord vars in } \hat{\alpha}^- = .} \quad (\text{UVar}^-)$$

$\boxed{\text{ord vars in } P = \vec{\alpha}}$

$$\frac{}{\text{ord vars in } \hat{\alpha}^+ = .} \quad (\text{UVar}^+)$$

3.1.2 Quantifier Normalization

$$\boxed{\mathbf{nf}(N) = M}$$

$$\begin{array}{c} \overline{\mathbf{nf}(\alpha^-) = \alpha^-} \quad (\text{VAR}^-) \\ \frac{\mathbf{nf}(P) = Q}{\mathbf{nf}(\uparrow P) = \uparrow Q} \quad (\uparrow) \\ \frac{\mathbf{nf}(P) = Q \quad \mathbf{nf}(N) = M}{\mathbf{nf}(P \rightarrow N) = Q \rightarrow M} \quad (\rightarrow) \\ \frac{\mathbf{nf}(N) = N' \quad \text{ord}\{\overrightarrow{\alpha^+}\} \text{ in } N' = \overrightarrow{\alpha^{+'}}}{\mathbf{nf}(\forall \overrightarrow{\alpha^+}.N) = \forall \overrightarrow{\alpha^{+'}}.N'} \quad (\forall) \end{array}$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\begin{array}{c} \overline{\mathbf{nf}(\alpha^+) = \alpha^+} \quad (\text{VAR}^+) \\ \frac{\mathbf{nf}(N) = M}{\mathbf{nf}(\downarrow N) = \downarrow M} \quad (\downarrow) \\ \frac{\mathbf{nf}(P) = P' \quad \text{ord}\{\overrightarrow{\alpha^-}\} \text{ in } P' = \overrightarrow{\alpha^{-'}}}{\mathbf{nf}(\exists \overrightarrow{\alpha^-}.P) = \exists \overrightarrow{\alpha^{-'}}.P'} \quad (\exists) \end{array}$$

$$\boxed{\mathbf{nf}(N) = M}$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^-) = \hat{\alpha}^-} \quad (\text{UVAR}^-)$$

$$\overline{\mathbf{nf}(\hat{\alpha}^+) = \hat{\alpha}^+} \quad (\text{UVAR}^+)$$

3.2 Unification

$$\boxed{\Theta \models N \stackrel{u}{\simeq} M \Rightarrow \hat{\sigma}} \quad \text{Negative unification}$$

$$\begin{array}{c} \overline{\Theta \models \alpha^- \stackrel{u}{\simeq} \alpha^- \Rightarrow \cdot} \quad \text{UNVAR} \\ \frac{\Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}}{\Theta \models \uparrow P \stackrel{u}{\simeq} \uparrow Q \Rightarrow \hat{\sigma}} \quad \text{USHIFTU} \\ \frac{\Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}_1 \quad \Theta \models N \stackrel{u}{\simeq} M \Rightarrow \hat{\sigma}_2}{\Theta \models P \rightarrow N \stackrel{u}{\simeq} Q \rightarrow M \Rightarrow \hat{\sigma}_1 \ \& \ \hat{\sigma}_2} \quad \text{UARROW} \\ \frac{\Theta \models N \stackrel{u}{\simeq} M \Rightarrow \hat{\sigma}}{\Theta \models \forall \overrightarrow{\alpha^+}.N \stackrel{u}{\simeq} \forall \overrightarrow{\alpha^+}.M \Rightarrow \hat{\sigma}} \quad \text{UFORALL} \\ \frac{\hat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Theta \models \hat{\alpha}^- \stackrel{u}{\simeq} N \Rightarrow (\Delta \vdash \hat{\alpha}^- : \approx N)} \quad \text{UNUVAR} \end{array}$$

$$\boxed{\Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}} \quad \text{Positive unification}$$

$$\begin{array}{c} \overline{\Theta \models \alpha^+ \stackrel{u}{\simeq} \alpha^+ \Rightarrow \cdot} \quad \text{UPVAR} \\ \frac{\Theta \models N \stackrel{u}{\simeq} M \Rightarrow \hat{\sigma}}{\Theta \models \downarrow N \stackrel{u}{\simeq} \downarrow M \Rightarrow \hat{\sigma}} \quad \text{USHIFTD} \\ \frac{\Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}}{\Theta \models \exists \overrightarrow{\alpha^-}.P \stackrel{u}{\simeq} \exists \overrightarrow{\alpha^-}.Q \Rightarrow \hat{\sigma}} \quad \text{UEXISTS} \\ \frac{\hat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Theta \models \hat{\alpha}^+ \stackrel{u}{\simeq} P \Rightarrow (\Delta \vdash \hat{\alpha}^+ : \approx P)} \quad \text{UPUVAR} \end{array}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma; \Theta \models N \leq M \Rightarrow \hat{\sigma}}$ Negative subtyping

$$\begin{array}{c}
\overline{\Gamma; \Theta \models \alpha^- \leq \alpha^- \Rightarrow \cdot} \quad \text{ANVAR} \\
\\
\frac{\Theta \models \mathbf{nf}(P) \stackrel{u}{\simeq} \mathbf{nf}(Q) \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \uparrow P \leq \uparrow Q \Rightarrow \hat{\sigma}} \quad \text{AShiftU} \\
\\
\frac{\Gamma; \Theta \models P \geq Q \Rightarrow \hat{\sigma}_1 \quad \Gamma; \Theta \models N \leq M \Rightarrow \hat{\sigma}_2}{\Gamma; \Theta \models P \rightarrow N \leq Q \rightarrow M \Rightarrow \hat{\sigma}_1 \ \& \ \hat{\sigma}_2} \quad \text{AArrow} \\
\\
\frac{\Gamma, \vec{\beta}^+; \Theta, \vec{\alpha}^+ \{ \Gamma, \vec{\beta}^+ \} \models [\vec{\alpha}^+ / \alpha^+] N \leq M \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \forall \alpha^+. N \leq \forall \beta^+. M \Rightarrow \hat{\sigma} \setminus \vec{\alpha}^+} \quad \text{AForall}
\end{array}$$

$\boxed{\Gamma; \Theta \models P \geq Q \Rightarrow \hat{\sigma}}$ Positive supertyping

$$\begin{array}{c}
\overline{\Gamma; \Theta \models \alpha^+ \geq \alpha^+ \Rightarrow \cdot} \quad \text{APVAR} \\
\\
\frac{\Theta \models \mathbf{nf}(N) \stackrel{u}{\simeq} \mathbf{nf}(M) \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \downarrow N \geq \downarrow M \Rightarrow \hat{\sigma}} \quad \text{AShiftD} \\
\\
\frac{\Gamma, \vec{\beta}^-; \Theta, \vec{\alpha}^- \{ \Gamma, \vec{\beta}^- \} \models [\vec{\alpha}^- / \alpha^-] P \geq Q \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \exists \alpha^-. P \geq \exists \beta^-. Q \Rightarrow \hat{\sigma}} \quad \text{AExists} \\
\\
\frac{\text{upgrade } \Gamma \vdash \mathbf{nf}(P) \text{ to } \Delta = Q}{\Gamma; \Theta \models \hat{\alpha}^+ \{ \Delta \} \geq P \Rightarrow (\Delta \vdash \hat{\alpha}^+ : \geq Q)} \quad \text{APUVar}
\end{array}$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restrict an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\Delta \vdash \hat{\alpha}^+ : \approx P$ or $\Delta \vdash \hat{\alpha}^- : \approx N$) or that it must be a (positive) supertype of a certain type ($\Delta \vdash \hat{\alpha}^+ : \geq P$).

Definition 1 (Matching Entries). *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.

$\boxed{e_1 \ \& \ e_2 = e_3}$ Unification Solution Entry Merge

$$\begin{array}{c}
\frac{\Gamma \vdash P_1 \vee P_2 = Q}{(\Gamma \vdash \hat{\alpha}^+ : \geq P_1) \ \& \ (\Gamma \vdash \hat{\alpha}^+ : \geq P_2) = (\Gamma \vdash \hat{\alpha}^+ : \geq Q)} \quad (\geq \ \& \ \geq) \\
\\
\frac{\Gamma; \cdot \models P \geq Q \Rightarrow \hat{\sigma}'}{(\Gamma \vdash \hat{\alpha}^+ : \approx P) \ \& \ (\Gamma \vdash \hat{\alpha}^+ : \geq Q) = (\Gamma \vdash \hat{\alpha}^+ : \approx P)} \quad (\simeq \ \& \ \geq) \\
\\
\frac{\Gamma; \cdot \models Q \geq P \Rightarrow \hat{\sigma}'}{(\Gamma \vdash \hat{\alpha}^+ : \geq P) \ \& \ (\Gamma \vdash \hat{\alpha}^+ : \approx Q) = (\Gamma \vdash \hat{\alpha}^+ : \approx Q)} \quad (\geq \ \& \ \simeq) \\
\\
\frac{}{(\Gamma \vdash \hat{\alpha}^+ : \approx P) \ \& \ (\Gamma \vdash \hat{\alpha}^+ : \approx P) = (\Gamma \vdash \hat{\alpha}^+ : \approx P)} \quad (\simeq \ \& \ \simeq^+) \\
\\
\frac{}{(\Gamma \vdash \hat{\alpha}^- : \approx N) \ \& \ (\Gamma \vdash \hat{\alpha}^- : \approx N) = (\Gamma \vdash \hat{\alpha}^- : \approx N)} \quad (\simeq \ \& \ \simeq^-)
\end{array}$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3. *foo*

$$\begin{aligned}
\hat{\sigma}_1 \ \& \ \hat{\sigma}_2 &= \{e_1 \ \& \ e_2 \mid e_1 \in \hat{\sigma}_1, e_2 \in \hat{\sigma}_2, \text{ s.t. } e_1 \text{ matches with } e_2\} \\
&\cup \{e_1 \mid e_1 \in \hat{\sigma}_1, \text{ s.t. } \forall e_2 \in \hat{\sigma}_2, e_1 \text{ does not match with } e_2\} \\
&\cup \{e_2 \mid e_2 \in \hat{\sigma}_2, \text{ s.t. } \forall e_1 \in \hat{\sigma}_1, e_2 \text{ does not match with } e_2\}
\end{aligned}$$

3.5 Least Upper Bound

$\boxed{\Gamma \models P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\begin{array}{c} \overline{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVAR} \\ \frac{\Gamma, \cdot \models \downarrow N \overset{a}{\simeq} \downarrow M \Rightarrow (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}. [\overrightarrow{\alpha^-} / \Xi] P} \quad \text{LUBSHIFT} \\ \frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q}{\Gamma \models \exists \overrightarrow{\alpha^-}. P_1 \vee \exists \overrightarrow{\beta^-}. P_2 = Q} \quad \text{LUBEXISTS} \end{array}$$

$\boxed{\text{upgrade} \Gamma \vdash P \text{ to } \Delta = Q}$

3.6 Antiunification

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}$

$$\begin{array}{c} \overline{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \Rightarrow (\cdot, \alpha^+, \cdot, \cdot)} \quad \text{AUPVAR} \\ \frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \Rightarrow (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \Rightarrow (\Xi, \downarrow M, \hat{\tau}_1, \hat{\tau}_2)} \quad \text{AUPSHIFT} \\ \frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \emptyset \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \exists \overrightarrow{\alpha^-}. P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}. P_2 \Rightarrow (\Xi, \exists \overrightarrow{\alpha^-}. Q, \hat{\tau}_1, \hat{\tau}_2)} \quad \text{AUPEXISTS} \end{array}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \Rightarrow (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}$

$$\begin{array}{c} \overline{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \Rightarrow (\Xi, \alpha^-, \cdot, \cdot)} \quad \text{AUNVAR} \\ \frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \Rightarrow (\Xi, \uparrow Q, \hat{\tau}_1, \hat{\tau}_2)} \quad \text{AUNSHIFT} \\ \frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \Rightarrow (\Xi_1, Q, \hat{\tau}_1, \hat{\tau}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \Rightarrow (\Xi_2, M, \hat{\tau}'_1, \hat{\tau}'_2)}{\Gamma \models P_1 \rightarrow N_1 \overset{a}{\simeq} P_2 \rightarrow N_2 \Rightarrow (\Xi_1 \cup \Xi_2, Q \rightarrow M, \hat{\tau}_1 \cup \hat{\tau}'_1, \hat{\tau}_2 \cup \hat{\tau}'_2)} \quad \text{AUNARROW} \\ \frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \overset{a}{\simeq} M \Rightarrow (\hat{\alpha}^-_{\{N,M\}}, \hat{\alpha}^-_{\{N,M\}}, (\hat{\alpha}^-_{\{N,M\}} : \approx N), (\hat{\alpha}^-_{\{N,M\}} : \approx M))} \quad \text{AUNAU} \end{array}$$

4 Proofs

4.1 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\overline{\{\mathbf{ord\ vars\ in\ } N\} \equiv \mathbf{vars} \cap \mathbf{fv\ } N}$	$\frac{N \simeq_1^D M}{\mathbf{ord\ vars\ in\ } N = \mathbf{ord\ vars\ in\ } M}$	—
Normalization	$\overline{N \simeq_1^D \mathbf{nf}(N)}$	$\frac{N \simeq_1^D M}{\mathbf{nf}(N) = \mathbf{nf}(M)}$	—
Uppgrade	$\frac{\mathbf{upgrade}\ \Gamma \vdash P \mathbf{to}\ \Delta = Q}{Q \text{ is sound } \left\{ \begin{array}{l} \Delta \vdash Q \\ \Gamma \vdash Q \geq_1 P \end{array} \right.}$	$\frac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\ \Gamma \vdash P \mathbf{to}\ \Delta = Q}$	$\frac{Q' \text{ is sound } \quad \mathbf{upgrade}\ \Gamma \vdash P \mathbf{to}\ \Delta = Q}{\Delta \vdash Q' \geq_1 Q}$
LUB	$\frac{\Gamma \models P_1 \vee P_2 = Q}{Q \text{ is sound } \left\{ \begin{array}{l} \Gamma \vdash Q \\ \Gamma \vdash Q \geq_1 P_1 \\ \Gamma \vdash Q \geq_1 P_2 \end{array} \right.}$	$\frac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \models P_1 \vee P_2 = Q}$	$\frac{Q' \text{ is sound } \quad \Gamma \models P_1 \vee P_2 = Q}{\Delta \vdash Q' \geq_1 Q}$
Anti-unification	$\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \left\{ \begin{array}{l} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat{\tau}_i : \Xi \\ [\hat{\tau}_i] Q = P_i \end{array} \right. \text{ is sound}}$	$\frac{\exists \text{ sound } (\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2)}{\exists (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \text{ s.t. } \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}$	$\frac{(\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2) \text{ is sound } \quad \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\exists \Gamma; \Xi \vdash \hat{\tau} : \Xi' \text{ s.t. } [\hat{\tau}] Q' = Q}$
Unification (matching)	$\frac{\Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}}{\hat{\sigma} \text{ is sound } \left\{ \begin{array}{l} \Theta \vdash \hat{\sigma} \\ [\hat{\sigma}] P = Q \end{array} \right.}$	$\frac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}}$	—
Equivalence (matching)	$\frac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq_1^D Q}{\Gamma \vdash P \simeq_1^{\leq} Q}$	$\frac{\Gamma \vdash P \simeq_1^{\leq} Q}{P \simeq_1^D Q}$	—
Subtyping	$\frac{\Gamma; \Theta \models N \leq M \Rightarrow \hat{\sigma}}{\hat{\sigma} \text{ is sound } \left\{ \begin{array}{l} \Theta \vdash \hat{\sigma} \\ \Gamma \vdash [\hat{\sigma}] N \leq_1 M \end{array} \right.}$	$\frac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Gamma; \Theta \models N \leq M \Rightarrow \hat{\sigma}}$	—

4.2 Variable Ordering

Definition 4 (Collision free bijection). *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is **collision free on sets P and Q** if and only if*

1. $\mu(P \cap A) \cap Q = \emptyset$
2. $\mu(Q \cap A) \cap P = \emptyset$

Lemma 1 (Soundness of variable ordering). *Variable ordering extracts precisely used free variables.*

- $\{\mathbf{ord\ vars\ in\ } N\} \equiv \mathbf{vars} \cap \mathbf{fv\ } N$ (as sets)
- + $\{\mathbf{ord\ vars\ in\ } P\} \equiv \mathbf{vars} \cap \mathbf{fv\ } P$ (as sets)

Proof. Straightforward mutual induction on $\mathbf{ord\ vars\ in\ } N = \vec{\alpha}$ and $\mathbf{ord\ vars\ in\ } P = \vec{\alpha}$ □

Corollary 1 (Additivity of ordering). *Variable ordering is additive (in terms of set union) with respect to its first argument.*

- $\{\mathbf{ord\ (vars}_1 \cup \mathbf{vars}_2) \mathbf{\ in\ } N\} \equiv \{\mathbf{ord\ vars}_1 \mathbf{\ in\ } N\} \cup \{\mathbf{ord\ vars}_2 \mathbf{\ in\ } N\}$ (as sets)
- + $\{\mathbf{ord\ (vars}_1 \cup \mathbf{vars}_2) \mathbf{\ in\ } P\} \equiv \{\mathbf{ord\ vars}_1 \mathbf{\ in\ } P\} \cup \{\mathbf{ord\ vars}_2 \mathbf{\ in\ } P\}$ (as sets)

Corollary 2 (Weakening of ordering). *Extending the first argument of the ordering with unused variables does not change the result.*

- $\mathbf{ord}(vars \cap \mathbf{fv} N) \mathbf{in} N = \mathbf{ord} vars \mathbf{in} N$
- + $\mathbf{ord}(vars \cap \mathbf{fv} P) \mathbf{in} P = \mathbf{ord} vars \mathbf{in} P$

Lemma 2 (Distributivity of renaming over variable ordering). *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

- *If μ is collision free on $vars$ and $\mathbf{fv} N$ then $[\mu](\mathbf{ord} vars \mathbf{in} N) = \mathbf{ord}([\mu]vars) \mathbf{in} [\mu]N$*
- + *If μ is collision free on $vars$ and $\mathbf{fv} P$ then $[\mu](\mathbf{ord} vars \mathbf{in} P) = \mathbf{ord}([\mu]vars) \mathbf{in} [\mu]P$*

Proof. Mutual induction on N and P .

Case 1. $N = \alpha^-$

let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$

$$\begin{aligned}
 \text{Then } [\mu](\mathbf{ord} vars \mathbf{in} N) &= [\mu](\mathbf{ord} vars \mathbf{in} \alpha^-) \\
 &= [\mu]\alpha^- && \text{by Rule (Var}_{\epsilon}^+) \\
 &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars) \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} \beta^- && \text{by Rule (Var}_{\epsilon}^+), \text{ because } \beta^- \in [\mu]vars \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^-
 \end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$

Notice that $[\mu](\mathbf{ord} vars \mathbf{in} N) = [\mu](\mathbf{ord} vars \mathbf{in} \alpha^-) = \cdot$ by Rule (Var_ε⁺). On the other hand, $\mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^- = \mathbf{ord} [\mu]vars \mathbf{in} \alpha^- = \cdot$. The latter equality is from Rule (Var_ε⁺), because μ is collision free on $vars$ and $\mathbf{fv} N$, so $\mathbf{fv} N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$

Then $[\mu](\mathbf{ord} vars \mathbf{in} N) = [\mu](\mathbf{ord} vars \mathbf{in} \alpha^-) = \cdot$ by Rule (Var_ε⁺). To prove that $\mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^- = \cdot$, we apply Rule (Var_ε⁺). Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

(i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ , which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.

(ii) Since μ is collision free on $vars$ and $\mathbf{fv} N$, $\mu(A \cap \mathbf{fv} N) \ni \mu(\alpha^-) \notin vars$.

d. $\alpha^- \notin A$ but $\alpha^- \in vars$

$\mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^- = \mathbf{ord} [\mu]vars \mathbf{in} \alpha^- = \alpha^-$. The latter is by Rule (Var_ε⁺), because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord} vars \mathbf{in} N) = [\mu](\mathbf{ord} vars \mathbf{in} \alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$

$$\begin{aligned}
 [\mu](\mathbf{ord} vars \mathbf{in} N) &= [\mu](\mathbf{ord} vars \mathbf{in} \uparrow P) \\
 &= [\mu](\mathbf{ord} vars \mathbf{in} P) && \text{by Rule } (\uparrow) \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]P && \text{by the induction hypothesis} \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} \uparrow [\mu]P && \text{by Rule } (\uparrow) \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]\uparrow P && \text{by the definition of substitution} \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]N
 \end{aligned}$$

Case 3. $N = P \rightarrow M$

$$\begin{aligned}
 [\mu](\mathbf{ord} vars \mathbf{in} N) &= [\mu](\mathbf{ord} vars \mathbf{in} P \rightarrow M) \\
 &= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \setminus \{\vec{\alpha}_1\})) && \text{where } \mathbf{ord} vars \mathbf{in} P = \vec{\alpha}_1 \text{ and } \mathbf{ord} vars \mathbf{in} M = \vec{\alpha}_2 \\
 &= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \setminus \{\vec{\alpha}_1\}) \\
 &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus [\mu]\{\vec{\alpha}_1\}) && \text{by induction on } \vec{\alpha}_2; \text{ the inductive step is similar to case 1. Notice that } \mu \text{ is} \\
 & && \text{collision free on } \{\vec{\alpha}_1\} \text{ and } \{\vec{\alpha}_2\} \text{ since } \{\vec{\alpha}_1\} \subseteq vars \text{ and } \{\vec{\alpha}_2\} \subseteq \mathbf{fv} N \\
 &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus \{[\mu]\vec{\alpha}_1\}) \\
 (\mathbf{ord} [\mu]vars \mathbf{in} [\mu]N) &= (\mathbf{ord} [\mu]vars \mathbf{in} [\mu]P \rightarrow [\mu]M) \\
 &= (\vec{\beta}_1, (\vec{\beta}_2 \setminus \{\vec{\beta}_1\})) && \text{where } \mathbf{ord} [\mu]vars \mathbf{in} [\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord} [\mu]vars \mathbf{in} [\mu]M = \vec{\beta}_2 \\
 & && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1, \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\
 &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus \{[\mu]\vec{\alpha}_1\})
 \end{aligned}$$

Case 4. $N = \forall \alpha^+ . M$

$$\begin{aligned}
[\mu](\mathbf{ord\,vars\,in}\,N) &= [\mu]\mathbf{ord\,vars\,in}\,\forall \alpha^+ . M \\
&= [\mu]\mathbf{ord\,vars\,in}\,M \\
&= \mathbf{ord}\,[\mu]\mathbf{vars\,in}\,[\mu]M \quad \text{by the induction hypothesis} \\
(\mathbf{ord}\,[\mu]\mathbf{vars\,in}\,[\mu]N) &= \mathbf{ord}\,[\mu]\mathbf{vars\,in}\,[\mu]\forall \alpha^+ . M \\
&= \mathbf{ord}\,[\mu]\mathbf{vars\,in}\,\forall \alpha^+ . [\mu]M \\
&= \mathbf{ord}\,[\mu]\mathbf{vars\,in}\,[\mu]M
\end{aligned}$$

□

Lemma 3 (Completeness of variable ordering). *Variable ordering is invariant under equivalence.*

- For $N \simeq_1^D M$ and any vars, $\mathbf{ord\,vars\,in}\,N = \mathbf{ord\,vars\,in}\,M$ (as lists)
- + For $P \simeq_1^D Q$ and any vars, $\mathbf{ord\,vars\,in}\,P = \mathbf{ord\,vars\,in}\,Q$ (as lists)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

□

4.3 Normalizaion

Lemma 4. *Set of free variables is invariant under equivalence.*

- If $N \simeq_1^D M$ then $\mathbf{fv}\,N \equiv \mathbf{fv}\,M$ (as sets)
- + If $P \simeq_1^D Q$ then $\mathbf{fv}\,P \equiv \mathbf{fv}\,Q$ (as sets)

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$

□

Lemma 5. *Free variables are not changed by the normalization*

- $\mathbf{fv}\,N \equiv \mathbf{fv}\,\mathbf{nf}\,(N)$
- + $\mathbf{fv}\,P \equiv \mathbf{fv}\,\mathbf{nf}\,(P)$

Proof. By straightforward induction on $\mathbf{nf}\,(N) = M$.

□

Lemma 6 (Commutativity of normalization and renaming). *Normalization of a term commutes with renaming.*

Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then

- $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$
- + $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Here equality means alpha-equivalence.

Proof. Mutual induction on N and P .

Case 1. $N = \alpha^-$

$\mathbf{nf}\,([\mu]N) = \mathbf{nf}\,([\mu]\alpha^-) = [\mu]\alpha^-$. The latter follows from the fact that $[\mu]\alpha^-$ is a variable, and thus, Rule (Var[−]) is applicable.
 $[\mu]\mathbf{nf}\,(N) = [\mu]\mathbf{nf}\,(\alpha^-) = [\mu]\alpha^-$.

Case 2. If the type is formed by \rightarrow , \uparrow , or \downarrow , the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \rightarrow M$ then

$$\begin{aligned}
\mathbf{nf}\,([\mu]N) &= \mathbf{nf}\,([\mu](P \rightarrow M)) \\
&= \mathbf{nf}\,([\mu]P \rightarrow [\mu]M) && \text{By the congruence of substitution} \\
&= \mathbf{nf}\,([\mu]P) \rightarrow \mathbf{nf}\,([\mu]M) && \text{By the congruence of normalization, i.e. Rule } (\rightarrow) \\
&= [\mu]\mathbf{nf}\,(P) \rightarrow [\mu]\mathbf{nf}\,(M) && \text{By the induction hypothesis} \\
&= [\mu](\mathbf{nf}\,(P) \rightarrow \mathbf{nf}\,(M)) && \text{By the congruence of substitution} \\
&= [\mu]\mathbf{nf}\,(P \rightarrow M) && \text{By the congruence of normalization} \\
&= [\mu]\mathbf{nf}\,(N)
\end{aligned}$$

Case 3. $N = \forall \alpha^+ . M$

$$\begin{aligned} [\mu] \mathbf{nf} (N) &= [\mu] \mathbf{nf} (\forall \alpha^+ . M) \\ &= [\mu] \forall \alpha^{+'} . \mathbf{nf} (M) \quad \text{Where } \mathbf{ord} \{ \alpha^+ \} \text{ in } \mathbf{nf} (M) = \alpha^{+'} \\ \mathbf{nf} ([\mu] N) &= \mathbf{nf} ([\mu] \forall \alpha^+ . M) \\ &= \mathbf{nf} (\forall \alpha^+ . [\mu] M) \quad \text{Assuming } \{ \alpha^+ \} \cap A = \emptyset \text{ and } \{ \alpha^+ \} \cap B = \emptyset \\ &= \forall \beta^+ . \mathbf{nf} ([\mu] M) \quad \text{Where } \mathbf{ord} \{ \alpha^+ \} \text{ in } \mathbf{nf} ([\mu] M) = \beta^+ \\ &= \forall \alpha^{+'} . \mathbf{nf} ([\mu] M) \quad \text{As } \beta^+ = \alpha^{+'} \text{ (see below)} \end{aligned}$$

Notice that μ is free of collisions on $\{ \alpha^+ \}$ and $\mathbf{fv} \mathbf{nf} (M)$ because

(i) $\mu(A \cap \{ \alpha^+ \}) \cap \mathbf{fv} \mathbf{nf} (M) = \emptyset \cap \mathbf{fv} \mathbf{nf} (M) = \emptyset$ and
(ii) $\mu(A \cap \mathbf{fv} \mathbf{nf} (M)) \cap \{ \alpha^+ \} \subseteq B \cap \{ \alpha^+ \} = \emptyset$

$$\begin{aligned} \overrightarrow{\beta^+} &= \mathbf{ord} \{ \alpha^+ \} \text{ in } \mathbf{nf} ([\mu] M) \\ &= \mathbf{ord} \{ \alpha^+ \} \text{ in } [\mu] \mathbf{nf} (M) \quad \text{By the induction hypothesis} \\ &= \mathbf{ord} \{ [\mu] \alpha^+ \} \text{ in } [\mu] \mathbf{nf} (M) \quad \text{Since } \{ \alpha^+ \} \cap A = \emptyset \\ &= [\mu] \mathbf{ord} \{ \alpha^+ \} \text{ in } \mathbf{nf} (M) \quad \text{by lemma 2} \\ &= \mathbf{ord} \{ \alpha^+ \} \text{ in } \mathbf{nf} (M) \quad \text{Since } \{ \mathbf{ord} \{ \alpha^+ \} \text{ in } \mathbf{nf} (M) \} \cap A \subseteq \{ \alpha^+ \} \cap A = \emptyset \\ &= \alpha^{+'} \end{aligned}$$

To show alpha-equivalence of $[\mu] \forall \alpha^{+'} . \mathbf{nf} (M)$ and $\forall \alpha^{+'} . \mathbf{nf} ([\mu] M)$, we can assume that $\{ \alpha^{+'} \} \cap A = \emptyset$, and $\{ \alpha^{+'} \} \cap B = \emptyset$. Then $[\mu] \forall \alpha^{+'} . \mathbf{nf} (M) = \forall \alpha^{+'} . [\mu] \mathbf{nf} (M) = \forall \alpha^{+'} . \mathbf{nf} ([\mu] M)$, the latter follows from the induction hypothesis.

Case 4. $P = \exists \alpha^- . Q$
Same as for case 3.

□

Lemma 7 (Soundness of quantifier normalization).

$$\begin{aligned} - & N \simeq_1^D \mathbf{nf} (N) \\ + & P \simeq_1^D \mathbf{nf} (P) \end{aligned}$$

Proof. Mutual induction on $\mathbf{nf} (N) = M$ and $\mathbf{nf} (P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var^-) and (Var^+)

By the corresponding equivalence rules.

Case 2. (\uparrow), (\downarrow), and (\rightarrow)

By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. (\forall), i.e. $\mathbf{nf} (\forall \alpha^+ . N) = \forall \alpha^{+'} . N'$

From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 4, $\mathbf{fv} N \equiv \mathbf{fv} N'$. Then by lemma 1, $\{ \alpha^{+'} \} \equiv \{ \alpha^+ \} \cap \mathbf{fv} N' \equiv \{ \alpha^+ \} \cap \mathbf{fv} N$, and thus, $\{ \alpha^{+'} \} \cap \mathbf{fv} N' \equiv \{ \alpha^+ \} \cap \mathbf{fv} N$.

To prove $\forall \alpha^+ . N \simeq_1^D \forall \alpha^{+'} . N'$, it suffices to provide a bijection $\mu : \{ \alpha^{+'} \} \cap \mathbf{fv} N' \leftrightarrow \{ \alpha^+ \} \cap \mathbf{fv} N$ such that $N \simeq_1^D [\mu] N'$. Since these sets are equal, we take $\mu = id$.

Case 4. (\exists) Same as for case 3.

□

Lemma 8 (Completeness of quantified normalization). *Normalization returns the same representative for equivalent types.*

$$\begin{aligned} - & \text{ If } N \simeq_1^D M \text{ then } \mathbf{nf} (N) = \mathbf{nf} (M) \\ + & \text{ If } P \simeq_1^D Q \text{ then } \mathbf{nf} (P) = \mathbf{nf} (Q) \end{aligned}$$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. ($\forall \simeq_1^D$)

From the definition of the normalization,

- $\mathbf{nf}(\forall \alpha^+ . N) = \forall \alpha^{+'} . \mathbf{nf}(N)$ where $\alpha^{+'}$ is **ord** $\{\alpha^+\}$ in $\mathbf{nf}(N)$
- $\mathbf{nf}(\forall \beta^+ . M) = \forall \beta^{+'} . \mathbf{nf}(M)$ where $\beta^{+'}$ is **ord** $\{\beta^+\}$ in $\mathbf{nf}(M)$

Let us take $\mu : (\{\beta^+\} \cap \mathbf{fv} M) \leftrightarrow (\{\alpha^+\} \cap \mathbf{fv} N)$ from the inversion of the equivalence judgment. Notice that from lemmas 1 and 5, the domain and the codomain of μ can be written as $\mu : \{\beta^{+'}\} \leftrightarrow \{\alpha^{+'}\}$.

To show the alpha-equivalence of $\forall \alpha^{+'} . \mathbf{nf}(N)$ and $\forall \beta^{+'} . \mathbf{nf}(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}(M) = \mathbf{nf}(N)$ and (ii) $[\mu]\beta^{+'} = \alpha^{+'}$.

(i) $[\mu]\mathbf{nf}(M) = \mathbf{nf}([\mu]M) = \mathbf{nf}(N)$. The first equality holds by lemma 6, the second—by the induction hypothesis.

$$\begin{aligned}
\text{(ii) } [\mu]\beta^{+'} &= [\mu]\mathbf{ord}\{\beta^+\} \text{ in } \mathbf{nf}(M) && \text{by the definition of } \beta^{+'} \\
&= [\mu]\mathbf{ord}(\{\beta^+\} \cap \mathbf{fv} M) \text{ in } \mathbf{nf}(M) && \text{from lemma 5 and corollary 2} \\
&= \mathbf{ord}[\mu](\{\beta^+\} \cap \mathbf{fv} M) \text{ in } [\mu]\mathbf{nf}(M) && \text{by lemma 2, because } \{\alpha^+\} \cap \mathbf{fv} N \cap \mathbf{fv} \mathbf{nf}(M) \subseteq \{\alpha^+\} \cap \mathbf{fv} M = \emptyset \\
&&& \text{and } \{\alpha^+\} \cap \mathbf{fv} N \cap (\{\beta^+\} \cap \mathbf{fv} M) \subseteq \{\alpha^+\} \cap \mathbf{fv} M = \emptyset \\
&= \mathbf{ord}[\mu](\{\beta^+\} \cap \mathbf{fv} M) \text{ in } \mathbf{nf}(N) && \text{since } [\mu]\mathbf{nf}(M) = \mathbf{nf}(N) \text{ is proved} \\
&= \mathbf{ord}(\{\alpha^+\} \cap \mathbf{fv} N) \text{ in } \mathbf{nf}(N) && \text{because } \mu \text{ is a bijection between } \{\alpha^+\} \cap \mathbf{fv} N \text{ and } \{\beta^+\} \cap \mathbf{fv} M \\
&= \mathbf{ord}\{\alpha^+\} \text{ in } \mathbf{nf}(N) && \text{from lemma 5 and corollary 2} \\
&= \alpha^{+'} && \text{by the definition of } \alpha^{+'}
\end{aligned}$$

Case 2. ($\exists \simeq_1^D$) Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis. □

4.4 Upper Bounds

Lemma 9 (Shape of the Supertypes). *Let us define the set of upper bounds of a positive type $\mathbf{UB}(P)$ in the following way:*

$$\begin{array}{c}
\hline \Gamma \vdash P \qquad \qquad \qquad \mathbf{UB}(\Gamma \vdash P) \\
\hline
\Gamma \vdash \beta^+ \qquad \qquad \qquad \{\exists \alpha^- . \beta^+ \mid \text{for some } \alpha^-\} \\
\Gamma \vdash \exists \beta^- . P \qquad \qquad \qquad \mathbf{UB}(\Gamma \vdash P) \\
\Gamma \vdash \downarrow M \qquad \left\{ \begin{array}{l} \exists \alpha^- . \downarrow M' \mid [\vec{N}/\alpha^-] \downarrow M' \simeq_1^D \downarrow M \\ \text{for some } \Gamma \vdash N_i \end{array} \right\} \\
\text{Then } \mathbf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geq_1 P\}.
\end{array}$$

Lemma 10 (Normalized Shape of the Supertypes). *For a normalized positive type $P = \mathbf{nf}(P)$, let us define the set of normalized upper bounds in the following way:*

$$\begin{array}{c}
\hline \Gamma \vdash P \qquad \qquad \qquad \mathbf{NFUB}(\Gamma \vdash P) \\
\hline
\Gamma \vdash \beta^+ \qquad \qquad \qquad \{\beta^+\} \\
\Gamma \vdash \exists \beta^- . P \qquad \qquad \qquad \mathbf{NFUB}(\Gamma \vdash P) \\
\Gamma \vdash \downarrow M \qquad \left\{ \begin{array}{l} \exists \alpha^- . \downarrow M' \mid [\vec{N}/\alpha^-] \downarrow M' = \downarrow M \\ \text{for some } \Gamma \vdash N_i \end{array} \right\} \\
\text{hen } \mathbf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geq_1 P\}.
\end{array}$$

Lemma 11 (Soundness of the Least Upper Bound). *For types $\Gamma \vdash P_1$, and $\Gamma \vdash P_2$, if $\Gamma \models P_1 \vee P_2 = Q$ then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ and $\Gamma \vdash Q \geqslant_1 P_2$

Lemma 12 (Completeness of the Least Upper Bound). *For types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q'$ such that $\Gamma \vdash Q' \geqslant_1 P_1$ and $\Gamma \vdash Q' \geqslant_1 P_2$, there exists Q s.t. $\Gamma \models P_1 \vee P_2 = Q$, and $\Gamma \vdash Q' \geqslant_1 Q$*

Lemma 13 (Soundness of Upgrade). *For $\Delta \subseteq \Gamma$, suppose that $\mathbf{upgrade} \Gamma \vdash P \mathbf{to} \Delta = Q$. Then*

(i) $\Delta \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P$

Lemma 14 (Completeness of Upgrade). *For $\Delta \subseteq \Gamma$, $\Gamma \vdash P$ and $\Delta \vdash Q'$, such that $\Gamma \vdash Q' \geqslant_1 P$, there exists Q s.t. $\mathbf{upgrade} \Gamma \vdash P \mathbf{to} \Delta = Q$, and $\Delta \vdash Q' \geqslant_1 Q$.*