

α, β type variables
 n, m, i, j index variables

α^+, β^+	$::=$ α^+	positive variable
α^-, β^-	$::=$ α^-	negative variable
σ	$::=$ \cdot $P/a+$ $N/a-$ $\overrightarrow{P}/\overrightarrow{\alpha^+}$ $\overrightarrow{N}/\overrightarrow{\alpha^-}$ $\overrightarrow{\alpha^+}/\overrightarrow{\alpha^+}$ $\overrightarrow{\alpha^-}/\overrightarrow{\alpha^-}$ $\overrightarrow{\alpha^+}/\overrightarrow{\alpha^+}$ $\overrightarrow{\alpha^-}/\overrightarrow{\alpha^-}$ $vars_1/vars_2$ $\overrightarrow{\sigma_i}^i$	substitution concatenate
e	$::=$ $\hat{\alpha}^+ : \approx P$ $\hat{\alpha}^- : \approx N$ $\hat{\alpha}^+ : \geq P$	entry of a unification solution
$\hat{\sigma}$	$::=$ \cdot e $\hat{\sigma} \setminus \overrightarrow{\alpha^+}$ $\hat{\sigma} \setminus \overrightarrow{\alpha^-}$ $\hat{\sigma} \setminus \hat{\alpha}^+$ $\hat{\sigma} \setminus \hat{\alpha}^-$ $\hat{\sigma}_1 \cup \hat{\sigma}_2$ $\overrightarrow{\hat{\sigma}_i}^i$ $(\hat{\sigma})$ $\hat{\sigma}_1 \& \hat{\sigma}_2$	unification solution (substitution) concatenate S M
P, Q	$::=$ $a+$ $\downarrow N$ $\exists \alpha^-. P$ $[\sigma]P$	positive types M
N, M	$::=$ $a-$ $\uparrow P$ $\forall \alpha^+. N$ $P \rightarrow N$ $[\sigma]N$	negative types M

$\overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$	$::=$ $ \cdot$ $ \alpha^+$ $ \overrightarrow{\alpha^+}^i$	positive variable list empty list a variable concatenate lists
$\overrightarrow{\alpha^-}, \overrightarrow{\beta^-}$	$::=$ $ \cdot$ $ \alpha^-$ $ \overrightarrow{\alpha^-}^i$	negative variables empty list a variable concatenate lists
P, Q	$::=$ $ \alpha^+$ $ \downarrow N$ $ \exists \alpha^-. P$ $ [\sigma]P$ $ [\mu]P$ $ P_1 \vee P_2$ $ \mathbf{nf}(P')$	multi-quantified positive types $P \neq \exists \dots$ M M M M
N, M	$::=$ $ \alpha^-$ $ \uparrow P$ $ P \rightarrow N$ $ \forall \alpha^+. N$ $ [\sigma]N$ $ [\mu]N$ $ \mathbf{nf}(N')$	multi-quantified negative types $N \neq \forall \dots$ M M M
\vec{P}	$::=$ $ \cdot$ $ P$ $ \overrightarrow{P}^i$	list of positive types empty list a singel type concatenate lists
\vec{N}	$::=$ $ \cdot$ $ N$ $ \overrightarrow{N}^i$	list of negative types empty list a singel type concatenate lists
Γ	$::=$ $ \cdot$ $ vars$ $ \overrightarrow{\alpha^+}$ $ \overrightarrow{\alpha^-}$ $ \overrightarrow{\Gamma}^i$ $ (\Gamma)$	declarative type context empty context list of variables list of variables concatenate contexts S
$\vec{\alpha}, \vec{\beta}$	$::=$ $ \cdot$ $ \alpha^+$	ordered positive or negative variables empty list list of variables

	$\vec{\alpha}^-$		list of variables
	$\vec{\alpha}_1 \setminus vars$		setminus
	$\vec{\alpha}_i^i$		concatenate contexts
	$(\vec{\alpha})$	S	parenthesis
	$[\mu]\vec{\alpha}$	M	apply moving to list
	ord $vars$ in P	M	
	ord $vars$ in N	M	
	ord $vars$ in P	M	
	ord $vars$ in N	M	
$vars$	$::=$		set of variables
	\emptyset		empty set
	fv P		free variables
	fv N		free variables
	fv P		free variables
	fv N		free variables
	$vars_1 \cap vars_2$		set intersection
	$vars_1 \cup vars_2$		set union
	$vars_1 \setminus vars_2$		set complement
	mv P		movable variables
	mv N		movable variables
	uv N		unification variables
	uv P		unification variables
	fv N		free variables
	fv P		free variables
	$(vars)$	S	parenthesis
	Γ		context
	$\{\vec{\alpha}\}$		ordered list of variables
	$[\mu]vars$	M	apply moving to varset
μ	$::=$		
	\cdot		empty moving
	$\tilde{\alpha}_1^+ \mapsto \tilde{\alpha}_2^+$		Positive unit substitution
	$\tilde{\alpha}_1^- \mapsto \tilde{\alpha}_2^-$		Positive unit substitution
	$\mu_1 \cup \mu_2$	M	Set-like union of movings
	$\overline{\mu_i}^i$		concatenate movings
	$\mu vars$	M	restriction on a set
	μ^{-1}	M	inversion
n	$::=$		cohort index
	0		
	$n + 1$		
$\tilde{\alpha}^+$	$::=$		positive movable variable
	$\tilde{\alpha}^{+n}$		
$\tilde{\alpha}^-$	$::=$		negative movable variable
	$\tilde{\alpha}^{-n}$		

$\overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$	$::=$	\cdot $\tilde{\alpha}^+$ $\overrightarrow{\alpha^{+n}}$ $\overrightarrow{\alpha^+}^i$ α^+_i	positive movable variable list empty list a variable from a non-movable variable concatenate lists
$\overrightarrow{\alpha^-}, \overrightarrow{\beta^-}$	$::=$	\cdot $\tilde{\alpha}^-$ $\overrightarrow{\alpha^{-n}}$ $\overrightarrow{\alpha^-}^i$ α^-_i	negative movable variable list empty list a variable from a non-movable variable concatenate lists
P, Q	$::=$	α^+ $\tilde{\alpha}^+$ $\downarrow N$ $\exists \alpha^-. P$ $[\sigma]P$ $[\mu]P$	multi-quantified positive types with movable variables M M
N, M	$::=$	α^- $\tilde{\alpha}^-$ $\uparrow P$ $P \rightarrow N$ $\forall \alpha^+. N$ $[\sigma]N$ $[\mu]N$	multi-quantified negative types with movable variables M M
$\hat{\alpha}^+$	$::=$	$\hat{\alpha}^+$	positive unification variable
$\hat{\alpha}^-$	$::=$	$\hat{\alpha}^-$ $\hat{\alpha}^-_{\{N, M\}}$	negative unification variable
$\overrightarrow{\hat{\alpha}^+}, \overrightarrow{\hat{\beta}^+}$	$::=$	\cdot $\hat{\alpha}^+$ $\overrightarrow{\hat{\alpha}^+ \{vars\}}$ $\hat{\alpha}^+$ $\overrightarrow{\hat{\alpha}^+}^i$ $\hat{\alpha}^+_i$	positive unification variable list empty list a variable from a normal variable from a normal variable, context unspecified concatenate lists
$\overrightarrow{\hat{\alpha}^-}, \overrightarrow{\hat{\beta}^-}$	$::=$	\cdot $\hat{\alpha}^-$ $\overrightarrow{\hat{\alpha}^- \{vars\}}$ $\hat{\alpha}^-$ $\overrightarrow{\hat{\alpha}^-}^i$ $\hat{\alpha}^-_i$	negative unification variable list empty list a variable from a normal variable from a normal variable, context unspecified concatenate lists

P, Q	$::=$	a positive algorithmic type (potentially with metavariables)
	α^+	
	$\tilde{\alpha}^+$	
	$\hat{\alpha}^+\{vars\}$	
	$\downarrow N$	
	$\xrightarrow{\quad} \exists \alpha^-. P$	
	$[\sigma]P$	M
	$[\mu]P$	M
	$\mathbf{nf}(P')$	M

N, M	$::=$	a negative algorithmic type (potentially with metavariables)
	α^-	
	$\tilde{\alpha}^-$	
	$\hat{\alpha}^-\{vars\}$	
	$\hat{\alpha}^-$	
	$\uparrow P$	
	$P \rightarrow N$	
	$\xrightarrow{\quad} \forall \alpha^+. N$	
	$[\sigma]N$	M
	$[\mu]N$	M
	$\mathbf{nf}(N')$	M

<i>terminals</i>	$::=$	
	\exists	
	\forall	
	\uparrow	
	\downarrow	
	\rightarrow	
	\leftrightarrow	
	\in	
	\notin	
	\cdot	
	\top	
	\leq	
	\geq	
	\mathbb{R}	
	\subset	
	\supset	
	\setminus	
	\cap	
	\mapsto	
	\mathbb{R}^s	
	\mathbb{R}^a	
	\emptyset	
	\top	
	\perp	
	\neq	
	\equiv_n	
	$<$	
	\Downarrow	

<i>formula</i>	$::=$ <ul style="list-style-type: none"> <i>judgement</i> $formula_1 \dots formula_n$ $\mu : vars_1 \leftrightarrow vars_2$ μ is bijective $\hat{\sigma}$ is functional $\hat{\sigma}_1 \in \hat{\sigma}_2$ $vars_1 \subseteq vars_2$ $vars_1 = vars_2$ $vars$ is fresh $\alpha^- \notin vars$ $\alpha^+ \notin vars$ $\alpha^- \in vars$ $\alpha^+ \in vars$ if any other rule is not applicable $N \neq M$ $P \neq Q$ 	
<i>E1A</i>	$::=$ <ul style="list-style-type: none"> $n \models N \simeq_1^A M \models \mu$ $n \models P \simeq_1^A Q \models \mu$ 	Negative multi-quantified type equivalence (algorithm 1) Positive multi-quantified type equivalence (algorithm 1)
<i>A</i>	$::=$ <ul style="list-style-type: none"> $\Gamma \models \overline{N} \leqslant M \models \hat{\sigma}$ $\Gamma \models \overline{P} \geqslant Q \models \hat{\sigma}$ 	Negative subtyping Positive supertyping
<i>E1</i>	$::=$ <ul style="list-style-type: none"> $N \simeq_1^D M$ $P \simeq_1^D Q$ 	Negative multi-quantified type equivalence Positive multi-quantified type equivalence
<i>D1</i>	$::=$ <ul style="list-style-type: none"> $\Gamma \vdash N \simeq_1^{\leq} M$ $\Gamma \vdash P \simeq_1^{\leq} Q$ $\Gamma \vdash N \leqslant_1 M$ $\Gamma \vdash P \geqslant_1 Q$ 	Negative equivalence on MQ types Positive equivalence on MQ types Negative subtyping Positive supertyping
<i>D0</i>	$::=$ <ul style="list-style-type: none"> $\Gamma \vdash N \simeq_0^{\leq} M$ $\Gamma \vdash P \simeq_0^{\leq} Q$ $\Gamma \vdash N \leqslant_0 M$ $\Gamma \vdash P \geqslant_0 Q$ 	Negative equivalence Positive equivalence Negative subtyping Positive supertyping
<i>LUBF</i>	$::=$ <ul style="list-style-type: none"> $P_1 \vee P_2 == Q$ ord <i>vars</i> in $\overline{P} == \vec{\alpha}$ ord <i>vars</i> in $\overline{N} == \vec{\alpha}$ ord <i>vars</i> in $P == \vec{\alpha}$ ord <i>vars</i> in $N == \vec{\alpha}$ nf (N') $== N$ 	

	$ \begin{array}{l} \quad \mathbf{nf} (P') === P \\ \quad \mathbf{nf} (N') === N \\ \quad \mathbf{nf} (P') === P \\ \quad \hat{\sigma}_1 \& \hat{\sigma}_2 === \hat{\sigma} \end{array} $	
<i>LUB</i>	$ \begin{array}{l} ::= \\ \quad P_1 \vee P_2 = Q \end{array} $	Least Upper Bound (Least Common Supertype)
<i>AU</i>	$ \begin{array}{l} ::= \\ \quad \Gamma \vdash P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (Q, \hat{\sigma}_1, \hat{\sigma}_2) \\ \quad \Gamma \vdash N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (M, \hat{\sigma}_1, \hat{\sigma}_2) \end{array} $	
<i>Order</i>	$ \begin{array}{l} ::= \\ \quad \mathbf{ord} \, vars \mathbf{in} \, N = \vec{\alpha} \\ \quad \mathbf{ord} \, vars \mathbf{in} \, P = \vec{\alpha} \\ \quad \mathbf{ord} \, vars \mathbf{in} \, N = \vec{\alpha} \\ \quad \mathbf{ord} \, vars \mathbf{in} \, P = \vec{\alpha} \end{array} $	
<i>Nrm</i>	$ \begin{array}{l} ::= \\ \quad \mathbf{nf} (N) = M \\ \quad \mathbf{nf} (P) = Q \\ \quad \mathbf{nf} (N) = M \\ \quad \mathbf{nf} (P) = Q \end{array} $	
<i>SM</i>	$ \begin{array}{l} ::= \\ \quad e_1 \& e_2 = e_3 \\ \quad \hat{\sigma}_1 \& \hat{\sigma}_2 = \hat{\sigma}_3 \end{array} $	Unification Solution Entry Merge Merge unification solutions
<i>U</i>	$ \begin{array}{l} ::= \\ \quad N \stackrel{u}{\simeq} M \Rightarrow \hat{\sigma} \\ \quad P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma} \end{array} $	Negative unification Positive unification
<i>WF</i>	$ \begin{array}{l} ::= \\ \quad \Gamma \vdash N \\ \quad \Gamma \vdash P \\ \quad \Gamma \vdash N \\ \quad \Gamma \vdash P \end{array} $	Negative type well-formedness Positive type well-formedness Negative type well-formedness Positive type well-formedness
<i>judgement</i>	$ \begin{array}{l} ::= \\ \quad E1A \\ \quad A \\ \quad E1 \\ \quad D1 \\ \quad D0 \\ \quad LUB \\ \quad AU \\ \quad Order \\ \quad Nrm \\ \quad SM \end{array} $	

		U
		WF
$user_syntax$	$::=$	
		α
		n
		α^+
		α^-
		σ
		e
		$\hat{\sigma}$
		P
		N
		$\xrightarrow{\alpha^+}$
		$\xrightarrow{\alpha^-}$
		P
		N
		\vec{P}
		\vec{N}
		Γ
		$\vec{\alpha}$
		$vars$
		μ
		n
		$\tilde{\alpha}^+$
		$\tilde{\alpha}^-$
		$\rightrightarrows_{\alpha^+}$
		$\rightrightarrows_{\alpha^-}$
		P
		N
		$\hat{\alpha}^+$
		$\hat{\alpha}^-$
		$\rightrightarrows_{\alpha^+}$
		$\rightrightarrows_{\alpha^-}$
		P
		N
		$terminals$
		$formula$

$n \models N \simeq_1^A M \Rightarrow \mu$ Negative multi-quantified type equivalence (algorithmic)

$$\begin{array}{c}
\frac{}{n \models \alpha^- \simeq_1^A \alpha^- \Rightarrow \cdot} \quad \text{E1ANVAR} \\
\\
\frac{n \models P \simeq_1^A Q \Rightarrow \mu}{n \models \uparrow P \simeq_1^A \uparrow Q \Rightarrow \mu} \quad \text{E1ASHIFTU} \\
\\
\frac{n \models P \simeq_1^A Q \Rightarrow \mu_1 \quad n \models N \simeq_1^A M \Rightarrow \mu_2 \quad \mu_1 \cup \mu_2 \text{ is bijective}}{n \models P \rightarrow N \simeq_1^A Q \rightarrow M \Rightarrow \mu_1 \cup \mu_2} \quad \text{E1AARROW}
\end{array}$$

$$\frac{n+1 \models [\overrightarrow{\alpha^{+n}/\alpha^+}]N \simeq_1^A [\overrightarrow{\beta^{+n}/\beta^+}]M \Rightarrow \mu}{n \models \forall \alpha^+. N \simeq_1^A \forall \beta^+. M \Rightarrow \mu|_{\mathbf{mv} M}} \quad \text{E1AFORALL}$$

$$\frac{}{n \models \tilde{\alpha}^{-n} \simeq_1^A \tilde{\beta}^{-n} \Rightarrow \tilde{\beta}^{-n} \mapsto \tilde{\alpha}^{-n}} \quad \text{E1ANMVAR}$$

$$\boxed{n \models P \simeq_1^A Q \Rightarrow \mu} \quad \text{Positive multi-quantified type equivalence (algorithmic)}$$

$$\frac{}{n \models \alpha^+ \simeq_1^A \alpha^+ \Rightarrow \cdot} \quad \text{E1APVAR}$$

$$\frac{n \models N \simeq_1^A M \Rightarrow \mu}{n \models \downarrow N \simeq_1^A \downarrow M \Rightarrow \mu} \quad \text{E1ASHIFTD}$$

$$\frac{n+1 \models [\overrightarrow{\alpha^{-n}/\alpha^-}]P \simeq_1^A [\overrightarrow{\beta^{-n}/\beta^-}]Q \Rightarrow \mu}{n \models \exists \alpha^-. P \simeq_1^A \exists \beta^-. Q \Rightarrow \mu|_{\mathbf{mv} Q}} \quad \text{E1AEXISTS}$$

$$\frac{}{n \models \tilde{\alpha}^{+n} \simeq_1^A \tilde{\beta}^{+n} \Rightarrow \tilde{\beta}^{+n} \mapsto \tilde{\alpha}^{+n}} \quad \text{E1APMVAR}$$

$$\boxed{\Gamma \models N \leq M \Rightarrow \hat{\sigma}} \quad \text{Negative subtyping}$$

$$\frac{}{\Gamma \models \alpha^- \leq \alpha^- \Rightarrow \cdot} \quad \text{ANVAR}$$

$$\frac{\mathbf{nf}(P) \stackrel{u}{\simeq} \mathbf{nf}(Q) \Rightarrow \hat{\sigma}}{\Gamma \models \uparrow P \leq \uparrow Q \Rightarrow \hat{\sigma}} \quad \text{ASHIFTU}$$

$$\frac{\Gamma \models P \geq Q \Rightarrow \hat{\sigma}_1 \quad \Gamma \models N \leq M \Rightarrow \hat{\sigma}_2}{\Gamma \models P \rightarrow N \leq Q \rightarrow M \Rightarrow \hat{\sigma}_1 \& \hat{\sigma}_2} \quad \text{AARROW}$$

$$\frac{\Gamma, \beta^+ \models [\hat{\alpha}^+ \{ \Gamma, \beta^+ \} / \alpha^+] N \leq M \Rightarrow \hat{\sigma}}{\Gamma \models \forall \alpha^+. N \leq \forall \beta^+. M \Rightarrow \hat{\sigma} \setminus \hat{\alpha}^+} \quad \text{AFORALL}$$

$$\boxed{\Gamma \models P \geq Q \Rightarrow \hat{\sigma}} \quad \text{Positive supertyping}$$

$$\frac{}{\Gamma \models \alpha^+ \geq \alpha^+ \Rightarrow \cdot} \quad \text{APVAR}$$

$$\frac{\mathbf{nf}(N) \stackrel{u}{\simeq} \mathbf{nf}(M) \Rightarrow \hat{\sigma}}{\Gamma \models \downarrow N \geq \downarrow M \Rightarrow \hat{\sigma}} \quad \text{ASHIFTD}$$

$$\frac{\Gamma, \beta^- \models [\hat{\alpha}^- \{ \Gamma, \beta^- \} / \alpha^-] P \geq Q \Rightarrow \hat{\sigma}}{\Gamma \models \exists \alpha^-. P \geq \exists \beta^-. Q \Rightarrow \hat{\sigma}} \quad \text{AEXISTS}$$

$$\frac{\mathbf{nf}(P) = P' \quad \text{vars}_1 = \mathbf{fv} P' \setminus \text{vars} \quad \text{vars}_2 \text{ is fresh}}{\Gamma \models \hat{\alpha}^+ \{ \text{vars} \} \geq P \Rightarrow (\hat{\alpha}^+ : \geq P' \vee [\text{vars}_2 / \text{vars}_1] P')} \quad \text{APUVAR}$$

$$\boxed{N \simeq_1^D M} \quad \text{Negative multi-quantified type equivalence}$$

$$\frac{}{\alpha^- \simeq_1^D \alpha^-} \quad \text{E1NVAR}$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad \text{E1SHIFTU}$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \rightarrow N \simeq_1^D Q \rightarrow M} \quad \text{E1ARROW}$$

$$\frac{\overrightarrow{\alpha}^+ \cap \mathbf{fv} M = \emptyset \quad \mu : (\overrightarrow{\beta}^+ \cap \mathbf{fv} M) \leftrightarrow (\overrightarrow{\alpha}^+ \cap \mathbf{fv} N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha}^+. N \simeq_1^D \forall \overrightarrow{\beta}^+. M} \quad \text{E1FORALL}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\overrightarrow{\alpha}^+ \simeq_1^D \overrightarrow{\alpha}^+} \quad \text{E1PVAR}$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad \text{E1SHIFTD}$$

$$\frac{\overrightarrow{\alpha}^- \cap \mathbf{fv} Q = \emptyset \quad \mu : (\overrightarrow{\beta}^- \cap \mathbf{fv} Q) \leftrightarrow (\overrightarrow{\alpha}^- \cap \mathbf{fv} P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha}^-. P \simeq_1^D \exists \overrightarrow{\beta}^-. Q} \quad \text{E1EXISTS}$$

$\boxed{\Gamma \vdash N \simeq_1^{\leq} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leq_1 M \quad \Gamma \vdash M \leq_1 N}{\Gamma \vdash N \simeq_1^{\leq} M} \quad \text{D1NDEF}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leq} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash Q \geq_1 P}{\Gamma \vdash P \simeq_1^{\leq} Q} \quad \text{D1PDEF}$$

$\boxed{\Gamma \vdash N \leq_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leq_1 \alpha^-} \quad \text{D1NVAR}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leq} Q}{\Gamma \vdash \uparrow P \leq_1 \uparrow Q} \quad \text{D1SHIFTU}$$

$$\frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash N \leq_1 M}{\Gamma \vdash P \rightarrow N \leq_1 Q \rightarrow M} \quad \text{D1ARROW}$$

$$\frac{\Gamma, \overrightarrow{\beta}^+ \vdash P_i \quad \Gamma, \overrightarrow{\beta}^+ \vdash [\overrightarrow{P}/\overrightarrow{\alpha}^+]N \leq_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha}^+. N \leq_1 \forall \overrightarrow{\beta}^+. M} \quad \text{D1FORALL}$$

$\boxed{\Gamma \vdash P \geq_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geq_1 \alpha^+} \quad \text{D1PVAR}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leq} M}{\Gamma \vdash \downarrow N \geq_1 \downarrow M} \quad \text{D1SHIFTD}$$

$$\frac{\Gamma, \overrightarrow{\beta}^- \vdash N_i \quad \Gamma, \overrightarrow{\beta}^- \vdash [\overrightarrow{N}/\overrightarrow{\alpha}^-]P \geq_1 Q'}{\Gamma \vdash \exists \overrightarrow{\alpha}^-. P \geq_1 \exists \overrightarrow{\beta}^-. Q} \quad \text{D1EXISTS L}$$

$\boxed{\Gamma \vdash N \simeq_0^{\leq} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leq_0 M \quad \Gamma \vdash M \leq_0 N}{\Gamma \vdash N \simeq_0^{\leq} M} \quad \text{D0NDEF}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leq} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geq_0 Q \quad \Gamma \vdash Q \geq_0 P}{\Gamma \vdash P \simeq_0^{\leq} Q} \quad \text{D0PDEF}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$

Negative subtyping

$$\begin{array}{c}
\frac{}{\Gamma \vdash a- \leqslant_0 a-} \text{D0NVAR} \\
\frac{\Gamma \vdash P \simeq_0^{\leq} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \text{D0SHIFTU} \\
\frac{\Gamma \vdash P \quad \Gamma \vdash [P/a+]N \leqslant_0 M \quad M \neq \forall \beta^+. M'}{\Gamma \vdash \forall \alpha^+. N \leqslant_0 M} \text{D0FORALLL} \\
\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+. M} \text{D0FORALLR} \\
\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \rightarrow N \leqslant_0 Q \rightarrow M} \text{D0ARROW}
\end{array}$$

 $\boxed{\Gamma \vdash P \geqslant_0 Q}$

Positive supertyping

$$\begin{array}{c}
\frac{}{\Gamma \vdash a+ \geqslant_0 a+} \text{D0PVAR} \\
\frac{\Gamma \vdash N \simeq_0^{\leq} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \text{D0SHIFTD} \\
\frac{\Gamma \vdash N \quad \Gamma \vdash [N/a-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-. Q'}{\Gamma \vdash \exists \alpha^-. P \geqslant_0 Q} \text{D0EXISTSL} \\
\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-. Q} \text{D0EXISTSR}
\end{array}$$

 $\boxed{P_1 \vee P_2}$
 $\boxed{\text{ord vars in } P}$
 $\boxed{\text{ord vars in } N}$
 $\boxed{\text{ord vars in } P}$
 $\boxed{\text{ord vars in } N}$
 $\boxed{\text{nf}(N')}$
 $\boxed{\text{nf}(P')}$
 $\boxed{\text{nf}(N')}$

$$\boxed{\mathbf{nf}(P')}$$

$$\boxed{\hat{\sigma}_1 \& \hat{\sigma}_2}$$

$$\boxed{P_1 \vee P_2 = Q} \quad \text{Least Upper Bound (Least Common Supertype)}$$

$$\frac{\overline{\alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVAR}}{\frac{(\mathbf{fv} N \cup \mathbf{fv} M) \models \downarrow N \stackrel{a}{\simeq} \downarrow M \models (P, \hat{\sigma}_1, \hat{\sigma}_2) \quad \text{LUBSHIFT}}{\downarrow N \vee \downarrow M = \exists \alpha^-. [\overrightarrow{\alpha^-} / \mathbf{uv} P] P}} \quad \text{LUBEXISTS}$$

$$\frac{\overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \emptyset}{\exists \alpha^-. P_1 \vee \exists \beta^-. P_2 = P_1 \vee P_2} \quad \text{LUBEXISTS}$$

$$\boxed{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (Q, \hat{\sigma}_1, \hat{\sigma}_2)}$$

$$\frac{}{\Gamma \models \alpha^+ \stackrel{a}{\simeq} \alpha^+ \models (\alpha^+, \cdot, \cdot)} \quad \text{AUPVAR}$$

$$\frac{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \models (M, \hat{\sigma}_1, \hat{\sigma}_2)}{\Gamma \models \downarrow N_1 \stackrel{a}{\simeq} \downarrow N_2 \models (\downarrow M, \hat{\sigma}_1, \hat{\sigma}_2)} \quad \text{AUPSHIFT}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \emptyset \quad \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (Q, \hat{\sigma}_1, \hat{\sigma}_2)}{\Gamma \models \exists \alpha^-. P_1 \stackrel{a}{\simeq} \exists \alpha^-. P_2 \models (\exists \alpha^-. Q, \hat{\sigma}_1, \hat{\sigma}_2)} \quad \text{AUPEXISTS}$$

$$\boxed{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \models (M, \hat{\sigma}_1, \hat{\sigma}_2)}$$

$$\frac{}{\Gamma \models \alpha^- \stackrel{a}{\simeq} \alpha^- \models (\alpha^-, \cdot, \cdot)} \quad \text{AUNVAR}$$

$$\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (Q, \hat{\sigma}_1, \hat{\sigma}_2)}{\Gamma \models \uparrow P_1 \stackrel{a}{\simeq} \uparrow P_2 \models (\uparrow Q, \hat{\sigma}_1, \hat{\sigma}_2)} \quad \text{AUNSHIFT}$$

$$\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \models (Q, \hat{\sigma}_1, \hat{\sigma}_2) \quad \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \models (M, \hat{\sigma}'_1, \hat{\sigma}'_2)}{\Gamma \models P_1 \rightarrow N_1 \stackrel{a}{\simeq} P_2 \rightarrow N_2 \models (Q \rightarrow M, \hat{\sigma}_1 \cup \hat{\sigma}'_1, \hat{\sigma}_2 \cup \hat{\sigma}'_2)} \quad \text{AUNARROW}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \stackrel{a}{\simeq} M \models (\hat{\alpha}_{\{N, M\}}^-, (\hat{\alpha}_{\{N, M\}}^- : \approx N), (\hat{\alpha}_{\{N, M\}}^- : \approx M))} \quad \text{AUNAU}$$

$$\boxed{\mathbf{ord} \, \mathbf{vars} \, \mathbf{in} \, N = \vec{\alpha}}$$

$$\boxed{\mathbf{ord} \, \mathbf{vars} \, \mathbf{in} \, P = \vec{\alpha}}$$

$$\boxed{\mathbf{ord} \, \mathbf{vars} \, \mathbf{in} \, N = \vec{\alpha}}$$

$$\frac{\alpha^- \in \mathbf{vars}}{\mathbf{ord} \, \mathbf{vars} \, \mathbf{in} \, \alpha^- = \alpha^-} \quad \text{ONVARIN}$$

$$\frac{\alpha^- \notin \mathbf{vars}}{\mathbf{ord} \, \mathbf{vars} \, \mathbf{in} \, \alpha^- = \cdot} \quad \text{ONVARNIN}$$

$$\frac{}{\text{ord vars in } \hat{\alpha}^-\{vars'\} = .} \quad \text{ONUVar}$$

$$\frac{\text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \uparrow P = \vec{\alpha}} \quad \text{OShiftU}$$

$$\frac{\text{ord vars in } P = \vec{\alpha}_1 \quad \text{ord vars in } N = \vec{\alpha}_2}{\text{ord vars in } P \rightarrow N = \vec{\alpha}_1, (\vec{\alpha}_2 \setminus \{\vec{\alpha}_1\})} \quad \text{OArrow}$$

$$\frac{vars \cap \vec{\alpha}^+ = \emptyset \quad \text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \forall \vec{\alpha}^+. N = \vec{\alpha}} \quad \text{OForall}$$

$$\boxed{\text{ord vars in } P = \vec{\alpha}}$$

$$\frac{\alpha^+ \in vars}{\text{ord vars in } \alpha^+ = \alpha^+} \quad \text{OPVarIn}$$

$$\frac{\alpha^+ \notin vars}{\text{ord vars in } \alpha^+ = .} \quad \text{OPVarNiN}$$

$$\frac{}{\text{ord vars in } \hat{\alpha}^+\{vars'\} = .} \quad \text{OPUVar}$$

$$\frac{\text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \downarrow N = \vec{\alpha}} \quad \text{OShiftD}$$

$$\frac{vars \cap \vec{\alpha}^- = \emptyset \quad \text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \exists \vec{\alpha}^-. P = \vec{\alpha}} \quad \text{OExists}$$

$$\boxed{\text{nf}(N) = M}$$

$$\boxed{\text{nf}(P) = Q}$$

$$\boxed{\text{nf}(N) = M}$$

$$\frac{}{\text{nf}(\alpha^-) = \alpha^-} \quad \text{NRMNVar}$$

$$\frac{}{\text{nf}(\hat{\alpha}^-\{vars'\}) = \hat{\alpha}^-\{vars'\}} \quad \text{NRMNUVar}$$

$$\frac{\text{nf}(P) = Q}{\text{nf}(\uparrow P) = \uparrow Q} \quad \text{NRMShiftU}$$

$$\frac{\text{nf}(P) = Q \quad \text{nf}(N) = M}{\text{nf}(P \rightarrow N) = Q \rightarrow M} \quad \text{NRMArrow}$$

$$\frac{\text{nf}(N) = N' \quad \text{ord } \vec{\alpha}^+ \text{ in } N' = \vec{\alpha}^{+'}}{\text{nf}(\forall \vec{\alpha}^+. N) = \forall \vec{\alpha}^{+'}. N'} \quad \text{NRMForall}$$

$$\boxed{\text{nf}(P) = Q}$$

$$\frac{}{\text{nf}(\alpha^+) = \alpha^+} \quad \text{NRMPVar}$$

$$\frac{}{\text{nf}(\hat{\alpha}^+\{vars'\}) = \hat{\alpha}^+\{vars'\}} \quad \text{NRMPUVar}$$

$$\frac{\text{nf}(N) = M}{\text{nf}(\downarrow N) = \downarrow M} \quad \text{NRMShiftD}$$

$$\frac{\text{nf}(P) = P' \quad \text{ord } \vec{\alpha}^- \text{ in } P' = \vec{\alpha}^{-'}}{\text{nf}(\exists \vec{\alpha}^-. P) = \exists \vec{\alpha}^{-'}. P'} \quad \text{NRMExists}$$

$$\boxed{e_1 \& e_2 = e_3}$$

Unification Solution Entry Merge

$$\begin{array}{c} \frac{}{\hat{\alpha}^+ : \geq P \& \hat{\alpha}^+ : \geq Q = \hat{\alpha}^+ : \geq P \vee Q} \text{ SMEPSUPSUP} \\ \frac{\mathbf{fv} P \cup \mathbf{fv} Q \models \boxed{P} \succcurlyeq Q \models \hat{\sigma}'}{\hat{\alpha}^+ : \approx P \& \hat{\alpha}^+ : \geq Q = \hat{\alpha}^+ : \approx P} \text{ SMEPEQSUP} \\ \frac{\mathbf{fv} P \cup \mathbf{fv} Q \models \boxed{Q} \succcurlyeq P \models \hat{\sigma}'}{\hat{\alpha}^+ : \geq P \& \hat{\alpha}^+ : \approx Q = \hat{\alpha}^+ : \approx Q} \text{ SMEPSUPEQ} \\ \frac{}{\hat{\alpha}^+ : \approx P \& \hat{\alpha}^+ : \approx P = \hat{\alpha}^+ : \approx P} \text{ SMEPEQEQ} \\ \frac{}{\hat{\alpha}^- : \approx N \& \hat{\alpha}^- : \approx N = \hat{\alpha}^- : \approx N} \text{ SMENEQEQ} \end{array}$$

$$\boxed{\hat{\sigma}_1 \& \hat{\sigma}_2 = \hat{\sigma}_3}$$

Merge unification solutions

$$\begin{array}{c} \frac{}{\cdot \& \hat{\sigma} = \hat{\sigma}} \text{ SEMPTY} \\ \frac{(\hat{\alpha}^+ : \approx P) \in \hat{\sigma}_2 \quad \hat{\sigma}_1 \& (\hat{\sigma}_2 \setminus \hat{\alpha}^+) = \hat{\sigma}_3}{(\hat{\alpha}^+ : \approx P, \hat{\sigma}_1) \& \hat{\sigma}_2 = (\hat{\alpha}^+ : \approx P, \hat{\sigma}_3)} \text{ SMPEQEQ} \\ \frac{(\hat{\alpha}^+ : \geq Q) \in \hat{\sigma}_2 \quad \hat{\sigma}_1 \& (\hat{\sigma}_2 \setminus \hat{\alpha}^+) = \hat{\sigma}_3}{(\hat{\alpha}^+ : \geq P, \hat{\sigma}_1) \& \hat{\sigma}_2 = (\hat{\alpha}^+ : \geq P \vee Q, \hat{\sigma}_3)} \text{ SMPSUPSUP} \\ \frac{(\hat{\alpha}^+ : \approx Q) \in \hat{\sigma}_2 \quad \mathbf{fv} Q \cup \mathbf{fv} P \models \boxed{Q} \succcurlyeq P \models \hat{\sigma}' \quad \hat{\sigma}_1 \& (\hat{\sigma}_2 \setminus \hat{\alpha}^+) = \hat{\sigma}_3}{(\hat{\alpha}^+ : \geq P, \hat{\sigma}_1) \& \hat{\sigma}_2 = (\hat{\alpha}^+ : \approx Q, \hat{\sigma}_3)} \text{ SMPSUPEQ} \\ \frac{(\hat{\alpha}^+ : \geq Q) \in \hat{\sigma}_2 \quad \mathbf{fv} Q \cup \mathbf{fv} P \models \boxed{P} \succcurlyeq Q \models \hat{\sigma}' \quad \hat{\sigma}_1 \& (\hat{\sigma}_2 \setminus \hat{\alpha}^+) = \hat{\sigma}_3}{(\hat{\alpha}^+ : \approx P, \hat{\sigma}_1) \& \hat{\sigma}_2 = (\hat{\alpha}^+ : \approx P, \hat{\sigma}_3)} \text{ SMPEQSUP} \\ \frac{(\hat{\alpha}^- : \approx N) \in \hat{\sigma}_2 \quad \hat{\sigma}_1 \& (\hat{\sigma}_2 \setminus \hat{\alpha}^-) = \hat{\sigma}_3}{(\hat{\alpha}^- : \approx N, \hat{\sigma}_1) \& \hat{\sigma}_2 = (\hat{\alpha}^- : \approx N, \hat{\sigma}_3)} \text{ SMNEQEQ} \end{array}$$

$$\boxed{N \overset{u}{\preceq} M \models \hat{\sigma}}$$

Negative unification

$$\begin{array}{c} \frac{}{\alpha^- \overset{u}{\preceq} \alpha^- \models \cdot} \text{ UNVAR} \\ \frac{\boxed{P} \overset{u}{\preceq} Q \models \hat{\sigma}}{\uparrow P \overset{u}{\preceq} \uparrow Q \models \hat{\sigma}} \text{ USHIFTU} \\ \frac{\boxed{P} \overset{u}{\preceq} Q \models \hat{\sigma}_1 \quad \boxed{N} \overset{u}{\preceq} M \models \hat{\sigma}_2}{\boxed{P} \rightarrow \boxed{N} \overset{u}{\preceq} Q \rightarrow M \models \hat{\sigma}_1 \& \hat{\sigma}_2} \text{ UARROW} \\ \frac{\boxed{N} \overset{u}{\preceq} M \models \hat{\sigma}}{\forall \alpha^+. \boxed{N} \overset{u}{\preceq} \forall \alpha^+. M \models \hat{\sigma}} \text{ UFORALL} \\ \frac{\mathbf{fv} N \subseteq \text{vars}}{\hat{\alpha}^- \{ \text{vars} \} \overset{u}{\preceq} N \models \hat{\alpha}^- : \approx N} \text{ UNUVAR} \end{array}$$

$$\boxed{P \overset{u}{\preceq} Q \models \hat{\sigma}}$$

Positive unification

$$\begin{array}{c}
\frac{}{\alpha^+ \stackrel{u}{\simeq} \alpha^+ \Rightarrow \cdot} \text{UPVAR} \\
\\
\frac{N \stackrel{u}{\simeq} M \Rightarrow \hat{\sigma}}{\downarrow N \stackrel{u}{\simeq} \downarrow M \Rightarrow \hat{\sigma}} \text{USHIFTD} \\
\\
\frac{P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}}{\exists \alpha^-. \overrightarrow{P} \stackrel{u}{\simeq} \exists \alpha^-. \overrightarrow{Q} \Rightarrow \hat{\sigma}} \text{UEXISTS} \\
\\
\frac{\mathbf{fv} P \subseteq \mathit{vars}}{\hat{\alpha}^+ \{\mathit{vars}\} \stackrel{u}{\simeq} P \Rightarrow \hat{\alpha}^+ : \approx P} \text{UPUVAR}
\end{array}$$

$\boxed{\Gamma \vdash N}$ Negative type well-formedness
 $\boxed{\Gamma \vdash P}$ Positive type well-formedness
 $\boxed{\Gamma \vdash N}$ Negative type well-formedness
 $\boxed{\Gamma \vdash P}$ Positive type well-formedness

Definition rules: 94 good 0 bad
Definition rule clauses: 165 good 0 bad