1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{lll}
P,\ Q & ::= & \text{positive types} \\
& \mid \quad \alpha^+ & \\
& \mid \quad \downarrow N & \\
& \mid \quad \exists \alpha^-.P &
\end{array}
$$

$$
\begin{array}{lll}
N,\ M & ::= & \text{negative types} \\
& \mid \quad \alpha^- & \\
& \mid \quad \uparrow P & \\
& \mid \quad \forall \alpha^+.N & \\
& \mid \quad P \to N &
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$
\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \quad \text{D0NDef}
$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$
\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \quad \text{D0PDef}
$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$
\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \quad \text{D0NVar}
$$

$$
\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \quad \text{D0ShiftU}
$$

$$
\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \quad \text{D0ForallL}
$$

$$
\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \quad \text{D0ForallR}
$$

$$
\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \quad \text{D0Arrow}
$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$
\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \quad \text{D0PVar}
$$

$$
\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \quad \text{D0ShiftD}
$$

$$
\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \quad \text{D0ExistsL}
$$

$$
\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \quad \text{D0ExistsR}
$$

2 Multi-Quantified System

2.1 Grammar

P, Q	::=		multi-quantified positive types
	$\mid$	α^+	
	$\mid$	$\downarrow N$	
	$\mid$	$\exists\overrightarrow{\alpha^-}.P$	$P \neq \exists \ldots$
	$\mid$	(P) S	

N, M	::=		multi-quantified negative types
	$\mid$	α^-	
	$\mid$	$\uparrow P$	
	$\mid$	$P \to N$	
	$\mid$	$\forall\overrightarrow{\alpha^+}.N$	$N \neq \forall \ldots$
	$\mid$	(N) S	

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq^{\leqslant}_1 M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq^{\leqslant}_1 M} \quad \text{D1N\textsc{Def}}$$

$\boxed{\Gamma \vdash P \simeq^{\leqslant}_1 Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq^{\leqslant}_1 Q} \quad \text{D1P\textsc{Def}}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad (\textsc{Var}^{-\leqslant_1})$$

$$\frac{\Gamma \vdash P \simeq^{\leqslant}_1 Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \quad (\uparrow^{\leqslant_1})$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \quad (\to^{\leqslant_1})$$

$$\frac{\mathbf{fv}\, N \cap \{\overrightarrow{\beta^+}\} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M} \quad (\forall^{\leqslant_1})$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad (\textsc{Var}^{+\geqslant_1})$$

$$\frac{\Gamma \vdash N \simeq^{\leqslant}_1 M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \quad (\downarrow^{\geqslant_1})$$

$$\frac{\mathbf{fv}\, P \cap \{\overrightarrow{\beta^-}\} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant_1 \exists\overrightarrow{\beta^-}.Q} \quad (\exists^{\geqslant_1})$$

2.3 Declarative Equivalence

$\boxed{N \simeq^D_1 M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq^D_1 \alpha^-} \quad (\textsc{Var}^{-\simeq^D_1})$$

$$\frac{P \simeq^D_1 Q}{\uparrow P \simeq^D_1 \uparrow Q} \quad (\uparrow^{\simeq^D_1})$$

$$\frac{P \simeq^D_1 Q \quad N \simeq^D_1 M}{P \to N \simeq^D_1 Q \to M} \quad (\to^{\simeq^D_1})$$

$$\frac{\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\, M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\text{Var}^+)$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\{\overrightarrow{\beta^-}\} \cap \mathbf{fv}\, Q) \leftrightarrow (\{\overrightarrow{\alpha^-}\} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3   Algorithm

3.1   Normalization

3.1.1   Ordering

$\boxed{\mathbf{ord}\ vars\,\mathbf{in}\ N = \overrightarrow{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\ vars\,\mathbf{in}\ \alpha^- = \alpha^-} \quad (\text{Var}_\in^-)$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\ vars\,\mathbf{in}\ \alpha^- = \cdot} \quad (\text{Var}_{\notin}^-)$$

$$\frac{\mathbf{ord}\ vars\,\mathbf{in}\ P = \overrightarrow{\alpha}}{\mathbf{ord}\ vars\,\mathbf{in}\ {\uparrow}P = \overrightarrow{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\ vars\,\mathbf{in}\ P = \overrightarrow{\alpha}_1 \quad \mathbf{ord}\ vars\,\mathbf{in}\ N = \overrightarrow{\alpha}_2}{\mathbf{ord}\ vars\,\mathbf{in}\ P \to N = \overrightarrow{\alpha}_1, (\overrightarrow{\alpha}_2 \backslash \{\overrightarrow{\alpha}_1\})} \quad (\to)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^+}\} = \varnothing \quad \mathbf{ord}\ vars\,\mathbf{in}\ N = \overrightarrow{\alpha}}{\mathbf{ord}\ vars\,\mathbf{in}\ \forall \overrightarrow{\alpha^+}.N = \overrightarrow{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\ vars\,\mathbf{in}\ P = \overrightarrow{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\ vars\,\mathbf{in}\ \alpha^+ = \alpha^+} \quad (\text{Var}_\in^+)$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\ vars\,\mathbf{in}\ \alpha^+ = \cdot} \quad (\text{Var}_{\notin}^+)$$

$$\frac{\mathbf{ord}\ vars\,\mathbf{in}\ N = \overrightarrow{\alpha}}{\mathbf{ord}\ vars\,\mathbf{in}\ {\downarrow}N = \overrightarrow{\alpha}} \quad (\downarrow)$$

$$\frac{vars \cap \{\overrightarrow{\alpha^-}\} = \varnothing \quad \mathbf{ord}\ vars\,\mathbf{in}\ P = \overrightarrow{\alpha}}{\mathbf{ord}\ vars\,\mathbf{in}\ \exists \overrightarrow{\alpha^-}.P = \overrightarrow{\alpha}} \quad (\exists)$$

$\boxed{\mathbf{ord}\ vars\,\mathbf{in}\ N = \overrightarrow{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\,\mathbf{in}\ \widehat{\alpha}^- = \cdot} \quad (\text{UVar}^-)$$

$\boxed{\mathbf{ord}\ vars\,\mathbf{in}\ P = \overrightarrow{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\,\mathbf{in}\ \widehat{\alpha}^+ = \cdot} \quad (\text{UVar}^+)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\ \ (\text{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\ \ (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\ \ (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,N' = \overrightarrow{\alpha^{+\prime}}}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.N'}\ \ (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\ \ (\text{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\ \ (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,P' = \overrightarrow{\alpha^{-\prime}}}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^{-\prime}}.P'}\ \ (\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\ \ (\text{UVar}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\ \ (\text{UVar}^+)$$

3.2 Unification

$\boxed{\Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Theta \vDash \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\ \ \text{UNVar}$$

$$\frac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \vDash \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\ \ \text{UShiftU}$$

$$\frac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Theta \vDash P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1\,\&\,\widehat{\sigma}_2}\ \ \text{UArrow}$$

$$\frac{\Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \vDash \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\ \ \text{UForall}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Theta \vDash \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)}\ \ \text{UNUVar}$$

$\boxed{\Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Theta \vDash \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\ \ \text{UPVar}$$

$$\frac{\Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \vDash \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\ \ \text{UShiftD}$$

$$\frac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \vDash \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\ \ \text{UExists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Theta \vDash \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)}\ \ \text{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma; \Theta \vDash \alpha^- \leqslant \alpha^- \dashv \cdot} \quad \text{ANVar}$$

$$\frac{\Theta \vDash \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad \text{AShiftU}$$

$$\frac{\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \vDash P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \And \widehat{\sigma}_2} \quad \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vDash [\overrightarrow{\widehat{\alpha}^+/\alpha^+}]\,N \leqslant M \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}} \quad \text{AForall}$$

$\boxed{\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma; \Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad \text{APVar}$$

$$\frac{\Theta \vDash \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \quad \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vDash [\overrightarrow{\widehat{\alpha}^-/\alpha^-}]\,P \geqslant Q \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}} \quad \text{AExists}$$

$$\frac{\mathbf{upgrade}\,\Gamma \vdash \mathbf{nf}\,(P)\,\mathbf{to}\,\Delta = Q}{\Gamma; \Theta \vDash \widehat{\alpha}^+\{\Delta\} \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)} \quad \text{APUVar}$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restrict an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\Delta \vdash \widehat{\alpha}^+ :\approx P$ or $\Delta \vdash \widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\Delta \vdash \widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries). *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.
$\boxed{e_1 \And e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\Gamma \vDash P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1) \And (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)} \quad (\geqslant \And \geqslant)$$

$$\frac{\Gamma; \cdot \vDash P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \And (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)} \quad (\simeq \And \geqslant)$$

$$\frac{\Gamma; \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P) \And (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)} \quad (\geqslant \And \simeq)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \And (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)} \quad (\simeq \And \simeq^+)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N) \And (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)} \quad (\simeq \And \simeq^-)$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3. $\widehat{\sigma}_1 \And \widehat{\sigma}_2 = \{e_1 \And e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, s.t.\ e_1 \text{ matches with } e_2\}$
$\qquad\qquad \cup \{e_1 \mid e_1 \in \widehat{\sigma}_1,\ s.t.\ \forall e_2 \in \widehat{\sigma}_2, e_1 \text{ does not match with } e_2\}$
$\qquad\qquad \cup \{e_2 \mid e_2 \in \widehat{\sigma}_2,\ s.t.\ \forall e_1 \in \widehat{\sigma}_1, e_1 \text{ does not match with } e_2\}$

3.5 Least Upper Bound

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+} \ \ (\text{Var}^\vee)$$

$$\frac{\Gamma, \cdot \vDash \downarrow N \overset{a}{\simeq} \downarrow M \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.\overrightarrow{[\alpha^-/\Xi]}\, P} \ \ (\downarrow^\vee)$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q} \ \ (\exists^\vee)$$

$\boxed{\textbf{upgrade}\,\Gamma \vdash P \,\textbf{to}\, \Delta = Q}$

3.6 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \ \ \text{AUPVar}$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \dashv (\Xi, \downarrow M, \widehat{\tau}_1, \widehat{\tau}_2)} \ \ \text{AUPShift}$$

$$\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists \overrightarrow{\alpha^-}.Q, \widehat{\tau}_1, \widehat{\tau}_2)} \ \ \text{AUPExists}$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)} \ \ \text{AUNVar}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)} \ \ \text{AUNShift}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}_1', \widehat{\tau}_2')}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}_1', \widehat{\tau}_2 \cup \widehat{\tau}_2')} \ \ \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}_{\{N,M\}}^-, \widehat{\alpha}_{\{N,M\}}^-, (\widehat{\alpha}_{\{N,M\}}^- :\approx N), (\widehat{\alpha}_{\{N,M\}}^- :\approx M))} \ \ \text{AUNAU}$$

4 Proofs

4.1 Declarative Subtyping

Lemma 1 (Free Variable Propagation)**.** *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context Γ,*

- $-$ *if $\Gamma \vdash N \leqslant_1 M$ then $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$*

- $+$ *if $\Gamma \vdash P \geqslant_1 Q$ then $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$*

Proof. Mutual induction on $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash P \geqslant_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
It is self-evident that $\{\alpha^-\} \subseteq \{\alpha^-\}$.

Case 2. $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq_1^\leqslant Q$), we have $\Gamma \vdash P \geqslant_1 Q$. Then by the induction hypothesis, $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$. The desired inclusion inclusion holds, since $\mathbf{fv}(\uparrow P) = \mathbf{fv}(P)$ and $\mathbf{fv}(\uparrow Q) = \mathbf{fv}(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant_1 Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$ and $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$. Then $\mathbf{fv}(P \to N) = \mathbf{fv}(P) \cup \mathbf{fv}(N) \subseteq \mathbf{fv}(Q) \cup \mathbf{fv}(M) = \mathbf{fv}(Q \to M)$.

Case 4. $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M$

$\quad \mathbf{fv}\, \forall \overrightarrow{\alpha^+}.N \subseteq \mathbf{fv}\,([\overrightarrow{P/\alpha^+}]N) \setminus \{\overrightarrow{\beta^+}\}$ $\quad$ here $\{\overrightarrow{\beta^+}\}$ is excluded by the premise $\mathbf{fv}\, N \cap \{\overrightarrow{\beta^+}\} = \varnothing$

$\quad\quad\quad\quad \subseteq \mathbf{fv}\, M \setminus \{\overrightarrow{\beta^+}\}$ $\quad\quad\quad\quad$ by the induction hypothesis, $\mathbf{fv}\,([\overrightarrow{P/\alpha^+}]N) \subseteq \mathbf{fv}\, M$

$\quad\quad\quad\quad \subseteq \mathbf{fv}\, \forall \overrightarrow{\beta^+}.M$

Case 5. The positive cases are symmetric.

$\square$

4.2 Substitution

Lemma 2 (Substitution strengthening)**.** *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

$+$ *if $\Gamma_1 \vdash P$ then $[\sigma]P = [\sigma|_{\mathbf{fv}\, P}]P$,*

$-$ *if $\Gamma_1 \vdash N$ then $[\sigma]N = [\sigma|_{\mathbf{fv}\, N}]N$*

Proof. `Ilya:  todo` $\square$

4.3 Type well-formedness

Lemma 3 (Well-formedness agrees with substitution)**.** *Suppose that $\Gamma_1 \vdash \sigma : \Gamma_2$. Then*

$+$ $\Gamma_1, \Gamma_2 \vdash P \Leftrightarrow \Gamma_1 \vdash [\sigma]P$

$-$ $\Gamma_1, \Gamma_2 \vdash N \Leftrightarrow \Gamma_1 \vdash [\sigma]N$

Lemma 4 (Equivalent Contexts)**.** *In the well-formedness judgment, only used variables matter:*

$+$ *if $\{\Gamma_1\} \cap \mathbf{fv}\, P = \{\Gamma_2\} \cap \mathbf{fv}\, P$ then $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$,*

$-$ *if $\{\Gamma_1\} \cap \mathbf{fv}\, N = \{\Gamma_2\} \cap \mathbf{fv}\, N$ then $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$.*

Proof. By simple mutual induction on P and Q. $\square$

4.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\overline{\{\mathbf{ord}\,vars\,\mathbf{in}\,N\} \equiv vars \cap \mathbf{fv}\,N}$	$\dfrac{N \simeq_1^D M}{\mathbf{ord}\,vars\,\mathbf{in}\,N = \mathbf{ord}\,vars\,\mathbf{in}\,M}$	—
Normalization	$\overline{N \simeq_1^D \mathbf{nf}\,(N)}$	$\dfrac{N \simeq_1^D M}{\mathbf{nf}\,(N) = \mathbf{nf}\,(M)}$	—
Equivalence	$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq_1^D Q}{\Gamma \vdash P \simeq_1^{\leqslant} Q}$	$\dfrac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{P \simeq_1^D Q}$	—
Uppgrade	$\dfrac{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{Q \text{ is sound} \begin{cases} \Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$	$Q' \text{ is sound}$ $\dfrac{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \models P_1 \vee P_2 = Q}{Q \text{ is sound} \begin{cases} \Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2 \end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \models P_1 \vee P_2 = Q}$	$Q' \text{ is sound}$ $\dfrac{\Gamma \models P_1 \vee P_2 = Q}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2)}{(\Xi, Q, \hat\tau_1, \hat\tau_2) \text{ is sound} \begin{cases} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat\tau_i : \Xi \\ [\hat\tau_i]\,Q = P_i \end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \hat\tau_1', \hat\tau_2')}{\begin{array}{c}\exists (\Xi, Q, \hat\tau_1, \hat\tau_2) \text{ s.t.} \\ \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2)\end{array}}$	$(\Xi', Q', \hat\tau_1', \hat\tau_2') \text{ is sound}$ $\dfrac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \hat\tau_1, \hat\tau_2)}{\exists \Gamma; \Xi \vdash \hat\tau : \Xi' \text{ s.t. } [\hat\tau]\,Q' = Q}$
Unification (matching)	$\dfrac{\Theta \models P \stackrel{u}{\simeq} Q \dashv \hat\sigma}{\hat\sigma \text{ is sound} \begin{cases} \Theta \vdash \hat\sigma \\ [\hat\sigma]\,P = Q \end{cases}}$	$\dfrac{\exists \text{ sound } \hat\sigma'}{\exists \hat\sigma \text{ s.t. } \Theta \models P \stackrel{u}{\simeq} Q \dashv \hat\sigma}$	—
Subtyping	$\dfrac{\Gamma; \Theta \models N \leqslant M \dashv \hat\sigma}{\hat\sigma \text{ is sound} \begin{cases} \Theta \vdash \hat\sigma \\ \Gamma \vdash [\hat\sigma]\,N \leqslant_1 M \end{cases}}$	$\dfrac{\exists \text{ sound } \hat\sigma'}{\exists \hat\sigma \text{ s.t. } \Gamma; \Theta \models N \leqslant M \dashv \hat\sigma}$	—

4.5 Anti-unification

Lemma 5 (Soundness of the anti-unification algorithm)**.**

Lemma 6 (Completeness of the anti-unification algorithm)**.**

Lemma 7 (Initiality of the anti-unification algorithm)**.**

4.6 Variable Ordering

Definition 4 (Collision free bijection)**.** *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is **collision free on sets** P and Q if and only if*

 1. $\mu(P \cap A) \cap Q = \varnothing$

 2. $\mu(Q \cap A) \cap P = \varnothing$

Lemma 8 (Soundness of variable ordering)**.** *Variable ordering extracts precisely used free variables.*

 − $\{\mathbf{ord}\,vars\,\mathbf{in}\,N\} \equiv vars \cap \mathbf{fv}\,N$ *(as sets)*

 + $\{\mathbf{ord}\,vars\,\mathbf{in}\,P\} \equiv vars \cap \mathbf{fv}\,P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\,vars\,\mathbf{in}\,N = \vec{\alpha}$ and $\mathbf{ord}\,vars\,\mathbf{in}\,P = \vec{\alpha}$ □

Corollary 1 (Additivity of ordering)**.** *Variable ordering is additive (in terms of set union) with respect to its first argument.*

– $\{\mathbf{ord}\,(vars_1 \cup vars_2)\,\mathbf{in}\,N\} \equiv \{\mathbf{ord}\,vars_1\,\mathbf{in}\,N\} \cup \{\mathbf{ord}\,vars_2\,\mathbf{in}\,N\}$ *(as sets)*

+ $\{\mathbf{ord}\,(vars_1 \cup vars_2)\,\mathbf{in}\,P\} \equiv \{\mathbf{ord}\,vars_1\,\mathbf{in}\,P\} \cup \{\mathbf{ord}\,vars_2\,\mathbf{in}\,P\}$ *(as sets)*

Corollary 2 (Weakening of ordering). *Extending the first argument of the ordering with unused variables does not change the result.*

– $\mathbf{ord}\,(vars \cap \mathbf{fv}\,N)\,\mathbf{in}\,N = \mathbf{ord}\,vars\,\mathbf{in}\,N$

+ $\mathbf{ord}\,(vars \cap \mathbf{fv}\,P)\,\mathbf{in}\,P = \mathbf{ord}\,vars\,\mathbf{in}\,P$

Lemma 9 (Distributivity of renaming over variable ordering). *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

– *If μ is collision free on vars and $\mathbf{fv}\,N$ then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]N$*

+ *If μ is collision free on vars and $\mathbf{fv}\,P$ then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]P$*

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

    a. $\alpha^- \in A$ and $\alpha^- \in vars$
    Then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-)$

$$= [\mu]\alpha^- \qquad\qquad \text{by Rule } (\mathrm{Var}^+_\in)$$
$$= \beta^- \qquad\qquad \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars)$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\beta^- \qquad \text{by Rule } (\mathrm{Var}^+_\in), \text{ because } \beta^- \in [\mu]vars$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^-$$

    b. $\alpha^- \notin A$ and $\alpha^- \notin vars$
    Notice that $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_{\notin})$. On the other hand, $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_{\notin})$, because μ is collision free on $vars$ and $\mathbf{fv}\,N$, so $\mathbf{fv}\,N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

    c. $\alpha^- \in A$ but $\alpha^- \notin vars$
    Then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_{\notin})$. To prove that $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_{\notin})$. Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

     (i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.
     (ii) Since μ is collision free on $vars$ and $\mathbf{fv}\,N$, $\mu(A \cap \mathbf{fv}\,N) \ni \mu(\alpha^-) \notin vars$.

    d. $\alpha^- \notin A$ but $\alpha^- \in vars$
    $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_\in)$, because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$
$[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\uparrow P)$

$$= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) \qquad \text{by Rule } (\uparrow)$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P \qquad \text{by the induction hypothesis}$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\uparrow[\mu]P \qquad \text{by Rule } (\uparrow)$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\uparrow P \qquad \text{by the definition of substitution}$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N$$

Case 3. $N = P \to M$
$[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P \to M)$

$$= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})) \qquad \text{where } \mathbf{ord}\,vars\,\mathbf{in}\,P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\,vars\,\mathbf{in}\,M = \vec{\alpha}_2$$
$$= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})$$
$$= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\{\vec{\alpha}_1\}) \qquad \text{by induction on } \vec{\alpha}_2; \text{ the inductive step is similar to case 1. Notice that } \mu \text{ is}$$
$$\qquad\qquad\qquad \text{collision free on } \{\vec{\alpha}_1\} \text{ and } \{\vec{\alpha}_2\} \text{ since } \{\vec{\alpha}_1\} \subseteq vars \text{ and } \{\vec{\alpha}_2\} \subseteq \mathbf{fv}\,N$$
$$= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})$$
$(\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N) = (\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P \to [\mu]M)$
$$= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \{\vec{\beta}_1\})) \qquad\qquad \text{where } \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M = \vec{\beta}_2$$
$$\qquad\qquad\qquad \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1, \vec{\beta}_2 = [\mu]\vec{\alpha}_2,$$
$$= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})$$

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Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu]\mathbf{ord}\,vars\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.M$$
$$= [\mu]\mathbf{ord}\,vars\,\mathbf{in}\,M$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M \quad \text{by the induction hypothesis}$$
$$(\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N) = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\forall\overrightarrow{\alpha^+}.M$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.[\mu]M$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M$$

$\square$

Lemma 10 (Ordering is not affected by independent substitutions). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2, and vars is a set of variables disjoint with both Γ_1 and Γ_2. Then*

- $\mathbf{ord}\,vars\,\mathbf{in}\,[\sigma]N = \mathbf{ord}\,vars\,\mathbf{in}\,N$

+ $\mathbf{ord}\,vars\,\mathbf{in}\,[\sigma]P = \mathbf{ord}\,vars\,\mathbf{in}\,P$

Proof. `Ilya:  Should be easy` $\square$

Lemma 11 (Completeness of variable ordering). *Variable ordering is invariant under equivalence. For arbitrary vars,*

- *If $N \simeq_1^D M$ then $\mathbf{ord}\,vars\,\mathbf{in}\,N = \mathbf{ord}\,vars\,\mathbf{in}\,M$ (as lists)*

+ *If $P \simeq_1^D Q$ then $\mathbf{ord}\,vars\,\mathbf{in}\,P = \mathbf{ord}\,vars\,\mathbf{in}\,Q$ (as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. $\square$

4.7   Normaliztaion

Lemma 12. *Set of free variables is invariant under equivalence.*

- *If $N \simeq_1^D M$ then $\mathbf{fv}\,N \equiv \mathbf{fv}\,M$ (as sets)*

+ *If $P \simeq_1^D Q$ then $\mathbf{fv}\,P \equiv \mathbf{fv}\,Q$ (as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ $\square$

Lemma 13. *Free variables are not changed by the normalization*

- $\mathbf{fv}\,N \equiv \mathbf{fv}\,\mathbf{nf}\,(N)$

+ $\mathbf{fv}\,P \equiv \mathbf{fv}\,\mathbf{nf}\,(P)$

Proof. By straightforward induction on $\mathbf{nf}\,(N) = M$. $\square$

Lemma 14 (Soundness of quantifier normalization).

- $N \simeq_1^D \mathbf{nf}\,(N)$

+ $P \simeq_1^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var^-) and (Var^+)
By the corresponding equivalence rules.

Case 2. $(\uparrow)$, $(\downarrow)$, and $(\rightarrow)$
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. $(\forall)$, i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.\,N) = \forall\overrightarrow{\alpha^{+\prime}}.\,N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 12, $\mathbf{fv}\,N \equiv \mathbf{fv}\,N'$. Then by lemma 8, $\{\overrightarrow{\alpha^{+\prime}}\} \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$, and thus, $\{\overrightarrow{\alpha^{+\prime}}\} \cap \mathbf{fv}\,N' \equiv \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$.
To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^{+\prime}}.N'$, it suffices to provide a bijection $\mu : \{\overrightarrow{\alpha^{+\prime}}\} \cap \mathbf{fv}\,N' \leftrightarrow \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. $(\exists)$ Same as for case 3.

$\square$

Corollary 3 (Normalization preserves ordering)**.** *For any vars,*

- $\mathbf{ord}\,vars\,\mathbf{in}\,\mathbf{nf}\,(N) = \mathbf{ord}\,vars\,\mathbf{in}\,M$

+ $\mathbf{ord}\,vars\,\mathbf{in}\,\mathbf{nf}\,(P) = \mathbf{ord}\,vars\,\mathbf{in}\,Q$

Proof. Immediately from lemmas 11 and 14. $\square$

Lemma 15 (Distributivity of normalization over substitution)**.** *Normalization of a term distributes over substitution. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$, *i.e.* σ *maps variables from* Γ_1 *into types taking free variables from* Γ_2. *Then*

- $\mathbf{nf}\,([\sigma]N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$

+ $\mathbf{nf}\,([\sigma]P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P)$

where $\mathbf{nf}\,(\sigma)$ *means pointwise normalization:* $[\mathbf{nf}\,(\sigma)]\alpha^- = \mathbf{nf}\,([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.

Case 2. $P = \alpha^+$
Similar to case 1.

Case 3. If the type is formed by $\to$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \to M$ then
$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma](P \to M))$

$\begin{aligned}
&= \mathbf{nf}\,([\sigma]P \to [\sigma]M) && \text{By the congruence of substitution} \\
&= \mathbf{nf}\,([\sigma]P) \to \mathbf{nf}\,([\sigma]M) && \text{By the congruence of normalization, i.e. Rule } (\to) \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) \to [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M) && \text{By the induction hypothesis} \\
&= [\mathbf{nf}\,(\sigma)](\mathbf{nf}\,(P) \to \mathbf{nf}\,(M)) && \text{By the congruence of substitution} \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P \to M) && \text{By the congruence of normalization} \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)
\end{aligned}$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.M)$

$\quad = [\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,(M) \quad$ Where $\overrightarrow{\alpha^{+\prime}} = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M) = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M$ (the latter is by corollary 3)

$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\forall\overrightarrow{\alpha^+}.M)$

$\begin{aligned}
&= \mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\sigma]M) && \text{Assuming } \{\overrightarrow{\alpha^+}\} \cap \{\Gamma_1\} = \varnothing \text{ and } \{\overrightarrow{\alpha^+}\} \cap \{\Gamma_2\} = \varnothing \\
&= \forall\overrightarrow{\beta^+}.\mathbf{nf}\,([\sigma]M) && \text{Where } \overrightarrow{\beta^+} = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,([\sigma]M) = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\sigma]M \text{ (the latter is by corollary 3)} \\
&= \forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,([\sigma]M) && \text{By lemma 10, } \overrightarrow{\beta^+} = \overrightarrow{\alpha^{+\prime}} \text{ since } \{\overrightarrow{\alpha^+}\} \text{ is disjoint with } \Gamma_1 \text{ and } \Gamma_2 \\
&= \forall\overrightarrow{\alpha^{+\prime}}.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M) && \text{By the induction hypothesis}
\end{aligned}$

To show alpha-equivalence of $[\mathbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^{+\prime}}.[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M)$, we can assume that $\{\overrightarrow{\alpha^{+\prime}}\} \cap \{\Gamma_1\} = \varnothing$, and $\{\overrightarrow{\alpha^{+\prime}}\} \cap \{\Gamma_2\} = \varnothing$.

Case 5. $P = \exists\overrightarrow{\alpha^-}.Q$
Same as for case 4.

$\square$

Corollary 4 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that* μ *is a bijection between two sets of variables* $\mu : A \leftrightarrow B$. *Then*

- $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$

+ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Proof. Immediately from lemma 15, after noticing that $\mathbf{nf}\,(\mu) = \mu$. $\qquad\square$

Lemma 16 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

- $-$ *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$
- $+$ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. $(\forall^{\simeq_1^D})$

From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+}}{}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^{+}}{}'$ is $\mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^{+}}{}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^{+}}{}'$ is $\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \leftrightarrow (\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 8 and 13, the domain and the codomain of μ can be written as $\mu : \{\overrightarrow{\beta^{+}}{}'\} \leftrightarrow \{\overrightarrow{\alpha^{+}}{}'\}$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^{+}}{}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^{+}}{}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^{+}}{}' = \overrightarrow{\alpha^{+}}{}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by corollary 4, the second—by the induction hypothesis.

(ii) $[\mu]\overrightarrow{\beta^{+}}{}' = [\mu]\mathbf{ord}\,\{\overrightarrow{\beta^+}\}\,\mathbf{in}\,\mathbf{nf}\,(M)$ 　　　by the definition of $\overrightarrow{\beta^{+}}{}'$

$\quad = [\mu]\mathbf{ord}\,(\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M)$ 　　from lemma 13 and corollary 2

$\quad = \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M)$ 　by lemma 9, because $\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing$

　　　　　　　　　　　　　　　　　　　　 and $\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N \cap (\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M) \subseteq \{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,M = \varnothing$

$\quad = \mathbf{ord}\,[\mu](\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N)$ 　since $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ is proved

$\quad = \mathbf{ord}\,(\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N)$ 　　because μ is a bijection between $\{\overrightarrow{\alpha^+}\} \cap \mathbf{fv}\,N$ and $\{\overrightarrow{\beta^+}\} \cap \mathbf{fv}\,M$

$\quad = \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,\mathbf{nf}\,(N)$ 　　　　　from lemma 13 and corollary 2

$\quad = \overrightarrow{\alpha^{+}}{}'$ 　　　　　　　　　　　　by the definition of $\overrightarrow{\alpha^{+}}{}'$

Case 2. $(\exists^{\simeq_1^D})$ Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

$\qquad\square$

Lemma 17 (Idempotence of normalization)**.** *Normalization is idempotent*

- $-$ $\mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$
- $+$ $\mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

Proof. By applying lemma 16 to lemma 14. $\qquad\square$

Lemma 18. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized.*
Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then

$+$ $[\sigma]P$ *is normal* $\iff$ $\begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \text{is normal} \\ P & \text{is normal} \end{cases}$

$-$ $[\sigma]N$ *is normal* $\iff$ $\begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \text{is normal} \\ N & \text{is normal} \end{cases}$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$

Then N is always normal, and the normality of $\sigma|_{\{\alpha^-\}}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$[\sigma](P \to M) \text{ is normal} \iff [\sigma]P \to [\sigma]M \text{ is normal} \qquad \text{by the substitution congruence}$$

$$\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases} \qquad \text{by congruence of normality } \texttt{Ilya: lemma?}$$

$$\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}(M)} & \text{is normal} \end{cases} \qquad \text{by the induction hypothesis}$$

$$\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}(P) \cup \mathbf{fv}(M)} & \text{is normal} \end{cases} \iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}(P \to M)} & \text{is normal} \end{cases}$$

Case 3. $N = {\uparrow}P$

By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$[\sigma](\forall\overrightarrow{\alpha^+}.M) \text{ is normal} \iff (\forall\overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} \qquad \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$$

$$\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by the definition of normalization}$$

$$\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by lemma 10}$$

$$\iff \begin{cases} \sigma|_{\mathbf{fv}(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord}\,\{\overrightarrow{\alpha^+}\}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by the induction hypothesis}$$

$$\iff \begin{cases} \sigma|_{\mathbf{fv}(\forall\overrightarrow{\alpha^+}.M)} \text{ is normal} \\ \forall\overrightarrow{\alpha^+}.M \text{ is normal} \end{cases} \qquad \begin{array}{l} \text{since } \mathbf{fv}(\forall\overrightarrow{\alpha^+}.M) = \mathbf{fv}(M); \\ \text{by the definition of normalization} \end{array}$$

Case 5. $P = \ldots$

The positive cases are done in the same way as the negative ones.

$\square$

4.8 Equivalence

Lemma 19 (Type well-formedness is invariant under equivalence). *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,

− *if* $N \simeq_1^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. `Ilya: todo` $\square$

Corollary 5 (Normalization preserves well-formedness).

+ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}(P)$,

− $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}(N)$

Proof. Immediately from lemmas 14 and 19. $\square$

Corollary 6 (Normalization preserves well-formedness of substitution).
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}(\sigma) : \Gamma_1$

Lemma 20 (Soundness of equivalence). *Declarative equivalence implies mutual subtyping.*

+ *if* $\Gamma \vdash P$, $\Gamma \vdash Q$, *and* $P \simeq_1^D Q$ *then* $\Gamma \vdash P \simeq_1^\leqslant Q$,

$-$ if $\Gamma \vdash N$, $\Gamma \vdash M$, and $N \simeq_1^D M$ then $\Gamma \vdash N \simeq_1^{\leqslant} M$.

Proof. `Ilya:  todo` □

Lemma 21 (Completeness of equivalence). *Mutual subtyping implies declarative equivalence.*

$+$ *if* $\Gamma \vdash P \simeq_1^{\leqslant} Q$ *then* $P \simeq_1^D Q$,

$-$ *if* $\Gamma \vdash N \simeq_1^{\leqslant} M$ *then* $N \simeq_1^D M$.

Proof. `Ilya:  todo` □

4.9 Upper Bounds

Lemma 22 (Decomposition of the quantifier rule). *`Ilya:  move somewhere`* *Whenever the quantifier rule (Rule ($\exists^{\geqslant_1}$) or Rule ($\forall^{\leqslant_1}$)) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.*

$-$ *If* $\Gamma \vdash N \leqslant_1 \forall\overrightarrow{\beta^+}.M$ *then* $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant_1 M$.

$+$ *If* $\Gamma \vdash P \geqslant_1 \exists\overrightarrow{\beta^-}.Q$ *then* $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant_1 Q$.

Lemma 23 (Characterization of the Supertypes). *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists\overrightarrow{\alpha^-}.\beta^+ \mid for\ \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)\ not\ using\ \overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \ \middle\vert \ \begin{array}{l} for\ \overrightarrow{\alpha^-},\ M',\ and\ \vec{N}\ s.t. \\ \Gamma \vdash N_i,\ \Gamma, \overrightarrow{\alpha^-} \vdash M',\ and\ [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Then the last rule that is applied to infer $\Gamma \vdash Q \geqslant_1 \beta^+$ must be either Rule ($\mathsf{Var}^{+\geqslant_1}$) or Rule ($\exists^{\geqslant_1}$). The former case means that $Q = \beta^+$. In the latter case, $Q = \exists\overrightarrow{\alpha^-}.Q'$, where Q' has no outer existential quantifiers. Then by inversion of Rule ($\exists^{\geqslant_1}$), $\Gamma \vdash [\vec{N}/\overrightarrow{\alpha^-}]Q' \geqslant_1 \beta^+$ for some $\vec{N}$. This time, to infer this judgment, only Rule ($\mathsf{Var}^{+\geqslant_1}$) is applicable, which means that $Q' = \beta^+$, and then $Q = \exists\overrightarrow{\alpha^-}.\beta^+$.

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$
Then if $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'$, then by lemma 22, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$, and $\mathbf{fv}\,Q \cap \{\overrightarrow{\beta^-}\} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule ($\exists^{\geqslant_1}$). This way, $\{Q \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$ s.t. $\mathbf{fv}\,(Q) \cap \{\overrightarrow{\beta^-}\} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = \downarrow M$
Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\not\exists}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

     a. $Q \neq \exists\overrightarrow{\beta^-}.Q'$
        Then the last applied rule to infer $\Gamma \vdash Q \geqslant_1 \downarrow M$ must be Rule ($\downarrow^{\geqslant_1}$), which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq_1^{\leqslant} M$, then by lemma 21 and Rule ($\downarrow^{\simeq_1^D}$), $\downarrow M' \simeq_1^D \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq_1^D \downarrow M\} = \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\not\exists}$.
        In the other direction, $\downarrow M' \simeq_1^D \downarrow M \Rightarrow \Gamma \vdash \downarrow M' \simeq_1^{\leqslant} \downarrow M$ by lemma 20, since $\Gamma \vdash \downarrow M'$ by lemma 19
$$\Rightarrow \Gamma \vdash \downarrow M' \geqslant_1 \downarrow M \quad \text{by inversion}$$

     b. $Q = \exists\overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)
        Then the last rule applied to infer $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q' \geqslant_1 \downarrow M$ must be Rule ($\exists^{\geqslant_1}$). Inversion of this rule gives us $\Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]Q' \geqslant_1 \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\vec{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.a, $[\vec{N}/\overrightarrow{\beta^-}]Q' \simeq_1^D \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists\overrightarrow{\beta^-}.\downarrow M' \in \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).

In the other direction, $[\vec{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^D \downarrow M \Rightarrow \Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq_1^{\leqslant} \downarrow M$ by lemma 20, since $\Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]\downarrow M'$ by lemma 19

$$\Rightarrow \Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]\downarrow M' \geqslant_1 \downarrow M \quad \text{by inversion}$$

$$\Rightarrow \Gamma \vdash \exists\overrightarrow{\beta^-}.\downarrow M' \geqslant_1 \downarrow M \qquad \text{by Rule } (\exists^{\geqslant_1})$$

$\square$

Lemma 24 (Characterization of the Normalized Supertypes). *For a normalized positive type $P = \mathbf{nf}\,(P)$, let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \text{and } \vec{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\, \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \end{array} \right\}$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Then from lemma 23, $\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\beta^+) \mid \text{ for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

$\mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') = \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ *not using* $\overrightarrow{\beta^-}$

$$\begin{aligned}
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta^-} &&\text{by the induction hypothesis} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} &&\text{because } \mathbf{fv}\,\mathbf{nf}\,(Q) = \mathbf{fv}\,Q \text{ by lemma 13} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} &&\text{by lemma 23} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} &&\text{by the definition of } \mathsf{UB} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} &&\text{by lemma 23}
\end{aligned}$$

Case 3. $P = \downarrow M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 1. *Suppose that $\nu : A \to A$ is an idempotent function, P is a predicate on A, $F : A \to B$ is a function. Then*

$$\{F(\nu x) \mid x \in A \text{ s.t. } P(\nu x)\} =$$
$$= \{F(x) \mid x \in A \text{ s.t. } \nu x = x \text{ and } P(x)\}.$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 2. *For functions F and ν, and predicates P and Q,*

$$\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } P(x)\} =$$
$$= \{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } (\exists x' \in A \text{ s.t. } P(x') \text{ and } \nu x' = \nu x)\}.$$

$$\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \downarrow M\} =$$

$$= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \downarrow M)\} \qquad\qquad\qquad \text{by lemma 23}$$

$$= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \text{and } \overrightarrow{N} \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash N_i, \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \qquad \text{by the definition of UB}$$

$$= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \text{and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \qquad \text{we reassigned the substitution } \overrightarrow{N}/\overrightarrow{\alpha^-} \text{ as } \sigma$$

$$= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \text{and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \qquad \text{by lemma 2}$$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \qquad \text{by the definition of normalization}$$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } \mathbf{nf}\,([\sigma|_{\mathbf{fv}\,M'}]\downarrow M') = \mathbf{nf}\,(\downarrow M) \end{array} \right\} \qquad \begin{array}{l}\text{from lemmas 14 and 16, equivalence of types can be} \\ \text{replaced with the equality of their normal forms}\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\mathbf{nf}\,(\downarrow M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})]\downarrow\mathbf{nf}\,(M') = \downarrow\mathbf{nf}\,(M) \end{array} \right\} \qquad \text{by congruence of normalization and lemma 15}$$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} \qquad \begin{array}{l}\text{by lemma 18, } \downarrow M' \text{ and } \sigma|_{\mathbf{fv}\,M'} \text{ are already normal,} \\ \text{since the result of the substitution is normal;} \\ M \text{ is normal by assumption}\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ (\exists\sigma' \text{ s.t. } \Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,(\downarrow M')} = \sigma'|_{\mathbf{fv}\,(\downarrow M')}) \\ \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \text{ and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} \qquad \begin{array}{l}\text{We apply observation 2 (with } \nu\sigma = \sigma|_{\mathbf{fv}\,M'}, \text{ and} \\ P(\sigma) = \Gamma \vdash \sigma : \overrightarrow{\alpha^-})\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv}\,M'} : \overrightarrow{\alpha^-}', \; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]\downarrow M' = \downarrow M \end{array} \right\} \qquad \begin{array}{l}\text{Notice that} \\ \text{"}\exists\sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,(\downarrow M')} = \sigma'|_{\mathbf{fv}\,(\downarrow M')})\text{"} \\ \text{is equivalent to } \Gamma \vdash \sigma|_{\mathbf{fv}\,(\downarrow M')} : \overrightarrow{\alpha^-}'\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, \overrightarrow{\alpha^-},\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}', \; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} \qquad \begin{array}{l}\text{We apply observation 1 to the restriction of } \sigma, \text{ and} \\ \text{remove } \sigma|_{\mathbf{fv}\,M'} = \sigma \text{ as it follows from } \Gamma \vdash \sigma : \overrightarrow{\alpha^-}'\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, \overrightarrow{\alpha^-}',\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}', \; \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} \qquad \text{by lemma 4, since } \{\Gamma, \overrightarrow{\alpha^-}\} \cap \mathbf{fv}\,M' = \{\Gamma, \overrightarrow{\alpha^-}'\} \cap \mathbf{fv}\,M'$$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, M',\, \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-}' \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}', \; \mathbf{ord}\,\{\overrightarrow{\alpha^-}'\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}' \\ \text{and } [\sigma]\downarrow M' = \downarrow M \end{array} \right\} \qquad \text{We apply observation 1 to the ordering of } \overrightarrow{\alpha^-}$$

$$= \left\{ \exists\overrightarrow{\alpha^-}'.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}',\, M',\, \text{and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}'\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i, \; \Gamma, \overrightarrow{\alpha^-}' \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}']\downarrow M' = \downarrow M \end{array} \right\} \qquad \text{By reassigning } \sigma \text{ explicitly as } \overrightarrow{N}/\overrightarrow{\alpha^-}'$$

$$= \mathsf{NFUB}(\downarrow M) \qquad\qquad\qquad\qquad \text{by definition}$$

$$\square$$

Observation 3. *Upper bounds of a type do not depend on the context as soon as the type are well-formed in it.*
 If $\Gamma_1 \vdash M$ and $\Gamma_2 \vdash M$ then $\mathsf{UB}(\Gamma_1 \vdash M) = \mathsf{UB}(\Gamma \vdash M)$ and $\mathsf{NFUB}(\Gamma_1 \vdash M) = \mathsf{NFUB}(\Gamma \vdash M)$

Proof. We prove both inclusions by induction on $\Gamma_1 \vdash M$. Notice that if $[\sigma]M' \simeq_1^D M$ and $\Gamma_2 \vdash M$ then the types from the range of $\sigma|_{\mathbf{fv}\,M'}$ are well-formed in 2 `Ilya: lemma`. $\square$

Lemma 25 (Soundness of the Least Upper Bound). *For types $\Gamma \vdash P_1$, and $\Gamma \vdash P_2$, if $\Gamma \models P_1 \vee P_2 = Q$ then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Proof. Induction on $\Gamma \models P_1 \vee P_2 = Q$.

Case 1. $\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+$

Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule (Var$^{+\geqslant_1}$).

Case 2. $\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q$

Then by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applied to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q$. Then

(i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,

(ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$,

(iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}(Q) \cap \{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}\} = \mathbf{fv}(Q) \cap \{\Gamma\}$ (and then apply lemma 4). The inclusion right-to-left is self-evident. To show $\mathbf{fv}(Q) \cap \{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}\} \subseteq \mathbf{fv}(Q) \cap \{\Gamma\}$, we prove that $\mathbf{fv}(Q) \subseteq \{\Gamma\}$

$$\mathbf{fv}(Q) \subseteq \mathbf{fv}\,P_1 \cap \mathbf{fv}\,P_2 \qquad \text{by lemma 1}$$

$$\subseteq (\{\Gamma, \overrightarrow{\alpha^-}\}\backslash\{\overrightarrow{\beta^-}\}) \cap (\{\Gamma, \overrightarrow{\beta^-}\}\backslash\{\overrightarrow{\alpha^-}\}) \qquad \begin{array}{l} \text{since } \Gamma \vdash \exists\overrightarrow{\alpha^-}.P_1, \ \mathbf{fv}(P_1) \subseteq \{\Gamma, \overrightarrow{\alpha^-}\} = \{\Gamma, \overrightarrow{\alpha^-}\}\backslash\{\overrightarrow{\beta^-}\} \\ \text{(the latter is because by the Barendregt's convention,} \\ \{\Gamma, \overrightarrow{\alpha^-}\} \cap \{\overrightarrow{\beta^-}\} = \varnothing); \text{ similarly, } \mathbf{fv}(P_2) \subseteq \{\Gamma, \overrightarrow{\beta^-}\}\backslash\{\overrightarrow{\alpha^-}\} \end{array}$$

$$\subseteq \{\Gamma\}$$

To show $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\alpha^-}.P_1$, we apply Rule ($\exists^{\geqslant_1}$). Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant_1 P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \models \downarrow N \vee \downarrow M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P$ By the inversion, $\Gamma, \cdot \models \downarrow N \overset{a}{\simeq} \downarrow M \dashv (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)$. Then by lemma 5,

(i) $\Gamma; \Xi \vdash P$, then by **??**,
$$\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]\,P \tag{1}$$

(ii) $\Gamma; \cdot \vdash \hat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \hat{\tau}_2 : \Xi$. Assuming that $\Xi = \widehat{\beta}_1^-, \ldots, \widehat{\beta}_n^-$, the antiunification solutions $\hat{\tau}_1$ and $\hat{\tau}_2$ can be put explicitly as $\hat{\tau}_1 = (\widehat{\beta}_1^- :\approx N_1, \ldots, \widehat{\beta}_n^- :\approx N_n)$, and $\hat{\tau}_2 = (\widehat{\beta}_1^- :\approx M_1, \ldots, \widehat{\beta}_n^- :\approx M_n)$. Then

$$\hat{\tau}_1 = (\overrightarrow{N/\alpha^-}) \circ (\overrightarrow{\alpha^-/\Xi}) \text{ (as substitutions)} \tag{2}$$

$$\hat{\tau}_2 = (\overrightarrow{M/\alpha^-}) \circ (\overrightarrow{\alpha^-/\Xi}) \text{ (as substitutions)} \tag{3}$$

(iii) $[\hat{\tau}_1]\,Q = P_1$ and $[\hat{\tau}_2]\,Q = P_1$, which, by 2 and 3, means

$$[\overrightarrow{N/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P = \downarrow N \tag{4}$$

$$[\overrightarrow{M/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P = \downarrow M \tag{5}$$

Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-/\Xi}]\,P$ follows directly from 1.

To show $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 \downarrow N$, we apply Rule ($\exists^{\geqslant_1}$), instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{N}$. Then $\Gamma \vdash [\overrightarrow{N/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 \downarrow N$ follows from 4 and reflexivity of subtyping (**??**).

Analogously, instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{M}$, gives us $\Gamma \vdash [\overrightarrow{M/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 \downarrow M$ (from 5), and hence, $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 \downarrow M$.

$\square$

Lemma 26 (Completeness of the Least Upper Bound)**.** *For normalized types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q$ such that $\Gamma \vdash Q \geqslant_1 P_1$ and $\Gamma \vdash Q \geqslant_1 P_2$, there exists Q' s.t. $\Gamma \models P_1 \vee P_2 = Q'$.*

Proof. Induction on the pair (P_1, P_2). From lemma 24, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$. Let us consider the cases what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_1$).

Case 1. $P_1 = \exists \overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists \overrightarrow{\beta^-}_2.Q_2$ where $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then $Q \in \mathsf{UB}(\Gamma \vdash \exists \overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{UB}(\Gamma \vdash \exists \overrightarrow{\beta^-}_2.Q_2)$

$\qquad \subseteq \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_2 \vdash Q_2)$ $\qquad\qquad\qquad$ from the definition of UB

$\qquad = \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_2)$ $\qquad\quad$ by observation 3, weakening and exchange

$\qquad = \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1\} \cap \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2\}$ $\quad$ by lemma 23,

meaning that $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$. Then after one step, the algorithm terminates by the induction hypothesis. In other words, $\exists Q'$ s.t. $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vDash Q_1 \vee Q_2 = Q'$, and thus, Rule $(\exists^\vee)$ is applicable.

Case 2. $P_1 = \alpha^+$ and $P_2 = \downarrow N$

Then the set of common upper bounds of $\downarrow N$ and α^+ is empty, and thus, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$ gives a contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \downarrow N)$

$\qquad = \{\exists \overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists \overrightarrow{\beta^-}.\downarrow M' \mid \cdots\}$ $\quad$ by the definition of UB

$\qquad = \varnothing$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ since $\alpha^+ \neq \downarrow M'$ for any M'

Case 3. $P_1 = \downarrow N$ and $P_2 = \alpha^+$

Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)

Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \beta^+)$

$\qquad = \{\exists \overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists \overrightarrow{\beta^-}.\beta^+ \mid \cdots\}$ $\quad$ by the definition of UB

$\qquad = \varnothing$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ since $\alpha^+ \neq \beta^+$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Then the algorithm terminates in one step (Rule $(\mathrm{Var}^\vee)$): $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$.

Case 6. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$

Then on the next step, the algorithm tries to anti-unify $\downarrow M_1$ and $\downarrow M_2$. By lemma 6, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$\mathbf{nf}\,(Q) \in \mathsf{NFUB}(\Gamma \vdash \exists \overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{NFUB}(\Gamma \vdash \exists \overrightarrow{\beta^-}_2.Q_2)$

$$= \left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\, \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_1 \end{array} \right\}$$

$$\cap$$

$$\left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \vec{N}_1,\, \Gamma \vdash \vec{N}_2,\, \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_2 \end{array} \right\}$$

$$= \left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \vec{N}_1 \text{ and } \vec{N}_2 \text{ s.t. } \mathbf{ord}\,\{\overrightarrow{\alpha^-}\}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \vec{N}_1,\, \Gamma \vdash \vec{N}_2,\, \Gamma, \overrightarrow{\alpha^-} \vdash M', [\vec{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_1, \text{ and } [\vec{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_2 \end{array} \right\}$$

The fact that the latter set is non-empty means that there exist $\overrightarrow{\alpha^-}, M', \vec{N}_1$ and $\vec{N}_2$ such that

  (i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$,

  (ii) $\Gamma \vdash \vec{N}_1$ and $\Gamma \vdash \vec{N}_1$,

  (iii) $[\vec{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_1$ and $[\vec{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M_2$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative antiunification variable $\hat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\hat{\alpha}^-}$. Let us show that $(\overrightarrow{\alpha^-}, [\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M', \vec{N}_1/\overrightarrow{\hat{\alpha}^-}, \vec{N}_2/\overrightarrow{\hat{\alpha}^-})$ is a sound anti-unifier of $\downarrow M_1$ and $\downarrow M_2$ in context Γ:

- $\overrightarrow{\hat{\alpha}^-}$ is negative by construction,

- $\Gamma; \overrightarrow{\hat{\alpha}^-} \vdash [\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M'$ because $\Gamma, \overrightarrow{\alpha^-} \vdash \downarrow M'$ Ilya: lemma!,

- $\Gamma; \cdot \vdash (\vec{N}_1/\overrightarrow{\hat{\alpha}^-}) : \overrightarrow{\hat{\alpha}^-}$ because $\Gamma \vdash \vec{N}_1$ and $\Gamma; \cdot \vdash (\vec{N}_2/\overrightarrow{\hat{\alpha}^-}) : \overrightarrow{\hat{\alpha}^-}$ because $\Gamma \vdash \vec{N}_2$,

- $[\overrightarrow{N_1}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{N_1}/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_1$; analogously, $[\overrightarrow{N_2}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = i[\overrightarrow{N_2}/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_2$.

Then by the completeness of the anti-unification (lemma 6), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it.

$\square$

Lemma 27 (Initiality of the Least Upper Bound). *For normalized types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q$ such that $\Gamma \vdash Q \geqslant_1 P_1$ and $\Gamma \vdash Q \geqslant_1 P_2$, If $\Gamma \models P_1 \vee P_2 = Q'$ then $\Gamma \vdash Q \geqslant_1 Q'$.*

Proof. By induction on a pair (P_1, P_2), similarly to the proof of lemma 26.

Let us consider the cases what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_1$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$ where $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then by the same reasoning as in case 1 of the proof of lemma 26, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$.

On the other hand, the inversion of $\Gamma \models \exists\overrightarrow{\beta^-}_1.Q_1 \vee \exists\overrightarrow{\beta^-}_2.Q_2 = Q'$ gives us $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \models Q_1 \vee Q_2 = Q'$. Hence, by the induction hypothesis, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$.

Since both Q and Q' are sound, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$ can be strengthened to $\Gamma \vdash Q \geqslant_1 Q'$.

Ilya: lemma!

Case 2. $(P_1 = \alpha^+ \text{ and } P_2 = {\downarrow}N)$ or $(P_1 = {\downarrow}N \text{ and } P_2 = \alpha^+)$ or $(P_1 = \alpha^+ \text{ and } P_2 = \beta^+)$

By the same argument as in case 2 of the proof of lemma 26, the set of common supertypes of P_1 and P_2 is empty, hence contradiction.

Case 3. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Since $Q \in \mathsf{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists\overrightarrow{\alpha^-}.\alpha^+$. Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant_1 \alpha^+$ by Rule ($\exists^{\geqslant_1}$): $\overrightarrow{\alpha^-}$ can be instantiated with arbitrary negative types (for example $\forall\beta^+.{\uparrow}\beta^+$), since the substitution for unused variables does not change the term $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule ($\mathrm{Var}^{+\geqslant_1}$).

Case 4. $P_1 = {\downarrow}M_1$ and $P_2 = {\downarrow}M_2$

By the same reasoning as in case 6 of the proof of lemma 26, $\mathbf{nf}(Q) = \exists\overrightarrow{\alpha^-}.{\downarrow}M'$ for some $\overrightarrow{\alpha^-}$ and ${\downarrow}M'$ such that there exist $\overrightarrow{N}_1$ and $\overrightarrow{N}_2$ such that:

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$,

(ii) $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma \vdash \overrightarrow{N}_1$,

(iii) $[\overrightarrow{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_1$ and $[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}M_2$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative antiunification variable $\widehat{\alpha}^-$, and denote the list of these variables as $\widehat{\overrightarrow{\alpha^-}}$. As shown in case 6 of the proof of lemma 26, $(\widehat{\overrightarrow{\alpha^-}}, [\widehat{\overrightarrow{\alpha^-}}/\overrightarrow{\alpha^-}]{\downarrow}M', \overrightarrow{N}_1/\widehat{\overrightarrow{\alpha^-}}, \overrightarrow{N}_2/\widehat{\overrightarrow{\alpha^-}})$ is a sound anti-unifier of ${\downarrow}M_1$ and ${\downarrow}M_2$ in context Γ.

By the inversion of $\Gamma \models {\downarrow}M_1 \vee {\downarrow}M_2 = Q'$, we conclude that $Q' = \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, where $(\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$ is the result of the antiunification of ${\downarrow}M_1$ and ${\downarrow}M_2$ in context Γ.

Then by the initiality of the anti-unification (lemma 7), there exisits $\widehat{\tau}$ such that $\Gamma; \Xi \vdash \widehat{\tau} : \widehat{\overrightarrow{\alpha^-}}$ and $[\widehat{\tau}][\widehat{\overrightarrow{\alpha^-}}/\overrightarrow{\alpha^-}]{\downarrow}M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\alpha^-}/\widehat{\overrightarrow{\alpha^-}}$, (ii) $\widehat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi][\widehat{\tau}][\widehat{\overrightarrow{\alpha^-}}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi]P$. In particular, from the reflexivity of subtyping: $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]{\downarrow}M' \geqslant_1 [\overrightarrow{\beta^-}/\Xi]P$.

It allows us to show $\Gamma \vdash \mathbf{nf}(Q) \geqslant_1 Q'$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.{\downarrow}M' \geqslant_1 \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, by applying Rule ($\exists^{\geqslant_1}$), instantiating $\overrightarrow{\alpha^-}$ with respect to σ. Finally, $\Gamma \vdash Q \geqslant_1 Q'$ since $\Gamma \vdash \mathbf{nf}(Q) \simeq_1^{\leqslant} Q$, and equivalence implies subtyping by Ilya: lemma.

$\square$

Lemma 28 (Soundness of Upgrade). *For $\Delta \subseteq \Gamma$, suppose that $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$.*

(i) $\Delta \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P$

Lemma 29 (Completeness of Upgrade). *For $\Delta \subseteq \Gamma$, $\Gamma \vdash P$ and $\Delta \vdash Q'$, such that $\Gamma \vdash Q' \geqslant_1 P$, there exists Q s.t. $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$, and $\Delta \vdash Q' \geqslant_1 Q$.*