α, β, α, β, γ, δ	type variables
n, m, i, j	index variables

$n,\ m$	$::=$			cohort index
	$\vert$	0		
	$\vert$	$n+1$		
$\alpha^+,\ \beta^+,\ \gamma^+,\ \delta^+$	$::=$			positive variable
	$\vert$	α^+		
	$\vert$	α^{+n}		
	$\vert$	$\beta^\pm$		
$\alpha^-,\ \beta^-,\ \gamma^-,\ \delta^-$	$::=$			negative variable
	$\vert$	α^-		
	$\vert$	α^{-n}		
	$\vert$	$\alpha^\pm$		
$\alpha^\pm,\ \beta^\pm$	$::=$			positive or negative variable
	$\vert$	$\alpha^\pm$		
	$\vert$	$\alpha^{\pm n}$		
σ	$::=$			substitution
	$\vert$	$\cdot$		
	$\vert$	id		
	$\vert$	P/α^+		
	$\vert$	N/α^-		
	$\vert$	$\vec{P}/\overrightarrow{\alpha^+}$		
	$\vert$	$\vec{N}/\overrightarrow{\alpha^-}$		
	$\vert$	$\mathbf{pmas}\,/\overrightarrow{\alpha^+}$		
	$\vert$	$\mathbf{nmas}\,/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	$\vert$	$\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha^-}}$		
	$\vert$	μ		
	$\vert$	$\widehat{\sigma}$		
	$\vert$	$\sigma_1 \circ \sigma_2$		
	$\vert$	$\vec{\alpha}_1/\vec{\alpha}_2$		
	$\vert$	(σ)	S	
	$\vert$	$\overline{\sigma_i}^{\,i}$		concatenate
	$\vert$	$\mathbf{nf}\,(\sigma')$	M	
	$\vert$	$\sigma'\vert_{vars}$	M	
e	$::=$			entry of a unification solution
	$\vert$	$\widehat{\alpha}^+ :\approx P$		
	$\vert$	$\widehat{\alpha}^- :\approx N$		
	$\vert$	$\widehat{\alpha}^+ :\geqslant P$		
	$\vert$	(e)	S	
	$\vert$	$\widehat{\sigma}(\widehat{\alpha}^+)$	M	
	$\vert$	$\widehat{\sigma}(\widehat{\alpha}^-)$	M	
	$\vert$	$e_1 \ \&\ e_2$	M	
$\widehat{\sigma}$	$::=$			unification solution (substitution)

	$\mid$	$\cdot$		
	$\mid$	e		
	$\mid$	$\widehat{\sigma}\backslash vars$		
	$\mid$	$\widehat{\sigma}_1 \cup \widehat{\sigma}_2$		
	$\mid$	$\overline{\widehat{\sigma}_i}^{\,i}$		concatenate
	$\mid$	$(\widehat{\sigma})$	S	
	$\mid$	$\widehat{\sigma}_1 \ \& \ \widehat{\sigma}_2$	M	
$\widehat{\tau}$	$::=$			anti-unification substitution
	$\mid$	$\cdot$		
	$\mid$	$\widehat{\alpha}^- :\approx N$		
	$\mid$	$\widehat{\alpha}^- :\approx N$		
	$\mid$	$\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}$		
	$\mid$	$\overrightarrow{N}/\overrightarrow{\widehat{\alpha}^-}$		
	$\mid$	$\widehat{\tau}_1 \cup \widehat{\tau}_2$		
	$\mid$	$\overline{\widehat{\tau}_i}^{\,i}$		concatenate
	$\mid$	$(\widehat{\tau})$	S	
$P,\ Q$	$::=$			positive types
	$\mid$	α^+		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists \alpha^-.P$		
	$\mid$	$[\sigma]P$	M	
$N,\ M$	$::=$			negative types
	$\mid$	α^-		
	$\mid$	$\uparrow P$		
	$\mid$	$\forall \alpha^+.N$		
	$\mid$	$P \to N$		
	$\mid$	$[\sigma]N$	M	
$\overrightarrow{\alpha^+},\ \overrightarrow{\beta^+},\ \overrightarrow{\gamma^+},\ \overrightarrow{\delta^+}$	$::=$			positive variable list
	$\mid$	$\cdot$		empty list
	$\mid$	α^+		a variable
	$\mid$	$\overrightarrow{\alpha^+}$		a variable
	$\mid$	$\overline{\overrightarrow{\alpha^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^-},\ \overrightarrow{\beta^-},\ \overrightarrow{\gamma^-},\ \overrightarrow{\delta^-}$	$::=$			negative variables
	$\mid$	$\cdot$		empty list
	$\mid$	α^-		a variable
	$\mid$	$\overrightarrow{\alpha^-}$		variables
	$\mid$	$\overline{\overrightarrow{\alpha^-}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^\pm},\ \overrightarrow{\beta^\pm},\ \overrightarrow{\gamma^\pm},\ \overrightarrow{\delta^\pm}$	$::=$			positive or negative variable list
	$\mid$	$\cdot$		empty list
	$\mid$	$\alpha^\pm$		a variable
	$\mid$	$\overrightarrow{\mathbf{pa}}$		variables
	$\mid$	$\overline{\overrightarrow{\alpha^\pm}_i}^{\,i}$		concatenate lists

$P,\ Q$	$::=$			multi-quantified positive types
	$\mid$	α^{+}		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists\overrightarrow{\alpha^{-}}.P$		$P \neq \exists\ldots$
	$\mid$	$[\sigma]P$	M	
	$\mid$	$[\widehat{\tau}]P$	M	
	$\mid$	$[\widehat{\sigma}]P$	M	
	$\mid$	$[\mu]P$	M	
	$\mid$	(P)	S	
	$\mid$	$\mathbf{nf}\,(P')$	M	
$N,\ M$	$::=$			multi-quantified negative types
	$\mid$	α^{-}		
	$\mid$	$\uparrow P$		
	$\mid$	$P \to N$		
	$\mid$	$\forall\overrightarrow{\alpha^{+}}.N$		$N \neq \forall\ldots$
	$\mid$	$[\sigma]N$	M	
	$\mid$	$[\mu]N$	M	
	$\mid$	$[\widehat{\sigma}]N$	M	
	$\mid$	(N)	S	
	$\mid$	$\mathbf{nf}\,(N')$	M	
$\vec{P},\ \vec{Q}$	$::=$			list of positive types
	$\mid$	$\cdot$		empty list
	$\mid$	P		a singel type
	$\mid$	$\overline{\vec{P}_i}^{\,i}$		concatenate lists
	$\mid$	$\mathbf{nf}\,(\vec{P}')$	M	
$\vec{N},\ \vec{M}$	$::=$			list of negative types
	$\mid$	$\cdot$		empty list
	$\mid$	N		a singel type
	$\mid$	$\overline{\vec{N}_i}^{\,i}$		concatenate lists
	$\mid$	$\mathbf{nf}\,(\vec{N}')$	M	
$\Delta,\ \Gamma$	$::=$			declarative type context
	$\mid$	$\cdot$		empty context
	$\mid$	$\overrightarrow{\alpha^{+}}$		list of variables
	$\mid$	$\overrightarrow{\alpha^{-}}$		list of variables
	$\mid$	$\overrightarrow{\alpha^{\pm}}$		list of variables
	$\mid$	$vars$		
	$\mid$	$\overline{\Gamma_i}^{\,i}$		concatenate contexts
	$\mid$	(Γ)	S	
Θ	$::=$			unification type variable context
	$\mid$	$\cdot$		empty context
	$\mid$	$\overrightarrow{\widetilde{\alpha}^{+}}$		list of variables
	$\mid$	$\overrightarrow{\widehat{\alpha}^{-}}$		list of variables
	$\mid$	$vars$		
	$\mid$	$\overline{\Theta_i}^{\,i}$		concatenate contexts
	$\mid$	(Θ)	S	

	$\mid$	$\Theta\vert_{vars}$		leave only those variables that are in the set
Ξ	$::=$			anti-unification type variable context
	$\mid$	$\cdot$		empty context
	$\mid$	$\overrightarrow{\widetilde{\alpha}^{+}}$		list of variables
	$\mid$	$\overrightarrow{\widehat{\alpha}^{-}}$		list of variables
	$\mid$	$\overline{\Xi_i}^{\,i}$		concatenate contexts
	$\mid$	(Ξ)	S	
	$\mid$	$\Xi_1 \cup \Xi_2$		
$\vec{\alpha},\ \vec{\beta}$	$::=$			ordered positive or negative variables
	$\mid$	$\cdot$		empty list
	$\mid$	$\overrightarrow{\alpha^{+}}$		list of variables
	$\mid$	$\overrightarrow{\alpha^{-}}$		list of variables
	$\mid$	$\overrightarrow{\alpha^{\pm}}$		list of variables
	$\mid$	$\overrightarrow{\widetilde{\alpha}^{+}}$		list of variables
	$\mid$	$\overrightarrow{\widehat{\alpha}^{-}}$		list of variables
	$\mid$	$\vec{\alpha}_1 \backslash vars$		setminus
	$\mid$	Γ		context
	$\mid$	$vars$		
	$\mid$	$\overline{\vec{\alpha}_i}^{\,i}$		concatenate contexts
	$\mid$	$(\vec{\alpha})$	S	parenthesis
	$\mid$	$[\mu]\vec{\alpha}$	M	apply moving to list
	$\mid$	$\mathbf{ord}\ vars\ \mathbf{in}\ P$	M	
	$\mid$	$\mathbf{ord}\ vars\ \mathbf{in}\ N$	M	
	$\mid$	$\mathbf{ord}\ vars\ \mathbf{in}\ P$	M	
	$\mid$	$\mathbf{ord}\ vars\ \mathbf{in}\ N$	M	
$vars$	$::=$			set of variables
	$\mid$	$\varnothing$		empty set
	$\mid$	$\mathbf{fv}\ P$		free variables
	$\mid$	$\mathbf{fv}\ N$		free variables
	$\mid$	$\mathbf{fv}\ \mathbf{imP}$		free variables
	$\mid$	$\mathbf{fv}\ \mathbf{imN}$		free variables
	$\mid$	$vars_1 \cap vars_2$		set intersection
	$\mid$	$vars_1 \cup vars_2$		set union
	$\mid$	$vars_1 \backslash vars_2$		set complement
	$\mid$	$\mathbf{mv}\ \mathbf{imP}$		movable variables
	$\mid$	$\mathbf{mv}\ \mathbf{imN}$		movable variables
	$\mid$	$\mathbf{uv}\ N$		unification variables
	$\mid$	$\mathbf{uv}\ P$		unification variables
	$\mid$	$\mathbf{fv}\ N$		free variables
	$\mid$	$\mathbf{fv}\ P$		free variables
	$\mid$	$(vars)$	S	parenthesis
	$\mid$	$\vec{\alpha}$		ordered list of variables
	$\mid$	$[\mu]vars$	M	apply moving to varset
μ	$::=$			
	$\mid$	$\cdot$		empty moving
	$\mid$	$pma1 \mapsto pma2$		Positive unit substitution

	|	$nma1 \mapsto nma2$		Positive unit substitution
	|	$\mu_1 \cup \mu_2$	M	Set-like union of movings
	|	$\mu_1 \circ \mu_2$	M	Composition
	|	$\overline{\mu_i}^{\,i}$		concatenate movings
	|	$\mu\|_{vars}$	M	restriction on a set
	|	μ^{-1}	M	inversion
	|	$\mathbf{nf}\,(\mu')$	M	
$\widehat{\alpha}^{\pm}$	::=			positive/negative unification variable
	|	$\widehat{\alpha}^{\pm}$		
$\widehat{\alpha}^{+}$	::=			positive unification variable
	|	$\widehat{\alpha}^{+}$		
	|	$\widehat{\alpha}^{+}\{\Delta\}$		
	|	$\widehat{\alpha}^{\pm}$		
$\widehat{\alpha}^{-},\ \widehat{\beta}^{-}$	::=			negative unification variable
	|	$\widehat{\alpha}^{-}$		
	|	$\widehat{\alpha}^{-}_{\{N,M\}}$		
	|	$\widehat{\alpha}^{-}\{\Delta\}$		
	|	$\widehat{\alpha}^{\pm}$		
$\overrightarrow{\widetilde{\alpha}^{+}},\ \overrightarrow{\widetilde{\beta}^{+}}$	::=			positive unification variable list
	|	$\cdot$		empty list
	|	$\widehat{\alpha}^{+}$		a variable
	|	$\overrightarrow{\widehat{\alpha}^{+}\{\Delta\}}$		from a normal variable
	|	$\overrightarrow{\widehat{\alpha}^{+}}$		from a normal variable, context unspecified
	|	$\overline{\overrightarrow{\widetilde{\alpha}^{+}}_i}^{\,i}$		concatenate lists
$\overrightarrow{\widehat{\alpha}^{-}},\ \overrightarrow{\widehat{\beta}^{-}}$	::=			negative unification variable list
	|	$\cdot$		empty list
	|	$\widehat{\alpha}^{-}$		a variable
	|	Ξ		from an antiunification context
	|	$\overrightarrow{\widehat{\alpha}^{-}\{\Delta\}}$		from a normal variable
	|	$\overrightarrow{\widehat{\alpha}^{-}}$		from a normal variable, context unspecified
	|	$\overline{\overrightarrow{\widehat{\alpha}^{-}}_i}^{\,i}$		concatenate lists
$P,\ Q$	::=			a positive algorithmic type (potentially with metavariables)
	|	α^{+}		
	|	**pma**		
	|	$\widehat{\alpha}^{+}$		
	|	$\downarrow N$		
	|	$\exists \overrightarrow{\alpha^{-}}.P$		
	|	$[\sigma]P$	M	
	|	$[\widehat{\tau}]P$	M	
	|	$[\mu]P$	M	
	|	(P)	S	
	|	$\mathbf{nf}\,(P')$	M	

N, M	::=			a negative algorithmic type (potentially with metavariables)
	|	α^-		
	|	$\widehat{\alpha}^-$		
	|	$\uparrow P$		
	|	$P \rightarrow N$		
	|	$\forall \overrightarrow{\alpha^+}. N$		
	|	$[\sigma] N$	M	
	|	$[\mu] N$	M	
	|	(N)	S	
	|	$\mathbf{nf}\,(N')$	M	
auSol	::=			
	|	$(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$		
terminals	::=			
	|	$\exists$		
	|	$\forall$		
	|	$\uparrow$		
	|	$\downarrow$		
	|	$\rightarrow$		
	|	$\leftrightarrow$		
	|	$\in$		
	|	$\notin$		
	|	$\cdot$		
	|	$\vdash$		
	|	$\leqslant$		
	|	$\geqslant$		
	|	$\simeq$		
	|	$\cup$		
	|	$\cap$		
	|	$\setminus$		
	|	$\subseteq$		
	|	$\mapsto$		
	|	$\stackrel{u}{\simeq}$		
	|	$\stackrel{a}{\simeq}$		
	|	$\varnothing$		
	|	$\circ$		
	|	$\Rightarrow$		
	|	$\models$		
	|	$\dashv$		
	|	$\neq$		
	|	$\equiv_n$		
	|	$\vee$		
	|	$\Downarrow$		
	|	$:\geqslant$		
	|	$:\simeq$		
formula	::=			
	|	*judgement*		
	|	$formula_1 \quad .. \quad formula_n$		

| $\mu : \mathit{vars}_1 \leftrightarrow \mathit{vars}_2$
| μ **is bijective**
| $\widehat{\sigma}$ **is functional**
| $\widehat{\sigma}_1 \in \widehat{\sigma}_2$
| $\mathit{vars}_1 \subseteq \mathit{vars}_2$
| $\mathit{vars}_1 = \mathit{vars}_2$
| vars **is fresh**
| $\alpha^- \notin \mathit{vars}$
| $\alpha^+ \notin \mathit{vars}$
| $\alpha^- \in \mathit{vars}$
| $\alpha^+ \in \mathit{vars}$
| $\widehat{\alpha}^- \in \Theta$
| $\widehat{\alpha}^+ \in \Theta$
| if any other rule is not applicable
| $\vec{\alpha}_1 = \vec{\alpha}_2$
| $N \neq M$
| $P \neq Q$

A ::=
| $\Gamma; \Theta \models N \leqslant M \dashv \widehat{\sigma}$ — Negative subtyping
| $\Gamma; \Theta \models P \geqslant Q \dashv \widehat{\sigma}$ — Positive supertyping

AU ::=
| $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$
| $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$

$E1$ ::=
| $N \simeq_1^D M$ — Negative multi-quantified type equivalence
| $P \simeq_1^D Q$ — Positive multi-quantified type equivalence
| $P \simeq Q$

$D1$ ::=
| $\Gamma \vdash N \simeq_1^{\leqslant} M$ — Negative equivalence on MQ types
| $\Gamma \vdash P \simeq_1^{\leqslant} Q$ — Positive equivalence on MQ types
| $\Gamma \vdash N \leqslant_1 M$ — Negative subtyping
| $\Gamma \vdash P \geqslant_1 Q$ — Positive supertyping
| $\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1$ — Equivalence of substitutions

$D0$ ::=
| $\Gamma \vdash N \simeq_0^{\leqslant} M$ — Negative equivalence
| $\Gamma \vdash P \simeq_0^{\leqslant} Q$ — Positive equivalence
| $\Gamma \vdash N \leqslant_0 M$ — Negative subtyping
| $\Gamma \vdash P \geqslant_0 Q$ — Positive supertyping

EQ ::=
| $N = M$ — Negative type equality (alpha-equivalence)
| $P = Q$ — Positive type equuality (alphha-equivalence)
| $P = Q$

LUBF	::=		
	|	$\textbf{ord}\ \mathit{vars}\ \textbf{in}\ P\ ===\ \vec{\alpha}$	
	|	$\textbf{ord}\ \mathit{vars}\ \textbf{in}\ N\ ===\ \vec{\alpha}$	
	|	$\textbf{ord}\ \mathit{vars}\ \textbf{in}\ P\ ===\ \vec{\alpha}$	
	|	$\textbf{ord}\ \mathit{vars}\ \textbf{in}\ N\ ===\ \vec{\alpha}$	
	|	$\textbf{nf}\ (N')\ ===\ N$	
	|	$\textbf{nf}\ (P')\ ===\ P$	
	|	$\textbf{nf}\ (N')\ ===\ N$	
	|	$\textbf{nf}\ (P')\ ===\ P$	
	|	$\textbf{nf}\ (\vec{N}')\ ===\ \vec{N}$	
	|	$\textbf{nf}\ (\vec{P}')\ ===\ \vec{P}$	
	|	$\textbf{nf}\ (\sigma')\ ===\ \sigma$	
	|	$\textbf{nf}\ (\mu')\ ===\ \mu$	
	|	$\sigma'\|_{vars}$	
	|	$e_1\ \&\ e_2$	
	|	$\widehat{\sigma}_1\ \&\ \widehat{\sigma}_2$	
LUB	::=		
	|	$\Gamma \models P_1 \vee P_2 = Q$	Least Upper Bound (Least Common Supertype)
	|	$\textbf{upgrade}\ \Gamma \vdash P\ \textbf{to}\ \Delta = Q$	
Nrm	::=		
	|	$\textbf{nf}\ (N) = M$	
	|	$\textbf{nf}\ (P) = Q$	
	|	$\textbf{nf}\ (N) = M$	
	|	$\textbf{nf}\ (P) = Q$	
Order	::=		
	|	$\textbf{ord}\ \mathit{vars}\ \textbf{in}\ N = \vec{\alpha}$	
	|	$\textbf{ord}\ \mathit{vars}\ \textbf{in}\ P = \vec{\alpha}$	
	|	$\textbf{ord}\ \mathit{vars}\ \textbf{in}\ N = \vec{\alpha}$	
	|	$\textbf{ord}\ \mathit{vars}\ \textbf{in}\ P = \vec{\alpha}$	
SM	::=		
	|	$\Theta \vdash e_1\ \&\ e_2 = e_3$	Unification Solution Entry Merge
	|	$\Theta \vdash \widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3$	Merge unification solutions
SWEAK	::=		
	|	$\Theta \vdash e_1 \Rightarrow e_2$	Weakening of unification solution entries
	|	$\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2$	Weakening of unification solutions
U	::=		
	|	$\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}$	Negative unification
	|	$\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}$	Positive unification
WF	::=		
	|	$\Gamma \vdash N$	Negative type well-formedness
	|	$\Gamma \vdash P$	Positive type well-formedness
	|	$\Gamma \vdash N$	Negative type well-formedness

	$\vert$	$\Gamma \vdash P$	Positive type well-formedness
	$\vert$	$\Gamma \vdash \overrightarrow{N}$	Negative type list well-formedness
	$\vert$	$\Gamma \vdash \overrightarrow{P}$	Positive type list well-formedness
	$\vert$	$\Gamma; \Theta \vdash N$	Negative unification type well-formedness
	$\vert$	$\Gamma; \Theta \vdash P$	Positive unifiication type well-formedness
	$\vert$	$\Gamma; \Xi \vdash P$	Positive anti-unification type well-formedness
	$\vert$	$\Gamma; \Xi_2 \vdash \widehat{\tau} : \Xi_1$	Antiunification substitution well-formedness
	$\vert$	$\widehat{\sigma} : \Theta$	Unification substitution well-formedness
	$\vert$	$\Gamma \vdash^{\supseteq} \Theta$	Unification context well-formedness
	$\vert$	$\Gamma_1 \vdash \sigma : \Gamma_2$	Substitution well-formedness
judgement	::=		
	$\vert$	*A*	
	$\vert$	*AU*	
	$\vert$	*E1*	
	$\vert$	*D1*	
	$\vert$	*D0*	
	$\vert$	*EQ*	
	$\vert$	*LUB*	
	$\vert$	*Nrm*	
	$\vert$	*Order*	
	$\vert$	*SM*	
	$\vert$	*SWEAK*	
	$\vert$	*U*	
	$\vert$	*WF*	
user_syntax	::=		
	$\vert$	α	
	$\vert$	n	
	$\vert$	n	
	$\vert$	α^+	
	$\vert$	α^-	
	$\vert$	$\alpha^\pm$	
	$\vert$	σ	
	$\vert$	e	
	$\vert$	$\widehat{\sigma}$	
	$\vert$	$\widehat{\tau}$	
	$\vert$	P	
	$\vert$	N	
	$\vert$	$\overrightarrow{\alpha^+}$	
	$\vert$	$\overrightarrow{\alpha^-}$	
	$\vert$	$\overrightarrow{\alpha^\pm}$	
	$\vert$	P	
	$\vert$	N	
	$\vert$	$\overrightarrow{P}$	
	$\vert$	$\overrightarrow{N}$	
	$\vert$	Γ	
	$\vert$	Θ	
	$\vert$	Ξ	
	$\vert$	$\overrightarrow{\alpha}$	

$$
\begin{array}{ll}
| & vars \\
| & \mu \\
| & \widehat{\alpha}^{\pm} \\
| & \widehat{\alpha}^{+} \\
| & \widehat{\alpha}^{-} \\
| & \overrightarrow{\widehat{\alpha}^{+}} \\
| & \overrightarrow{\widehat{\alpha}^{-}} \\
| & P \\
| & N \\
| & auSol \\
| & terminals \\
| & formula
\end{array}
$$

$\boxed{\Gamma;\, \Theta \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma;\, \Theta \models \alpha^- \leqslant \alpha^- \dashv \cdot}\ \text{ANVar}$$

$$\frac{\Gamma; \Theta \models \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma;\, \Theta \models {\uparrow}P \leqslant {\uparrow}Q \dashv \widehat{\sigma}}\ \text{AShiftU}$$

$$\frac{\Gamma;\, \Theta \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma;\, \Theta \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma;\, \Theta \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\ \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+};\, \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \models [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma;\, \Theta \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}}\ \text{AForall}$$

$\boxed{\Gamma;\, \Theta \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma;\, \Theta \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \text{APVar}$$

$$\frac{\Gamma; \Theta \models \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma;\, \Theta \models {\downarrow}N \geqslant {\downarrow}M \dashv \widehat{\sigma}}\ \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-};\, \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \models [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma;\, \Theta \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}}\ \text{AExists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{\Gamma;\, \Theta \models \widehat{\alpha}^+ \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant Q)}\ \text{APUVar}$$

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)}\ \text{AUPVar}$$

$$\frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models {\downarrow}N_1 \overset{a}{\simeq} {\downarrow}N_2 \dashv (\Xi, {\downarrow}M, \widehat{\tau}_1, \widehat{\tau}_2)}\ \text{AUPShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \models \exists\overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists\overrightarrow{\alpha^-}.Q, \widehat{\tau}_1, \widehat{\tau}_2)}\ \text{AUPExists}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)} \quad \text{AUNVar}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad \text{AUNShift}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)} \quad \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, \widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad \text{AUNAU}$$

$\boxed{N \simeq^D_1 M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq^D_1 \alpha^-} \quad \text{E1NVar}$$

$$\frac{P \simeq^D_1 Q}{\uparrow P \simeq^D_1 \uparrow Q} \quad \text{E1ShiftU}$$

$$\frac{P \simeq^D_1 Q \quad N \simeq^D_1 M}{P \to N \simeq^D_1 Q \to M} \quad \text{E1Arrow}$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq^D_1 [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq^D_1 \forall \overrightarrow{\beta^+}.M} \quad \text{E1Forall}$$

$\boxed{P \simeq^D_1 Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq^D_1 \alpha^+} \quad \text{E1PVar}$$

$$\frac{N \simeq^D_1 M}{\downarrow N \simeq^D_1 \downarrow M} \quad \text{E1ShiftD}$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq^D_1 [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq^D_1 \exists \overrightarrow{\beta^-}.Q} \quad \text{E1Exists}$$

$\boxed{P \simeq Q}$

$\boxed{\Gamma \vdash N \simeq^{\leqslant}_1 M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq^{\leqslant}_1 M} \quad \text{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq^{\leqslant}_1 Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq^{\leqslant}_1 Q} \quad \text{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad \text{D1NVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \quad \text{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \text{ D1Arrow}$$

$$\frac{\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M} \text{ D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \text{ D1PVar}$$

$$\frac{\text{<<multiple parses>>}}{\Gamma \vdash {\downarrow} N \geqslant_1 {\downarrow} M} \text{ D1ShiftD}$$

$$\frac{\mathbf{fv}\, P \cap \overrightarrow{\beta^-} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q} \text{ D1Exists}$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1}$ Equivalence of substitutions

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \text{ D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \text{ D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \text{ D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash {\uparrow} P \leqslant_0 {\uparrow} Q} \text{ D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \text{ D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \text{ D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \text{ D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \text{ D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash {\downarrow} N \geqslant_0 {\downarrow} M} \text{ D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \text{ D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \text{ D0ExistsR}$$

$\boxed{N = M}$ Negative type equality (alpha-equivalence)

$\boxed{P = Q}$ Positive type equuality (alphha-equivalence)

$\boxed{P = Q}$
$\boxed{\textbf{ord}\ vars\ \textbf{in}\ P}$

$\boxed{\textbf{ord}\ vars\ \textbf{in}\ N}$

$\boxed{\textbf{ord}\ vars\ \textbf{in}\ P}$

$\boxed{\textbf{ord}\ vars\ \textbf{in}\ N}$

$\boxed{\textbf{nf}\ (N')}$

$\boxed{\textbf{nf}\ (P')}$

$\boxed{\textbf{nf}\ (N')}$

$\boxed{\textbf{nf}\ (P')}$

$\boxed{\textbf{nf}\ (\overrightarrow{N'})}$

$\boxed{\textbf{nf}\ (\overrightarrow{P'})}$

$\boxed{\textbf{nf}\ (\sigma')}$

$\boxed{\textbf{nf}\ (\mu')}$

$\boxed{\sigma'|_{vars}}$

$\boxed{e_1\ \&\ e_2}$

$\boxed{\widehat{\sigma}_1\ \&\ \widehat{\sigma}_2}$

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+} \quad \textsc{LUBVar}$$

$$\frac{\Gamma, \cdot \vDash \mathbf{nf}\,(\downarrow N) \overset{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P} \quad \textsc{LUBShift}$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q} \quad \textsc{LUBExists}$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

$$\frac{\Gamma = \Delta, \overrightarrow{\alpha^{\pm}} \quad \overrightarrow{\beta^{\pm}}\,\mathbf{is\,fresh} \quad \overrightarrow{\gamma^{\pm}}\,\mathbf{is\,fresh} \qquad \Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vDash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P = Q}{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q} \quad \textsc{LUBUpgrade}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-} \quad \textsc{NrmNVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\uparrow P) = \uparrow Q} \quad \textsc{NrmShiftU}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(P \to N) = Q \to M} \quad \textsc{NrmArrow}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\forall \overrightarrow{\alpha^+}.N) = \forall \overrightarrow{\alpha^+}'.N'} \quad \textsc{NrmForall}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+} \quad \textsc{NrmPVar}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\downarrow N) = \downarrow M} \quad \textsc{NrmShiftD}$$

$$\frac{\texttt{<<multiple parses>>}}{\mathbf{nf}\,(\exists \overrightarrow{\alpha^-}.P) = \exists \overrightarrow{\alpha^-}'.P'} \quad \textsc{NrmExists}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-} \quad \textsc{NrmNUVar}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+} \quad \textsc{NrmPUVar}$$

$\boxed{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,N = \vec{\alpha}}$

$$\frac{\alpha^- \in \mathit{vars}}{\mathbf{ord}\,\mathit{vars}\,\mathbf{in}\,\alpha^- = \alpha^-} \quad \textsc{ONVarIn}$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \cdot}\ \text{ONVarNIn}$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \uparrow P = \vec{\alpha}}\ \text{OShiftU}$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ vars\ \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)}\ \text{OArrow}$$

$$\frac{vars \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}}\ \text{OForall}$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \alpha^+}\ \text{OPVarIn}$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \cdot}\ \text{OPVarNIn}$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \downarrow N = \vec{\alpha}}\ \text{OShiftD}$$

$$\frac{vars \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \exists \overrightarrow{\alpha^-}.P = \vec{\alpha}}\ \text{OExists}$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^- = \cdot}\ \text{ONUVar}$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^+ = \cdot}\ \text{OPUVar}$$

$\boxed{\Theta \vdash e_1\ \&\ e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{\widehat{\alpha}^+\{\Gamma\} \in \Theta \quad \Gamma \models P_1 \vee P_2 = Q}{\Theta \vdash (\widehat{\alpha}^+ :\geqslant P_1)\ \&\ (\widehat{\alpha}^+ :\geqslant P_2) = (\widehat{\alpha}^+ :\geqslant Q)}\ \text{SMESupSup}$$

$$\frac{\widehat{\alpha}^+\{\Gamma\} \in \Theta \quad \Gamma; \cdot \models P \geqslant Q \dashv \widehat{\sigma}'}{\Theta \vdash (\widehat{\alpha}^+ :\approx P)\ \&\ (\widehat{\alpha}^+ :\geqslant Q) = (\widehat{\alpha}^+ :\approx P)}\ \text{SMEEqSup}$$

$$\frac{\widehat{\alpha}^+\{\Gamma\} \in \Theta \quad \Gamma; \cdot \models Q \geqslant P \dashv \widehat{\sigma}'}{\Theta \vdash (\widehat{\alpha}^+ :\geqslant P)\ \&\ (\widehat{\alpha}^+ :\approx Q) = (\widehat{\alpha}^+ :\approx Q)}\ \text{SMESupEq}$$

$$\frac{}{\Theta \vdash (\widehat{\alpha}^+ :\approx P)\ \&\ (\widehat{\alpha}^+ :\approx P) = (\widehat{\alpha}^+ :\approx P)}\ \text{SMEPEqEq}$$

$$\frac{}{\Theta \vdash (\widehat{\alpha}^- :\approx N)\ \&\ (\widehat{\alpha}^- :\approx N) = (\widehat{\alpha}^- :\approx N)}\ \text{SMENEqEq}$$

$\boxed{\Theta \vdash \widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$\boxed{\Theta \vdash e_1 \Rightarrow e_2}$ Weakening of unification solution entries

$$\frac{\widehat{\alpha}^+\{\Gamma\} \in \Theta \quad \Gamma \vdash P_1 \geqslant_1 P_2}{\Theta \vdash (\widehat{\alpha}^+ :\geqslant P_1) \Rightarrow (\widehat{\alpha}^+ :\geqslant P_2)}\ \text{SWeakESupSup}$$

$$\frac{\widehat{\alpha}^+\{\Gamma\} \in \Theta \quad \Gamma \vdash P_1 \geqslant_1 P_2}{\Theta \vdash (\widehat{\alpha}^+ :\approx P_1) \Rightarrow (\widehat{\alpha}^+ :\geqslant P_2)}\ \text{SWeakEEqSup}$$

$$\frac{\texttt{<<multiple parses>>}}{\Theta \vdash (\widehat{\alpha}^{+} :\approx P_1) \Rightarrow (\widehat{\alpha}^{+} :\approx P_2)} \quad \textsc{SWeakeEqEq}$$

$\boxed{\Theta \vdash \widehat{\sigma}_1 \Rightarrow \widehat{\sigma}_2}$ Weakening of unification solutions

$\boxed{\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Gamma; \Theta \models \alpha^{-} \overset{u}{\simeq} \alpha^{-} \dashv \cdot} \quad \textsc{UNVar}$$

$$\frac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma; \Theta \models \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}} \quad \textsc{UShiftU}$$

$$\frac{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \models P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2} \quad \textsc{UArrow}$$

$$\frac{\Gamma, \overrightarrow{\alpha^{+}}; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma; \Theta \models \forall \overrightarrow{\alpha^{+}}. N \overset{u}{\simeq} \forall \overrightarrow{\alpha^{+}}. M \dashv \widehat{\sigma}} \quad \textsc{UForall}$$

$$\frac{\widehat{\alpha}^{-}\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma; \Theta \models \widehat{\alpha}^{-} \overset{u}{\simeq} N \dashv (\widehat{\alpha}^{-} :\approx N)} \quad \textsc{UNUVar}$$

$\boxed{\Gamma; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Gamma; \Theta \models \alpha^{+} \overset{u}{\simeq} \alpha^{+} \dashv \cdot} \quad \textsc{UPVar}$$

$$\frac{\Gamma; \Theta \models N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Gamma; \Theta \models \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}} \quad \textsc{UShiftD}$$

$$\frac{\Gamma, \overrightarrow{\alpha^{-}}; \Theta \models P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Gamma; \Theta \models \exists \overrightarrow{\alpha^{-}}. P \overset{u}{\simeq} \exists \overrightarrow{\alpha^{-}}. Q \dashv \widehat{\sigma}} \quad \textsc{UExists}$$

$$\frac{\widehat{\alpha}^{+}\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma; \Theta \models \widehat{\alpha}^{+} \overset{u}{\simeq} P \dashv (\widehat{\alpha}^{+} :\approx P)} \quad \textsc{UPUVar}$$

$\boxed{\Gamma \vdash N}$ Negative type well-formedness
$\boxed{\Gamma \vdash P}$ Positive type well-formedness
$\boxed{\Gamma \vdash N}$ Negative type well-formedness
$\boxed{\Gamma \vdash P}$ Positive type well-formedness
$\boxed{\Gamma \vdash \vec{N}}$ Negative type list well-formedness
$\boxed{\Gamma \vdash \vec{P}}$ Positive type list well-formedness
$\boxed{\Gamma; \Theta \vdash N}$ Negative unification type well-formedness
$\boxed{\Gamma; \Theta \vdash P}$ Positive unifiication type well-formedness
$\boxed{\Gamma; \Xi \vdash P}$ Positive anti-unification type well-formedness
$\boxed{\Gamma; \Xi_2 \vdash \widehat{\tau} : \Xi_1}$ Antiunification substitution well-formedness
$\boxed{\widehat{\sigma} : \Theta}$ Unification substitution well-formedness
$\boxed{\Gamma \vdash^{\supseteq} \Theta}$ Unification context well-formedness
$\boxed{\Gamma_1 \vdash \sigma : \Gamma_2}$ Substitution well-formedness

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Definition rules:         75 good    8 bad
Definition rule clauses: 138 good    8 bad
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