α, β	type variables
n, m, i, j	index variables

$\alpha^+,\ \beta^+$	$::=$		positive variable
	$\vert$	α^+	
$\alpha^-,\ \beta^-$	$::=$		negative variable
	$\vert$	α^-	
σ	$::=$		substitution
	$\vert$	$\cdot$	
	$\vert$	$P/a+$	
	$\vert$	$N/a-$	
	$\vert$	$\overrightarrow{P}/\overrightarrow{\alpha^+}$	
	$\vert$	$\overrightarrow{N}/\overrightarrow{\alpha^-}$	
	$\vert$	$\overrightarrow{\widetilde{\alpha}^+}/\overrightarrow{\alpha^+}$	
	$\vert$	$\overrightarrow{\widetilde{\alpha}^-}/\overrightarrow{\alpha^-}$	
	$\vert$	$\overrightarrow{\widetilde{\alpha}^+}/\overrightarrow{\alpha^+}$	
	$\vert$	$\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}$	
	$\vert$	$vars_1/vars_2$	
	$\vert$	$\overline{\sigma_i}^{\,i}$	concatenate
e	$::=$		entry of a unification solution
	$\vert$	$\widehat{\alpha}^+ :\approx P$	
	$\vert$	$\widehat{\alpha}^- :\approx N$	
	$\vert$	$\widehat{\alpha}^+ :\geqslant P$	
$\widehat{\sigma}$	$::=$		unification solution (substitution)
	$\vert$	$\cdot$	
	$\vert$	e	
	$\vert$	$\widehat{\sigma}\backslash\overrightarrow{\widetilde{\alpha}^+}$	
	$\vert$	$\widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^-}$	
	$\vert$	$\widehat{\sigma}\backslash\widehat{\alpha}^+$	
	$\vert$	$\widehat{\sigma}\backslash\widehat{\alpha}^-$	
	$\vert$	$\overline{\widehat{\sigma}_i}^{\,i}$	concatenate
	$\vert$	$(\widehat{\sigma})$ S	
	$\vert$	$\widehat{\sigma}_1 \& \widehat{\sigma}_2$ M	
$P,\ Q$	$::=$		positive types
	$\vert$	$a+$	
	$\vert$	$\downarrow N$	
	$\vert$	$\exists\alpha^-.P$	
	$\vert$	$[\sigma]P$ M	
$N,\ M$	$::=$		negative types
	$\vert$	$a-$	
	$\vert$	$\uparrow P$	
	$\vert$	$\forall\alpha^+.N$	
	$\vert$	$P \to N$	
	$\vert$	$[\sigma]N$ M	
$\overrightarrow{\alpha^+},\ \overrightarrow{\beta^+}$	$::=$		positive variable list

	$\vert$	$\cdot$		empty list
	$\vert$	α^+		a variable
	$\vert$	$\overline{\overrightarrow{\alpha^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^-},\ \overrightarrow{\beta^-}$	$::=$			negative variables
	$\vert$	$\cdot$		empty list
	$\vert$	α^-		a variable
	$\vert$	$\overline{\overrightarrow{\alpha^-}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$			multi-quantified positive types
	$\vert$	α^+		
	$\vert$	$\downarrow N$		
	$\vert$	$\exists\overrightarrow{\alpha^-}.P$		$P \neq \exists\ldots$
	$\vert$	$[\sigma]P$	M	
	$\vert$	$[\mu]P$	M	
	$\vert$	$P_1 \vee P_2$	M	
$N,\ M$	$::=$			multi-quantified negative types
	$\vert$	α^-		
	$\vert$	$\uparrow P$		
	$\vert$	$P \to N$		
	$\vert$	$\forall\overrightarrow{\alpha^+}.N$		$N \neq \forall\ldots$
	$\vert$	$[\sigma]N$	M	
	$\vert$	$[\mu]N$	M	
$\vec{P}$	$::=$			list of positive types
	$\vert$	$\cdot$		empty list
	$\vert$	P		a singel type
	$\vert$	$\overline{\vec{P}_i}^{\,i}$		concatenate lists
$\vec{N}$	$::=$			list of negative types
	$\vert$	$\cdot$		empty list
	$\vert$	N		a singel type
	$\vert$	$\overline{\vec{N}_i}^{\,i}$		concatenate lists
Γ	$::=$			declarative type context
	$\vert$	$\cdot$		empty context
	$\vert$	$vars$		
	$\vert$	$\overrightarrow{\alpha^+}$		list of variables
	$\vert$	$\overrightarrow{\alpha^-}$		list of variables
	$\vert$	$\overline{\Gamma_i}^{\,i}$		concatenate contexts
	$\vert$	(Γ)	S	
	$\vert$	$\Gamma_1 \cup \Gamma_2$		
$\vec{\alpha}$	$::=$			ordered positive or negative variables
	$\vert$	$\cdot$		empty set
	$\vert$	$\overrightarrow{\alpha^+}$		list of variables
	$\vert$	$\overrightarrow{\alpha^-}$		list of variables
	$\vert$	$\vec{\alpha}_1 \backslash \vec{\alpha}_2$		setminus

	|	$\overline{\overrightarrow{\alpha}_i}^{\,i}$		concatenate contexts
	|	$(\overrightarrow{\alpha})$	S	parenthesis
$vars$	::=			set of variables
	|	$\varnothing$		empty set
	|	$\mathbf{fv}\, P$		free variables
	|	$\mathbf{fv}\, N$		free variables
	|	$\mathbf{fv}\, P$		free variables
	|	$\mathbf{fv}\, N$		free variables
	|	$vars_1 \cap vars_2$		set intersection
	|	$vars_1 \cup vars_2$		set union
	|	$vars_1 \backslash vars_2$		set complement
	|	$\mathbf{mv}\, P$		movable variables
	|	$\mathbf{mv}\, N$		movable variables
	|	$\mathbf{uv}\, N$		unification variables
	|	$\mathbf{uv}\, P$		unification variables
	|	$(vars)$	S	parenthesis
	|	Γ		context
	|	$\overrightarrow{\alpha}$		ordered list of variables
μ	::=			
	|	$\cdot$		empty moving
	|	$\widetilde{\alpha}_1^+ \mapsto \widetilde{\alpha}_2^+$		Positive unit substitution
	|	$\widetilde{\alpha}_1^- \mapsto \widetilde{\alpha}_2^-$		Positive unit substitution
	|	$\mu_1 \cup \mu_2$	M	Set-like union of movings
	|	$\overline{\mu_i}^{\,i}$		concatenate movings
	|	$\mu\vert_{vars}$	M	restriction on a set
n	::=			cohort index
	|	0		
	|	$n+1$		
$\widetilde{\alpha}^+$	::=			positive movable variable
	|	$\widetilde{\alpha}^{+n}$		
$\widetilde{\alpha}^-$	::=			negative movable variable
	|	$\widetilde{\alpha}^{-n}$		
$\overrightarrow{\widetilde{\alpha}^+},\ \overrightarrow{\widetilde{\beta}^+}$	::=			positive movable variable list
	|	$\cdot$		empty list
	|	$\widetilde{\alpha}^+$		a variable
	|	$\overrightarrow{\widetilde{\alpha^{+n}}}$		from a non-movable variable
	|	$\overline{\overrightarrow{\widetilde{\alpha}^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\widetilde{\alpha}^-},\ \overrightarrow{\widetilde{\beta}^-}$	::=			negatiive movable variable list
	|	$\cdot$		empty list
	|	$\widetilde{\alpha}^-$		a variable
	|	$\overrightarrow{\widetilde{\alpha^{-n}}}$		from a non-movable variable
	|	$\overline{\overrightarrow{\widetilde{\alpha}^-}_i}^{\,i}$		concatenate lists

$P,\ Q$	$::=$			multi-quantified positive types with movable variables
	$\mid$	α^+		
	$\mid$	$\widetilde{\alpha}^+$		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists \overrightarrow{\alpha^-}.P$		
	$\mid$	$[\sigma]P$	M	
	$\mid$	$[\mu]P$	M	
$N,\ M$	$::=$			multi-quantified negative types with movable variables
	$\mid$	α^-		
	$\mid$	$\widetilde{\alpha}^-$		
	$\mid$	$\uparrow P$		
	$\mid$	$P \to N$		
	$\mid$	$\forall \overrightarrow{\alpha^+}.N$		
	$\mid$	$[\sigma]N$	M	
	$\mid$	$[\mu]N$	M	
$\widehat{\alpha}^+$	$::=$			positive unification variable
	$\mid$	$\widehat{\alpha}^+$		
$\widehat{\alpha}^-$	$::=$			negative unification variable
	$\mid$	$\widehat{\alpha}^-$		
$\overrightarrow{\widetilde{\alpha^+}},\ \overrightarrow{\widetilde{\beta^+}}$	$::=$			positive unification variable list
	$\mid$	$\cdot$		empty list
	$\mid$	$\widehat{\alpha}^+$		a variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^+\{vars\}}$		from a normal variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^+}$		from a normal variable, context unspecified
	$\mid$	$\overline{\overrightarrow{\widetilde{\alpha^+}}_i}^{\,i}$		concatenate lists
$\overrightarrow{\widehat{\alpha^-}},\ \overrightarrow{\widehat{\beta^-}}$	$::=$			negative unification variable list
	$\mid$	$\cdot$		empty list
	$\mid$	$\widehat{\alpha}^-$		a variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^-\{vars\}}$		from a normal variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^-}$		from a normal variable, context unspecified
	$\mid$	$\overline{\overrightarrow{\widehat{\alpha^-}}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$			a positive algorithmic type (potentially with metavariables)
	$\mid$	α^+		
	$\mid$	$\widetilde{\alpha}^+$		
	$\mid$	$\widehat{\alpha}^+\{vars\}$		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists \overrightarrow{\alpha^-}.P$		
	$\mid$	$[\sigma]P$	M	
	$\mid$	$[\mu]P$	M	
$N,\ M$	$::=$			a negative algorithmic type (potentially with metavariables)
	$\mid$	α^-		
	$\mid$	$\widetilde{\alpha}^-$		

	$\mid$	$\widehat{\alpha}^-\{vars\}$	
	$\mid$	$\uparrow P$	
	$\mid$	$P \rightarrow N$	
	$\mid$	$\forall \overrightarrow{\alpha^+}. N$	
	$\mid$	$[\sigma] N$	M
	$\mid$	$[\mu] N$	M
terminals	$::=$		
	$\mid$	$\exists$	
	$\mid$	$\forall$	
	$\mid$	$\uparrow$	
	$\mid$	$\downarrow$	
	$\mid$	$\rightarrow$	
	$\mid$	$\leftrightarrow$	
	$\mid$	$\in$	
	$\mid$	$\notin$	
	$\mid$	$\cdot$	
	$\mid$	$\vdash$	
	$\mid$	$\leqslant$	
	$\mid$	$\geqslant$	
	$\mid$	$\simeq$	
	$\mid$	$\cup$	
	$\mid$	$\cap$	
	$\mid$	$\setminus$	
	$\mid$	$\subseteq$	
	$\mid$	$\mapsto$	
	$\mid$	$\stackrel{u}{\simeq}$	
	$\mid$	$\stackrel{a}{\simeq}$	
	$\mid$	$\varnothing$	
	$\mid$	$\vDash$	
	$\mid$	$\dashv$	
	$\mid$	$\neq$	
	$\mid$	$\equiv_n$	
	$\mid$	$\vee$	
	$\mid$	$\Downarrow$	
formula	$::=$		
	$\mid$	$judgement$	
	$\mid$	$formula_1 \quad .. \quad formula_n$	
	$\mid$	$\mu : vars_1 \leftrightarrow vars_2$	
	$\mid$	μ **is bijective**	
	$\mid$	$\widehat{\sigma}$ **is functional**	
	$\mid$	$\widehat{\sigma}_1 \in \widehat{\sigma}_2$	
	$\mid$	$vars_1 \subseteq vars_2$	
	$\mid$	$vars_1 = vars_2$	
	$\mid$	$vars$ **is fresh**	
	$\mid$	$\alpha^- \notin vars$	
	$\mid$	$\alpha^+ \notin vars$	
	$\mid$	$\alpha^- \in vars$	
	$\mid$	$\alpha^+ \in vars$	

	$\mid$ $N \neq M$	
	$\mid$ $P \neq Q$	
E1A	$::=$	
	$\mid$ $n \models N \simeq_1^A M \dashv \mu$	Negative multi-quantified type equivalence (algorithmic)
	$\mid$ $n \models P \simeq_1^A Q \dashv \mu$	Positive multi-quantified type equivalence (algorithmic)
A	$::=$	
	$\mid$ $\Gamma \models N \leqslant M \dashv \widehat{\sigma}$	Negative subtyping
	$\mid$ $\Gamma \models P \geqslant Q \dashv \widehat{\sigma}$	Positive supertyping
E1	$::=$	
	$\mid$ $N \simeq_1^D M$	Negative multi-quantified type equivalence
	$\mid$ $P \simeq_1^D Q$	Positive multi-quantified type equivalence
D1	$::=$	
	$\mid$ $\Gamma \vdash N \simeq_1^{\leqslant} M$	Negative equivalence on MQ types
	$\mid$ $\Gamma \vdash P \simeq_1^{\leqslant} Q$	Positive equivalence on MQ types
	$\mid$ $\Gamma \vdash N \leqslant_1 M$	Negative subtyping
	$\mid$ $\Gamma \vdash P \geqslant_1 Q$	Positive supertyping
D0	$::=$	
	$\mid$ $\Gamma \vdash N \simeq_0^{\leqslant} M$	Negative equivalence
	$\mid$ $\Gamma \vdash P \simeq_0^{\leqslant} Q$	Positive equivalence
	$\mid$ $\Gamma \vdash N \leqslant_0 M$	Negative subtyping
	$\mid$ $\Gamma \vdash P \geqslant_0 Q$	Positive supertyping
LUBF	$::=$	
	$\mid$ $P_1 \vee P_2 === Q$	
	$\mid$ $\widehat{\sigma}_1 \& \widehat{\sigma}_2 === \widehat{\sigma}$	
LUB	$::=$	
	$\mid$ $P_1 \vee P_2 = Q$	Least Upper Bound (Least Common Supertype)
AU	$::=$	
	$\mid$ $n \models P_1 \stackrel{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2); \mu$	
	$\mid$ $n \models N_1 \stackrel{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2); \mu$	
Order	$::=$	
	$\mid$ $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \mathit{vars}'$	
	$\mid$ $\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \mathit{vars}'$	
Nrm	$::=$	
	$\mid$ $N \Downarrow M$	
	$\mid$ $P \Downarrow Q$	
SM	$::=$	
	$\mid$ $e_1 \& e_2 = e_3$	Unification Solution Entry Merge
	$\mid$ $\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3$	Merge unification solutions

U	$::=$		
	$\|$	$n \models \boxed{N} \overset{u}{\simeq} M \dashv \mu;\ \widehat{\sigma}$	Negative unification
	$\|$	$n \models \boxed{P} \overset{u}{\simeq} Q \dashv \mu;\ \widehat{\sigma}$	Positive unification
WF	$::=$		
	$\|$	$\Gamma \vdash N$	Negative type well-formedness
	$\|$	$\Gamma \vdash P$	Positive type well-formedness
	$\|$	$\Gamma \vdash N$	Negative type well-formedness
	$\|$	$\Gamma \vdash P$	Positive type well-formedness
$judgement$	$::=$		
	$\|$	$E1A$	
	$\|$	A	
	$\|$	$E1$	
	$\|$	$D1$	
	$\|$	$D0$	
	$\|$	LUB	
	$\|$	AU	
	$\|$	$Order$	
	$\|$	Nrm	
	$\|$	SM	
	$\|$	U	
	$\|$	WF	
$user\_syntax$	$::=$		
	$\|$	α	
	$\|$	n	
	$\|$	α^+	
	$\|$	α^-	
	$\|$	σ	
	$\|$	e	
	$\|$	$\widehat{\sigma}$	
	$\|$	P	
	$\|$	N	
	$\|$	$\overrightarrow{\alpha^+}$	
	$\|$	$\overrightarrow{\alpha^-}$	
	$\|$	P	
	$\|$	N	
	$\|$	$\overrightarrow{P}$	
	$\|$	$\overrightarrow{N}$	
	$\|$	Γ	
	$\|$	$\overrightarrow{\alpha}$	
	$\|$	$vars$	
	$\|$	μ	
	$\|$	n	
	$\|$	$\widetilde{\alpha}^+$	
	$\|$	$\widetilde{\alpha}^-$	
	$\|$	$\overrightarrow{\widetilde{\alpha}^+}$	
	$\|$	$\overrightarrow{\widetilde{\alpha}^-}$	
	$\|$	P	

$|\quad N$
$|\quad \widehat{\alpha}^+$
$|\quad \widehat{\alpha}^-$
$|\quad \overrightarrow{\widetilde{\alpha}^+}$
$|\quad \overrightarrow{\widehat{\alpha}^-}$
$|\quad P$
$|\quad N$
$|\quad \textit{terminals}$
$|\quad \textit{formula}$

$\boxed{n \models N \simeq_1^A M \dashv \mu}$ Negative multi-quantified type equivalence (algorithmic)

$$\frac{}{n \models \alpha^- \simeq_1^A \alpha^- \dashv \cdot} \quad \text{E1ANVar}$$

$$\frac{n \models P \simeq_1^A Q \dashv \mu}{n \models \uparrow P \simeq_1^A \uparrow Q \dashv \mu} \quad \text{E1AShiftU}$$

$$\frac{n \models P \simeq_1^A Q \dashv \mu_1 \quad n \models N \simeq_1^A M \dashv \mu_2 \quad \mu_1 \cup \mu_2 \ \textbf{is bijective}}{n \models P \to N \simeq_1^A Q \to M \dashv \mu_1 \cup \mu_2} \quad \text{E1AArrow}$$

$$\frac{n+1 \models [\overrightarrow{\widetilde{\alpha}^{+n}}/\overrightarrow{\alpha^+}]N \simeq_1^A [\overrightarrow{\widetilde{\beta}^{+n}}/\overrightarrow{\beta^+}]M \dashv \mu}{n \models \forall \overrightarrow{\alpha^+}.N \simeq_1^A \forall \overrightarrow{\beta^+}.M \dashv \mu|_{\mathbf{mv}\, M}} \quad \text{E1AForall}$$

$$\frac{}{n \models \widetilde{\alpha}^{-n} \simeq_1^A \widetilde{\beta}^{-n} \dashv \widetilde{\beta}^{-n} \mapsto \widetilde{\alpha}^{-n}} \quad \text{E1ANMVar}$$

$\boxed{n \models P \simeq_1^A Q \dashv \mu}$ Positive multi-quantified type equivalence (algorithmic)

$$\frac{}{n \models \alpha^+ \simeq_1^A \alpha^+ \dashv \cdot} \quad \text{E1APVar}$$

$$\frac{n \models N \simeq_1^A M \dashv \mu}{n \models \downarrow N \simeq_1^A \downarrow M \dashv \mu} \quad \text{E1AShiftD}$$

$$\frac{n+1 \models [\overrightarrow{\widetilde{\alpha}^{-n}}/\overrightarrow{\alpha^-}]P \simeq_1^A [\overrightarrow{\widetilde{\beta}^{-n}}/\overrightarrow{\beta^-}]Q \dashv \mu}{n \models \exists \overrightarrow{\alpha^-}.P \simeq_1^A \exists \overrightarrow{\beta^-}.Q \dashv \mu|_{\mathbf{mv}\, Q}} \quad \text{E1AExists}$$

$$\frac{}{n \models \widetilde{\alpha}^{+n} \simeq_1^A \widetilde{\beta}^{+n} \dashv \widetilde{\beta}^{+n} \mapsto \widetilde{\alpha}^{+n}} \quad \text{E1APMVar}$$

$\boxed{\Gamma \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma \models \alpha^- \leqslant \alpha^- \dashv \cdot} \quad \text{ANVar}$$

$$\frac{0 \models P \stackrel{u}{\simeq} Q \dashv \mu; \widehat{\sigma}}{\Gamma \models \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad \text{AShiftU}$$

$$\frac{\Gamma \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2} \quad \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \models [\overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}/\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma \models \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^+}} \quad \text{AForall}$$

$\boxed{\Gamma \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \textsc{APVar}$$

$$\frac{0 \models N \overset{u}{\simeq} M \dashv \mu; \widehat{\sigma}}{\Gamma \models {\downarrow} N \geqslant {\downarrow} M \dashv \widehat{\sigma}}\ \textsc{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \models [\overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}/\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma \models \exists \overrightarrow{\alpha^-}.P \geqslant \exists \overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}}\ \textsc{AExists}$$

$$\frac{vars_1 = \mathbf{fv}\, P \backslash vars \quad vars_2\ \textbf{is fresh}}{\Gamma \models \widehat{\alpha}^+\{vars\} \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant P \vee [vars_2/vars_1]P)}\ \textsc{APUVar}$$

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-}\ \textsc{E1NVar}$$

$$\frac{P \simeq_1^D Q}{{\uparrow} P \simeq_1^D {\uparrow} Q}\ \textsc{E1ShiftU}$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M}\ \textsc{E1Arrow}$$

$$\frac{\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M}\ \textsc{E1Forall}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+}\ \textsc{E1PVar}$$

$$\frac{N \simeq_1^D M}{{\downarrow} N \simeq_1^D {\downarrow} M}\ \textsc{E1ShiftD}$$

$$\frac{\mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q}\ \textsc{E1Exists}$$

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M}\ \textsc{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q}\ \textsc{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-}\ \textsc{D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash {\uparrow} P \leqslant_1 {\uparrow} Q}\ \textsc{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M}\ \textsc{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M} \quad \text{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad \text{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \quad \text{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q'}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q} \quad \text{D1ExistsL}$$

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \quad \text{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \quad \text{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash a- \leqslant_0 a-} \quad \text{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \quad \text{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/a+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \quad \text{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \quad \text{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \quad \text{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash a+ \geqslant_0 a+} \quad \text{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \quad \text{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/a-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \quad \text{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \quad \text{D0ExistsR}$$

$\boxed{P_1 \vee P_2}$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2}$

$\boxed{P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\alpha^+ \vee \alpha^+ = \alpha^+}\ \textsc{LUBVar}$$

$$\frac{0 \models {\downarrow}N \overset{a}{\simeq} {\downarrow}M \dashv (P, \hat{\sigma}_1, \hat{\sigma}_2); \mu}{{\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\mathbf{uv}\ P]P}\ \textsc{LUBShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing}{\exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = P_1 \vee P_2}\ \textsc{LUBExists}$$

$\boxed{n \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \hat{\sigma}_1, \hat{\sigma}_2); \mu}$

$$\frac{}{n \models \tilde{\alpha}^{+n} \overset{a}{\simeq} \tilde{\beta}^{+n} \dashv (\alpha^+, \cdot, \cdot); \tilde{\beta}^{+n} \mapsto \tilde{\alpha}^{+n}}\ \textsc{AUPPVar}$$

$\boxed{n \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \hat{\sigma}_1, \hat{\sigma}_2); \mu}$

$$\frac{}{n \models \tilde{\alpha}^{-n} \overset{a}{\simeq} \tilde{\beta}^{-n} \dashv (\alpha^-, \cdot, \cdot); \tilde{\beta}^{-n} \mapsto \tilde{\alpha}^{-n}}\ \textsc{AUNNVar}$$

$$\frac{\begin{array}{c} n \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \hat{\sigma}_1, \hat{\sigma}_2); \mu \\ n \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \hat{\sigma}'_1, \hat{\sigma}'_2); \mu' \end{array}}{n \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (Q \to M, \ , \cdot); \mu \cup \mu'}\ \textsc{AUNArrow}$$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \mathit{vars}'}$

$$\frac{\alpha^- \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^- = \alpha^-}\ \textsc{ONVarIn}$$

$$\frac{\alpha^- \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^- = \cdot}\ \textsc{ONVarNIn}$$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \hat{\alpha}^-\{\mathit{vars}'\} = \cdot}\ \textsc{ONUVar}$$

$$\frac{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ {\uparrow}P = \vec{\alpha}}\ \textsc{OShiftU}$$

$$\frac{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)}\ \textsc{OArrow}$$

$$\frac{\mathbf{ord}\ (\mathit{vars} \backslash \overrightarrow{\alpha^+})\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.N = \vec{\alpha}}\ \textsc{OForall}$$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \mathit{vars}'}$

$$\frac{\alpha^+ \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^+ = \alpha^+}\ \textsc{OPVarIn}$$

$$\frac{\alpha^+ \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^+ = \alpha^+}\ \textsc{OPVarIn}$$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \hat{\alpha}^+\{\mathit{vars}'\} = \cdot}\ \textsc{OPUVar}$$

$$\frac{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ {\downarrow}N = \vec{\alpha}}\ \textsc{OShiftD}$$

$$\frac{\mathbf{ord}\,(vars\backslash\overrightarrow{\alpha^+})\,\mathbf{in}\,N = \vec{\alpha}}{\mathbf{ord}\,vars\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.N = \vec{\alpha}}\ \textsc{OForall}$$

$\boxed{N \Downarrow M}$

$$\frac{}{\alpha^- \Downarrow \alpha^-}\ \textsc{NrmNVar}$$

$$\frac{}{\widehat{\alpha}^-\{vars\} \Downarrow \widehat{\alpha}^-\{vars\}}\ \textsc{NrmNUVar}$$

$$\frac{P \Downarrow Q}{\uparrow P \Downarrow \uparrow Q}\ \textsc{NrmShiftU}$$

$$\frac{P \Downarrow Q \quad N \Downarrow M}{P \to N \Downarrow Q \to M}\ \textsc{NrmArrow}$$

$$\frac{N \Downarrow N' \quad \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\forall\overrightarrow{\alpha^+}.N \Downarrow \forall\overrightarrow{\alpha^+}'.N'}\ \textsc{NrmForall}$$

$\boxed{P \Downarrow Q}$

$$\frac{}{\alpha^+ \Downarrow \alpha^+}\ \textsc{NrmPVar}$$

$$\frac{}{\widehat{\alpha}^+\{vars\} \Downarrow \widehat{\alpha}^+\{vars\}}\ \textsc{NrmPUVar}$$

$$\frac{N \Downarrow M}{\downarrow N \Downarrow \downarrow M}\ \textsc{NrmShiftD}$$

$$\frac{P \Downarrow P' \quad \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\exists\overrightarrow{\alpha^-}.P \Downarrow \exists\overrightarrow{\alpha^-}'.P'}\ \textsc{NrmExists}$$

$\boxed{e_1 \& e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{}{\widehat{\alpha}^+ :\geqslant P \& \widehat{\alpha}^+ :\geqslant Q = \widehat{\alpha}^+ :\geqslant P \vee Q}\ \textsc{SMEPSupSup}$$

$$\frac{\mathbf{fv}\,P \cup \mathbf{fv}\,Q \vDash P \geqslant Q \dashv \widehat{\sigma}'}{\widehat{\alpha}^+ :\approx P \& \widehat{\alpha}^+ :\geqslant Q = \widehat{\alpha}^+ :\approx P}\ \textsc{SMEPEqSup}$$

$$\frac{\mathbf{fv}\,P \cup \mathbf{fv}\,Q \vDash Q \geqslant P \dashv \widehat{\sigma}'}{\widehat{\alpha}^+ :\geqslant P \& \widehat{\alpha}^+ :\approx Q = \widehat{\alpha}^+ :\approx Q}\ \textsc{SMEPSupEq}$$

$$\frac{0 \vDash P \simeq_1^A Q \dashv \mu}{\widehat{\alpha}^+ :\approx P \& \widehat{\alpha}^+ :\approx Q = \widehat{\alpha}^+ :\approx Q}\ \textsc{SMEPEqEq}$$

$$\frac{0 \vDash N \simeq_1^A M \dashv \mu}{\widehat{\alpha}^- :\approx N \& \widehat{\alpha}^- :\approx M = \widehat{\alpha}^+ :\approx Q}\ \textsc{SMENEqEq}$$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$$\frac{}{\cdot \& \widehat{\sigma} = \widehat{\sigma}}\ \textsc{SMEmpty}$$

$$\frac{(\widehat{\alpha}^+ :\approx Q) \in \widehat{\sigma}_2 \quad 0 \vDash P \simeq_1^A Q \dashv \mu \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)}\ \textsc{SMPEqEq}$$

$$\frac{(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\geqslant P \vee Q, \widehat{\sigma}_3)}\ \textsc{SMPSupSup}$$

$$\frac{(\widehat{\alpha}^{+} :\approx Q) \in \widehat{\sigma}_2 \quad \mathbf{fv}\, Q \cup \mathbf{fv}\, P \vDash Q \geqslant P \dashv \widehat{\sigma}' \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^{+}) = \widehat{\sigma}_3}{(\widehat{\alpha}^{+} :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^{+} :\approx Q, \widehat{\sigma}_3)} \ \textsc{SMPSupEq}$$

$$\frac{(\widehat{\alpha}^{+} :\geqslant Q) \in \widehat{\sigma}_2 \quad \mathbf{fv}\, Q \cup \mathbf{fv}\, P \vDash P \geqslant Q \dashv \widehat{\sigma}' \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^{+}) = \widehat{\sigma}_3}{(\widehat{\alpha}^{+} :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^{+} :\approx P, \widehat{\sigma}_3)} \ \textsc{SMPEqSup}$$

$$\frac{(\widehat{\alpha}^{-} :\approx M) \in \widehat{\sigma}_2 \quad 0 \vDash N \simeq_1^{A} M \dashv \mu \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^{-}) = \widehat{\sigma}_3}{(\widehat{\alpha}^{-} :\approx N, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^{-} :\approx N, \widehat{\sigma}_3)} \ \textsc{SMNEqEq}$$

$\boxed{n \vDash N \overset{u}{\simeq} M \dashv \mu;\ \widehat{\sigma}}$ Negative unification

$$\frac{}{n \vDash \alpha^{-} \overset{u}{\simeq} \alpha^{-} \dashv \cdot;\ \cdot} \ \textsc{UNVar}$$

$$\frac{n \vDash P \overset{u}{\simeq} Q \dashv \mu;\ \widehat{\sigma}}{n \vDash \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \mu;\ \widehat{\sigma}} \ \textsc{UShiftU}$$

$$\frac{n \vDash P \overset{u}{\simeq} Q \dashv \mu_1;\ \widehat{\sigma}_1 \quad n \vDash N \overset{u}{\simeq} M \dashv \mu_2;\ \widehat{\sigma}_2 \quad \mu_1 \cup \mu_2\ \textbf{is bijective}}{n \vDash P \to N \overset{u}{\simeq} Q \to M \dashv \mu_1 \cup \mu_2;\ \widehat{\sigma}_1 \& \widehat{\sigma}_2} \ \textsc{UArrow}$$

$$\frac{n+1 \vDash [\overrightarrow{\widetilde{\alpha}^{+n}} / \overrightarrow{\alpha^{+}}] N \overset{u}{\simeq} [\overrightarrow{\widetilde{\beta}^{+n}} / \overrightarrow{\beta^{+}}] M \dashv \mu;\ \widehat{\sigma}}{n \vDash \forall \overrightarrow{\alpha^{+}}. N \overset{u}{\simeq} \forall \overrightarrow{\beta^{+}}. M \dashv \mu|_{\mathbf{mv}\, M};\ \widehat{\sigma}} \ \textsc{UForall}$$

$$\frac{}{n \vDash \widetilde{\alpha}^{-n} \overset{u}{\simeq} \widetilde{\beta}^{-n} \dashv \widetilde{\beta}^{-n} \mapsto \widetilde{\alpha}^{-n};\ \cdot} \ \textsc{UNMVar}$$

$$\frac{\mathbf{fv}\, N \subseteq \mathit{vars} \quad \mathbf{mv}\, N = \varnothing}{n \vDash \widehat{\alpha}^{-}\{\mathit{vars}\} \overset{u}{\simeq} N \dashv \cdot;\ \widehat{\alpha}^{-} :\approx N} \ \textsc{UNUVar}$$

$\boxed{n \vDash P \overset{u}{\simeq} Q \dashv \mu;\ \widehat{\sigma}}$ Positive unification

$$\frac{}{n \vDash \alpha^{+} \overset{u}{\simeq} \alpha^{+} \dashv \cdot;\ \cdot} \ \textsc{UPVar}$$

$$\frac{n \vDash N \overset{u}{\simeq} M \dashv \mu;\ \widehat{\sigma}}{n \vDash \downarrow N \overset{u}{\simeq} \downarrow M \dashv \mu;\ \widehat{\sigma}} \ \textsc{UShiftD}$$

$$\frac{n+1 \vDash [\overrightarrow{\widetilde{\alpha}^{-n}} / \overrightarrow{\alpha^{-}}] P \overset{u}{\simeq} [\overrightarrow{\widetilde{\beta}^{-n}} / \overrightarrow{\beta^{-}}] Q \dashv \mu;\ \widehat{\sigma}}{n \vDash \exists \overrightarrow{\alpha^{-}}. P \overset{u}{\simeq} \exists \overrightarrow{\beta^{-}}. Q \dashv \mu|_{\mathbf{mv}\, Q};\ \widehat{\sigma}} \ \textsc{UExists}$$

$$\frac{}{n \vDash \widetilde{\alpha}^{+n} \overset{u}{\simeq} \widetilde{\beta}^{+n} \dashv \widetilde{\beta}^{+n} \mapsto \widetilde{\alpha}^{+n};\ \cdot} \ \textsc{UPMVar}$$

$$\frac{\mathbf{fv}\, P \subseteq \mathit{vars} \quad \mathbf{mv}\, P = \varnothing}{n \vDash \widehat{\alpha}^{+}\{\mathit{vars}\} \overset{u}{\simeq} P \dashv \cdot;\ \widehat{\alpha}^{+} :\approx P} \ \textsc{UPUVar}$$

$\boxed{\Gamma \vdash N}$ Negative type well-formedness

$\boxed{\Gamma \vdash P}$ Positive type well-formedness

$\boxed{\Gamma \vdash N}$ Negative type well-formedness

$\boxed{\Gamma \vdash P}$ Positive type well-formedness

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Definition rules:         92 good    0 bad
Definition rule clauses: 163 good    0 bad
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