

1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$P, Q ::=$ positive types

- α^+
- $\downarrow N$
- $\exists \alpha^-. P$

$N, M ::=$ negative types

- α^-
- $\uparrow P$
- $\forall \alpha^+. N$
- $P \rightarrow N$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leq} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leq_0 M \quad \Gamma \vdash M \leq_0 N}{\Gamma \vdash N \simeq_0^{\leq} M} \quad \text{D0NDEF}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leq} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geq_0 Q \quad \Gamma \vdash Q \geq_0 P}{\Gamma \vdash P \simeq_0^{\leq} Q} \quad \text{D0PDEF}$$

$\boxed{\Gamma \vdash N \leq_0 M}$ Negative subtyping

$$\begin{array}{c} \overline{\Gamma \vdash \alpha^- \leq_0 \alpha^-} \quad \text{D0NVAR} \\ \frac{\Gamma \vdash P \simeq_0^{\leq} Q}{\Gamma \vdash \uparrow P \leq_0 \uparrow Q} \quad \text{D0SHIFTU} \\ \frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leq_0 M \quad M \neq \forall \beta^+. M'}{\Gamma \vdash \forall \alpha^+. N \leq_0 M} \quad \text{D0FORALLL} \\ \frac{\Gamma, \alpha^+ \vdash N \leq_0 M}{\Gamma \vdash N \leq_0 \forall \alpha^+. M} \quad \text{D0FORALLR} \\ \frac{\Gamma \vdash P \geq_0 Q \quad \Gamma \vdash N \leq_0 M}{\Gamma \vdash P \rightarrow N \leq_0 Q \rightarrow M} \quad \text{D0ARROW} \end{array}$$

$\boxed{\Gamma \vdash P \geq_0 Q}$ Positive supertyping

$$\begin{array}{c} \overline{\Gamma \vdash \alpha^+ \geq_0 \alpha^+} \quad \text{D0PVAR} \\ \frac{\Gamma \vdash N \simeq_0^{\leq} M}{\Gamma \vdash \downarrow N \geq_0 \downarrow M} \quad \text{D0SHIFTD} \\ \frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geq_0 Q \quad Q \neq \exists \alpha^-. Q'}{\Gamma \vdash \exists \alpha^-. P \geq_0 Q} \quad \text{D0EXISTSL} \\ \frac{\Gamma, \alpha^- \vdash P \geq_0 Q}{\Gamma \vdash P \geq_0 \exists \alpha^-. Q} \quad \text{D0EXISTSR} \end{array}$$

2 Multi-Quantified System

2.1 Grammar

P, Q	$::=$	multi-quantified positive types
	α^+	
	$\downarrow N$	
	$\exists \overrightarrow{\alpha^+}.P$	$P \neq \exists \dots$
	(P)	S
N, M	$::=$	multi-quantified negative types
	α^-	
	$\uparrow P$	
	$P \rightarrow N$	
	$\forall \overrightarrow{\alpha^+}.N$	$N \neq \forall \dots$
	(N)	S

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^\leq M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leq_1 M \quad \Gamma \vdash M \leq_1 N}{\Gamma \vdash N \simeq_1^\leq M} \quad (\simeq_1^\leq -)$$

$\boxed{\Gamma \vdash P \simeq_1^\leq Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash Q \geq_1 P}{\Gamma \vdash P \simeq_1^\leq Q} \quad (\simeq_1^\leq +)$$

$\boxed{\Gamma \vdash N \leq_1 M}$ Negative subtyping

$$\begin{array}{c} \overline{\Gamma \vdash \alpha^- \leq_1 \alpha^-} \quad (\text{VAR}^{-\leq_1}) \\ \frac{\Gamma \vdash P \simeq_1^\leq Q}{\Gamma \vdash \uparrow P \leq_1 \uparrow Q} \quad (\uparrow^{\leq_1}) \\ \frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash N \leq_1 M}{\Gamma \vdash P \rightarrow N \leq_1 Q \rightarrow M} \quad (\rightarrow^{\leq_1}) \\ \frac{\text{fv } N \cap \overrightarrow{\beta^+} = \emptyset \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leq_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leq_1 \forall \overrightarrow{\beta^+}.M} \quad (\forall^{\leq_1}) \end{array}$$

$\boxed{\Gamma \vdash P \geq_1 Q}$ Positive supertyping

$$\begin{array}{c} \overline{\Gamma \vdash \alpha^+ \geq_1 \alpha^+} \quad (\text{VAR}^{+\geq_1}) \\ \frac{\Gamma \vdash N \simeq_1^\leq M}{\Gamma \vdash \downarrow N \geq_1 \downarrow M} \quad (\downarrow^{\geq_1}) \\ \frac{\text{fv } P \cap \overrightarrow{\beta^-} = \emptyset \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geq_1 Q}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geq_1 \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\geq_1}) \end{array}$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^\leq \sigma_2 : \Gamma_1}$ Equivalence of substitutions

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\begin{array}{c} \overline{\alpha^- \simeq_1^D \alpha^-} \quad (\text{VAR}^{-\simeq_1^D}) \\ \frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad (\uparrow^{\simeq_1^D}) \\ \frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \rightarrow N \simeq_1^D Q \rightarrow M} \quad (\rightarrow^{\simeq_1^D}) \end{array}$$

$$\frac{\vec{\alpha}^+ \cap \mathbf{fv} M = \emptyset \quad \mu : (\vec{\beta}^+ \cap \mathbf{fv} M) \leftrightarrow (\vec{\alpha}^+ \cap \mathbf{fv} N) \quad N \simeq_1^D [\mu]M}{\forall \vec{\alpha}^+. N \simeq_1^D \forall \vec{\beta}^+. M} \quad (\forall \simeq_1^D)$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{\overline{\alpha^+ \simeq_1^D \alpha^+} \quad (\text{VAR}^+ \simeq_1^D)}{\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow \simeq_1^D)} \quad \frac{\vec{\alpha}^- \cap \mathbf{fv} Q = \emptyset \quad \mu : (\vec{\beta}^- \cap \mathbf{fv} Q) \leftrightarrow (\vec{\alpha}^- \cap \mathbf{fv} P) \quad P \simeq_1^D [\mu]Q}{\exists \vec{\alpha}^-. P \simeq_1^D \exists \vec{\beta}^-. Q} \quad (\exists \simeq_1^D)$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\text{ord vars in } N = \vec{\alpha}}$

$$\frac{\alpha^- \in \text{vars}}{\text{ord vars in } \alpha^- = \alpha^-} \quad (\text{VAR}_{\in}^-)$$

$$\frac{\alpha^- \notin \text{vars}}{\text{ord vars in } \alpha^- = .} \quad (\text{VAR}_{\notin}^-)$$

$$\frac{\text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \uparrow P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\text{ord vars in } P = \vec{\alpha}_1 \quad \text{ord vars in } N = \vec{\alpha}_2}{\text{ord vars in } P \rightarrow N = \vec{\alpha}_1, (\vec{\alpha}_2 \setminus \vec{\alpha}_1)} \quad (\rightarrow)$$

$$\frac{\text{vars} \cap \vec{\alpha}^+ = \emptyset \quad \text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \forall \vec{\alpha}^+. N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\text{ord vars in } P = \vec{\alpha}}$

$$\frac{\alpha^+ \in \text{vars}}{\text{ord vars in } \alpha^+ = \alpha^+} \quad (\text{VAR}_{\in}^+)$$

$$\frac{\alpha^+ \notin \text{vars}}{\text{ord vars in } \alpha^+ = .} \quad (\text{VAR}_{\notin}^+)$$

$$\frac{\text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \downarrow N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{\text{vars} \cap \vec{\alpha}^- = \emptyset \quad \text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \exists \vec{\alpha}^-. P = \vec{\alpha}} \quad (\exists)$$

$\boxed{\text{ord vars in } N = \vec{\alpha}}$

$$\frac{}{\text{ord vars in } \hat{\alpha}^- = .} \quad (\text{UVar}^-)$$

$\boxed{\text{ord vars in } P = \vec{\alpha}}$

$$\frac{}{\text{ord vars in } \hat{\alpha}^+ = .} \quad (\text{UVar}^+)$$

3.1.2 Quantifier Normalization

$$\boxed{\mathbf{nf}(N) = M}$$

$$\begin{array}{c} \overline{\mathbf{nf}(\alpha^-) = \alpha^-} \quad (\text{VAR}^-) \\ \frac{\mathbf{nf}(P) = Q}{\mathbf{nf}(\uparrow P) = \uparrow Q} \quad (\uparrow) \\ \frac{\mathbf{nf}(P) = Q \quad \mathbf{nf}(N) = M}{\mathbf{nf}(P \rightarrow N) = Q \rightarrow M} \quad (\rightarrow) \\ \frac{\mathbf{nf}(N) = N' \quad \text{ord } \vec{\alpha}^+ \text{ in } N' = \vec{\alpha}^{+'}}{\mathbf{nf}(\forall \alpha^+. N) = \forall \alpha^{+'}. N'} \quad (\forall) \end{array}$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\begin{array}{c} \overline{\mathbf{nf}(\alpha^+) = \alpha^+} \quad (\text{VAR}^+) \\ \frac{\mathbf{nf}(N) = M}{\mathbf{nf}(\downarrow N) = \downarrow M} \quad (\downarrow) \\ \frac{\mathbf{nf}(P) = P' \quad \text{ord } \vec{\alpha}^- \text{ in } P' = \vec{\alpha}^{-'}}{\mathbf{nf}(\exists \alpha^-. P) = \exists \alpha^{-'}. P'} \quad (\exists) \end{array}$$

$$\boxed{\mathbf{nf}(N) = M}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^-) = \hat{\alpha}^-} \quad (\text{UVAR}^-)$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^+) = \hat{\alpha}^+} \quad (\text{UVAR}^+)$$

3.2 Unification

$$\boxed{\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow \hat{\sigma}} \quad \text{Negative unification}$$

$$\begin{array}{c} \overline{\Gamma; \Theta \models \alpha^- \stackrel{u}{\simeq} \alpha^- \Rightarrow \cdot} \quad \text{UNVAR} \\ \frac{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \uparrow P \stackrel{u}{\simeq} \uparrow Q \Rightarrow \hat{\sigma}} \quad \text{USHIFTU} \\ \frac{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}_1 \quad \Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow \hat{\sigma}_2}{\Gamma; \Theta \models P \rightarrow N \stackrel{u}{\simeq} Q \rightarrow M \Rightarrow \hat{\sigma}_1 \ \& \ \hat{\sigma}_2} \quad \text{UARROW} \\ \frac{\Gamma, \vec{\alpha}^+; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \forall \alpha^+. N \stackrel{u}{\simeq} \forall \alpha^+. M \Rightarrow \hat{\sigma}} \quad \text{UFORALL} \\ \frac{\hat{\alpha}^- \{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma; \Theta \models \hat{\alpha}^- \stackrel{u}{\simeq} N \Rightarrow (\hat{\alpha}^- : \approx N)} \quad \text{UNUVAR} \end{array}$$

$$\boxed{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}} \quad \text{Positive unification}$$

$$\begin{array}{c} \overline{\Gamma; \Theta \models \alpha^+ \stackrel{u}{\simeq} \alpha^+ \Rightarrow \cdot} \quad \text{UPVAR} \\ \frac{\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \downarrow N \stackrel{u}{\simeq} \downarrow M \Rightarrow \hat{\sigma}} \quad \text{USHIFTD} \\ \frac{\Gamma, \vec{\alpha}^-; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \exists \alpha^-. P \stackrel{u}{\simeq} \exists \alpha^-. Q \Rightarrow \hat{\sigma}} \quad \text{UEXISTS} \\ \frac{\hat{\alpha}^+ \{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma; \Theta \models \hat{\alpha}^+ \stackrel{u}{\simeq} P \Rightarrow (\hat{\alpha}^+ : \approx P)} \quad \text{UPUVAR} \end{array}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma; \Theta \models N \leq M \Rightarrow \hat{\sigma}}$ Negative subtyping

$$\begin{array}{c}
\overline{\Gamma; \Theta \models \alpha^- \leq \alpha^- \Rightarrow} \quad (\text{VAR}^- \leq) \\
\frac{\Gamma; \Theta \models \mathbf{nf}(P) \stackrel{u}{\approx} \mathbf{nf}(Q) \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \uparrow P \leq \uparrow Q \Rightarrow \hat{\sigma}} \quad (\uparrow \leq) \\
\frac{\Gamma; \Theta \models P \geq Q \Rightarrow \hat{\sigma}_1 \quad \Gamma; \Theta \models N \leq M \Rightarrow \hat{\sigma}_2}{\Gamma; \Theta \models P \rightarrow N \leq Q \rightarrow M \Rightarrow \hat{\sigma}_1 \ \& \ \hat{\sigma}_2} \quad (\rightarrow \leq) \\
\frac{\Gamma, \vec{\beta}^+; \Theta, \hat{\alpha}^+ \{ \Gamma, \vec{\beta}^+ \} \models [\hat{\alpha}^+ / \alpha^+] N \leq M \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \forall \alpha^+. N \leq \forall \beta^+. M \Rightarrow \hat{\sigma} \setminus \hat{\alpha}^+} \quad (\forall \leq)
\end{array}$$

$\boxed{\Gamma; \Theta \models P \geq Q \Rightarrow \hat{\sigma}}$ Positive supertyping

$$\begin{array}{c}
\overline{\Gamma; \Theta \models \alpha^+ \geq \alpha^+ \Rightarrow} \quad (\text{VAR}^+ \geq) \\
\frac{\Gamma; \Theta \models \mathbf{nf}(N) \stackrel{u}{\approx} \mathbf{nf}(M) \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \downarrow N \geq \downarrow M \Rightarrow \hat{\sigma}} \quad (\downarrow \geq) \\
\frac{\Gamma, \vec{\beta}^-; \Theta, \hat{\alpha}^- \{ \Gamma, \vec{\beta}^- \} \models [\hat{\alpha}^- / \alpha^-] P \geq Q \Rightarrow \hat{\sigma}}{\Gamma; \Theta \models \exists \alpha^-. P \geq \exists \beta^-. Q \Rightarrow \hat{\sigma} \setminus \hat{\alpha}^-} \quad (\exists \geq) \\
\frac{\hat{\alpha}^+ \{ \Delta \} \in \Theta \quad \mathbf{upgrade} \Gamma \vdash P \mathbf{to} \Delta = Q}{\Gamma; \Theta \models \hat{\alpha}^+ \geq P \Rightarrow (\hat{\alpha}^+ : \geq Q)} \quad (\text{UVar} \geq)
\end{array}$$

3.4 Unification Solution Weakening

Unification solution is represented by a list of unification solution entries. Each entry restricts an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\hat{\alpha}^+ \approx P$ or $\hat{\alpha}^- \approx N$) or that it must be a (positive) supertype of a certain type ($\hat{\alpha}^+ \geq P$).

Definition 1 (Matching Entries). *We call two entries matching if they are restricting the same unification variable.*

Unification solutions are preordered by the weakening relation. First, let us define the weakening on the unification solution entries.

Definition 2.

$\boxed{\Gamma \vdash e_1 \Rightarrow e_2}$ Weakening of unification solution entries

$$\begin{array}{c}
\frac{\Gamma \vdash P_1 \geq_1 P_2}{\Gamma \vdash (\hat{\alpha}^+ : \geq P_1) \Rightarrow (\hat{\alpha}^+ : \geq P_2)} \quad (\geq \Rightarrow^+ \geq) \\
\frac{\Gamma \vdash P_1 \geq_1 P_2}{\Gamma \vdash (\hat{\alpha}^+ : \approx P_1) \Rightarrow (\hat{\alpha}^+ : \geq P_2)} \quad (\approx \Rightarrow^+ \geq) \\
\frac{\Gamma \vdash P_1 \approx_1^{\leq} P_2}{\Gamma \vdash (\hat{\alpha}^+ : \approx P_1) \Rightarrow (\hat{\alpha}^+ : \approx P_2)} \quad (\approx \Rightarrow^+ \approx) \\
\frac{\Gamma \vdash N_1 \approx_1^{\leq} N_2}{\Gamma \vdash (\hat{\alpha}^- : \approx N_1) \Rightarrow (\hat{\alpha}^- : \approx N_2)} \quad (\approx \Rightarrow^- \approx)
\end{array}$$

Notice that $\Gamma \vdash e_1 \Rightarrow e_2$ means that e_1 and e_2 are matching. And matching is an equivalence relation on the set of unification solutions.

Next, we lift the weakening relation to unification solutions. Informally, $\Theta \vdash \hat{\sigma}_1 \Rightarrow \hat{\sigma}_2$ means that for any entry of $\hat{\sigma}_2$, there is a matching entry in $\hat{\sigma}_1$ that is stronger than it.

Definition 3. *Assuming $\hat{\sigma}_2 : \Theta$ and $\hat{\sigma}_1 : \Theta' \supseteq \Theta$, we say $\Theta \vdash \hat{\sigma}_1 \Rightarrow \hat{\sigma}_2$ iff $\forall e_2 \in \hat{\sigma}_2, \exists e_1 \in \hat{\sigma}_1$ s.t.*

1. e_1 and e_2 are matching, i.e. restricting the same unification variable $\hat{\alpha}^\pm$;
2. $\Theta(\hat{\alpha}^\pm) \vdash e_1 \Rightarrow e_2$ (where $\Theta(\hat{\alpha}^\pm)$ is the context corresponding to $\hat{\alpha}^\pm$ in Θ).

3.5 Unification Solution Merge

Two matching entries can be merged in the following way:

Definition 4.

$\boxed{\Gamma \vdash e_1 \& e_2 = e_3}$ *Unification Solution Entry Merge*

$$\begin{array}{c}
\frac{\Gamma \models P_1 \vee P_2 = Q}{\Gamma \vdash (\hat{\alpha}^+ : \geq P_1) \& (\hat{\alpha}^+ : \geq P_2) = (\hat{\alpha}^+ : \geq Q)} \quad (\geq \&^+ \geq) \\
\\
\frac{\Gamma; \cdot \models \textcolor{gray}{P} \succcurlyeq Q \Rightarrow \hat{\sigma}'}{\Gamma \vdash (\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \geq Q) = (\hat{\alpha}^+ : \approx P)} \quad (\simeq \&^+ \geq) \\
\\
\frac{\Gamma; \cdot \models \textcolor{gray}{Q} \succcurlyeq P \Rightarrow \hat{\sigma}'}{\Gamma \vdash (\hat{\alpha}^+ : \geq P) \& (\hat{\alpha}^+ : \approx Q) = (\hat{\alpha}^+ : \approx Q)} \quad (\geq \&^+ \simeq) \\
\\
\frac{\mathbf{nf}(P) = \mathbf{nf}(P')}{\Gamma \vdash (\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \approx P') = (\hat{\alpha}^+ : \approx P)} \quad (\simeq \&^+ \simeq) \\
\\
\frac{\mathbf{nf}(N) = \mathbf{nf}(N')}{\Gamma \vdash (\hat{\alpha}^- : \approx N_1) \& (\hat{\alpha}^- : \approx N') = (\hat{\alpha}^- : \approx N)} \quad (\simeq \&^- \simeq)
\end{array}$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 5. $\hat{\sigma}_1 \& \hat{\sigma}_2 = \{e_1 \& e_2 \mid e_1 \in \hat{\sigma}_1, e_2 \in \hat{\sigma}_2, \text{ s.t. } e_1 \text{ matches with } e_2\}$
 $\cup \{e_1 \mid e_1 \in \hat{\sigma}_1, \text{ s.t. } \forall e_2 \in \hat{\sigma}_2, e_1 \text{ does not match with } e_2\}$
 $\cup \{e_2 \mid e_2 \in \hat{\sigma}_2, \text{ s.t. } \forall e_1 \in \hat{\sigma}_1, e_2 \text{ does not match with } e_2\}$

3.6 Least Upper Bound

$\boxed{\Gamma \models P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\begin{array}{c}
\frac{}{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \quad (\text{VAR}^\vee) \\
\\
\frac{\Gamma, \cdot \models \mathbf{nf}(\downarrow N) \stackrel{a}{\simeq} \mathbf{nf}(\downarrow M) \Rightarrow (\Xi, \textcolor{gray}{P}, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vdash \downarrow N \vee \downarrow M = \exists \alpha^-. [\alpha^- / \Xi] \textcolor{gray}{P}} \quad (\downarrow^\vee) \\
\\
\frac{\Gamma, \alpha^-, \beta^- \models P_1 \vee P_2 = Q}{\Gamma \vdash \exists \alpha^-. P_1 \vee \exists \beta^-. P_2 = Q} \quad (\exists^\vee)
\end{array}$$

$\boxed{\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q}$

$$\frac{\begin{array}{c} \Gamma = \Delta, \alpha^\pm \quad \beta^\pm \text{ is fresh} \quad \gamma^\pm \text{ is fresh} \\ \Delta, \beta^\pm, \gamma^\pm \models [\beta^\pm / \alpha^\pm] P \vee [\gamma^\pm / \alpha^\pm] P = Q \end{array}}{\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q} \quad (\text{UPG})$$

3.7 Antiunification

$\boxed{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \textcolor{gray}{Q}, \hat{\tau}_1, \hat{\tau}_2)}$

$$\begin{array}{c}
\frac{}{\Gamma \models \alpha^+ \stackrel{a}{\simeq} \alpha^+ \Rightarrow (\cdot, \alpha^+, \cdot, \cdot)} \quad \text{AUPVAR} \\
\\
\frac{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \textcolor{gray}{M}, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N_1 \stackrel{a}{\simeq} \downarrow N_2 \Rightarrow (\Xi, \downarrow \textcolor{gray}{M}, \hat{\tau}_1, \hat{\tau}_2)} \quad \text{AUPSHIFT} \\
\\
\frac{\alpha^- \cap \Gamma = \emptyset \quad \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \textcolor{gray}{Q}, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \exists \alpha^-. P_1 \stackrel{a}{\simeq} \exists \alpha^-. P_2 \Rightarrow (\Xi, \exists \alpha^-. \textcolor{gray}{Q}, \hat{\tau}_1, \hat{\tau}_2)} \quad \text{AUPEXISTS}
\end{array}$$

$\boxed{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \textcolor{gray}{M}, \hat{\tau}_1, \hat{\tau}_2)}$

$$\frac{}{\Gamma \models \alpha^- \stackrel{a}{\simeq} \alpha^- \Rightarrow (\Xi, \alpha^-, \cdot, \cdot)} \quad \text{AUNVAR}$$

$$\begin{array}{c}
\frac{\Gamma \vdash P_1 \overset{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vdash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \Rightarrow (\Xi, \uparrow Q, \hat{\tau}_1, \hat{\tau}_2)} \text{AUNSHIFT} \\
\\
\frac{\Gamma \vdash P_1 \overset{a}{\simeq} P_2 \Rightarrow (\Xi_1, Q, \hat{\tau}_1, \hat{\tau}_2) \quad \Gamma \vdash N_1 \overset{a}{\simeq} N_2 \Rightarrow (\Xi_2, M, \hat{\tau}'_1, \hat{\tau}'_2)}{\Gamma \vdash P_1 \rightarrow N_1 \overset{a}{\simeq} P_2 \rightarrow N_2 \Rightarrow (\Xi_1 \cup \Xi_2, Q \rightarrow M, \hat{\tau}_1 \cup \hat{\tau}'_1, \hat{\tau}_2 \cup \hat{\tau}'_2)} \text{AUNARROW} \\
\\
\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vdash N \overset{a}{\simeq} M \Rightarrow (\hat{\alpha}_{\{N, M\}}^-, \hat{\alpha}_{\{N, M\}}^-, (\hat{\alpha}_{\{N, M\}}^- : \approx N), (\hat{\alpha}_{\{N, M\}}^- : \approx M))} \text{AUNAU}
\end{array}$$

4 Proofs

4.1 Declarative Subtyping

Lemma 1 (Free Variable Propagation). *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context Γ ,*

- *if $\Gamma \vdash N \leq_1 M$ then $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$*
- *if $\Gamma \vdash P \geq_1 Q$ then $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$*

Proof. Mutual induction on $\Gamma \vdash N \leq_1 M$ and $\Gamma \vdash P \geq_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leq_1 \alpha^-$

It is self-evident that $\alpha^- \subseteq \alpha^-$.

Case 2. $\Gamma \vdash \uparrow P \leq_1 \uparrow Q$ From the inversion (and unfolding $\Gamma \vdash P \overset{\leq_1}{\simeq} Q$), we have $\Gamma \vdash P \geq_1 Q$. Then by the induction hypothesis, $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$. The desired inclusion holds, since $\mathbf{fv}(\uparrow P) = \mathbf{fv}(P)$ and $\mathbf{fv}(\uparrow Q) = \mathbf{fv}(Q)$.

Case 3. $\Gamma \vdash P \rightarrow N \leq_1 Q \rightarrow M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$ and $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$. Then $\mathbf{fv}(P \rightarrow N) = \mathbf{fv}(P) \cup \mathbf{fv}(N) \subseteq \mathbf{fv}(Q) \cup \mathbf{fv}(M) = \mathbf{fv}(Q \rightarrow M)$.

Case 4. $\Gamma \vdash \forall \alpha^+. N \leq_1 \forall \beta^+. M$
 $\mathbf{fv} \forall \alpha^+. N \subseteq \mathbf{fv} ([\vec{P}/\alpha^+]N) \setminus \beta^+$ here β^+ is excluded by the premise $\mathbf{fv} N \cap \beta^+ = \emptyset$
 $\subseteq \mathbf{fv} M \setminus \beta^+$ by the induction hypothesis, $\mathbf{fv} ([\vec{P}/\alpha^+]N) \subseteq \mathbf{fv} M$
 $\subseteq \mathbf{fv} \forall \beta^+. M$

Case 5. The positive cases are symmetric.

□

Corollary 1 (Free Variables of mutual subtypes).

- *If $\Gamma \vdash N \overset{\leq_1}{\simeq} M$ then $\mathbf{fv} N = \mathbf{fv} M$,*
- *If $\Gamma \vdash P \overset{\geq_1}{\simeq} Q$ then $\mathbf{fv} P = \mathbf{fv} Q$*

Lemma 2 (Subtypes and supertypes of a variable). *Assuming $\Gamma \vdash \alpha^-$, $\Gamma \vdash \alpha^+$, $\Gamma \vdash N$, and $\Gamma \vdash P$,*

- *if $\Gamma \vdash P \geq_1 \alpha^+$ or $\Gamma \vdash \alpha^+ \geq_1 P$ then $P = \exists \alpha^+. \alpha^+$ (for some potentially empty α^+)*
- *if $\Gamma \vdash N \leq_1 \alpha^-$ or $\Gamma \vdash \alpha^- \leq_1 N$ then $N = \forall \alpha^+. \alpha^-$ (for some potentially empty α^+)*

Proof. We prove by induction on the tree inferring $\Gamma \vdash P \geq_1 \alpha^+$ or $\Gamma \vdash \alpha^+ \geq_1 P$ or $\Gamma \vdash N \leq_1 \alpha^-$ or $\Gamma \vdash \alpha^- \leq_1 N$. Let us consider which of these judgments the tree is inferring.

Case 1. $\Gamma \vdash P \geq_1 \alpha^+$

If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\text{Var}^+ \geq_1$), which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists \geq_1$). By inverting this rule, $P = \exists \alpha^+. P'$ where P' does not start with \exists and $\Gamma \vdash [\vec{N}/\alpha^+]P' \geq_1 \alpha^+$ for some $\Gamma, \vec{N} \vdash N_i$.

By the induction hypothesis, $[\vec{N}/\alpha^+]P' = \exists \beta^+. \alpha^+$. Notice that P' must be a variable, because P' does not start with \exists , nor does it start with \uparrow (otherwise, $[\vec{N}/\alpha^+]P'$ would also started with \uparrow and would not be equal to $\exists \beta^+. \alpha^+$). Since P' is a *positive* variable, $[\vec{N}/\alpha^+]P' = P'$, and then $P' = \exists \beta^+. \alpha^+$ means that $P' = \alpha^+$. This way, $P = \exists \alpha^+. P' = \exists \alpha^+. \alpha^+$, as required.

Case 2. $\Gamma \vdash \alpha^+ \geq_1 P$

If the size of the inference tree is 1 then the only rule that can infer it is Rule $(\text{Var}^{+\geq_1})$, which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule (\exists^{\geq_1}) . By inverting this rule, $P = \exists \beta^{\rightarrow}.Q$ where and $\Gamma, \beta^{\rightarrow} \vdash \alpha^+ \geq_1 Q$.

By the induction hypothesis, $Q = \exists \beta'^{\rightarrow}.\alpha^+$. This way, $P = \exists \beta^{\rightarrow}.Q = \exists \beta^{\rightarrow}.\exists \beta'^{\rightarrow}.\alpha^+$, as required.

Case 3. The negative cases ($\Gamma \vdash N \leq_1 \alpha^-$ and $\Gamma \vdash \alpha^- \leq_1 N$) are proved analogously.

□

Corollary 2 (Variables have no proper subtypes and supertypes). *Assuming that all mentioned types are well-formed in Γ ,*

$$\begin{aligned} \Gamma \vdash P \geq_1 \alpha^+ &\iff P = \exists \beta^{\rightarrow}.\alpha^+ \iff \Gamma \vdash P \simeq_1^{\leq} \alpha^+ \iff P \simeq_1^D \alpha^+ \\ \Gamma \vdash \alpha^+ \geq_1 P &\iff P = \exists \beta^{\rightarrow}.\alpha^+ \iff \Gamma \vdash P \simeq_1^{\leq} \alpha^+ \iff P \simeq_1^D \alpha^+ \\ \Gamma \vdash N \leq_1 \alpha^- &\iff N = \forall \beta^{\rightarrow}.\alpha^- \iff \Gamma \vdash N \simeq_1^{\leq} \alpha^- \iff N \simeq_1^D \alpha^- \\ \Gamma \vdash \alpha^- \leq_1 N &\iff N = \forall \beta^{\rightarrow}.\alpha^- \iff \Gamma \vdash N \simeq_1^{\leq} \alpha^- \iff N \simeq_1^D \alpha^- \end{aligned}$$

Proof. Notice that $\Gamma \vdash \exists \alpha^{\rightarrow}.\alpha^+ \simeq_1^{\leq} \alpha^+$ and $\exists \alpha^{\rightarrow}.\alpha^+ \simeq \alpha^+$ and apply lemma 2. **Ilya:** fix

□

Corollary 3 (Transitivity of subtyping). *Assuming the types are well-formed in Γ ,*

- if $\Gamma \vdash N_1 \leq_1 N_2$ and $\Gamma \vdash N_2 \leq_1 N_3$ then $\Gamma \vdash N_1 \leq_1 N_3$,
- + if $\Gamma \vdash P_1 \geq_1 P_2$ and $\Gamma \vdash P_2 \geq_1 P_3$ then $\Gamma \vdash P_1 \geq_1 P_3$.

Corollary 4 (Transitivity of equivalence). *Assuming the types are well-formed in Γ ,*

- if $\Gamma \vdash N_1 \simeq_1^{\leq} N_2$ and $\Gamma \vdash N_2 \simeq_1^{\leq} N_3$ then $\Gamma \vdash N_1 \simeq_1^{\leq} N_3$,
- + if $\Gamma \vdash P_1 \simeq_1^{\leq} P_2$ and $\Gamma \vdash P_2 \simeq_1^{\leq} P_3$ then $\Gamma \vdash P_1 \simeq_1^{\leq} P_3$.

4.2 Substitution

Lemma 3 (Substitution strengthening). *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

- + if $\Gamma_1 \vdash P$ then $[\sigma]P = [\sigma|_{\text{fv } P}]P$,
- if $\Gamma_1 \vdash N$ then $[\sigma]N = [\sigma|_{\text{fv } N}]N$

Proof. **Ilya:** todo

□

Lemma 4 (Substitution preserves subtyping). *Suppose that $\Gamma \vdash \sigma : \Gamma_1$. Then*

- + if $\Gamma_1 \vdash P$, $\Gamma_1 \vdash Q$, and $\Gamma_1 \vdash P \geq_1 Q$ then $\Gamma \vdash [\sigma]P \geq_1 [\sigma]Q$
- if $\Gamma_1 \vdash N$, $\Gamma_1 \vdash M$, and $\Gamma_1 \vdash N \leq_1 M$ then $\Gamma \vdash [\sigma]N \leq_1 [\sigma]M$

Proof. **Ilya:** todo

□

Corollary 5 (Substitution preserves equivalence). *Suppose that $\Gamma \vdash \sigma : \Gamma_1$. Then*

- + if $\Gamma_1 \vdash P$, $\Gamma_1 \vdash Q$, and $\Gamma_1 \vdash P \simeq_1^{\leq} Q$ then $\Gamma \vdash [\sigma]P \simeq_1^{\leq} [\sigma]Q$
- if $\Gamma_1 \vdash N$, $\Gamma_1 \vdash M$, and $\Gamma_1 \vdash N \simeq_1^{\leq} M$ then $\Gamma \vdash [\sigma]N \simeq_1^{\leq} [\sigma]M$

4.3 Type well-formedness

Lemma 5 (Well-formedness agrees with substitution). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

- + $\Gamma, \Gamma_1 \vdash P \Leftrightarrow \Gamma, \Gamma_2 \vdash [\sigma]P$
- $\Gamma, \Gamma_1 \vdash N \Leftrightarrow \Gamma, \Gamma_2 \vdash [\sigma]N$

Proof. Ilya: todo □

Corollary 6. *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

- + $\Gamma_1, \Gamma_2 \vdash P \Leftrightarrow \Gamma_2 \vdash [\sigma]P$
- $\Gamma_1, \Gamma_2 \vdash N \Leftrightarrow \Gamma_2 \vdash [\sigma]N$

Lemma 6 (Equivalent Contexts). *In the well-formedness judgment, only used variables matter:*

- + if $\Gamma_1 \cap \mathbf{fv} P = \Gamma_2 \cap \mathbf{fv} P$ then $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$,
- if $\Gamma_1 \cap \mathbf{fv} N = \Gamma_2 \cap \mathbf{fv} N$ then $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$.

Proof. By simple mutual induction on P and Q . □

Corollary 7. *Suppose that all the types below are well-formed in Γ and $\Gamma' \subseteq \Gamma$. Then*

- + $\Gamma \vdash P \simeq_1^{\leq} Q$ implies $\Gamma' \vdash P \iff \Gamma' \vdash Q$
- $\Gamma \vdash N \simeq_1^{\leq} M$ implies $\Gamma' \vdash N \iff \Gamma' \vdash M$

Proof. From lemma 6 and corollary 1. □

4.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\overline{\mathbf{ord} \text{ vars in } N \equiv \text{vars} \cap \mathbf{fv} N}$	$\frac{N \simeq_1^D M}{\mathbf{ord} \text{ vars in } N = \mathbf{ord} \text{ vars in } M}$	—
Normalization	$\overline{N \simeq_1^D \mathbf{nf}(N)}$	$\frac{N \simeq_1^D M}{\mathbf{nf}(N) = \mathbf{nf}(M)}$	—
Equivalence	$\frac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq_1^D Q}{\Gamma \vdash P \simeq_1^{\leq} Q}$	$\frac{\Gamma \vdash P \simeq_1^{\leq} Q}{P \simeq_1^D Q}$	—
Uppgrade	$\frac{\mathbf{upgrade} \Gamma \vdash P \text{ to } \Delta = Q}{Q \text{ is sound } \left\{ \begin{array}{l} \Delta \vdash Q \\ \Gamma \vdash Q \geq_1 P \end{array} \right.}$	$\frac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade} \Gamma \vdash P \text{ to } \Delta = Q}$	$\frac{Q' \text{ is sound } \quad \mathbf{upgrade} \Gamma \vdash P \text{ to } \Delta = Q}{\Delta \vdash Q' \geq_1 Q}$
LUB	$\frac{\Gamma \models P_1 \vee P_2 = Q}{Q \text{ is sound } \left\{ \begin{array}{l} \Gamma \vdash Q \\ \Gamma \vdash Q \geq_1 P_1 \\ \Gamma \vdash Q \geq_1 P_2 \end{array} \right.}$	$\frac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \models P_1 \vee P_2 = Q}$	$\frac{Q' \text{ is sound } \quad \Gamma \models P_1 \vee P_2 = Q}{\Delta \vdash Q' \geq_1 Q}$
Anti-unification	$\frac{\Gamma \models P_1 \simeq P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \left\{ \begin{array}{l} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat{\tau}_i : \Xi \\ [\hat{\tau}_i] Q = P_i \end{array} \right. \text{ is sound}}$	$\frac{\exists \text{ sound } (\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2)}{\exists (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \text{ s.t. } \Gamma \models P_1 \simeq P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}$	$\frac{(\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2) \text{ is sound } \quad \Gamma \models P_1 \simeq P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\exists \Gamma; \Xi \vdash \hat{\tau} : \Xi' \text{ s.t. } [\hat{\tau}] Q' = Q}$
Unification (matching)	$\frac{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}}{\hat{\sigma} \text{ is sound } \left\{ \begin{array}{l} \Gamma \vdash \hat{\sigma} : \Theta \\ [\hat{\sigma}] P = Q \end{array} \right.}$	$\frac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow \hat{\sigma}}$	—
Subtyping	$\frac{\Gamma; \Theta \models N \leq M \Rightarrow \hat{\sigma}}{\hat{\sigma} \text{ is sound } \left\{ \begin{array}{l} \Gamma \vdash \hat{\sigma} : \Theta \\ \Gamma \vdash [\hat{\sigma}] N \leq_1 M \end{array} \right.}$	$\frac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Gamma; \Theta \models N \leq M \Rightarrow \hat{\sigma}}$	—

4.5 Variable Ordering

Definition 6 (Collision free bijection). *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is **collision free on sets P and Q** if and only if*

1. $\mu(P \cap A) \cap Q = \emptyset$
2. $\mu(Q \cap A) \cap P = \emptyset$

Lemma 7 (Soundness of variable ordering). *Variable ordering extracts precisely used free variables.*

- **ord vars in N** \equiv $\text{vars} \cap \mathbf{fv} N$ (as sets)
- + **ord vars in P** \equiv $\text{vars} \cap \mathbf{fv} P$ (as sets)

Proof. Straightforward mutual induction on **ord vars in N** $= \vec{\alpha}$ and **ord vars in P** $= \vec{\alpha}$ □

Corollary 8 (Additivity of ordering). *Variable ordering is additive (in terms of set union) with respect to its first argument.*

- **ord** ($\text{vars}_1 \cup \text{vars}_2$) **in N** \equiv **ord** vars_1 **in N** \cup **ord** vars_2 **in N** (as sets)
- + **ord** ($\text{vars}_1 \cup \text{vars}_2$) **in P** \equiv **ord** vars_1 **in P** \cup **ord** vars_2 **in P** (as sets)

Corollary 9 (Weakening of ordering). *Extending the first argument of the ordering with unused variables does not change the result.*

- **ord** ($\text{vars} \cap \mathbf{fv} N$) **in N** $=$ **ord vars in N**
- + **ord** ($\text{vars} \cap \mathbf{fv} P$) **in P** $=$ **ord vars in P**

Lemma 8 (Distributivity of renaming over variable ordering). *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

- If μ is collision free on vars and $\mathbf{fv} N$ then $[\mu](\mathbf{ord vars in } N) = \mathbf{ord} ([\mu]\text{vars}) \mathbf{in } [\mu]N$
- + If μ is collision free on vars and $\mathbf{fv} P$ then $[\mu](\mathbf{ord vars in } P) = \mathbf{ord} ([\mu]\text{vars}) \mathbf{in } [\mu]P$

Proof. Mutual induction on N and P .

Case 1. $N = \alpha^-$

let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in \text{vars}$

$$\begin{aligned}
 \text{Then } [\mu](\mathbf{ord vars in } N) &= [\mu](\mathbf{ord vars in } \alpha^-) \\
 &= [\mu]\alpha^- && \text{by Rule (Var}_{\epsilon}^+) \\
 &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]\text{vars}) \\
 &= \mathbf{ord} [\mu]\text{vars in } \beta^- && \text{by Rule (Var}_{\epsilon}^+), \text{ because } \beta^- \in [\mu]\text{vars} \\
 &= \mathbf{ord} [\mu]\text{vars in } [\mu]\alpha^-
 \end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin \text{vars}$

Notice that $[\mu](\mathbf{ord vars in } N) = [\mu](\mathbf{ord vars in } \alpha^-) = \cdot$ by Rule (Var $_{\epsilon}^+$). On the other hand, $\mathbf{ord} [\mu]\text{vars in } [\mu]\alpha^- = \mathbf{ord} [\mu]\text{vars in } \alpha^- = \cdot$. The latter equality is from Rule (Var $_{\epsilon}^+$), because μ is collision free on vars and $\mathbf{fv} N$, so $\mathbf{fv} N \ni \alpha^- \notin \mu(A \cap \text{vars}) \cup \text{vars} \supseteq [\mu]\text{vars}$.

c. $\alpha^- \in A$ but $\alpha^- \notin \text{vars}$

Then $[\mu](\mathbf{ord vars in } N) = [\mu](\mathbf{ord vars in } \alpha^-) = \cdot$ by Rule (Var $_{\epsilon}^+$). To prove that $\mathbf{ord} [\mu]\text{vars in } [\mu]\alpha^- = \cdot$, we apply Rule (Var $_{\epsilon}^+$). Let us show that $[\mu]\alpha^- \notin [\mu]\text{vars}$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]\text{vars} \subseteq \mu(A \cap \text{vars}) \cup \text{vars}$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap \text{vars}) \cup \text{vars}$.

- (i) If there is an element $x \in A \cap \text{vars}$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ , which contradicts with $\alpha^- \notin \text{vars}$. This way, $\mu(\alpha^-) \notin \mu(A \cap \text{vars})$.
- (ii) Since μ is collision free on vars and $\mathbf{fv} N$, $\mu(A \cap \mathbf{fv} N) \ni \mu(\alpha^-) \notin \text{vars}$.

d. $\alpha^- \notin A$ but $\alpha^- \in \text{vars}$

$\mathbf{ord} [\mu]\text{vars in } [\mu]\alpha^- = \mathbf{ord} [\mu]\text{vars in } \alpha^- = \alpha^-$. The latter is by Rule (Var $_{\epsilon}^+$), because $\alpha^- = [\mu]\alpha^- \in [\mu]\text{vars}$ since $\alpha^- \in \text{vars}$. On the other hand, $[\mu](\mathbf{ord vars in } N) = [\mu](\mathbf{ord vars in } \alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$

$$\begin{aligned}
[\mu](\mathbf{ord\ vars\ in}\ N) &= [\mu](\mathbf{ord\ vars\ in}\ \uparrow P) \\
&= [\mu](\mathbf{ord\ vars\ in}\ P) && \text{by Rule } (\uparrow) \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]P && \text{by the induction hypothesis} \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ \uparrow[\mu]P && \text{by Rule } (\uparrow) \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]\uparrow P && \text{by the definition of substitution} \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]N
\end{aligned}$$

Case 3. $N = P \rightarrow M$

$$\begin{aligned}
[\mu](\mathbf{ord\ vars\ in}\ N) &= [\mu](\mathbf{ord\ vars\ in}\ P \rightarrow M) \\
&= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \setminus \vec{\alpha}_1)) && \text{where } \mathbf{ord\ vars\ in}\ P = \vec{\alpha}_1 \text{ and } \mathbf{ord\ vars\ in}\ M = \vec{\alpha}_2 \\
&= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \setminus \vec{\alpha}_1) \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus [\mu]\vec{\alpha}_1) && \text{by induction on } \vec{\alpha}_2; \text{ the inductive step is similar to case 1. Notice that } \mu \text{ is} \\
&&& \text{collision free on } \vec{\alpha}_1 \text{ and } \vec{\alpha}_2 \text{ since } \vec{\alpha}_1 \subseteq \mathbf{vars} \text{ and } \vec{\alpha}_2 \subseteq \mathbf{fv}\ N \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus [\mu]\vec{\alpha}_1) \\
(\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]N) &= (\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]P \rightarrow [\mu]M) \\
&= (\vec{\beta}_1, (\vec{\beta}_2 \setminus \vec{\beta}_1)) && \text{where } \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]M = \vec{\beta}_2 \\
&&& \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1, \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus [\mu]\vec{\alpha}_1)
\end{aligned}$$

Case 4. $N = \forall \vec{\alpha}^+. M$

$$\begin{aligned}
[\mu](\mathbf{ord\ vars\ in}\ N) &= [\mu]\mathbf{ord\ vars\ in}\ \forall \vec{\alpha}^+. M \\
&= [\mu]\mathbf{ord\ vars\ in}\ M \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]M && \text{by the induction hypothesis} \\
(\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]N) &= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]\forall \vec{\alpha}^+. M \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ \forall \vec{\alpha}^+. [\mu]M \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]M
\end{aligned}$$

□

Lemma 9 (Ordering is not affected by independent substitutions). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2 , and $vars$ is a set of variables disjoint with both Γ_1 and Γ_2 . Then*

- $\mathbf{ord\ vars\ in}\ [\sigma]N = \mathbf{ord\ vars\ in}\ N$
- + $\mathbf{ord\ vars\ in}\ [\sigma]P = \mathbf{ord\ vars\ in}\ P$

Proof. Ilya: Should be easy

□

Lemma 10 (Completeness of variable ordering). *Variable ordering is invariant under equivalence. For arbitrary $vars$,*

- If $N \simeq_1^D M$ then $\mathbf{ord\ vars\ in}\ N = \mathbf{ord\ vars\ in}\ M$ (as lists)
- + If $P \simeq_1^D Q$ then $\mathbf{ord\ vars\ in}\ P = \mathbf{ord\ vars\ in}\ Q$ (as lists)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

□

4.6 Normaliztaion

Lemma 11. *Set of free variables is invariant under equivalence.*

- If $N \simeq_1^D M$ then $\mathbf{fv}\ N \equiv \mathbf{fv}\ M$ (as sets)
- + If $P \simeq_1^D Q$ then $\mathbf{fv}\ P \equiv \mathbf{fv}\ Q$ (as sets)

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$

□

Lemma 12. *Free variables are not changed by the normalization*

- $\mathbf{fv}\ N \equiv \mathbf{fv}\ \mathbf{nf}\ (N)$

$$+ \mathbf{fv} P \equiv \mathbf{fv} \mathbf{nf} (P)$$

Proof. By straightforward induction on $\mathbf{nf} (N) = M$. □

Lemma 13 (Soundness of quantifier normalization).

$$\begin{aligned} - N &\simeq_1^D \mathbf{nf} (N) \\ + P &\simeq_1^D \mathbf{nf} (P) \end{aligned}$$

Proof. Mutual induction on $\mathbf{nf} (N) = M$ and $\mathbf{nf} (P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var^-) and (Var^+)

By the corresponding equivalence rules.

Case 2. (\uparrow), (\downarrow), and (\rightarrow)

By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. (\forall), i.e. $\mathbf{nf} (\forall \alpha^+. N) = \forall \alpha^{+'}. N'$

From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 11, $\mathbf{fv} N \equiv \mathbf{fv} N'$. Then by lemma 7, $\alpha^{+'} \equiv \alpha^+ \cap \mathbf{fv} N' \equiv \alpha^+ \cap \mathbf{fv} N$, and thus, $\alpha^{+'} \cap \mathbf{fv} N' \equiv \alpha^+ \cap \mathbf{fv} N$.

To prove $\forall \alpha^+. N \simeq_1^D \forall \alpha^{+'}. N'$, it suffices to provide a bijection $\mu : \alpha^{+'} \cap \mathbf{fv} N' \leftrightarrow \alpha^+ \cap \mathbf{fv} N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. (\exists) Same as for case 3. □

Corollary 10 (Normalization preserves ordering). *For any vars,*

$$\begin{aligned} - \mathbf{ord} \text{ vars in } \mathbf{nf} (N) &= \mathbf{ord} \text{ vars in } M \\ + \mathbf{ord} \text{ vars in } \mathbf{nf} (P) &= \mathbf{ord} \text{ vars in } Q \end{aligned}$$

Proof. Immediately from lemmas 10 and 13. □

Lemma 14 (Distributivity of normalization over substitution). *Normalization of a term distributes over substitution. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2 . Then*

$$\begin{aligned} - \mathbf{nf} ([\sigma]N) &= [\mathbf{nf} (\sigma)]\mathbf{nf} (N) \\ + \mathbf{nf} ([\sigma]P) &= [\mathbf{nf} (\sigma)]\mathbf{nf} (P) \end{aligned}$$

where $\mathbf{nf} (\sigma)$ means pointwise normalization: $[\mathbf{nf} (\sigma)]\alpha^- = \mathbf{nf} ([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P .

Case 1. $N = \alpha^-$

$$\begin{aligned} \mathbf{nf} ([\sigma]N) &= \mathbf{nf} ([\sigma]\alpha^-) = [\mathbf{nf} (\sigma)]\alpha^- \\ [\mathbf{nf} (\sigma)]\mathbf{nf} (N) &= [\mathbf{nf} (\sigma)]\mathbf{nf} (\alpha^-) = [\mathbf{nf} (\sigma)]\alpha^- \end{aligned}$$

Case 2. $P = \alpha^+$

Similar to case 1.

Case 3. If the type is formed by \rightarrow , \uparrow , or \downarrow , the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \rightarrow M$ then

$$\begin{aligned} \mathbf{nf} ([\sigma]N) &= \mathbf{nf} ([\sigma](P \rightarrow M)) \\ &= \mathbf{nf} ([\sigma]P \rightarrow [\sigma]M) && \text{By the congruence of substitution} \\ &= \mathbf{nf} ([\sigma]P) \rightarrow \mathbf{nf} ([\sigma]M) && \text{By the congruence of normalization, i.e. Rule } (\rightarrow) \\ &= [\mathbf{nf} (\sigma)]\mathbf{nf} (P) \rightarrow [\mathbf{nf} (\sigma)]\mathbf{nf} (M) && \text{By the induction hypothesis} \\ &= [\mathbf{nf} (\sigma)](\mathbf{nf} (P) \rightarrow \mathbf{nf} (M)) && \text{By the congruence of substitution} \\ &= [\mathbf{nf} (\sigma)]\mathbf{nf} (P \rightarrow M) && \text{By the congruence of normalization} \\ &= [\mathbf{nf} (\sigma)]\mathbf{nf} (N) \end{aligned}$$

Case 4. $N = \forall \vec{\alpha}^+. M$

$$[\mathbf{nf}(\sigma)]\mathbf{nf}(N) = [\mathbf{nf}(\sigma)]\mathbf{nf}(\forall \vec{\alpha}^+. M)$$

$$= [\mathbf{nf}(\sigma)]\forall \vec{\alpha}^{+'}. \mathbf{nf}(M) \quad \text{Where } \vec{\alpha}^{+'} = \mathbf{ord} \vec{\alpha}^+ \text{ in } \mathbf{nf}(M) = \mathbf{ord} \vec{\alpha}^+ \text{ in } M \text{ (the latter is by corollary 10)}$$

$$\mathbf{nf}([\sigma]N) = \mathbf{nf}([\sigma]\forall \vec{\alpha}^+. M)$$

$$= \mathbf{nf}(\forall \vec{\alpha}^+. [\sigma]M) \quad \text{Assuming } \vec{\alpha}^+ \cap \Gamma_1 = \emptyset \text{ and } \vec{\alpha}^+ \cap \Gamma_2 = \emptyset$$

$$= \forall \vec{\beta}^+. \mathbf{nf}([\sigma]M) \quad \text{Where } \vec{\beta}^+ = \mathbf{ord} \vec{\alpha}^+ \text{ in } \mathbf{nf}([\sigma]M) = \mathbf{ord} \vec{\alpha}^+ \text{ in } [\sigma]M \text{ (the latter is by corollary 10)}$$

$$= \forall \vec{\alpha}^{+'}. \mathbf{nf}([\sigma]M) \quad \text{By lemma 9, } \vec{\beta}^+ = \vec{\alpha}^{+'} \text{ since } \vec{\alpha}^+ \text{ is disjoint with } \Gamma_1 \text{ and } \Gamma_2$$

$$= \forall \vec{\alpha}^{+'}. [\mathbf{nf}(\sigma)]\mathbf{nf}(M) \quad \text{By the induction hypothesis}$$

To show alpha-equivalence of $[\mathbf{nf}(\sigma)]\forall \vec{\alpha}^{+'}. \mathbf{nf}(M)$ and $\forall \vec{\alpha}^{+'}. [\mathbf{nf}(\sigma)]\mathbf{nf}(M)$, we can assume that $\vec{\alpha}^{+'} \cap \Gamma_1 = \emptyset$, and $\vec{\alpha}^{+'} \cap \Gamma_2 = \emptyset$.

Case 5. $P = \exists \vec{\alpha}^-. Q$

Same as for case 4.

□

Corollary 11 (Commutativity of normalization and renaming). *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

$$- \mathbf{nf}([\mu]N) = [\mu]\mathbf{nf}(N)$$

$$+ \mathbf{nf}([\mu]P) = [\mu]\mathbf{nf}(P)$$

Proof. Immediately from lemma 14, after noticing that $\mathbf{nf}(\mu) = \mu$.

□

Lemma 15 (Completeness of quantified normalization). *Normalization returns the same representative for equivalent types.*

$$- \text{If } N \simeq_1^D M \text{ then } \mathbf{nf}(N) = \mathbf{nf}(M)$$

$$+ \text{If } P \simeq_1^D Q \text{ then } \mathbf{nf}(P) = \mathbf{nf}(Q)$$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. $(\forall \vec{\alpha}^+)$

From the definition of the normalization,

$$\bullet \mathbf{nf}(\forall \vec{\alpha}^+. N) = \forall \vec{\alpha}^{+'}. \mathbf{nf}(N) \text{ where } \vec{\alpha}^{+'} \text{ is } \mathbf{ord} \vec{\alpha}^+ \text{ in } \mathbf{nf}(N)$$

$$\bullet \mathbf{nf}(\forall \vec{\beta}^+. M) = \forall \vec{\beta}^{+'}. \mathbf{nf}(M) \text{ where } \vec{\beta}^{+'} \text{ is } \mathbf{ord} \vec{\beta}^+ \text{ in } \mathbf{nf}(M)$$

Let us take $\mu : (\vec{\beta}^+ \cap \mathbf{fv} M) \leftrightarrow (\vec{\alpha}^+ \cap \mathbf{fv} N)$ from the inversion of the equivalence judgment. Notice that from lemmas 7 and 12, the domain and the codomain of μ can be written as $\mu : \vec{\beta}^{+'} \leftrightarrow \vec{\alpha}^{+'}$.

To show the alpha-equivalence of $\forall \vec{\alpha}^{+'}. \mathbf{nf}(N)$ and $\forall \vec{\beta}^{+'}. \mathbf{nf}(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}(M) = \mathbf{nf}(N)$ and (ii) $[\mu]\vec{\beta}^{+'} = \vec{\alpha}^{+'}$.

(i) $[\mu]\mathbf{nf}(M) = \mathbf{nf}([\mu]M) = \mathbf{nf}(N)$. The first equality holds by corollary 11, the second—by the induction hypothesis.

$$\begin{aligned} \text{(ii) } [\mu]\vec{\beta}^{+'} &= [\mu]\mathbf{ord} \vec{\beta}^+ \text{ in } \mathbf{nf}(M) && \text{by the definition of } \vec{\beta}^{+'} \\ &= [\mu]\mathbf{ord} (\vec{\beta}^+ \cap \mathbf{fv} M) \text{ in } \mathbf{nf}(M) && \text{from lemma 12 and corollary 9} \\ &= \mathbf{ord} [\mu](\vec{\beta}^+ \cap \mathbf{fv} M) \text{ in } [\mu]\mathbf{nf}(M) && \text{by lemma 8, because } \vec{\alpha}^+ \cap \mathbf{fv} N \cap \mathbf{fv} \mathbf{nf}(M) \subseteq \vec{\alpha}^+ \cap \mathbf{fv} M = \emptyset \\ &&& \text{and } \vec{\alpha}^+ \cap \mathbf{fv} N \cap (\vec{\beta}^+ \cap \mathbf{fv} M) \subseteq \vec{\alpha}^+ \cap \mathbf{fv} M = \emptyset \\ &= \mathbf{ord} [\mu](\vec{\beta}^+ \cap \mathbf{fv} M) \text{ in } \mathbf{nf}(N) && \text{since } [\mu]\mathbf{nf}(M) = \mathbf{nf}(N) \text{ is proved} \\ &= \mathbf{ord} (\vec{\alpha}^+ \cap \mathbf{fv} N) \text{ in } \mathbf{nf}(N) && \text{because } \mu \text{ is a bijection between } \vec{\alpha}^+ \cap \mathbf{fv} N \text{ and } \vec{\beta}^+ \cap \mathbf{fv} M \\ &= \mathbf{ord} \vec{\alpha}^+ \text{ in } \mathbf{nf}(N) && \text{from lemma 12 and corollary 9} \\ &= \vec{\alpha}^{+'} && \text{by the definition of } \vec{\alpha}^{+'} \end{aligned}$$

Case 2. ($\exists^{\approx^P_1}$) Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis. □

Lemma 16 (Idempotence of normalization). *Normalization is idempotent*

- $\mathbf{nf}(\mathbf{nf}(N)) = \mathbf{nf}(N)$
- + $\mathbf{nf}(\mathbf{nf}(P)) = \mathbf{nf}(P)$

Proof. By applying lemma 15 to lemma 13. □

Lemma 17. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized.*

Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then

$$\begin{aligned}
 + \quad [\sigma]P \text{ is normal} &\iff \begin{cases} \sigma|_{\mathbf{fv}(P)} & \text{is normal} \\ P & \text{is normal} \end{cases} \\
 - \quad [\sigma]N \text{ is normal} &\iff \begin{cases} \sigma|_{\mathbf{fv}(N)} & \text{is normal} \\ N & \text{is normal} \end{cases}
 \end{aligned}$$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$

Then N is always normal, and the normality of $\sigma|_{\alpha^-}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \rightarrow M$

$$\begin{aligned}
 [\sigma](P \rightarrow M) \text{ is normal} &\iff [\sigma]P \rightarrow [\sigma]M \text{ is normal} && \text{by the substitution congruence} \\
 &\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases} && \text{by congruence of normality Ilya: lemma?} \\
 &\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}(M)} & \text{is normal} \end{cases} && \text{by the induction hypothesis} \\
 &\iff \begin{cases} P \rightarrow M & \text{is normal} \\ \sigma|_{\mathbf{fv}(P) \cup \mathbf{fv}(M)} & \text{is normal} \end{cases} \iff \begin{cases} P \rightarrow M & \text{is normal} \\ \sigma|_{\mathbf{fv}(P \rightarrow M)} & \text{is normal} \end{cases}
 \end{aligned}$$

Case 3. $N = \uparrow P$

By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall \alpha^+. M$

$$\begin{aligned}
 [\sigma](\forall \alpha^+. M) \text{ is normal} &\iff (\forall \alpha^+. [\sigma]M) \text{ is normal} && \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \emptyset \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \emptyset \\
 &\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord} \, \overrightarrow{\alpha^+} \text{ in } [\sigma]M = \overrightarrow{\alpha^+} \end{cases} && \text{by the definition of normalization} \\
 &\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord} \, \overrightarrow{\alpha^+} \text{ in } M = \overrightarrow{\alpha^+} \end{cases} && \text{by lemma 9} \\
 &\iff \begin{cases} \sigma|_{\mathbf{fv}(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord} \, \overrightarrow{\alpha^+} \text{ in } M = \overrightarrow{\alpha^+} \end{cases} && \text{by the induction hypothesis} \\
 &\iff \begin{cases} \sigma|_{\mathbf{fv}(\forall \alpha^+. M)} \text{ is normal} \\ \forall \alpha^+. M \text{ is normal} \end{cases} && \begin{array}{l} \text{since } \mathbf{fv}(\forall \alpha^+. M) = \mathbf{fv}(M); \\ \text{by the definition of normalization} \end{array}
 \end{aligned}$$

Case 5. $P = \dots$

The positive cases are done in the same way as the negative ones. □

4.7 Equivalence

Lemma 18 (Declarative equivalence is transitive).

- + if $P_1 \simeq_1^D P_2$ and $P_2 \simeq_1^D P_3$ then $P_1 \simeq_1^D P_3$,
- if $N_1 \simeq_1^D N_2$ and $N_2 \simeq_1^D N_3$ then $N_1 \simeq_1^D N_3$.

Proof. Ilya: should be easy to do by induction since the types are getting smaller □

Lemma 19 (Algorithmization of declarative equivalence). *Declarative equivalence is equality of normal forms.*

- + $P \simeq_1^D Q \iff \mathbf{nf}(P) = \mathbf{nf}(Q)$,
- $N \simeq_1^D M \iff \mathbf{nf}(N) = \mathbf{nf}(M)$.

Proof.

- + Let us prove both directions separately.
 - \Rightarrow exactly by lemma 15,
 - \Leftarrow from lemma 13, we know $P \simeq_1^D \mathbf{nf}(P) = \mathbf{nf}(Q) \simeq_1^D Q$, then by transitivity (lemma 18), $P \simeq_1^D Q$.
- The proof is exactly the same. □

Lemma 20 (Type well-formedness is invariant under equivalence). *Mutual subtyping implies declarative equivalence.*

- + if $P \simeq_1^D Q$ then $\Gamma \vdash P \iff \Gamma \vdash Q$,
- if $N \simeq_1^D M$ then $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. Ilya: todo □

Corollary 12 (Normalization preserves well-formedness).

- + $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}(P)$,
- $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}(N)$

Proof. Immediately from lemmas 13 and 20. □

Corollary 13 (Normalization preserves well-formedness of substitution).

$$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}(\sigma) : \Gamma_1$$

Lemma 21 (Soundness of equivalence). *Declarative equivalence implies mutual subtyping.*

- + if $\Gamma \vdash P, \Gamma \vdash Q$, and $P \simeq_1^D Q$ then $\Gamma \vdash P \preceq_1^\leq Q$,
- if $\Gamma \vdash N, \Gamma \vdash M$, and $N \simeq_1^D M$ then $\Gamma \vdash N \preceq_1^\leq M$.

Proof. We prove it by mutual induction on $P \simeq_1^D Q$ and $N \simeq_1^D M$.

Case 1. $\alpha^- \simeq_1^D \alpha^-$

Then $\Gamma \vdash \alpha^- \preceq_1^\leq \alpha^-$ by Rule (Var^{−≤₁}), which immediately implies $\Gamma \vdash \alpha^- \preceq_1^\leq \alpha^-$ by Rule (\preceq_1^\leq −).

Case 2. $\uparrow P \simeq_1^D \uparrow Q$

Then by inversion of Rule (\uparrow^{\leq_1}), $P \simeq_1^D Q$, and from the induction hypothesis, $\Gamma \vdash P \preceq_1^\leq Q$, and (by symmetry) $\Gamma \vdash Q \preceq_1^\leq P$.

When Rule (\uparrow^{\leq_1}) is applied to $\Gamma \vdash P \preceq_1^\leq Q$, it gives us $\Gamma \vdash \uparrow P \preceq_1^\leq \uparrow Q$; when it is applied to $\Gamma \vdash Q \preceq_1^\leq P$, we obtain $\Gamma \vdash \uparrow Q \preceq_1^\leq \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \preceq_1^\leq \uparrow Q$.

Case 3. $P \rightarrow N \simeq_1^D Q \rightarrow M$

Then by inversion of Rule (\rightarrow^{\leq_1}), $P \simeq_1^D Q$ and $N \simeq_1^D M$. By the induction hypothesis, $\Gamma \vdash P \preceq_1^\leq Q$ and $\Gamma \vdash N \preceq_1^\leq M$, which means by inversion: (i) $\Gamma \vdash P \succeq_1^\geq Q$, (ii) $\Gamma \vdash Q \succeq_1^\geq P$, (iii) $\Gamma \vdash N \preceq_1^\leq M$, (iv) $\Gamma \vdash M \preceq_1^\leq N$. Applying Rule (\rightarrow^{\leq_1}) to (i) and (iii), we obtain $\Gamma \vdash P \rightarrow N \preceq_1^\leq Q \rightarrow M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \rightarrow M \preceq_1^\leq P \rightarrow N$. Together, it implies $\Gamma \vdash P \rightarrow N \preceq_1^\leq Q \rightarrow M$.

Case 4. $\forall \vec{\alpha}^+. N \simeq_1^D \forall \vec{\beta}^+. M$

Then by inversion, there exists bijection $\mu : (\vec{\beta}^+ \cap \mathbf{fv} M) \leftrightarrow (\vec{\alpha}^+ \cap \mathbf{fv} N)$, such that $N \simeq_1^D [\mu]M$. By the induction hypothesis, $\Gamma, \vec{\alpha}^+ \vdash N \simeq_1^{\leq} [\mu]M$. From corollary 5 and the fact that μ is bijective, we also have $\Gamma, \vec{\beta}^+ \vdash [\mu^{-1}]N \simeq_1^{\leq} M$.

Let us construct a substitution $\vec{\alpha}^+ \vdash \vec{P}/\vec{\beta}^+ : \vec{\beta}^+$ by extending μ with arbitrary positive types on $\vec{\beta}^+ \setminus \mathbf{fv} M$.

Notice that $[\mu]M = [\vec{P}/\vec{\beta}^+]M$, and therefore, $\Gamma, \vec{\alpha}^+ \vdash N \simeq_1^{\leq} [\mu]M$ implies $\Gamma, \vec{\alpha}^+ \vdash [\vec{P}/\vec{\beta}^+]M \leq_1 N$. Then by Rule (\forall^{\leq_1}) , $\Gamma \vdash \forall \vec{\beta}^+. M \leq_1 \forall \vec{\alpha}^+. N$.

Analogously, we construct the substitution from μ^{-1} , and use it to instantiate $\vec{\alpha}^+$ in the application of Rule (\forall^{\leq_1}) to infer $\Gamma \vdash \forall \vec{\alpha}^+. N \leq_1 \forall \vec{\beta}^+. M$.

This way, $\Gamma \vdash \forall \vec{\beta}^+. M \leq_1 \forall \vec{\alpha}^+. N$ and $\Gamma \vdash \forall \vec{\alpha}^+. N \leq_1 \forall \vec{\beta}^+. M$ gives us $\Gamma \vdash \forall \vec{\beta}^+. M \simeq_1^{\leq} \forall \vec{\alpha}^+. N$.

Case 5. For the cases of the positive types, the proofs are symmetric. □

Corollary 14 (Normalization is sound w.r.t. subtyping-induced equivalence).

- + if $\Gamma \vdash P$ then $\Gamma \vdash P \simeq_1^{\leq} \mathbf{nf}(P)$,
- if $\Gamma \vdash N$ then $\Gamma \vdash N \simeq_1^{\leq} \mathbf{nf}(N)$.

Proof. Immediately from lemmas 13 and 21 and corollary 12. □

Lemma 22 (Subtyping induced by disjoint substitutions). *If two disjoint substitutions induce subtyping, they are degenerate (so is the subtyping). Suppose that $\Gamma \vdash \sigma_1 : \Gamma_1$ and $\Gamma \vdash \sigma_2 : \Gamma_2$, where $\Gamma_i \subseteq \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then*

- assuming $\Gamma \vdash N$, $\Gamma \vdash [\sigma_1]N \leq_1 [\sigma_2]N$ implies $\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \mathbf{fv} N$
- + assuming $\Gamma \vdash P$, $\Gamma \vdash [\sigma_1]P \geq_1 [\sigma_2]P$ implies $\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \mathbf{fv} P$

Proof. Proof by induction on $\Gamma \vdash N$ (and mutually on $\Gamma \vdash P$).

Case 1. $N = \alpha^-$

Then $\Gamma \vdash [\sigma_1]N \leq_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\alpha^- \leq_1 [\sigma_2]\alpha^-$. Let us consider the following cases:

- a. $\alpha^- \notin \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then $\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \alpha^-$ holds immediately, since $[\sigma_i]\alpha^- = [\text{id}]\alpha^- = \alpha^-$ and $\Gamma \vdash \alpha^- \simeq_1^{\leq} \alpha^-$.
- b. $\alpha^- \in \Gamma_1$ and $\alpha^- \in \Gamma_2$
This case is not possible by assumption: $\Gamma_1 \cap \Gamma_2 = \emptyset$.
- c. $\alpha^- \in \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then we have $\Gamma \vdash [\sigma_1]\alpha^- \leq_1 \alpha^-$, which by corollary 2 means $\Gamma \vdash [\sigma_1]\alpha^- \simeq_1^{\leq} \alpha^-$, and hence, $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \text{id} : \alpha^-$.
 $\Gamma \vdash \sigma_2 \simeq_1^{\leq} \text{id} : \alpha^-$ holds since $[\sigma_2]\alpha^- = \alpha^-$, similarly to case 1.a.
- d. $\alpha^- \notin \Gamma_1$ and $\alpha^- \in \Gamma_2$
Then we have $\Gamma \vdash \alpha^- \leq_1 [\sigma_2]\alpha^-$, which by corollary 2 means $\Gamma \vdash \alpha^- \simeq_1^{\leq} [\sigma_2]\alpha^-$, and hence, $\Gamma \vdash \sigma_2 \simeq_1^{\leq} \text{id} : \alpha^-$.
 $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \text{id} : \alpha^-$ holds since $[\sigma_1]\alpha^- = \alpha^-$, similarly to case 1.a.

Case 2. $N = \forall \vec{\alpha}^+. M$

Then by inversion, $\Gamma, \vec{\alpha}^+ \vdash M$. $\Gamma \vdash [\sigma_1]N \leq_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\forall \vec{\alpha}^+. M \leq_1 [\sigma_2]\forall \vec{\alpha}^+. M$. By the congruence of substitution and by the inversion of Rule (\forall^{\leq_1}) , $\Gamma, \vec{\alpha}^+ \vdash [\vec{Q}/\vec{\alpha}^+][\sigma_1]M \leq_1 [\sigma_2]M$, where $\Gamma, \vec{\alpha}^+ \vdash Q_i$. Let us denote the (Kleisli) composition of σ_1 and $\vec{Q}/\vec{\alpha}^+$ as σ'_1 , noting that $\Gamma, \vec{\alpha}^+ \vdash \sigma'_1 : \Gamma_1, \vec{\alpha}^+$, and $\Gamma_1, \vec{\alpha}^+ \cap \Gamma_2 = \emptyset$.

Let us apply the induction hypothesis to M and the substitutions σ'_1 and σ_2 with $\Gamma, \vec{\alpha}^+ \vdash [\sigma'_1]M \leq_1 [\sigma_2]M$ to obtain:

$$\Gamma, \vec{\alpha}^+ \vdash \sigma'_1 \simeq_1^{\leq} \text{id} : \mathbf{fv} M \quad (1)$$

$$\Gamma, \vec{\alpha}^+ \vdash \sigma_2 \simeq_1^{\leq} \text{id} : \mathbf{fv} M \quad (2)$$

Then $\Gamma \vdash \sigma_2 \simeq_1^{\leq} \text{id} : \mathbf{fv} \forall \vec{\alpha}^+. M$ holds by strengthening of 2: for any $\beta^\pm \in \mathbf{fv} \forall \vec{\alpha}^+. M = \mathbf{fv} M \setminus \vec{\alpha}^+$, $\Gamma, \vec{\alpha}^+ \vdash [\sigma_2]\beta^\pm \simeq_1^{\leq} \beta^\pm$ is strengthened to $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leq} \beta^\pm$, because $\mathbf{fv} [\sigma_2]\beta^\pm = \mathbf{fv} \beta^\pm = \{\beta^\pm\} \subseteq \Gamma$.

To show that $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \text{id} : \mathbf{fv} \forall \vec{\alpha}^+. M$, let us take an arbitrary $\beta^\pm \in \mathbf{fv} \forall \vec{\alpha}^+. M = \mathbf{fv} M \setminus \vec{\alpha}^+$.

$$\begin{aligned}
\beta^\pm &= [\text{id}]\beta^\pm && \text{by definition of id} \\
&\simeq_1^\leq [\sigma'_1]\beta^\pm && \text{by 1} \\
&= [\vec{Q}/\vec{\alpha}^\pm][\sigma_1]\beta^\pm && \text{by definition of } \sigma'_1 \\
&= [\sigma_1]\beta^\pm && \text{because } \vec{\alpha}^\pm \cap \mathbf{fv}[\sigma_1]\beta^\pm \subseteq \vec{\alpha}^\pm \cap \Gamma = \emptyset \\
\text{This way, } \Gamma \vdash [\sigma_1]\beta^\pm &\simeq_1^\leq \beta^\pm \text{ for any } \beta^\pm \in \mathbf{fv} \forall \vec{\alpha}^\pm. M \text{ and thus, } \Gamma \vdash \sigma_1 \simeq_1^\leq \text{id} : \mathbf{fv} \forall \vec{\alpha}^\pm. M.
\end{aligned}$$

Case 3. $N = P \rightarrow M$

Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. $\Gamma \vdash [\sigma_1]N \leq_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1](P \rightarrow M) \leq_1 [\sigma_2](P \rightarrow M)$, then by congruence of substitution, $\Gamma \vdash [\sigma_1]P \rightarrow [\sigma_1]M \leq_1 [\sigma_2]P \rightarrow [\sigma_2]M$, then by inversion $\Gamma \vdash [\sigma_1]P \geq_1 [\sigma_2]P$ and $\Gamma \vdash [\sigma_1]M \leq_1 [\sigma_2]M$.

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geq_1 [\sigma_2]P$ and to $\Gamma \vdash [\sigma_1]M \leq_1 [\sigma_2]M$, we obtain (respectively):

$$\Gamma \vdash \sigma_i \simeq_1^\leq \text{id} : \mathbf{fv} P \quad (3)$$

$$\Gamma \vdash \sigma_i \simeq_1^\leq \text{id} : \mathbf{fv} M \quad (4)$$

Noting that $\mathbf{fv}(P \rightarrow M) = \mathbf{fv} P \cup \mathbf{fv} M$, we combine eqs. (3) and (4) to conclude: $\Gamma \vdash \sigma_i \simeq_1^\leq \text{id} : \mathbf{fv}(P \rightarrow M)$.

Case 4. $N = \uparrow P$

Then by inversion, $\Gamma \vdash P$. $\Gamma \vdash [\sigma_1]N \leq_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\uparrow P \leq_1 [\sigma_2]\uparrow P$, then by congruence of substitution and by inversion, $\Gamma \vdash [\sigma_1]P \geq_1 [\sigma_2]P$

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geq_1 [\sigma_2]P$, we obtain $\Gamma \vdash \sigma_i \simeq_1^\leq \text{id} : \mathbf{fv} P$. Since $\mathbf{fv} \uparrow P = \mathbf{fv} P$, we can conclude: $\Gamma \vdash \sigma_i \simeq_1^\leq \text{id} : \mathbf{fv} \uparrow P$.

Case 5. The positive cases are proved symmetrically. □

Corollary 15 (Substitution cannot induce proper subtypes or supertypes). *Assuming all mentioned types are well-formed in Γ and σ is a substitution $\Gamma \vdash \sigma : \Gamma$,*

$$\begin{aligned}
\Gamma \vdash [\sigma]N \leq_1 N &\Rightarrow \Gamma \vdash [\sigma]N \simeq_1^\leq N \text{ and } \Gamma \vdash \sigma \simeq_1^\leq \text{id} : \mathbf{fv} N \\
\Gamma \vdash N \leq_1 [\sigma]N &\Rightarrow \Gamma \vdash N \simeq_1^\leq [\sigma]N \text{ and } \Gamma \vdash \sigma \simeq_1^\leq \text{id} : \mathbf{fv} N \\
\Gamma \vdash [\sigma]P \geq_1 P &\Rightarrow \Gamma \vdash [\sigma]P \simeq_1^\leq P \text{ and } \Gamma \vdash \sigma \simeq_1^\leq \text{id} : \mathbf{fv} P \\
\Gamma \vdash P \geq_1 [\sigma]P &\Rightarrow \Gamma \vdash P \simeq_1^\leq [\sigma]P \text{ and } \Gamma \vdash \sigma \simeq_1^\leq \text{id} : \mathbf{fv} P
\end{aligned}$$

Lemma 23. *Assuming that the mentioned types (P , Q , N , and M) are well-formed in Γ and that the substitutions (σ_1 and σ_2) have signature $\Gamma \vdash \sigma_i : \Gamma$,*

- + if $\Gamma \vdash [\sigma_1]P \geq_1 Q$ and $\Gamma \vdash [\sigma_2]Q \geq_1 P$
then there exists a bijection $\mu : \mathbf{fv} P \leftrightarrow \mathbf{fv} Q$ such that $\Gamma \vdash \sigma_1 \simeq_1^\leq \mu : \mathbf{fv} P$ and $\Gamma \vdash \sigma_2 \simeq_1^\leq \mu^{-1} : \mathbf{fv} Q$;
- if $\Gamma \vdash [\sigma_1]N \leq_1 M$ and $\Gamma \vdash [\sigma_2]N \leq_1 M$
then there exists a bijection $\mu : \mathbf{fv} N \leftrightarrow \mathbf{fv} M$ such that $\Gamma \vdash \sigma_1 \simeq_1^\leq \mu : \mathbf{fv} N$ and $\Gamma \vdash \sigma_2 \simeq_1^\leq \mu^{-1} : \mathbf{fv} M$.

Proof.

- + Applying σ_2 to both sides of $\Gamma \vdash [\sigma_1]P \geq_1 Q$ (by ??), we have: $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geq_1 [\sigma_2]Q$. Composing it with $\Gamma \vdash [\sigma_2]Q \geq_1 P$ (by transitivity ??), we have $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geq_1 P$. Then by corollary 15, $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^\leq \text{id} : \mathbf{fv} P$.

By a symmetric argument, we also have: $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^\leq \text{id} : \mathbf{fv} Q$.

Now, we prove that $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^\leq \text{id} : \mathbf{fv} P$ and $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^\leq \text{id} : \mathbf{fv} Q$ implies that σ_1 and σ_1 are (equivalent to) mutually inverse bijections.

To do so, it suffices to prove that

- (i) for any $\alpha^\pm \in \mathbf{fv} P$ there exists $\beta^\pm \in \mathbf{fv} Q$ such that $\Gamma \vdash [\sigma_1]\alpha^\pm \simeq_1^\leq \beta^\pm$ and $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^\leq \alpha^\pm$; and
- (ii) for any $\beta^\pm \in \mathbf{fv} Q$ there exists $\alpha^\pm \in \mathbf{fv} P$ such that $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^\leq \alpha^\pm$ and $\Gamma \vdash [\sigma_1]\alpha^\pm \simeq_1^\leq \beta^\pm$.

Then these correspondences between $\mathbf{fv} P$ and $\mathbf{fv} Q$ are mutually inverse functions, since for any β^\pm there can be at most one α^\pm such that $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^\leq \alpha^\pm$ (and vice versa).

(i) Let us take $\alpha^\pm \in \mathbf{fv} P$.

(a) if α^\pm is positive ($\alpha^\pm = \alpha^+$), from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^+ \simeq_1^{\rightarrow} \alpha^+$, by corollary 2, we have $[\sigma_2][\sigma_1]\alpha^+ = \exists \overrightarrow{\beta^-}.\alpha^+$.

What shape can $[\sigma_1]\alpha^+$ have? It cannot be $\exists \overrightarrow{\alpha^-}.\downarrow N$ (for potentially empty α^-), because the outer constructor \downarrow would remain after the substitution σ_2 , whereas $\exists \overrightarrow{\beta^-}.\alpha^+$ does not have \downarrow . The only case left is $[\sigma_1]\alpha^+ = \exists \overrightarrow{\alpha^-}.\gamma^+$.

Notice that $\Gamma \vdash \exists \overrightarrow{\alpha^-}.\gamma^+ \simeq_1^{\rightarrow} \gamma^+$, meaning that $\Gamma \vdash [\sigma_1]\alpha^+ \simeq_1^{\rightarrow} \gamma^+$. Also notice that $[\sigma_2]\exists \overrightarrow{\alpha^-}.\gamma^+ = \exists \overrightarrow{\beta^-}.\alpha^+$ implies $\Gamma \vdash [\sigma_2]\gamma^+ \simeq_1^{\rightarrow} \alpha^+$.

(b) if α^\pm is negative ($\alpha^\pm = \alpha^-$) from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^- \simeq_1^{\rightarrow} \alpha^-$, by corollary 2, we have $[\sigma_2][\sigma_1]\alpha^- = \forall \overrightarrow{\beta^+}.\alpha^-$.

What shape can $[\sigma_1]\alpha^-$ have? It cannot be $\forall \overrightarrow{\alpha^+}.\uparrow P$ nor $\forall \overrightarrow{\alpha^+}.P \rightarrow M$ (for potentially empty $\overrightarrow{\alpha^+}$), because the outer constructor (\rightarrow or \uparrow), remaining after the substitution σ_2 , is however absent in the resulting $\forall \overrightarrow{\beta^+}.\alpha^-$. Hence, the only case left is $[\sigma_1]\alpha^- = \forall \overrightarrow{\alpha^+}.\gamma^-$. Notice that $\Gamma \vdash \gamma^- \simeq_1^{\rightarrow} \forall \overrightarrow{\alpha^+}.\gamma^-$, meaning that $\Gamma \vdash [\sigma_1]\alpha^- \simeq_1^{\rightarrow} \gamma^-$. Also notice that $[\sigma_2]\forall \overrightarrow{\alpha^+}.\gamma^- = \forall \overrightarrow{\beta^+}.\alpha^-$ implies $\Gamma \vdash [\sigma_2]\gamma^- \simeq_1^{\rightarrow} \alpha^-$.

(ii) The proof is symmetric: We swap P and Q , σ_1 and σ_2 , and exploit $\Gamma \vdash [\sigma_1][\sigma_2]\alpha^\pm \simeq_1^{\rightarrow} \alpha^\pm$ instead of $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^\pm \simeq_1^{\rightarrow} \alpha^\pm$.

– The proof is symmetric to the positive case.

□

Lemma 24 (Equivalence of polymorphic types).

- For $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N$ and $\Gamma \vdash \forall \overrightarrow{\beta^+}.M$,
if $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \simeq_1^{\rightarrow} \forall \overrightarrow{\beta^+}.M$ then there exists a bijection $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv} M \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv} N$ such that $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\rightarrow} [\mu]N$,
- + For $\Gamma \vdash \exists \overrightarrow{\alpha^-}.P$ and $\Gamma \vdash \exists \overrightarrow{\beta^-}.Q$,
if $\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \simeq_1^{\rightarrow} \exists \overrightarrow{\beta^-}.Q$ then there exists a bijection $\mu : \overrightarrow{\beta^-} \cap \mathbf{fv} Q \leftrightarrow \overrightarrow{\alpha^-} \cap \mathbf{fv} P$ such that $\Gamma, \overrightarrow{\beta^-} \vdash P \simeq_1^{\rightarrow} [\mu]Q$.

Proof.

– First, by α -conversion, we ensure $\overrightarrow{\alpha^+} \cap \mathbf{fv} M = \emptyset$ and $\overrightarrow{\beta^+} \cap \mathbf{fv} N = \emptyset$. By inversion, $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \simeq_1^{\rightarrow} \forall \overrightarrow{\beta^+}.M$ implies

1. $\Gamma, \overrightarrow{\beta^+} \vdash [\sigma_1]N \leq_1 M$ for $\Gamma, \overrightarrow{\beta^+} \vdash \sigma_1 : \overrightarrow{\alpha^+}$ and
2. $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]M \leq_1 N$ for $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 : \overrightarrow{\beta^+}$.

To apply lemma 23, we weaken and rearrange the contexts, and extend the substitutions to act as identity outside of their initial domain:

1. $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \leq_1 M$ for $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 : \Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$ and
2. $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \leq_1 N$ for $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 : \Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$.

Then from lemma 23, there exists a bijection $\mu : \mathbf{fv} M \leftrightarrow \mathbf{fv} N$ such that $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq_1^{\rightarrow} \mu : \mathbf{fv} M$ and $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 \simeq_1^{\rightarrow} \mu^{-1} : \mathbf{fv} N$.

Let us show that if we restrict the domain of μ to $\overrightarrow{\beta^+}$, its range will be contained in $\overrightarrow{\alpha^+}$. Let us take $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv} M$ and assume $[\mu]\gamma^+ \notin \overrightarrow{\alpha^+}$. Then since $\Gamma, \overrightarrow{\beta^+} \vdash \sigma_1 : \overrightarrow{\alpha^+}$, σ_1 acts as identity outside of $\overrightarrow{\alpha^+}$, i.e. $[\sigma_1][\mu]\gamma^+ = [\mu]\gamma^+$. Since $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 \simeq_1^{\rightarrow} \mu^{-1} : \mathbf{fv} N$, application of σ_1 is equivalent to application of μ^{-1} , then $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}][\mu]\gamma^+ \simeq_1^{\rightarrow} [\mu]\gamma^+$, i.e. $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \gamma^+ \simeq_1^{\rightarrow} [\mu]\gamma^+$, which means $\gamma^+ \in \mathbf{fv} [\mu]\gamma^+ \subseteq \mathbf{fv} N$. By assumption, $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv} M$, i.e. $\overrightarrow{\beta^+} \cap \mathbf{fv} N \neq \emptyset$, hence contradiction.

By ??, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq_1^{\rightarrow} \mu|_{\overrightarrow{\beta^+}} : \mathbf{fv} M$ implies $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \simeq_1^{\rightarrow} [\mu|_{\overrightarrow{\beta^+}}]M$. By similar reasoning, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \simeq_1^{\rightarrow} [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N$.

This way,

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N \leq_1 M \quad (5)$$

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu|_{\overrightarrow{\beta^+}}]M \leq_1 N \quad (6)$$

By applying $\mu|_{\overrightarrow{\beta^+}}$ to both sides of 5 (??) and contracting $\mu^{-1}|_{\overrightarrow{\alpha^+}} \circ \mu|_{\overrightarrow{\beta^+}} = \mu|_{\overrightarrow{\beta^+}}^{-1} \circ \mu|_{\overrightarrow{\beta^+}} = \text{id}$, we have: $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \leq_1 [\mu|_{\overrightarrow{\beta^+}}]M$, which together with 6 means $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \simeq_1^{\rightarrow} [\mu|_{\overrightarrow{\beta^+}}]M$, and by strengthening, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^{\rightarrow} [\mu|_{\overrightarrow{\beta^+}}]M$. Symmetrically, $\Gamma, \overrightarrow{\beta^+} \vdash M \simeq_1^{\rightarrow} [\mu|_{\overrightarrow{\beta^+}}^{-1}]N$.

- + The proof is symmetric to the proof of the negative case.

□

Lemma 25 (Completeness of equivalence). *Mutual subtyping implies declarative equivalence. Assuming all the types below are well-formed in Γ :*

- + if $\Gamma \vdash P \simeq_1^{\leq} Q$ then $P \simeq_1^D Q$,
- if $\Gamma \vdash N \simeq_1^{\leq} M$ then $N \simeq_1^D M$.

Proof. – Induction on the sum of sizes of N and M . By inversion, $\Gamma \vdash N \simeq_1^{\leq} M$ means $\Gamma \vdash N \leq_1 M$ and $\Gamma \vdash M \leq_1 N$. Let us consider the last rule that forms $\Gamma \vdash N \leq_1 M$:

Case 1. Rule (Var^{\leq_1}) i.e. $\Gamma \vdash N \leq_1 M$ is of the form $\Gamma \vdash \alpha^- \leq_1 \alpha^-$
Then $N \simeq_1^D M$ (i.e. $\alpha^- \simeq_1^D \alpha^-$) holds immediately by Rule ($\text{Var}^{\simeq_1^D}$).

Case 2. Rule (\uparrow^{\leq_1}) i.e. $\Gamma \vdash N \leq_1 M$ is of the form $\Gamma \vdash \uparrow P \leq_1 \uparrow Q$
Then by inversion, $\Gamma \vdash P \simeq_1^{\leq} Q$, and by induction hypothesis, $P \simeq_1^D Q$. Then $N \simeq_1^D M$ (i.e. $\uparrow P \simeq_1^D \uparrow Q$) holds by Rule ($\uparrow^{\simeq_1^D}$).

Case 3. Rule (\rightarrow^{\leq_1}) i.e. $\Gamma \vdash N \leq_1 M$ is of the form $\Gamma \vdash P \rightarrow N' \leq_1 Q \rightarrow M'$
Then by inversion, $\Gamma \vdash P \geq_1 Q$ and $\Gamma \vdash N' \leq_1 M'$. Notice that $\Gamma \vdash M \leq_1 N$ is of the form $\Gamma \vdash Q \rightarrow M' \leq_1 P \rightarrow N'$, which by inversion means $\Gamma \vdash Q \geq_1 P$ and $\Gamma \vdash M' \leq_1 N'$.
This way, $\Gamma \vdash Q \simeq_1^{\leq} P$ and $\Gamma \vdash M' \simeq_1^{\leq} N'$. Then by induction hypothesis, $Q \simeq_1^D P$ and $M' \simeq_1^D N'$. Then $N \simeq_1^D M$ (i.e. $P \rightarrow N' \simeq_1^D Q \rightarrow M'$) holds by Rule ($\rightarrow^{\simeq_1^D}$).

Case 4. Rule (\forall^{\leq_1}) i.e. $\Gamma \vdash N \leq_1 M$ is of the form $\Gamma \vdash \forall \alpha^+. N' \leq_1 \forall \beta^+. M'$
Then by ??, $\Gamma \vdash \forall \alpha^+. N' \simeq_1^{\leq} \forall \beta^+. M'$ means that there exists a bijection $\mu : \beta^+ \cap \text{fv } M' \leftrightarrow \alpha^+ \cap \text{fv } N'$ such that $\Gamma, \alpha^+ \vdash [\mu]M' \simeq_1^{\leq} N'$.
Notice that the application of bijection μ to M' does not change its size (which is less than the size of M), hence the induction hypothesis applies. This way, $[\mu]M' \simeq_1^D N'$ (and by symmetry, $N' \simeq_1^D [\mu]M'$) holds by induction. Then we apply Rule ($\forall^{\simeq_1^D}$) to get $\forall \alpha^+. N' \simeq_1^D \forall \beta^+. M'$, i.e. $N \simeq_1^D M$.

- + The proof is symmetric to the proof of the negative case.

□

Corollary 16 (Normalization is complete w.r.t. subtyping-induced equivalence). *Assuming all the types below are well-formed in Γ :*

- + if $\Gamma \vdash P \simeq_1^{\leq} Q$ then $\mathbf{nf}(P) = \mathbf{nf}(Q)$,
- if $\Gamma \vdash N \simeq_1^{\leq} M$ then $\mathbf{nf}(N) = \mathbf{nf}(M)$.

Proof. Immediately from lemmas 15 and 25. □

Lemma 26 (Algorithmization of subtyping-induced equivalence). *Mutual subtyping is equality of normal forms. Assuming all the types below are well-formed in Γ :*

- + $\Gamma \vdash P \simeq_1^{\leq} Q \iff \mathbf{nf}(P) = \mathbf{nf}(Q)$,
- $\Gamma \vdash N \simeq_1^{\leq} M \iff \mathbf{nf}(N) = \mathbf{nf}(M)$.

Proof. Let us prove the positive case, the negative case is symmetric. We prove both directions of \iff separately:

\Rightarrow exactly corollary 16;

\Leftarrow by lemmas 19 and 21.

□

4.8 Upgrade

Let us consider a type P well-formed in Γ . Some of its Γ -supertypes are also well-formed in a smaller context $\Delta \subseteq \Gamma$. The upgrade is the operation that returns the least of such supertypes.

Lemma 27 (Soundness of Upgrade). *Assuming P is well-formed in $\Gamma = \Delta, \alpha^\pm$, if $\mathbf{upgrade} \Gamma \vdash P \mathbf{to} \Delta = Q$ then*

1. $\Delta \vdash Q$
2. $\Gamma \vdash Q \geq_1 P$

Proof. By inversion, $\mathbf{upgrade} \Gamma \vdash P \mathbf{to} \Delta = Q$ means that for fresh $\vec{\beta}^\pm$ and $\vec{\gamma}^\pm$, $\Delta, \vec{\beta}^\pm, \vec{\gamma}^\pm \models [\vec{\beta}^\pm/\alpha^\pm]P \vee [\vec{\gamma}^\pm/\alpha^\pm]P = Q$. Then by the soundness of the least upper bound (lemma 32),

1. $\Delta, \vec{\beta}^\pm, \vec{\gamma}^\pm \vdash Q$,
2. $\Delta, \vec{\beta}^\pm, \vec{\gamma}^\pm \vdash Q \geq_1 [\vec{\beta}^\pm/\alpha^\pm]P$, and
3. $\Delta, \vec{\beta}^\pm, \vec{\gamma}^\pm \vdash Q \geq_1 [\vec{\gamma}^\pm/\alpha^\pm]P$.

$$\begin{aligned}
 \mathbf{fv} Q &\subseteq \mathbf{fv} [\vec{\beta}^\pm/\alpha^\pm]P \cap \mathbf{fv} [\vec{\gamma}^\pm/\alpha^\pm]P && \text{Since by lemma 1, } \mathbf{fv} Q \subseteq \mathbf{fv} [\vec{\beta}^\pm/\alpha^\pm]P \text{ and } \mathbf{fv} Q \subseteq \mathbf{fv} [\vec{\gamma}^\pm/\alpha^\pm]P \\
 &\subseteq ((\mathbf{fv} P \setminus \alpha^\pm) \cup \vec{\beta}^\pm) \cap ((\mathbf{fv} P \setminus \alpha^\pm) \cup \vec{\gamma}^\pm) \\
 &= (\mathbf{fv} P \setminus \alpha^\pm) \cap (\mathbf{fv} P \setminus \alpha^\pm) && \text{since } \vec{\beta}^\pm \text{ and } \vec{\gamma}^\pm \text{ are fresh} \\
 &= \mathbf{fv} P \setminus \alpha^\pm \\
 &\subseteq \Gamma \setminus \alpha^\pm && \text{since } P \text{ is well-formed in } \Gamma \\
 &\subseteq \Delta
 \end{aligned}$$

This way, by lemma 6, $\Delta \vdash Q$.

Let us apply $\vec{\alpha}^\pm/\vec{\beta}^\pm$ —the inverse of the substitution $\vec{\beta}^\pm/\alpha^\pm$ to both sides of $\Delta, \vec{\beta}^\pm, \vec{\gamma}^\pm \vdash Q \geq_1 [\vec{\beta}^\pm/\alpha^\pm]P$ and by ??, get $\Delta, \vec{\alpha}^\pm, \vec{\gamma}^\pm \vdash [\vec{\alpha}^\pm/\vec{\beta}^\pm]Q \geq_1 P$. Notice that $\Delta \vdash Q$ implies that $\mathbf{fv} Q \cap \vec{\beta}^\pm = \emptyset$, then by ??, $[\vec{\alpha}^\pm/\vec{\beta}^\pm]Q = Q$, and thus $\Delta, \vec{\alpha}^\pm, \vec{\gamma}^\pm \vdash Q \geq_1 P$. By context strengthening, $\Delta, \vec{\alpha}^\pm \vdash Q \geq_1 P$. \square

Lemma 28 (Completeness and Initiality of Upgrade). *The upgrade returns the least Γ -supertype of P well-formed in Δ . Assuming P is well-formed in $\Gamma = \Delta, \alpha^\pm$, For any Q' such that*

1. $\Delta \vdash Q'$ and
2. $\Gamma \vdash Q' \geq_1 P$,

The result of the upgrade algorithm Q exists ($\mathbf{upgrade} \Gamma \vdash P \mathbf{to} \Delta = Q$) and satisfies $\Delta \vdash Q' \geq_1 Q$.

Proof. Let us consider fresh (not intersecting with Γ) $\vec{\beta}^\pm$ and $\vec{\gamma}^\pm$.

If we apply substitution $\vec{\beta}^\pm/\alpha^\pm$ to both sides of $\Delta, \vec{\alpha}^\pm \vdash Q' \geq_1 P$, we have $\Delta, \vec{\beta}^\pm \vdash [\vec{\beta}^\pm/\alpha^\pm]Q' \geq_1 [\vec{\beta}^\pm/\alpha^\pm]P$, which by ??, since Q' is well-formed in Δ , simplifies to $\Delta, \vec{\beta}^\pm \vdash Q' \geq_1 [\vec{\beta}^\pm/\alpha^\pm]P$.

Analogously, if we apply substitution $\vec{\gamma}^\pm/\alpha^\pm$ to both sides of $\Delta, \vec{\alpha}^\pm \vdash Q' \geq_1 P$, we have $\Delta, \vec{\gamma}^\pm \vdash Q' \geq_1 [\vec{\gamma}^\pm/\alpha^\pm]P$.

This way, Q' is a common supertype of $[\vec{\beta}^\pm/\alpha^\pm]P$ and $[\vec{\gamma}^\pm/\alpha^\pm]P$ in context $\Delta, \vec{\beta}^\pm, \vec{\gamma}^\pm$. It means that we can apply the completeness of the least upper bound (lemma 33):

1. there exists Q s.t. $\Gamma \models [\vec{\beta}^\pm/\alpha^\pm]P \vee [\vec{\gamma}^\pm/\alpha^\pm]P = Q$
2. $\Gamma \vdash Q' \geq_1 Q$.

The former means that the upgrade algorithm terminates and returns Q . The latter means that since both Q' and Q are well-formed in Δ , by ??, $\Delta \vdash Q' \geq_1 Q$. \square

4.9 Upper Bounds

Lemma 29 (Decomposition of the quantifier rule). *Ilya: move somewhere* Whenever the quantifier rule (Rule $(\exists^{\geq 1})$ or Rule $(\forall^{\leq 1})$) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.

- If $\Gamma \vdash N \leq_1 \forall \vec{\beta}^+.M$ then $\Gamma, \vec{\beta}^+ \vdash N \leq_1 M$.
- + If $\Gamma \vdash P \geq_1 \exists \vec{\beta}^-.Q$ then $\Gamma, \vec{\beta}^- \vdash P \geq_1 Q$.

Lemma 30 (Characterization of the Supertypes). Let us define the set of upper bounds of a positive type $\text{UB}(P)$ in the following way:

$\Gamma \vdash P$	$\text{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists \vec{\alpha}^-. \beta^+ \mid \text{for } \vec{\alpha}^-\}$
$\Gamma \vdash \exists \vec{\beta}^-.Q$	$\text{UB}(\Gamma, \vec{\beta}^- \vdash Q) \text{ not using } \vec{\beta}^-$
$\Gamma \vdash \downarrow M$	$\left\{ \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t.} \\ \Gamma \vdash N_i, \Gamma, \vec{\alpha}^- \vdash M', \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' \simeq_1^D \downarrow M \end{array} \right\}$
Then $\text{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geq_1 P\}$.	

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Immediately from lemma 2

Case 2. $P = \exists \vec{\beta}^-.P'$

Then if $\Gamma \vdash Q \geq_1 \exists \vec{\beta}^-.P'$, then by lemma 29, $\Gamma, \vec{\beta}^- \vdash Q \geq_1 P'$, and $\mathbf{fv} Q \cap \vec{\beta}^- = \emptyset$ by the the Barendregt's convention. The other direction holds by Rule $(\exists^{\geq 1})$. This way, $\{Q \mid \Gamma \vdash Q \geq_1 \exists \vec{\beta}^-.P'\} = \{Q \mid \Gamma, \vec{\beta}^- \vdash Q \geq_1 P' \text{ s.t. } \mathbf{fv}(Q) \cap \vec{\beta}^- = \emptyset\}$. From the induction hypothesis, the latter is equal to $\text{UB}(\Gamma, \vec{\beta}^- \vdash P')$ not using $\vec{\beta}^-$, i.e. $\text{UB}(\Gamma \vdash \exists \vec{\beta}^-.P')$.

Case 3. $P = \downarrow M$

Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|\#$) and upper bounds with outer quantifiers ($|\exists$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists \vec{\beta}^-.Q'$

Then the last applied rule to infer $\Gamma \vdash Q \geq_1 \downarrow M$ must be Rule $(\downarrow^{\geq 1})$, which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq_1^< M$, then by lemma 25 and Rule $(\downarrow^{\simeq 1^D})$, $\downarrow M' \simeq_1^D \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq_1^D \downarrow M\} = \text{UB}(\Gamma \vdash \downarrow M)|\#$.

In the other direction, $\downarrow M' \simeq_1^D \downarrow M \Rightarrow \Gamma \vdash \downarrow M' \simeq_1^< \downarrow M$ by lemma 21, since $\Gamma \vdash \downarrow M'$ by lemma 20

$\Rightarrow \Gamma \vdash \downarrow M' \geq_1 \downarrow M$ by inversion

b. $Q = \exists \vec{\beta}^-.Q'$ (for non-empty $\vec{\beta}^-$)

Then the last rule applied to infer $\Gamma \vdash \exists \vec{\beta}^-.Q' \geq_1 \downarrow M$ must be Rule $(\exists^{\geq 1})$. Inversion of this rule gives us $\Gamma \vdash [\vec{N}/\vec{\beta}^-]Q' \geq_1 \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\vec{N}/\vec{\beta}^-]Q'$ has no outer quantifiers. Thus from case 3.a, $[\vec{N}/\vec{\beta}^-]Q' \simeq_1^D \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists \vec{\beta}^-. \downarrow M' \in \text{UB}(\Gamma \vdash \downarrow M)|\exists$ (notice that $\vec{\beta}^-$ is not empty).

In the other direction, $[\vec{N}/\vec{\beta}^-] \downarrow M' \simeq_1^D \downarrow M \Rightarrow \Gamma \vdash [\vec{N}/\vec{\beta}^-] \downarrow M' \simeq_1^< \downarrow M$ by lemma 21, since $\Gamma \vdash [\vec{N}/\vec{\beta}^-] \downarrow M'$ by lemma 20

$\Rightarrow \Gamma \vdash [\vec{N}/\vec{\beta}^-] \downarrow M' \geq_1 \downarrow M$ by inversion

$\Rightarrow \Gamma \vdash \exists \vec{\beta}^-. \downarrow M' \geq_1 \downarrow M$ by Rule $(\exists^{\geq 1})$

□

Lemma 31 (Characterization of the Normalized Supertypes). For a normalized positive type $P = \mathbf{nf}(P)$, let us define the set of normalized upper bounds in the following way:

$\Gamma \vdash P$	$\text{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists \vec{\beta}^-.P$	$\text{NFUB}(\Gamma, \vec{\beta}^- \vdash P) \text{ not using } \vec{\beta}^-$
$\Gamma \vdash \downarrow M$	$\left\{ \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \Gamma \vdash N_i, \Gamma, \vec{\alpha}^- \vdash M', \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' = \downarrow M \end{array} \right\}$

Then $\text{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Then from lemma 30, $\{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}(\overrightarrow{\exists \alpha^-}.\beta^+) \mid \text{for some } \overrightarrow{\alpha^-} = \{\beta^+\}\}$

Case 2. $P = \overrightarrow{\exists \beta^-}.P'$

$$\begin{aligned}
\text{NFUB}(\Gamma \vdash \overrightarrow{\exists \beta^-}.P') &= \text{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ not using } \overrightarrow{\beta^-} \\
&= \{\mathbf{nf}(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothesis} \\
&= \{\mathbf{nf}(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv} Q \cap \overrightarrow{\beta^-} = \emptyset\} && \text{because } \mathbf{fv} \mathbf{nf}(Q) = \mathbf{fv} Q \text{ by lemma 12} \\
&= \{\mathbf{nf}(Q) \mid Q \in \text{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv} Q \cap \overrightarrow{\beta^-} = \emptyset\} && \text{by lemma 30} \\
&= \{\mathbf{nf}(Q) \mid Q \in \text{UB}(\Gamma \vdash \overrightarrow{\exists \beta^-}.P')\} && \text{by the definition of UB} \\
&= \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 \overrightarrow{\exists \beta^-}.P'\} && \text{by lemma 30}
\end{aligned}$$

Case 3. $P = \downarrow M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 1. Suppose that $\nu : A \rightarrow A$ is an idempotent function, P is a predicate on A , $F : A \rightarrow B$ is a function. Then

$$\begin{aligned}
&\{F(\nu x) \mid x \in A \text{ s.t. } P(\nu x)\} = \\
&= \{F(x) \mid x \in A \text{ s.t. } \nu x = x \text{ and } P(x)\}.
\end{aligned}$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 2. For functions F and ν , and predicates P and Q ,

$$\begin{aligned}
&\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } P(x)\} = \\
&= \{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } (\exists x' \in A \text{ s.t. } P(x') \text{ and } \nu x' = \nu x)\}.
\end{aligned}$$

Observation 3. There exist positive and negative types well-formed in empty context, hence, a type substitution can be extended to an arbitrary domain (if its values on the domain extension are irrelevant). Specifically, Suppose that $\text{vars}_1 \subseteq \text{vars}_2$. Then $\Gamma \vdash \sigma|_{\text{vars}_1} : \text{vars}_1$ implies $\exists \sigma' \text{ s.t. } \Gamma \vdash \sigma' : \text{vars}_2$ and $\sigma|_{\text{vars}_1} = \sigma'|_{\text{vars}_1}$.

$$\begin{aligned}
& \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \gg_1 \downarrow M\} = \\
& = \{\mathbf{nf}(Q) \mid Q \in \mathbf{UB}(\Gamma \vdash \downarrow M)\} \\
& = \left\{ \mathbf{nf}(\exists \vec{\alpha}^-. \downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash N_i, \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \\
& = \left\{ \mathbf{nf}(\exists \vec{\alpha}^-. \downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } [\sigma] \downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \\
& = \left\{ \mathbf{nf}(\exists \vec{\alpha}^-. \downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } [\sigma|_{\mathbf{fv} M'}] \downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^{-'} . \mathbf{nf}(\downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^{-'}, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^{-'} \\ \text{and } [\sigma|_{\mathbf{fv} M'}] \downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^{-'} . \mathbf{nf}(\downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^{-'}, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^{-'} \\ \text{and } \mathbf{nf}([\sigma|_{\mathbf{fv} M'}] \downarrow M') = \mathbf{nf}(\downarrow M) \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^{-'} . \mathbf{nf}(\downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^{-'}, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^{-'} \\ \text{and } [\mathbf{nf}(\sigma|_{\mathbf{fv} M'})] \downarrow \mathbf{nf}(M') = \downarrow \mathbf{nf}(M) \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^{-'} . \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^{-'}, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^{-'} \\ \text{and } [\sigma|_{\mathbf{fv} M'}] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^{-'} . \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^{-'}, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ (\exists \sigma' \text{ s.t. } \Gamma \vdash \sigma' : \vec{\alpha}^- \text{ and } \sigma|_{\mathbf{fv}(\downarrow M')} = \sigma'|_{\mathbf{fv}(\downarrow M')}) \\ \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^{-'} \text{ and } [\sigma|_{\mathbf{fv} M'}] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^{-'} . \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^{-'}, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv} M'} : \vec{\alpha}^{-'}, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^{-'} \\ \text{and } [\sigma|_{\mathbf{fv} M'}] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^{-'} . \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^{-'}, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^{-'}, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^{-'} \\ \text{and } [\sigma] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^{-'} . \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^{-'}, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^{-'} \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^{-'}, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^{-'} \\ \text{and } [\sigma] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^{-'} . \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^{-'}, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^{-'} \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^{-'}, \text{ and } \mathbf{ord} \vec{\alpha}^{-'} \text{ in } M' = \vec{\alpha}^{-'} \\ \text{and } [\sigma] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^{-'} . \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^{-'}, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord} \vec{\alpha}^{-'} \text{ in } M' = \vec{\alpha}^{-'}, \\ \Gamma \vdash N_i, \Gamma, \vec{\alpha}^{-'} \vdash M', \text{ and } [\vec{N}/\vec{\alpha}^{-'}] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \mathbf{NFUB}(\downarrow M)
\end{aligned}$$

by lemma 30

by the definition of UB

we reassigned the substitution $\vec{N}/\vec{\alpha}^-$ as σ

by lemma 3

by the definition of normalization

from lemmas 13 and 15, equivalence of types can be replaced with the equality of their normal forms

by congruence of normalization and lemma 14

by lemma 17, $\downarrow M'$ and $\sigma|_{\mathbf{fv} M'}$ are already normal, since the result of the substitution is normal; M is normal by assumption

We apply observation 2 (with $\nu\sigma = \sigma|_{\mathbf{fv} M'}$, and $P(\sigma) = \Gamma \vdash \sigma : \vec{\alpha}^-$)

Notice that
“ $\exists \sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \vec{\alpha}^- \text{ and } \sigma|_{\mathbf{fv}(\downarrow M')} = \sigma'|_{\mathbf{fv}(\downarrow M')})$ ”
is equivalent to $\Gamma \vdash \sigma|_{\mathbf{fv}(\downarrow M')} : \vec{\alpha}^{-'} \text{ (observation 3)}$

We apply observation 1 to the restriction of σ , and remove $\sigma|_{\mathbf{fv} M'} = \sigma$ as it follows from $\Gamma \vdash \sigma : \vec{\alpha}^{-'}$

by lemma 6, since $\Gamma, \vec{\alpha}^- \cap \mathbf{fv} M' = \Gamma, \vec{\alpha}^{-'} \cap \mathbf{fv} M'$

We apply observation 1 to the ordering of $\vec{\alpha}^-$

By reassigning σ explicitly as $\vec{N}/\vec{\alpha}^{-'}$

by definition

□

Observation 4. Upper bounds of a type do not depend on the context as soon as the type are well-formed in it.

If $\Gamma_1 \vdash M$ and $\Gamma_2 \vdash M$ then $\mathbf{UB}(\Gamma_1 \vdash M) = \mathbf{UB}(\Gamma_2 \vdash M)$ and $\mathbf{NFUB}(\Gamma_1 \vdash M) = \mathbf{NFUB}(\Gamma_2 \vdash M)$

Proof. We prove both inclusions by induction on $\Gamma_1 \vdash M$. Notice that if $[\sigma]M' \simeq_1^D M$ and $\Gamma_2 \vdash M$ then the types from the range of $\sigma|_{\mathbf{fv} M'}$ are well-formed in 2 **Ilya: lemma**. □

Lemma 32 (Soundness of the Least Upper Bound). For types $\Gamma \vdash P_1$, and $\Gamma \vdash P_2$, if $\Gamma \models P_1 \vee P_2 = Q$ then

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geq_1 P_1$ and $\Gamma \vdash Q \geq_1 P_2$

Proof. Induction on $\Gamma \models P_1 \vee P_2 = Q$.

Case 1. $\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+$

Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geq_1 \alpha^+$ by Rule (Var⁺ \geq_1).

Case 2. $\Gamma \models \exists \vec{\alpha}^-. P_1 \vee \exists \vec{\beta}^-. P_2 = Q$

Then by inversion of $\Gamma \vdash \exists \vec{\alpha}^-. P_i$ and weakening, $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \vdash P_i$, hence, the induction hypothesis applies to $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \models P_1 \vee P_2 = Q$. Then

- (i) $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \vdash Q$,
- (ii) $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \vdash Q \geq_1 P_1$,
- (iii) $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \vdash Q \geq_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}(Q) \cap \Gamma, \vec{\alpha}^-, \vec{\beta}^- = \mathbf{fv}(Q) \cap \Gamma$ (and then apply lemma 6). The inclusion right-to-left is self-evident. To show $\mathbf{fv}(Q) \cap \Gamma, \vec{\alpha}^-, \vec{\beta}^- \subseteq \mathbf{fv}(Q) \cap \Gamma$, we prove that $\mathbf{fv}(Q) \subseteq \Gamma$

$$\begin{aligned} \mathbf{fv}(Q) &\subseteq \mathbf{fv} P_1 \cap \mathbf{fv} P_2 && \text{by lemma 1} \\ &\subseteq (\Gamma, \vec{\alpha}^- \setminus \vec{\beta}^-) \cap (\Gamma, \vec{\beta}^- \setminus \vec{\alpha}^-) && \begin{array}{l} \text{since } \Gamma \vdash \exists \vec{\alpha}^-. P_1, \mathbf{fv}(P_1) \subseteq \Gamma, \vec{\alpha}^- = \Gamma, \vec{\alpha}^- \setminus \vec{\beta}^- \\ \text{(the latter is because by the Barendregt's convention,} \\ \Gamma, \vec{\alpha}^- \cap \vec{\beta}^- = \emptyset); \text{ similarly, } \mathbf{fv}(P_2) \subseteq \Gamma, \vec{\beta}^- \setminus \vec{\alpha}^- \end{array} \\ &\subseteq \Gamma \end{aligned}$$

To show $\Gamma \vdash Q \geq_1 \exists \vec{\alpha}^-. P_1$, we apply Rule ($\exists \geq_1$). Then $\Gamma, \vec{\alpha}^- \vdash Q \geq_1 P_1$ holds since $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \vdash Q \geq_1 P_1$ (by the induction hypothesis), $\Gamma, \vec{\alpha}^- \vdash Q$ (by weakening), and $\Gamma, \vec{\alpha}^- \vdash P_1$.

Judgment $\Gamma \vdash Q \geq_1 \exists \vec{\beta}^-. P_2$ is proved symmetrically.

Case 3. $\Gamma \models \downarrow N \vee \downarrow M = \exists \vec{\alpha}^-. [\vec{\alpha}^- / \Xi] P$. By the inversion, $\Gamma, \cdot \models \mathbf{nf}(\downarrow N) \stackrel{a}{\simeq} \mathbf{nf}(\downarrow M) = (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)$. Then by the soundness of anti-unification (??),

(i) $\Gamma; \Xi \vdash P$, then by ??,

$$\Gamma, \vec{\alpha}^- \vdash [\vec{\alpha}^- / \Xi] P \quad (7)$$

(ii) $\Gamma; \cdot \vdash \hat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \hat{\tau}_2 : \Xi$. Assuming that $\Xi = \hat{\beta}_1^-, \dots, \hat{\beta}_n^-$, the antiunification solutions $\hat{\tau}_1$ and $\hat{\tau}_2$ can be put explicitly as $\hat{\tau}_1 = (\hat{\beta}_1^- : \approx N_1, \dots, \hat{\beta}_n^- : \approx N_n)$, and $\hat{\tau}_2 = (\hat{\beta}_1^- : \approx M_1, \dots, \hat{\beta}_n^- : \approx M_n)$. Then

$$\hat{\tau}_1 = (\vec{N} / \vec{\alpha}^-) \circ (\vec{\alpha}^- / \Xi) \quad (8)$$

$$\hat{\tau}_2 = (\vec{M} / \vec{\alpha}^-) \circ (\vec{\alpha}^- / \Xi) \quad (9)$$

(iii) $[\hat{\tau}_1] Q = P_1$ and $[\hat{\tau}_2] Q = P_1$, which, by 8 and 9, means

$$[\vec{N} / \vec{\alpha}^-][\vec{\alpha}^- / \Xi] P = \mathbf{nf}(\downarrow N) \quad (10)$$

$$[\vec{M} / \vec{\alpha}^-][\vec{\alpha}^- / \Xi] P = \mathbf{nf}(\downarrow M) \quad (11)$$

Then $\Gamma \vdash \exists \vec{\alpha}^-. [\vec{\alpha}^- / \Xi] P$ follows directly from 7.

To show $\Gamma \vdash \exists \vec{\alpha}^-. [\vec{\alpha}^- / \Xi] P \geq_1 \downarrow N$, we apply Rule ($\exists \geq_1$), instantiating $\vec{\alpha}^-$ with \vec{N} . Then $\Gamma \vdash [\vec{N} / \vec{\alpha}^-][\vec{\alpha}^- / \Xi] P \geq_1 \downarrow N$ follows from 10 and since $\Gamma \vdash \mathbf{nf}(\downarrow N) \geq_1 \downarrow N$ (by corollary 14).

Analogously, instantiating $\vec{\alpha}^-$ with \vec{M} , gives us $\Gamma \vdash [\vec{M} / \vec{\alpha}^-][\vec{\alpha}^- / \Xi] P \geq_1 \downarrow M$ (from 11), and hence, $\Gamma \vdash \exists \vec{\alpha}^-. [\vec{\alpha}^- / \Xi] P \geq_1 \downarrow M$. \square

Lemma 33 (Completeness and Initiality of the Least Upper Bound). *For types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q$ such that $\Gamma \vdash Q \geq_1 P_1$ and $\Gamma \vdash Q \geq_1 P_2$, there exists Q' s.t. $\Gamma \models P_1 \vee P_2 = Q'$ and $\Gamma \vdash Q \geq_1 Q'$.*

Proof. Induction on the pair (P_1, P_2) . From lemma 31, $Q \in \mathbf{UB}(\Gamma \vdash P_1) \cap \mathbf{UB}(\Gamma \vdash P_2)$. Let us consider the cases of what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_i$).

Case 1. $P_1 = \exists \vec{\beta}^-_1. Q_1$, $P_2 = \exists \vec{\beta}^-_2. Q_2$, where either $\vec{\beta}^-_1$ or $\vec{\beta}^-_2$ is not empty

Then $Q \in \text{UB}(\Gamma \vdash \exists \vec{\beta}^-_1. Q_1) \cap \text{UB}(\Gamma \vdash \exists \vec{\beta}^-_2. Q_2)$
 $\subseteq \text{UB}(\Gamma, \vec{\beta}^-_1 \vdash Q_1) \cap \text{UB}(\Gamma, \vec{\beta}^-_2 \vdash Q_2)$ from the definition of UB
 $= \text{UB}(\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q_1) \cap \text{UB}(\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q_2)$ by observation 4, weakening and exchange
 $= \{Q' \mid \Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q' \geq_1 Q_1\} \cap \{Q' \mid \Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q' \geq_1 Q_2\}$ by lemma 30,
 meaning that $\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q \geq_1 Q_1$ and $\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q \geq_1 Q_2$. Then the next step of the algorithm—the recursive call $\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q_1 \vee Q_2 = Q'$ terminates by the induction hypothesis, and moreover, $\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q \geq_1 Q'$. This way, the result of the algorithm is Q' , i.e. $\Gamma \vdash P_1 \vee P_2 = Q'$.

Since both Q and Q' are sound, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q \geq_1 Q'$ can be strengthened to $\Gamma \vdash Q \geq_1 Q'$ by ??.

Case 2. $P_1 = \alpha^+$ and $P_2 = \downarrow N$

Then the set of common upper bounds of $\downarrow N$ and α^+ is empty, and thus, $Q \in \text{UB}(\Gamma \vdash P_1) \cap \text{UB}(\Gamma \vdash P_2)$ gives a contradiction:
 $Q \in \text{UB}(\Gamma \vdash \alpha^+) \cap \text{UB}(\Gamma \vdash \downarrow N)$
 $= \{\exists \vec{\alpha}^-. \alpha^+ \mid \dots\} \cap \{\exists \vec{\beta}^-. \downarrow M' \mid \dots\}$ by the definition of UB
 $= \emptyset$ since $\alpha^+ \neq \downarrow M'$ for any M'

Case 3. $P_1 = \downarrow N$ and $P_2 = \alpha^+$

Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)

Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$Q \in \text{UB}(\Gamma \vdash \alpha^+) \cap \text{UB}(\Gamma \vdash \beta^+)$
 $= \{\exists \vec{\alpha}^-. \alpha^+ \mid \dots\} \cap \{\exists \vec{\beta}^-. \beta^+ \mid \dots\}$ by the definition of UB
 $= \emptyset$ since $\alpha^+ \neq \beta^+$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Then the algorithm terminates in one step (Rule (Var^v)) and the result is α^+ , i.e. $\Gamma \vdash \alpha^+ \vee \alpha^+ = \alpha^+$.

Since $Q \in \text{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists \vec{\alpha}^-. \alpha^+$. Then $\Gamma \vdash \exists \vec{\alpha}^-. \alpha^+ \geq_1 \alpha^+$ by Rule ($\exists \geq_1$): $\vec{\alpha}^-$ can be instantiated with arbitrary negative types (for example $\forall \beta^+. \uparrow \beta^+$), since the substitution for unused variables does not change the term $[\vec{N}/\vec{\alpha}^-] \alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geq_1 \alpha^+$ by Rule (Var⁺ \geq_1).

Case 6. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$

Then on the next step, the algorithm tries to anti-unify $\mathbf{nf}(\downarrow M_1)$ and $\mathbf{nf}(\downarrow M_2)$. By ??, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$$\begin{aligned} \mathbf{nf}(Q) &\in \text{NFUB}(\Gamma \vdash \mathbf{nf}(\downarrow M_1)) \cap \text{NFUB}(\Gamma \vdash \mathbf{nf}(\downarrow M_2)) \\ &= \text{NFUB}(\Gamma \vdash \downarrow \mathbf{nf}(M_1)) \cap \text{NFUB}(\Gamma \vdash \downarrow \mathbf{nf}(M_2)) \\ &= \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \Gamma \vdash N_i, \Gamma, \vec{\alpha}^- \vdash M', \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_1) \end{array} \right\} \\ &= \cap \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \Gamma \vdash \vec{N}_1, \Gamma \vdash \vec{N}_2, \Gamma, \vec{\alpha}^- \vdash M', \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_2) \end{array} \right\} \\ &= \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \vec{N}_1 \text{ and } \vec{N}_2 \text{ s.t. } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \Gamma \vdash \vec{N}_1, \Gamma \vdash \vec{N}_2, \Gamma, \vec{\alpha}^- \vdash M', [\vec{N}_1/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_1), \text{ and } [\vec{N}_2/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_2) \end{array} \right\} \end{aligned}$$

The fact that the latter set is non-empty means that there exist $\vec{\alpha}^-$, M' , \vec{N}_1 and \vec{N}_2 such that

- (i) $\Gamma, \vec{\alpha}^- \vdash M'$ (notice that M' is normal)
- (ii) $\Gamma \vdash \vec{N}_1$ and $\Gamma \vdash \vec{N}_2$,
- (iii) $[\vec{N}_1/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_1)$ and $[\vec{N}_2/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_2)$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative anti-unification variable $\hat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\hat{\alpha}^-}$. Let us show that $(\overrightarrow{\hat{\alpha}^-}, [\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M', \overrightarrow{N_1}/\overrightarrow{\hat{\alpha}^-}, \overrightarrow{N_2}/\overrightarrow{\hat{\alpha}^-})$ is a sound anti-unifier of $\mathbf{nf}(\downarrow M_1)$ and $\mathbf{nf}(\downarrow M_2)$ in context Γ :

- $\overrightarrow{\hat{\alpha}^-}$ is negative by construction,
- $\Gamma; \overrightarrow{\hat{\alpha}^-} \vdash [\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M'$ because $\Gamma, \overrightarrow{\alpha^-} \vdash \downarrow M'$ **Ilya: lemma!**,
- $\Gamma; \cdot \vdash (\overrightarrow{N_1}/\overrightarrow{\hat{\alpha}^-}) : \overrightarrow{\hat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N_1}$ and $\Gamma; \cdot \vdash (\overrightarrow{N_2}/\overrightarrow{\hat{\alpha}^-}) : \overrightarrow{\hat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N_2}$,
- $[\overrightarrow{N_1}/\overrightarrow{\hat{\alpha}^-}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{N_1}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow \mathbf{nf}(M_1) = \mathbf{nf}(\downarrow M_1)$.
- $[\overrightarrow{N_2}/\overrightarrow{\hat{\alpha}^-}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{N_2}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow \mathbf{nf}(M_2) = \mathbf{nf}(\downarrow M_2)$.

Then by the completeness of the anti-unification (??), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it, i.e. $Q' = \exists \overrightarrow{\beta^-}. [\overrightarrow{\beta^-}/\Xi] P$, where $(\Xi, P, \hat{\tau}_1, \hat{\tau}_2)$ is the result of the anti-unification of $\mathbf{nf}(\downarrow M_1)$ and $\mathbf{nf}(\downarrow M_2)$ in context Γ .

Moreover, ?? also says that the found solution is initial, i.e. there exists $\hat{\tau}$ such that $\Gamma; \Xi \vdash \hat{\tau} : \overrightarrow{\hat{\alpha}^-}$ and $[\hat{\tau}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}$, (ii) $\hat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]\downarrow M' = [\overrightarrow{\beta^-}/\Xi][\hat{\tau}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{\beta^-}/\Xi]P$. In particular, from the reflexivity of subtyping: $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]\downarrow M' \geq_1 [\overrightarrow{\beta^-}/\Xi]P$.

It allows us to show $\Gamma \vdash \mathbf{nf}(Q) \geq_1 Q'$, i.e. $\Gamma \vdash \exists \overrightarrow{\alpha^-}. \downarrow M' \geq_1 \exists \overrightarrow{\beta^-}. [\overrightarrow{\beta^-}/\Xi]P$, by applying Rule ($\exists \geq_1$), instantiating $\overrightarrow{\alpha^-}$ with respect to σ . Finally, $\Gamma \vdash Q \geq_1 Q'$ since $\Gamma \vdash \mathbf{nf}(Q) \simeq_1^s Q$, and equivalence implies subtyping by **Ilya: lemma**.

□

4.10 Unification

Lemma 34 (Soundness of Unification).

- + For normalized P and Q such that $\Gamma; \Theta \vdash P$ and $\Gamma \vdash Q$,
if $\Gamma; \Theta \models P \simeq^u Q \Rightarrow \hat{\sigma}$ then $\hat{\sigma} : \Theta|_{\mathbf{uv} P}$, $\hat{\sigma}$ is equivalence-only, and $[\hat{\sigma}]P = Q$.
- For normalized N and M such that $\Gamma; \Theta \vdash N$ and $\Gamma \vdash M$,
if $\Gamma; \Theta \models N \simeq^u M \Rightarrow \hat{\sigma}$ then $\hat{\sigma} : \Theta|_{\mathbf{uv} N}$, $\hat{\sigma}$ is equivalence-only, and $[\hat{\sigma}]N = M$.

Lemma 35 (Completeness of Unification).

- + For normalized P and Q such that $\Gamma; \Theta \vdash P$ and $\Gamma \vdash Q$,
assume there exists $\hat{\sigma} : \Theta|_{\mathbf{uv} P}$ such that $[\hat{\sigma}]P = Q$. Then $\Gamma; \Theta \models N \simeq^u M \Rightarrow \hat{\sigma}$.
- For normalized N and M such that $\Gamma; \Theta \vdash N$ and $\Gamma \vdash M$,
assume there exists $\hat{\sigma} : \Theta|_{\mathbf{uv} N}$ such that $[\hat{\sigma}]N = M$. Then $\Gamma; \Theta \models N \simeq^u M \Rightarrow \hat{\sigma}$.

4.11 Anti-unification

Lemma 36 (Soundness of Anti-Unification).

- +
-

Lemma 37 (Completeness of Anti-Unification).

- +
-

4.12 Solution Merge

Lemma 38. *Given a fixed context Γ , weakening forms a preorder on the set of entries well-formed in Γ .*

Proof.

- Reflexivity: $\Gamma \vdash e \simeq e$ then $\Gamma \vdash e \Rightarrow e$. Let us consider the shape of e . In all cases, there is a rule inferring $\Gamma \vdash e \Rightarrow e$, since $\Gamma \vdash P \simeq_1^< Q$ implies $\Gamma \vdash P \geq_1 Q$ by inversion.
- Transitivity: $\Gamma \vdash e_1 \Rightarrow e_2$ and $\Gamma \vdash e_2 \Rightarrow e_3$ implies $\Gamma \vdash e_1 \Rightarrow e_3$. It follows immediately by considering the rules inferring $\Gamma \vdash e_1 \Rightarrow e_2$ and $\Gamma \vdash e_2 \Rightarrow e_3$, and applying the transitivity of subtyping (corollary 3).

□

Lemma 39 (Solution Weakening forms a preorder). *Let us consider a set of pairs $(\Theta, \hat{\sigma})$ such that $\hat{\sigma} : \Theta$. Then the relation defined as $(\Theta_1, \hat{\sigma}_1) \Rightarrow (\Theta_2, \hat{\sigma}_2)$ iff $\Theta_1 \supseteq \Theta_2$ and $\Theta_2 \vdash \hat{\sigma}_1 \Rightarrow \hat{\sigma}_2$ forms a preorder.*

Proof.

- Reflexivity: $(\Theta, \hat{\sigma}) \Rightarrow (\Theta, \hat{\sigma})$.
It is clear that $\Theta \supseteq \Theta$. $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}$ holds since for any e in $\hat{\sigma}$ restricting $\hat{\alpha}^\pm$, $\Theta(\hat{\alpha}^\pm) \vdash e \Rightarrow e$ by lemma 38
- Transitivity: If $(\Theta_1, \hat{\sigma}_1) \Rightarrow (\Theta_2, \hat{\sigma}_2)$ and $(\Theta_2, \hat{\sigma}_2) \Rightarrow (\Theta_3, \hat{\sigma}_3)$ then $(\Theta_1, \hat{\sigma}_1) \Rightarrow (\Theta_3, \hat{\sigma}_3)$.
It is clear that since $\Theta_1 \subseteq \Theta_2$ and $\Theta_2 \subseteq \Theta_3$ then $\Theta_1 \subseteq \Theta_3$. Let us consider $e_3 \in \hat{\sigma}_3$. Then there exists $e_2 \in \hat{\sigma}_2$ such that $\Theta_3 \vdash e_2 \Rightarrow e_3$. Also there exists $e_1 \in \hat{\sigma}_1$ such that $\Theta_2 \vdash e_1 \Rightarrow e_2$, and by weakening, $\Theta_3 \vdash e_1 \Rightarrow e_2$. Then by transitivity of solution entry weakening (lemma 38), $\Theta_3 \vdash e_1 \Rightarrow e_3$.

□

Corollary 17 (Solution Weakening is transitive). *If $\Theta \vdash \hat{\sigma}_1 \Rightarrow \hat{\sigma}_2$ and $\Theta \vdash \hat{\sigma}_2 \Rightarrow \hat{\sigma}_3$ then $\Theta \vdash \hat{\sigma}_1 \Rightarrow \hat{\sigma}_3$.*

Lemma 40 (Soundness of Merge of Unification Solution Entries). *For a fixed context Γ , suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$, where $\Gamma \vdash e_i$ is an equivalence-shaped restriction (i.e. it has shape $\hat{\alpha}^+ : \approx P$ or $\hat{\alpha}^- : \approx N$ but not $\hat{\alpha}^+ : \geq P$). If $e_1 \& e_2$ is defined then*

1. $\Gamma \vdash e_1 \& e_2$
2. $\Gamma \vdash e_1 \& e_2 \Rightarrow e_1$
3. $\Gamma \vdash e_1 \& e_2 \Rightarrow e_2$

Proof. Let us consider the rule forming $\Gamma \vdash e_1 \& e_2 = e$.

Case 1. Rule $(\simeq \&^+ \simeq)$, i.e. $\Gamma \vdash e_1 \& e_2 = e$ has form $\Gamma \vdash (\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \approx P') = (\hat{\alpha}^+ : \approx P)$ and $\mathbf{nf}(P) = \mathbf{nf}(P')$. Then

1. $\Gamma \vdash e$, i.e. $\Gamma \vdash \hat{\alpha}^+ : \approx P$ holds by assumption;
2. $\Gamma \vdash (\hat{\alpha}^+ : \approx P) \Rightarrow (\hat{\alpha}^+ : \approx P)$ holds by reflexivity (lemma 38);
3. $\Gamma \vdash (\hat{\alpha}^+ : \approx P) \Rightarrow (\hat{\alpha}^+ : \approx P')$ holds because $\mathbf{nf}(P) = \mathbf{nf}(P')$ implies $\Gamma \vdash P \simeq_1^< P'$ (by lemma 26), and thus, Rule $(\simeq \&^+ \simeq)$ applies.

Case 2. Rule $(\simeq \&^- \simeq)$. The negative case is proved in exactly the same way.

□

Lemma 41 (Soundness of Merge of Unification Solutions). *Suppose that $\hat{\sigma}_1 : \Theta|_{vars_1}$ and $\hat{\sigma}_2 : \Theta|_{vars_2}$ are unification solutions (i.e. $\hat{\sigma}_1$ and $\hat{\sigma}_2$ can only have equivalence-shaped restrictions). If $\hat{\sigma}_1 \& \hat{\sigma}_2$ is defined then*

1. $\hat{\sigma}_1 \& \hat{\sigma}_2 : \Theta|_{vars_1 \cup vars_2}$,
2. $\Theta|_{vars_1} \vdash \hat{\sigma}_1 \& \hat{\sigma}_2 \Rightarrow \hat{\sigma}_1$, and
3. $\Theta|_{vars_2} \vdash \hat{\sigma}_1 \& \hat{\sigma}_2 \Rightarrow \hat{\sigma}_2$.

Proof. Let us prove the properties separately:

1. $\hat{\sigma}_1 \& \hat{\sigma}_2 : \Theta|_{vars_1 \cup vars_2}$. It suffices to prove the following two properties:

- The set of variables of the entries of $\hat{\sigma}_1 \& \hat{\sigma}_2$ coincides with $vars_1 \cup vars_2$
By definition, $\hat{\sigma}_1 \& \hat{\sigma}_2$ consists of three parts: entries of $\hat{\sigma}_1$ that do not have matching entries of $\hat{\sigma}_2$, entries of $\hat{\sigma}_2$ that do not have matching entries of $\hat{\sigma}_1$, and the merge of matching entries. It means that $\mathbf{dom}(\hat{\sigma}_1 \& \hat{\sigma}_2) = \mathbf{dom}(\hat{\sigma}_1) \setminus \mathbf{dom}(\hat{\sigma}_2) \cup \mathbf{dom}(\hat{\sigma}_2) \setminus \mathbf{dom}(\hat{\sigma}_1) \cup \mathbf{dom}(\hat{\sigma}_1) \cap \mathbf{dom}(\hat{\sigma}_2) = \mathbf{dom}(\hat{\sigma}_1) \cup \mathbf{dom}(\hat{\sigma}_2)$, which, since $\hat{\sigma}_i : \Theta|_{vars_i}$, is equal to $vars_1 \cup vars_2$.

- Each entry of $\hat{\sigma}_1$ & $\hat{\sigma}_2$ restricting $\hat{\alpha}^\pm$ is well-formed in the corresponding context $\Theta(\hat{\alpha}^\pm)$.
Let us consider an arbitrary entry e of $\hat{\sigma}_1$ & $\hat{\sigma}_2$ restricting $\hat{\alpha}^\pm$. Then there are three cases:
 - (a) e the entry is from $\hat{\sigma}_1$ and does not have a matching entry in $\hat{\sigma}_2$, i.e. $\hat{\alpha}^\pm \in \mathbf{dom}(\hat{\sigma}_1) \setminus \mathbf{dom}(\hat{\sigma}_2)$. Then e is well-formed in $\Theta|_{\text{vars}_1}$ by assumption.
 - (b) e the entry is from $\hat{\sigma}_2$ and does not have a matching entry in $\hat{\sigma}_1$. This case is symmetric.
 - (c) e is the merge of two matching entries $e_1 \in \hat{\sigma}_1$ and $e_2 \in \hat{\sigma}_2$ restricting $\hat{\alpha}^\pm$. Since $\hat{\sigma}_1 : \Theta|_{\text{vars}_1}$ and $\hat{\sigma}_2 : \Theta|_{\text{vars}_2}$, $\hat{\alpha}^\pm \in \mathbf{dom}(\Theta|_{\text{vars}_1}) \cap \mathbf{dom}(\Theta|_{\text{vars}_2})$, i.e. there is an entry $\hat{\alpha}^\pm\{\Gamma\} \in \Theta$, and e_1 and e_2 are well-formed in Γ . Then by lemma 40, $\Gamma \vdash e_1 \& e_2$, where $\Gamma = \Theta(\hat{\alpha}^\pm)$.
- 2. $\Theta|_{\text{vars}_1} \vdash \hat{\sigma}_1 \& \hat{\sigma}_2 \Rightarrow \hat{\sigma}_1$ We need to show that for every entry e from $\hat{\sigma}_1$, there is an entry e' from $\hat{\sigma}_1 \& \hat{\sigma}_2$ such that $\Theta|_{\text{vars}_1} \vdash e' \Rightarrow e$. Let us consider an arbitrary e from $\hat{\sigma}_1$ restricting $\hat{\alpha}^\pm$. Then there are two cases:
 - e does not have a matching entry in $\hat{\sigma}_2$. Then e is also in $\hat{\sigma}_1 \& \hat{\sigma}_2$ and $\Theta|_{\text{vars}_1} \vdash e \Rightarrow e$ by reflexivity (lemma 38).
 - e has a matching entry e' in $\hat{\sigma}_2$. Then $e \& e'$ is in $\hat{\sigma}_1 \& \hat{\sigma}_2$ and $\Theta|_{\text{vars}_1} \vdash e \& e' \Rightarrow e$ by lemma 40.
- 3. $\Theta|_{\text{vars}_2} \vdash \hat{\sigma}_1 \& \hat{\sigma}_2 \Rightarrow \hat{\sigma}_2$ is proved analogously. □

Lemma 42 (Soundness of Solution Entry Merge). *For a fixed context Γ , suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$. If $e_1 \& e_2$ is defined then*

1. $\Gamma \vdash e_1 \& e_2$
2. $\Gamma \vdash e_1 \& e_2 \Rightarrow e_1$
3. $\Gamma \vdash e_1 \& e_2 \Rightarrow e_2$

Proof. Let us consider the rule forming $\Gamma \vdash e_1 \& e_2 = e$.

Case 1. Rule $(\simeq \&^+ \simeq)$ or Rule $(\simeq \&^- \simeq)$ The proof is analogous to the proof of lemma 40.

Case 2. Rule $(\geq \&^+ \geq)$ Then e_1 is $\hat{\alpha}^+ : \geq P_1$, e_2 is $\hat{\alpha}^+ : \geq P_2$, and $e_1 \& e_2$ is $\hat{\alpha}^+ : \geq Q$ where Q is the least upper bound of P_1 and P_2 . Then by lemma 32,

- $\Gamma \vdash Q$, and hence $\Gamma \vdash \hat{\alpha}^+ : \geq Q$, that is $\Gamma \vdash e_1 \& e_2$,
- $\Gamma \vdash Q \geq_1 P_1$, and hence $\Gamma \vdash \hat{\alpha}^+ : \geq Q \Rightarrow \hat{\alpha}^+ : \geq P_1$, that is $\Gamma \vdash e_1 \& e_2 \Rightarrow e_1$,
- $\Gamma \vdash Q \geq_1 P_2$, and hence $\Gamma \vdash e_1 \& e_2 \Rightarrow e_2$.

Case 3. Rule $(\geq \&^+ \simeq)$ Then e_1 is $\hat{\alpha}^+ : \geq P$, e_2 is $\hat{\alpha}^+ : \approx Q$, where $\Gamma; \cdot \models Q \geq P \Rightarrow \hat{\sigma}'$, and the resulting $e_1 \& e_2$ is equal to e_2 , that is $\hat{\alpha}^+ : \approx Q$.

- By assumption, $\Gamma \vdash Q$, and hence $\Gamma \vdash \hat{\alpha}^+ : \approx Q$, that is $\Gamma \vdash e_1 \& e_2$.
- Since $\mathbf{uv}(Q) = \emptyset$, $\Gamma; \cdot \models Q \geq P \Rightarrow \hat{\sigma}'$ implies $\Gamma \vdash Q \geq_1 P$ by the soundness of positive subtyping (lemma 46), which means $\Gamma \vdash \hat{\alpha}^+ : \approx Q \Rightarrow \hat{\alpha}^+ : \geq P$, that is $\Gamma \vdash e_1 \& e_2 \Rightarrow e_1$.
- $\Gamma \vdash \hat{\alpha}^+ : \approx Q \Rightarrow \hat{\alpha}^+ : \approx Q$ by reflexivity (lemma 38), and hence $\Gamma \vdash e_1 \& e_2 \Rightarrow e_2$.

Case 4. Rule $(\simeq \&^+ \geq)$ The proof is analogous to the previous case. □

Lemma 43 (Soundness of Solution Merge). *Suppose that $\hat{\sigma}_1 : \Theta_1$ and $\hat{\sigma}_2 : \Theta_2$ and $\hat{\sigma}_1 \& \hat{\sigma}_2$ is defined. Then*

1. $\hat{\sigma}_1 \& \hat{\sigma}_2 : \Theta_1 \cup \Theta_2$,
2. $\Theta_1 \vdash \hat{\sigma}_1 \& \hat{\sigma}_2 \Rightarrow \hat{\sigma}_1$, and
3. $\Theta_2 \vdash \hat{\sigma}_1 \& \hat{\sigma}_2 \Rightarrow \hat{\sigma}_2$.

Proof. The proof repeats the proof of lemma 41, but uses lemma 42 instead of lemma 40. □

Lemma 44 (Completeness and Initiality of Solution Merge). *Suppose that $\hat{\sigma}_1 : \Theta_1$ and $\hat{\sigma}_2 : \Theta_2$ and there exists $\hat{\sigma}$ such that $\Theta_1 \vdash \hat{\sigma}_1 \Rightarrow \hat{\sigma}$ and $\Theta_2 \vdash \hat{\sigma}_2 \Rightarrow \hat{\sigma}$. Then $\hat{\sigma}_1 \& \hat{\sigma}_2$ is defined and $\Theta_1 \cup \Theta_2 \vdash \hat{\sigma}_1 \& \hat{\sigma}_2 \Rightarrow \hat{\sigma}$.*

Lemma 45 (Solution Weakening is Monotonous). *if $\hat{\sigma}_1 : \Theta$ and $\hat{\sigma}_2 : \Theta$ and $\hat{\sigma}_1 \subseteq \hat{\sigma}_2$ then $\Theta \vdash \hat{\sigma}_2 \Rightarrow \hat{\sigma}_1$.*

4.13 Subtyping Algorithm

Lemma 46 (Soundness of Positive Subtyping). *If $\Gamma \vdash^\geq \Theta$, $\Gamma \vdash Q$, and $\Gamma; \Theta \vdash P$ then*

$\Gamma; \Theta \models P \geq Q \equiv \hat{\sigma} \text{ implies } \hat{\sigma} : \Theta|_{\mathbf{uv} P}$ and for any $\hat{\sigma}'$ such that $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}$, $\Gamma \vdash [\hat{\sigma}']P \geq_1 Q$.

Proof. We prove it by induction on $\Gamma; \Theta \models P \geq Q \equiv \hat{\sigma}$. Let us consider the last rule to infer this judgment.

Case 1. Rule (UVar $^\geq$) then $\Gamma; \Theta \models P \geq Q \equiv \hat{\sigma}$ has shape $\Gamma; \Theta \models \hat{\alpha}^+ \geq P' \equiv (\hat{\alpha}^+ : \geq Q')$ where $\hat{\alpha}^+ \{\Delta\} \in \Theta$ and **upgrade** $\Gamma \vdash P' \text{ to } \Delta = Q'$.

Notice that $\hat{\alpha}^+ \{\Delta\} \in \Theta$ and $\Gamma \vdash^\geq \Theta$ implies $\Gamma = \Delta, \bar{\alpha}^\pm$ for some $\bar{\alpha}^\pm$, hence, the soundness of upgrade (lemma 27) is applicable:

1. $\Delta \vdash Q'$ and
2. $\Gamma \vdash Q \geq_1 P$.

Since $\hat{\alpha}^+ \{\Delta\} \in \Theta|_{\mathbf{uv} \hat{\alpha}^+}$ and $\Delta \vdash Q'$, it is clear that $(\hat{\alpha}^+ : \geq Q') : \Theta|_{\mathbf{uv} \hat{\alpha}^+}$.

It is left to show that $\Gamma \vdash [\hat{\sigma}']\hat{\alpha}^+ \geq_1 P'$ for any $\hat{\sigma}'$ s.t. $\Theta \vdash \hat{\sigma}' \Rightarrow (\hat{\alpha}^+ : \geq Q')$. The latter weakening statement means that either $\hat{\sigma}' \ni \hat{\alpha}^+ : \geq Q''$ or $\hat{\sigma}' \ni (\hat{\alpha}^+ : \approx Q'')$ for $\Delta \vdash Q'' \geq_1 Q'$. In any case, $\Delta \vdash [\hat{\sigma}']\hat{\alpha}^+ \geq_1 Q$. By weakening the context to Γ and combining this judgment transitively (??) with $\Gamma \vdash Q \geq_1 P$, we have $\Gamma \vdash [\hat{\sigma}']\hat{\alpha}^+ \geq_1 P$, as required.

Case 2. Rule (Var $^\geq$) then $\Gamma; \Theta \models P \geq Q \equiv \hat{\sigma}$ has shape $\Gamma; \Theta \models \alpha^+ \geq \alpha^+ \equiv \cdot$. Then $\mathbf{uv} \alpha^+ = \emptyset$, and $\hat{\sigma} = \cdot : \cdot$ satisfies $\hat{\sigma} : \Theta|_{\emptyset}$. Since $\mathbf{uv} \alpha^+ = \emptyset$, application of any unification solution $\hat{\sigma}'$ does not change α^+ , i.e. $[\hat{\sigma}']\alpha^+ = \alpha^+$. Therefore, $\Gamma \vdash [\hat{\sigma}']\alpha^+ \geq_1 \alpha^+$ holds by Rule (Var $^\leq_1$).

Case 3. Rule (\downarrow^\geq) then $\Gamma; \Theta \models P \geq Q \equiv \hat{\sigma}$ has shape $\Gamma; \Theta \models \downarrow N \geq \downarrow M \equiv \hat{\sigma}$.

Then the next step of the algorithm is the unification of $\mathbf{nf}(N)$ and $\mathbf{nf}(M)$. By the soundness of the unification algorithm (lemma 34), it returns an equivalence-only solution $\hat{\sigma}$ such that $\hat{\sigma} : \mathbf{uv} N$. By ??, since $\hat{\sigma}$ is equivalence-only and $\Gamma \vdash^\geq \Theta$, $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}$ means $\Gamma \vdash \hat{\sigma}' \simeq_1 \hat{\sigma} : \mathbf{uv} N$ as substitutions. $[\hat{\sigma}]\mathbf{nf}(N) = \mathbf{nf}(M)$ implies $\Gamma \vdash [\hat{\sigma}]\mathbf{nf}(N) \simeq_1 \mathbf{nf}(M)$, and then $\Gamma \vdash [\hat{\sigma}']\mathbf{nf}(N) \simeq_1 \mathbf{nf}(M)$. **Ilya: add lemmas**

Let us rewrite the left-hand side and the right-hand side of $\Gamma \vdash [\hat{\sigma}']\mathbf{nf}(N) \simeq_1 \mathbf{nf}(M)$ by transitivity of equivalence (corollary 4). By corollaries 5 and 14, $\Gamma \vdash [\hat{\sigma}']\mathbf{nf}(N) \simeq_1 [\hat{\sigma}']N$. By corollary 14, $\Gamma \vdash \mathbf{nf}(M) \simeq_1 M$. This way, we have $\Gamma \vdash [\hat{\sigma}']N \simeq_1 M$. Then by Rule (\uparrow^{\leq_1}) and congruence of substitution, $\Gamma \vdash [\hat{\sigma}']\downarrow N \geq_1 \downarrow M$.

Case 4. Rule (\exists^\geq) then $\Gamma; \Theta \models P \geq Q \equiv \hat{\sigma}$ has shape $\Gamma; \Theta \models \exists \bar{\alpha}^\pm. P' \geq \exists \bar{\beta}^\pm. Q' \equiv \hat{\sigma}$ s.t. either $\bar{\alpha}^\pm$ or $\bar{\beta}^\pm$ is not empty.

Then the algorithm creates fresh unification variables $\hat{\alpha}^\pm \{\Gamma, \bar{\beta}^\pm\}$, substitutes the old $\bar{\alpha}^\pm$ with them in P' , and makes the recursive call: $\Gamma, \bar{\beta}^\pm; \Theta, \hat{\alpha}^\pm \{\Gamma, \bar{\beta}^\pm\} \models [\hat{\alpha}^\pm / \bar{\alpha}^\pm] P' \geq Q' \equiv \hat{\sigma}'$, returning as the result $\hat{\sigma} = \hat{\sigma}' \setminus \hat{\alpha}^\pm$.

Notice that $\Gamma, \bar{\beta}^\pm \vdash^\geq \Theta, \hat{\alpha}^\pm \{\Gamma, \bar{\beta}^\pm\}$, $\Gamma, \bar{\beta}^\pm \vdash Q'$, and $\Gamma, \bar{\beta}^\pm; \Theta, \hat{\alpha}^\pm \{\Gamma, \bar{\beta}^\pm\} \vdash [\hat{\alpha}^\pm / \bar{\alpha}^\pm] P'$, so the induction hypothesis is applicable, that is $\hat{\sigma}' : \Theta|_{\mathbf{uv} [\hat{\alpha}^\pm / \bar{\alpha}^\pm] P'}$, and $\Gamma, \bar{\beta}^\pm \vdash [\hat{\sigma}'_2][\hat{\alpha}^\pm / \bar{\alpha}^\pm] P' \geq_1 Q'$ for any $\hat{\sigma}'_2$ s.t. $\Theta \vdash \hat{\sigma}'_2 \Rightarrow \hat{\sigma}'$.

Since the domain of $\hat{\sigma}'$ is $\mathbf{uv} [\hat{\alpha}^\pm / \bar{\alpha}^\pm] N'$, the domain of $\hat{\sigma} = \hat{\sigma}' \setminus \hat{\alpha}^\pm$ is $\mathbf{uv} [\hat{\alpha}^\pm / \bar{\alpha}^\pm] N \setminus \hat{\alpha}^\pm = \mathbf{uv} N' = \mathbf{uv} \forall \bar{\alpha}^\pm. N'$, this way, $\hat{\sigma} : \Theta|_{\mathbf{uv} \forall \bar{\alpha}^\pm. N'}$, as required.

It is left to show that $\Gamma \vdash [\hat{\sigma}'_2]\forall \bar{\alpha}^\pm. N' \leq_1 \forall \bar{\beta}^\pm. M'$ for any $\hat{\sigma}'_2$ s.t. $\Theta \vdash \hat{\sigma}'_2 \Rightarrow \hat{\sigma}$. Let us consider an arbitrary such $\hat{\sigma}'_2$. Let us construct $\hat{\sigma}'_2$, extending $\hat{\sigma}'_2$ to the domain $\mathbf{uv} [\hat{\alpha}^\pm / \bar{\alpha}^\pm] N'$ with the values of $\hat{\sigma}'$, i.e. $\hat{\sigma}'_2 : \Theta|_{\mathbf{uv} [\hat{\alpha}^\pm / \bar{\alpha}^\pm] N'}$, and

$$\hat{\sigma}'_2(\hat{\beta}^\pm) = \begin{cases} \hat{\sigma}'_2(\hat{\beta}^\pm) & \text{if } \hat{\beta}^\pm \in \mathbf{uv} N' \\ \hat{\sigma}'(\hat{\beta}^\pm) & \text{if } \hat{\beta}^\pm \in \bar{\alpha}^\pm \end{cases}$$

, where the application of the unification solution to a variable returns the corresponding *unification entry*. Notice that $\hat{\sigma}'_2|_{\mathbf{uv} N'} = \hat{\sigma}'_2$. It is easy to see that $\Theta \vdash \hat{\sigma}'_2 \Rightarrow \hat{\sigma}'$:

1. if $\hat{\beta}^\pm \in \mathbf{uv} N'$ then $\hat{\sigma}'_2(\hat{\beta}^\pm) = \hat{\sigma}'_2(\hat{\beta}^\pm) \Rightarrow \hat{\sigma}(\hat{\beta}^\pm) = \hat{\sigma}'(\hat{\beta}^\pm)$;
2. it $\hat{\beta}^\pm \in \bar{\alpha}^\pm$ then $\hat{\sigma}'_2(\hat{\beta}^\pm) = \hat{\sigma}'(\hat{\beta}^\pm) \Rightarrow \hat{\sigma}'(\hat{\beta}^\pm)$,

which means that the induction hypothesis can be applied to $\hat{\sigma}'_2$, i.e. $\Gamma, \bar{\beta}^\pm \vdash [\hat{\sigma}'_2][\hat{\alpha}^\pm / \bar{\alpha}^\pm] N' \leq_1 M'$.

Notice that $[\hat{\sigma}'_2][\hat{\alpha}^\pm / \bar{\alpha}^\pm] N' = [\hat{\sigma}'_2|_{\bar{\alpha}^\pm} \circ \hat{\alpha}^\pm / \bar{\alpha}^\pm][\hat{\sigma}'_2|_{\mathbf{uv} N'}] N'$ by substitution properties **Ilya: todo**

$$= [\hat{\sigma}'_2|_{\bar{\alpha}^\pm} \circ \hat{\alpha}^\pm / \bar{\alpha}^\pm][\hat{\sigma}'_2] N' \quad \text{since } \hat{\sigma}'_2|_{\mathbf{uv} N'} = \hat{\sigma}'_2.$$

Also notice that the domain of $\hat{\sigma}'_2|_{\bar{\alpha}^\pm} \circ \hat{\alpha}^\pm / \bar{\alpha}^\pm$ is $\bar{\alpha}^\pm$, and the range is $\Gamma, \bar{\beta}^\pm$, i.e. $\Gamma, \bar{\beta}^\pm \vdash \hat{\sigma}'_2|_{\bar{\alpha}^\pm} \circ \hat{\alpha}^\pm / \bar{\alpha}^\pm : \bar{\alpha}^\pm$, which means that we can apply Rule (\forall^{\leq_1}) to $\Gamma, \bar{\beta}^\pm \vdash [\hat{\sigma}'_2|_{\bar{\alpha}^\pm} \circ \hat{\alpha}^\pm / \bar{\alpha}^\pm][\hat{\sigma}'_2] N' \leq_1 M'$ to get $\Gamma \vdash [\hat{\sigma}'_2]\forall \bar{\alpha}^\pm. N' \leq_1 \forall \bar{\beta}^\pm. M'$, as required.

Lemma 47 (Soundness of Negative Subtyping). *If $\Gamma \vdash^\exists \Theta$, $\Gamma \vdash M$, and $\Gamma; \Theta \vdash N$ then $\Gamma; \Theta \models N \leq M \Leftrightarrow \hat{\sigma} : \Theta|_{\mathbf{uv} N}$ and for any $\hat{\sigma}'$ such that $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}$, $\Gamma \vdash [\hat{\sigma}']N \leq_1 M$*

Proof. We prove it by induction on $\Gamma; \Theta \models N \leq M \Leftrightarrow \hat{\sigma}$. Let us consider the last rule to infer this judgment.

Case 1. $\Gamma; \Theta \models \alpha^- \leq \alpha^- \Leftrightarrow \cdot$

Then $\mathbf{uv} \alpha^- = \emptyset$, and $\hat{\sigma} = \cdot$ satisfies $\hat{\sigma} : \Theta|_{\emptyset}$. Since $\mathbf{uv} \alpha^- = \emptyset$, application of any unification solution $\hat{\sigma}'$ does not change α^- , i.e. $[\hat{\sigma}']\alpha^- = \alpha^-$. Therefore, $\Gamma \vdash [\hat{\sigma}']\alpha^- \leq_1 \alpha^-$ holds by Rule (Var $^{\leq_1}$).

Case 2. $\Gamma; \Theta \models \uparrow P \leq \uparrow Q \Leftrightarrow \hat{\sigma}$

Then the next step of the algorithm is the unification of $\mathbf{nf}(P)$ and $\mathbf{nf}(Q)$. By the soundness of the unification algorithm (lemma 34), it returns an equivalence-only solution $\hat{\sigma}$ such that $\hat{\sigma} : \mathbf{uv} P$. By ??, since $\hat{\sigma}$ is equivalence-only and $\Gamma \vdash^\exists \Theta$, $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}$ means $\Gamma \vdash \hat{\sigma}' \simeq_1^{\leq} \hat{\sigma} : \mathbf{uv} P$ as substitutions.

$[\hat{\sigma}]\mathbf{nf}(P) = \mathbf{nf}(Q)$ implies $\Gamma \vdash [\hat{\sigma}]\mathbf{nf}(P) \simeq_1^{\leq} \mathbf{nf}(Q)$, and then $\Gamma \vdash [\hat{\sigma}']\mathbf{nf}(P) \simeq_1^{\leq} \mathbf{nf}(Q)$. **Ilya: add lemmas**

Let us rewrite the left-hand side and the right-hand side of $\Gamma \vdash [\hat{\sigma}']\mathbf{nf}(P) \simeq_1^{\leq} \mathbf{nf}(Q)$ by transitivity of equivalence (corollary 4). By corollaries 5 and 14, $\Gamma \vdash [\hat{\sigma}']\mathbf{nf}(P) \simeq_1^{\leq} [\hat{\sigma}']P$. By corollary 14, $\Gamma \vdash \mathbf{nf}(Q) \simeq_1^{\leq} Q$. This way, we have $\Gamma \vdash [\hat{\sigma}']P \simeq_1^{\leq} Q$. Then by Rule (\uparrow^{\leq_1}) and congruence of substitution, $\Gamma \vdash [\hat{\sigma}']\uparrow P \leq_1 \uparrow Q$.

Case 3. $\Gamma; \Theta \models P \rightarrow N' \leq Q \rightarrow M' \Leftrightarrow \hat{\sigma}$

The next step of the algorithm is two recursive calls: $\Gamma; \Theta \models P \geq Q \Leftrightarrow \hat{\sigma}_1$ and $\Gamma; \Theta \models N' \leq M' \Leftrightarrow \hat{\sigma}_2$. By the soundness of positive subtyping (lemma 46) and induction hypothesis, respectively

1. $\hat{\sigma}_1 : \Theta|_{\mathbf{uv} P}$ and $\Gamma \vdash [\hat{\sigma}_1]P \geq_1 Q$ for any $\hat{\sigma}'_1$ s.t. $\Theta \vdash \hat{\sigma}'_1 \Rightarrow \hat{\sigma}_1$
2. $\hat{\sigma}_2 : \Theta|_{\mathbf{uv} N'}$ and $\Gamma \vdash [\hat{\sigma}_2]N' \leq_1 M'$ for any $\hat{\sigma}'_2$ s.t. $\Theta \vdash \hat{\sigma}'_2 \Rightarrow \hat{\sigma}_2$

Then the algorithm merges two unification solutions $\hat{\sigma}_1$ and $\hat{\sigma}_2$. By lemma 43, since $\mathbf{uv} P \cup \mathbf{uv} N' = \mathbf{uv}(P \rightarrow N')$, we have $\hat{\sigma}_1 \& \hat{\sigma}_2 : \Theta|_{\mathbf{uv}(P \rightarrow N')}$, and also $\Theta \vdash \hat{\sigma}_1 \& \hat{\sigma}_2 \Rightarrow \hat{\sigma}_1$ and $\Theta \vdash \hat{\sigma}_1 \& \hat{\sigma}_2 \Rightarrow \hat{\sigma}_2$. By the transitivity of solution weakening (corollary 17), $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}_1 \& \hat{\sigma}_2$ implies $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}_1$ and $\Theta \vdash \hat{\sigma}' \Rightarrow \hat{\sigma}_2$.

The application of the induction hypothesis gives us $\Gamma \vdash [\hat{\sigma}']P \geq_1 Q$ and $\Gamma \vdash [\hat{\sigma}']N' \leq_1 M'$. Finally, by Rule (\rightarrow^{\leq_1}), $\Gamma \vdash [\hat{\sigma}'](P \rightarrow N') \leq_1 Q \rightarrow M'$.

Case 4. $\Gamma; \Theta \models \forall \vec{\alpha}^+. N' \leq \forall \vec{\beta}^+. M' \Leftrightarrow \hat{\sigma}$ s.t. either $\vec{\alpha}^+$ or $\vec{\beta}^+$ is not empty

Then the algorithm creates fresh unification variables $\hat{\alpha}^+ \{\Gamma, \vec{\beta}^+\}$, substitutes the old $\vec{\alpha}^+$ with them in N' , and makes the recursive call: $\Gamma, \vec{\beta}^+; \Theta, \hat{\alpha}^+ \{\Gamma, \vec{\beta}^+\} \models [\hat{\alpha}^+ / \vec{\alpha}^+] N' \leq M' \Leftrightarrow \hat{\sigma}'$, returning as the resulting solution $\hat{\sigma} = \hat{\sigma}' \setminus \hat{\alpha}^+$.

Notice that $\Gamma, \vec{\beta}^+ \vdash^\exists \Theta, \hat{\alpha}^+ \{\Gamma, \vec{\beta}^+\}$, $\Gamma, \vec{\beta}^+ \vdash M'$, and $\Gamma, \vec{\beta}^+; \Theta, \hat{\alpha}^+ \{\Gamma, \vec{\beta}^+\} \vdash [\hat{\alpha}^+ / \vec{\alpha}^+] N'$, so the induction hypothesis is applicable, that is $\hat{\sigma}' : \Theta|_{\mathbf{uv} [\hat{\alpha}^+ / \vec{\alpha}^+] N'}$ and $\Gamma, \vec{\beta}^+ \vdash [\hat{\sigma}_2'] [\hat{\alpha}^+ / \vec{\alpha}^+] N' \leq_1 M'$ for any $\hat{\sigma}'_2$ s.t. $\Theta \vdash \hat{\sigma}'_2 \Rightarrow \hat{\sigma}'$.

Since the domain of $\hat{\sigma}'$ is $\mathbf{uv} [\hat{\alpha}^+ / \vec{\alpha}^+] N'$, the domain of $\hat{\sigma} = \hat{\sigma}' \setminus \hat{\alpha}^+$ is $\mathbf{uv} [\hat{\alpha}^+ / \vec{\alpha}^+] N' \setminus \hat{\alpha}^+ = \mathbf{uv} N' = \mathbf{uv} \forall \vec{\alpha}^+. N'$, this way, $\hat{\sigma} : \Theta|_{\mathbf{uv} \forall \vec{\alpha}^+. N'}$, as required.

It is left to show that $\Gamma \vdash [\hat{\sigma}_2'] \forall \vec{\alpha}^+. N' \leq_1 \forall \vec{\beta}^+. M'$ for any $\hat{\sigma}_2$ s.t. $\Theta \vdash \hat{\sigma}_2 \Rightarrow \hat{\sigma}$. Let us consider an arbitrary such $\hat{\sigma}_2$. Let us construct $\hat{\sigma}'_2$, extending $\hat{\sigma}_2$ to the domain $\mathbf{uv} [\hat{\alpha}^+ / \vec{\alpha}^+] N'$ with the values of $\hat{\sigma}'$, i.e. $\hat{\sigma}'_2 : \Theta|_{\mathbf{uv} [\hat{\alpha}^+ / \vec{\alpha}^+] N'}$, and

$$\hat{\sigma}'_2(\hat{\beta}^\pm) = \begin{cases} \hat{\sigma}_2(\hat{\beta}^\pm) & \text{if } \hat{\beta}^\pm \in \mathbf{uv} N' \\ \hat{\sigma}'(\hat{\beta}^\pm) & \text{if } \hat{\beta}^\pm \in \hat{\alpha}^+ \end{cases}$$

, where the application of the unification solution to a variable returns the corresponding *unification entry*. Notice that $\hat{\sigma}'_2|_{\mathbf{uv} N'} = \hat{\sigma}_2$. It is easy to see that $\Theta \vdash \hat{\sigma}'_2 \Rightarrow \hat{\sigma}'$:

1. if $\hat{\beta}^\pm \in \mathbf{uv} N'$ then $\hat{\sigma}'_2(\hat{\beta}^\pm) = \hat{\sigma}_2(\hat{\beta}^\pm) \Rightarrow \hat{\sigma}(\hat{\beta}^\pm) = \hat{\sigma}'(\hat{\beta}^\pm)$;
2. if $\hat{\beta}^\pm \in \hat{\alpha}^+$ then $\hat{\sigma}'_2(\hat{\beta}^\pm) = \hat{\sigma}'(\hat{\beta}^\pm) \Rightarrow \hat{\sigma}'(\hat{\beta}^\pm)$,

which means that the induction hypothesis can be applied to $\hat{\sigma}'_2$, i.e. $\Gamma, \vec{\beta}^+ \vdash [\hat{\sigma}'_2] [\hat{\alpha}^+ / \vec{\alpha}^+] N' \leq_1 M'$.

Notice that $[\hat{\sigma}'_2] [\hat{\alpha}^+ / \vec{\alpha}^+] N' = [\hat{\sigma}'_2|_{\hat{\alpha}^+} \circ \hat{\alpha}^+ / \vec{\alpha}^+] [\hat{\sigma}'_2|_{\mathbf{uv} N'}] N'$ by substitution properties **Ilya: todo**

$$= [\hat{\sigma}'_2|_{\hat{\alpha}^+} \circ \hat{\alpha}^+ / \vec{\alpha}^+] [\hat{\sigma}_2] N' \quad \text{since } \hat{\sigma}'_2|_{\mathbf{uv} N'} = \hat{\sigma}_2.$$

Also notice that the domain of $\hat{\sigma}'_2|_{\hat{\alpha}^+} \circ \hat{\alpha}^+ / \vec{\alpha}^+$ is $\vec{\alpha}^+$, and the range is $\Gamma, \vec{\beta}^+$, i.e. $\Gamma, \vec{\beta}^+ \vdash \hat{\sigma}'_2|_{\hat{\alpha}^+} \circ \hat{\alpha}^+ / \vec{\alpha}^+ : \vec{\alpha}^+$, which means that we can apply Rule (\forall^{\leq_1}) to $\Gamma, \vec{\beta}^+ \vdash [\hat{\sigma}'_2|_{\hat{\alpha}^+} \circ \hat{\alpha}^+ / \vec{\alpha}^+] [\hat{\sigma}_2] N' \leq_1 M'$ to get $\Gamma \vdash [\hat{\sigma}_2] \forall \vec{\alpha}^+. N' \leq_1 \forall \vec{\beta}^+. M'$, as required.

□

Theorem 1 (Soundness of Subtyping Algorithm).

Proof.

□

Lemma 48 (Completeness of the Positive Subtyping). *Suppose that $\Gamma \vdash^\exists \Theta$, $\Gamma \vdash Q$ and $\Gamma; \Theta \vdash P$ and there exists $\hat{\sigma} : \Theta|_{\mathbf{uv} P}$ such that $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$. Then there exists $\hat{\sigma}'$ such that $\Gamma; \Theta \models P \geq Q = \hat{\sigma}'$ and $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'$.*

Proof. We prove it by induction on $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$. Let us consider the last rule used in the derivation of $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$, but first consider the base case for the substitution $[\hat{\sigma}]P$:

Case 1. $P = \exists \beta^{\rightarrow} . \hat{\alpha}^+$ (for potentially empty β^{\rightarrow})

Then by assumption, there exists $\hat{\sigma} : \Theta|_{\mathbf{uv} P}$ such that $\Gamma \vdash \exists \beta^{\rightarrow} . [\hat{\sigma}]\hat{\alpha}^+ \geq_1 Q$. $\hat{\sigma} : \Theta|_{\mathbf{uv} \hat{\alpha}^+}$ means that $\hat{\sigma}$ is either $(\hat{\alpha}^+ : \approx P')$ or $(\hat{\alpha}^+ : \geq P')$, where $\Delta \vdash P'$ and $\hat{\alpha}^+ \{\Delta\} \in \Theta$. $\Gamma \vdash \exists \beta^{\rightarrow} . [\hat{\sigma}]\hat{\alpha}^+ \geq_1 Q$ means that $\Gamma \vdash P' \geq_1 Q$ because multiple inversions of Rule $(\exists \geq_1)$ gives us $\Gamma \vdash [\hat{\sigma}]\hat{\alpha}^+ \geq_1 Q$ since $\beta^{\rightarrow} \cap \mathbf{fv} [\hat{\sigma}]\hat{\alpha}^+ = \emptyset$.

In the algorithm, after multiple applications of Rule $(\exists \geq)$ the type $\exists \beta^{\rightarrow} . \hat{\alpha}^+$ is reduced to $\hat{\alpha}^+$. Next, the algorithm tries to apply Rule $(\mathbf{UVar} \geq)$ and the resulting solution is $\hat{\sigma}' = (\hat{\alpha}^+ : \geq Q')$ where $\mathbf{upgrade} \Gamma \vdash Q \text{ to } \Delta = Q'$.

Why does the upgrade procedure terminates? Because P' satisfies the pre-conditions of the completeness of the upgrade (lemma 28) :

- $\Delta \vdash P'$ because $P' = [\hat{\sigma}]\hat{\alpha}^+$, $\hat{\sigma} : \Theta|_{\mathbf{uv} P}$ and $\hat{\alpha}^+ \{\Delta\} \in \Theta|_{\mathbf{uv} P}$,
- $\Gamma \vdash P' \geq_1 Q$ as noted before

Moreover, completeness of the upgrade also gives us $\Gamma \vdash P' \geq_1 Q'$ and further, we strengthen it to $\Delta \vdash P' \geq_1 Q'$ (since by the soundness of the upgrade (lemma 27), $\Delta \vdash Q'$).

It means that $\Theta \vdash (\hat{\alpha}^+ : \approx P') \Rightarrow (\hat{\alpha}^+ : \geq Q')$ and $\Theta \vdash (\hat{\alpha}^+ : \geq P') \Rightarrow (\hat{\alpha}^+ : \geq Q')$, which means that in any case, $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'$.

Case 2. $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$ is derived by Rule (\mathbf{Var}^{\leq_1}) , i.e. $[\hat{\sigma}]N = M = \alpha^- = N$ (the latter equality holds because $N \neq \hat{\alpha}^-$).

Notice that $\hat{\sigma} : \Theta|_{\mathbf{uv} \alpha^-} = \hat{\sigma} : \Theta|_{\emptyset} = \cdot$.

The algorithm applies Rule (\mathbf{Var}^{\leq}) and infers $\hat{\sigma}' = \cdot$, i.e. $\Gamma; \Theta \models \alpha^- \leq \alpha^- = \cdot$.

Since $\Theta \vdash \cdot \Rightarrow \cdot$, we have $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'$.

Case 3. The positive case when $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$ is derived by Rule $(\downarrow \geq_1)$ is symmetric to the corresponding negative case 1.

Notice that the algorithm does not make a recursive call in this case but rather invoke the unification. This way, the proof does not apply the induction hypothesis, and hence, the absence of negative unification variables in a negative type ($\hat{\alpha}^- \notin N$ for $\downarrow N = P$) is not required.

Case 4. The positive case when $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$ is derived by Rule $(\exists \geq_1)$ is symmetric to the corresponding negative case 3.

Notice that we only consider the case when the substitution $[\hat{\sigma}]P$ results in the existential type $\exists \alpha^{\rightarrow} . P'$ (for $P' \neq \exists \dots$) by congruence, i.e. $P = \exists \alpha^{\rightarrow} . P''$ (for $P'' \neq \exists \dots$) and $[\hat{\sigma}]P'' = P'$. This is because the case when $P = \exists \beta^{\rightarrow} . \hat{\alpha}^+$ has been covered (case 1), and thus, the substitution $\hat{\sigma}$ must preserve all the outer quantifiers of P and does not generate any new ones.

Case 5. The positive case when $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$ is derived by Rule (\mathbf{Var}^{\geq_1}) is symmetric to the corresponding negative case 2.

Notice that $[\hat{\sigma}]P = \beta^+$ implies $P = \beta^+$ because the case when $P = \hat{\alpha}^+$ has been covered (case 1).

□

Lemma 49 (Completeness of the Negative Subtyping). *Suppose that $\Gamma \vdash^\exists \Theta$, $\Gamma \vdash M$, $\Gamma; \Theta \vdash N$, N does not contain negative unification variables ($\hat{\alpha}^- \notin \mathbf{uv} N$) and there exists $\hat{\sigma} : \Theta|_{\mathbf{uv} N}$ such that $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$. Then there exists $\hat{\sigma}'$ such that $\Gamma; \Theta \models N \leq M = \hat{\sigma}'$ and $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'$.*

Proof. We prove it by induction on $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$. Let us consider the last rule used in the derivation of $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$.

Case 1. $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$ is derived by Rule $(\uparrow \leq_1)$, i.e. $[\hat{\sigma}]N = \uparrow[\hat{\sigma}]P$ (since the substitution $[\hat{\sigma}]N$ must preserve the top-level constructor of $N \neq \hat{\alpha}^-$), and $M = \uparrow Q$, and by inversion, $\Gamma \vdash [\hat{\sigma}]P \leq_1 Q$.

Since both types start with \uparrow , the algorithm tries to apply Rule $(\uparrow \leq)$: $\Gamma; \Theta \models \uparrow P \leq \uparrow Q = \hat{\sigma}'$. The premise of this rule is the unification of $\mathbf{nf}(P)$ and $\mathbf{nf}(Q)$: $\Gamma; \Theta \models \mathbf{nf}(P) \stackrel{u}{\approx} \mathbf{nf}(Q) = \hat{\sigma}'$. Let us show that the unification successfully terminates and returns $\hat{\sigma}' = \mathbf{nf}(\hat{\sigma})$.

To demonstrate that the unification terminates, we apply the completeness of the unification algorithm (lemma 35). In order to do that, we need to provide a unifier of $\mathbf{nf}(P)$ and $\mathbf{nf}(Q)$. Thankfully, $\mathbf{nf}(\hat{\sigma})$ does it.

- $\mathbf{nf}(P)$ and $\mathbf{nf}(Q)$ are normalized
- $\Gamma; \Theta \vdash \mathbf{nf}(P)$ because $\Gamma; \Theta \vdash P$ (??)
- $\Gamma \vdash \mathbf{nf}(Q)$ because $\Gamma \vdash Q$ (corollary 12)
- $\mathbf{nf}(\hat{\sigma}) : \Theta|_{\mathbf{uv} \mathbf{nf}(P)}$ because $\hat{\sigma} : \Theta|_{\mathbf{uv} P}$ (??)
- $\Gamma \vdash [\hat{\sigma}]P \simeq_1^< Q \Rightarrow [\hat{\sigma}]P \simeq_1^D Q$ by lemma 25
- $\Rightarrow \mathbf{nf}([\hat{\sigma}]P) = \mathbf{nf}(Q)$ by lemma 15
- $\Rightarrow [\mathbf{nf}(\hat{\sigma})]\mathbf{nf}(P) = \mathbf{nf}(Q)$ by lemma 14

Then by the completeness of the unification, $\Gamma; \Theta \models \mathbf{nf}(P) \stackrel{u}{\simeq} \mathbf{nf}(Q) = \mathbf{nf}(\hat{\sigma})$. This way, the subtyping algorithm terminates and the resulting solution is $\mathbf{nf}(\hat{\sigma})$.

It is left to note that $\Theta \vdash \hat{\sigma} \Rightarrow \mathbf{nf}(\hat{\sigma})$, by ??, since $\Theta \vdash \hat{\sigma} \simeq \mathbf{nf}(\hat{\sigma})$

Case 2. $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$ is derived by Rule (\rightarrow^{\leq_1}) , i.e. $[\hat{\sigma}]N = [\hat{\sigma}]P \rightarrow [\hat{\sigma}]N'$ and $M = Q \rightarrow M'$, and by inversion, $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$ and $\Gamma \vdash [\hat{\sigma}]N' \leq_1 M'$.

Let us consider restrictions of $\hat{\sigma}$ to the set of unification variables in P and N' : $\hat{\sigma}|_{\mathbf{uv} P} : \Theta|_{\mathbf{uv} P}$ and $\hat{\sigma}|_{\mathbf{uv} N'} : \Theta|_{\mathbf{uv} N'}$. Notice that $[\hat{\sigma}]P = [\hat{\sigma}|_{\mathbf{uv} P}]P$ and $[\hat{\sigma}]N' = [\hat{\sigma}|_{\mathbf{uv} N'}]N'$.

Let us apply the induction hypothesis to $\Gamma \vdash [\hat{\sigma}|_{\mathbf{uv} P}]P \geq_1 Q$ and $\Gamma \vdash [\hat{\sigma}|_{\mathbf{uv} N'}]N' \leq_1 M'$ (notice that since $\mathbf{uv} N' \subseteq \mathbf{uv} N$, there are no negative unification variables in N') to obtain $\hat{\sigma}'_1$ and $\hat{\sigma}'_2$ such that

1. $\Gamma; \Theta \models P \geq Q = \hat{\sigma}'_1$ and $\Theta \vdash \hat{\sigma}|_{\mathbf{uv} P} \Rightarrow \hat{\sigma}'_1$
2. $\Gamma; \Theta \models P \geq Q = \hat{\sigma}'_2$ and $\Theta \vdash \hat{\sigma}|_{\mathbf{uv} N'} \Rightarrow \hat{\sigma}'_2$

This way, the algorithm applies Rule (\rightarrow^{\leq}) and terminates returning $\hat{\sigma}'_1$ & $\hat{\sigma}'_2$ as the result.

It is left to show that $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'_1$ & $\hat{\sigma}'_2$. Since $\hat{\sigma}|_{\mathbf{uv} P} \subseteq \hat{\sigma}$, by lemma 45, $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}|_{\mathbf{uv} P}$, and then by transitivity (corollary 17), $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'_1$. Analogously, $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'_2$. Then since by lemma 44, $\hat{\sigma}'_1$ & $\hat{\sigma}'_2$ is the weakest restriction implying both $\hat{\sigma}'_1$ and $\hat{\sigma}'_2$, we have $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'_1$ & $\hat{\sigma}'_2$, as required.

Case 3. $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$ is derived by Rule (\forall^{\leq_1}) , i.e. $[\hat{\sigma}]N = \forall \alpha^+. [\hat{\sigma}]N'$ (assuming α^+ does not intersect with the range of $\hat{\sigma}$) and $M = \forall \beta^+. M'$, where either α^+ or β^+ is non-empty.

By inversion, $\Gamma, \beta^+ \vdash [\sigma][\hat{\sigma}]N' \leq_1 M'$ for $\Gamma, \beta^+ \vdash \sigma : \alpha^+$. Since σ and $\hat{\sigma}$ have disjoint domains, and the range of one does not intersect with the domain of the other, they commute, i.e. $\Gamma, \beta^+ \vdash [\hat{\sigma}][\sigma]N' \leq_1 M'$ (notice that the tree inferring this judgement is a proper subtree of the tree inferring $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$).

On the next step, the algorithm creates fresh (disjoint with $\mathbf{uv} N'$) unification variables $\hat{\alpha}^+$, replaces α^+ with them in $[\hat{\sigma}]N'$, and makes the recursive call: $\Gamma, \beta^+; \Theta, \hat{\alpha}^+ \{ \Gamma, \beta^+ \} \models N_0 \leq M = \hat{\sigma}_1$, (where $N_0 = [\hat{\alpha}^+ / \alpha^+]N'$), returning $\hat{\sigma}_1 \setminus \hat{\alpha}^+$ as the result.

Notice that $[\hat{\sigma}][\sigma][\hat{\alpha}^+ / \alpha^+]N_0 = [\hat{\sigma}][\sigma]N'$, and thus, $\Gamma, \beta^+ \vdash [\hat{\sigma}][\sigma][\hat{\alpha}^+ / \alpha^+]N_0 \leq_1 M'$. Let us combine $\hat{\sigma}$ and $\sigma \circ \hat{\alpha}^+ / \hat{\alpha}^+$ into one $\hat{\sigma}_0$ an unification solution for N_0 :

$$\hat{\sigma}_0(\hat{\beta}^\pm) = \begin{cases} \hat{\beta}^\pm : \approx [\sigma]\alpha_i^+ & \text{if } \hat{\beta}^\pm = \hat{\alpha}_i^+ \in \hat{\alpha}^+ \\ \hat{\sigma}(\hat{\beta}^\pm) & \text{if } \hat{\beta}^\pm \in \mathbf{uv} N' \end{cases}$$

Let us show that the induction hypothesis is applicable to $\Gamma, \beta^+ \vdash [\hat{\sigma}_0]N_0 \leq_1 M'$.

- $\hat{\sigma}_0 : \Theta|_{\mathbf{uv} N'}, \hat{\alpha}^+ \{ \Gamma, \beta^+ \}$. Notice that for every $\hat{\alpha}_i^+$, the type corresponding to the entry $\hat{\sigma}_0(\hat{\alpha}_i^+)$ is well-formed in Γ, β^+ since $\Gamma, \beta^+ \vdash \sigma : \alpha_i^+$; and for every $\hat{\beta}^\pm \in \mathbf{uv} N'$, the type corresponding to the entry $\hat{\sigma}_0(\hat{\beta}^\pm)$ is well-formed in context $\Theta(\hat{\beta}^\pm)$ since $\hat{\sigma} : \Theta|_{\mathbf{uv} N'}$.
- $\Gamma, \beta^+ \vdash [\hat{\sigma}_0]N_0 \leq_1 M'$. This is because $[\hat{\sigma}_0]N_0 = [\hat{\sigma}][\sigma][\hat{\alpha}^+ / \alpha^+]N_0 = [\hat{\sigma}][\sigma]N'$.
- $\Gamma, \beta^+ \vdash \supset \Theta, \hat{\alpha}^+ \{ \Gamma, \beta^+ \}$ since $\Gamma, \beta^+ \vdash \supset \Theta$ and $\Gamma, \beta^+ \subseteq \Gamma, \beta^+$.
- $\Gamma, \beta^+; \Theta, \hat{\alpha}^+ \{ \Gamma, \beta^+ \} \vdash N_0$
- There are no negative unification variables in N_0 because $\mathbf{uv} N_0 \subseteq \mathbf{uv} N \cup \hat{\alpha}^+$.

This way, we apply the induction hypothesis to $\Gamma, \vec{\beta}^+ \vdash [\hat{\sigma}_0] N_0 \leq_1 M'$ and infer that there exists $\hat{\sigma}'_0 : (\Theta, \hat{\alpha}^+ \{\Gamma, \vec{\beta}^+\})|_{\mathbf{uv} N_0}$ such that $\Gamma, \vec{\beta}^+; \Theta, \hat{\alpha}^+ \{\Gamma, \vec{\beta}^+\} \models N_0 \leq M \Rightarrow \hat{\sigma}'_0$, and $\Theta, \hat{\alpha}^+ \{\Gamma, \vec{\beta}^+\} \vdash \hat{\sigma}'_0 \Rightarrow \hat{\sigma}_0$.

This way, the algorithm terminates with the result $\hat{\sigma}'_0 \setminus \vec{\alpha}^+$ and it is left to show $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'_0 \setminus \vec{\alpha}^+$. Notice that $\hat{\sigma}'_0 \setminus \vec{\alpha}^+ : \Theta|_{\mathbf{uv} N'}$. This way, to show $\Theta \vdash \hat{\sigma} \Rightarrow \hat{\sigma}'_0 \setminus \vec{\alpha}^+$, it suffices to consider an unification variable $\hat{\beta}^\pm \mathbf{uv} N'$ and show that $\Gamma, \vec{\beta}^+ \vdash \hat{\sigma}(\hat{\beta}^\pm) \Rightarrow \hat{\sigma}'_0(\hat{\beta}^\pm)$. It holds by the reflexivity of weakening (??) since $\hat{\sigma}'_0(\hat{\beta}^\pm) = \hat{\sigma}(\hat{\beta}^\pm)$.

□