1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
 & | & a+ & \\
 & | & \downarrow N & \\
 & | & \exists \alpha^-.P & \\
\end{array}
$$

$$
\begin{array}{llll}
N,\ M & ::= & & \text{negative types} \\
 & | & a- & \\
 & | & \uparrow P & \\
 & | & \forall \alpha^+.N & \\
 & | & P \to N & \\
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \ \text{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \ \text{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash a- \leqslant_0 a-} \ \text{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \ \text{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/a+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \ \text{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \ \text{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \ \text{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash a+ \geqslant_0 a+} \ \text{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \ \text{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/a-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \ \text{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \ \text{D0ExistsR}$$

2 Multi-Quantified System

2.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{multi-quantified positive types} \\
 & \mid & \alpha^+ & \\
 & \mid & \downarrow N & \\
 & \mid & \exists \overrightarrow{\alpha^-}.P & P \neq \exists \ldots
\end{array}
$$

$$
\begin{array}{llll}
N,\ M & ::= & & \text{multi-quantified negative types} \\
 & \mid & \alpha^- & \\
 & \mid & \uparrow P & \\
 & \mid & P \to N & \\
 & \mid & \forall \overrightarrow{\alpha^+}.N & N \neq \forall \ldots
\end{array}
$$

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \ \text{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \ \text{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \ \text{D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \ \text{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M} \ \text{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M} \ \text{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \ \text{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \ \text{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q'}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q} \ \text{D1ExistsL}$$

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-} \ (\text{Var}^{-\simeq_1^D})$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \ (\uparrow^{\simeq_1^D})$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M} \ (\to^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M} \ (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\mathrm{VAR}^+)$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\; vars\; \mathbf{in}\; N = \vec{\alpha}}$
$\boxed{\mathbf{ord}\; vars\; \mathbf{in}\; P = \vec{\alpha}}$
$\boxed{\mathbf{ord}\; vars\; \mathbf{in}\; N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\; vars\; \mathbf{in}\; \alpha^- = \alpha^-} \quad (\mathrm{VAR}^-_{\in})$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\; vars\; \mathbf{in}\; \alpha^- = \cdot} \quad (\mathrm{VAR}^-_{\notin})$$

$$\frac{}{\mathbf{ord}\; vars\; \mathbf{in}\; \widehat{\alpha}^-\{vars'\} = \cdot} \quad (\mathrm{UVAR}^-)$$

$$\frac{\mathbf{ord}\; vars\; \mathbf{in}\; P = \vec{\alpha}}{\mathbf{ord}\; vars\; \mathbf{in}\; \uparrow P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\; vars\; \mathbf{in}\; P = \vec{\alpha}_1 \quad \mathbf{ord}\; vars\; \mathbf{in}\; N = \vec{\alpha}_2}{\mathbf{ord}\; vars\; \mathbf{in}\; P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})} \quad (\to)$$

$$\frac{vars \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\; vars\; \mathbf{in}\; N = \vec{\alpha}}{\mathbf{ord}\; vars\; \mathbf{in}\; \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\; vars\; \mathbf{in}\; P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\; vars\; \mathbf{in}\; \alpha^+ = \alpha^+} \quad (\mathrm{VAR}^+_{\in})$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\; vars\; \mathbf{in}\; \alpha^+ = \cdot} \quad (\mathrm{VAR}^+_{\notin})$$

$$\frac{}{\mathbf{ord}\; vars\; \mathbf{in}\; \widehat{\alpha}^+\{vars'\} = \cdot} \quad (\mathrm{UVAR}^+)$$

$$\frac{\mathbf{ord}\; vars\; \mathbf{in}\; N = \vec{\alpha}}{\mathbf{ord}\; vars\; \mathbf{in}\; \downarrow N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{vars \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\; vars\; \mathbf{in}\; P = \vec{\alpha}}{\mathbf{ord}\; vars\; \mathbf{in}\; \exists \overrightarrow{\alpha^-}.P = \vec{\alpha}} \quad (\exists)$$

3.1.2 Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$
$\boxed{\mathbf{nf}\,(P) = Q}$
$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-} \quad (\mathrm{VAR}^-)$$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-\{vars\}) = \widehat{\alpha}^-\{vars\}} \quad (\mathrm{UVAR}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q} \quad (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M} \quad (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.N'} \quad (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+} \quad (\textsc{Var}^+)$$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+\{vars\}) = \widehat{\alpha}^+\{vars\}} \quad (\textsc{UVar}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M} \quad (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}'.P'} \quad (\exists)$$

3.2 Unification

$\boxed{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot} \quad \text{UNVar}$$

$$\frac{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}} \quad \text{UShiftU}$$

$$\frac{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2} \quad \text{UArrow}$$

$$\frac{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}} \quad \text{UForall}$$

$$\frac{\mathbf{fv}\,N \subseteq vars}{\widehat{\alpha}^-\{vars\} \overset{u}{\simeq} N \dashv \widehat{\alpha}^- :\approx N} \quad \text{UNUVar}$$

$\boxed{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot} \quad \text{UPVar}$$

$$\frac{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}} \quad \text{UShiftD}$$

$$\frac{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}} \quad \text{UExists}$$

$$\frac{\mathbf{fv}\,P \subseteq vars}{\widehat{\alpha}^+\{vars\} \overset{u}{\simeq} P \dashv \widehat{\alpha}^+ :\approx P} \quad \text{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma \models N \leqslant M \dashv \hat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma \models \alpha^- \leqslant \alpha^- \dashv \cdot}\ \textsc{ANVar}$$

$$\frac{\mathbf{nf}\,(P) \stackrel{u}{\simeq} \mathbf{nf}\,(Q) \dashv \hat{\sigma}}{\Gamma \models {\uparrow}P \leqslant {\uparrow}Q \dashv \hat{\sigma}}\ \textsc{AShiftU}$$

$$\frac{\Gamma \models P \geqslant Q \dashv \hat{\sigma}_1 \quad \Gamma \models N \leqslant M \dashv \hat{\sigma}_2}{\Gamma \models P \to N \leqslant Q \to M \dashv \hat{\sigma}_1 \& \hat{\sigma}_2}\ \textsc{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \models [\overrightarrow{\hat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}}/\overrightarrow{\alpha^+}]N \leqslant M \dashv \hat{\sigma}}{\Gamma \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \hat{\sigma}\backslash\overrightarrow{\hat{\alpha}^+}}\ \textsc{AForall}$$

$\boxed{\Gamma \models P \geqslant Q \dashv \hat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \textsc{APVar}$$

$$\frac{\mathbf{nf}\,(N) \stackrel{u}{\simeq} \mathbf{nf}\,(M) \dashv \hat{\sigma}}{\Gamma \models {\downarrow}N \geqslant {\downarrow}M \dashv \hat{\sigma}}\ \textsc{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \models [\overrightarrow{\hat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv \hat{\sigma}}{\Gamma \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \hat{\sigma}}\ \textsc{AExists}$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathit{vars}_1 = \mathbf{fv}\,P'\backslash\mathit{vars} \quad \mathit{vars}_2\ \textbf{is fresh}}{\Gamma \models \hat{\alpha}^+\{\mathit{vars}\} \geqslant P \dashv (\hat{\alpha}^+ :\geqslant P' \vee [\mathit{vars}_2/\mathit{vars}_1]P')}\ \textsc{APUVar}$$

3.4 Unification Solution Merge

$\boxed{e_1 \& e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{}{\hat{\alpha}^+ :\geqslant P \& \hat{\alpha}^+ :\geqslant Q = \hat{\alpha}^+ :\geqslant P \vee Q}\ \textsc{SMEPSupSup}$$

$$\frac{\mathbf{fv}\,P \cup \mathbf{fv}\,Q \models P \geqslant Q \dashv \hat{\sigma}'}{\hat{\alpha}^+ :\approx P \& \hat{\alpha}^+ :\geqslant Q = \hat{\alpha}^+ :\approx P}\ \textsc{SMEPEqSup}$$

$$\frac{\mathbf{fv}\,P \cup \mathbf{fv}\,Q \models Q \geqslant P \dashv \hat{\sigma}'}{\hat{\alpha}^+ :\geqslant P \& \hat{\alpha}^+ :\approx Q = \hat{\alpha}^+ :\approx Q}\ \textsc{SMEPSupEq}$$

$$\frac{}{\hat{\alpha}^+ :\approx P \& \hat{\alpha}^+ :\approx P = \hat{\alpha}^+ :\approx P}\ \textsc{SMEPEqEq}$$

$$\frac{}{\hat{\alpha}^- :\approx N \& \hat{\alpha}^- :\approx N = \hat{\alpha}^- :\approx N}\ \textsc{SMENEqEq}$$

$\boxed{\hat{\sigma}_1 \& \hat{\sigma}_2 = \hat{\sigma}_3}$ Merge unification solutions

$$\frac{}{\cdot \& \hat{\sigma} = \hat{\sigma}}\ \textsc{SMEmpty}$$

$$\frac{\begin{array}{c}(\hat{\alpha}^+ :\approx P) \in \hat{\sigma}_2 \\ \hat{\sigma}_1 \& (\hat{\sigma}_2 \backslash \hat{\alpha}^+) = \hat{\sigma}_3\end{array}}{(\hat{\alpha}^+ :\approx P, \hat{\sigma}_1) \& \hat{\sigma}_2 = (\hat{\alpha}^+ :\approx P, \hat{\sigma}_3)}\ \textsc{SMPEqEq}$$

$$\frac{(\hat{\alpha}^+ :\geqslant Q) \in \hat{\sigma}_2 \quad \hat{\sigma}_1 \& (\hat{\sigma}_2 \backslash \hat{\alpha}^+) = \hat{\sigma}_3}{(\hat{\alpha}^+ :\geqslant P, \hat{\sigma}_1) \& \hat{\sigma}_2 = (\hat{\alpha}^+ :\geqslant P \vee Q, \hat{\sigma}_3)}\ \textsc{SMPSupSup}$$

$$\frac{\begin{array}{l}(\hat{\alpha}^+ :\approx Q) \in \hat{\sigma}_2 \quad \mathbf{fv}\,Q \cup \mathbf{fv}\,P \models Q \geqslant P \dashv \hat{\sigma}' \\ \hat{\sigma}_1 \& (\hat{\sigma}_2 \backslash \hat{\alpha}^+) = \hat{\sigma}_3\end{array}}{(\hat{\alpha}^+ :\geqslant P, \hat{\sigma}_1) \& \hat{\sigma}_2 = (\hat{\alpha}^+ :\approx Q, \hat{\sigma}_3)}\ \textsc{SMPSupEq}$$

$$\frac{\begin{array}{l}(\hat{\alpha}^+ :\geqslant Q) \in \hat{\sigma}_2 \quad \mathbf{fv}\,Q \cup \mathbf{fv}\,P \models P \geqslant Q \dashv \hat{\sigma}' \\ \hat{\sigma}_1 \& (\hat{\sigma}_2 \backslash \hat{\alpha}^+) = \hat{\sigma}_3\end{array}}{(\hat{\alpha}^+ :\approx P, \hat{\sigma}_1) \& \hat{\sigma}_2 = (\hat{\alpha}^+ :\approx P, \hat{\sigma}_3)}\ \textsc{SMPEqSup}$$

$$\frac{\begin{array}{c}(\hat{\alpha}^- :\approx N) \in \hat{\sigma}_2 \\ \hat{\sigma}_1 \& (\hat{\sigma}_2 \backslash \hat{\alpha}^-) = \hat{\sigma}_3\end{array}}{(\hat{\alpha}^- :\approx N, \hat{\sigma}_1) \& \hat{\sigma}_2 = (\hat{\alpha}^- :\approx N, \hat{\sigma}_3)}\ \textsc{SMNEqEq}$$

3.5 Least Upper Bound

$\boxed{P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\alpha^+ \vee \alpha^+ = \alpha^+}\ \text{LUBVar}$$

$$\frac{(\mathbf{fv}\, N \cup \mathbf{fv}\, M) \vDash {\downarrow} N \overset{a}{\simeq} {\downarrow} M \dashv (P, \widehat{\sigma}_1, \widehat{\sigma}_2)}{{\downarrow} N \vee {\downarrow} M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\mathbf{uv}\, P]P}\ \text{LUBShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing}{\exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = P_1 \vee P_2}\ \text{LUBExists}$$

3.6 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\alpha^+, \cdot, \cdot)}\ \text{AUPVar}$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \vDash {\downarrow} N_1 \overset{a}{\simeq} {\downarrow} N_2 \dashv ({\downarrow} M, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \text{AUPShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\exists \overrightarrow{\alpha^-}.Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \text{AUPExists}$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\alpha^-, \cdot, \cdot)}\ \text{AUNVar}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \vDash {\uparrow} P_1 \overset{a}{\simeq} {\uparrow} P_2 \dashv ({\uparrow} Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \text{AUNShift}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}'_1, \widehat{\sigma}'_2)}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (Q \to M, \widehat{\sigma}_1 \cup \widehat{\sigma}'_1, \widehat{\sigma}_2 \cup \widehat{\sigma}'_2)}\ \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))}\ \text{AUNAU}$$

4 Proofs

4.1 Normaliztaion

4.1.1 Auxiliary properties

Lemma 1. *Set of free variables is invariant under equivalence.*

− *If* $N \simeq^D_1 M$ *then* $\mathbf{fv}\, N \equiv \mathbf{fv}\, M$ *(as sets)*

+ *If* $P \simeq^D_1 Q$ *then* $\mathbf{fv}\, P \equiv \mathbf{fv}\, Q$ *(as sets)*

Proof. Straightforward mutual induction on $N \simeq^D_1 M$ and $P \simeq^D_1 Q$ □

Lemma 2 (Distributivity of renaming over variable ordering)**.** *Suppose that* μ *is a bijection between two sets of variables* $\mu : A \leftrightarrow B$*, and* B *is disjoint with* $vars$*.*

− *If* B *is disjoint with* $\mathbf{fv}\, N$ *then* $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = \mathbf{ord}\, ([\mu]vars)\, \mathbf{in}\, [\mu]N$

+ *If* B *is disjoint with* $\mathbf{fv}\, P$ *then* $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, P) = \mathbf{ord}\, ([\mu]vars)\, \mathbf{in}\, [\mu]P$

Proof. Mutual induction on N and P.

Case 1: $N = \alpha^-$
let us consider four cases:

1. $\alpha^- \in A$ and $\alpha^- \in vars$. Then
$$\begin{aligned}[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-)\\ &= [\mu]\alpha^- && \text{by Rule } (\mathrm{Var}^+_{\in})\\ &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars)\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \beta^- && \text{by Rule } (\mathrm{Var}^+_{\in})\text{, because } \beta^- \in [\mu]vars\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^-\end{aligned}$$

2. $\alpha^- \notin A$ and $\alpha^- \notin vars$. Notice that $\alpha^- \notin B$, because B is disjoint with $\mathbf{fv}\, N$. Then $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_{\notin})$. On the other hand, $\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- = \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}^+_{\notin})$, because $\alpha^- \notin B \cup vars \supseteq [\mu]vars$.

3. $\alpha^- \in A$ but $\alpha^- \notin vars$. Then $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = \cdot$ by Rule $(\mathrm{Var}^+_{\notin})$. To prove that $\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}^+_{\notin})$. Let us show that $[\mu]\alpha^- \notin [\mu]vars$. If there is an element $x \in vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. On the other hand, $[\mu]\alpha^- \in B$, and hence, $[\mu]\alpha^- \notin vars$.

4. $\alpha^- \notin A$ but $\alpha^- \in vars$. $\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\alpha^- = \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}^+_{\notin})$, because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) = [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2: $N = \uparrow P$
$$\begin{aligned}[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, \uparrow P)\\ &= [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, P) && \text{by Rule } (\uparrow)\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]P && \text{by the induction hypothesis}\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \uparrow[\mu]P && \text{by Rule } (\uparrow)\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\uparrow P && \text{by the definition of substitution}\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]N\end{aligned}$$

Case 3: $N = P \to M$
$$\begin{aligned}[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu](\mathbf{ord}\, vars\, \mathbf{in}\, P \to M)\\ &= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})) && \text{where } \mathbf{ord}\, vars\, \mathbf{in}\, P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\, vars\, \mathbf{in}\, M = \vec{\alpha}_2\\ &= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \{\vec{\alpha}_1\})\\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\{\vec{\alpha}_1\}) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to Case 1}\\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})\end{aligned}$$
On the other hand,
$$\begin{aligned}(\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]N) &= (\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]P \to [\mu]M)\\ &= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \{\vec{\beta}_1\})) && \text{where } \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]M = \vec{\beta}_2\\ & && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1,\ \vec{\beta}_2 = [\mu]\vec{\alpha}_2,\\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash \{[\mu]\vec{\alpha}_1\})\end{aligned}$$

Case 4: $N = \forall\overrightarrow{\alpha^+}.M$
$$\begin{aligned}[\mu](\mathbf{ord}\, vars\, \mathbf{in}\, N) &= [\mu]\mathbf{ord}\, vars\, \mathbf{in}\, \forall\overrightarrow{\alpha^+}.M\\ &= [\mu]\mathbf{ord}\, vars\, \mathbf{in}\, M\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]M && \text{by the induction hypothesis}\end{aligned}$$
On the other hand,
$$\begin{aligned}(\mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]N) &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]\forall\overrightarrow{\alpha^+}.M\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, \forall\overrightarrow{\alpha^+}.[\mu]M\\ &= \mathbf{ord}\, [\mu]vars\, \mathbf{in}\, [\mu]M\end{aligned}$$

□

Lemma 3 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming.*
Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then

$-$ $\mathbf{nf}\,([\mu]N) = [\mu]\mathbf{nf}\,(N)$

$+$ $\mathbf{nf}\,([\mu]P) = [\mu]\mathbf{nf}\,(P)$

Here equality means alpha-equivalence.

Proof. Mutual induction on N and P.

Write a little bit about the exists/forall case

$\square$

4.1.2 Soundness

Lemma 4 (Soundness of variable ordering). *Variable ordering extracts precisely used free variables.*

$-$ $\{\mathbf{ord}\, vars\, \mathbf{in}\; N\} \equiv vars \cap \mathbf{fv}\; N$ *(as sets)*

$+$ $\{\mathbf{ord}\, vars\, \mathbf{in}\; P\} \equiv vars \cap \mathbf{fv}\; P$ *(as sets)*

Proof. Straightforward mutual induction on $\mathbf{ord}\, vars\, \mathbf{in}\; N = \vec{\alpha}$ and $\mathbf{ord}\, vars\, \mathbf{in}\; P = \vec{\alpha}$ $\square$

Corollary 1 (Additivity of ordering). *Variable ordering is additive (in terms of set union) with respect to its first argument.*

$-$ $\{\mathbf{ord}\,(vars_1 \cup vars_2)\, \mathbf{in}\, N\} \equiv \{\mathbf{ord}\, vars_1\, \mathbf{in}\, N\} \cup \{\mathbf{ord}\, vars_2\, \mathbf{in}\, N\}$ *(as sets)*

$+$ $\{\mathbf{ord}\,(vars_1 \cup vars_2)\, \mathbf{in}\, P\} \equiv \{\mathbf{ord}\, vars_1\, \mathbf{in}\, P\} \cup \{\mathbf{ord}\, vars_2\, \mathbf{in}\, P\}$ *(as sets)*

Corollary 2 (Weakening of ordering). *Extending the first argument of the ordering with unused variables does not change the result.*

$-$ $\mathbf{ord}\,(vars \cap \mathbf{fv}\, N)\, \mathbf{in}\, N = \mathbf{ord}\, vars\, \mathbf{in}\, N$

$+$ $\mathbf{ord}\,(vars \cap \mathbf{fv}\, P)\, \mathbf{in}\, P = \mathbf{ord}\, vars\, \mathbf{in}\, P$

Lemma 5 (Soundness of quantifier normalization). *Normalization respects equivalence.*

$-$ $N \simeq_1^D \mathbf{nf}\,(N)$

$+$ $P \simeq_1^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

- (Var^-) and (Var^+) by the corresponding equivalence rules.
- ($\uparrow$), ($\downarrow$), and ($\rightarrow$) by the induction hypothesis and the corresponding congruent equivalence rules.
- ($\forall$) From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 1, $\mathbf{fv}\, N \equiv \mathbf{fv}\, N'$. Then by lemma 4, $\overrightarrow{\alpha^{+}}' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$, and thus, $\overrightarrow{\alpha^{+}}' \cap \mathbf{fv}\, N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$.

  To prove $\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\alpha^{+}}'.N'$, it suffices to provide a bijection $\mu : \overrightarrow{\alpha^{+}}' \cap \mathbf{fv}\, N' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.
- ($\exists$) Same as for ($\forall$).

$\square$

Corollary 3. *Free variables are not changed by the normalization*

$-$ $\mathbf{fv}\, N \equiv \mathbf{fv}\, \mathbf{nf}\,(N)$

$+$ $\mathbf{fv}\, P \equiv \mathbf{fv}\, \mathbf{nf}\,(P)$

Proof. Immediately from lemmas 1 and 5. $\square$

4.1.3 Completeness

Lemma 6 (Completeness of variable ordering). *Variable ordering is invariant under equivalence.*

$-$ *For* $N \simeq_1^D M$ *and any vars, if* $\mathbf{ord}\, vars\, \mathbf{in}\; N = \vec{\alpha}_1$ *and* $\mathbf{ord}\, vars\, \mathbf{in}\; M = \vec{\alpha}_2$*, then* $\vec{\alpha}_1 = \vec{\alpha}_2$ *(as lists)*

$+$ *For* $P \simeq_1^D Q$ *and any vars, if* $\mathbf{ord}\, vars\, \mathbf{in}\; P = \vec{\alpha}_1$ *and* $\mathbf{ord}\, vars\, \mathbf{in}\; Q = \vec{\alpha}_2$*, then* $\vec{\alpha}_1 = \vec{\alpha}_2$ *(as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. $\square$

Lemma 7 (Completeness of quantified normalization). *Normalization returns the same representative for equivalent types.*

$-$ *If* $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

+ *If* $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

- $(\forall^{\simeq_1^D})$

  From the definition of the normalization,

  - $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N)$
  - $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M)$

  Let us take $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that by lemma 4 and corollary 3, the domain and codomain of μ can be written as $\mu : \overrightarrow{\beta^+}' \leftrightarrow \overrightarrow{\alpha^+}'$.

  To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

  (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by lemma 3, the second—by the induction hypothesis.

  (ii)
  $$\begin{aligned}
  [\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}' \\
  &= [\mu]\mathbf{ord}\,(\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by corollaries 2 and 3} \\
  &= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 2, because } \overrightarrow{\alpha^+}' \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \\
  & && \text{and } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \\
  &= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved} \\
  &= \mathbf{ord}\,(\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \text{ and } \overrightarrow{\beta^+} \cap \mathbf{fv}\,M \\
  &= \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{by corollaries 2 and 3} \\
  &= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'
  \end{aligned}$$

- $(\exists^{\simeq_1^D})$ Same as for $(\forall^{\simeq_1^D})$.

- Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□