1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

$$
\begin{array}{llll}
P,\ Q & ::= & & \text{positive types} \\
& | & \alpha^+ & \\
& | & \downarrow N & \\
& | & \exists \alpha^-.P & \\
\end{array}
$$

$$
\begin{array}{llll}
N,\ M & ::= & & \text{negative types} \\
& | & \alpha^- & \\
& | & \uparrow P & \\
& | & \forall \alpha^+.N & \\
& | & P \to N & \\
\end{array}
$$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$
\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \quad \text{D0NDef}
$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$
\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q} \quad \text{D0PDef}
$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$
\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \quad \text{D0NVar}
$$

$$
\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \quad \text{D0ShiftU}
$$

$$
\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \quad \text{D0ForallL}
$$

$$
\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \quad \text{D0ForallR}
$$

$$
\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \quad \text{D0Arrow}
$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$
\frac{}{\Gamma \vdash \alpha^+ \geqslant_0 \alpha^+} \quad \text{D0PVar}
$$

$$
\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \quad \text{D0ShiftD}
$$

$$
\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \quad \text{D0ExistsL}
$$

$$
\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \quad \text{D0ExistsR}
$$

2 Multi-Quantified System

2.1 Grammar

$$
\begin{array}{lll}
P,\ Q & ::= & \text{multi-quantified positive types} \\
& | & \alpha^+ \\
& | & {\downarrow}N \\
& | & \overrightarrow{\exists\alpha^-}.P \qquad\quad P \neq \exists\ldots \\
& | & (P) \qquad\ \ \mathsf{S}
\end{array}
$$

$$
\begin{array}{lll}
N,\ M & ::= & \text{multi-quantified negative types} \\
& | & \alpha^- \\
& | & {\uparrow}P \\
& | & P \rightarrow N \\
& | & \overrightarrow{\forall\alpha^+}.N \qquad\quad N \neq \forall\ldots \\
& | & (N) \qquad\ \ \mathsf{S}
\end{array}
$$

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$
\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \quad (\simeq_1^{\leqslant}{}^-)
$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$
\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \quad (\simeq_1^{\leqslant}{}^+)
$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$
\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad (\textsc{Var}^{-\leqslant_1})
$$

$$
\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash {\uparrow}P \leqslant_1 {\uparrow}Q} \quad (\uparrow^{\leqslant_1})
$$

$$
\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \rightarrow N \leqslant_1 Q \rightarrow M} \quad (\rightarrow^{\leqslant_1})
$$

$$
\frac{\mathbf{fv}\,N \cap \overrightarrow{\beta^+} = \varnothing \quad \Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \overrightarrow{\forall\alpha^+}.N \leqslant_1 \overrightarrow{\forall\beta^+}.M} \quad (\forall^{\leqslant_1})
$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$
\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad (\textsc{Var}^{+\geqslant_1})
$$

$$
\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash {\downarrow}N \geqslant_1 {\downarrow}M} \quad (\downarrow^{\geqslant_1})
$$

$$
\frac{\mathbf{fv}\,P \cap \overrightarrow{\beta^-} = \varnothing \quad \Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q}{\Gamma \vdash \overrightarrow{\exists\alpha^-}.P \geqslant_1 \overrightarrow{\exists\beta^-}.Q} \quad (\exists^{\geqslant_1})
$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1}$ Equivalence of substitutions

2.3 Declarative Equivalence

$\boxed{N \simeq_1^{D} M}$ Negative multi-quantified type equivalence

$$
\frac{}{\alpha^- \simeq_1^{D} \alpha^-} \quad (\textsc{Var}^{-\simeq_1^{D}})
$$

$$
\frac{P \simeq_1^{D} Q}{{\uparrow}P \simeq_1^{D} {\uparrow}Q} \quad (\uparrow^{\simeq_1^{D}})
$$

$$
\frac{P \simeq_1^{D} Q \quad N \simeq_1^{D} M}{P \rightarrow N \simeq_1^{D} Q \rightarrow M} \quad (\rightarrow^{\simeq_1^{D}})
$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N) \quad N \simeq_1^D [\mu]M}{\overrightarrow{\forall \alpha^+}.N \simeq_1^D \overrightarrow{\forall \beta^+}.M} \quad (\forall^{\simeq_1^D})$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\mathrm{VAR}^{+\simeq_1^D})$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow^{\simeq_1^D})$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\,Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\,Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\,P) \quad P \simeq_1^D [\mu]Q}{\overrightarrow{\exists \alpha^-}.P \simeq_1^D \overrightarrow{\exists \beta^-}.Q} \quad (\exists^{\simeq_1^D})$$

$\boxed{P \simeq Q}$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\mathbf{ord}\ vars\,\mathbf{in}\ N = \vec{\alpha}}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\ vars\,\mathbf{in}\ \alpha^- = \alpha^-} \quad (\mathrm{VAR}_\in^-)$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\ vars\,\mathbf{in}\ \alpha^- = \cdot} \quad (\mathrm{VAR}_{\notin}^-)$$

$$\frac{\mathbf{ord}\ vars\,\mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\,\mathbf{in}\ {\uparrow}P = \vec{\alpha}} \quad (\uparrow)$$

$$\frac{\mathbf{ord}\ vars\,\mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ vars\,\mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ vars\,\mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)} \quad (\to)$$

$$\frac{vars \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\ vars\,\mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\,\mathbf{in}\ \overrightarrow{\forall \alpha^+}.N = \vec{\alpha}} \quad (\forall)$$

$\boxed{\mathbf{ord}\ vars\,\mathbf{in}\ P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\ vars\,\mathbf{in}\ \alpha^+ = \alpha^+} \quad (\mathrm{VAR}_\in^+)$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\ vars\,\mathbf{in}\ \alpha^+ = \cdot} \quad (\mathrm{VAR}_{\notin}^+)$$

$$\frac{\mathbf{ord}\ vars\,\mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\,\mathbf{in}\ {\downarrow}N = \vec{\alpha}} \quad (\downarrow)$$

$$\frac{vars \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\ vars\,\mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\,\mathbf{in}\ \overrightarrow{\exists \alpha^-}.P = \vec{\alpha}} \quad (\exists)$$

$\boxed{\mathbf{ord}\ vars\,\mathbf{in}\ N = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\,\mathbf{in}\ \widehat{\alpha}^- = \cdot} \quad (\mathrm{UVAR}^-)$$

$\boxed{\mathbf{ord}\ vars\,\mathbf{in}\ P = \vec{\alpha}}$

$$\frac{}{\mathbf{ord}\ vars\,\mathbf{in}\ \widehat{\alpha}^+ = \cdot} \quad (\mathrm{UVAR}^+)$$

3.1.2   Quantifier Normalization

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-}\ \ (\text{Var}^-)$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q}\ \ (\uparrow)$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M}\ \ (\to)$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,N' = \overrightarrow{\alpha^{+\prime}}}{\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.N'}\ \ (\forall)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+}\ \ (\text{Var}^+)$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M}\ \ (\downarrow)$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,P' = \overrightarrow{\alpha^{-\prime}}}{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^{-\prime}}.P'}\ \ (\exists)$$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-) = \widehat{\alpha}^-}\ \ (\text{UVar}^-)$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+) = \widehat{\alpha}^+}\ \ (\text{UVar}^+)$$

3.2   Unification

$\boxed{\Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\Theta \vDash \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\ \ \text{UNVar}$$

$$\frac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \vDash \uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}}\ \ \text{UShiftU}$$

$$\frac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{\Theta \vDash P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \,\&\, \widehat{\sigma}_2}\ \ \text{UArrow}$$

$$\frac{\Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \vDash \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv \widehat{\sigma}}\ \ \text{UForall}$$

$$\frac{\widehat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Theta \vDash \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\Delta \vdash \widehat{\alpha}^- :\approx N)}\ \ \text{UNUVar}$$

$\boxed{\Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\Theta \vDash \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\ \ \text{UPVar}$$

$$\frac{\Theta \vDash N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\Theta \vDash \downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}}\ \ \text{UShiftD}$$

$$\frac{\Theta \vDash P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\Theta \vDash \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv \widehat{\sigma}}\ \ \text{UExists}$$

$$\frac{\widehat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Theta \vDash \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\Delta \vdash \widehat{\alpha}^+ :\approx P)}\ \ \text{UPUVar}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma; \Theta \vDash \alpha^- \leqslant \alpha^- \dashv \cdot} \quad \text{ANVAR}$$

$$\frac{\Theta \vDash \mathbf{nf}\,(P) \stackrel{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad \text{ASHIFTU}$$

$$\frac{\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma; \Theta \vDash N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma; \Theta \vDash P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2} \quad \text{AARROW}$$

$$\frac{\Gamma, \overrightarrow{\beta^+}; \Theta, \overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}} \vDash [\overrightarrow{\widehat{\alpha}^+/\alpha^+}]\,N \leqslant M \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \forall \overrightarrow{\alpha^+}.N \leqslant \forall \overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}} \quad \text{AFORALL}$$

$\boxed{\Gamma; \Theta \vDash P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma; \Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot} \quad \text{APVAR}$$

$$\frac{\Theta \vDash \mathbf{nf}\,(N) \stackrel{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \quad \text{ASHIFTD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}} \vDash [\overrightarrow{\widehat{\alpha}^-/\alpha^-}]\,P \geqslant Q \dashv \widehat{\sigma}}{\Gamma; \Theta \vDash \exists \overrightarrow{\alpha^-}.P \geqslant \exists \overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}} \quad \text{AEXISTS}$$

$$\frac{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}{\Gamma; \Theta \vDash \widehat{\alpha}^+\{\Delta\} \geqslant P \dashv (\Delta \vdash \widehat{\alpha}^+ :\geqslant Q)} \quad \text{APUVAR}$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restrict an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\Delta \vdash \widehat{\alpha}^+ :\approx P$ or $\Delta \vdash \widehat{\alpha}^- :\approx N$) or that it must be a (positive) supertype of a certain type ($\Delta \vdash \widehat{\alpha}^+ :\geqslant P$).

Definition 1 (Matching Entries). *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.
$\boxed{e_1 \;\&\; e_2 = e_3}$ *Unification Solution Entry Merge*

$$\frac{\Gamma \vDash P_1 \vee P_2 = Q}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_1) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\geqslant P_2) = (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q)} \quad (\geqslant \& \geqslant)$$

$$\frac{\Gamma; \cdot \vDash P \geqslant Q \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\geqslant Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)} \quad (\simeq \& \geqslant)$$

$$\frac{\Gamma; \cdot \vDash Q \geqslant P \dashv \widehat{\sigma}'}{(\Gamma \vdash \widehat{\alpha}^+ :\geqslant P) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\approx Q) = (\Gamma \vdash \widehat{\alpha}^+ :\approx Q)} \quad (\geqslant \& \simeq)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^+ :\approx P) \;\&\; (\Gamma \vdash \widehat{\alpha}^+ :\approx P) = (\Gamma \vdash \widehat{\alpha}^+ :\approx P)} \quad (\simeq \& \simeq^+)$$

$$\frac{}{(\Gamma \vdash \widehat{\alpha}^- :\approx N) \;\&\; (\Gamma \vdash \widehat{\alpha}^- :\approx N) = (\Gamma \vdash \widehat{\alpha}^- :\approx N)} \quad (\simeq \& \simeq^-)$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3. $\widehat{\sigma}_1 \;\&\; \widehat{\sigma}_2 = \{e_1 \;\&\; e_2 \mid e_1 \in \widehat{\sigma}_1, e_2 \in \widehat{\sigma}_2, s.t.\ e_1\ matching\ with\ e_2\}$
$\cup \{e_1 \mid e_1 \in \widehat{\sigma}_1,\ s.t.\ \forall e_2 \in \widehat{\sigma}_2, e_1\ does\ not\ match\ with\ e_2\}$
$\cup \{e_2 \mid e_2 \in \widehat{\sigma}_2,\ s.t.\ \forall e_1 \in \widehat{\sigma}_1, e_1\ does\ not\ match\ with\ e_2\}$

3.5 Least Upper Bound

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+} \quad (\text{VAR}^\vee)$$

$$\frac{\Gamma, \cdot \vDash \mathbf{nf}\,(\downarrow N) \overset{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists\overrightarrow{\alpha^-}.\overrightarrow{[\alpha^-/\Xi]}\,P} \quad (\downarrow^\vee)$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q} \quad (\exists^\vee)$$

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P \,\mathbf{to}\,\Delta = Q}$

$$\frac{\Gamma = \Delta, \overrightarrow{\alpha^\pm} \quad \overrightarrow{\beta^\pm}\,\mathbf{is\,fresh} \quad \overrightarrow{\gamma^\pm}\,\mathbf{is\,fresh}}{\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vDash \overrightarrow{[\beta^\pm/\alpha^\pm]}P \vee \overrightarrow{[\gamma^\pm/\alpha^\pm]}P = Q}{\mathbf{upgrade}\,\Gamma \vdash P \,\mathbf{to}\,\Delta = Q} \quad (\text{UPG})$$

3.6 Antiunification

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \quad \text{AUPVAR}$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vDash \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \dashv (\Xi, \downarrow M, \hat{\tau}_1, \hat{\tau}_2)} \quad \text{AUPSHIFT}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vDash \exists\overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists\overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists\overrightarrow{\alpha^-}.Q, \hat{\tau}_1, \hat{\tau}_2)} \quad \text{AUPEXISTS}$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\Xi, \alpha^-, \cdot, \cdot)} \quad \text{AUNVAR}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \hat{\tau}_1, \hat{\tau}_2)} \quad \text{AUNSHIFT}$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \hat{\tau}_1, \hat{\tau}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \hat{\tau}_1', \hat{\tau}_2')}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \hat{\tau}_1 \cup \hat{\tau}_1', \hat{\tau}_2 \cup \hat{\tau}_2')} \quad \text{AUNARROW}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\hat{\alpha}^-_{\{N,M\}}, \hat{\alpha}^-_{\{N,M\}}, (\hat{\alpha}^-_{\{N,M\}} :\approx N), (\hat{\alpha}^-_{\{N,M\}} :\approx M))} \quad \text{AUNAU}$$

4 Proofs

4.1 Declarative Subtyping

Lemma 1 (Free Variable Propagation)**.** *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context Γ,*

- *$-$ if $\Gamma \vdash N \leqslant_1 M$ then $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$*

- *$+$ if $\Gamma \vdash P \geqslant_1 Q$ then $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$*

Proof. Mutual induction on $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash P \geqslant_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
It is self-evident that $\alpha^- \subseteq \alpha^-$.

Case 2. $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq^\leqslant_1 Q$), we have $\Gamma \vdash P \geqslant_1 Q$. Then by the induction hypothesis, $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$. The desired inclusion inclusion holds, since $\mathbf{fv}\,(\uparrow P) = \mathbf{fv}\,(P)$ and $\mathbf{fv}\,(\uparrow Q) = \mathbf{fv}\,(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant_1 Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$ and $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$. Then $\mathbf{fv}\,(P \to N) = \mathbf{fv}\,(P) \cup \mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(Q) \cup \mathbf{fv}\,(M) = \mathbf{fv}\,(Q \to M)$.

Case 4. $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant_1 \forall\overrightarrow{\beta^+}.M$

$$\mathbf{fv}\,\forall\overrightarrow{\alpha^+}.N \subseteq \mathbf{fv}\,([\overrightarrow{P/\alpha^+}]N) \setminus \overrightarrow{\beta^+} \quad \text{here } \overrightarrow{\beta^+} \text{ is excluded by the premise } \mathbf{fv}\,N \cap \overrightarrow{\beta^+} = \varnothing$$

$$\subseteq \mathbf{fv}\,M \setminus \overrightarrow{\beta^+} \qquad \text{by the induction hypothesis, } \mathbf{fv}\,([\overrightarrow{P/\alpha^+}]N) \subseteq \mathbf{fv}\,M$$

$$\subseteq \mathbf{fv}\,\forall\overrightarrow{\beta^+}.M$$

Case 5. The positive cases are symmetric.

$\square$

Corollary 1 (Free Variables of mutual subtypes)**.**

$-$ If $\Gamma \vdash N \simeq^{\leqslant}_1 M$ then $\mathbf{fv}\,N = \mathbf{fv}\,M$,

$+$ If $\Gamma \vdash P \simeq^{\leqslant}_1 Q$ then $\mathbf{fv}\,P = \mathbf{fv}\,Q$

Lemma 2 (Subtypes and supertypes of a variable)**.** *Assuming* $\Gamma \vdash \alpha^-$, $\Gamma \vdash \alpha^+$, $\Gamma \vdash N$, *and* $\Gamma \vdash P$,

$+$ *if* $\Gamma \vdash P \geqslant_1 \alpha^+$ *or* $\Gamma \vdash \alpha^+ \geqslant_1 P$ *then* $P = \exists\overrightarrow{\alpha^-}.\alpha^+$ *(for some potentially empty* $\overrightarrow{\alpha^-}$*)*

$-$ *if* $\Gamma \vdash N \leqslant_1 \alpha^-$ *or* $\Gamma \vdash \alpha^- \leqslant_1 N$ *then* $N = \forall\overrightarrow{\alpha^+}.\alpha^-$ *(for some potentially empty* $\overrightarrow{\alpha^+}$*)*

Proof. We prove by induction on the tree inferring $\Gamma \vdash P \geqslant_1 \alpha^+$ or $\Gamma \vdash \alpha^+ \geqslant_1 P$ or or $\Gamma \vdash N \leqslant_1 \alpha^-$ or $\Gamma \vdash \alpha^- \leqslant_1 N$. Let us consider which of these judgments the tree is inferring.

Case 1. $\Gamma \vdash P \geqslant_1 \alpha^+$

If the size of the inference tree is 1 then the only rule that can infer it is Rule $(\mathrm{Var}^{+\geqslant_1})$, which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule $(\exists^{\geqslant_1})$. By inverting this rule, $P = \exists\overrightarrow{\alpha^-}.P'$ where P' does not start with $\exists$ and $\Gamma \vdash [\overrightarrow{N/\alpha^-}]P' \geqslant_1 \alpha^+$ for some $\Gamma, \overrightarrow{\beta^-} \vdash N_i$.

By the induction hypothesis, $[\overrightarrow{N/\alpha^-}]P' = \exists\overrightarrow{\beta^-}.\alpha^+$. Notice that P' must be a variable, because P' does not start with $\exists$, nor does it start with $\uparrow$ (otherwise, $[\overrightarrow{N/\alpha^-}]P'$ would also started with $\uparrow$ and would not be equal to $\exists\overrightarrow{\beta^-}.\alpha^+$). Since P' is a *positive* variable, $[\overrightarrow{N/\alpha^-}]P' = P'$, and then $P' = \exists\overrightarrow{\beta^-}.\alpha^+$ means that $P' = \alpha^+$. This way, $P = \exists\overrightarrow{\alpha^-}.P' = \exists\overrightarrow{\alpha^-}.\alpha^+$, as required.

Case 2. $\Gamma \vdash \alpha^+ \geqslant_1 P$

If the size of the inference tree is 1 then the only rule that can infer it is Rule $(\mathrm{Var}^{+\geqslant_1})$, which implies that $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule $(\exists^{\geqslant_1})$. By inverting this rule, $P = \exists\overrightarrow{\beta^-}.Q$ where and $\Gamma, \overrightarrow{\beta^-} \vdash \alpha^+ \geqslant_1 Q$.

By the induction hypothesis, $Q = \exists\overrightarrow{\beta^-}'.\alpha^+$. This way, $P = \exists\overrightarrow{\beta^-}.Q = \exists\overrightarrow{\beta^-}.\exists\overrightarrow{\beta^-}'.\alpha^+$, as required.

Case 3. The negative cases ($\Gamma \vdash N \leqslant_1 \alpha^-$ and $\Gamma \vdash \alpha^- \leqslant_1 N$) are proved analogously.

$\square$

Corollary 2 (Variables have no proper subtypes and supertypes)**.** *Assuming that all mentioned types are well-formed in* Γ,

$$\Gamma \vdash P \geqslant_1 \alpha^+ \iff P = \exists\overrightarrow{\beta^-}.\alpha^+ \iff \Gamma \vdash P \simeq^{\leqslant}_1 \alpha^+ \iff P \simeq^D_1 \alpha^+$$

$$\Gamma \vdash \alpha^+ \geqslant_1 P \iff P = \exists\overrightarrow{\beta^-}.\alpha^+ \iff \Gamma \vdash P \simeq^{\leqslant}_1 \alpha^+ \iff P \simeq^D_1 \alpha^+$$

$$\Gamma \vdash N \leqslant_1 \alpha^- \iff N = \forall\overrightarrow{\beta^+}.\alpha^- \iff \Gamma \vdash N \simeq^{\leqslant}_1 \alpha^- \iff N \simeq^D_1 \alpha^-$$

$$\Gamma \vdash \alpha^- \leqslant_1 N \iff N = \forall\overrightarrow{\beta^+}.\alpha^- \iff \Gamma \vdash N \simeq^{\leqslant}_1 \alpha^- \iff N \simeq^D_1 \alpha^-$$

Proof. Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \simeq^{\leqslant}_1 \alpha^+$ and $\exists\overrightarrow{\alpha^-}.\alpha^+ \simeq \alpha^+$ and apply lemma 2. **Ilya: fix**

$\square$

4.2 Substitution

Lemma 3 (Substitution strengthening). *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

+ *if $\Gamma_1 \vdash P$ then $[\sigma]P = [\sigma|_{\mathbf{fv}\,P}]P$,*

− *if $\Gamma_1 \vdash N$ then $[\sigma]N = [\sigma|_{\mathbf{fv}\,N}]N$*

Proof. `Ilya:  todo` □

Lemma 4 (Substitution preserves subtyping). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

+ *if $\Gamma, \Gamma_1 \vdash P$, $\Gamma, \Gamma_1 \vdash Q$, and $\Gamma, \Gamma_1 \vdash P \geqslant_1 Q$ then $\Gamma, \Gamma_2 \vdash [\sigma]P \geqslant_1 [\sigma]Q$*

− *if $\Gamma, \Gamma_1 \vdash N$, $\Gamma, \Gamma_1 \vdash M$, and $\Gamma, \Gamma_1 \vdash N \leqslant_1 M$ then $\Gamma, \Gamma_2 \vdash [\sigma]N \leqslant_1 [\sigma]M$*

Proof. `Ilya:  todo` □

Corollary 3 (Substitution preserves subtyping). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

+ *if $\Gamma, \Gamma_1 \vdash P$, $\Gamma, \Gamma_1 \vdash Q$, and $\Gamma, \Gamma_1 \vdash P \simeq_1^\leqslant Q$ then $\Gamma, \Gamma_2 \vdash [\sigma]P \simeq_1^\leqslant [\sigma]Q$*

− *if $\Gamma, \Gamma_1 \vdash N$, $\Gamma, \Gamma_1 \vdash M$, and $\Gamma, \Gamma_1 \vdash N \simeq_1^\leqslant M$ then $\Gamma, \Gamma_2 \vdash [\sigma]N \simeq_1^\leqslant [\sigma]M$*

4.3 Type well-formedness

Lemma 5 (Well-formedness agrees with substitution). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

+ $\Gamma, \Gamma_1 \vdash P \;\Leftrightarrow\; \Gamma, \Gamma_2 \vdash [\sigma]P$

− $\Gamma, \Gamma_1 \vdash N \;\Leftrightarrow\; \Gamma, \Gamma_2 \vdash [\sigma]N$

Proof. `Ilya:  todo` □

Corollary 4. *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

+ $\Gamma_1, \Gamma_2 \vdash P \;\Leftrightarrow\; \Gamma_2 \vdash [\sigma]P$

− $\Gamma_1, \Gamma_2 \vdash N \;\Leftrightarrow\; \Gamma_2 \vdash [\sigma]N$

Lemma 6 (Equivalent Contexts). *In the well-formedness judgment, only used variables matter:*

+ *if $\Gamma_1 \cap \mathbf{fv}\,P = \Gamma_2 \cap \mathbf{fv}\,P$ then $\Gamma_1 \vdash P \;\Longleftrightarrow\; \Gamma_2 \vdash P$,*

− *if $\Gamma_1 \cap \mathbf{fv}\,N = \Gamma_2 \cap \mathbf{fv}\,N$ then $\Gamma_1 \vdash N \;\Longleftrightarrow\; \Gamma_2 \vdash N$.*

Proof. By simple mutual induction on P and Q. □

Corollary 5. *Suppose that all the types below are well-formed in Γ and $\Gamma' \subseteq \Gamma$. Then*

+ $\Gamma \vdash P \simeq_1^\leqslant Q$ *implies* $\Gamma' \vdash P \;\Longleftrightarrow\; \Gamma' \vdash Q$

− $\Gamma \vdash N \simeq_1^\leqslant M$ *implies* $\Gamma' \vdash N \;\Longleftrightarrow\; \Gamma' \vdash M$

Proof. From lemma 6 and corollary 1. □

4.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\overline{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N \equiv \mathit{vars} \cap \mathbf{fv}\ N}$	$\dfrac{N \simeq^D_1 M}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ M}$	—
Normalization	$\overline{N \simeq^D_1 \mathbf{nf}(N)}$	$\dfrac{N \simeq^D_1 M}{\mathbf{nf}(N) = \mathbf{nf}(M)}$	—
Equivalence	$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq^D_1 Q}{\Gamma \vdash P \simeq^\leqslant_1 Q}$	$\dfrac{\Gamma \vdash P \simeq^\leqslant_1 Q}{P \simeq^D_1 Q}$	—
Uppgrade	$\dfrac{\mathbf{upgrade}\ \Gamma \vdash P\ \mathbf{to}\ \Delta = Q}{Q\text{ is sound } \begin{cases}\Delta \vdash Q \\ \Gamma \vdash Q \geqslant_1 P\end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \mathbf{upgrade}\ \Gamma \vdash P\ \mathbf{to}\ \Delta = Q}$	$\dfrac{Q' \text{ is sound}}{\mathbf{upgrade}\ \Gamma \vdash P\ \mathbf{to}\ \Delta = Q}{\Delta \vdash Q' \geqslant_1 Q}$
LUB	$\dfrac{\Gamma \models P_1 \vee P_2 = Q}{Q\text{ is sound } \begin{cases}\Gamma \vdash Q \\ \Gamma \vdash Q \geqslant_1 P_1 \\ \Gamma \vdash Q \geqslant_1 P_2\end{cases}}$	$\dfrac{\exists \text{ sound } Q'}{\exists Q \text{ s.t. } \Gamma \models P_1 \vee P_2 = Q}$	$\dfrac{Q' \text{ is sound}}{\Gamma \models P_1 \vee P_2 = Q}{\Delta \vdash Q' \geqslant_1 Q}$
Anti-unification	$\dfrac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \text{ is sound } \begin{cases}\Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat{\tau}_i : \Xi \\ [\hat{\tau}_i]\,Q = P_i\end{cases}}$	$\dfrac{\exists \text{ sound } (\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2)}{\exists (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \text{ s.t.}}$ $\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$	$(\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2) \text{ is sound}$ $\dfrac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\exists \Gamma; \Xi \vdash \hat{\tau} : \Xi' \text{ s.t. } [\hat{\tau}]\,Q' = Q}$
Unification (matching)	$\dfrac{\Theta \models P \overset{u}{\simeq} Q \dashv \hat{\sigma}}{\hat{\sigma}\text{ is sound } \begin{cases}\Theta \vdash \hat{\sigma} \\ [\hat{\sigma}]\,P = Q\end{cases}}$	$\dfrac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Theta \models P \overset{u}{\simeq} Q \dashv \hat{\sigma}}$	—
Subtyping	$\dfrac{\Gamma; \Theta \models N \leqslant M \dashv \hat{\sigma}}{\hat{\sigma}\text{ is sound } \begin{cases}\Theta \vdash \hat{\sigma} \\ \Gamma \vdash [\hat{\sigma}]\,N \leqslant_1 M\end{cases}}$	$\dfrac{\exists \text{ sound } \hat{\sigma}'}{\exists \hat{\sigma} \text{ s.t. } \Gamma; \Theta \models N \leqslant M \dashv \hat{\sigma}}$	—

4.5 Anti-unification

Lemma 7 (Soundness of the anti-unification algorithm)**.**

Lemma 8 (Completeness of the anti-unification algorithm)**.**

Lemma 9 (Initiality of the anti-unification algorithm)**.**

4.6 Variable Ordering

Definition 4 (Collision free bijection)**.** *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is **collision free on sets** P and Q if and only if*

1. *$\mu(P \cap A) \cap Q = \varnothing$*

2. *$\mu(Q \cap A) \cap P = \varnothing$*

Lemma 10 (Soundness of variable ordering)**.** *Variable ordering extracts precisely used free variables.*

- $-$ **ord** vars **in** $N \equiv \mathit{vars} \cap$ **fv** N *(as sets)*

- $+$ **ord** vars **in** $P \equiv \mathit{vars} \cap$ **fv** P *(as sets)*

Proof. Straightforward mutual induction on **ord** vars **in** $N = \vec{\alpha}$ and **ord** vars **in** $P = \vec{\alpha}$ ☐

Corollary 6 (Additivity of ordering)**.** *Variable ordering is additive (in terms of set union) with respect to its first argument.*

- $\mathbf{ord}\,(vars_1 \cup vars_2)\,\mathbf{in}\,N \equiv \mathbf{ord}\,vars_1\,\mathbf{in}\,N \cup \mathbf{ord}\,vars_2\,\mathbf{in}\,N$ *(as sets)*

+ $\mathbf{ord}\,(vars_1 \cup vars_2)\,\mathbf{in}\,P \equiv \mathbf{ord}\,vars_1\,\mathbf{in}\,P \cup \mathbf{ord}\,vars_2\,\mathbf{in}\,P$ *(as sets)*

Corollary 7 (Weakening of ordering)**.** *Extending the first argument of the ordering with unused variables does not change the result.*

- $\mathbf{ord}\,(vars \cap \mathbf{fv}\,N)\,\mathbf{in}\,N = \mathbf{ord}\,vars\,\mathbf{in}\,N$

+ $\mathbf{ord}\,(vars \cap \mathbf{fv}\,P)\,\mathbf{in}\,P = \mathbf{ord}\,vars\,\mathbf{in}\,P$

Lemma 11 (Distributivity of renaming over variable ordering)**.** *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

- *If μ is collision free on vars and $\mathbf{fv}\,N$ then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]N$*

+ *If μ is collision free on vars and $\mathbf{fv}\,P$ then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) = \mathbf{ord}\,([\mu]vars)\,\mathbf{in}\,[\mu]P$*

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$

let us consider four cases:

    a. $\alpha^- \in A$ and $\alpha^- \in vars$

Then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-)$

$$
\begin{aligned}
&= [\mu]\alpha^- && \text{by Rule } (\mathrm{Var}_\in^+)\\
&= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars)\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\beta^- && \text{by Rule } (\mathrm{Var}_\in^+), \text{ because } \beta^- \in [\mu]vars\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^-
\end{aligned}
$$

    b. $\alpha^- \notin A$ and $\alpha^- \notin vars$

Notice that $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}_{\notin}^+)$. On the other hand, $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \cdot$ The latter equality is from Rule $(\mathrm{Var}_{\notin}^+)$, because μ is collision free on $vars$ and $\mathbf{fv}\,N$, so $\mathbf{fv}\,N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

    c. $\alpha^- \in A$ but $\alpha^- \notin vars$

Then $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = \cdot$ by Rule $(\mathrm{Var}_{\notin}^+)$. To prove that $\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \cdot$, we apply Rule $(\mathrm{Var}_{\notin}^+)$. Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

      (i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.

      (ii) Since μ is collision free on $vars$ and $\mathbf{fv}\,N$, $\mu(A \cap \mathbf{fv}\,N) \ni \mu(\alpha^-) \notin vars$.

    d. $\alpha^- \notin A$ but $\alpha^- \in vars$

$\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule $(\mathrm{Var}_\in^+)$, because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = {\uparrow}P$

$[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,{\uparrow}P)$

$$
\begin{aligned}
&= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) && \text{by Rule } ({\uparrow})\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P && \text{by the induction hypothesis}\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,{\uparrow}[\mu]P && \text{by Rule } ({\uparrow})\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]{\uparrow}P && \text{by the definition of substitution}\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N
\end{aligned}
$$

Case 3. $N = P \to M$

$[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P \to M)$

$$
\begin{aligned}
&= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2\backslash\vec{\alpha}_1)) && \text{where } \mathbf{ord}\,vars\,\mathbf{in}\,P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\,vars\,\mathbf{in}\,M = \vec{\alpha}_2\\
&= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2\backslash\vec{\alpha}_1)\\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2\backslash[\mu]\vec{\alpha}_1) && \text{by induction on } \vec{\alpha}_2; \text{ the inductive step is similar to case 1. Notice that } \mu \text{ is}\\
& && \text{collision free on } \vec{\alpha}_1 \text{ and } \vec{\alpha}_2 \text{ since } \vec{\alpha}_1 \subseteq vars \text{ and } \vec{\alpha}_2 \subseteq \mathbf{fv}\,N\\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2\backslash[\mu]\vec{\alpha}_1)
\end{aligned}
$$

$(\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N) = (\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P \to [\mu]M)$

$$
\begin{aligned}
&= (\vec{\beta}_1, (\vec{\beta}_2\backslash\vec{\beta}_1)) && \text{where } \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M = \vec{\beta}_2\\
& && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1, \vec{\beta}_2 = [\mu]\vec{\alpha}_2,\\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2\backslash[\mu]\vec{\alpha}_1)
\end{aligned}
$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu]\mathbf{ord}\,vars\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.M$$
$$= [\mu]\mathbf{ord}\,vars\,\mathbf{in}\,M$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M \quad\text{by the induction hypothesis}$$
$$(\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N) = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\forall\overrightarrow{\alpha^+}.M$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.[\mu]M$$
$$= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M$$

$\square$

Lemma 12 (Ordering is not affected by independent substitutions). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2, and vars is a set of variables disjoint with both Γ_1 and Γ_2. Then*

  − $\mathbf{ord}\,vars\,\mathbf{in}\,[\sigma]N = \mathbf{ord}\,vars\,\mathbf{in}\,N$

  + $\mathbf{ord}\,vars\,\mathbf{in}\,[\sigma]P = \mathbf{ord}\,vars\,\mathbf{in}\,P$

Proof. `Ilya: Should be easy` $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 13 (Completeness of variable ordering). *Variable ordering is invariant under equivalence. For arbitrary vars,*

  − *If $N \simeq_1^D M$ then $\mathbf{ord}\,vars\,\mathbf{in}\,N = \mathbf{ord}\,vars\,\mathbf{in}\,M$ (as lists)*

  + *If $P \simeq_1^D Q$ then $\mathbf{ord}\,vars\,\mathbf{in}\,P = \mathbf{ord}\,vars\,\mathbf{in}\,Q$ (as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4.7 Normaliztaion

Lemma 14. *Set of free variables is invariant under equivalence.*

  − *If $N \simeq_1^D M$ then $\mathbf{fv}\,N \equiv \mathbf{fv}\,M$ (as sets)*

  + *If $P \simeq_1^D Q$ then $\mathbf{fv}\,P \equiv \mathbf{fv}\,Q$ (as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$ $\qquad\qquad\qquad\qquad$ $\square$

Lemma 15. *Free variables are not changed by the normalization*

  − $\mathbf{fv}\,N \equiv \mathbf{fv}\,\mathbf{nf}\,(N)$

  + $\mathbf{fv}\,P \equiv \mathbf{fv}\,\mathbf{nf}\,(P)$

Proof. By straightforward induction on $\mathbf{nf}\,(N) = M$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 16 (Soundness of quantifier normalization).

  − $N \simeq_1^D \mathbf{nf}\,(N)$

  + $P \simeq_1^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var^-) and (Var^+)
By the corresponding equivalence rules.

Case 2. $(\uparrow)$, $(\downarrow)$, and $(\rightarrow)$
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. $(\forall)$, i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.N'$
From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 14, $\mathbf{fv}\,N \equiv \mathbf{fv}\,N'$. Then by lemma 10, $\overrightarrow{\alpha^{+\prime}} \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$, and thus, $\overrightarrow{\alpha^{+\prime}} \cap \mathbf{fv}\,N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$.
To prove $\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\alpha^{+\prime}}.N'$, it suffices to provide a bijection $\mu : \overrightarrow{\alpha^{+\prime}} \cap \mathbf{fv}\,N' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. $(\exists)$ Same as for case 3.

$\square$

Corollary 8 (Normalization preserves ordering)**.** *For any vars,*

- **ord** *vars* **in nf** (N) = **ord** *vars* **in** M

+ **ord** *vars* **in nf** (P) = **ord** *vars* **in** Q

Proof. Immediately from lemmas 13 and 16. $\square$

Lemma 17 (Distributivity of normalization over substitution)**.** *Normalization of a term distributes over substitution. Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$, *i.e.* σ *maps variables from* Γ_1 *into types taking free variables from* Γ_2. *Then*

- **nf** $([\sigma]N)$ = $[\textbf{nf}\,(\sigma)]\textbf{nf}\,(N)$

+ **nf** $([\sigma]P)$ = $[\textbf{nf}\,(\sigma)]\textbf{nf}\,(P)$

where **nf** (σ) *means pointwise normalization:* $[\textbf{nf}\,(\sigma)]\alpha^- = \textbf{nf}\,([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\quad$ **nf** $([\sigma]N)$ = **nf** $([\sigma]\alpha^-)$ = $[\textbf{nf}\,(\sigma)]\alpha^-$.
$[\textbf{nf}\,(\sigma)]\textbf{nf}\,(N)$ = $[\textbf{nf}\,(\sigma)]\textbf{nf}\,(\alpha^-)$ = $[\textbf{nf}\,(\sigma)]\alpha^-$.

Case 2. $P = \alpha^+$
Similar to case 1.

Case 3. If the type is formed by $\to$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \to M$ then
$\textbf{nf}\,([\sigma]N)$ = $\textbf{nf}\,([\sigma](P \to M))$

$\quad\quad$ = $\textbf{nf}\,([\sigma]P \to [\sigma]M)$ $\quad\quad\quad\quad\quad\quad$ By the congruence of substitution

$\quad\quad$ = $\textbf{nf}\,([\sigma]P) \to \textbf{nf}\,([\sigma]M)$ $\quad\quad\quad\quad$ By the congruence of normalization, i.e. Rule $(\to)$

$\quad\quad$ = $[\textbf{nf}\,(\sigma)]\textbf{nf}\,(P) \to [\textbf{nf}\,(\sigma)]\textbf{nf}\,(M)$ $\quad$ By the induction hypothesis

$\quad\quad$ = $[\textbf{nf}\,(\sigma)](\textbf{nf}\,(P) \to \textbf{nf}\,(M))$ $\quad\quad\quad$ By the congruence of substitution

$\quad\quad$ = $[\textbf{nf}\,(\sigma)]\textbf{nf}\,(P \to M)$ $\quad\quad\quad\quad\quad$ By the congruence of normalization

$\quad\quad$ = $[\textbf{nf}\,(\sigma)]\textbf{nf}\,(N)$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$
$[\textbf{nf}\,(\sigma)]\textbf{nf}\,(N)$ = $[\textbf{nf}\,(\sigma)]\textbf{nf}\,(\forall\overrightarrow{\alpha^+}.M)$

$\quad\quad\quad$ = $[\textbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^{+\prime}}.\textbf{nf}\,(M)$ $\quad$ Where $\overrightarrow{\alpha^{+\prime}}$ = **ord** $\overrightarrow{\alpha^+}$ **in nf** (M) = **ord** $\overrightarrow{\alpha^+}$ **in** M (the latter is by corollary 8)

$\textbf{nf}\,([\sigma]N)$ = $\textbf{nf}\,([\sigma]\forall\overrightarrow{\alpha^+}.M)$

$\quad\quad$ = $\textbf{nf}\,(\forall\overrightarrow{\alpha^+}.[\sigma]M)$ $\quad\quad\quad\quad$ Assuming $\overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing$ and $\overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$

$\quad\quad$ = $\forall\overrightarrow{\beta^+}.\textbf{nf}\,([\sigma]M)$ $\quad\quad\quad\quad$ Where $\overrightarrow{\beta^+}$ = **ord** $\overrightarrow{\alpha^+}$ **in nf** $([\sigma]M)$ = **ord** $\overrightarrow{\alpha^+}$ **in** $[\sigma]M$ (the latter is by corollary 8)

$\quad\quad$ = $\forall\overrightarrow{\alpha^{+\prime}}.\textbf{nf}\,([\sigma]M)$ $\quad\quad\quad\quad$ By lemma 12, $\overrightarrow{\beta^+} = \overrightarrow{\alpha^{+\prime}}$ since $\overrightarrow{\alpha^+}$ is disjoint with Γ_1 and Γ_2

$\quad\quad$ = $\forall\overrightarrow{\alpha^{+\prime}}.[\textbf{nf}\,(\sigma)]\textbf{nf}\,(M)$ $\quad$ By the induction hypothesis

To show alpha-equivalence of $[\textbf{nf}\,(\sigma)]\forall\overrightarrow{\alpha^{+\prime}}.\textbf{nf}\,(M)$ and $\forall\overrightarrow{\alpha^{+\prime}}.[\textbf{nf}\,(\sigma)]\textbf{nf}\,(M)$, we can assume that $\overrightarrow{\alpha^{+\prime}} \cap \Gamma_1 = \varnothing$, and $\overrightarrow{\alpha^{+\prime}} \cap \Gamma_2 = \varnothing$.

Case 5. $P = \exists\overrightarrow{\alpha^-}.Q$
Same as for case 4.

$\square$

Corollary 9 (Commutativity of normalization and renaming)**.** *Normalization of a term commutes with renaming. Suppose that* μ *is a bijection between two sets of variables* $\mu : A \leftrightarrow B$. *Then*

- **nf** $([\mu]N)$ = $[\mu]\textbf{nf}\,(N)$

+ **nf** $([\mu]P)$ = $[\mu]\textbf{nf}\,(P)$

Proof. Immediately from lemma 17, after noticing that $\mathbf{nf}\,(\mu) = \mu$. □

Lemma 18 (Completeness of quantified normalization)**.** *Normalization returns the same representative for equivalent types.*

   − If $N \simeq_1^D M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$

   + If $P \simeq_1^D Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

   Case 1. $(\forall^{\simeq_1^P})$

     From the definition of the normalization,

- $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ where $\overrightarrow{\alpha^+}'$ is $\mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N)$
- $\mathbf{nf}\,(\forall\overrightarrow{\beta^+}.M) = \forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$ where $\overrightarrow{\beta^+}'$ is $\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M)$

Let us take $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)$ from the inversion of the equivalence judgment. Notice that from lemmas 10 and 15, the domain and the codomain of μ can be written as $\mu : \overrightarrow{\beta^+}' \leftrightarrow \overrightarrow{\alpha^+}'$.

To show the alpha-equivalence of $\forall\overrightarrow{\alpha^+}'.\mathbf{nf}\,(N)$ and $\forall\overrightarrow{\beta^+}'.\mathbf{nf}\,(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ and (ii) $[\mu]\overrightarrow{\beta^+}' = \overrightarrow{\alpha^+}'$.

(i) $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,([\mu]M) = \mathbf{nf}\,(N)$. The first equality holds by corollary 9, the second—by the induction hypothesis.

(ii)
$$\begin{aligned}
[\mu]\overrightarrow{\beta^+}' &= [\mu]\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by the definition of } \overrightarrow{\beta^+}'\\
&= [\mu]\mathbf{ord}\,(\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M) && \text{from lemma 15 and corollary 7}\\
&= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M) && \text{by lemma 11, because } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing\\
& && \text{and } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing\\
&= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{since } [\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N) \text{ is proved}\\
&= \mathbf{ord}\,(\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{because } \mu \text{ is a bijection between } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \text{ and } \overrightarrow{\beta^+} \cap \mathbf{fv}\,M\\
&= \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N) && \text{from lemma 15 and corollary 7}\\
&= \overrightarrow{\alpha^+}' && \text{by the definition of } \overrightarrow{\alpha^+}'
\end{aligned}$$

   Case 2. $(\exists^{\simeq_1^D})$ Same as for case 1.

   Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□

Lemma 19 (Idempotence of normalization)**.** *Normalization is idempotent*

   − $\mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$

   + $\mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

Proof. By applying lemma 18 to lemma 16. □

Lemma 20. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized.*
  Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then

   + $[\sigma]P$ *is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \text{is normal}\\ P & \text{is normal} \end{cases}$

   − $[\sigma]N$ *is normal* $\iff \begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \text{is normal}\\ N & \text{is normal} \end{cases}$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$

Then N is always normal, and the normality of $\sigma|_{\alpha^-}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$[\sigma](P \to M) \text{ is normal} \iff [\sigma]P \to [\sigma]M \text{ is normal} \qquad \text{by the substitution congruence}$$

$$\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases} \qquad \text{by congruence of normality } \texttt{Ilya:\ \ lemma?}$$

$$\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(M)} & \text{is normal} \end{cases} \qquad \text{by the induction hypothesis}$$

$$\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P) \cup \mathbf{fv}\,(M)} & \text{is normal} \end{cases} \iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P \to M)} & \text{is normal} \end{cases}$$

Case 3. $N = \,\uparrow\! P$

By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall \overrightarrow{\alpha^+}.M$

$$[\sigma](\forall \overrightarrow{\alpha^+}.M) \text{ is normal} \iff (\forall \overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} \qquad \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$$

$$\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by the definition of normalization}$$

$$\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by lemma 12}$$

$$\iff \begin{cases} \sigma|_{\mathbf{fv}\,(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by the induction hypothesis}$$

$$\iff \begin{cases} \sigma|_{\mathbf{fv}\,(\forall \overrightarrow{\alpha^+}.M)} \text{ is normal} & \text{since } \mathbf{fv}\,(\forall \overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M); \\ \forall \overrightarrow{\alpha^+}.M \text{ is normal} & \text{by the definition of normalization} \end{cases}$$

Case 5. $P = \ldots$

The positive cases are done in the same way as the negative ones. $\qquad\qquad\square$

4.8   Equivalence

Lemma 21 (Type well-formedness is invariant under equivalence)**.** *Mutual subtyping implies declarative equivalence.*

+ *if $P \simeq_1^D Q$ then $\Gamma \vdash P \iff \Gamma \vdash Q$,*

− *if $N \simeq_1^D M$ then $\Gamma \vdash N \iff \Gamma \vdash M$*

Proof. `Ilya:  todo` $\qquad\qquad\square$

Corollary 10 (Normalization preserves well-formedness)**.**

+ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}\,(P)$,

− $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}\,(N)$

Proof. Immediately from lemmas 16 and 21. $\qquad\qquad\square$

Corollary 11 (Normalization preserves well-formedness of substitution)**.**
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}\,(\sigma) : \Gamma_1$

Lemma 22 (Soundness of equivalence)**.** *Declarative equivalence implies mutual subtyping.*

+ *if $\Gamma \vdash P$, $\Gamma \vdash Q$, and $P \simeq_1^D Q$ then $\Gamma \vdash P \simeq_1^\leqslant Q$,*

− *if $\Gamma \vdash N$, $\Gamma \vdash M$, and $N \simeq_1^D M$ then $\Gamma \vdash N \simeq_1^\leqslant M$.*

Proof. We prove it by mutual induction on $P \simeq_1^D Q$ and $N \simeq_1^D M$.

Case 1. $\alpha^- \simeq_1^D \alpha^-$

Then $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$ by Rule ($\text{Var}^{-\leqslant_1}$), which immediately implies $\Gamma \vdash \alpha^- \simeq_1^\leqslant \alpha^-$ by Rule ($\simeq_1^\leqslant{}^-$).

Case 2. $\uparrow P \simeq_1^D \uparrow Q$

Then by inversion of Rule ($\uparrow^{\leqslant_1}$), $P \simeq_1^D Q$, and from the induction hypothesis, $\Gamma \vdash P \simeq_1^\leqslant Q$, and (by symmetry) $\Gamma \vdash Q \simeq_1^\leqslant P$.
When Rule ($\uparrow^{\leqslant_1}$) is applied to $\Gamma \vdash P \simeq_1^\leqslant Q$, it gives us $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$; when it is applied to $\Gamma \vdash Q \simeq_1^\leqslant P$, we obtain $\Gamma \vdash \uparrow Q \leqslant_1 \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \simeq_1^\leqslant \uparrow Q$.

Case 3. $P \to N \simeq_1^D Q \to M$

Then by inversion of Rule ($\to^{\leqslant_1}$), $P \simeq_1^D Q$ and $N \simeq_1^D M$. By the induction hypothesis, $\Gamma \vdash P \simeq_1^\leqslant Q$ and $\Gamma \vdash N \simeq_1^\leqslant M$, which means by inversion: (i) $\Gamma \vdash P \geqslant_1 Q$, (ii) $\Gamma \vdash Q \geqslant_1 P$, (iii) $\Gamma \vdash N \leqslant_1 M$, (iv) $\Gamma \vdash M \leqslant_1 N$. Applying Rule ($\to^{\leqslant_1}$) to (i) and (iii), we obtain $\Gamma \vdash P \to N \leqslant_1 Q \to M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \to M \leqslant_1 P \to N$. Together, it implies $\Gamma \vdash P \to N \simeq_1^\leqslant Q \to M$.

Case 4. $\forall \overrightarrow{\alpha^+}.N \simeq_1^D \forall \overrightarrow{\beta^+}.M$

Then by inversion, there exists bijection $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N)$, such that $N \simeq_1^D [\mu]M$. By the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^\leqslant [\mu]M$. From corollary 3 and the fact that μ is bijective, we also have $\Gamma, \overrightarrow{\beta^+} \vdash [\mu^{-1}]N \simeq_1^\leqslant M$.

Let us construct a subsitution $\overrightarrow{\alpha^+} \vdash \overrightarrow{P}/\overrightarrow{\beta^+} : \overrightarrow{\beta^+}$ by extending μ with arbitrary positive types on $\overrightarrow{\beta^+}\backslash \mathbf{fv}\, M$.

Notice that $[\mu]M = [\overrightarrow{P}/\overrightarrow{\beta^+}]M$, and therefore, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^\leqslant [\mu]M$ implies $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{P}/\overrightarrow{\beta^+}]M \leqslant_1 N$. Then by Rule ($\forall^{\leqslant_1}$), $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \leqslant_1 \forall \overrightarrow{\alpha^+}.N$.

Analogously, we construct the substitution from μ^{-1}, and use it to instantiate $\overrightarrow{\alpha^+}$ in the application of Rule ($\forall^{\leqslant_1}$) to infer $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M$.

This way, $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \leqslant_1 \forall \overrightarrow{\alpha^+}.N$ and $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M$ gives us $\Gamma \vdash \forall \overrightarrow{\beta^+}.M \simeq_1^\leqslant \forall \overrightarrow{\alpha^+}.N$.

Case 5. For the cases of the positive types, the proofs are symmetric.

$\square$

Corollary 12.

+ *if $\Gamma \vdash P$ then $\Gamma \vdash P \simeq_1^\leqslant \mathbf{nf}\,(P)$,*

− *if $\Gamma \vdash N$ then $\Gamma \vdash N \simeq_1^\leqslant \mathbf{nf}\,(N)$.*

Proof. Immediately from lemmas 16 and 22 and corollary 10. $\square$

Lemma 23 (Subtyping induced by disjoint substitutions). *If two disjoint substitutions induce subtyping, they are degenerate (so is the subtyping). Suppose that $\Gamma \vdash \sigma_1 : \Gamma_1$ and $\Gamma \vdash \sigma_2 : \Gamma_1$, where $\Gamma_i \subseteq \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$. Then*

− *assuming $\Gamma \vdash N$, $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ implies $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\, N$*

+ *assuming $\Gamma \vdash P$, $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ implies $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \mathbf{fv}\, P$*

Proof. Proof by induciton on $\Gamma \vdash N$ (and mutually on $\Gamma \vdash P$).

Case 1. $N = \alpha^-$

Then $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\alpha^- \leqslant_1 [\sigma_2]\alpha^-$. Let us consider the following cases:

    a. $\alpha^- \notin \Gamma_1$ and $\alpha^- \notin \Gamma_2$
        Then $\Gamma \vdash \sigma_i \simeq_1^\leqslant \mathsf{id} : \alpha^-$ holds immediately, since $[\sigma_i]\alpha^- = [\mathsf{id}]\alpha^- = \alpha^-$ and $\Gamma \vdash \alpha^- \simeq_1^\leqslant \alpha^-$.

    b. $\alpha^- \in \Gamma_1$ and $\alpha^- \in \Gamma_2$
        This case is not possible by assumption: $\Gamma_1 \cap \Gamma_2 = \varnothing$.

    c. $\alpha^- \in \Gamma_1$ and $\alpha^- \notin \Gamma_2$
        Then we have $\Gamma \vdash [\sigma_1]\alpha^- \leqslant_1 \alpha^-$, which by corollary 2 means $\Gamma \vdash [\sigma_1]\alpha^- \simeq_1^\leqslant \alpha^-$, and hence, $\Gamma \vdash \sigma_1 \simeq_1^\leqslant \mathsf{id} : \alpha^-$.
        $\Gamma \vdash \sigma_2 \simeq_1^\leqslant \mathsf{id} : \alpha^-$ holds since $[\sigma_2]\alpha^- = \alpha^-$, similarly to case 1.*a.*

    d. $\alpha^- \notin \Gamma_1$ and $\alpha^- \in \Gamma_2$
        Then we have $\Gamma \vdash \alpha^- \leqslant_1 [\sigma_2]\alpha^-$, which by corollary 2 means $\Gamma \vdash \alpha^- \simeq_1^\leqslant [\sigma_2]\alpha^-$, and hence, $\Gamma \vdash \sigma_2 \simeq_1^\leqslant \mathsf{id} : \alpha^-$.
        $\Gamma \vdash \sigma_1 \simeq_1^\leqslant \mathsf{id} : \alpha^-$ holds since $[\sigma_1]\alpha^- = \alpha^-$, similarly to case 1.*a.*

Case 2. $N = \forall \overrightarrow{\alpha^+}.M$

Then by inversion, $\Gamma, \overrightarrow{\alpha^+} \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\forall\overrightarrow{\alpha^+}.M \leqslant_1 [\sigma_2]\forall\overrightarrow{\alpha^+}.M$. By the congruence of substitution and by the inversion of Rule $(\forall^{\leqslant_1})$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{Q/\alpha^+}][\sigma_1]M \leqslant_1 [\sigma_2]M$, where $\Gamma, \overrightarrow{\alpha^+} \vdash Q_i$. Let us denote the (Kleisli) composition of σ_1 and $\overrightarrow{Q/\alpha^+}$ as σ_1', noting that $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' : \Gamma_1, \overrightarrow{\alpha^+}$, and $\Gamma_1, \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$.

Let us apply the induction hypothesis to M and the substitutions σ_1' and σ_2 with $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_1']M \leqslant_1 [\sigma_2]M$ to obtain:

$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{1}$$

$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{2}$$

Then $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ holds by strengthening of 2: for any $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M \backslash \overrightarrow{\alpha^+}$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$ is strengthened to $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$, because $\mathbf{fv}\,[\sigma_2]\beta^\pm = \mathbf{fv}\,\beta^\pm = \{\beta^\pm\} \subseteq \Gamma$.

To show that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$, let us take an arbitrary $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M \backslash \overrightarrow{\alpha^+}$.

$\begin{aligned}
\beta^\pm &= [\mathsf{id}]\beta^\pm && \text{by definition of } \mathsf{id} \\
&\simeq_1^{\leqslant} [\sigma_1']\beta^\pm && \text{by 1} \\
&= [\overrightarrow{Q/\alpha^+}][\sigma_1]\beta^\pm && \text{by definition of } \sigma_1' \\
&= [\sigma_1]\beta^\pm && \text{because } \overrightarrow{\alpha^+} \cap \mathbf{fv}\,[\sigma_1]\beta^\pm \subseteq \overrightarrow{\alpha^+} \cap \Gamma = \varnothing
\end{aligned}$

This way, $\Gamma \vdash [\sigma_1]\beta^\pm \simeq_1^{\leqslant} \beta^\pm$ for any $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ and thus, $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$.

Case 3. $N = P \to M$

Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1](P \to M) \leqslant_1 [\sigma_2](P \to M)$, then by congruence of substitution, $\Gamma \vdash [\sigma_1]P \to [\sigma_1]M \leqslant_1 [\sigma_2]P \to [\sigma_2]M$, then by inversion $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ and $\Gamma \vdash [\sigma_1]M \leqslant_1 [\sigma_2]M$.

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$ and to $\Gamma \vdash [\sigma_1]M \leqslant_1 [\sigma_2]M$, we obtain (respectively):

$$\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P \tag{3}$$

$$\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{4}$$

Noting that $\mathbf{fv}\,(P \to M) = \mathbf{fv}\, P \cup \mathbf{fv}\, M$, we combine eqs. (3) and (4) to conclude: $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\,(P \to M)$.

Case 4. $N = {\uparrow}P$

Then by inversion, $\Gamma \vdash P$. $\Gamma \vdash [\sigma_1]N \leqslant_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]{\uparrow}P \leqslant_1 [\sigma_2]{\uparrow}P$, then by congruence of substitution and by inversion, $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant_1 [\sigma_2]P$, we obtain $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$. Since $\mathbf{fv}\,{\uparrow}P = \mathbf{fv}\, P$, we can conclude: $\Gamma \vdash \sigma_i \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\,{\uparrow}P$.

Case 5. The positive cases are proved symmetrically.

$\square$

Corollary 13 (Substitution cannot induce proper subtypes or supertypes)**.** *Assuming all mentioned types are well-formed in Γ and σ is a substitution $\Gamma \vdash \sigma : \Gamma$,*

$$\Gamma \vdash [\sigma]N \leqslant_1 N \;\Rightarrow\; \Gamma \vdash [\sigma]N \simeq_1^{\leqslant} N \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, N$$

$$\Gamma \vdash N \leqslant_1 [\sigma]N \;\Rightarrow\; \Gamma \vdash N \simeq_1^{\leqslant} [\sigma]N \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, N$$

$$\Gamma \vdash [\sigma]P \geqslant_1 P \;\Rightarrow\; \Gamma \vdash [\sigma]P \simeq_1^{\leqslant} P \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$$

$$\Gamma \vdash P \geqslant_1 [\sigma]P \;\Rightarrow\; \Gamma \vdash P \simeq_1^{\leqslant} [\sigma]P \text{ and } \Gamma \vdash \sigma \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$$

Lemma 24. *Asssuming that the mentioned types (P, Q, N, and M) are well-formed in Γ and that the substitutions (σ_1 and σ_2) have signature $\Gamma \vdash \sigma_i : \Gamma$,*

+ *if $\Gamma \vdash [\sigma_1]P \geqslant_1 Q$ and $\Gamma \vdash [\sigma_2]Q \geqslant_1 P$*
  then there exists a bijection $\mu : \mathbf{fv}\, P \leftrightarrow \mathbf{fv}\, Q$ such that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, P$ and $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, Q$;

− *if $\Gamma \vdash [\sigma_1]N \leqslant_1 M$ and $\Gamma \vdash [\sigma_2]N \leqslant_1 M$*
  then there exists a bijection $\mu : \mathbf{fv}\, N \leftrightarrow \mathbf{fv}\, M$ such that $\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \mu : \mathbf{fv}\, N$ and $\Gamma \vdash \sigma_2 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\, M$.

Proof.

+ Applying σ_2 to both sides of $\Gamma \vdash [\sigma_1]P \geqslant_1 Q$ (by **??**), we have: $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant_1 [\sigma_2]Q$. Composing it with $\Gamma \vdash [\sigma_2]Q \geqslant_1 P$ (by transitivity **??**), we have $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant_1 P$. Then by corollary 13, $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\,P$.

By a symmetric argument, we also have: $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\,Q$.

Now, we prove that $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\,P$ and $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^{\leqslant} \mathsf{id} : \mathbf{fv}\,Q$ implies that σ_1 and σ_1 are (equivalent to) mutually inverse bijections.

To do so, it suffices to prove that

  (i) for any $\alpha^{\pm} \in \mathbf{fv}\,P$ there exists $\beta^{\pm} \in \mathbf{fv}\,Q$ such that $\Gamma \vdash [\sigma_1]\alpha^{\pm} \simeq_1^{\leqslant} \beta^{\pm}$ and $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$; and

  (ii) for any $\beta^{\pm} \in \mathbf{fv}\,Q$ there exists $\alpha^{\pm} \in \mathbf{fv}\,P$ such that $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$ and $\Gamma \vdash [\sigma_1]\alpha^{\pm} \simeq_1^{\leqslant} \beta^{\pm}$.

Then the these correspondences between $\mathbf{fv}\,P$ and $\mathbf{fv}\,Q$ are mutually inverse functions, since for any $\beta^{\pm}$ there can be at most one $\alpha^{\pm}$ such that $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$ (and vice versa).

  (i) Let us take $\alpha^{\pm} \in \mathbf{fv}\,P$.

   (a) if $\alpha^{\pm}$ is positive ($\alpha^{\pm} = \alpha^{+}$), from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{+} \simeq_1^{\leqslant} \alpha^{+}$, by corollary 2, we have $[\sigma_2][\sigma_1]\alpha^{+} = \exists\overrightarrow{\beta^{-}}.\alpha^{+}$.
    What shape can $[\sigma_1]\alpha^{+}$ have? It cannot be $\exists\overrightarrow{\alpha^{-}}.{\downarrow}N$ (for potentially empty $\overrightarrow{\alpha^{-}}$), because the outer constructor ${\downarrow}$ would remain after the substitution σ_2, whereas $\exists\overrightarrow{\beta^{-}}.\alpha^{+}$ does not have ${\downarrow}$. The only case left is $[\sigma_1]\alpha^{+} = \exists\overrightarrow{\alpha^{-}}.\gamma^{+}$.
    Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.\gamma^{+} \simeq_1^{\leqslant} \gamma^{+}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{+} \simeq_1^{\leqslant} \gamma^{+}$. Also notice that $[\sigma_2]\exists\overrightarrow{\alpha^{-}}.\gamma^{+} = \exists\overrightarrow{\beta^{-}}.\alpha^{+}$ implies $\Gamma \vdash [\sigma_2]\gamma^{+} \simeq_1^{\leqslant} \alpha^{+}$.

   (b) if $\alpha^{\pm}$ is negative ($\alpha^{\pm} = \alpha^{-}$) from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{-} \simeq_1^{\leqslant} \alpha^{-}$, by corollary 2, we have $[\sigma_2][\sigma_1]\alpha^{-} = \forall\overrightarrow{\beta^{+}}.\alpha^{-}$.
    What shape can $[\sigma_1]\alpha^{-}$ have? It cannot be $\forall\overrightarrow{\alpha^{+}}.{\uparrow}P$ nor $\forall\overrightarrow{\alpha^{+}}.P \to M$ (for potentially empty $\overrightarrow{\alpha^{+}}$), because the outer constructor ($\to$ or ${\uparrow}$), remaining after the substitution σ_2, is however absent in the resulting $\forall\overrightarrow{\beta^{+}}.\alpha^{-}$. Hence, the only case left is $[\sigma_1]\alpha^{-} = \forall\overrightarrow{\alpha^{+}}.\gamma^{-}$ Notice that $\Gamma \vdash \gamma^{-} \simeq_1^{\leqslant} \forall\overrightarrow{\alpha^{+}}.\gamma^{-}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{-} \simeq_1^{\leqslant} \gamma^{-}$. Also notice that $[\sigma_2]\forall\overrightarrow{\alpha^{+}}.\gamma^{-} = \forall\overrightarrow{\beta^{+}}.\alpha^{-}$ implies $\Gamma \vdash [\sigma_2]\gamma^{-} \simeq_1^{\leqslant} \alpha^{-}$.

  (ii) The proof is symmetric: We swap P and Q, σ_1 and σ_2, and exploit $\Gamma \vdash [\sigma_1][\sigma_2]\alpha^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$ instead of $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{\pm} \simeq_1^{\leqslant} \alpha^{\pm}$.

− The proof is symmetric to the positive case.

$\square$

Lemma 25 (Equivalence of polymorphic types).

 − *For $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N$ and $\Gamma \vdash \forall\overrightarrow{\beta^{+}}.M$,*
   if $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N \simeq_1^{\leqslant} \forall\overrightarrow{\beta^{+}}.M$ then there exists a bijection $\mu : \overrightarrow{\beta^{+}} \cap \mathbf{fv}\,M \leftrightarrow \overrightarrow{\alpha^{+}} \cap \mathbf{fv}\,N$ such that $\Gamma, \overrightarrow{\alpha^{+}} \vdash N \simeq_1^{\leqslant} [\mu]N$,

 + *For $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.P$ and $\Gamma \vdash \exists\overrightarrow{\beta^{-}}.Q$,*
   if $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.P \simeq_1^{\leqslant} \exists\overrightarrow{\beta^{-}}.Q$ then there exists a bijection $\mu : \overrightarrow{\beta^{-}} \cap \mathbf{fv}\,Q \leftrightarrow \overrightarrow{\alpha^{-}} \cap \mathbf{fv}\,P$ such that $\Gamma, \overrightarrow{\beta^{-}} \vdash P \simeq_1^{\leqslant} [\mu]Q$.

Proof.

 − First, by α-conversion, we ensure $\overrightarrow{\alpha^{+}} \cap \mathbf{fv}\,M = \varnothing$ and $\overrightarrow{\beta^{+}} \cap \mathbf{fv}\,N = \varnothing$. By inversion, $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N \simeq_1^{\leqslant} \forall\overrightarrow{\beta^{+}}.M$ implies

  1. $\Gamma, \overrightarrow{\beta^{+}} \vdash [\sigma_1]N \leqslant_1 M$ for $\Gamma, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \overrightarrow{\alpha^{+}}$ and

  2. $\Gamma, \overrightarrow{\alpha^{+}} \vdash [\sigma_2]M \leqslant_1 N$ for $\Gamma, \overrightarrow{\alpha^{+}} \vdash \sigma_2 : \overrightarrow{\beta^{+}}$.

To apply lemma 24, we weaken and rearrange the contexts, and extend the substitutions to act as identity outside of their initial domain:

  1. $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\sigma_1]N \leqslant_1 M$ for $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}}$ and

  2. $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\sigma_2]M \leqslant_1 N$ for $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_2 : \Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}}$.

Then from lemma 24, there exists a bijection $\mu : \mathbf{fv}\,M \leftrightarrow \mathbf{fv}\,N$ such that $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_2 \simeq_1^{\leqslant} \mu : \mathbf{fv}\,M$ and $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\,N$.

Let us show that if we restrict the domain of μ to $\overrightarrow{\beta^{+}}$, its range will be contained in $\overrightarrow{\alpha^{+}}$. Let us take $\gamma^{+} \in \overrightarrow{\beta^{+}} \cap \mathbf{fv}\,M$ and assume $[\mu]\gamma^{+} \notin \overrightarrow{\alpha^{+}}$. Then since $\Gamma, \overrightarrow{\beta^{+}} \vdash \sigma_1 : \overrightarrow{\alpha^{+}}$, σ_1 acts as identity outside of $\overrightarrow{\alpha^{+}}$, i.e. $[\sigma_1][\mu]\gamma^{+} = [\mu]\gamma^{+}$. Since $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash \sigma_1 \simeq_1^{\leqslant} \mu^{-1} : \mathbf{fv}\,N$, application of σ_1 is equivalent to application of μ^{-1}, then $\Gamma, \overrightarrow{\alpha^{+}}, \overrightarrow{\beta^{+}} \vdash [\mu^{-1}][\mu]\gamma^{+} \simeq_1^{\leqslant} [\mu]\gamma^{+}$, i.e.

$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \gamma^+ \simeq_1^\leqslant [\mu]\gamma^+$, which means $\gamma^+ \in \mathbf{fv}\,[\mu]\gamma^+ \subseteq \mathbf{fv}\,N$. By assumption, $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv}\,M$, i.e. $\overrightarrow{\beta^+} \cap \mathbf{fv}\,N \neq \varnothing$, hence contradiction.

By **??**, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq_1^\leqslant \mu|_{\overrightarrow{\beta^+}} : \mathbf{fv}\,M$ implies $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \simeq_1^\leqslant [\mu|_{\overrightarrow{\beta^+}}]M$. By similar reasoning, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \simeq_1^\leqslant [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N$.

This way,

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N \leqslant_1 M \tag{5}$$

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu|_{\overrightarrow{\beta^+}}]M \leqslant_1 N \tag{6}$$

By applying $\mu|_{\overrightarrow{\beta^+}}$ to both sides of 5 (**??**) and contracting $\mu^{-1}|_{\overrightarrow{\alpha^+}} \circ \mu|_{\overrightarrow{\beta^+}} = \mu|_{\overrightarrow{\beta^+}}^{-1} \circ \mu|_{\overrightarrow{\beta^+}} = \mathsf{id}$, we have: $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \leqslant_1 [\mu|_{\overrightarrow{\beta^+}}]M$, which together with 6 means $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \simeq_1^\leqslant [\mu|_{\overrightarrow{\beta^+}}]M$, and by strengthening, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq_1^\leqslant [\mu|_{\overrightarrow{\beta^+}}]M$. Symmetrically, $\Gamma, \overrightarrow{\beta^+} \vdash M \simeq_1^\leqslant [\mu|_{\overrightarrow{\beta^+}}^{-1}]N$.

- $+$ The proof is symmetric to the proof of the negative case.

$\square$

Lemma 26 (Completeness of equivalence). *Mutual subtyping implies declarative equivalence. Assuming all the types below are well-formed in Γ:*

$+$ *if $\Gamma \vdash P \simeq_1^\leqslant Q$ then $P \simeq_1^D Q$,*

$-$ *if $\Gamma \vdash N \simeq_1^\leqslant M$ then $N \simeq_1^D M$.*

Proof. $-$ Induction on the sum of sizes of N and M. By inversion, $\Gamma \vdash N \simeq_1^\leqslant M$ means $\Gamma \vdash N \leqslant_1 M$ and $\Gamma \vdash M \leqslant_1 N$. Let us consider the last rule that forms $\Gamma \vdash N \leqslant_1 M$:

Case 1. Rule $(\mathrm{Var}^{-\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \alpha^- \leqslant_1 \alpha^-$
Then $N \simeq_1^D M$ (i.e. $\alpha^- \simeq_1^D \alpha^-$) holds immediately by Rule $(\mathrm{Var}^{-\simeq_1^D})$.

Case 2. Rule $(\uparrow^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q$
Then by inversion, $\Gamma \vdash P \simeq_1^\leqslant Q$, and by induction hypothesis, $P \simeq_1^D Q$. Then $N \simeq_1^D M$ (i.e. $\uparrow P \simeq_1^D \uparrow Q$) holds by Rule $(\uparrow^{\simeq_1^D})$.

Case 3. Rule $(\rightarrow^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash P \rightarrow N' \leqslant_1 Q \rightarrow M'$
Then by inversion, $\Gamma \vdash P \geqslant_1 Q$ and $\Gamma \vdash N' \leqslant_1 M'$. Notice that $\Gamma \vdash M \leqslant_1 N$ is of the form $\Gamma \vdash Q \rightarrow M' \leqslant_1 P \rightarrow N'$, which by inversion means $\Gamma \vdash Q \geqslant_1 P$ and $\Gamma \vdash M' \leqslant_1 N'$.
This way, $\Gamma \vdash Q \simeq_1^\leqslant P$ and $\Gamma \vdash M' \simeq_1^\leqslant N'$. Then by induction hypothesis, $Q \simeq_1^D P$ and $M' \simeq_1^D N'$. Then $N \simeq_1^D M$ (i.e. $P \rightarrow N' \simeq_1^D Q \rightarrow M'$) holds by Rule $(\rightarrow^{\simeq_1^D})$.

Case 4. Rule $(\forall^{\leqslant_1})$ i.e. $\Gamma \vdash N \leqslant_1 M$ is of the form $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant_1 \forall\overrightarrow{\beta^+}.M'$
Then by **??**, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \simeq_1^\leqslant \forall\overrightarrow{\beta^+}.M'$ means that there exists a bijection $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv}\,M' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N'$ such that $\Gamma, \overrightarrow{\alpha^+} \vdash [\mu]M' \simeq_1^\leqslant N'$.
Notice that the application of bijection μ to M' does not change its size (which is less than the size of M), hence the induction hypothesis applies. This way, $[\mu]M' \simeq_1^D N'$ (and by symmetry, $N' \simeq_1^D [\mu]M'$) holds by induction. Then we apply Rule $(\forall^{\simeq_1^D})$ to get $\forall\overrightarrow{\alpha^+}.N' \simeq_1^D \forall\overrightarrow{\beta^+}.M'$, i.e. $N \simeq_1^D M$.

$+$ The proof is symmetric to the proof of the negative case.

$\square$

4.9 Upgrade

Let us consider a type P well-formed in Γ. Some of its Γ-supertypes are also well-formed in a smaller context $\Delta \subseteq \Gamma$. The upgrade is the operation that returns the least of such supertypes.

Lemma 27 (Soundness of Upgrade). *Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^\pm}$, if $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ then*

1. $\Delta \vdash Q$

2. $\Gamma \vdash Q \geqslant_1 P$

Proof. By inversion, $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ means that for fresh $\overrightarrow{\beta^{\pm}}$ and $\overrightarrow{\gamma^{\pm}}$, $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vDash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P = Q$. Then by the soundness of the least upper bound (lemma 32),

1. $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q$,

2. $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$, and

3. $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$.

$$
\begin{aligned}
\mathbf{fv}\,Q &\subseteq \mathbf{fv}\,[\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \cap \mathbf{fv}\,[\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P && \text{Since by lemma 1, } \mathbf{fv}\,Q \subseteq \mathbf{fv}\,[\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \text{ and } \mathbf{fv}\,Q \subseteq \mathbf{fv}\,[\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \\
&\subseteq ((\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) \cup \overrightarrow{\beta^{\pm}}) \cap ((\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) \cup \overrightarrow{\gamma^{\pm}}) \\
&= (\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) \cap (\mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}}) && \text{since } \overrightarrow{\beta^{\pm}} \text{ and } \overrightarrow{\gamma^{\pm}} \text{ are fresh} \\
&= \mathbf{fv}\,P \backslash \overrightarrow{\alpha^{\pm}} \\
&\subseteq \Gamma \backslash \overrightarrow{\alpha^{\pm}} && \text{since } P \text{ is well-formed in } \Gamma \\
&\subseteq \Delta
\end{aligned}
$$

This way, by lemma 6, $\Delta \vdash Q$.

Let us apply $\overrightarrow{\alpha^{\pm}}/\overrightarrow{\beta^{\pm}}$—the inverse of the substitution $\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$ and by **??**, get $\Delta, \overrightarrow{\alpha^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash [\overrightarrow{\alpha^{\pm}}/\overrightarrow{\beta^{\pm}}]Q \geqslant_1 P$. Notice that $\Delta \vdash Q$ implies that $\mathbf{fv}\,Q \cap \overrightarrow{\beta^{\pm}} = \varnothing$, then by **??**, $[\overrightarrow{\alpha^{\pm}}/\overrightarrow{\beta^{\pm}}]Q = Q$, and thus $\Delta, \overrightarrow{\alpha^{\pm}}, \overrightarrow{\gamma^{\pm}} \vdash Q \geqslant_1 P$. By context strengthening, $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q \geqslant_1 P$. $\qquad\square$

Lemma 28 (Completeness and Initiality of Upgrade). *The upgrade returns the least Γ-supertype of P well-formed in Δ. Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$,*
For any Q' such that

1. *$\Delta \vdash Q'$ and*

2. *$\Gamma \vdash Q' \geqslant_1 P$,*

The result of the upgrade algorithm Q exists ($\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$) and satisfies $\Delta \vdash Q' \geqslant_1 Q$.

Proof. Let us consider fresh (not intersecting with Γ) $\overrightarrow{\beta^{\pm}}$ and $\overrightarrow{\gamma^{\pm}}$.

If we apply substitution $\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q' \geqslant_1 P$, we have $\Delta, \overrightarrow{\beta^{\pm}} \vdash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]Q' \geqslant_1 [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$, which by **??**, since Q' is well-formed in Δ, simplifies to $\Delta, \overrightarrow{\beta^{\pm}} \vdash Q' \geqslant_1 [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$.

Analogously, if we apply substitution $\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q' \geqslant_1 P$, we have $\Delta, \overrightarrow{\gamma^{\pm}} \vdash Q' \geqslant_1 [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$.

This way, Q is a common supertype of $[\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$ and $[\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$ in context $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}}$. This way, we can apply the completeness of the least upper bound (lemma 33):

1. there exists Q s.t. $\Gamma \vDash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P = Q'$

2. $\Gamma \vdash Q' \geqslant_1 Q$.

The former means that the upgrade algorithm terminates and returns Q. The latter means that since both Q' and Q are well-formed in Δ, by **??**, $\Delta \vdash Q' \geqslant_1 Q$. $\qquad\square$

4.10 Upper Bounds

Lemma 29 (Decomposition of the quantifier rule). *Ilya: move somewhere* *Whenever the quantifier rule (Rule ($\exists^{\geqslant_1}$) or Rule ($\forall^{\leqslant_1}$)) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.*

$-$ *If $\Gamma \vdash N \leqslant_1 \forall\overrightarrow{\beta^{+}}.M$ then $\Gamma, \overrightarrow{\beta^{+}} \vdash N \leqslant_1 M$.*

$+$ *If $\Gamma \vdash P \geqslant_1 \exists\overrightarrow{\beta^{-}}.Q$ then $\Gamma, \overrightarrow{\beta^{-}} \vdash P \geqslant_1 Q$.*

Lemma 30 (Characterization of the Supertypes). *Let us define the set of upper bounds of a positive type $\mathsf{UB}(P)$ in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists\overrightarrow{\alpha^-}.\beta^+ \ \mid \ for \ \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q) \ not \ using \ \overrightarrow{\beta^-}$

$$\Gamma \vdash \downarrow M \qquad \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \ \middle| \ \begin{array}{l} for \ \overrightarrow{\alpha^-}, \ M', \ and \ \vec{N} \ s.t. \\ \Gamma \vdash N_i, \ \Gamma, \overrightarrow{\alpha^-} \vdash M', \ and \ [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq^D_1 \downarrow M \end{array} \right\}$$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \ \mid \ \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Immediately from lemma 2

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

Then if $\Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'$, then by lemma 29, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'$, and $\mathbf{fv}\, Q \cap \overrightarrow{\beta^-} = \varnothing$ by the the Barendregt's convention. The other direction holds by Rule ($\exists^{\geqslant_1}$). This way, $\{Q \mid \Gamma \vdash Q \geqslant_1 \exists\overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,(Q) \cap \overrightarrow{\beta^-} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = \downarrow M$

Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\not\exists}$) and upper bounds with outer quantifiers ($|_\exists$). We prove that for both of these groups, the restricted sets are equal.

    a. $Q \neq \exists\overrightarrow{\beta^-}.Q'$

        Then the last applied rule to infer $\Gamma \vdash Q \geqslant_1 \downarrow M$ must be Rule ($\downarrow^{\geqslant_1}$), which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq^{\leqslant}_1 M$, then by lemma 26 and Rule ($\downarrow^{\simeq^D_1}$), $\downarrow M' \simeq^D_1 \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq^D_1 \downarrow M\} = \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\not\exists}$.

        In the other direction, $\downarrow M' \simeq^D_1 \downarrow M \Rightarrow \Gamma \vdash \downarrow M' \simeq^{\leqslant}_1 \downarrow M$ by lemma 22, since $\Gamma \vdash \downarrow M'$ by lemma 21
$$\Rightarrow \Gamma \vdash \downarrow M' \geqslant_1 \downarrow M \quad \text{by inversion}$$

    b. $Q = \exists\overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)

        Then the last rule applied to infer $\Gamma \vdash \exists\overrightarrow{\beta^-}.Q' \geqslant_1 \downarrow M$ must be Rule ($\exists^{\geqslant_1}$). Inversion of this rule gives us $\Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]Q' \geqslant_1 \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\vec{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.a, $[\vec{N}/\overrightarrow{\beta^-}]Q' \simeq^D_1 \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists\overrightarrow{\beta^-}.\downarrow M' \in \mathsf{UB}(\Gamma \vdash \downarrow M)|_\exists$ (notice that $\overrightarrow{\beta^-}$ is not empty).

        In the other direction, $[\vec{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq^D_1 \downarrow M \Rightarrow \Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq^{\leqslant}_1 \downarrow M$ by lemma 22, since $\Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]\downarrow M'$ by lemma 21
$$\Rightarrow \Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]\downarrow M' \geqslant_1 \downarrow M \quad \text{by inversion}$$
$$\Rightarrow \Gamma \vdash \exists\overrightarrow{\beta^-}.\downarrow M' \geqslant_1 \downarrow M \qquad \text{by Rule } (\exists^{\geqslant_1})$$

$\square$

Lemma 31 (Characterization of the Normalized Supertypes). *For a normalized positive type* $P = \mathbf{nf}\,(P)$, *let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P) \ not \ using \ \overrightarrow{\beta^-}$

$$\Gamma \vdash \downarrow M \qquad \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \ \middle| \ \begin{array}{l} for \ \overrightarrow{\alpha^-}, \ M', \ and \ \vec{N} \ s.t. \ \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i, \ \Gamma, \overrightarrow{\alpha^-} \vdash M', \ and \ [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \end{array} \right\}$$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \ \mid \ \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Then from lemma 30, $\{\mathbf{nf}\,(Q) \ \mid \ \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\beta^+) \ \mid \ \text{for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists \overrightarrow{\beta}.P'$

$\mathsf{NFUB}(\Gamma \vdash \exists \overrightarrow{\beta}.P') = \mathsf{NFUB}(\Gamma, \overrightarrow{\beta} \vdash P')$ not using $\overrightarrow{\beta}$

$$
\begin{aligned}
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta} \vdash Q \geqslant_1 P'\} \text{ not using } \overrightarrow{\beta} && \text{by the induction hypothesis} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}\,(Q) = \mathbf{fv}\,Q \text{ by lemma 15} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta} = \varnothing\} && \text{by lemma 30} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists \overrightarrow{\beta}.P')\} && \text{by the definition of } \mathsf{UB} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \exists \overrightarrow{\beta}.P'\} && \text{by lemma 30}
\end{aligned}
$$

Case 3. $P = \downarrow M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 1. *Suppose that $\nu : A \to A$ is an idempotent function, P is a predicate on A, $F : A \to B$ is a function. Then*

$$
\{F(\nu x) \mid x \in A \text{ s.t. } P(\nu x)\} =
$$
$$
= \{F(x) \mid x \in A \text{ s.t. } \nu x = x \text{ and } P(x)\}.
$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 2. *For functions F and ν, and predicates P and Q,*

$$
\{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } P(x)\} =
$$
$$
= \{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } (\exists x' \in A \text{ s.t. } P(x') \text{ and } \nu x' = \nu x)\}.
$$

Observation 3. *There exist positive and negative types well-formed in empty context, hence, a type substitution can be extended to an arbitrary domain (if its values on the domain extension are irrelevant). Specifically, Suppose that $vars_1 \subseteq vars_2$. Then $\Gamma \vdash \sigma|_{vars_1} : vars_1$ implies $\exists\, \sigma'$ s.t. $\Gamma \vdash \sigma' : vars_2$ and $\sigma|_{vars_1} = \sigma'|_{vars_1}$.*

$$\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant_1 \downarrow M\} =$$

$$= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \downarrow M)\} \qquad\qquad \text{by lemma 30}$$

$$= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \text{ and } \overrightarrow{N} \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash N_i, \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} \qquad \text{by the definition of UB}$$

$$= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} \qquad \text{we reassigned the substitution } \overrightarrow{N}/\overrightarrow{\alpha^-} \text{ as } \sigma$$

$$= \left\{ \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.{\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \text{ and } \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} \qquad \text{by lemma 3}$$

$$= \left\{ \exists\overrightarrow{\alpha^{-\prime}}.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-}, M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \mathbf{ord}\,\overrightarrow{\alpha^-} \text{ in } M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' \simeq_1^D {\downarrow}M \end{array} \right\} \qquad \text{by the definition of normalization}$$

$$= \left\{ \exists\overrightarrow{\alpha^{-\prime}}.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-}, M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \mathbf{ord}\,\overrightarrow{\alpha^-} \text{ in } M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } \mathbf{nf}\,([\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M') = \mathbf{nf}\,({\downarrow}M) \end{array} \right\} \qquad \begin{array}{l}\text{from lemmas 16 and 18, equivalence of types can be} \\ \text{replaced with the equality of their normal forms}\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^{-\prime}}.\mathbf{nf}\,({\downarrow}M') \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-}, M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \mathbf{ord}\,\overrightarrow{\alpha^-} \text{ in } M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})]{\downarrow}\mathbf{nf}\,(M') = {\downarrow}\mathbf{nf}\,(M) \end{array} \right\} \qquad \text{by congruence of normalization and lemma 17}$$

$$= \left\{ \exists\overrightarrow{\alpha^{-\prime}}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-}, M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^-}, \mathbf{ord}\,\overrightarrow{\alpha^-} \text{ in } M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} \qquad \begin{array}{l}\text{by lemma 20, } {\downarrow}M' \text{ and } \sigma|_{\mathbf{fv}\,M'} \text{ are already normal,} \\ \text{since the result of the substitution is normal;} \\ M \text{ is normal by assumption}\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^{-\prime}}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-}, M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ (\exists\sigma' \text{ s.t. } \Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')}) \\ \mathbf{ord}\,\overrightarrow{\alpha^-} \text{ in } M' = \overrightarrow{\alpha^{-\prime}} \text{ and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} \qquad \begin{array}{l}\text{We apply observation 2 (with } \nu\sigma = \sigma|_{\mathbf{fv}\,M'}, \text{ and} \\ P(\sigma) = \Gamma \vdash \sigma : \overrightarrow{\alpha^-})\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^{-\prime}}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-}, M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv}\,M'} : \overrightarrow{\alpha^{-\prime}}, \mathbf{ord}\,\overrightarrow{\alpha^-} \text{ in } M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma|_{\mathbf{fv}\,M'}]{\downarrow}M' = {\downarrow}M \end{array} \right\} \qquad \begin{array}{l}\text{Notice that} \\ \text{``}\exists\sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \overrightarrow{\alpha^-} \text{ and } \sigma|_{\mathbf{fv}\,({\downarrow}M')} = \sigma'|_{\mathbf{fv}\,({\downarrow}M')})\text{''} \\ \text{is equivalent to } \Gamma \vdash \sigma|_{\mathbf{fv}\,({\downarrow}M')} : \overrightarrow{\alpha^{-\prime}} \text{ (observation 3)}\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^{-\prime}}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-}, M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^-} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^{-\prime}}, \mathbf{ord}\,\overrightarrow{\alpha^-} \text{ in } M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} \qquad \begin{array}{l}\text{We apply observation 1 to the restriction of } \sigma, \text{ and} \\ \text{remove } \sigma|_{\mathbf{fv}\,M'} = \sigma \text{ as it follows from } \Gamma \vdash \sigma : \overrightarrow{\alpha^{-\prime}}\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^{-\prime}}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, \overrightarrow{\alpha^-}, M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^{-\prime}} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^{-\prime}}, \mathbf{ord}\,\overrightarrow{\alpha^-} \text{ in } M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} \qquad \begin{array}{l}\text{by lemma 6, since } \Gamma, \overrightarrow{\alpha^-} \cap \mathbf{fv}\,M' = \Gamma, \overrightarrow{\alpha^{-\prime}} \cap \mathbf{fv}\,M'\end{array}$$

$$= \left\{ \exists\overrightarrow{\alpha^{-\prime}}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, M', \sigma \text{ s.t. } \Gamma, \overrightarrow{\alpha^{-\prime}} \vdash M', \\ \Gamma \vdash \sigma : \overrightarrow{\alpha^{-\prime}}, \mathbf{ord}\,\overrightarrow{\alpha^{-\prime}} \text{ in } M' = \overrightarrow{\alpha^{-\prime}} \\ \text{and } [\sigma]{\downarrow}M' = {\downarrow}M \end{array} \right\} \qquad \text{We apply observation 1 to the ordering of } \overrightarrow{\alpha^-}$$

$$= \left\{ \exists\overrightarrow{\alpha^{-\prime}}.{\downarrow}M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^{-\prime}}, M', \text{ and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^{-\prime}} \text{ in } M' = \overrightarrow{\alpha^{-\prime}}, \\ \Gamma \vdash N_i, \Gamma, \overrightarrow{\alpha^{-\prime}} \vdash M', \text{ and } [\overrightarrow{N}/\overrightarrow{\alpha^{-\prime}}]{\downarrow}M' = {\downarrow}M \end{array} \right\} \qquad \text{By reassigning } \sigma \text{ explicitly as } \overrightarrow{N}/\overrightarrow{\alpha^{-\prime}}$$

$$= \mathsf{NFUB}({\downarrow}M) \qquad\qquad \text{by definition}$$

$\square$

Observation 4. *Upper bounds of a type do not depend on the context as soon as the type are well-formed in it.*
If $\Gamma_1 \vdash M$ and $\Gamma_2 \vdash M$ then $\mathsf{UB}(\Gamma_1 \vdash M) = \mathsf{UB}(\Gamma \vdash M)$ and $\mathsf{NFUB}(\Gamma_1 \vdash M) = \mathsf{NFUB}(\Gamma \vdash M)$

Proof. We prove both inclusions by induction on $\Gamma_1 \vdash M$. Notice that if $[\sigma]M' \simeq_1^D M$ and $\Gamma_2 \vdash M$ then the types from the range of $\sigma|_{\mathbf{fv}\,M'}$ are well-formed in 2 `Ilya: lemma`. $\square$

Lemma 32 (Soundness of the Least Upper Bound). *For types $\Gamma \vdash P_1$, and $\Gamma \vdash P_2$, if $\Gamma \models P_1 \vee P_2 = Q$ then*

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ *and* $\Gamma \vdash Q \geqslant_1 P_2$

Proof. Induction on $\Gamma \models P_1 \vee P_2 = Q$.

Case 1. $\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+$
Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule (Var$^{+\geqslant_1}$).

Case 2. $\Gamma \models \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q$
Then by inversion of $\Gamma \vdash \exists \overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applies to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q$. Then

(i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,

(ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$,

(iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}\,(Q) \cap \Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} = \mathbf{fv}\,(Q) \cap \Gamma$ (and then apply lemma 6). The inclusion right-to-left is self-evident. To show $\mathbf{fv}\,(Q) \cap \Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \subseteq \mathbf{fv}\,(Q) \cap \Gamma$, we prove that $\mathbf{fv}\,(Q) \subseteq \Gamma$

$$\mathbf{fv}\,(Q) \subseteq \mathbf{fv}\,P_1 \cap \mathbf{fv}\,P_2 \qquad \text{by lemma 1}$$

$$\subseteq (\Gamma, \overrightarrow{\alpha^-}\backslash\overrightarrow{\beta^-}) \cap (\Gamma, \overrightarrow{\beta^-}\backslash\overrightarrow{\alpha^-}) \qquad \begin{array}{l} \text{since } \Gamma \vdash \exists \overrightarrow{\alpha^-}.P_1, \ \ \mathbf{fv}\,(P_1) \subseteq \Gamma, \overrightarrow{\alpha^-} = \Gamma, \overrightarrow{\alpha^-}\backslash\overrightarrow{\beta^-} \\ \text{(the latter is because by the Barendregt's convention,} \\ \Gamma, \overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing); \text{ similarly, } \mathbf{fv}\,(P_2) \subseteq \Gamma, \overrightarrow{\beta^-}\backslash\overrightarrow{\alpha^-} \end{array}$$

$$\subseteq \Gamma$$

To show $\Gamma \vdash Q \geqslant_1 \exists \overrightarrow{\alpha^-}.P_1$, we apply Rule ($\exists^{\geqslant_1}$). Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant_1 P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant_1 \exists \overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \models \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]\,P$. By the inversion, $\Gamma, \cdot \models \mathbf{nf}\,(\downarrow N) \overset{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \hat{\tau}_1, \hat{\tau}_2)$. Then by the soundness of anti-unification (lemma 7),

(i) $\Gamma; \Xi \vdash P$, then by **??**,

$$\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]\,P \tag{7}$$

(ii) $\Gamma; \cdot \vdash \hat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \hat{\tau}_2 : \Xi$. Assuming that $\Xi = \hat{\beta}_1^-, .., \hat{\beta}_n^-$, the antiunification solutions $\hat{\tau}_1$ and $\hat{\tau}_2$ can be put explicitly as $\hat{\tau}_1 = (\hat{\beta}_1^- :\approx N_1, .., \hat{\beta}_n^- :\approx N_n)$, and $\hat{\tau}_2 = (\hat{\beta}_1^- :\approx M_1, .., \hat{\beta}_n^- :\approx M_n)$. Then

$$\hat{\tau}_1 = (\overrightarrow{N/\alpha^-}) \circ (\overrightarrow{\alpha^-/\Xi}) \tag{8}$$

$$\hat{\tau}_2 = (\overrightarrow{M/\alpha^-}) \circ (\overrightarrow{\alpha^-/\Xi}) \tag{9}$$

(iii) $[\hat{\tau}_1]\,Q = P_1$ and $[\hat{\tau}_2]\,Q = P_1$, which, by 8 and 9, means

$$[\overrightarrow{N/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P = \mathbf{nf}\,(\downarrow N) \tag{10}$$

$$[\overrightarrow{M/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P = \mathbf{nf}\,(\downarrow M) \tag{11}$$

Then $\Gamma \vdash \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-/\Xi}]\,P$ follows directly from 7.

To show $\Gamma \vdash \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 \downarrow N$, we apply Rule ($\exists^{\geqslant_1}$), instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{N}$. Then $\Gamma \vdash [\overrightarrow{N/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 \downarrow N$ follows from 10 and since $\Gamma \vdash \mathbf{nf}\,(\downarrow N) \geqslant_1 \downarrow N$ (by corollary 12).

Analogously, instantiating $\overrightarrow{\alpha^-}$ with $\overrightarrow{M}$, gives us $\Gamma \vdash [\overrightarrow{M/\alpha^-}][\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 \downarrow M$ (from 11), and hence, $\Gamma \vdash \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-/\Xi}]\,P \geqslant_1 \downarrow M$.

□

Lemma 33 (Completeness and Initiality of the Least Upper Bound). *For types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q$ such that $\Gamma \vdash Q \geqslant_1 P_1$ and $\Gamma \vdash Q \geqslant_1 P_2$, there exists Q' s.t. $\Gamma \models P_1 \vee P_2 = Q'$ and $\Gamma \vdash Q \geqslant_1 Q'$.*

Proof. Induction on the pair (P_1, P_2). From lemma 31, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$. Let us consider the cases of what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_i$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$, where either $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then $Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_2.Q_2)$

$\qquad \subseteq \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_2 \vdash Q_2)$ $\qquad\qquad$ from the definition of UB

$\qquad = \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_2)$ $\qquad$ by observation 4, weakening and exchange

$\qquad = \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant_1 Q_1\} \cap \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant_1 Q_2\}$ $\quad$ by lemma 30,

meaning that $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q_2$. Then the next step of the algorithm—the recursive call $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vDash Q_1 \vee Q_2 = Q'$ terminates by the induction hypothesis, and moreover, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$. This way, the result of the algorithm is Q', i.e. $\Gamma \vDash P_1 \vee P_2 = Q'$.

Since both Q and Q' are sound, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant_1 Q'$ can be strengthened to $\Gamma \vdash Q \geqslant_1 Q'$ by **??**.

Case 2. $P_1 = \alpha^+$ and $P_2 = \downarrow N$

Then the set of common upper bounds of $\downarrow N$ and α^+ is empty, and thus, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$ gives a contradiction:
$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \downarrow N)$

$\qquad = \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\downarrow M' \mid \cdots\}$ $\quad$ by the definition of UB

$\qquad = \varnothing$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ since $\alpha^+ \neq \downarrow M'$ for any M'

Case 3. $P_1 = \downarrow N$ and $P_2 = \alpha^+$

Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)

Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \beta^+)$

$\qquad = \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\beta^+ \mid \cdots\}$ $\quad$ by the definition of UB

$\qquad = \varnothing$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ since $\alpha^+ \neq \beta^+$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Then the algorithm terminates in one step (Rule $(\mathrm{Var}^\vee)$) and the result is α^+, i.e. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$.

Since $Q \in \mathsf{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists\overrightarrow{\alpha^-}.\alpha^+$. Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant_1 \alpha^+$ by Rule $(\exists^{\geqslant_1})$: $\overrightarrow{\alpha^-}$ can be instantiated with arbitrary negative types (for example $\forall\beta^+.\uparrow\beta^+$), since the substitution for unused variables does not change the term $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule $(\mathrm{Var}^{+\geqslant_1})$.

Case 6. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$

Then on the next step, the algorithm tries to anti-unify $\mathbf{nf}\,(\downarrow M_1)$ and $\mathbf{nf}\,(\downarrow M_2)$. By lemma 8, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$\mathbf{nf}\,(Q) \in \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,(\downarrow M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,(\downarrow M_2))$

$\qquad = \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}\,(M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}\,(M_2))$

$$
= \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \text{and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i,\, \Gamma, \overrightarrow{\alpha^-} \vdash M',\, \text{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1) \end{array} \right\}
$$
$$
\cap
$$
$$
\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \text{and } \overrightarrow{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \overrightarrow{N}_1,\, \Gamma \vdash \overrightarrow{N}_2,\, \Gamma, \overrightarrow{\alpha^-} \vdash M',\, \text{and } [\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2) \end{array} \right\}
$$
$$
= \left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-},\, M',\, \overrightarrow{N}_1 \text{ and } \overrightarrow{N}_2 \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \overrightarrow{N}_1,\, \Gamma \vdash \overrightarrow{N}_2,\, \Gamma, \overrightarrow{\alpha^-} \vdash M',\, [\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1),\, \text{and } [\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2) \end{array} \right\}
$$

The fact that the latter set is non-empty means that there exist $\overrightarrow{\alpha^-}, M', \overrightarrow{N}_1$ and $\overrightarrow{N}_2$ such that

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$ (notice that M' is normal)

(ii) $\Gamma \vdash \overrightarrow{N}_1$ and $\Gamma \vdash \overrightarrow{N}_1$,

(iii) $[\overrightarrow{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1)$ and $[\overrightarrow{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2)$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative anti-unification variable $\hat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\hat{\alpha}^-}$. Let us show that $(\overrightarrow{\hat{\alpha}^-},\ [\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M',\ \vec{N}_1/\overrightarrow{\hat{\alpha}^-},\ \vec{N}_2/\overrightarrow{\hat{\alpha}^-})$ is a sound anti-unifier of $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$ in context Γ:

- $\overrightarrow{\hat{\alpha}^-}$ is negative by construction,
- $\Gamma; \overrightarrow{\hat{\alpha}^-} \vdash [\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M'$ because $\Gamma, \overrightarrow{\alpha^-} \vdash {\downarrow}M'$ `Ilya:  lemma!`,
- $\Gamma; \cdot \vdash (\vec{N}_1/\overrightarrow{\hat{\alpha}^-}) : \overrightarrow{\hat{\alpha}^-}$ because $\Gamma \vdash \vec{N}_1$ and $\Gamma; \cdot \vdash (\vec{N}_2/\overrightarrow{\hat{\alpha}^-}) : \overrightarrow{\hat{\alpha}^-}$ because $\Gamma \vdash \vec{N}_2$,
- $[\vec{N}_1/\overrightarrow{\hat{\alpha}^-}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\vec{N}_1/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_1) = \mathbf{nf}\,({\downarrow}M_1)$.
- $[\vec{N}_2/\overrightarrow{\hat{\alpha}^-}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\vec{N}_2/\overrightarrow{\alpha^-}]{\downarrow}M' = {\downarrow}\mathbf{nf}\,(M_2) = \mathbf{nf}\,({\downarrow}M_2)$.

Then by the completeness of the anti-unification (lemma 8), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it, i.e. $Q' = \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]\,P$, where $(\Xi,\,P,\,\hat{\tau}_1,\,\hat{\tau}_2)$ is the result of the anti-unification of $\mathbf{nf}\,({\downarrow}M_1)$ and $\mathbf{nf}\,({\downarrow}M_2)$ in context Γ.

Moreover, lemma 8 also says that the found solution is initial, i.e. there exists $\hat{\tau}$ such that $\Gamma; \Xi \vdash \hat{\tau} : \overrightarrow{\hat{\alpha}^-}$ and $[\hat{\tau}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}$, (ii) $\hat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi][\hat{\tau}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi]\,P$. In particular, from the reflexivity of subtyping: $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]{\downarrow}M' \geqslant_1 [\overrightarrow{\beta^-}/\Xi]\,P$.

It allows us to show $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant_1 Q'$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.{\downarrow}M' \geqslant_1 \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]\,P$, by applying Rule $(\exists^{\geqslant_1})$, instantiating $\overrightarrow{\alpha^-}$ with respect to σ. Finally, $\Gamma \vdash Q \geqslant_1 Q'$ since $\Gamma \vdash \mathbf{nf}\,(Q) \simeq_1^{\leqslant} Q$, and equivalence implies subtyping by `Ilya:  lemma`.

$\square$