Contents

1 DECLARATIVE TYPE SYSTEMS

1.1 Grammar

We assume that there is an infinite set of positive and negative *type* variables. Positive type variables are denoted as α^+, β^+, γ^+, etc. Negative type variables are denoted as α^-, β^-, γ^-, etc. We assume there is an infinite set of *term* variables, which are denoted as x, y, z, etc. A list of objects (variables, types or terms) is denoted by an overline arrow. For instance, $\overrightarrow{\alpha^+}$ is a list of positive type variables, $\overrightarrow{\beta^-}$ is a list of negative type variables, $\vec{v}$ is a list of values, which are arguments of a function. $\mathbf{fv}\,(P)$ and $\mathbf{fv}\,(N)$ denote the set of free variables in a type P and N, respectively.

Definition 1 (Declarative Types).

Negative declarative types

$$\begin{array}{lll} N,\ M,\ K & ::= & \\ & \mid & \alpha^- \\ & \mid & \uparrow P \\ & \mid & P \to N \\ & \mid & \forall \overrightarrow{\alpha^+}.N \end{array}$$

Positive declarative types

$$\begin{array}{lll} P,\ Q,\ R & ::= & \\ & \mid & \alpha^+ \\ & \mid & \downarrow N \\ & \mid & \exists \overrightarrow{\alpha^-}.P \end{array}$$

1.2 Equalities

For simplicity, we assume alpha-equivalent terms equal. This way, we assume that substitutions do not capture bound variables. Besides, we equate $\forall \overrightarrow{\alpha^+}.\forall \overrightarrow{\beta^+}.N$ with $\forall \overrightarrow{\alpha^+}, \overrightarrow{\beta^+}.N$, as well as $\exists \overrightarrow{\alpha^-}.\exists \overrightarrow{\beta^-}.P$ with $\exists \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}.P$, and lift these equations transitively and congruently to the whole system.

1.3 Contexts and Well-formedness

Definition 2 (Declarative Type Context).
Declarative type context Γ *is represented by a set of type variables. The concatenation* Γ_1, Γ_2 *means the union of two contexts* $\Gamma_1 \cup \Gamma_2$.

$\Gamma \vdash P$ and $\Gamma \vdash N$ denote that the type is well-formed in the context Γ, which, in fact, means that each free type variable of the type is contained in Γ (it will be shown later in lemmas 3 and 4).

Notice that checking the well-formedness of a type is an *algorithmic* procedure, in which both the context and the type are considered inputs. In other words, it is syntax-directed and mode-correct (according to [Dunfield et al. 2020]), which means that

Algorithm 1 (Type Well-formedness).

Negative type well-formedness

$\boxed{\Gamma \vdash N}$

$$\frac{\alpha^- \in \Gamma}{\Gamma \vdash \alpha^-}\ (\mathit{VAR}_-^{WF})$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash \uparrow P}\ (\uparrow^{WF})$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash N}{\Gamma \vdash P \to N}\ (\to^{WF})$$

$$\frac{\Gamma, \overrightarrow{\alpha^+} \vdash N}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N}\ (\forall^{WF})$$

Positive type well-formedness

$\boxed{\Gamma \vdash P}$

$$\frac{\alpha^+ \in \Gamma}{\Gamma \vdash \alpha^+} \quad (VAR_+^{WF})$$

$$\frac{\Gamma \vdash N}{\Gamma \vdash \downarrow N} \quad (\downarrow^{WF})$$

$$\frac{\Gamma, \overrightarrow{\alpha^-} \vdash P}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P} \quad (\exists^{WF})$$

1.4 Substitutions

Definition 3 (Substitution). *Substitutions (denoted as σ) are represented by total functions form variables to types, preserving the polarity.*

Algorithm 2 (Substitution Application). *Substitution application is denoted as $[\sigma]P$ and $[\sigma]N$. It is defined naturally as follows:*

$[\sigma]\alpha^+ = \sigma(\alpha^+)$;
$[\sigma]\alpha^- = \sigma(\alpha^-)$;
$[\sigma]\downarrow N = \downarrow[\sigma]N$;
$[\sigma]\uparrow P = \uparrow[\sigma]P$;
$[\sigma](P \to N) = [\sigma]P \to [\sigma]N$.

$[\sigma]\exists\overrightarrow{\alpha^-}.Q = \exists\overrightarrow{\alpha^-}.[\sigma]Q$,
$[\sigma]\forall\overrightarrow{\alpha^+}.N = \forall\overrightarrow{\alpha^+}.[\sigma]N$ *(here we assume that $\overrightarrow{\alpha^-}$ and $\overrightarrow{\alpha^+}$ are lists of fresh variables, that is the variable capture never happens);*

Definition 4 (Substitution Signature). *The signature $\Gamma' \vdash \sigma : \Gamma$ means that*

(1) for any $\alpha^\pm \in \Gamma, \Gamma' \vdash [\sigma]\alpha^\pm$; and
(2) for any $\alpha^\pm \notin \Gamma', [\sigma]\alpha^\pm = \alpha^\pm$.

A substitution can be restricted to a set of variables. The restricted substitution is define as expected.

Definition 5 (Subsitution Restriction). *The specification $\sigma|_{vars}$ is defined as a function such that*

(1) $\sigma|_{vars}(\alpha^\pm) = \sigma(\alpha^\pm)$, if $\alpha^\pm \in vars$; and
(2) $\sigma|_{vars}(\alpha^\pm) = \alpha^\pm$, if $\alpha^\pm \notin vars$.

Two substitutions can be composed in two ways: $\sigma_2 \circ \sigma_1$ corresponds to a consecutive application of σ_1 and σ_2, while $\sigma_2 \ll \sigma_1$ depends on a signature of σ_1 and modifies σ_1 by applying σ_2 to its results on the domain.

Definition 6 (Substitution Composition). *$\sigma_2 \circ \sigma_1$ is defined as a function such that $\sigma_2 \circ \sigma_1(\alpha^\pm) = \sigma_2(\sigma_1(\alpha^\pm))$.*

Definition 7 (Monadic Substitution Composition). *Suppose that $\Gamma' \vdash \sigma_1 : \Gamma$. Then we define $\sigma_2 \ll \sigma_1$ as $(\sigma_2 \circ \sigma_1)|_\Gamma$.*

Notice that the result of $\sigma_2 \ll \sigma_1$ depends on the specification of σ_1, which is not unique. However, we assume that the used specification clear from the context of the proof.

Definition 8 (Equivalent Substitutions). *The substitution equivalence judgement $\Gamma' \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \Gamma$ indicates that on the domain Γ, the result of σ_1 and σ_2 are equivalent in context Γ'. Formally, for any $\alpha^\pm \in \Gamma, \Gamma' \vdash [\sigma_1]\alpha^\pm \simeq^{\leqslant} [\sigma_2]\alpha^\pm$.*

Sometimes it is convenient to construct substitution explicitly mapping each variable from a list (or a set) to a type. Such substitutions are denoted as $\vec{P}/\overrightarrow{\alpha^+}$ and $\vec{N}/\overrightarrow{\alpha^-}$, where $\vec{P}$ and $\vec{N}$ are lists of the corresponding types.

Definition 9 (Explicit Substitution).

– *Suppose that* $\overrightarrow{\alpha^-}$ *is a list of negative type variables, and* $\vec{N}$ *is a list of negative types of the same length. Then* $\vec{N}/\overrightarrow{\alpha^-}$ *denotes a substitution such that*
  (1) for $\alpha_i^+ \in \overrightarrow{\alpha^-}$, $[\vec{N}/\overrightarrow{\alpha^-}]\alpha_i^+ = N_i$;
  (2) for $\beta^+ \notin \overrightarrow{\alpha^-}$, $[\vec{N}/\overrightarrow{\alpha^-}]\beta^+ = \beta^+$.

\+ *Positive explicit substitution* $\vec{P}/\overrightarrow{\alpha^+}$ *is defined symmetrically.*

1.5 Declarative Subtyping

Subtyping is one of the key mechanism of our system. It realizes the polymorphism: abstract $\forall$ and $\exists$ types can be used where concrete types are expected, exactly because the subtyping relation between them.

Definition 10.

Negative subtyping
$\boxed{\Gamma \vdash N \leqslant M}$

$$\frac{}{\Gamma \vdash \alpha^- \leqslant \alpha^-}\ (\textsc{Var}_-^{\leqslant})$$

$$\frac{\Gamma \vdash P \simeq^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant \uparrow Q}\ (\uparrow^{\leqslant})$$

$$\frac{\Gamma \vdash P \geqslant Q \quad \Gamma \vdash N \leqslant M}{\Gamma \vdash P \to N \leqslant Q \to M}\ (\to^{\leqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash \sigma : \overrightarrow{\alpha^+} \quad \Gamma, \overrightarrow{\beta^+} \vdash [\sigma]N \leqslant M}{\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M}\ (\forall^{\leqslant})$$

Positive supertyping
$\boxed{\Gamma \vdash P \geqslant Q}$

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant \alpha^+}\ (\textsc{Var}_+^{\geqslant})$$

$$\frac{\Gamma \vdash N \simeq^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant \downarrow M}\ (\downarrow^{\geqslant})$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-} \quad \Gamma, \overrightarrow{\beta^-} \vdash [\sigma]P \geqslant Q}{\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q}\ (\exists^{\geqslant})$$

Negative subtyping-induced equivalence
$\boxed{\Gamma \vdash N \simeq^{\leqslant} M}$

$$\frac{\Gamma \vdash N \leqslant M \quad \Gamma \vdash M \leqslant N}{\Gamma \vdash N \simeq^{\leqslant} M}\ (\simeq_-^{\leqslant})$$

Positive subtyping-induced equivalence
$\boxed{\Gamma \vdash P \simeq^{\leqslant} Q}$

$$\frac{\Gamma \vdash P \geqslant Q \quad \Gamma \vdash Q \geqslant P}{\Gamma \vdash P \simeq^{\leqslant} Q}\ (\simeq_+^{\leqslant})$$

The following observations about the declarative subtyping are worth noting:

- Rule ($\textsc{Var}_-^{\leqslant}$) and Rule ($\textsc{Var}_+^{\geqslant}$) make the subtyping reflexive on variables (and further, on any type).
- Rule ($\to^{\leqslant}$) is standard: the arrow is covariant on the resulting type and contravariant on the argument type.
- Rule ($\downarrow^{\geqslant}$) and Rule ($\uparrow^{\leqslant}$) are non-standard: the subtyping is *invariant* for shifts. This way, the subtyping of shifted types in one direction implies the subtyping in the opposite direction. Although this rule restricts the subtyping relation, it makes the system decidable.
- Rule ($\forall^{\leqslant}$) and Rule ($\exists^{\geqslant}$) are the only non-algorithmic rules: the substitution for the quantified variable is not specified, those, these rules 'drive' the subtyping relation.

In the next section, we present the sound and complete algorithm checking whether one type is a subtype of another according to definition 10.

2 ALGORITHMIC TYPE SYSTEM

2.1 Grammar

In the algorithmic system, we extend the grammar of types by adding positive and negative *algorithmic variables* ($\widehat{\alpha}^+, \widehat{\beta}^+, \widehat{\gamma}^+$, etc. and $\widehat{\alpha}^-, \widehat{\beta}^-, \widehat{\gamma}^-$, etc.). They represent the unknown types, which will be inferred by the algorithm. This way, we add two base cases to the grammar of positive and negative types, and use highlight to denote that the type can potentially contain algorithmic variables.

Definition 11 (Algorithmic Types).

Negative algorithmic type

$$
\begin{array}{lll}
N,\ M & ::= & \\
 & \mid & \widehat{\alpha}^- \\
 & \mid & \alpha^- \\
 & \mid & \uparrow P \\
 & \mid & P \to N \\
 & \mid & \forall \overrightarrow{\alpha^+}.N
\end{array}
$$

Positive algorithmic type

$$
\begin{array}{lll}
P,\ Q & ::= & \\
 & \mid & \widehat{\alpha}^+ \\
 & \mid & \alpha^+ \\
 & \mid & \downarrow N \\
 & \mid & \exists \overrightarrow{\alpha^-}.P
\end{array}
$$

2.2 Fresh Variable Selection

Both the subtyping and the type inference algorithm rely on the ability to select fresh, unused variables. For a set of variables *vars*, it is indicated as *vars* **are fresh** in the inference rules. We assume that the selection subroutine always succeeds and is deterministic. In other words, whenever it is called in an algorithmic inference rule, it returns the same result, uniquely determined by the input of this rule.

2.3 Variable Algorithmization

In several places of our algorithm, in particular, during algorithmic subtyping, we turn a declarative type into the algorithmic one via replacing certain type variables with fresh algorithmic variables. We call this procedure *variable algorithmization*, and define it as follows.

Definition 12 (Variable Algorithmization). *Suppose that $\overrightarrow{\alpha^-}$ is a list of negative type variables and $\overrightarrow{\widehat{\alpha}^-}$ is a list of negative algorithmic variables of the same length. Then $\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}$ is a substitution-like procedure replacing each $\alpha_i^- \in \overrightarrow{\alpha^-}$ in a type for $\widehat{\alpha}_i^- \in \overrightarrow{\widehat{\alpha}^-}$.*

Conversely, we have the opposite procedure turning algorithmic type variables into declarative type variables via *dealgorithmization*.

Definition 13 (Variable Dealgorithmization). *Suppose that $\overrightarrow{\widehat{\alpha}^-}$ is a list of negative algorithmic variables and $\overrightarrow{\alpha^-}$ is a list of negative type variables of the same length. Then $\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}$ is a substitution-like procedure replacing each $\widehat{\alpha}_i^- \in \overrightarrow{\widehat{\alpha}^-}$ in a type for $\alpha_i^- \in \overrightarrow{\alpha^-}$.*

2.4 Contexts and Well-formedness

Definition 14 (Algorithmic Type Context Ξ).
Algorithmic type context Ξ is represented by a set of algorithmic *type variables ($\widehat{\alpha}^+, \widehat{\alpha}^-, \widehat{\beta}^+, \ldots$). The concatenation Ξ_1, Ξ_2 means the union of two contexts $\Xi_1 \cup \Xi_2$.*

$\Gamma; \Xi \vdash P$ and $\Gamma; \Xi \vdash N$ are used to denote that the algorithmic type is well-formed in the contexts Γ and Ξ, which means that each algorithmic variable of the type is contained in Ξ, and each free declarative type variable of the type is contained in Γ.

Algorithm 3 (Algorithmic Type Well-formedness).

Negative algorithmic type well-formedness $\boxed{\Gamma; \Xi \vdash N}$

$$\frac{\alpha^- \in \Gamma}{\Gamma; \Xi \vdash \alpha^-} \ (VAR_-^{WF})$$

$$\frac{\widehat{\alpha}^- \in \Xi}{\Gamma; \Xi \vdash \widehat{\alpha}^-} \ (UVAR_-^{WF})$$

$$\frac{\Gamma; \Xi \vdash P}{\Gamma; \Xi \vdash \uparrow P} \ (\uparrow^{WF})$$

$$\frac{\Gamma; \Xi \vdash P \quad \Gamma; \Xi \vdash N}{\Gamma; \Xi \vdash P \to N} \ (\to^{WF})$$

$$\frac{\Gamma, \overrightarrow{\alpha^+}; \Xi \vdash N}{\Gamma; \Xi \vdash \forall \overrightarrow{\alpha^+}.N} \ (\forall^{WF})$$

Positive algorithmic type well-formedness $\boxed{\Gamma; \Xi \vdash P}$

$$\frac{\alpha^+ \in \Gamma}{\Gamma; \Xi \vdash \alpha^+} \ (VAR_+^{WF})$$

$$\frac{\widehat{\alpha}^+ \in \Xi}{\Gamma; \Xi \vdash \widehat{\alpha}^+} \ (UVAR_+^{WF})$$

$$\frac{\Gamma; \Xi \vdash N}{\Gamma; \Xi \vdash \downarrow N} \ (\downarrow^{WF})$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}; \Xi \vdash P}{\Gamma; \Xi \vdash \exists \overrightarrow{\alpha^-}.P} \ (\exists^{WF})$$

Algorithmic Type Context are used in the unification algorithm. In the subtyping algorithm, the context needs to remember additional information. In the subtyping context, each algorithmic variable is associated with a context it must be instantiated in (i.e. the context in which the type replacing the variable must be well-formed). This association is represented by *algorithmic subtyping context* Θ.

Definition 15 (Algorithmic Subtyping Context Θ).
Algorithmic Subtyping Context Θ is represented by a set of entries of form $\widehat{\alpha}^+\{\Gamma\}$ and $\widehat{\alpha}^-\{\Gamma\}$, where $\widehat{\alpha}^+$ and $\widehat{\alpha}^-$ are algorithmic variables, and Γ is a context in which they must be instantiated. We assume that no two entries associating the same variable appear in Θ.
dom (Θ) *denotes the set of variables appearing in* Θ*:* **dom** $(\Theta) = \{\widehat{\alpha}^\pm \mid \widehat{\alpha}^\pm\{\Gamma\} \in \Theta\}$. *If* $\widehat{\alpha}^\pm\{\Gamma\} \in \Theta$*, we denote* Γ *as* $\Theta(\widehat{\alpha}^\pm)$.

2.5 Subsitutions

Substitution that operates on algorithmic type variables is denoted as $\widehat{\sigma}$. It is defined as a total function from algorithmic type variables to *declarative* types, preserving the polarity.

The signature $\Theta \vdash \widehat{\sigma} : \Xi$ means that $\Xi \subseteq$ **dom** (Θ) and $\widehat{\sigma}$ maps each algorithmic variable from Ξ to a type well-formed in $\Theta(\widehat{\alpha}^\pm)$; and for each variable not appearing in **dom** (Θ), it acts as identity.

Definition 16 (Signature of Algorithmic Substitution).

- $\Theta \vdash \widehat{\sigma} : \Xi$ *means that*
  (1) for any $\widehat{\alpha}^\pm \in \Xi$*, there exists* Γ *such that* $\widehat{\alpha}^\pm\{\Gamma\} \in \Theta$ *and* $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^\pm$*;*
  (2) for any $\widehat{\alpha}^\pm \notin \Xi$*,* $[\widehat{\sigma}]\widehat{\alpha}^\pm = \widehat{\alpha}^\pm$*.*
- $\Gamma \vdash \widehat{\sigma} : \Xi$ *means that*
  (1) for any $\widehat{\alpha}^\pm \in \Xi$*,* $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^\pm$*;*
  (2) for any $\widehat{\alpha}^\pm \notin \Xi$*,* $[\widehat{\sigma}]\widehat{\alpha}^\pm = \widehat{\alpha}^\pm$*.*

In the anti-unification algorithm, we use another kind of substitution. In contrast to algorithmic substitution $\widehat{\sigma}$, it allows mapping algorithmic variables to *algorithmic* types. Additionally, anti-unification substitution is restricted to the *negative* segment of the language. Anti-unification substitution is denoted as $\widehat{\tau}$ and $\widehat{\rho}$.a

The pair of contexts Γ and Ξ, in which the results of an anti-unification substitution are formed, is fixed for this substitution. This way, $\Gamma; \Xi_2 \vdash \widehat{\tau} : \Xi_1$ means that $\widehat{\tau}$ maps each negative algorithmic variable appearing in Ξ_1 to a term well-formed in Γ and Ξ_2.

Definition 17 (Signature of Anti-unification substitution). $\Gamma; \Xi_2 \vdash \widehat{\tau} : \Xi_1$ *means that*
(1) for any $\widehat{\alpha}^- \in \Xi_1$, $\Gamma; \Xi_2 \vdash [\widehat{\tau}]\widehat{\alpha}^-$ *and*
(2) for any $\widehat{\alpha}^- \notin \Xi_1$, $[\widehat{\tau}]\widehat{\alpha}^- = \widehat{\alpha}^-$.

2.6 Equivalence and Normalization

The subtyping-induced equivalence (definition 10) is non-trivial: there are types that are subtypes of each other but not equal. For example, $\forall\alpha^+, \beta^+.\alpha^+ \to \uparrow\beta^+$ is a subtype and a supertype of $\forall\alpha^+, \beta^+.\beta^+ \to \uparrow\alpha^+$ and of, for example, $\forall\alpha^+, \beta^+.\beta^+ \to \uparrow\exists\gamma^-.\alpha^+$, although these types are not alpha-equivalent. For the subtyping algorithm, it is crucial to be able to check whether two types are equivalent, without checking mutual subtyping. For this purpose we define the normalization procedure, which allows us to uniformly choose the representative type of the equivalence class. This way, the equivalence checking is reduced to normalization and equality checking.

For clarification of the proofs and better understanding of the system, we introduce an intermediate relation—*declarative equivalence*. As will be shown in lemmas 29 and 34, this relation is equivalent to the subtyping-induced equivalence, but does not depend on it. Although this relation is not defined algorithmically, it gives the intuition of what types our system considers equivalent. Specifically, in addition to *alpha-equivalence*, our system allows for *reordering of adjacent quantifiers*, and *introduction/elimination of unused quantifiers*.

The non-trivial rules of the declarative equivalence are Rule ($\forall^{\simeq^D}$) and Rule ($\exists^{\simeq^D}$). Intuitively, the variable bijection μ reorders the quantifiers before the recursive call on the body of the quantified type. It will be covered formally in section 5.4.

Definition 18 (Declarative Type Equivalence).

Negative type equivalence $\boxed{N \simeq^D M}$

$$\frac{}{\alpha^- \simeq^D \alpha^-} \ (\mathrm{VAR}_-^{\simeq^D}) \qquad \frac{P \simeq^D Q}{\uparrow P \simeq^D \uparrow Q} \ (\uparrow^{\simeq^D})$$

$$\frac{P \simeq^D Q \quad N \simeq^D M}{P \to N \simeq^D Q \to M} \ (\to^{\simeq^D}) \qquad \frac{\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad \overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing \quad N \simeq^D [\mu]M}{\forall\overrightarrow{\alpha^+}.N \simeq^D \forall\overrightarrow{\beta^+}.M} \ (\forall^{\simeq^D})$$

Positive type equivalence $\boxed{P \simeq^D Q}$

$$\frac{}{\alpha^+ \simeq^D \alpha^+} \ (\mathrm{VAR}_+^{\simeq^D}) \qquad \frac{N \simeq^D M}{\downarrow N \simeq^D \downarrow M} \ (\downarrow^{\simeq^D})$$

$$\frac{\mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad \overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad P \simeq^D [\mu]Q}{\exists\overrightarrow{\alpha^-}.P \simeq^D \exists\overrightarrow{\beta^-}.Q} \ (\exists^{\simeq^D})$$

As the equivalence includes arbitrary reordering of quantified variables, the normalization procedure is needed to choose the canonical order. For this purpose, we introduce an auxiliary

procedure—variable ordering. Intuitively, **ord** *vars* **in** N returns a list of variables from *vars* in the order they appear in N.

Algorithm 4 (Variable Ordering).

variable ordering in a negative type

$\boxed{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ N = \vec{\alpha}}$

$$\frac{\alpha^- \in \mathit{vars}}{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ \alpha^- = \alpha^-}\ (\mathit{VAR}^{\mathit{ORD}}_{-\in})$$

$$\frac{\alpha^- \notin \mathit{vars}}{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ \alpha^- = \cdot}\ (\mathit{VAR}^{\mathit{ORD}}_{-\notin})$$

$$\frac{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ P = \vec{\alpha}}{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ \uparrow P = \vec{\alpha}}\ (\uparrow^{\mathit{ORD}})$$

$$\frac{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ P = \vec{\alpha}_1 \quad \textbf{ord}\ \mathit{vars}\ \textbf{in}\ N = \vec{\alpha}_2}{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)}\ (\to^{\mathit{ORD}})$$

$$\frac{\mathit{vars} \cap \overrightarrow{\alpha^+} = \varnothing \quad \textbf{ord}\ \mathit{vars}\ \textbf{in}\ N = \vec{\alpha}}{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}}\ (\forall^{\mathit{ORD}})$$

$\boxed{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ P = \vec{\alpha}}$

variable ordering in a positive type

$$\frac{\alpha^+ \in \mathit{vars}}{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ \alpha^+ = \alpha^+}\ (\mathit{VAR}^{\mathit{ORD}}_{+\in})$$

$$\frac{\alpha^+ \notin \mathit{vars}}{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ \alpha^+ = \cdot}\ (\mathit{VAR}^{\mathit{ORD}}_{+\notin})$$

$$\frac{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ N = \vec{\alpha}}{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ \downarrow N = \vec{\alpha}}\ (\downarrow^{\mathit{ORD}})$$

$$\frac{\mathit{vars} \cap \overrightarrow{\alpha^-} = \varnothing \quad \textbf{ord}\ \mathit{vars}\ \textbf{in}\ P = \vec{\alpha}}{\textbf{ord}\ \mathit{vars}\ \textbf{in}\ \exists \overrightarrow{\alpha^-}.P = \vec{\alpha}}\ (\exists^{\mathit{ORD}})$$

Analogously, the variable can be ordered in an algorithmic *type (***ord** *vars* **in** P *and* **ord** *vars* **in** N*). In these cases, we treat the algorithmic variables as if they were declarative variables.*

Next, we use the variable ordering in the normalization procedure. Specifically, normalization recursively traverses the type, and for each quantified case reorders the quantified variables in a canonical order dictated by algorithm 4, removing unused ones.

Algorithm 5 (Type Normalization).

$\boxed{\textbf{nf}\ (N) = M}$

$$\frac{}{\textbf{nf}\ (\alpha^-) = \alpha^-}\ (\mathit{VAR}^{\mathit{NF}}_{-})$$

$$\frac{\textbf{nf}\ (P) = Q}{\textbf{nf}\ (\uparrow P) = \uparrow Q}\ (\uparrow^{\mathit{NF}})$$

$$\frac{\textbf{nf}\ (P) = Q \quad \textbf{nf}\ (N) = M}{\textbf{nf}\ (P \to N) = Q \to M}\ (\to^{\mathit{NF}})$$

$$\frac{\textbf{nf}\ (N) = N' \quad \textbf{ord}\ \overrightarrow{\alpha^+}\ \textbf{in}\ N' = \overrightarrow{\alpha^{+\prime}}}{\textbf{nf}\ (\forall \overrightarrow{\alpha^+}.N) = \forall \overrightarrow{\alpha^{+\prime}}.N'}\ (\forall^{\mathit{NF}})$$

$\boxed{\textbf{nf}\ (P) = Q}$

$$\frac{}{\textbf{nf}\ (\alpha^+) = \alpha^+}\ (\mathit{VAR}^{\mathit{NF}}_{-})$$

$$\frac{\textbf{nf}\ (N) = M}{\textbf{nf}\ (\downarrow N) = \downarrow M}\ (\downarrow^{\mathit{NF}})$$

$$\frac{\textbf{nf}\ (P) = P' \quad \textbf{ord}\ \overrightarrow{\alpha^-}\ \textbf{in}\ P' = \overrightarrow{\alpha^{-\prime}}}{\textbf{nf}\ (\exists \overrightarrow{\alpha^-}.P) = \exists \overrightarrow{\alpha^{-\prime}}.P'}\ (\exists^{\mathit{NF}})$$

Analogously, we define normalization of algorithmic types by adding base cases:

$\boxed{\textbf{nf}\ (N) = M}$

$$\frac{}{\textbf{nf}\ (\widehat{\alpha^-}) = \widehat{\alpha^-}}\ (\mathit{UVAR}^{\mathit{NF}}_{-})$$

$\boxed{\mathbf{nf}\,(P) = Q}$ $\qquad\qquad$ $\dfrac{}{\mathbf{nf}\,(\widehat{\alpha}^{+}) = \widehat{\alpha}^{+}}\ (UVAR_{-}^{NF})$

Lemma 35 demonstrates that the equivalence of types is the same as the equality of their normal forms.

Theorem (Correctness of Normalization). *Assuming the types are well-formed in* Γ,

- $\Gamma \vdash N \simeq^{\leqslant} M$ *if and only if* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$;
+ $\Gamma \vdash P \simeq^{\leqslant} Q$ *if and only if* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$.

Algorithm 6 (Substitution Normalization). *For a substitution* σ*, we define* $\mathbf{nf}\,(\sigma)$ *as a substitution that maps* $\alpha^{\pm}$ *into* $\mathbf{nf}\,([\sigma]\alpha^{\pm})$.

The rest of this chapter is devoted to the central algorithm of the type system—the subtyping algorithm. Figure 1 shows the dependency graph of the subtyping algorithm. The nodes represent the algorithmic procedures, and the edge $A \longrightarrow B$ means that A uses B as a sub-procedure.

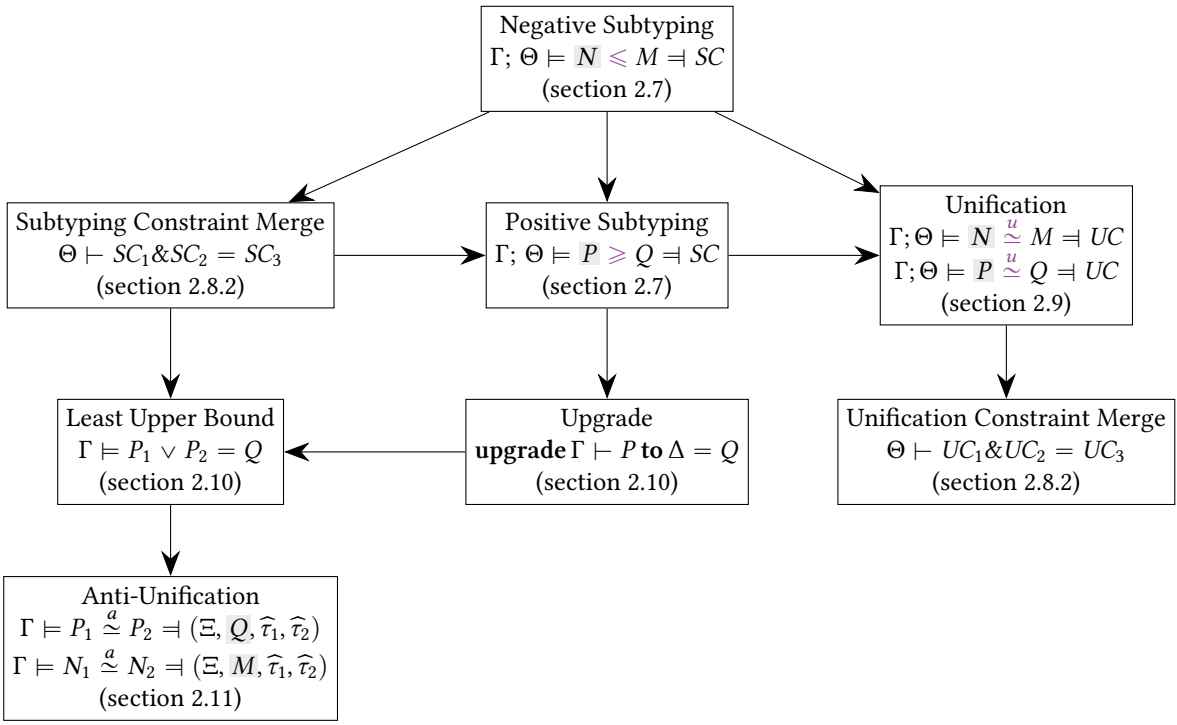

Fig. 1. Dependency graph of the subtyping algorithm

2.7 Subtyping

Now, we present the subtyping algorithm itself. Although the algorithm is presented as a single procedure, is important for the structure of the proof that the positive subtyping algorithm does not invoke the negative one. This way, the correctness of the positive subtyping will be proved independently, and used afterwards to prove the correctness of the negative subtyping.

Algorithm 7 (Subtyping).

Negative subtyping
$\boxed{\Gamma;\Theta \vDash N \leqslant M \dashv SC}$

$$\frac{}{\Gamma;\Theta \vDash \alpha^- \leqslant \alpha^- \dashv \cdot}\ (\textsc{Var}^{\leqslant}_{-})$$

$$\frac{\Gamma;\Theta \vDash \mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv UC}{\Gamma;\Theta \vDash \uparrow P \leqslant \uparrow Q \dashv UC}\ (\uparrow^{\leqslant})$$

$$\frac{\overrightarrow{\widehat{\alpha}^+}\ \textbf{are fresh} \quad \Gamma, \overrightarrow{\beta^+};\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma, \overrightarrow{\beta^+}\} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \leqslant M \dashv SC}{\Gamma;\Theta \vDash \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv SC \backslash \overrightarrow{\widehat{\alpha}^+}}\ (\forall^{\leqslant})$$

$$\frac{\Gamma;\Theta \vDash P \geqslant Q \dashv SC_1 \quad \Gamma;\Theta \vDash N \leqslant M \dashv SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC}{\Gamma;\Theta \vDash P \to N \leqslant Q \to M \dashv SC}\ (\to^{\leqslant})$$

Positive supertyping
$\boxed{\Gamma;\Theta \vDash P \geqslant Q \dashv SC}$

$$\frac{}{\Gamma;\Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ (\textsc{Var}^{\geqslant}_{+})$$

$$\frac{\Gamma;\Theta \vDash \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv UC}{\Gamma;\Theta \vDash \downarrow N \geqslant \downarrow M \dashv UC}\ (\downarrow^{\geqslant})$$

$$\frac{\overrightarrow{\widehat{\alpha}^-}\ \textbf{are fresh} \quad \Gamma, \overrightarrow{\beta^-};\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vDash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv SC}{\Gamma;\Theta \vDash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv SC \backslash \overrightarrow{\widehat{\alpha}^-}}\ (\exists^{\geqslant})$$

$$\frac{\textbf{upgrade}\,\Gamma \vdash P\ \textbf{to}\ \Theta(\widehat{\alpha}^+) = Q}{\Gamma;\Theta \vDash \widehat{\alpha}^+ \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant Q)}\ (\textsc{UVar}^{\geqslant})$$

The inputs of the subtyping algorithm are the declarative context Γ, the subtyping context Θ (it specifies in which contexts the algorithmic variables must be instantiated), and the types themselves: N and M for the negative case, and P and Q for the positive case. As one of the invariants, we require M and Q to be declarative (i.e. not containing algorithmic variables). The output of the algorithm is a set of *subtyping constraints* SC, which will be discussed in the next section.

Let us overview the inference rules of the subtyping algorithm.

- Rule ($\textsc{Var}^{\leqslant}_{-}$) and Rule ($\textsc{Var}^{\geqslant}_{+}$) are the base cases. They copy the corresponding declarative rules and ensure the reflexivity.
- Rule ($\textsc{UVar}^{\geqslant}$) is the only case generating subtyping constraints. In this case, we must ensure that the resulting constrains guarantees that the instantiation of $\widehat{\alpha}^+$ is a supertype of P. However, the obvious constraint $\widehat{\alpha}^+ :\geqslant P$ might be problematic if P is not well-formed in $\Theta(\widehat{\alpha}^+)$. For this reason, we use the *upgrade* procedure (it will be covered in section 2.10) to find the minimal supertype of P, which is well-formed in $\Theta(\widehat{\alpha}^+)$.
  Notice that this rule does not have a negative counterpart. This is because one one of the important invariants of the algorithm: in the negative subtyping, only positive algorithmic variables can occur in the types.
- Rule ($\downarrow^{\geqslant}$) and Rule ($\uparrow^{\leqslant}$) are the *shift* rules. According to the declarative system, shifted subtyping requires equivalence. In presence of the algorithmic variables, it means that the left and the right-hand sides of the subtyping must be unified. Hence, the shift rules invoke the unification algorithm, which will be discussed in section 2.9. The unification returns the minimal set of constraints UC, which is necessary and sufficient for the subtyping.
- Rule ($\to^{\leqslant}$). In this case, the algorithm makes two calls: a recursive call to the negative subtyping algorithm for the argument types, and a call to the positive subtyping algorithm for the result types. After that, the resulting constraints are merged using the *subtyping constraint merge* procedure, which is discussed in section 2.8.2.
- Rule ($\forall^{\leqslant}$) and Rule ($\exists^{\geqslant}$) are symmetric. These are the only places where the algorithmic variables are introduced. It is done by algorithmization (section 2.3) of the quantified variables: these variables are replaced by fresh algorithmic variables in the body of the quantified type, the algorithmic variables are added to the subtyping context Θ, after that the recursive call is made. Notice that the declarative context Γ is extended by the quantified variables from the right-hand side, which matches the declarative system.

Then soundness lemma (lemmas 77 and 83) and completeness (lemmas 78 and 84) of the algorithm together give us the following simplified theorem:

Theorem (Correctness of subtyping algorithm).
- $-$ $\Gamma; \cdot \models N \leqslant M \dashv \cdot$ *is equivalent to* $\Gamma \vdash N \leqslant M$;
- $+$ $\Gamma; \cdot \models P \geqslant Q \dashv \cdot$ *is equivalent to* $\Gamma \vdash P \geqslant Q$.

2.8 Constraints

Unification and subtyping algorithms are based on the constraint generation. The constraints are represented by set of constraint entries.

Definition 19 (Unification Constraint).
unification entry *(denoted as e) is an expression of shape* $\widehat{\alpha}^+ :\simeq P$ *or* $\widehat{\alpha}^- :\simeq N$;
unification constraint *(denoted as UC) is a set of unification constraint entries. We denote* $\{\widehat{\alpha}^\pm \mid e \in UC \text{ restricting } \widehat{\alpha}^\pm\}$ *as* $\mathbf{dom}(UC)$.

However, in the subtyping, we need to consider more general kind of constraints. Specifically, subtyping constraint entries can restrict a variable not only to be equivalent to a certain type, but also to be a supertype of a positive type.

Definition 20 (Subtyping Constraint).
subtyping entry *(denoted as e) is an expression of shape* $\widehat{\alpha}^+ :\geqslant P$, $\widehat{\alpha}^- :\simeq N$, *or* $\widehat{\alpha}^+ :\simeq P$;
subtyping constraint *(denoted as SC) is a set of subtyping constraint entries. We denote* $\{\widehat{\alpha}^\pm \mid e \in SC \text{ restricting } \widehat{\alpha}^\pm\}$ *as* $\mathbf{dom}(SC)$.

Definition 21 (Well-formed Constraint Entry). *We say that a constraint entry is well-formed in a context* Γ *if its associated type is well-formed in* Γ.

$\Gamma \vdash \widehat{\alpha}^+ :\geqslant P$ *iff* $\Gamma \vdash P$;
$\Gamma \vdash \widehat{\alpha}^+ :\simeq P$ *iff* $\Gamma \vdash P$;
$\Gamma \vdash \widehat{\alpha}^- :\simeq N$ *iff* $\Gamma \vdash N$.

Definition 22 (Well-formed Constraint). *We say that a constraint is well-formed in a subtyping context* Θ *if all its entries are well-formed in the corresponding elements of* Θ. *More formally,* $\Theta \vdash SC$ *holds iff for every* $e \in SC$, *such that* e *restricts* $\widehat{\alpha}^\pm$, *we have* $\Theta(\widehat{\alpha}^\pm) \vdash e$.
We write $\Theta \vdash SC : \Xi$ *to denote that* $\Theta \vdash SC$ *and* $\mathbf{dom}(SC) = \Xi$.
$\Theta \vdash UC$ *and* $\Theta \vdash UC : \Xi$ *are defined analogously.*

2.8.1 Constraint Satisfaction. A constraint entry restricts a type that can be assigned to a variable. We say that a type satisfies a constraint entry if it can be assigned to the variable restricted by the entry.

Definition 23 (Type Satisfying a Constraint Entry).

Negative constraint entry satisfaction
$\boxed{\Gamma \vdash N : e}$

$$\frac{\Gamma \vdash N \simeq^{\leqslant} M}{\Gamma \vdash N : (\widehat{\alpha}^- :\simeq M)} \quad (:\simeq^{SAT}_{-})$$

Positive constraint entry satisfaction
$\boxed{\Gamma \vdash P : e}$

$$\frac{\Gamma \vdash P \geqslant Q}{\Gamma \vdash P : (\widehat{\alpha}^+ :\geqslant Q)} \quad (:\geqslant^{SAT}_{+})$$

$$\frac{\Gamma \vdash P \simeq^{\leqslant} Q}{\Gamma \vdash P : (\widehat{\alpha}^+ :\simeq Q)} \quad (:\simeq^{SAT}_{+})$$

We say that a substitution satisfies a constraint—a set of constraint entries if each entry is satisfied by the type assigned to the variable by the substitution.

Definition 24 (Substitution Satisfying a Constraint). *We write $\Theta \vdash \widehat{\sigma} : SC$ to denote that a substitution $\widehat{\sigma}$ satisfies a constraint SC in a context Θ. It presumes that $\Theta \vdash SC$ and means that for any $e \in SC$, if e restricts $\widehat{\alpha}^{\pm}$, then $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$.*

Unification constraint satisfaction $\Theta \vdash \widehat{\sigma} : UC$ is defined analogously as a special case of subtyping constraint satisfaction.

Notice that $\Theta \vdash \widehat{\sigma} : SC$ does not imply the signature $\Theta \vdash \widehat{\sigma} : \mathbf{dom}\,(SC)$, because the latter also specifies $\widehat{\sigma}$ outside of the domain $\mathbf{dom}\,(SC)$ (see definition 16).

2.8.2 Constraint Merge. In this section, define the least upper bound for constraints, which we call *merge*. Intuitively, the merge of two constraints is the least constraint such that any substitution satisfying both constraints satisfies the merge as well. First, we define the merge of entries, and then extend it to the set of entries.

Definition 25 (Matching Entries). *We call two unification constraint entries or two subtyping constraint entries matching if they are restricting the same unification variable.*

Two matching entries formed in the same context Γ can be merged in the following way:

Algorithm 8 (Merge of Matching Constraint Entries).
Subtyping Constraint Entry Merge
$\boxed{\Gamma \vdash e_1 \,\&\, e_2 = e_3}$

$$\frac{\Gamma \vDash P_1 \vee P_2 = Q}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P_1) \,\&\, (\widehat{\alpha}^+ :\geqslant P_2) = (\widehat{\alpha}^+ :\geqslant Q)} \quad (\geqslant \&^+ \geqslant)$$

$$\frac{\Gamma; \cdot \vDash P \geqslant Q \dashv \cdot}{\Gamma \vdash (\widehat{\alpha}^+ :\simeq P) \,\&\, (\widehat{\alpha}^+ :\geqslant Q) = (\widehat{\alpha}^+ :\simeq P)} \quad (\simeq \&^+ \geqslant)$$

$$\frac{\Gamma; \cdot \vDash Q \geqslant P \dashv \cdot}{\Gamma \vdash (\widehat{\alpha}^+ :\geqslant P) \,\&\, (\widehat{\alpha}^+ :\simeq Q) = (\widehat{\alpha}^+ :\simeq Q)} \quad (\geqslant \&^+ \simeq)$$

$$\frac{\mathbf{nf}\,(P) = \mathbf{nf}\,(P')}{\Gamma \vdash (\widehat{\alpha}^+ :\simeq P) \,\&\, (\widehat{\alpha}^+ :\simeq P') = (\widehat{\alpha}^+ :\simeq P)} \quad (\simeq \&^+ \simeq)$$

$$\frac{\mathbf{nf}\,(N) = \mathbf{nf}\,(N')}{\Gamma \vdash (\widehat{\alpha}^- :\simeq N) \,\&\, (\widehat{\alpha}^- :\simeq N') = (\widehat{\alpha}^- :\simeq N)} \quad (\simeq \&^- \simeq)$$

- Rule $(\simeq \&^+ \simeq)$ and Rule $(\simeq \&^- \simeq)$ are symmetric cases. To merge two matching entries restricting a variable to be equivalent to certain types, we check that these types are equivalent to each other. To do so, it suffices to check for *equality* of their normal forms, as discussed in section 2.6. After that, we return the left-hand entry.
- Rule $(\simeq \&^+ \geqslant)$ and Rule $(\geqslant \&^+ \simeq)$ are also symmetric. In this case, since one of the entries requires the variable to be equal to a type, the resulting entry must also imply that. However, for the soundness, it is needed to ensure that the equating restriction is stronger than the subtyping restriction. For this purpose, the premise invokes the positive subtyping.
- Rule $(\geqslant \&^+ \geqslant)$ In this case, we find the least upper bound of the types from the input restrictions, and as the output, restrict the variable to be a supertype of the result. The least upper bound procedure will be discussed in section 2.10.

Unification constraint entries are a special case of subtyping constraint entries. They are merged using the same algorithm (algorithm 8). Notice that the merge of two matching unification constraint entries is a unification constraint entry.

Lemma 1 (Merge of Matching Unification Constraint Entries is well-defined). *Suppose that* $\Gamma \vdash e_1$ *and* $\Gamma \vdash e_2$ *are unification constraint entries. Then the merge of* e_1 *and* e_2 $\Gamma \vdash e_1 \& e_2 = e$ *according to algorithm 8, is a unification constraint entry.*

Proof. Since e_1 and e_2 are matching unification constraint entries, they have the shape $(\widehat{\alpha}^+ :\simeq P_1, \widehat{\alpha}^+ :\simeq P_2)$ or $(\widehat{\alpha}^- :\simeq N_1, \widehat{\alpha}^- :\simeq N_2)$. Then the merge of e_1 and e_2 can only be defined by Rule $(\simeq \&^+ \simeq)$ or Rule $(\simeq \&^- \simeq)$. In both cases the result, if it exists, is a unification constraint entry: in the first case, the result has shape $\widehat{\alpha}^+ :\simeq P_1$, in the second case, the result has shape $\widehat{\alpha}^- :\simeq N_1$. □

Algorithm 9 (Merge of Subtyping Constraints). *Suppose that* $\Theta \vdash SC_1$ *and* $\Theta \vdash SC_2$. *Then* $\Theta \vdash SC_1 \& SC_2 = SC$ *defines a set such that* $e \in SC$ *iff either*

- $e \in SC_1$ *and there is no matching* $e' \in SC_2$*; or*
- $e \in SC_2$ *and there is no matching* $e' \in SC_1$*; or*
- $\Theta(\widehat{\alpha}^{\pm}) \vdash e_1 \& e_2 = e$ *for some* $e_1 \in SC_1$ *and* $e_2 \in SC_2$ *such that* e_1 *matches with* e_2 *restricting variable* $\widehat{\alpha}^{\pm}$.

Unification constraints can be considered as a special case of subtyping constraints, and the merge of unification constraints is defined as the merge of subtyping constraints. Then it is easy to see that the merge of two unification constraints is a unification constraint.

Lemma 2 (Merge of Unification Constraints is well-defined). *Suppose that* $\Theta \vdash UC_1$ *and* $\Theta \vdash UC_2$ *are unification constraints. Then the merge of* UC_1 *and* UC_2 $\Theta \vdash UC_1 \& UC_2 = UC$ *according to algorithm 9, is a unification constraint.*

Proof. UC consists of unmatched entries of UC_1 and UC_2, which are *unification* constraint entries by assumption, and merge of matching entries, which also are *unification* constraint entries by lemma 1. □

Lemmas 80 and 82 show the correctness and initiality of the merge operation, which can be expressed in the following simplified theorem:

Theorem (Correctness of Constraint Merge). *A substitution* $\widehat{\sigma}$ *satisfying both constraints* SC_1 *and* SC_2 *if and only if it satisfies their merge.*

The unification constraint merge satisfies the same theorem, however, because the merge of unification constraint entries e_1 and e_2 always results in one of them, a stronger soundness property holds (see lemma 60):

Theorem (Soundness of Unification Constraint Merge). *If* $\Theta \vdash UC_1 \& UC_2 = UC$ *then* $UC = UC_1 \cup UC_2$.

2.9 Unification

The subtyping algorithm calls the following subtask: given two algorithmic types, we need to find the most general substitution for the algorithmic variables in these types, such that the resulting types are equivalent. This problem is known as *unification*.

In our case, the unification is restricted in the following way: first, before unifying the types, we normalize them, which allows us to reduce (non-trivial) equivalence to (trivial) equality; second, we preserve invariants which guarantee that one side of the unification is always declarative, which in fact, reduces the unification to the *matching* problem.

The unification procedure returns a set of minimal constraints, that must be satisfied by a substitution unifying the input types.

Algorithm 10 (Unification).

Negative unification

$\boxed{\Gamma;\Theta \vDash N \overset{u}{\simeq} M \dashv UC}$

$$\frac{}{\Gamma;\Theta \vDash \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot}\ (\textit{VAR}^{\overset{u}{\simeq}}_-)$$

$$\frac{\Gamma;\Theta \vDash P \overset{u}{\simeq} Q \dashv UC}{\Gamma;\Theta \vDash \uparrow P \overset{u}{\simeq} \uparrow Q \dashv UC}\ (\uparrow^{\overset{u}{\simeq}})$$

$$\frac{\Gamma;\Theta \vDash P \overset{u}{\simeq} Q \dashv UC_1 \quad \Gamma;\Theta \vDash N \overset{u}{\simeq} M \dashv UC_2}{\Gamma;\Theta \vDash P \to N \overset{u}{\simeq} Q \to M \dashv UC_1 \,\&\, UC_2}\ (\to^{\overset{u}{\simeq}})$$

$$\frac{\Gamma, \overrightarrow{\alpha^+};\Theta \vDash N \overset{u}{\simeq} M \dashv UC}{\Gamma;\Theta \vDash \forall\overrightarrow{\alpha^+}.N \overset{u}{\simeq} \forall\overrightarrow{\alpha^+}.M \dashv UC}\ (\forall^{\overset{u}{\simeq}})$$

$$\frac{\Theta(\widehat{\alpha}^-) \vdash N}{\Gamma;\Theta \vDash \widehat{\alpha}^- \overset{u}{\simeq} N \dashv (\widehat{\alpha}^- :\simeq N)}\ (\textit{UVAR}^{\overset{u}{\simeq}}_-)$$

Positive unification

$\boxed{\Gamma;\Theta \vDash P \overset{u}{\simeq} Q \dashv UC}$

$$\frac{}{\Gamma;\Theta \vDash \alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot}\ (\textit{VAR}^{\overset{u}{\simeq}}_+)$$

$$\frac{\Gamma;\Theta \vDash N \overset{u}{\simeq} M \dashv UC}{\Gamma;\Theta \vDash \downarrow N \overset{u}{\simeq} \downarrow M \dashv UC}\ (\downarrow^{\overset{u}{\simeq}})$$

$$\frac{\Gamma, \overrightarrow{\alpha^-};\Theta \vDash P \overset{u}{\simeq} Q \dashv UC}{\Gamma;\Theta \vDash \exists\overrightarrow{\alpha^-}.P \overset{u}{\simeq} \exists\overrightarrow{\alpha^-}.Q \dashv UC}\ (\exists^{\overset{u}{\simeq}})$$

$$\frac{\Theta(\widehat{\alpha}^+) \vdash P}{\Gamma;\Theta \vDash \widehat{\alpha}^+ \overset{u}{\simeq} P \dashv (\widehat{\alpha}^+ :\simeq P)}\ (\textit{UVAR}^{\overset{u}{\simeq}}_+)$$

- Rule $(\uparrow^{\overset{u}{\simeq}})$, Rule $(\downarrow^{\overset{u}{\simeq}})$, Rule $(\forall^{\overset{u}{\simeq}})$, and Rule $(\exists^{\overset{u}{\simeq}})$ are defined congruently. In the shift rules, the algorithm removes the outermost constructor. In the $\forall$ and $\exists$ rules, it removes the quantifiers, adding the quantified variables to the context Γ. Notice that Θ, which specifies the contexts in which the algorithmic variables must be instantiated, is not changed.
- Rule $(\textit{VAR}^{\overset{u}{\simeq}}_-)$ and Rule $(\textit{VAR}^{\overset{u}{\simeq}}_+)$ are the base cases. Since the sides are equal and free from algorithmic variables, the unification returns an empty constraint.
- Rule $(\textit{VAR}^{\overset{u}{\simeq}}_-)$ and Rule $(\textit{VAR}^{\overset{u}{\simeq}}_+)$ are symmetric cases constructing the constraints. When an algorithmic variable is unified with a type, we must check that the type is well-formed in the required context, and if it is, we return a constraint restricting the variable to be equivalent to that type.
- Rule $(\to^{\overset{u}{\simeq}})$. In this case, the algorithm makes two recursive calls: it unifies the arguments and the results of the arrows. After that, the resulting constraints are merged using the *unification constraint merge* procedure, which is discussed in section 2.8.2. Notice that UC_1 and UC_2 are guaranteed to be *unification* constraints, not arbitrary *subtyping* constraints: it is important for modularizing the proofs, since the properties of the *unification* constraint merge can be proved independently from the *subtyping* constraint merge.

2.10 Least Upper Bound

In this section, we present the algorithm finding the least common supertype of two positive types. It is used directly by the constraint merge procedure (section 2.8.2), and indirectly, through the type upgrade by positive subtyping (section 2.7). Perhaps, the least upper bound is the least intuitive part of the algorithm, and its correctness will be covered in section 6.8.

Algorithm 11 (The Least Upper Bound Algorithm).

Least Upper Bound

$\boxed{\Gamma \vDash P_1 \vee P_2 = Q}$

$$\frac{}{\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+} \quad (\textsc{Var}^\vee)$$

$$\frac{\Gamma \vDash \mathbf{nf}\,(\downarrow N) \overset{a}{\simeq} \mathbf{nf}\,(\downarrow M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N \vee \downarrow M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P} \quad (\downarrow^\vee)$$

$$\frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = Q} \quad (\exists^\vee)$$

- Rule ($\textsc{Var}^\vee$) The base case is trivial: the least upper bound of to equal variables is the variable itself.
- Rule ($\downarrow^\vee$) In case both sides of the least upper bound are shifted, the algorithm needs to find the anti-unifier of them. Intuitively, this is because in general, the upper bounds of $\downarrow N$ are $\exists \overrightarrow{\alpha^-}.P$ such that $\overrightarrow{\alpha^-}$ can be instantiated with some $\overrightarrow{M}$ so that $\Gamma \vdash [\overrightarrow{M}/\overrightarrow{\alpha^-}]P \simeq^{\leqslant} \downarrow N$ (see lemma 68).
- Rule ($\exists^\vee$) In this case, we move the quantified variables to the context Γ, and make a recursive call. It is important to make sure that $\overrightarrow{\alpha^-}$ and $\overrightarrow{\beta^-}$ are disjoint. In this case, it is guaranteed that the resulting $\mathbf{fv}\,(Q)$ will be free of $\overrightarrow{\alpha^-}$ and $\overrightarrow{\beta^-}$, and thus, the resulting type will be a supertype of both sides (it will be discussed in lemma 68).

In the positive subtyping algorithm (section 2.7), Rule ($\textsc{UVar}^{\geqslant}$) generates a restriction of a variable $\widehat{\alpha}^+$. On the one hand, this restriction must imply $\widehat{\alpha}^+ :\geqslant P$ for the subtyping to hold. On the other hand, the type used in this restriction must be well-formed in a potentially stronger (smaller) context than P.

To resolve this problem, we define the *upgrade* procedure, which for given $\Delta, \overrightarrow{\alpha^\pm}$, and $\Delta, \overrightarrow{\alpha^\pm} \vdash P$, finds $\Delta \vdash Q$—the least supertype of P among the types well-formed in Δ.

The trick is to make sure that the 'forbidden' variables $\overrightarrow{\alpha^\pm}$ are not used explicitly in the supertypes of P. For this purpose, we construct new types P_1 and P_2, in each of them replacing the forbidden variables with fresh variables $\overrightarrow{\beta^\pm}$ and $\overrightarrow{\gamma^\pm}$, and then find the least upper bound of P_1 and P_2. It turns out that this renaming forces the common types of P_1 and P_2 to be agnostic to $\overrightarrow{\alpha^\pm}$, and thus, the supertypes of P well-formed in Δ are exactly the common supertypes of P_1 and P_2. These properties are considered in more details in section 6.9.

Algorithm 12 (Type Upgrade).

$\boxed{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q}$

$$\frac{\Gamma = \Delta, \overrightarrow{\alpha^\pm} \quad \overrightarrow{\beta^\pm}\ \mathbf{are\ fresh} \quad \overrightarrow{\gamma^\pm}\ \mathbf{are\ fresh} \quad \Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vDash [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q}{\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q} \quad (\textsc{Upg})$$

Note on the Greatest Lower Bound. In contrast to the least upper bound, the general greatest lower bound does not exist in our system. For instance, consider a positive type P, together with its non-equivalent supertypes P_1 and $P_2 \not\simeq P_1$ (for example, $P = \downarrow\uparrow\downarrow\gamma^-$, $P_1 = \exists\alpha^-.\downarrow\uparrow\downarrow\alpha^-$, and $P_2 = \exists\alpha^-.\downarrow\alpha^-$). Then for arbitrary Q and N, let us consider the common subtypes of $A = Q \to \downarrow\uparrow Q \to \downarrow\uparrow Q \to N$ and $B = P \to \downarrow\uparrow P_1 \to \downarrow\uparrow P_2 \to N$. It is easy to see that $\forall\alpha^+.\forall\beta^+.\alpha^+ \to \downarrow\uparrow\alpha^+ \to \downarrow\uparrow\beta^+ \to N$ and $\forall\alpha^+.\forall\beta^+.\alpha^+ \to \downarrow\uparrow\beta^+ \to \downarrow\uparrow\alpha^+ \to N$ are both *maximal* common subtypes of A and B, and since they are not equivalent, none of them is the *greatest* one.

However, we designed the subtyping system in such a way that the greatest lower bound is not needed: the negative variables are always 'protected' by *invariant* shifts (↑ and ↓), and thus, the algorithm can only require a substitution of a negative variable to be *equivalent* to some type but never to be a *subtype*.

2.11 Anti-unification

Next, we define the anti-unification procedure, also known as the *most specific generalization*. As an input, it takes two declarative types (e.g., in the positive case P_1 and P_2) and a context Γ. and returns a type Q—the generalizer, containing negative placeholders (represented by algorithmic variables) from Ξ and two substitutions $\widehat{\tau}_1$ and $\widehat{\tau}_2$. The substitutions replace the placeholders with declarative types well-formed in Γ, such that $[\widehat{\tau}_1]Q = P_1$ and $[\widehat{\tau}_2]Q = P_2$. Moreover, the algorithm guarantees that Q is the most specific type with this property: any other generalizer can be turned into Q by some substitution $\widehat{\rho}$.

It is important to note the differences between the standard anti-unification and our version. First, we only allow the placeholders at *negative* positions, which means, for example, that α^+ and β^+ cannot be generalized. Second, the generated pair of substitutions $\widehat{\tau}_1$ and $\widehat{\tau}_2$ must replace the placeholders with types well-formed in a specified context Γ.

The anti-unification algorithm assumes that the input types are normalized. This way, anti-unification up-to-equality rather than anti-unification up-to-equivalence is sufficient.

Algorithm 13 (Anti-unification).

$\boxed{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\cdot, \alpha^+, \cdot, \cdot)} \quad (\mathit{VAR}^{\overset{a}{\simeq}}_+)$$

$$\frac{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 \dashv (\Xi, \downarrow M, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\downarrow^{\overset{a}{\simeq}})$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\Xi, \exists \overrightarrow{\alpha^-}.Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\exists^{\overset{a}{\simeq}})$$

$\boxed{\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}$

$$\frac{}{\Gamma \vDash \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\cdot, \alpha^-, \cdot, \cdot)} \quad (\mathit{VAR}^{\overset{a}{\simeq}}_-)$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\uparrow^{\overset{a}{\simeq}})$$

$$\frac{\overrightarrow{\alpha^+} \cap \Gamma = \varnothing \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)}{\Gamma \vDash \forall \overrightarrow{\alpha^+}.N_1 \overset{a}{\simeq} \forall \overrightarrow{\alpha^+}.N_2 \dashv (\Xi, \forall \overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)} \quad (\forall^{\overset{a}{\simeq}})$$

$$\frac{\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2) \quad \Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi_2, M, \widehat{\tau}'_1, \widehat{\tau}'_2)}{\Gamma \vDash P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}'_1, \widehat{\tau}_2 \cup \widehat{\tau}'_2)} \quad (\to^{\overset{a}{\simeq}})$$

$$\frac{\textit{if other rules are not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vDash N \overset{a}{\simeq} M \dashv (\widehat{\alpha}_{\{N,M\}}, \widehat{\alpha}_{\{N,M\}}, (\widehat{\alpha}_{\{N,M\}} \mapsto N), (\widehat{\alpha}_{\{N,M\}} \mapsto M))} \quad (\mathit{AU})$$

- Rule (VAR$^{\stackrel{a}{\simeq}}_{+}$) and Rule (VAR$^{\stackrel{a}{\simeq}}_{-}$) are the base cases. In this case, since the input types are equal, the algorithm returns this type as a generalizer, without generating any placeholders.
- Rule ($\downarrow^{\stackrel{a}{\simeq}}$), Rule ($\uparrow^{\stackrel{a}{\simeq}}$), Rule ($\forall^{\stackrel{a}{\simeq}}$), and Rule ($\exists^{\stackrel{a}{\simeq}}$) are defined congruently. In the shift rules, the algorithm removes the outermost constructor. In the $\forall$ and $\exists$ rules, it removes the quantifiers. Notice that the algorithm does not add the removed variables to the context Γ. This is because Γ is used to restrict the resulting anti-unification substitutions, and is fixed throughout the algorithm.
- Rule (AU) is the most important rule, since it generates the placeholders. This rule only applies if other negative rules failed. Because of that, the anti-unification procedure is *not* syntax-directed.
  The generated placeholder is indexed with a pair of types it is mapped to. It allows the algorithm to automatically unite the anti-unification solutions generated by the different branches of Rule ($\rightarrow^{\stackrel{a}{\simeq}}$).
  Notice that this rule does not have a positive counterpart, since we only allow negative placeholders.
- Rule ($\rightarrow^{\stackrel{a}{\simeq}}$) makes two recursive calls to the anti-unification procedure, and unites the results. Suppose that $\widehat{\tau}_1$ and $\widehat{\tau}_2$ are the substitutions generated by anti-unification of *argument* types of the arrow, and $\widehat{\tau}'_1$ and $\widehat{\tau}'_2$ are the substitutions generated by anti-unification of *result* types of the arrow. It is important that if $(\widehat{\tau}_1, \widehat{\tau}_2)$ and $(\widehat{\tau}'_1, \widehat{\tau}'_2)$ send some variables to the same pair of types, i.e., $[\widehat{\tau}_1]\widehat{\alpha}^- = [\widehat{\tau}'_1]\widehat{\beta}^-$ and $[\widehat{\tau}_2]\widehat{\alpha}^- = [\widehat{\tau}'_2]\widehat{\beta}^-$, then these variables are equal, i.e., $\widehat{\alpha}^- = \widehat{\beta}^-$. This property is guaranteed by Rule (AU): the name of the placeholder is determined by the pair of types it is mapped to.

3 DECLARATIVE TYPING

In the previous section, we presented the type system together with subtyping specification and the algorithm. In this section, we describe the language under this type system, together with the type inference specification and algorithm.

3.1 Grammar

First, we define the syntax of the language. The language combines System F with call-by-push-value style.

Definition 26 (Language Grammar).

computation terms			*value terms*		
$c,\ d$	$::=$		$v,\ w$	$::=$	
	$(c : N)$	*annotated computation*	x		*variable*
	$\lambda x : P.c$	*annotated abstraction*	$\{c\}$		*thunk*
	$\Lambda\alpha^{+}.c$	*annotated type abstraction*	$(v : P)$		*annotated value*
	return v	*computation embedding a value*			
	let $x = v; c$	*standard let-binding*			
	let $x : P = v(\overrightarrow{v}); c$	*annotated applicative let*			
	let $x = v(\overrightarrow{v}); c$	*unannotated applicative let*			
	$\textbf{let}^{\exists}(\overrightarrow{\alpha^{-}}, x) = v; c$	*unpack*			

Notice that the language does not have first-class applications: instead, we use applicative let bindings— constructions that bind a result of a fully applied function to a (positive) variable. In the call-by-push-value paradigm, it corresponds to monadic bind or do-notation. Typewise, these let-binders come in two forms: annotated and unannotated. The annotated let-binders **let** $x : P = v(\overrightarrow{v}); c$ requires the application to infer the annotated P, whereas the unannotated **let** $x = v(\overrightarrow{v}); c$ is used when the inferred type is unique.

A computation of a polymorphic type is constructed using $\Lambda\alpha^{+}.c$, however, the elimination of $\forall$ is implicit. Conversely, the existential types are constructed implicitly and eliminated using the standard unpack mechanism: $\textbf{let}^{\exists}(\overrightarrow{\alpha^{-}}, x) = v; c$.

Another dual pair of constructions are **return** v and $\{c\}$. The former allows us to embed a value in a pure computations. The latter, on the contrary, encapsulates a thunk of computation in a value.

Finally, the language has a number standard constructions: lambda-abstractions $\lambda x : P.c$, standard let-bindings **let** $x = v; c$, and type annotations that can be added to any value or computation: $(v : P)$ and $(c : N)$.

3.2 Declarative Type Inference

Next, we define the specification of the type inference for our language. First, we introduce variable context specifying the types of the variables in the scope of the current rule.

Definition 27 (Variable Context). *The variable typing context Φ is represented by a set of entries of the form $x : P$.*

The specification is represented by an inference system of three mutually recursive judgments: positive inference $\Gamma; \Phi \vdash v : P$, negative type inference $\Gamma; \Phi \vdash c : N$, and application type inference $\Gamma; \Phi \vdash N \bullet \overrightarrow{v} \Longrightarrow M$. In the premises, the inference rules also refer to the declarative subtyping (definition 10), type well-formedness (algorithm 1), and normalization (algorithm 5).

- $\Gamma; \Phi \vdash v \colon P$ (and symmetrically, $\Gamma; \Phi \vdash c \colon N$) means that under the type context Γ and the variable context Φ, for the value v, type P is inferrable. It guarantees that v is well-formed in Γ and Φ in the standard sense.
- $\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow\!\!> M$ is the application type inference judgement. It means that if a head of type N is applied to list of values $\vec{v}$, then the resulting computation can be typed as M.

Definition 28 (Declarative Type Inference).

Negative typing
$\boxed{\Gamma; \Phi \vdash c \colon N}$

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \vdash c \colon N}{\Gamma; \Phi \vdash \lambda x : P.c \colon P \to N} \ (\lambda^{INF})$$

$$\frac{\Gamma, \alpha^{+}; \Phi \vdash c \colon N}{\Gamma; \Phi \vdash \Lambda \alpha^{+}.c \colon \forall \alpha^{+}.N} \ (\Lambda^{INF})$$

$$\frac{\Gamma; \Phi \vdash v \colon P}{\Gamma; \Phi \vdash \mathbf{return}\ v \colon \uparrow P} \ (RET^{INF})$$

$$\frac{\Gamma; \Phi \vdash v \colon P \quad \Gamma; \Phi, x : P \vdash c \colon N}{\Gamma; \Phi \vdash \mathbf{let}\ x = v; c \colon N} \ (LET^{INF})$$

$$\frac{\Gamma; \Phi \vdash v \colon \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow\!\!> \uparrow Q \ \textit{unique} \quad \Gamma; \Phi, x : Q \vdash c \colon N}{\Gamma; \Phi \vdash \mathbf{let}\ x = v(\vec{v}); c \colon N} \ (LET^{INF}_{@})$$

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi \vdash v \colon \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow\!\!> \uparrow Q \quad \Gamma \vdash \uparrow Q \leqslant \uparrow P \quad \Gamma; \Phi, x : P \vdash c \colon N}{\Gamma; \Phi \vdash \mathbf{let}\ x : P = v(\vec{v}); c \colon N} \ (LET^{INF}_{:@})$$

$$\frac{\Gamma; \Phi \vdash v \colon \exists \overrightarrow{\alpha^{-}}.P \quad \mathbf{nf}\,(\exists \overrightarrow{\alpha^{-}}.P) = \exists \overrightarrow{\alpha^{-}}.P \quad \Gamma, \overrightarrow{\alpha^{-}}; \Phi, x : P \vdash c \colon N \quad \Gamma \vdash N}{\Gamma; \Phi \vdash \mathbf{let}^{\exists}(\overrightarrow{\alpha^{-}}, x) = v; c \colon N} \ (LET^{INF}_{\exists})$$

$$\frac{\Gamma \vdash M \quad \Gamma; \Phi \vdash c \colon N \quad \Gamma \vdash N \leqslant M}{\Gamma; \Phi \vdash (c : M) \colon M} \ (ANN^{INF}_{-})$$

$$\frac{\Gamma; \Phi \vdash c \colon N \quad \Gamma \vdash N \simeq^{\leqslant} N'}{\Gamma; \Phi \vdash c \colon N'} \ (\simeq^{INF}_{-})$$

Positive typing
$\boxed{\Gamma; \Phi \vdash v \colon P}$

$$\frac{x : P \in \Phi}{\Gamma; \Phi \vdash x \colon P} \ (VAR^{INF})$$

$$\frac{\Gamma; \Phi \vdash c \colon N}{\Gamma; \Phi \vdash \{c\} \colon \downarrow N} \ (\{\}^{INF})$$

$$\frac{\Gamma \vdash Q \quad \Gamma; \Phi \vdash v \colon P \quad \Gamma \vdash Q \geqslant P}{\Gamma; \Phi \vdash (v : Q) \colon Q} \ (ANN^{INF}_{+})$$

$$\frac{\Gamma; \Phi \vdash v \colon P \quad \Gamma \vdash P \simeq^{\leqslant} P'}{\Gamma; \Phi \vdash v \colon P'} \ (\simeq^{INF}_{+})$$

Application typing
$\boxed{\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow\!\!> M}$

$$\frac{\Gamma \vdash N \simeq^{\leqslant} N'}{\Gamma; \Phi \vdash N \bullet \cdot \Rightarrow\!\!> N'} \ (\varnothing^{INF}_{\bullet \Rightarrow\!\!>})$$

$$\frac{\Gamma; \Phi \vdash v \colon P \quad \Gamma \vdash Q \geqslant P \quad \Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow\!\!> M}{\Gamma; \Phi \vdash Q \to N \bullet v, \vec{v} \Rightarrow\!\!> M} \ (\to$$

$$\frac{\vec{v} \neq \cdot \quad \overrightarrow{\alpha^{+}} \neq \cdot \quad \Gamma \vdash \sigma : \overrightarrow{\alpha^{+}} \quad \Gamma; \Phi \vdash [\sigma]N \bullet \vec{v} \Rightarrow\!\!> M}{\Gamma; \Phi \vdash \forall \overrightarrow{\alpha^{+}}.N \bullet \vec{v} \Rightarrow\!\!> M} \ (\forall^{INF}_{\bullet \Rightarrow\!\!>})$$

Let us discuss selected rules of the declarative system:

- Rule (VAR$^{\text{INF}}$) says that the type of a variable is inferred from the context.
- Rule ($\{\}^{\text{INF}}$) says that the type of a thunk is inferred by shifting up the type of the contained computation. Symmetrically, Rule (RET$^{\text{INF}}$) infers the type of a return by shifting down the type of the contained value.
- Rule (ANN$^{\text{INF}}_{+}$) and Rule (ANN$^{\text{INF}}_{-}$) are symmetric. They allow the inferred type to be refined by annotating it with a supertype.
- Rule ($\simeq^{\text{INF}}_{-}$) and Rule ($\simeq^{\text{INF}}_{+}$) mean that th declarative system allows to infer any type from the equivalence class.
- Rule (LET$^{\text{INF}}_{\exists}$) is standard for existential types, and its first premise inferring the existential type of the value being unpacked. It is important however that the inferred existential type is normalized. This is because there might be multiple equivalent existential types with different order or even number of quantified variables, and to bind them, the algorithm needs to fix the canonical one.

- Rule ($\textsc{let}^{\textsc{inf}}_{:@}$) allows us to accommodate the applications with annotated let-bindings. The first premise infers the type of the head of the application, which must be a thunked computation. Then if after applying it to tha arguments, the resulting type can be equated to the annotated one, we infer the body of the let-binding in the context extended with the bound variable.
- Rule ($\textsc{let}^{\textsc{inf}}_{@}$) is similar to Rule ($\textsc{let}^{\textsc{inf}}_{:@}$), but it is used when the type of the application is unique, and thus, the annotation is redundant. Here $\Gamma;\Phi \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q$ unique means that if also $\Gamma;\Phi \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q'$ then $\Gamma \vdash Q \simeq^{\leqslant} Q'$

Let us discuss the rules of the application inference:

- Rule ($\varnothing^{\textsc{inf}}_{\bullet\Rrightarrow}$) is the base case. If the list of arguments is empty, the inferred type is the type of the head. However, we relax this specification by allowing it to infer any other equivalent type. The relaxation of this rule is enough to guarantee this property for the whole judgement: if $\Gamma;\Phi \vdash N \bullet \vec{v} \Rrightarrow M$ then $\Gamma;\Phi \vdash N \bullet \vec{v} \Rrightarrow M'$ for any equivalent M'.
- Rule ($\to^{\textsc{inf}}_{\bullet\Rrightarrow}$) is where the application type is inferred: if the head has an arrow type $Q \to N$, we are allowed to apply it as soon as as soon as the first argument has a type, which is a subtype of Q.
- Rule ($\forall^{\textsc{inf}}_{\bullet\Rrightarrow}$) is the rule ensuring the implicit elimination of the universal quantifiers. If we are applying a polymorphic computation, we can instantiate its quantified variables with any types, which is expressed by the substitution $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}$.

4 ALGORITHMIC TYPING

Next, we present the type inference algorithm, which is sound and complete with respect to the declarative specification (definition 28).

4.1 Algorithmic Type Inference

Mirroring the declarative typing, the algorithm is represented by an inference system of three mutually recursive judgments:

- $\Gamma;\Phi \vDash v\colon P$ and $\Gamma;\Phi \vDash c\colon N$ are the algorithmic versions of $\Gamma;\Phi \vdash v\colon P$ and $\Gamma;\Phi \vdash c\colon N$. In contrast with the declarative counterparts, they are deterministic, and guarantee that the inferred type is normalized.
- $\Gamma;\Phi;\Theta_1 \vDash N \bullet \vec{v} \Rrightarrow M \dashv \Theta_2; SC$ is the algorithmization of $\Gamma;\Phi \vdash N \bullet \vec{v} \Rrightarrow M$. Notice that N contains algorithmic variables, which are specified by the context Θ_1. Moreover, the inferred type M is also algorithmic, and can have several non-equivalent instantiations. To accommodate that, the algorithm also returns Θ_2 and SC specifying the variables used in $M\colon \Theta_2$ defines the contexts in which the variables must be instantiated, and SC imposes restrictions on the variables.

As subroutines, the algorithm calls subtyping (algorithm 7), type well-formedness (algorithm 1), constraint merge (section 2.8.2), normalization (algorithm 5), and constraint singularity which will be defined later in section 4.2. It also relies on basic set operations and the ability to deterministically choose fresh variables.

Algorithm 14.

Negative typing
$\boxed{\Gamma;\Phi \vDash c\colon N}$

$$\frac{\Gamma \vdash M \quad \Gamma;\Phi \vDash c\colon N \quad \Gamma;\cdot \vDash N \leqslant M \dashv \cdot}{\Gamma;\Phi \vDash (c : M)\colon \mathbf{nf}\,(M)} \; (ANN^{INF}_{-})$$

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \vDash c : N}{\Gamma; \Phi \vDash \lambda x : P.c : \mathbf{nf}\,(P \to N)}\ (\lambda^{INF})$$

$$\frac{\Gamma, \alpha^+; \Phi \vDash c : N}{\Gamma; \Phi \vDash \Lambda\alpha^+.c : \mathbf{nf}\,(\forall\alpha^+.N)}\ (\Lambda^{INF})$$

$$\frac{\Gamma; \Phi \vDash v : P}{\Gamma; \Phi \vDash \mathbf{return}\ v : \uparrow P}\ (RET^{INF})$$

$$\frac{\Gamma; \Phi \vDash v : P \quad \Gamma; \Phi, x : P \vDash c : N}{\Gamma; \Phi \vDash \mathbf{let}\ x = v; c : N}\ (LET^{INF})$$

$$\frac{\begin{array}{l}\Gamma \vdash P \quad \Gamma; \Phi \vDash v : \downarrow M \\ \Gamma; \Phi; \cdot \vDash M \bullet \vec{v} \Longrightarrow \uparrow Q \dashv \Theta; SC_1 \\ \Gamma; \Theta \vDash \uparrow Q \leqslant \uparrow P \dashv SC_2 \\ \Theta \vdash SC_1 \& SC_2 = SC \quad \Gamma; \Phi, x : P \vDash c : N\end{array}}{\Gamma; \Phi \vDash \mathbf{let}\ x : P = v(\vec{v}); c : N}\ (LET^{INF}_{:@})$$

$$\frac{\begin{array}{l}\Gamma; \Phi \vDash v : \downarrow M \quad \Gamma; \Phi; \cdot \vDash M \bullet \vec{v} \Longrightarrow \uparrow Q \dashv \Theta; SC \\ \mathbf{uv}\ Q = \mathbf{dom}\,(SC) \quad SC\ \mathbf{singular\ with}\ \widehat{\sigma} \\ \Gamma; \Phi, x : [\widehat{\sigma}]Q \vDash c : N\end{array}}{\Gamma; \Phi \vDash \mathbf{let}\ x = v(\vec{v}); c : N}\ (LET^{INF}_{@})$$

$$\frac{\Gamma; \Phi \vDash v : \exists\overrightarrow{\alpha^-}.P \quad \Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \vDash c : N \quad \Gamma \vdash N}{\Gamma; \Phi \vDash \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c : N}\ (LET^{INF}_{\exists})$$

Positive typing

$\boxed{\Gamma; \Phi \vDash v : P}$

$$\frac{x : P \in \Phi}{\Gamma; \Phi \vDash x : \mathbf{nf}\,(P)}\ (VAR^{INF})$$

$$\frac{\Gamma; \Phi \vDash c : N}{\Gamma; \Phi \vDash \{c\} : \downarrow N}\ (\{\}^{INF})$$

$$\frac{\Gamma \vdash Q \quad \Gamma; \Phi \vDash v : P \quad \Gamma; \cdot \vDash Q \geqslant P \dashv \cdot}{\Gamma; \Phi \vDash (v : Q) : \mathbf{nf}\,(Q)}\ (ANN^{INF}_{+})$$

Application typing

$\boxed{\Gamma; \Phi; \Theta_1 \vDash N \bullet \vec{v} \Longrightarrow M \dashv \Theta_2; SC}$

$$\frac{}{\Gamma; \Phi; \Theta \vDash N \bullet \cdot \Longrightarrow \mathbf{nf}\,(N) \dashv \Theta; \cdot}\ (\varnothing^{INF}_{\bullet\Longrightarrow})$$

$$\frac{\begin{array}{l}\Gamma; \Phi \vDash v : P \quad \Gamma; \Theta \vDash Q \geqslant P \dashv SC_1 \\ \Gamma; \Phi; \Theta \vDash N \bullet \vec{v} \Longrightarrow M \dashv \Theta'; SC_2 \\ \Theta \vdash SC_1 \& SC_2 = SC\end{array}}{\Gamma; \Phi; \Theta \vDash Q \to N \bullet v, \vec{v} \Longrightarrow M \dashv \Theta'; SC}\ (\to^{INF}_{\bullet\Longrightarrow})$$

$$\frac{\begin{array}{l}\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha^+}}\{\Gamma\} \vDash [\overrightarrow{\widehat{\alpha^+}}/\overrightarrow{\alpha^+}]N \bullet \vec{v} \Longrightarrow M \dashv \Theta'; SC \\ \overrightarrow{\widehat{\alpha^+}}\ \mathbf{are\ fresh} \quad \vec{v} \neq \cdot \quad \overrightarrow{\alpha^+} \neq \cdot\end{array}}{\Gamma; \Phi; \Theta \vDash \forall\overrightarrow{\alpha^+}.N \bullet \vec{v} \Longrightarrow M \dashv \Theta'; SC|_{\mathbf{uv}\,(N) \cup \mathbf{uv}\,(M)}}\ (LET^{INF}_{\forall})$$

Let us discuss the inference rules of the algorithm:

- Rule ($\textsc{Var}^{\textsc{inf}}$) infers the type of a variable by looking it up in the context and normalizing the result.
- Rule ($\{\}^{\textsc{inf}}$) and Rule ($\textsc{ret}^{\textsc{inf}}$) are similar to the declarative rules: they make a recursive call to type the body of the thunk or the return expression and put the shift on top of the result.
- Rule ($\textsc{ann}^{\textsc{inf}}_{+}$) and Rule ($\textsc{ann}^{\textsc{inf}}_{-}$) are symmetric. They make a recursive call to infer the type of the annotated expression, check that the inferred type is a subtype of the annotation, and return the normalized annotation.
- Rule ($\lambda^{\textsc{inf}}$) infers the type of a lambda-abstraction. It makes a recursive call to infer the type of the body in the extended context, and returns the corresponding arrow type. Notice that the algorithm also normalizes the result, which is because the annotation type P is allowed to be non-normalized.
- Rule ($\Lambda^{\textsc{inf}}$) infers the type of a big lambda. Similarly to the previous case, it makes a recursive call to infer the type of the body in the extended *type* context. After that, it returns the corresponding universal type. It is also required to normalize the result, because, for instance, α^+ might not occur in the body of the lambda, in which case the $\forall$ must be removed.
- Rule ($\textsc{let}^{\textsc{inf}}$) is defined in a standard way: it makes a recursive call to infer the type of the bound value, and then returns the type of the body in the extended context.
- Rule ($\textsc{let}^{\textsc{inf}}_{:@}$) is interpreted as follows. First, it infers the type of the head of the application, ensuring that it is a thunked computation $\downarrow M$; after that, it makes a recursive call to the application inference procedure, which returns the algorithmic type, whose instantiation to a declarative type must be associated with the bound variable x; then premise $\Gamma; \Theta \vDash \uparrow Q \leqslant \uparrow P \dashv SC_2$ together with $\Theta \vdash SC_1 \& SC_2 = SC$ check whether the instantiation to the

annotated type P is possible, and if it is, the algorithm infers the type of the body in the extended context, and returns it as the result.

- Rule ($\textsc{let}^{\textsc{inf}}_{:@}$) works similarly to Rule ($\textsc{let}^{\textsc{inf}}_{:@}$), However, since there is no annotation, instead of checking the instantiation to it, the algorithm checks that the inferred type $\uparrow Q$ is unique. It is the case if all the algorithmic variables of $\uparrow Q$ are sufficiently restricted by SC, which is checked by the combination of $\mathbf{uv}\ Q = \mathbf{dom}\ (SC)$ and SC **singular with** $\widehat{\sigma}$. Together, these two premises guarantee that the only possible instantiation of $\uparrow Q$ is $[\widehat{\sigma}]\, Q$.
- Rule ($\textsc{let}^{\textsc{inf}}_{\exists}$) works in the expected way. First, it infers the existential type $\exists \overrightarrow{\alpha^-}.P$ of the value being unpacked, and since the type is guaranteed to be normalized, binds the quantified variables with $\overrightarrow{\alpha^-}$. Then it infers the type of the body in the appropriately extended context, and checks that the inferred type does not depend on $\overrightarrow{\alpha^-}$ by checking well-formedness $\Gamma \vdash N$.

Finally, let us discuss the algorithmic rules of the application inference:

- Rule ($\varnothing^{\textsc{inf}}_{\bullet\Rightarrow}$) is the base case. If the list of arguments is empty, the inferred type is the type of the head, and the algorithm returns it after normalizing.
- Rule ($\rightarrow^{\textsc{inf}}_{\bullet\Rightarrow}$) is the main rule of the algorithmic application inference. If the head has an arrow type $Q \to N$, we find SC_1—the minimal constraint ensuring that Q is a supertype of the first argument's type. Then we make a recursive call applying N to the rest of the arguments, and merge the resulting constraint with SC_1
- Rule ($\forall^{\textsc{inf}}_{\bullet\Rightarrow}$), analogously to the declarative case, is the rule ensuring the implicit elimination of the universal quantifiers. This is the place where the algorithmic variables are generated. The algorithm simply replaces the quantified variables $\overrightarrow{\alpha^+}$ with fresh algorithmic variables $\overrightarrow{\widehat{\alpha}^+}$, and makes a recursive call in the extended context.

The correctness of the algorithm consists of its soundness and completeness, which is by mutual induction in lemmas 92 and 93. The simplified result is the following.

Theorem.

$-$ $\Gamma;\Phi \vDash c\colon N$ *implies* $\Gamma;\Phi \vdash c\colon N$, *and* $\Gamma;\Phi \vdash c\colon N$ *implies* $\Gamma;\Phi \vDash c\colon \mathbf{nf}\ (N)$;
$+$ $\Gamma;\Phi \vDash v\colon P$ *implies* $\Gamma;\Phi \vdash v\colon P$, *and* $\Gamma;\Phi \vdash v\colon P$ *implies* $\Gamma;\Phi \vDash v\colon \mathbf{nf}\ (P)$.

4.2 Constraint Singularity

The singularity algorithm used in Rule ($\textsc{let}^{\textsc{inf}}_{:@}$) of the algorithmic typing check wether the constraint SC is uniquely defines a substitution satisfying it, and if it does returns such a substitution as the result. To do that, we define a partial function SC **singular with** $\widehat{\sigma}$, taking a subtyping constraint SC and returning a substitution $\widehat{\sigma}$—the only possible solution of SC.

First, we define the notion of singularity on constraint entries. e **singular with** P and e **singular with** N are considered partial functions taking a constraint entry e and returning the type satisfying e if such a type is unique.

Algorithm 15 (Singular Constraint Entry).

$\boxed{e \textbf{ singular with } N}$ $\qquad$ $\boxed{e \textbf{ singular with } P}$

$$\frac{}{\widehat{\alpha}^- :\simeq N \textbf{ singular with nf } (N)}\ (\simeq^{SING}_-) \qquad \frac{}{\widehat{\alpha}^+ :\simeq P \textbf{ singular with nf } (P)}\ (\simeq^{SING}_+)$$

$$\frac{}{\widehat{\alpha}^{+} :\geqslant \exists\overrightarrow{\alpha^{-}}.\alpha^{+} \textbf{ singular with } \alpha^{+}} \; (:\geqslant \alpha^{SING}) \qquad \frac{\textbf{nf}\,(N) = \alpha_i^{-}}{\widehat{\alpha}^{+} :\geqslant \exists\overrightarrow{\alpha^{-}}.\downarrow N \textbf{ singular with } \exists\alpha^{-}.\downarrow\alpha^{-}} \; (:\geqslant \downarrow^{SING})$$

- Rule $(\simeq_{-}^{\text{SING}})$ and Rule $(\simeq_{+}^{\text{SING}})$ are symmetric. If the constraint entry says that a variable must be equivalent to a type T, then it is evidently singular, and the only (up-to-equivalence) type instantiating this variable could be T. This way, we return its normal form.
- Rule $(:\geqslant \alpha^{\text{SING}})$ implies that the only (normalized) solution of $\widehat{\alpha}^{+} :\geqslant \exists\overrightarrow{\alpha^{-}}.\alpha^{+}$ is α^{+} (it will be shown in lemma 19).
- Rule $(:\geqslant \downarrow^{\text{SING}})$ is perhaps the least obvious rule. In type $\exists\overrightarrow{\alpha^{-}}.\downarrow N$, if N is anything different (non-equivalent) to $\alpha_i^{-} \in \overrightarrow{\alpha^{-}}$, there are at least one proper supertype of $\exists\overrightarrow{\alpha^{-}}.\downarrow N$, since N can be abstracted over by an existential quantifier. Otherwise, any supertype of $\exists\overrightarrow{\alpha^{-}}.\downarrow\alpha_i^{-}$ is equivalent to it, and thus, the solution is unique.

Next, we extrapolate the singularity function on constraints—sets of constraint entries. We require SC to be a set of singular constraints, and the resulting substitution sends each variable from $\textbf{dom}\,(SC)$ to the unique type satisfying the corresponding constraint.

Algorithm 16. *SC* **singular with** *$\widehat{\sigma}$ means that*

(1) for any positive $e \in SC$, there exists P such that e **singular with** *P, and for any negative $e \in SC$, there exists N such that e* **singular with** *N;*

(2) $\widehat{\sigma}$ is defined as follows:

$$[\widehat{\sigma}]\widehat{\beta}^{+} = \begin{cases} P & \text{if there is } e \in \textbf{dom}\,(SC) \text{ restricting } \widehat{\beta}^{+} \text{ and } e \textbf{ singular with } P \\ \widehat{\beta}^{+} & \text{otherwise} \end{cases}$$

$$[\widehat{\sigma}]\widehat{\beta}^{-} = \begin{cases} N & \text{if there is } e \in \textbf{dom}\,(SC) \text{ restricting } \widehat{\beta}^{-} \text{ and } e \textbf{ singular with } N \\ \widehat{\beta}^{-} & \text{otherwise} \end{cases}$$

The correctness of the singularity algorithm is formulated as follows:

Theorem. *Suppose that SC is a subtyping constraint. Then SC* **singular with** *$\widehat{\sigma}$ holds if and only if $\widehat{\sigma}$ is the only (up-to-equivalence on $\textbf{dom}\,(SC)$) normalized substitution satisfying SC.*

5 PROPERTIES OF THE DECLARATIVE TYPE SYSTEM

5.1 Type Well-formedness

Lemma 3 (Soundness of type well-formedness).

+ *if* $\Gamma \vdash P$ *then* $\mathbf{fv}(P) \subseteq \Gamma$;
− *if* $\Gamma \vdash N$ *then* $\mathbf{fv}(N) \subseteq \Gamma$.

Proof. The proof is done by a simple structural induction on $\Gamma \vdash P$ and mutually, $\Gamma \vdash N$.

Case 1. $\Gamma \vdash \alpha^{\pm}$ means by inversion that $\alpha^{\pm} \in \Gamma$, that is, $\alpha^{\pm} = \mathbf{fv}(\alpha^{\pm}) \subseteq \Gamma$.
Case 2. $\Gamma \vdash Q \to M$ means by inversion that $\Gamma \vdash Q$ and $\Gamma \vdash M$. Then by the induction hypothesis, $\mathbf{fv}(Q) \subseteq \Gamma$ and $\mathbf{fv}(M) \subseteq \Gamma$, and hence, $\mathbf{fv}(Q \to M) = \mathbf{fv}(Q) \cup \mathbf{fv}(M) \subseteq \Gamma$.
Case 3. the cases when $P = \downarrow N'$ or $N = \uparrow P'$ are proven analogously.
Case 4. $\Gamma \vdash \forall \overrightarrow{\alpha^{+}}.M$ means by inversion that $\Gamma, \overrightarrow{\alpha^{+}} \vdash M$. Then by the induction hypothesis, $\mathbf{fv}(M) \subseteq \Gamma, \overrightarrow{\alpha^{+}}$, and hence, $\mathbf{fv}(\forall \overrightarrow{\alpha^{+}}.M) = \mathbf{fv}(M) \backslash \overrightarrow{\alpha^{+}} \subseteq \Gamma, \overrightarrow{\alpha^{+}} \backslash \overrightarrow{\alpha^{+}} = \Gamma$.
Case 5. The case $P = \exists \overrightarrow{\alpha^{-}}.Q$ is proven analogously.

□

Lemma 4 (Completeness of type well-formedness). *In the well-formedness judgment, only used variables matter:*

+ *if* $\Gamma_1 \cap \mathbf{fv}\, P = \Gamma_2 \cap \mathbf{fv}\, P$ *then* $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$,
− *if* $\Gamma_1 \cap \mathbf{fv}\, N = \Gamma_2 \cap \mathbf{fv}\, N$ *then* $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$.

Proof. By simple mutual induction on P and N. □

Corollary 1 (Context Strengthening).

+ *If* $\Gamma \vdash P$ *then* $\mathbf{fv}(P) \vdash P$;
− *If* $\Gamma \vdash N$ *then* $\mathbf{fv}(N) \vdash N$.

Proof. It follows from lemma 4 and lemma 3.

+ By lemma 3, $\mathbf{fv}(P) \subseteq \Gamma$, and hence, $\Gamma \cap \mathbf{fv}\, P = \mathbf{fv}\, P$, which makes lemma 4 applicable fore contexts Γ and $\mathbf{fv}(P)$.
− The negative case is proven analogously.

□

Corollary 2 (Well-formedness Context Weakening). *Suppose that* $\Gamma_1 \subseteq \Gamma_2$, *then*

+ *if* $\Gamma_1 \vdash P$ *then* $\Gamma_2 \vdash P$,
− *if* $\Gamma_1 \vdash N$ *then* $\Gamma_2 \vdash N$.

Proof. By lemma 3, $\Gamma_1 \vdash P$ implies $\mathbf{fv}(P) \subseteq \Gamma_1$, which means that $\mathbf{fv}(P) \subseteq \Gamma_2$, and thus, $\mathbf{fv}(P) = \mathbf{fv}(P) \cap \Gamma_1 = \mathbf{fv}(P) \cap \Gamma_2$. Then by lemma 4, $\Gamma_2 \vdash P$. The negative case is symmetric. □

Corollary 3. *Suppose that all the types below are well-formed in* Γ *and* $\Gamma' \subseteq \Gamma$. *Then*

+ $\Gamma \vdash P \simeq^{\leqslant} Q$ *implies* $\Gamma' \vdash P \iff \Gamma' \vdash Q$
− $\Gamma \vdash N \simeq^{\leqslant} M$ *implies* $\Gamma' \vdash N \iff \Gamma' \vdash M$

Proof. From lemma 4 and corollary 6. □

Lemma 5 (Well-formedness agrees with substitution). *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then*

+ $\Gamma, \Gamma_1 \vdash P$ *implies* $\Gamma, \Gamma_2 \vdash [\sigma]P$, *and*
− $\Gamma, \Gamma_1 \vdash N$ *implies* $\Gamma, \Gamma_2 \vdash [\sigma]N$.

Proof. We prove it by induction on $\Gamma, \Gamma_1 \vdash P$ and mutually, on $\Gamma, \Gamma_1 \vdash N$. Let us consider the last rule used in the derivation.

Case 1. Rule (VAR_+^{WF}), i.e. P is α^+.
By inversion, $\alpha^+ \in \Gamma, \Gamma_1$, then
- if $\alpha^+ \in \Gamma_1$ then $\Gamma_2 \vdash [\sigma]\alpha^+$, and by weakening (corollary 2), $\Gamma, \Gamma_2 \vdash [\sigma]\alpha^+$;
- if $\alpha^+ \in \Gamma \backslash \Gamma_1$ then $[\sigma]\alpha^+ = \alpha^+$, and by Rule ($\text{VAR}_+^{\text{WF}}$), $\Gamma, \Gamma_2 \vdash \alpha^+$.

Case 2. Rule ($\uparrow^{\text{WF}}$), i.e. P is $\downarrow N$.
Then $\Gamma, \Gamma_1 \vdash \downarrow N$ means $\Gamma, \Gamma_1 \vdash N$ by inversion, and by the induction hypothesis, $\Gamma, \Gamma_2 \vdash [\sigma]N$. Then by Rule ($\uparrow^{\text{WF}}$), $\Gamma, \Gamma_2 \vdash \downarrow[\sigma]N$, which by definition of substitution is rewritten as $\Gamma, \Gamma_2 \vdash [\sigma]\downarrow N$.

Case 3. Rule ($\exists^{\text{WF}}$), i.e. P is $\exists\overrightarrow{\alpha^-}.Q$.
Then $\Gamma, \Gamma_1 \vdash \exists\overrightarrow{\alpha^-}.Q$ means $\Gamma, \overrightarrow{\alpha^-}, \Gamma_1 \vdash Q$ by inversion, and by the induction hypothesis, $\Gamma, \overrightarrow{\alpha^-}, \Gamma_2 \vdash [\sigma]Q$. Then by Rule ($\exists^{\text{WF}}$), $\Gamma, \overrightarrow{\alpha^-}, \Gamma_2 \vdash \exists\overrightarrow{\alpha^-}.[\sigma]Q$, which by definition of substitution is rewritten as $\Gamma, \Gamma_2 \vdash [\sigma]\exists\overrightarrow{\alpha^-}.Q$.

Case 4. The negative cases are proved symmetrically.

□

5.2 Substitution

Lemma 6 (Substitution strengthening). *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that σ is a substitution, P and N are types. Then*

$+$ $[\sigma]P = [\sigma|_{\mathbf{fv}\,P}]P$,
$-$ $[\sigma]N = [\sigma|_{\mathbf{fv}\,N}]N$

Proof. First, we strengthen the statement by saying that one can restrict the substitution to an arbitrary superset of the free variables of the substitution subject:

$+$ $[\sigma]P = [\sigma|_{vars}]P$, for any $vars \supseteq \mathbf{fv}\,P$, and
$-$ $[\sigma]N = [\sigma|_{vars}]N$, for any $vars \supseteq \mathbf{fv}\,N$.

Then the proof is a straightforward induction on P and mutually, on N. For the base cases:

Case 1. $N = \alpha^-$
Then $[\sigma]\alpha^- = \sigma|_{vars}(\alpha^-)$ by definition, since $\alpha^- \in \mathbf{fv}\,\alpha^- \subseteq vars$.

Case 2. $N = P \to M$
Then $[\sigma](P \to M) = [\sigma]P \to [\sigma]M$ by definition. Since $\mathbf{fv}\,P \subseteq \mathbf{fv}\,(P \to M) \subseteq vars$, the induction hypothesis is applicable to $[\sigma]P$: $[\sigma]P = [\sigma|_{vars}]P$. Analogously, and $[\sigma]M = [\sigma|_{vars}]M$. Then $[\sigma](P \to M) = [\sigma|_{vars}]P \to [\sigma|_{vars}]M = [\sigma|_{vars}](P \to M)$.

Case 3. $N = \uparrow P$ is proved analogously to the previous case.

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$ (where $\overrightarrow{\alpha^+}$ is not empty)
Then $[\sigma]\forall\overrightarrow{\alpha^+}.M = \forall\overrightarrow{\alpha^+}.[\sigma]M$ by definition. Let us assume $\overrightarrow{\alpha^+}$ are fresh variables, it means that $\sigma(\alpha^\pm) = \alpha^\pm$ for any $\alpha^\pm \in \overrightarrow{\alpha^+}$, and thus, $[\sigma|_{vars}] = [\sigma|_{(vars \cup \overrightarrow{\alpha^+})}]$ immediately from the definition.
Since $vars \subseteq \mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.M) = \mathbf{fv}\,M \backslash \overrightarrow{\alpha^+}$, $vars \cup \overrightarrow{\alpha^+} \subseteq \mathbf{fv}\,(M)$. Then by the induction hypothesis, $[\sigma]M = [\sigma|_{(vars \cup \overrightarrow{\alpha^+})}]M$. Finally, $[\sigma]\forall\overrightarrow{\alpha^+}.M = \forall\overrightarrow{\alpha^+}.[\sigma|_{(vars \cup \overrightarrow{\alpha^+})}]M = \forall\overrightarrow{\alpha^+}.[\sigma|_{vars}]M = [\sigma|_{vars}]\forall\overrightarrow{\alpha^+}.M$.

Case 5. The positive cases are proves symmetrically.

□

Lemma 7 (Signature of a restricted substitution). *If* $\Gamma_2 \vdash \sigma : \Gamma_1$ *then* $\Gamma_2 \vdash \sigma|_{vars} : \Gamma_1 \cap vars$.

Proof. Let us take an arbitrary $\alpha^{\pm} \in \Gamma_1 \cap vars$. Since $\alpha^{\pm} \in \Gamma_1$, $\Gamma_2 \vdash [\sigma]\alpha^{\pm}$ by the signature of σ.
Let us take an arbitrary $\alpha^{\pm} \notin \Gamma_1 \cap vars$. If $\alpha^{\pm} \notin vars$ then $[\sigma|_{vars}]\alpha^{\pm} = \alpha^{\pm}$ by definition of restriction. If $\alpha^{\pm} \in vars \backslash \Gamma_1$ then $[\sigma|_{vars}]\alpha^{\pm} = [\sigma]\alpha^{\pm}$ by definition and $[\sigma]\alpha^{\pm} = \alpha^{\pm}$ by the signature of σ. □

Lemma 8. *Suppose that* σ *is a substitution with signature* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then if vars is disjoint from* Γ_1, *then* $\sigma|_{vars} = \mathrm{id}$.

Proof. Let us take an arbitrary $\alpha^{\pm}$. If $\alpha^{\pm} \notin vars$ then $[\sigma|_{vars}]\alpha^{\pm} = \alpha^{\pm}$ by definition.
If $\alpha^{\pm} \in vars$ then $\alpha^{\pm} \notin \Gamma_1$ by assumption. Then $[\sigma|_{vars}]\alpha^{\pm} = [\sigma]\alpha^{\pm}$ by definition of restricted substitution, and since $\Gamma_2 \vdash \sigma : \Gamma_1$, we have $[\sigma]\alpha^{\pm} = \alpha^{\pm}$. □

Corollary 4 (Application of a disjoint substitution). *Suppose that* σ *is a substitution with signature* $\Gamma_2 \vdash \sigma : \Gamma_1$. *Then*

+ *if* $\Gamma_1 \cap \mathbf{fv}(Q) = \varnothing$ *then* $[\sigma]Q = Q$;
− *if* $\Gamma_1 \cap \mathbf{fv}(N) = \varnothing$ *then* $[\sigma]N = N$.

Lemma 9 (Substitution range weakening). *Suppose that* $\Gamma_2 \subseteq \Gamma_2'$ *are contexts and* σ *is a substitution. Then* $\Gamma_2 \vdash \sigma : \Gamma_1$ *implies* $\Gamma_2' \vdash \sigma : \Gamma_1$.

Proof. For any $\alpha^{\pm} \in \Gamma_1$, $\Gamma_2 \vdash \sigma : \Gamma_1$ gives us $\Gamma_2 \vdash [\sigma]\alpha^{\pm}$, which can be weakened to $\Gamma_2' \vdash [\sigma]\alpha^{\pm}$ by corollary 2. This way, $\Gamma_2' \vdash \sigma : \Gamma_1$. □

Lemma 10. *Suppose that* $\Gamma' \subseteq \Gamma$, σ_1 *and* σ_2 *are substitutions of signature* $\Gamma \vdash \sigma_i : \Gamma'$. *Then*

+ *for a type* $\Gamma \vdash P$, *if* $\Gamma \vdash [\sigma_1]P \simeq^{\leqslant} [\sigma_2]P$ *then* $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \mathbf{fv}\, P \cap \Gamma'$;
− *for a type* $\Gamma \vdash N$, *if* $\Gamma \vdash [\sigma_1]N \simeq^{\leqslant} [\sigma_2]N$ *then* $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \mathbf{fv}\, N \cap \Gamma'$.

Proof. Let us make an additional assumption that σ_1, σ_2, and the mentioned types are normalized. If they are not, we normalize them first.
Notice that the normalization preserves the set of free variables (lemma 43), well-formedness (corollary 11), and equivalence (lemma 35), and distributes over substitution (lemma 45). This way, the assumed and desired properties are equivalent to their normalized versions.
We prove it by induction on the structure of P and mutually, N. Let us consider the shape of this type.

Case 1. $P = \alpha^{+} \in \Gamma'$. Then $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \mathbf{fv}\, P \cap \Gamma'$ means $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \alpha^{+}$, i.e. $\Gamma \vdash [\sigma_1]\alpha^{+} \simeq^{\leqslant} [\sigma_2]\alpha^{+}$, which holds by assumption.
Case 2. $P = \alpha^{+} \in \Gamma \backslash \Gamma'$. Then $\mathbf{fv}\, P \cap \Gamma' = \varnothing$, so $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \mathbf{fv}\, P \cap \Gamma'$ holds vacuously.
Case 3. $P = {\downarrow}N$. Then the induction hypothesis is applicable to type N:

(1) N is normalized,
(2) $\Gamma \vdash N$ by inversion of $\Gamma \vdash {\downarrow}N$,
(3) $\Gamma \vdash [\sigma_1]N \simeq^{\leqslant} [\sigma_2]N$ holds by inversion of $\Gamma \vdash [\sigma_1]{\downarrow}N \simeq^{\leqslant} [\sigma_2]{\downarrow}N$, i.e. $\Gamma \vdash {\downarrow}[\sigma_1]N \simeq^{\leqslant} {\downarrow}[\sigma_2]N$.

This way, we obtain $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \mathbf{fv}\, N \cap \Gamma'$, which implies the required equivalence since $\mathbf{fv}\, P \cap \Gamma' = \mathbf{fv}\, {\downarrow}N \cap \Gamma' = \mathbf{fv}\, N \cap \Gamma'$.
Case 4. $P = \exists \overrightarrow{\alpha^{-}}.Q$ Then the induction hypothesis is applicable to type Q well-formed in context $\Gamma, \overrightarrow{\alpha^{-}}$:

(1) $\Gamma' \subseteq \Gamma, \overrightarrow{\alpha^{-}}$ since $\Gamma' \subseteq \Gamma$,
(2) $\Gamma, \overrightarrow{\alpha^{-}} \vdash \sigma_i : \Gamma'$ by weakening,
(3) Q is normalized,

(4) $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.Q$,
(5) Notice that $[\sigma_i]\exists\overrightarrow{\alpha^-}.Q$ is normalized, and thus, $[\sigma_1]\exists\overrightarrow{\alpha^-}.Q \simeq^D [\sigma_2]\exists\overrightarrow{\alpha^-}.Q$ implies $[\sigma_1]\exists\overrightarrow{\alpha^-}.Q = [\sigma_2]\exists\overrightarrow{\alpha^-}.Q$ (by lemma 35).). This equality means $[\sigma_1]Q = [\sigma_2]Q$, which implies $\Gamma \vdash [\sigma_1]Q \simeq^{\leqslant} [\sigma_2]Q$.

Case 5. $N = P \to M$

□

Lemma 11 (Substitution composition well-formedness). *If* $\Gamma_1' \vdash \sigma_1 : \Gamma_1$ *and* $\Gamma_2' \vdash \sigma_2 : \Gamma_2$, *then* $\Gamma_1', \Gamma_2' \vdash \sigma_2 \circ \sigma_1 : \Gamma_1, \Gamma_2$.

Lemma 12 (Substitution monadic composition well-formedness). *If* $\Gamma_1' \vdash \sigma_1 : \Gamma_1$ *and* $\Gamma_2' \vdash \sigma_2 : \Gamma_2$, *then* $\Gamma_2' \vdash \sigma_2 \ll \sigma_1 : \Gamma_1$.

Lemma 13 (Substitution composition). *If* $\Gamma_1' \vdash \sigma_1 : \Gamma_1$, $\Gamma_2' \vdash \sigma_2 : \Gamma_2$, $\Gamma_1 \cap \Gamma_2' = \varnothing$ *and* $\Gamma_1 \cap \Gamma_2 = \varnothing$ *then* $\sigma_2 \circ \sigma_1 = (\sigma_2 \ll \sigma_1) \circ \sigma_2$.

Corollary 5 (Substitution composition commutativity). *If* $\Gamma_1' \vdash \sigma_1 : \Gamma_1$, $\Gamma_2' \vdash \sigma_2 : \Gamma_2$, *and* $\Gamma_1 \cap \Gamma_2 = \varnothing$, $\Gamma_1 \cap \Gamma_2' = \varnothing$, *and* $\Gamma_1' \cap \Gamma_2 = \varnothing$ *then* $\sigma_2 \circ \sigma_1 = \sigma_1 \circ \sigma_2$.

Proof. by lemma 13, $\sigma_2 \circ \sigma_1 = (\sigma_2 \ll \sigma_1) \circ \sigma_2$. Since the codomain of σ_1 is Γ_1', and it is disjoint with the domain of σ_2, $\sigma_2 \ll \sigma_1 = \sigma_1$. □

Lemma 14 (Substitution domain weakening). *If* $\Gamma_2 \vdash \sigma : \Gamma_1$ *then* $\Gamma_2, \Gamma' \vdash \sigma : \Gamma_1, \Gamma'$

Proof. If the variable $\alpha^\pm$ is in Γ_1 then $\Gamma_2 \vdash [\sigma]\alpha^\pm$ by assumption, and then $\Gamma_2, \Gamma' \vdash [\sigma]\alpha^\pm$ by weakening. If the variable $\alpha^\pm$ is in $\Gamma' \backslash \Gamma_1$ then $[\sigma]\alpha^\pm = \alpha^\pm \in \Gamma' \subseteq \Gamma_2, \Gamma'$, and thus, $\Gamma_2, \Gamma' \vdash \alpha^\pm$. □

Lemma 15 (Free variables after substitution). *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$, *then*

+ *for a type* P, *the free variables of* $[\sigma]P$ *are bounded in the following way:* $\mathbf{fv}(P)\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]P) \subseteq (\mathbf{fv}(P)\backslash\Gamma_1) \cup \Gamma_2$;
− *for a type* N, *the free variables of* $[\sigma]P$ *are bounded in the following way:* $\mathbf{fv}(N)\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]N) \subseteq (\mathbf{fv}(N)\backslash\Gamma_1) \cup \Gamma_2$.

Proof. We prove it by structural induction on P and mutually, on N.

Case 1. $P = \alpha^+$
If $\alpha^+ \in \Gamma_1$ then $\Gamma_2 \vdash [\sigma]\alpha^+$, and by lemma 3, $\mathbf{fv}([\sigma]\alpha^+) \subseteq \Gamma_2$. $\mathbf{fv}(\alpha^+)\backslash\Gamma_1 = \varnothing$, so $\mathbf{fv}([\sigma]P)\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]\alpha^+)$ vacuously.
If $\alpha^+ \notin \Gamma_1$ then $[\sigma]\alpha^+ = \alpha^+$, and $\mathbf{fv}([\sigma]\alpha^+) = \alpha^+ = \alpha^+\backslash\Gamma_1$.

Case 2. $P = \exists\overrightarrow{\alpha^-}.Q$
Then we need to show that $\mathbf{fv}([\sigma]P) = \mathbf{fv}([\sigma]Q)\backslash\overrightarrow{\alpha^-}$ is a subset of $(\mathbf{fv}(P)\backslash\Gamma_1) \cup \Gamma_2$ and a superset of $\mathbf{fv}(P)\backslash\Gamma_1$. Notice that $\mathbf{fv}(P) = \mathbf{fv}(Q)\backslash\overrightarrow{\alpha^-}$ by definition. This way, we need to show that $\mathbf{fv}(Q)\backslash\overrightarrow{\alpha^-}\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]Q)\backslash\overrightarrow{\alpha^-} \subseteq (\mathbf{fv}(Q)\backslash\overrightarrow{\alpha^-}\backslash\Gamma_1) \cup \Gamma_2$,
By the induction hypothesis, $\mathbf{fv}([\sigma]Q) \subseteq (\mathbf{fv}(Q)\backslash\Gamma_1) \cup \Gamma_2$. So for the second inclusion, it suffices to show that $((\mathbf{fv}(Q)\backslash\Gamma_1) \cup \Gamma_2)\backslash\overrightarrow{\alpha^-} \subseteq (\mathbf{fv}(Q)\backslash\overrightarrow{\alpha^-}\backslash\Gamma_1) \cup \Gamma_2$, which holds by set theoretical reasoning.
Also by the induction hypothesis, $\mathbf{fv}(Q)\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]Q)$, and thus, by subtracting $\overrightarrow{\alpha^-}$ from both sides, $\mathbf{fv}(Q)\backslash\overrightarrow{\alpha^-}\backslash\Gamma_1 \subseteq \mathbf{fv}([\sigma]Q)\backslash\overrightarrow{\alpha^-}$.

Case 3. The case $N = \forall\overrightarrow{\alpha^+}.M$ is proved analogously.

Case 4. $N = P \to M$
Then $\mathbf{fv}([\sigma]N) = \mathbf{fv}([\sigma]P) \cup \mathbf{fv}([\sigma]M)$. By the induction hypothesis,

(1) $\mathbf{fv}\,(P)\backslash\Gamma_1 \subseteq \mathbf{fv}\,([\sigma]P) \subseteq (\mathbf{fv}\,(P)\backslash\Gamma_1) \cup \Gamma_2$ and
(2) $\mathbf{fv}\,(M)\backslash\Gamma_1 \subseteq \mathbf{fv}\,([\sigma]M) \subseteq (\mathbf{fv}\,(M)\backslash\Gamma_1) \cup \Gamma_2$.

We unite these inclusions vertically and obtain $\mathbf{fv}\,(P)\backslash\Gamma_1 \cup \mathbf{fv}\,(M)\backslash\Gamma_1 \subseteq \mathbf{fv}\,([\sigma]N) \subseteq ((\mathbf{fv}\,(P)\backslash\Gamma_1) \cup \Gamma_2) \cup ((\mathbf{fv}\,(M)\backslash\Gamma_1) \cup \Gamma_2)$, which is equivalent to $(\mathbf{fv}\,(P) \cup \mathbf{fv}\,(M))\backslash\Gamma_1 \subseteq \mathbf{fv}\,([\sigma]N) \subseteq (\mathbf{fv}\,(P) \cup \mathbf{fv}\,(M))\backslash\Gamma_1 \cup \Gamma_2$. Since $\mathbf{fv}\,(P) \cup \mathbf{fv}\,(M) = \mathbf{fv}\,(N)$, $\mathbf{fv}\,(N)\backslash\Gamma_1 \subseteq \mathbf{fv}\,([\sigma]N) \subseteq (\mathbf{fv}\,(N)\backslash\Gamma_1) \cup \Gamma_2$.

Case 5. The cases when $P = \downarrow M$ and $N = \uparrow Q$ are proved analogously

□

Lemma 16 (Free variables of a variable image). *Suppose that σ is an arbitrary substitution, Then*

+ *if $\alpha^{\pm} \in \mathbf{fv}\,(P)$ then $\mathbf{fv}\,([\sigma]\alpha^{\pm}) \subseteq \mathbf{fv}\,([\sigma]P)$,*
− *if $\alpha^{\pm} \in \mathbf{fv}\,(N)$ then $\mathbf{fv}\,([\sigma]\alpha^{\pm}) \subseteq \mathbf{fv}\,([\sigma]N)$.*

Proof. By mutual induction on P and N. The base cases (when P or N is a variable) are trivial, since then $\alpha^{\pm} \in \mathbf{fv}\,(P)$ means $\alpha^{\pm} = P$ (and symmetrically for N). The congruent cases (when the type is formed by $\downarrow$, $\uparrow$, or $\to$) hold since $\alpha^{\pm}$ occurs in type means that it occurs in one of its parts, to which we apply the induction hypothesis.

Let us suppose that te type is $\exists\overrightarrow{\alpha^{-}}.Q$. Then $\alpha^{\pm} \in \mathbf{fv}\,(\exists\overrightarrow{\alpha^{-}}.Q)$ means $\alpha^{\pm} \in \mathbf{fv}\,(Q)$ and $\alpha^{\pm} \notin \overrightarrow{\alpha^{-}}$. Then by the induction hypothesis, $\mathbf{fv}\,([\sigma]\alpha^{\pm}) \subseteq \mathbf{fv}\,([\sigma]Q)$, and it is left to notice that $\mathbf{fv}\,([\sigma]\alpha^{\pm}) \cap \overrightarrow{\alpha^{-}} = \varnothing$, which we can ensure by alpha-equivalence. □

5.3 Declarative Subtyping

Lemma 17 (Free Variable Propagation). *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context Γ,*

- *− if $\Gamma \vdash N \leqslant M$ then $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$*
- *+ if $\Gamma \vdash P \geqslant Q$ then $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$*

Proof. Mutual induction on $\Gamma \vdash N \leqslant M$ and $\Gamma \vdash P \geqslant Q$.

Case 1. $\Gamma \vdash \alpha^{-} \leqslant \alpha^{-}$
It is self-evident that $\alpha^{-} \subseteq \alpha^{-}$.

Case 2. $\Gamma \vdash \uparrow P \leqslant \uparrow Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq^{\leqslant} Q$), we have $\Gamma \vdash P \geqslant Q$. Then by the induction hypothesis, $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$. The desired inclusion holds, since $\mathbf{fv}\,(\uparrow P) = \mathbf{fv}\,(P)$ and $\mathbf{fv}\,(\uparrow Q) = \mathbf{fv}\,(Q)$.

Case 3. $\Gamma \vdash P \to N \leqslant Q \to M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}\,(P) \subseteq \mathbf{fv}\,(Q)$ and $\mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(M)$. Then $\mathbf{fv}\,(P \to N) = \mathbf{fv}\,(P) \cup \mathbf{fv}\,(N) \subseteq \mathbf{fv}\,(Q) \cup \mathbf{fv}\,(M) = \mathbf{fv}\,(Q \to M)$.

Case 4. $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N \leqslant \forall\overrightarrow{\beta^{+}}.M$

$$\begin{aligned}
\mathbf{fv}\,\forall\overrightarrow{\alpha^{+}}.N &\subseteq \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^{+}}]N) \setminus \overrightarrow{\beta^{+}} && \text{here } \overrightarrow{\beta^{+}} \text{ is excluded by the premise } \mathbf{fv}\,N \cap \overrightarrow{\beta^{+}} = \varnothing \\
&\subseteq \mathbf{fv}\,M \setminus \overrightarrow{\beta^{+}} && \text{by the induction hypothesis, } \mathbf{fv}\,([\overrightarrow{P}/\overrightarrow{\alpha^{+}}]N) \subseteq \mathbf{fv}\,M \\
&\subseteq \mathbf{fv}\,\forall\overrightarrow{\beta^{+}}.M
\end{aligned}$$

Case 5. The positive cases are symmetric.

□

Corollary 6 (Free Variables of mutual subtypes).

− *If $\Gamma \vdash N \simeq^{\leqslant} M$ then $\mathbf{fv}\,N = \mathbf{fv}\,M$,*
+ *If $\Gamma \vdash P \simeq^{\leqslant} Q$ then $\mathbf{fv}\,P = \mathbf{fv}\,Q$*

Lemma 18 (Decomposition of quantifier rules). *Assuming that $\overrightarrow{\alpha^+}$, $\overrightarrow{\beta^+}$, $\overrightarrow{\alpha^-}$, and $\overrightarrow{\alpha^-}$ are disjoint from Γ,*

- $-_R$ *$\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M$ holds if and only if $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M$;*
- $+_R$ *$\Gamma \vdash P \geqslant \exists\overrightarrow{\beta^-}.Q$ holds if and only if $\Gamma, \overrightarrow{\beta^-} \vdash P \geqslant Q$;*
- $-_L$ *suppose $M \neq \forall\ldots$ then $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant M$ holds if and only if $\Gamma \vdash [\vec{P}/\overrightarrow{\alpha^+}]N \leqslant M$ for some $\Gamma \vdash \vec{P}$;*
- $+_L$ *suppose $Q \neq \exists\ldots$ then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant Q$ holds if and only if $\Gamma \vdash [\vec{N}/\overrightarrow{\alpha^-}]P \geqslant Q$ for some $\Gamma \vdash \vec{N}$.*

Proof.

$-_R$ Let us prove both directions.

- $\Rightarrow$ Let us assume $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M$. $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M$. Let us decompose M as $\forall\overrightarrow{\beta^{+\prime}}.M'$ where M' does not start with $\forall$, and decompose N as $\forall\overrightarrow{\alpha^+}.N'$ where N' does not start with $\forall$. If $\overrightarrow{\beta^+}$ is empty, then $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M$ holds by assumption. Otherwise, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.\forall\overrightarrow{\beta^{+\prime}}.M$ is inferred by Rule ($\forall^{\leqslant}$), and by inversion: $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\beta^{+\prime}} \vdash [\vec{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$ for some $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\beta^{+\prime}} \vdash \vec{P}$. Then again by Rule ($\forall^{\leqslant}$) with the same $\vec{P}$, $\Gamma, \overrightarrow{\beta^+} \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^{+\prime}}.M'$, that is $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M$.
- $\Leftarrow$ let us assume $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M$, and let us decompose N as $\forall\overrightarrow{\alpha^+}.N'$ where N' does not start with $\forall$, and M as $\forall\overrightarrow{\beta^{+\prime}}.M'$ where M' does not start with $\forall$. if $\overrightarrow{\alpha^+}$ and $\overrightarrow{\beta^{+\prime}}$ are empty then $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M$ is turned into $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M$ by Rule ($\forall^{\leqslant}$). Otherwise, $\Gamma, \overrightarrow{\beta^+} \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^{+\prime}}.M'$ is inferred by Rule ($\forall^{\leqslant}$), that is $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\beta^{+\prime}} \vdash [\vec{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$ for some $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\beta^{+\prime}} \vdash \vec{P}$. Then by Rule ($\forall^{\leqslant}$) again, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}, \overrightarrow{\beta^{+\prime}}.M'$, in other words, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.\forall\overrightarrow{\beta^{+\prime}}.M'$, that is $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M$.

$-_L$ Suppose $M \neq \forall\ldots$. Let us prove both directions.

- $\Rightarrow$ Let us assume $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant M$. then if $\overrightarrow{\alpha^+} = \cdot$, $\Gamma \vdash N \leqslant M$ holds immediately. Otherwise, let us decompose N as $\forall\overrightarrow{\alpha^{+\prime}}.N'$ where N' does not start with $\forall$. Then $\Gamma \vdash \forall\overrightarrow{\alpha^+}.\forall\overrightarrow{\alpha^{+\prime}}.N' \leqslant M'$ is inferred by Rule ($\forall^{\leqslant}$), and by inversion, there exist $\Gamma \vdash \vec{P}, \vec{P}'$ such that $\Gamma \vdash [\vec{P}/\overrightarrow{\alpha^+}][\vec{P}'/\overrightarrow{\alpha^{+\prime}}]N' \leqslant M'$ (the decomposition of substitutions is possible since $\overrightarrow{\alpha^+} \cap \Gamma = \varnothing$). Then by Rule ($\forall^{\leqslant}$) again, $\Gamma \vdash \forall\overrightarrow{\alpha^{+\prime}}.[\vec{P}'/\overrightarrow{\alpha^{+\prime}}]N' \leqslant M'$ (notice that $[\vec{P}'/\overrightarrow{\alpha^{+\prime}}]N'$ cannot start with $\forall$).
- $\Leftarrow$ Let us assume $\Gamma \vdash [\vec{P}/\overrightarrow{\alpha^+}]N \leqslant M$ for some $\Gamma \vdash \vec{P}$. let us decompose N as $\forall\overrightarrow{\alpha^{+\prime}}.N'$ where N' does not start with $\forall$. Then $\Gamma \vdash [\vec{P}/\overrightarrow{\alpha^+}]\forall\overrightarrow{\alpha^{+\prime}}.N' \leqslant M'$ or, equivalently, $\Gamma \vdash \forall\overrightarrow{\alpha^{+\prime}}.[\vec{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$ is inferred by Rule ($\forall^{\leqslant}$) (notice that $[\vec{P}/\overrightarrow{\alpha^+}]N'$ cannot start with $\forall$). By inversion, there exist $\Gamma \vdash \vec{P}'$ such that $\Gamma \vdash [\vec{P}'/\overrightarrow{\alpha^{+\prime}}][\vec{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$. Since $\overrightarrow{\alpha^{+\prime}}$ is disjoint from the free variables of $\vec{P}$ and from $\overrightarrow{\alpha^+}$, the composition of $\vec{P}'/\overrightarrow{\alpha^{+\prime}}$ and $\vec{P}/\overrightarrow{\alpha^+}$ can be joined into a single substitution well-formed in Γ. Then by Rule ($\forall^{\leqslant}$) again, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant M$.

$+$ The positive cases are proved symmetrically.

□

Corollary 7 (Redundant quantifier elimination).

- $-_L$ *Suppose that $\overrightarrow{\alpha^+} \cap \mathbf{fv}(N) = \varnothing$ then $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant M$ holds if and only if $\Gamma \vdash N \leqslant M$;*
- $-_R$ *Suppose that $\overrightarrow{\alpha^+} \cap \mathbf{fv}(M) = \varnothing$ then $\Gamma \vdash N \leqslant \forall\overrightarrow{\alpha^+}.M$ holds if and only if $\Gamma \vdash N \leqslant M$;*

$+_L$ *Suppose that* $\overrightarrow{\alpha^-} \cap \mathbf{fv}(P) = \varnothing$ *then* $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P \geqslant Q$ *holds if and only if* $\Gamma \vdash P \geqslant Q$.

$+_R$ *Suppose that* $\overrightarrow{\alpha^-} \cap \mathbf{fv}(Q) = \varnothing$ *then* $\Gamma \vdash P \geqslant \exists\overrightarrow{\alpha^-}.Q$ *holds if and only if* $\Gamma \vdash P \geqslant Q$.

Proof. $-_R$ Suppose that $\overrightarrow{\alpha^+} \cap \mathbf{fv}(M) = \varnothing$ then by lemma 18, $\Gamma \vdash N \leqslant \forall\overrightarrow{\alpha^+}.M$ is equivalent to $\Gamma, \overrightarrow{\alpha^+} \vdash N \leqslant M$, By lemma 4, since $\overrightarrow{\alpha^+} \cap \mathbf{fv}(N) = \varnothing$ and $\overrightarrow{\alpha^+} \cap \mathbf{fv}(M) = \varnothing$, $\Gamma, \overrightarrow{\alpha^+} \vdash N \leqslant M$ is equivalent to $\Gamma \vdash N \leqslant M$.

$-_L$ Suppose that $\overrightarrow{\alpha^+} \cap \mathbf{fv}(N) = \varnothing$. Let us decompose M as $\forall\overrightarrow{\beta^+}.M'$ where M' does not start with $\forall$. By lemma 18, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M'$ is equivalent to $\Gamma, \overrightarrow{\beta^+} \vdash \forall\overrightarrow{\alpha^+}.N \leqslant M'$, which is equivalent to existence of $\Gamma, \overrightarrow{\beta^+} \vdash \overrightarrow{P}$ such that $\Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant M'$. Since $[\overrightarrow{P}/\overrightarrow{\alpha^+}]N = N$, the latter is equivalent to $\Gamma, \overrightarrow{\beta^+} \vdash N \leqslant M'$, which is equivalent to $\Gamma \vdash N \leqslant \forall\overrightarrow{\beta^+}.M'$. $\Gamma, \overrightarrow{\beta^+} \vdash \overrightarrow{P}$ can be chosen arbitrary, for example, $\overrightarrow{P}_i = \exists\alpha^-.{\downarrow}\alpha^-$.

$+$ The positive cases are proved symmetrically.

□

Lemma 19 (Subtypes and supertypes of a variable). *Assuming* $\Gamma \vdash \alpha^-$, $\Gamma \vdash \alpha^+$, $\Gamma \vdash N$, *and* $\Gamma \vdash P$,

$+$ *if* $\Gamma \vdash P \geqslant \exists\overrightarrow{\alpha^-}.\alpha^+$ *or* $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant P$ *then* $P = \exists\overrightarrow{\beta^-}.\alpha^+$ *(for some potentially empty* $\overrightarrow{\beta^-}$*)*

$-$ *if* $\Gamma \vdash N \leqslant \forall\overrightarrow{\alpha^+}.\alpha^-$ *or* $\Gamma \vdash \forall\overrightarrow{\alpha^+}.\alpha^- \leqslant N$ *then* $N = \forall\overrightarrow{\beta^+}.\alpha^-$ *(for some potentially empty* $\overrightarrow{\beta^+}$*)*

Proof. We prove by induction on the tree inferring $\Gamma \vdash P \geqslant \exists\overrightarrow{\alpha^-}.\alpha^+$ or $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant P$ or or $\Gamma \vdash N \leqslant \forall\overrightarrow{\alpha^+}.\alpha^-$ or $\Gamma \vdash \forall\overrightarrow{\alpha^+}.\alpha^- \leqslant N$.

Let us consider which one of these judgments is inferred.

Case 1. $\Gamma \vdash P \geqslant \exists\overrightarrow{\alpha^-}.\alpha^+$

If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\text{Var}^{\geqslant}_{+}$), which implies that $\overrightarrow{\alpha^-}$ is empty and $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists^{\geqslant}$). By inverting this rule, $P = \exists\overrightarrow{\beta^-}.P'$ where P' does not start with $\exists$ and $\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{N}/\overrightarrow{\beta^-}]P' \geqslant \alpha^+$ for some $\Gamma, \overrightarrow{\alpha^-} \vdash N_i$.

By the induction hypothesis, $[\overrightarrow{N}/\overrightarrow{\beta^-}]P' = \exists\overrightarrow{\gamma^-}.\alpha^+$. What shape can P' have? As mentioned, it does not start with $\exists$, and it cannot start with $\uparrow$ (otherwise, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]P'$ would also start with $\uparrow$ and would not be equal to $\exists\overrightarrow{\beta^-}.\alpha^+$). This way, P' is a *positive* variable. As such, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]P' = P'$, and then $P' = \exists\overrightarrow{\gamma^-}.\alpha^+$ meaning that $\overrightarrow{\gamma^-}$ is empty and $P' = \alpha^+$. This way, $P = \exists\overrightarrow{\beta^-}.P' = \exists\overrightarrow{\beta^-}.\alpha^+$, as required.

Case 2. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.\alpha^+ \geqslant P$

If the size of the inference tree is 1 then the only rule that can infer it is Rule ($\text{Var}^{\geqslant}_{+}$), which implies that $\overrightarrow{\alpha^-}$ is empty and $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule ($\exists^{\geqslant}$). By inverting this rule, $P = \exists\overrightarrow{\beta^-}.Q$ where $\Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ \geqslant Q$ and Q does not start with $\exists$. Notice that since α^+ is positive, $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash \alpha^+ \geqslant Q$.

By the induction hypothesis, $Q = \exists\overrightarrow{\beta^-{}'}.\alpha^+$, and since Q does not start with $\exists$, $\overrightarrow{\beta^-{}'}$ is empty This way, $P = \exists\overrightarrow{\beta^-}.Q = \exists\overrightarrow{\beta^-}.\alpha^+$, as required.

Case 3. The negative cases ($\Gamma \vdash N \leqslant \forall\overrightarrow{\alpha^+}.\alpha^-$ and $\Gamma \vdash \forall\overrightarrow{\alpha^+}.\alpha^- \leqslant N$) are proved analogously.

□

Corollary 8 (Variables have no proper subtypes and supertypes). *Assuming that all mentioned types are well-formed in* Γ,

$$\Gamma \vdash P \geqslant \alpha^+ \iff P = \exists\overrightarrow{\beta^-}.\alpha^+ \iff \Gamma \vdash P \simeq^{\leqslant} \alpha^+ \iff P \simeq^{D} \alpha^+$$
$$\Gamma \vdash \alpha^+ \geqslant P \iff P = \exists\overrightarrow{\beta^-}.\alpha^+ \iff \Gamma \vdash P \simeq^{\leqslant} \alpha^+ \iff P \simeq^{D} \alpha^+$$
$$\Gamma \vdash N \leqslant \alpha^- \iff N = \forall\overrightarrow{\beta^+}.\alpha^- \iff \Gamma \vdash N \simeq^{\leqslant} \alpha^- \iff N \simeq^{D} \alpha^-$$
$$\Gamma \vdash \alpha^- \leqslant N \iff N = \forall\overrightarrow{\beta^+}.\alpha^- \iff \Gamma \vdash N \simeq^{\leqslant} \alpha^- \iff N \simeq^{D} \alpha^-$$

Proof. Notice that $\Gamma \vdash \exists\overrightarrow{\beta^-}.\alpha^+ \simeq^{\leqslant} \alpha^+$ and $\exists\overrightarrow{\beta^-}.\alpha^+ \simeq^{D} \alpha^+$ and apply lemma 19. □

Lemma 20 (Subtyping context irrelevance). *Suppose that all the mentioned types are well-formed in* Γ_1 *and* Γ_2. *Then*

+ $\Gamma_1 \vdash P \geqslant Q$ *is equivalent to* $\Gamma_2 \vdash P \geqslant Q$;
− $\Gamma_1 \vdash N \leqslant M$ *is equivalent to* $\Gamma_2 \vdash N \leqslant M$.

Proof. We prove it by induction on the size of $\Gamma_1 \vdash P \geqslant Q$ and mutually, the size of $\Gamma_1 \vdash N \leqslant M$.

All the cases except Rule ($\exists^{\geqslant}$) and Rule ($\forall^{\leqslant}$) are proven congruently: first, we apply the inversion to $\Gamma_1 \vdash P \geqslant Q$ to obtain the premises of the corresponding rule X, then we apply the induction hypothesis to each premise, and build the inference tree (with Γ_2) by the same rule X.

Suppose that the judgement is inferred by Rule ($\exists^{\geqslant}$). Then we are proving that $\Gamma_1 \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q$ implies $\Gamma_2 \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q$ (the other implication is proven symmetrically).

By inversion of $\Gamma_1 \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q$, we obtain σ such that $\Gamma_1, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $\Gamma_1, \overrightarrow{\beta^-} \vdash [\sigma]P \geqslant Q$. By lemma 17, $\mathbf{fv}([\sigma]P) \subseteq \mathbf{fv}(Q)$.

From the well-formedness statements $\Gamma_i \vdash \exists\overrightarrow{\alpha^-}.P$ and $\Gamma_i \vdash \exists\overrightarrow{\beta^-}.Q$ we have:

- $\Gamma_1, \overrightarrow{\alpha^-} \vdash P$, which also means $\Gamma_1, \overrightarrow{\beta^-} \vdash [\sigma]P$ by lemma 5;
- $\Gamma_2, \overrightarrow{\alpha^-} \vdash P$;
- $\Gamma_1, \overrightarrow{\beta^-} \vdash Q$; and
- $\Gamma_2, \overrightarrow{\beta^-} \vdash Q$, which means $\mathbf{fv}(Q) \subseteq \Gamma_2, \overrightarrow{\beta^-}$ by lemma 3, and combining it with $\mathbf{fv}([\sigma]P) \subseteq \mathbf{fv}(Q)$, we have $\mathbf{fv}([\sigma]P) \subseteq \Gamma_2, \overrightarrow{\beta^-}$.

Let us construct a substitution σ_0 in the following way:

$$\begin{cases} [\sigma_0]\alpha_i^- = [\sigma]\alpha_i^- & \text{for } \alpha_i^- \in \overrightarrow{\alpha^-} \cap \mathbf{fv}(P) \\ [\sigma_0]\alpha_i^- = \forall\gamma^+.\uparrow\gamma^+ & \text{for } \alpha_i^- \in \overrightarrow{\alpha^-} \backslash \mathbf{fv}(P) \\ [\sigma_0]\gamma^\pm = \gamma^\pm & \text{for any other } \gamma^\pm \end{cases}$$

Notice that

(1) $[\sigma_0]P = [\sigma]P$. Since $\sigma_0|_{\mathbf{fv}(P)} = \sigma|_{\mathbf{fv}(P)}$ as functions (which follows from the construction of σ_0 and the signature of σ), $[\sigma_0]P = [\sigma_0|_{\mathbf{fv}(P)}]P = [\sigma|_{\mathbf{fv}(P)}]P = [\sigma]P$ (where the first and the last equalities are by lemma 6).
(2) $\mathbf{fv}([\sigma]P) \vdash \sigma_0 : \overrightarrow{\alpha^-}$. To show that, let us consider α_i^-
   - if $\alpha_i^- \in \overrightarrow{\alpha^-} \backslash \mathbf{fv}(P)$ then $\cdot \vdash [\sigma_0]\alpha_i^-$, which can be weakened to $\mathbf{fv}([\sigma]P) \vdash [\sigma_0]\alpha_i^-$;

- if $\alpha_i^- \in \overrightarrow{\alpha^-} \cap \mathbf{fv}(P)$, we have $[\sigma_0]\alpha_i^- = [\sigma]\alpha_i^-$, and thus, by specification of σ, $\Gamma_1, \overrightarrow{\beta^+} \vdash [\sigma_0]\alpha_i^-$. By corollary 1, it means $\mathbf{fv}([\sigma_0]\alpha_i^-) \vdash [\sigma_0]\alpha_i^-$, which we weaken (corollary 2) to $\mathbf{fv}([\sigma]P) \vdash [\sigma_0]\alpha_i^-$ (since $\mathbf{fv}([\sigma_0]\alpha_i^-) \subseteq \mathbf{fv}([\sigma_0]P)$ by lemma 16, and $[\sigma_0]P = [\sigma]P$, as noted above).

By corollary 1, $\Gamma_1, \overrightarrow{\beta^-} \vdash [\sigma]P$ implies $\mathbf{fv}([\sigma]P) \vdash [\sigma]P$, which, since $\mathbf{fv}([\sigma]P) \subseteq \Gamma_2, \overrightarrow{\beta^-}$, is weakened to $\Gamma_2, \overrightarrow{\beta^-} \vdash [\sigma]P$. and rewritten as $\Gamma_2, \overrightarrow{\beta^-} \vdash [\sigma_0]P$.

Notice that the premises of the induction hold:

(1) $\Gamma_i, \overrightarrow{\beta^-} \vdash [\sigma_0]P$,
(2) $\Gamma_i, \overrightarrow{\beta^-} \vdash Q$, and
(3) $\Gamma_1, \overrightarrow{\beta^-} \vdash [\sigma_0]P \geqslant Q$, notice that the tree inferring this judgement is the same tree inferring $\Gamma_1, \overrightarrow{\beta^-} \vdash [\sigma]P \geqslant Q$ (since $[\sigma_0]P = [\sigma]P$), i.e., it is a subtree of $\Gamma_1 \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q$.

This way, by the induction hypothesis, $\Gamma_2, \overrightarrow{\beta^-} \vdash [\sigma_0]P \geqslant Q$. Combining it with $\Gamma_2, \overrightarrow{\beta^-} \vdash \sigma_0 : \overrightarrow{\alpha^-}$ by Rule ($\exists^{\geqslant}$), we obtain $\Gamma_2 \vdash \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q$.

The case of $\Gamma_1 \vdash \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M$ is symmetric. □

Lemma 21 (Weakening of subtyping context). *Suppose* Γ_1 *and* Γ_2 *are contexts and* $\Gamma_1 \subseteq \Gamma_2$. *Then*

+ $\Gamma_1 \vdash P \geqslant Q$ *implies* $\Gamma_2 \vdash P \geqslant Q$;
− $\Gamma_1 \vdash N \leqslant M$ *implies* $\Gamma_2 \vdash N \leqslant M$.

Proof. □

Lemma 22 (Reflexivity of subtyping). *Assuming all the types are well-formed in* Γ,

− $\Gamma \vdash N \leqslant N$
+ $\Gamma \vdash P \geqslant P$

Proof. Let us prove it by the size of N and mutually, P.

Case 1. $N = \alpha^-$
Then $\Gamma \vdash \alpha^- \leqslant \alpha^-$ is inferred immediately by Rule ($\mathrm{VAR}_-^{\leqslant}$).

Case 2. $N = \forall\overrightarrow{\alpha^+}.N'$ where $\overrightarrow{\alpha^+}$ is not empty
First, we rename $\overrightarrow{\alpha^+}$ to fresh $\overrightarrow{\beta^+}$ in $\forall\overrightarrow{\alpha^+}.N'$ to avoid name clashes: $\forall\overrightarrow{\alpha^+}.N' = \forall\overrightarrow{\beta^+}.[\overrightarrow{\alpha^+}/\overrightarrow{\beta^+}]N'$.
Then to infer $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N' \leqslant \forall\overrightarrow{\beta^+}.[\overrightarrow{\alpha^+}/\overrightarrow{\beta^+}]N'$ we can apply Rule ($\forall^{\leqslant}$), instantiating $\overrightarrow{\alpha^+}$ with $\overrightarrow{\beta^+}$:

- $\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing$ by choice of $\overrightarrow{\beta^+}$,
- $\Gamma, \overrightarrow{\beta^+} \vdash \beta_i^+$,
- $\Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{\beta^+}/\overrightarrow{\alpha^+}]N' \leqslant [\overrightarrow{\beta^+}/\overrightarrow{\alpha^+}]N'$ by the induction hypothesis, since the size of $[\overrightarrow{\beta^+}/\overrightarrow{\alpha^+}]N'$ is equal to the size of N', which is smaller than the size of $N = \forall\overrightarrow{\alpha^+}.N'$.

Case 3. $N = P \to M$
Then $\Gamma \vdash P \to M \leqslant P \to M$ is inferred by Rule ($\to^{\leqslant}$), since $\Gamma \vdash P \geqslant P$ and $\Gamma \vdash M \leqslant M$ hold the induction hypothesis.

Case 4. $N = \uparrow P$
Then $\Gamma \vdash \uparrow P \leqslant \uparrow P$ is inferred by Rule ($\uparrow^{\leqslant}$), since $\Gamma \vdash P \geqslant P$ holds by the induction hypothesis.

Case 5. The positive cases are symmetric to the negative ones.

□

Lemma 23 (Substitution preserves subtyipng). *Suppose that all mentioned types are well-formed in* Γ_1*, and* σ *is a substitution* $\Gamma_2 \vdash \sigma : \Gamma_1$.

- $-$ *If* $\Gamma_1 \vdash N \leqslant M$ *then* $\Gamma_2 \vdash [\sigma]N \leqslant [\sigma]M$.
- $+$ *If* $\Gamma_1 \vdash P \geqslant Q$ *then* $\Gamma_2 \vdash [\sigma]P \geqslant [\sigma]Q$.

Proof. We prove it by induction on the size of the derivation of $\Gamma_1 \vdash N \leqslant M$ and mutually, $\Gamma_1 \vdash P \geqslant Q$. Let us consider the last rule used in the derivation:

Case 1. Rule (Var$^{\leqslant}_{-}$). Then by inversion, $N = \alpha^-$ and $M = \alpha^-$. By reflexivity of subtyping (lemma 22), we have $\Gamma_2 \vdash [\sigma]\alpha^- \leqslant [\sigma]\alpha^-$, i.e. $\Gamma_2 \vdash [\sigma]N \leqslant [\sigma]M$, as required.

Case 2. Rule ($\forall^{\leqslant}$). Then by inversion, $N = \forall\overrightarrow{\alpha^+}.N'$, $M = \forall\overrightarrow{\beta^+}.M'$, where $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is not empty. Moreover, $\Gamma_1, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$ for some $\Gamma_1, \overrightarrow{\beta^+} \vdash \overrightarrow{P}$, and $\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing$.
Notice that since the derivation of $\Gamma_1, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant M'$ is a subderivation of the derivation of $\Gamma \vdash N \leqslant M$, its size is smaller, and hence, the induction hypothesis applies ($\Gamma_1, \overrightarrow{\beta^+} \vdash \sigma : \Gamma_1, \overrightarrow{\beta^+}$ by lemma 14) : $\Gamma_2, \overrightarrow{\beta^+} \vdash [\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant [\sigma]M'$.
Notice that by convention, $\overrightarrow{\alpha^+}$ and $\overrightarrow{\beta^+}$ are fresh, and thus, $[\sigma]\forall\overrightarrow{\alpha^+}.N' = \forall\overrightarrow{\alpha^+}.[\sigma]N'$ and $[\sigma]\forall\overrightarrow{\beta^+}.M' = \forall\overrightarrow{\beta^+}.[\sigma]M'$, which means that the required $\Gamma_2, \Gamma \vdash [\sigma]\forall\overrightarrow{\alpha^+}.N' \leqslant [\sigma]\forall\overrightarrow{\beta^+}.M'$ is rewritten as $\Gamma_2, \Gamma \vdash \forall\overrightarrow{\alpha^+}.[\sigma]N' \leqslant \forall\overrightarrow{\beta^+}.[\sigma]M'$.
To infer it, we apply Rule ($\forall^{\leqslant}$), instantiating α_i^+ with $[\sigma]P_i$:

- $\mathbf{fv}\,[\sigma]N \cap \overrightarrow{\beta^+} = \varnothing$;
- $\Gamma_2, \Gamma, \overrightarrow{\beta^+} \vdash [\sigma]P_i$, by lemma 5 since from the inversion, $\Gamma_1, \Gamma, \overrightarrow{\beta^+} \vdash P_i$;
- $\Gamma, \overrightarrow{\beta^+} \vdash [[\sigma]\overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N' \leqslant [\sigma]M'$ holds by lemma 13: Since $\overrightarrow{\alpha^+}$ is fresh, it is disjoint with the domain and the codomain of σ (Γ_1 and Γ_2), and thus, $[\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' = [\sigma \ll \overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N' = [[\sigma]\overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N'$. Then $\Gamma_2, \Gamma, \overrightarrow{\beta^+} \vdash [\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leqslant [\sigma]M'$ holds by the induction hypothesis.

Case 3. Rule ($\to^{\leqslant}$). Then by inversion, $N = P \to N_1$, $M = Q \to M_1$, $\Gamma \vdash P \geqslant Q$, and $\Gamma \vdash N_1 \leqslant M_1$. And by the induction hypothesis, $\Gamma' \vdash [\sigma]P \geqslant [\sigma]Q$ and $\Gamma' \vdash [\sigma]N_1 \leqslant [\sigma]M_1$. Then $\Gamma' \vdash [\sigma]N \leqslant [\sigma]M$, i.e. $\Gamma' \vdash [\sigma]P \to [\sigma]N_1 \leqslant [\sigma]Q \to [\sigma]M_1$, is inferred by Rule ($\to^{\leqslant}$).

Case 4. Rule ($\uparrow^{\leqslant}$). Then by inversion, $N = \uparrow P$, $M = \uparrow Q$, and $\Gamma \vdash P \simeq^{\leqslant} Q$, which by inversion means that $\Gamma \vdash P \geqslant Q$ and $\Gamma \vdash Q \geqslant P$. Then the induction hypothesis applies, and we have $\Gamma' \vdash [\sigma]P \geqslant [\sigma]Q$ and $\Gamma' \vdash [\sigma]Q \geqslant [\sigma]P$. Then by sequential application of Rule ($\simeq^{\leqslant}_{-}$) and Rule ($\uparrow^{\leqslant}$) to these judgments, we have $\Gamma' \vdash \uparrow[\sigma]P \leqslant \uparrow[\sigma]Q$, i.e. $\Gamma' \vdash [\sigma]N \leqslant [\sigma]M$, as required.

Case 5. The positive cases are proved symmetrically.

□

Corollary 9 (Substitution preserves subtyping induced equivalence). *Suppose that* $\Gamma \vdash \sigma : \Gamma_1$. *Then*

- $+$ *if* $\Gamma_1 \vdash P, \Gamma_1 \vdash Q$, *and* $\Gamma_1 \vdash P \simeq^{\leqslant} Q$ *then* $\Gamma \vdash [\sigma]P \simeq^{\leqslant} [\sigma]Q$
- $-$ *if* $\Gamma_1 \vdash N, \Gamma_1 \vdash M$, *and* $\Gamma_1 \vdash N \simeq^{\leqslant} M$ *then* $\Gamma \vdash [\sigma]N \simeq^{\leqslant} [\sigma]M$

Lemma 24 (Transitivity of subtyping). *Assuming the types are well-formed in* Γ,

- $-$ *if* $\Gamma \vdash N_1 \leqslant N_2$ *and* $\Gamma \vdash N_2 \leqslant N_3$ *then* $\Gamma \vdash N_1 \leqslant N_3$,
- $+$ *if* $\Gamma \vdash P_1 \geqslant P_2$ *and* $\Gamma \vdash P_2 \geqslant P_3$ *then* $\Gamma \vdash P_1 \geqslant P_3$.

Proof. To prove it, we formulate a stronger property, which will imply the required one, taking $\sigma = \Gamma \vdash \mathrm{id} : \Gamma$.

Assuming all the types are well-formed in Γ,

- if $\Gamma \vdash N \leqslant M_1, \Gamma \vdash M_2 \leqslant K$, and for $\Gamma' \vdash \sigma : \Gamma$, $[\sigma]M_1 = [\sigma]M_2$ then $\Gamma' \vdash [\sigma]N \leqslant [\sigma]K$
+ if $\Gamma \vdash P \geqslant Q_1, \Gamma \vdash Q_2 \geqslant R$, and for $\Gamma' \vdash \sigma : \Gamma$, $[\sigma]Q_1 = [\sigma]Q_2$ then $\Gamma' \vdash [\sigma]P \geqslant [\sigma]R$

We prove it by induction on $\mathsf{size}(\Gamma \vdash N \leqslant M_1) + \mathsf{size}(\Gamma \vdash M_2 \leqslant K)$ and mutually, on $\mathsf{size}(\Gamma \vdash P \geqslant Q_1) + \mathsf{size}(\Gamma \vdash Q_2 \geqslant R)$.

First, let us consider the 3 important cases.

Case 1. Let us consider the case when $M_1 = \forall \overrightarrow{\beta^+}_1.\alpha^-$. Then by lemma 19, $\Gamma \vdash N \leqslant M_1$ means that $N = \forall \overrightarrow{\alpha^+}.\alpha^-$. $[\sigma]M_1 = [\sigma]M_2$ means that $\forall \overrightarrow{\beta^+}_1.[\sigma]\alpha^- = [\sigma]M_2$. Applying σ to both sides of $\Gamma \vdash M_2 \leqslant K$ (by lemma 23), we obtain $\Gamma' \vdash [\sigma]M_2 \leqslant [\sigma]K$, that is $\Gamma' \vdash \forall \overrightarrow{\beta^+}_1.[\sigma]\alpha^- \leqslant [\sigma]K$. Since $\mathbf{fv}([\sigma]\alpha^-) \subseteq \Gamma, \alpha^-$, it is disjoint from $\overrightarrow{\alpha^+}$ and $\overrightarrow{\beta^+}_1$, This way, by corollary 7, $\Gamma' \vdash \forall \overrightarrow{\beta^+}_1.[\sigma]\alpha^- \leqslant [\sigma]K$ is equivalent to $\Gamma' \vdash [\sigma]\alpha^- \leqslant [\sigma]K$, which is equivalent to $\Gamma' \vdash \forall \overrightarrow{\alpha^+}.[\sigma]\alpha^- \leqslant [\sigma]K$, that is $\Gamma' \vdash [\sigma]N \leqslant [\sigma]K$.

Case 2. Let us consider the case when $M_2 = \forall \overrightarrow{\beta^+}_2.\alpha^-$. This case is symmetric to the previous one. Notice that lemma 19 and corollary 7 are agnostic to the side on which the the quantifiers occur, and thus, the proof stays the same.

Case 3. Let us decompose the types, by extracting the outer quantifiers:

- $N = \forall \overrightarrow{\alpha^+}.N'$, where $N' \neq \forall \ldots$,
- $M_1 = \forall \overrightarrow{\beta^+}_1.M_1'$, where $M_1' \neq \forall \ldots$,
- $M_2 = \forall \overrightarrow{\beta^+}_2.M_2'$, where $M_2' \neq \forall \ldots$,
- $K = \forall \overrightarrow{\gamma^+}.K'$, where $K' \neq \forall \ldots$.

and assume that at least one of $\overrightarrow{\alpha^+}$, $\overrightarrow{\beta^+}_1$, $\overrightarrow{\beta^+}_2$, and $\overrightarrow{\gamma^+}$ is not empty. Since $[\sigma]M_1 = [\sigma]M_2$, we have $\forall \overrightarrow{\beta^+}_1.[\sigma]M_1' = \forall \overrightarrow{\beta^+}_2.[\sigma]M_2'$, and since M_i' are not variables (which was covered by the previous cases) and do not start with $\forall$, $[\sigma]M_i'$ do not start with $\forall$ either, which means $\overrightarrow{\beta^+}_1 = \overrightarrow{\beta^+}_2$ and $[\sigma]M_1' = [\sigma]M_2'$. Let us rename $\overrightarrow{\beta^+}_1$ and $\overrightarrow{\beta^+}_2$ to $\overrightarrow{\beta^+}$. Then $M_1 = \forall \overrightarrow{\beta^+}.M_1'$ and $M_2 = \forall \overrightarrow{\beta^+}.M_2'$.

By lemma 18 applied twice to $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N' \leqslant \forall \overrightarrow{\beta^+}.M_1'$ and to $\Gamma \vdash \forall \overrightarrow{\beta^+}.M_2' \leqslant \forall \overrightarrow{\gamma^+}.K'$, we have the following:

(1) $\Gamma, \overrightarrow{\beta^+} \vdash [\vec{P}/\overrightarrow{\alpha^+}]N' \leqslant M_1'$ for some $\Gamma, \overrightarrow{\beta^+} \vdash \vec{P}$;
(2) $\Gamma, \overrightarrow{\gamma^+} \vdash [\vec{Q}/\overrightarrow{\beta^+}]M_2' \leqslant K'$ for some $\Gamma, \overrightarrow{\gamma^+} \vdash \vec{Q}$.

And since at least one of $\overrightarrow{\alpha^+}$, $\overrightarrow{\beta^+}$, and $\overrightarrow{\gamma^+}$ is not empty, either $\Gamma \vdash N \leqslant M_1$ or $\Gamma \vdash M_2 \leqslant K$ is inferred by Rule $(\forall^{\leqslant})$, meaning that either $\Gamma, \overrightarrow{\beta^+} \vdash [\vec{P}/\overrightarrow{\alpha^+}]N' \leqslant M_1'$ is a proper subderivation of $\Gamma \vdash N \leqslant M_1$ or $\Gamma, \overrightarrow{\gamma^+} \vdash [\vec{Q}/\overrightarrow{\beta^+}]M_2' \leqslant K'$ is a proper subderivation of $\Gamma \vdash M_2 \leqslant K$.

Notice that we can weaken and rearrange the contexts without changing the sizes of the derivations: $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\gamma^+} \vdash [\vec{P}/\overrightarrow{\alpha^+}]N' \leqslant M_1'$ and $\Gamma, \overrightarrow{\beta^+}, \overrightarrow{\gamma^+} \vdash [\vec{Q}/\overrightarrow{\beta^+}]M_2' \leqslant K'$. This way, the sum of the sizes of these derivations is smaller than the sum of the sizes of $\Gamma \vdash N \leqslant M_1$ and $\Gamma \vdash M_2 \leqslant K$. Let us apply the induction hypothesis to these derivations, with the substitution $\Gamma', \overrightarrow{\gamma^+} \vdash \sigma \circ (\vec{Q}/\overrightarrow{\beta^+}) : \Gamma, \overrightarrow{\beta^+}, \overrightarrow{\gamma^+}$ (lemma 14). To apply the induction hypothesis,

it is left to show that $\sigma \circ (\vec{Q}/\overrightarrow{\beta^+})$ unifies M_1' and $[\vec{Q}/\overrightarrow{\beta^+}]M_2'$:

$$\begin{aligned}
[\sigma \circ \vec{Q}/\overrightarrow{\beta^+}]M_1' &= [\sigma][\vec{Q}/\overrightarrow{\beta^+}]M_1' \\
&= [[\sigma]\vec{Q}/\overrightarrow{\beta^+}][\sigma]M_2' && \text{by lemma 13} \\
&= [[\sigma]\vec{Q}/\overrightarrow{\beta^+}][\sigma]M_2' && \text{Since } [\sigma]M_1' = [\sigma]M_2' \\
&= [\sigma][\vec{Q}/\overrightarrow{\beta^+}]M_2' && \text{by lemma 13} \\
&= [\sigma][\vec{Q}/\overrightarrow{\beta^+}][\vec{Q}/\overrightarrow{\beta^+}]M_2' && \text{Since } \Gamma, \overrightarrow{\gamma^+} \vdash \vec{Q}, \text{ and } (\Gamma, \overrightarrow{\gamma^+}) \cap \overrightarrow{\beta^+} = \varnothing \\
&= [\sigma \circ \vec{Q}/\overrightarrow{\beta^+}][\vec{Q}/\overrightarrow{\beta^+}]M_2'
\end{aligned}$$

This way the induction hypothesis gives us $\Gamma', \overrightarrow{\gamma^+} \vdash [\sigma][\vec{Q}/\overrightarrow{\beta^+}][\vec{P}/\overrightarrow{\alpha^+}]N' \leqslant [\sigma][\vec{Q}/\overrightarrow{\beta^+}]K'$, and since $\Gamma, \overrightarrow{\gamma^+} \vdash K'$, $[\vec{Q}/\overrightarrow{\beta^+}]K' = K'$, that is $\Gamma', \overrightarrow{\gamma^+} \vdash [\sigma][\vec{Q}/\overrightarrow{\beta^+}][\vec{P}/\overrightarrow{\alpha^+}]N' \leqslant [\sigma]K'$. Let us rewrite the substitution that we apply to N':

$$\begin{aligned}
[\sigma \circ \vec{Q}/\overrightarrow{\beta^+} \circ \vec{P}/\overrightarrow{\alpha^+}]N' &= [(\sigma \lessdot \vec{Q}/\overrightarrow{\beta^+}) \circ \sigma \circ \vec{P}/\overrightarrow{\alpha^+}]N' && \text{by lemma 13} \\
&= [(\sigma \lessdot \vec{Q}/\overrightarrow{\beta^+}) \circ (\sigma \lessdot \vec{P}/\overrightarrow{\alpha^+}) \circ \sigma]N' && \text{by lemma 13} \\
&= [(((\sigma \lessdot \vec{Q}/\overrightarrow{\beta^+}) \circ \sigma) \lessdot \vec{P}/\overrightarrow{\alpha^+}) \circ \sigma]N' && \text{Since } \mathbf{fv}([\sigma]N') \cap \overrightarrow{\beta^+} = \varnothing \\
&= [((\sigma \circ \vec{Q}/\overrightarrow{\beta^+}) \lessdot \vec{P}/\overrightarrow{\alpha^+}) \circ \sigma]N' && \text{by lemma 13} \\
&= [(\sigma \circ \vec{Q}/\overrightarrow{\beta^+}) \lessdot \vec{P}/\overrightarrow{\alpha^+}][\sigma]N'
\end{aligned}$$

Notice that $(\sigma \circ \vec{Q}/\overrightarrow{\beta^+}) \lessdot \vec{P}/\overrightarrow{\alpha^+}$ is a substitution that turns α_i^+ into $[\sigma \circ \vec{Q}/\overrightarrow{\beta^+}]P_i$, where $\Gamma', \overrightarrow{\gamma^+} \vdash [\sigma \circ \vec{Q}/\overrightarrow{\beta^+}]P_i$. This way, $\Gamma', \overrightarrow{\gamma^+} \vdash [(\sigma \circ \vec{Q}/\overrightarrow{\beta^+}) \lessdot \vec{P}/\overrightarrow{\alpha^+}][\sigma]N' \leqslant [\sigma]K'$ means $\Gamma \vdash \forall\overrightarrow{\alpha^+}.[\sigma]N' \leqslant \forall\overrightarrow{\gamma^+}.[\sigma]K'$ by lemma 18, that is $\Gamma \vdash [\sigma]N \leqslant [\sigma]K$, as required.

Now, we can assume that neither $\Gamma \vdash N \leqslant M_1$ nor $\Gamma \vdash M_2 \leqslant K$ is inferred by Rule ($\forall^{\leqslant}$), and that neither M_1 nor M_2 have the same outer constructor. Let us consider the shape of M_1.

Case 1. $M_1 = \alpha^-$ this case has been considered;

Case 2. $M_1 = \forall\overrightarrow{\beta^+}.M_1'$ this case has been considered;

Case 3. $M_1 = \uparrow Q_1$. Then as noted above, $[\sigma]M_1 = [\sigma]M_2$ means that $M_2 = \uparrow Q_2$ and $[\sigma]Q_1 = [\sigma]Q_2$. Moreover, $\Gamma \vdash N \leqslant \uparrow Q_1$ can only be inferred by Rule ($\uparrow^{\leqslant}$), and thus, $N = \uparrow P$, and by inversion, $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash Q_1 \geqslant P$. Analogously, $\Gamma \vdash \uparrow Q_2 \leqslant K$ means that $K = \uparrow R$, $\Gamma \vdash Q_2 \geqslant R$, and $\Gamma \vdash R \geqslant Q_2$.
Notice that the derivations of $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash Q_1 \geqslant P$ are proper sub-derivations of $\Gamma \vdash N \leqslant M_1$, and the derivations of $\Gamma \vdash Q_2 \geqslant R$ and $\Gamma \vdash R \geqslant Q_2$ are proper sub-derivations of $\Gamma \vdash M_2 \leqslant K$. This way, the induction hypothesis is applicable:
- applying the induction hypothesis to $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash Q_2 \geqslant R$ with $\Gamma' \vdash \sigma : \Gamma$ unifying Q_1 and Q_2, we obtain $\Gamma' \vdash [\sigma]P \geqslant [\sigma]R$;
- applying the induction hypothesis to $\Gamma \vdash R \geqslant Q_2$ and $\Gamma \vdash Q_1 \geqslant P$ with $\Gamma' \vdash \sigma : \Gamma$ unifying Q_2 and Q_1, we obtain $\Gamma' \vdash [\sigma]R \geqslant [\sigma]P$.

This way, by Rule ($\uparrow^{\leqslant}$), $\Gamma' \vdash [\sigma]N \leqslant [\sigma]K$, as required.

Case 4. $M_1 = Q_1 \to M_1'$. Then as noted above, $[\sigma]M_1 = [\sigma]M_2$ means that $M_2 = Q_2 \to M_2'$, $[\sigma]Q_1 = [\sigma]Q_2$, and $[\sigma]M_1' = [\sigma]M_2'$. Moreover, $\Gamma \vdash N \leqslant Q_1 \to M_1'$ can only be inferred by Rule ($\to^{\leqslant}$), and thus, $N = P \to N'$, and by inversion, $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash N' \leqslant M_1'$. Analogously, $\Gamma \vdash Q_2 \to M_2' \leqslant K$ means that $K = R \to K'$, $\Gamma \vdash Q_2 \geqslant R$, and $\Gamma \vdash M_2' \leqslant K'$.

Notice that the derivations of $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash N' \leqslant M_1'$ are proper sub-derivations of $\Gamma \vdash P \to N' \leqslant Q_1 \to M_1'$, and the derivations of $\Gamma \vdash Q_2 \geqslant R$ and $\Gamma \vdash M_2' \leqslant K'$ are proper sub-derivations of $\Gamma \vdash Q_2 \to M_2' \leqslant R \to K'$. This way, the induction hypothesis is applicable:

- applying the induction hypothesis to $\Gamma \vdash P \geqslant Q_1$ and $\Gamma \vdash Q_2 \geqslant R$ with $\Gamma' \vdash \sigma : \Gamma$ unifying Q_1 and Q_2, we obtain $\Gamma' \vdash [\sigma]P \geqslant [\sigma]R$;
- applying the induction hypothesis to $\Gamma \vdash N' \leqslant M_1'$ and $\Gamma \vdash M_2' \leqslant K'$ with $\Gamma' \vdash \sigma : \Gamma$ unifying M_1' and M_2', we obtain $\Gamma' \vdash [\sigma]N' \leqslant [\sigma]K'$.

This way, by Rule $(\to^{\leqslant})$, $\Gamma' \vdash [\sigma]P \to [\sigma]N' \leqslant [\sigma]R \to [\sigma]K'$, that is $\Gamma' \vdash [\sigma]N \leqslant [\sigma]K$, as required.

After that we consider all the analogous positive cases, and prove them symmetrically. □

Corollary 10 (Transitivity of equivalence). *Assuming the types are well-formed in* Γ,

– *if* $\Gamma \vdash N_1 \simeq^{\leqslant} N_2$ *and* $\Gamma \vdash N_2 \simeq^{\leqslant} N_3$ *then* $\Gamma \vdash N_1 \simeq^{\leqslant} N_3$,
+ *if* $\Gamma \vdash P_1 \simeq^{\leqslant} P_2$ *and* $\Gamma \vdash P_2 \simeq^{\leqslant} P_3$ *then* $\Gamma \vdash P_1 \simeq^{\leqslant} P_3$.

5.4 Equivalence

Lemma 25 (Declarative Equivalence is invariant under bijections). *Suppose* μ *is a bijection* $\mu : vars_1 \leftrightarrow vars_2$, *then*

+ $P_1 \simeq^{D} P_2$ *implies* $[\mu]P_1 \simeq^{D} [\mu]P_2$, *and there exists an inference tree of* $[\mu]P_1 \simeq^{D} [\mu]P_2$ *with the same shape as the one inferring* $P_1 \simeq^{D} P_2$;
– $N_1 \simeq^{D} N_2$ *implies* $[\mu]N_1 \simeq^{D} [\mu]N_2$, *and there exists an inference tree of* $[\mu]N_1 \simeq^{D} [\mu]N_2$ *with the same shape as the one inferring* $N_1 \simeq^{D} N_2$.

Proof. We prove it by induction on $P_1 \simeq^{D} P_2$ and mutually, on $N_1 \simeq^{D} N_2$. Let us consider the last rule used in the derivation.

Case 1. Rule $(\forall^{\simeq^{D}})$

Then we decompose N_1 as $\forall\overrightarrow{\alpha^{+}}_1.M_1$ and N_2 as $\forall\overrightarrow{\alpha^{+}}_2.M_2$, where M_1 and M_2 do not start with $\forall$-quantifiers. where $|\overrightarrow{\alpha^{+}}_1| + |\overrightarrow{\alpha^{+}}_2| > 0$. By convention, let us assume that $\overrightarrow{\alpha^{+}}_1$ and $\overrightarrow{\alpha^{+}}_2$ are disjoint form $vars_2$ and $vars_1$.

By inversion, $\overrightarrow{\alpha^{+}}_1 \cap \mathbf{fv}\, M_2 = \varnothing$ and $M_1 \simeq^{D} [\mu']M_2$ for some bijection $\mu' : (\overrightarrow{\alpha^{+}}_2 \cap \mathbf{fv}\, M_2) \leftrightarrow (\overrightarrow{\alpha^{+}}_1 \cap \mathbf{fv}\, M_1)$. Then let us apply the induction hypothesis to $M_1 \simeq^{D} [\mu']M_2$ to obtain $[\mu]M_1 \simeq^{D} [\mu][\mu']M_2$ inferred by the tree of the same shape as $M_1 \simeq^{D} [\mu']M_2$.

Notice that $[\mu]M_1$ and $[\mu]M_2$ do not start with $\forall$, That is $[\mu]\forall\overrightarrow{\alpha^{+}}_1.M_1 \simeq^{D} [\mu]\forall\overrightarrow{\alpha^{+}}_2.M_2$, rewritten as $\forall\overrightarrow{\alpha^{+}}_1.[\mu]M_1 \simeq^{D} \forall\overrightarrow{\alpha^{+}}_2.[\mu]M_2$, can be inferred by Rule $(\forall^{\simeq^{D}})$:

(1) $\overrightarrow{\alpha^{+}}_1$ is disjoint from $vars_2 \cup \mathbf{fv}\, M_2 \subseteq \mathbf{fv}\, [\mu]M_2$;
(2) $[\mu]M_1 \simeq^{D} [\mu'][\mu]M_2$ because $[\mu'][\mu]M_2 = [\mu][\mu']M_2$ (by corollary 5: $\mu' : (\overrightarrow{\alpha^{+}}_2 \cap \mathbf{fv}\, M_2) \leftrightarrow (\overrightarrow{\alpha^{+}}_1 \cap \mathbf{fv}\, M_1)$, $\mu : vars_1 \leftrightarrow vars_2$, $vars_1$ is disjoint from $\overrightarrow{\alpha^{+}}_2$ and $\overrightarrow{\alpha^{+}}_1$; $\overrightarrow{\alpha^{+}}_2$ is disjoint from $vars_1$ and $vars_2$)

Notice that it is the same rule as the one inferring $N_1 \simeq^{D} N_2$, and thus, the shapes of the trees are the same.

Case 2. Rule $(\text{Var}^{\simeq^{D}}_{-})$

Then $N_1 = N_2 = \alpha^{-}$, and the required $[\mu]\alpha^{-} = [\mu]\alpha^{-}$ is also inferred by Rule $(\text{Var}^{\simeq^{D}}_{-})$, since $[\mu]\alpha^{-}$ is a variable.

Case 3. Rule ($\rightarrow^{\simeq^D}$)
Then we are proving that $P_1 \rightarrow M_1 \simeq^D P_2 \rightarrow M_2$ implies $[\mu](P_1 \rightarrow M_1) \simeq^D [\mu](P_2 \rightarrow M_2)$ (preserving the tree structure).
By inversion, we have $P_1 \simeq^D P_2$ and $M_1 \simeq^D M_2$, and thus, by the induction hypothesis, $[\mu]P_1 \simeq^D [\mu]P_2$ and $[\mu]M_1 \simeq^D [\mu]M_2$. Then $[\mu](P_1 \rightarrow M_1) \simeq^D [\mu](P_2 \rightarrow M_2)$, or in other words, $[\mu]P_1 \rightarrow [\mu]M_1 \simeq^D [\mu]P_2 \rightarrow [\mu]M_2$, is inferred by the same rule—Rule ($\rightarrow^{\simeq^D}$).
Case 4. Rule ($\uparrow^{\simeq^D}$) This case is done by similar congruent arguments as the previous one.
Case 5. The positive cases are proved symmetrically.

□

Lemma 26 (Declarative equivalence is transitive).

+ *if* $P_1 \simeq^D P_2$ *and* $P_2 \simeq^D P_3$ *then* $P_1 \simeq^D P_3$,
− *if* $N_1 \simeq^D N_2$ *and* $N_2 \simeq^D N_3$ *then* $N_1 \simeq^D N_3$.

Proof. We prove it by $\text{size}(P_1 \simeq^D P_2) + \text{size}(P_2 \simeq^D P_3)$ and mutually, $\text{size}(N_1 \simeq^D N_2) + \text{size}(N_2 \simeq^D N_3)$, where by size, we mean the size of the nodes in the corresponding inference tree.

Case 1. First, let us consider the case when either $N_1 \simeq^D N_2$ or $N_2 \simeq^D N_3$ is inferred by Rule ($\forall^{\simeq^D}$). Let us decompose N_1, N_2, and N_3 as follows: $N_1 = \forall\overrightarrow{\alpha^+}_1.M_1$, $N_2 = \forall\overrightarrow{\alpha^+}_2.M_2$, and $N_3 = \forall\overrightarrow{\alpha^+}_3.M_3$.
Then by inversion of $\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D \forall\overrightarrow{\alpha^+}_2.M_2$ (or if $\overrightarrow{\alpha^+}_1$ and $\overrightarrow{\alpha^+}_2$ are both empty, by assumption) :
(1) $\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\, M_2 = \varnothing$ and
(2) there exists a bijection on variables $\mu_1 : (\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, M_2) \leftrightarrow (\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\, M_1)$ such that $M_1 \simeq^D [\mu_1]M_2$.
Analogously, $\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D \forall\overrightarrow{\alpha^+}_2.M_2$ implies:
(1) $\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, M_3 = \varnothing$ and
(2) $M_2 \simeq^D [\mu_2]M_3$ for some bijection $\mu_2 : (\overrightarrow{\alpha^+}_3 \cap \mathbf{fv}\, M_3) \leftrightarrow (\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, M_2)$.
Notice that either $M_1 \simeq^D [\mu_1]M_2$ is inferred by a proper sub-tree of $\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D \forall\overrightarrow{\alpha^+}_2.M_2$ or $M_2 \simeq^D [\mu_2]M_3$ is inferred by a proper sub-tree of $\forall\overrightarrow{\alpha^+}_2.M_2 \simeq^D \forall\overrightarrow{\alpha^+}_3.M_3$.
Then by lemma 25, $[\mu_1]M_2 \simeq^D [\mu_1 \circ \mu_2]M_3$ and moreover, $\text{size}([\mu_1]M_2 \simeq^D [\mu_1 \circ \mu_2]M_3) = \text{size}(M_2 \simeq^D [\mu_2]M_3)$.
Since at least one of the trees inferring $M_1 \simeq^D [\mu_1]M_2$ and $M_2 \simeq^D [\mu_2]M_3$ is a proper sub-tree of the corresponding original tree, $\text{size}(M_1 \simeq^D [\mu_1]M_2) + \text{size}(M_2 \simeq^D [\mu_2]M_3) < \text{size}(\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D \forall\overrightarrow{\alpha^+}_2.M_2) + \text{size}(\forall\overrightarrow{\alpha^+}_2.M_2 \simeq^D \forall\overrightarrow{\alpha^+}_3.M_3)$, i.e., the induction hypothesis is applicable.
By the induction hypothesis, $M_1 \simeq^D [\mu_1 \circ \mu_2]M_3$. Where $\mu_1 \circ \mu_2$ is a bijection on variables $\mu_1 \circ \mu_2 : (\overrightarrow{\alpha^+}_3 \cap \mathbf{fv}\, M_3) \leftrightarrow (\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\, M_1)$. Then $\forall\overrightarrow{\alpha^+}_1.M_1 \simeq^D \forall\overrightarrow{\alpha^+}_3.M_3$ by Rule ($\forall^{\simeq^D}$).
Once this case has been considered, we can assume that neither $N_1 \simeq^D N_2$ nor $N_2 \simeq^D N_3$ is inferred by Rule ($\forall^{\simeq^D}$).
Case 2. $N_1 \simeq^D N_2$ is inferred by Rule ($\textsc{Var}_{-}^{\simeq^D}$)
Then $N_1 = N_2 = \alpha^-$, and thus, $N_1 \simeq^D N_3$ holds since $N_2 \simeq^D N_3$.
Case 3. $N_1 \simeq^D N_2$ is inferred by Rule ($\rightarrow^{\simeq^D}$)
Then $N_1 = P_1 \rightarrow M_1$ and $N_2 = P_2 \rightarrow M_2$, and by inversion, $P_1 \simeq^D P_2$ and $M_1 \simeq^D M_2$.
Moreover, since N_3 does not start with $\forall$, $N_2 \simeq^D N_3$ is also inferred by Rule ($\rightarrow^{\simeq^D}$), which means that $N_3 = P_3 \rightarrow M_3$, $P_2 \simeq^D P_3$, and $M_2 \simeq^D M_3$.

Then by the induction hypothesis, $P_1 \simeq^D P_3$ and $M_1 \simeq^D M_3$, and thus, $P_1 \to M_1 \simeq^D P_3 \to M_3$ by Rule $(\to^{\simeq^D})$.

Case 4. $N_1 \simeq^D N_2$ is inferred by Rule $(\to^{\simeq^D})$

For this case, the reasoning is the same as for the previous one.

Case 5. The positive cases are proved symmetrically.

□

Lemma 27 (Type well-formedness is invariant under equivalence). *Mutual subtyping implies declarative equivalence.*

$+$ *if* $P \simeq^D Q$ *then* $\Gamma \vdash P \iff \Gamma \vdash Q$,
$-$ *if* $N \simeq^D M$ *then* $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. We prove it by induction on $P \simeq^D Q$ and mutually, on $N \simeq^D M$. Let us consider the last rule used in the derivation.

Case 1. Rule $(\text{Var}^{\simeq^D}_{-})$, that is $N \simeq^D M$ has shape $\alpha^- \simeq^D \alpha^-$.

Than $\Gamma \vdash P \iff \Gamma \vdash Q$ is rewritten as $\Gamma \vdash \alpha^- \iff \Gamma \vdash \alpha^-$, which holds trivially.

Case 2. Rule $(\uparrow^{\simeq^D})$, that is $N \simeq^D M$ has shape $\uparrow P \simeq^D \uparrow Q$.

By inversion, $P \simeq^D Q$, and by the induction hypothesis, $\Gamma \vdash P \iff \Gamma \vdash Q$. Also notice that $\Gamma \vdash \uparrow P \iff \Gamma \vdash P$ and $\Gamma \vdash \uparrow Q \iff \Gamma \vdash Q$ by inversion and Rule $(\uparrow^{\text{WF}})$. This way, $\Gamma \vdash \uparrow P \iff \Gamma \vdash P \iff \Gamma \vdash Q \iff \Gamma \vdash \uparrow Q$.

Case 3. Rule $(\to^{\simeq^D})$, that is $N \simeq^D M$ has shape $P \to N' \simeq^D Q \to M'$.

Then by inversion, $P \simeq^D Q$ and $N' \simeq^D M'$, and by the induction hypothesis, $\Gamma \vdash P \iff \Gamma \vdash Q$ and $\Gamma \vdash N' \iff \Gamma \vdash M'$. $\Gamma \vdash P \to N' \iff \Gamma \vdash P$ and $\Gamma \vdash N'$ by inversion and Rule $(\to^{\text{V}}$

$\iff \Gamma \vdash Q$ and $\Gamma \vdash M'$ as noted above

$\iff \Gamma \vdash Q \to M'$ by Rule $(\to^{\text{WF}})$ and inversi

Case 4. Rule $(\forall^{\simeq^D})$, that is $N \simeq^D M$ has shape $\forall\overrightarrow{\alpha^+}.N' \simeq^D \forall\overrightarrow{\beta^+}.M'$.

By inversion, $\forall\overrightarrow{\alpha^+}.N' \simeq^D \forall\overrightarrow{\beta^+}.M'$ means that $\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing$ and that there exists a bijection on variables $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M') \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N')$ such that $N' \simeq^D [\mu]M'$.

By inversion and Rule $(\forall^{\text{WF}})$, $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N'$ is equivalent to $\Gamma, \overrightarrow{\alpha^+} \vdash N'$, and by lemma 4, it is equivalent to $\Gamma, (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N') \vdash N'$, which, by the induction hypothesis, is equivalent to $\Gamma, (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N') \vdash [\mu]M'$.

Analogously, $\Gamma \vdash \forall\overrightarrow{\beta^+}.M'$ is equivalent to $\Gamma, (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M') \vdash M'$. By lemma 5, it implies $\Gamma, (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M') \vdash [\mu]M'$. And vice versa, $\Gamma, (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M') \vdash [\mu]M'$ implies $\Gamma, (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M') \vdash [\mu^{-1}][\mu]M'$.

This way, both $\Gamma \vdash \forall\overrightarrow{\alpha^+}.N'$ and $\Gamma \vdash \forall\overrightarrow{\beta^+}.M'$ are equivalent to $\Gamma, (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N') \vdash [\mu]M'$.

Case 5. For the cases of the positive types, the proofs are symmetric.

□

Corollary 11 (Normalization preserves well-formedness).

$+$ $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}\,(P)$,
$-$ $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}\,(N)$

Proof. Immediately from lemmas 27 and 44. □

Corollary 12 (Normalization preserves well-formedness of substitution).
$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}\,(\sigma) : \Gamma_1$

Proof. Let us prove the forward direction. Suppose that $\alpha^{\pm} \in \Gamma_1$. Let us show that $\Gamma_2 \vdash [\mathbf{nf}\,(\sigma)]\alpha^{\pm}$. By the definition of substitution normalization, $[\mathbf{nf}\,(\sigma)]\alpha^{\pm} = \mathbf{nf}\,([\sigma]\alpha^{\pm})$. Then by corollary 11, to show $\Gamma_2 \vdash \mathbf{nf}\,([\sigma]\alpha^{\pm})$, it suffices to show $\Gamma_2 \vdash [\sigma]\alpha^{\pm}$, which holds by the assumption $\Gamma_2 \vdash \sigma : \Gamma_1$.

The backward direction is proved analogously. □

Lemma 28 (Normalization preserves substitution signature). *Suppose that σ is a substitution, Γ_1 and Γ_2 are contexts. Then $\Gamma_2 \vdash \sigma : \Gamma_1$ implies $\Gamma_2 \vdash \mathbf{nf}\,(\sigma) : \Gamma_1$.*

Proof. Suppose that $\alpha^{\pm} \in \Gamma_1$. Then by corollary 11, $\Gamma_2 \vdash \mathbf{nf}\,([\sigma]\alpha^{\pm}) = [\mathbf{nf}\,(\sigma)]\alpha^{\pm}$ is equivalent to $\Gamma_2 \vdash [\sigma]\alpha^{\pm}$.

Suppose that $\alpha^{\pm} \notin \Gamma_1$. $\Gamma_2 \vdash \sigma : \Gamma_1$ means that $[\sigma]\alpha^{\pm} = \alpha^{\pm}$, and then $[\mathbf{nf}\,(\sigma)]\alpha^{\pm} = \mathbf{nf}\,([\sigma]\alpha^{\pm}) = \mathbf{nf}\,(\alpha^{\pm}) = \alpha^{\pm}$. □

Lemma 29 (Soundness of equivalence). *Declarative equivalence implies mutual subtyping.*

+ *if $\Gamma \vdash P$, $\Gamma \vdash Q$, and $P \simeq^{D} Q$ then $\Gamma \vdash P \simeq^{\leqslant} Q$,*
− *if $\Gamma \vdash N$, $\Gamma \vdash M$, and $N \simeq^{D} M$ then $\Gamma \vdash N \simeq^{\leqslant} M$.*

Proof. We prove it by mutual induction on $P \simeq^{D} Q$ and $N \simeq^{D} M$.

Case 1. $\alpha^{-} \simeq^{D} \alpha^{-}$

Then $\Gamma \vdash \alpha^{-} \leqslant \alpha^{-}$ by Rule ($\mathrm{Var}^{\leqslant}_{-}$), which immediately implies $\Gamma \vdash \alpha^{-} \simeq^{\leqslant} \alpha^{-}$ by Rule ($\simeq^{\leqslant}_{-}$).

Case 2. $\uparrow P \simeq^{D} \uparrow Q$

Then by inversion of Rule ($\uparrow^{\leqslant}$), $P \simeq^{D} Q$, and from the induction hypothesis, $\Gamma \vdash P \simeq^{\leqslant} Q$, and (by symmetry) $\Gamma \vdash Q \simeq^{\leqslant} P$.

When Rule ($\uparrow^{\leqslant}$) is applied to $\Gamma \vdash P \simeq^{\leqslant} Q$, it gives us $\Gamma \vdash \uparrow P \leqslant \uparrow Q$; when it is applied to $\Gamma \vdash Q \simeq^{\leqslant} P$, we obtain $\Gamma \vdash \uparrow Q \leqslant \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \simeq^{\leqslant} \uparrow Q$.

Case 3. $P \to N \simeq^{D} Q \to M$

Then by inversion of Rule ($\to^{\leqslant}$), $P \simeq^{D} Q$ and $N \simeq^{D} M$. By the induction hypothesis, $\Gamma \vdash P \simeq^{\leqslant} Q$ and $\Gamma \vdash N \simeq^{\leqslant} M$, which means by inversion: (i) $\Gamma \vdash P \geqslant Q$, (ii) $\Gamma \vdash Q \geqslant P$, (iii) $\Gamma \vdash N \leqslant M$, (iv) $\Gamma \vdash M \leqslant N$. Applying Rule ($\to^{\leqslant}$) to (i) and (iii), we obtain $\Gamma \vdash P \to N \leqslant Q \to M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \to M \leqslant P \to N$. Together, it implies $\Gamma \vdash P \to N \simeq^{\leqslant} Q \to M$.

Case 4. $\forall \overrightarrow{\alpha^{+}}.N \simeq^{D} \forall \overrightarrow{\beta^{+}}.M$

Then by inversion, there exists bijection $\mu : (\overrightarrow{\beta^{+}} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^{+}} \cap \mathbf{fv}\,N)$, such that $N \simeq^{D} [\mu]M$. By the induction hypothesis, $\Gamma, \overrightarrow{\alpha^{+}} \vdash N \simeq^{\leqslant} [\mu]M$. From corollary 9 and the fact that μ is bijective, we also have $\Gamma, \overrightarrow{\beta^{+}} \vdash [\mu^{-1}]N \simeq^{\leqslant} M$.

Let us construct a substitution $\overrightarrow{\alpha^{+}} \vdash \overrightarrow{P}/\overrightarrow{\beta^{+}} : \overrightarrow{\beta^{+}}$ by extending μ with arbitrary positive types on $\overrightarrow{\beta^{+}} \backslash \mathbf{fv}\,M$.

Notice that $[\mu]M = [\overrightarrow{P}/\overrightarrow{\beta^{+}}]M$, and therefore, $\Gamma, \overrightarrow{\alpha^{+}} \vdash N \simeq^{\leqslant} [\mu]M$ implies $\Gamma, \overrightarrow{\alpha^{+}} \vdash [\overrightarrow{P}/\overrightarrow{\beta^{+}}]M \leqslant N$. Then by Rule ($\forall^{\leqslant}$), $\Gamma \vdash \forall \overrightarrow{\beta^{+}}.M \leqslant \forall \overrightarrow{\alpha^{+}}.N$.

Analogously, we construct the substitution from μ^{-1}, and use it to instantiate $\overrightarrow{\alpha^{+}}$ in the application of Rule ($\forall^{\leqslant}$) to infer $\Gamma \vdash \forall \overrightarrow{\alpha^{+}}.N \leqslant \forall \overrightarrow{\beta^{+}}.M$.

This way, $\Gamma \vdash \forall \overrightarrow{\beta^{+}}.M \leqslant \forall \overrightarrow{\alpha^{+}}.N$ and $\Gamma \vdash \forall \overrightarrow{\alpha^{+}}.N \leqslant \forall \overrightarrow{\beta^{+}}.M$ gives us $\Gamma \vdash \forall \overrightarrow{\beta^{+}}.M \simeq^{\leqslant} \forall \overrightarrow{\alpha^{+}}.N$.

Case 5. For the cases of the positive types, the proofs are symmetric.

□

Corollary 13 (Normalization is sound w.r.t. subtyping-induced equivalence).

+ *if* $\Gamma \vdash P$ *then* $\Gamma \vdash P \simeq^{\leqslant} \mathbf{nf}(P)$,
− *if* $\Gamma \vdash N$ *then* $\Gamma \vdash N \simeq^{\leqslant} \mathbf{nf}(N)$.

Proof. Immediately from lemmas 29 and 44 and corollary 11. □

Corollary 14 (Normalization preserves subtyping). *Assuming all the types are well-formed in context* Γ,

+ $\Gamma \vdash P \geqslant Q \iff \Gamma \vdash \mathbf{nf}(P) \geqslant \mathbf{nf}(Q)$,
− $\Gamma \vdash N \leqslant M \iff \Gamma \vdash \mathbf{nf}(N) \leqslant \mathbf{nf}(M)$.

Proof.

+ ⇒ Let us assume $\Gamma \vdash P \geqslant Q$. By corollary 13, $\Gamma \vdash P \simeq^{\leqslant} \mathbf{nf}(P)$ and $\Gamma \vdash Q \simeq^{\leqslant} \mathbf{nf}(Q)$, in particular, by inversion, $\Gamma \vdash \mathbf{nf}(P) \geqslant P$ and $\Gamma \vdash Q \geqslant \mathbf{nf}(Q)$. Then by transitivity of subtyping (lemma 24), $\Gamma \vdash \mathbf{nf}(P) \geqslant \mathbf{nf}(Q)$.
  ⇐ Let us assume $\Gamma \vdash \mathbf{nf}(P) \geqslant \mathbf{nf}(Q)$. Also by corollary 13 and inversion, $\Gamma \vdash P \geqslant \mathbf{nf}(P)$ and $\Gamma \vdash \mathbf{nf}(Q) \geqslant Q$. Then by the transitivity, $\Gamma \vdash P \geqslant Q$.
− The negative case is proved symmetrically.

□

Lemma 30 (Subtyping induced by disjoint substitutions). *If two disjoint substitutions induce subtyping, they are degenerate (so is the subtyping). Suppose that* $\Gamma \vdash \sigma_1 : \Gamma_1$ *and* $\Gamma \vdash \sigma_2 : \Gamma_1$, *where* $\Gamma_i \subseteq \Gamma$ *and* $\Gamma_1 \cap \Gamma_2 = \varnothing$. *Then*

− *assuming* $\Gamma \vdash N, \Gamma \vdash [\sigma_1]N \leqslant [\sigma_2]N$ *implies* $\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathrm{id} : \mathbf{fv}\, N$
+ *assuming* $\Gamma \vdash P, \Gamma \vdash [\sigma_1]P \geqslant [\sigma_2]P$ *implies* $\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathrm{id} : \mathbf{fv}\, P$

Proof. Proof by induciton on $\Gamma \vdash N$ (and mutually on $\Gamma \vdash P$).

Case 1. $N = \alpha^-$
Then $\Gamma \vdash [\sigma_1]N \leqslant [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\alpha^- \leqslant [\sigma_2]\alpha^-$. Let us consider the following cases:

a. $\alpha^- \notin \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then $\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathrm{id} : \alpha^-$ holds immediately, since $[\sigma_i]\alpha^- = [\mathrm{id}]\alpha^- = \alpha^-$ and $\Gamma \vdash \alpha^- \simeq^{\leqslant} \alpha^-$.

b. $\alpha^- \in \Gamma_1$ and $\alpha^- \in \Gamma_2$
This case is not possible by assumption: $\Gamma_1 \cap \Gamma_2 = \varnothing$.

c. $\alpha^- \in \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then we have $\Gamma \vdash [\sigma_1]\alpha^- \leqslant \alpha^-$, which by corollary 8 means $\Gamma \vdash [\sigma_1]\alpha^- \simeq^{\leqslant} \alpha^-$, and hence, $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mathrm{id} : \alpha^-$.
$\Gamma \vdash \sigma_2 \simeq^{\leqslant} \mathrm{id} : \alpha^-$ holds since $[\sigma_2]\alpha^- = \alpha^-$, similarly to case 1.*a*.

d. $\alpha^- \notin \Gamma_1$ and $\alpha^- \in \Gamma_2$
Then we have $\Gamma \vdash \alpha^- \leqslant [\sigma_2]\alpha^-$, which by corollary 8 means $\Gamma \vdash \alpha^- \simeq^{\leqslant} [\sigma_2]\alpha^-$, and hence, $\Gamma \vdash \sigma_2 \simeq^{\leqslant} \mathrm{id} : \alpha^-$.
$\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mathrm{id} : \alpha^-$ holds since $[\sigma_1]\alpha^- = \alpha^-$, similarly to case 1.*a*.

Case 2. $N = \forall \overrightarrow{\alpha^+}.M$
Then by inversion, $\Gamma, \overrightarrow{\alpha^+} \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\forall\overrightarrow{\alpha^+}.M \leqslant [\sigma_2]\forall\overrightarrow{\alpha^+}.M$. By the congruence of substitution and by the inversion of Rule ($\forall^{\leqslant}$), $\Gamma, \overrightarrow{\alpha^+} \vdash [\overrightarrow{Q}/\overrightarrow{\alpha^+}][\sigma_1]M \leqslant [\sigma_2]M$, where $\Gamma, \overrightarrow{\alpha^+} \vdash Q_i$. Let us denote the (Kleisli) composition of σ_1 and $\overrightarrow{Q}/\overrightarrow{\alpha^+}$ as σ_1', noting that $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' : \Gamma_1, \overrightarrow{\alpha^+}$, and $(\Gamma_1, \overrightarrow{\alpha^+}) \cap \Gamma_2 = \varnothing$.

Let us apply the induction hypothesis to M and the substitutions σ_1' and σ_2 with $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_1']M \leqslant [\sigma_2]M$ to obtain:

$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_1' \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{1}$$

$$\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{2}$$

Then $\Gamma \vdash \sigma_2 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ holds by strengthening of 2: for any $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M \backslash \overrightarrow{\alpha^+}$, $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]\beta^\pm \simeq^{\leqslant} \beta^\pm$ is strengthened to $\Gamma \vdash [\sigma_2]\beta^\pm \simeq^{\leqslant} \beta^\pm$, because $\mathbf{fv}\, [\sigma_2]\beta^\pm = \mathbf{fv}\, \beta^\pm = \{\beta^\pm\} \subseteq \Gamma$.

To show that $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$, let us take an arbitrary $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M = \mathbf{fv}\, M \backslash \overrightarrow{\alpha^+}$.

$\beta^\pm = [\mathsf{id}]\beta^\pm$ by definition of id

$\simeq^{\leqslant} [\sigma_1']\beta^\pm$ by 1

$= [\vec{Q}/\overrightarrow{\alpha^+}][\sigma_1]\beta^\pm$ by definition of σ_1'

$= [\sigma_1]\beta^\pm$ because $\overrightarrow{\alpha^+} \cap \mathbf{fv}\, [\sigma_1]\beta^\pm \subseteq \overrightarrow{\alpha^+} \cap \Gamma = \varnothing$

This way, $\Gamma \vdash [\sigma_1]\beta^\pm \simeq^{\leqslant} \beta^\pm$ for any $\beta^\pm \in \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$ and thus, $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, \forall\overrightarrow{\alpha^+}.M$.

Case 3. $N = P \to M$

Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. $\Gamma \vdash [\sigma_1]N \leqslant [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1](P \to M) \leqslant [\sigma_2](P \to M)$, then by congruence of substitution, $\Gamma \vdash [\sigma_1]P \to [\sigma_1]M \leqslant [\sigma_2]P \to [\sigma_2]M$, then by inversion $\Gamma \vdash [\sigma_1]P \geqslant [\sigma_2]P$ and $\Gamma \vdash [\sigma_1]M \leqslant [\sigma_2]M$.

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant [\sigma_2]P$ and to $\Gamma \vdash [\sigma_1]M \leqslant [\sigma_2]M$, we obtain (respectively):

$$\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, P \tag{3}$$

$$\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, M \tag{4}$$

Noting that $\mathbf{fv}\, (P \to M) = \mathbf{fv}\, P \cup \mathbf{fv}\, M$, we combine eqs. (3) and (4) to conclude: $\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, (P \to M)$.

Case 4. $N = \uparrow P$

Then by inversion, $\Gamma \vdash P$. $\Gamma \vdash [\sigma_1]N \leqslant [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\uparrow P \leqslant [\sigma_2]\uparrow P$, then by congruence of substitution and by inversion, $\Gamma \vdash [\sigma_1]P \geqslant [\sigma_2]P$

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geqslant [\sigma_2]P$, we obtain $\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$.

Since $\mathbf{fv}\, \uparrow P = \mathbf{fv}\, P$, we can conclude: $\Gamma \vdash \sigma_i \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, \uparrow P$.

Case 5. The positive cases are proved symmetrically.

□

Corollary 15 (Substitution cannot induce proper subtypes or supertypes). *Assuming all mentioned types are well-formed in* Γ *and* σ *is a substitution* $\Gamma \vdash \sigma : \Gamma$,

$$\Gamma \vdash [\sigma]N \leqslant N \;\Rightarrow\; \Gamma \vdash [\sigma]N \simeq^{\leqslant} N \text{ and } \Gamma \vdash \sigma \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, N$$

$$\Gamma \vdash N \leqslant [\sigma]N \;\Rightarrow\; \Gamma \vdash N \simeq^{\leqslant} [\sigma]N \text{ and } \Gamma \vdash \sigma \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, N$$

$$\Gamma \vdash [\sigma]P \geqslant P \;\Rightarrow\; \Gamma \vdash [\sigma]P \simeq^{\leqslant} P \text{ and } \Gamma \vdash \sigma \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$$

$$\Gamma \vdash P \geqslant [\sigma]P \;\Rightarrow\; \Gamma \vdash P \simeq^{\leqslant} [\sigma]P \text{ and } \Gamma \vdash \sigma \simeq^{\leqslant} \mathsf{id} : \mathbf{fv}\, P$$

Lemma 31. *Assuming that the mentioned types (P, Q, N, and M) are well-formed in* Γ *and that the substitutions (*σ_1 *and* σ_2*) have signature* $\Gamma \vdash \sigma_i : \Gamma$,

+ *if* $\Gamma \vdash [\sigma_1]P \geqslant Q$ *and* $\Gamma \vdash [\sigma_2]Q \geqslant P$
  then there exists a bijection $\mu : \mathbf{fv}\, P \leftrightarrow \mathbf{fv}\, Q$ *such that* $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mu : \mathbf{fv}\, P$ *and* $\Gamma \vdash \sigma_2 \simeq^{\leqslant} \mu^{-1} : \mathbf{fv}\, Q$;
− *if* $\Gamma \vdash [\sigma_1]N \leqslant M$ *and* $\Gamma \vdash [\sigma_2]N \leqslant M$
  then there exists a bijection $\mu : \mathbf{fv}\, N \leftrightarrow \mathbf{fv}\, M$ *such that* $\Gamma \vdash \sigma_1 \simeq^{\leqslant} \mu : \mathbf{fv}\, N$ *and* $\Gamma \vdash \sigma_2 \simeq^{\leqslant} \mu^{-1} : \mathbf{fv}\, M$.

Proof.

+ Applying σ_2 to both sides of $\Gamma \vdash [\sigma_1]P \geqslant Q$ (by lemma 23), we have: $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant [\sigma_2]Q$. Composing it with $\Gamma \vdash [\sigma_2]Q \geqslant P$ by transitivity (lemma 24), we have $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geqslant P$. Then by corollary 15, $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq^{\leqslant} \mathrm{id} : \mathbf{fv}\, P$. By a symmetric argument, we also have: $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq^{\leqslant} \mathrm{id} : \mathbf{fv}\, Q$.
  Now, we prove that $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq^{\leqslant} \mathrm{id} : \mathbf{fv}\, P$ and $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq^{\leqslant} \mathrm{id} : \mathbf{fv}\, Q$ implies that σ_1 and σ_1 are (equivalent to) mutually inverse bijections.
  To do so, it suffices to prove that
  (i) for any $\alpha^{\pm} \in \mathbf{fv}\, P$ there exists $\beta^{\pm} \in \mathbf{fv}\, Q$ such that $\Gamma \vdash [\sigma_1]\alpha^{\pm} \simeq^{\leqslant} \beta^{\pm}$ and $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq^{\leqslant} \alpha^{\pm}$; and
  (ii) for any $\beta^{\pm} \in \mathbf{fv}\, Q$ there exists $\alpha^{\pm} \in \mathbf{fv}\, P$ such that $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq^{\leqslant} \alpha^{\pm}$ and $\Gamma \vdash [\sigma_1]\alpha^{\pm} \simeq^{\leqslant} \beta^{\pm}$.
  Then these correspondences between $\mathbf{fv}\, P$ and $\mathbf{fv}\, Q$ are mutually inverse functions, since for any $\beta^{\pm}$ there can be at most one $\alpha^{\pm}$ such that $\Gamma \vdash [\sigma_2]\beta^{\pm} \simeq^{\leqslant} \alpha^{\pm}$ (and vice versa).
  (i) Let us take $\alpha^{\pm} \in \mathbf{fv}\, P$.
    (a) if $\alpha^{\pm}$ is positive ($\alpha^{\pm} = \alpha^{+}$), from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{+} \simeq^{\leqslant} \alpha^{+}$, by corollary 8, we have $[\sigma_2][\sigma_1]\alpha^{+} = \exists\overrightarrow{\beta^{-}}.\alpha^{+}$.
    What shape can $[\sigma_1]\alpha^{+}$ have? It cannot be $\exists\overrightarrow{\alpha^{-}}.{\downarrow}N$ (for potentially empty $\overrightarrow{\alpha^{-}}$), because the outer constructor $\downarrow$ would remain after the substitution σ_2, whereas $\exists\overrightarrow{\beta^{-}}.\alpha^{+}$ does not have $\downarrow$. The only case left is $[\sigma_1]\alpha^{+} = \exists\overrightarrow{\alpha^{-}}.\gamma^{+}$.
    Notice that $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}.\gamma^{+} \simeq^{\leqslant} \gamma^{+}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{+} \simeq^{\leqslant} \gamma^{+}$. Also notice that $[\sigma_2]\exists\overrightarrow{\alpha^{-}}.\gamma^{+} = \exists\overrightarrow{\beta^{-}}.\alpha^{+}$ implies $\Gamma \vdash [\sigma_2]\gamma^{+} \simeq^{\leqslant} \alpha^{+}$.
    (b) if $\alpha^{\pm}$ is negative ($\alpha^{\pm} = \alpha^{-}$) from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{-} \simeq^{\leqslant} \alpha^{-}$, by corollary 8, we have $[\sigma_2][\sigma_1]\alpha^{-} = \forall\overrightarrow{\beta^{+}}.\alpha^{-}$.
    What shape can $[\sigma_1]\alpha^{-}$ have? It cannot be $\forall\overrightarrow{\alpha^{+}}.{\uparrow}P$ nor $\forall\overrightarrow{\alpha^{+}}.P \to M$ (for potentially empty $\overrightarrow{\alpha^{+}}$), because the outer constructor ($\to$ or $\uparrow$), remaining after the substitution σ_2, is however absent in the resulting $\forall\overrightarrow{\beta^{+}}.\alpha^{-}$. Hence, the only case left is $[\sigma_1]\alpha^{-} = \forall\overrightarrow{\alpha^{+}}.\gamma^{-}$ Notice that $\Gamma \vdash \gamma^{-} \simeq^{\leqslant} \forall\overrightarrow{\alpha^{+}}.\gamma^{-}$, meaning that $\Gamma \vdash [\sigma_1]\alpha^{-} \simeq^{\leqslant} \gamma^{-}$. Also notice that $[\sigma_2]\forall\overrightarrow{\alpha^{+}}.\gamma^{-} = \forall\overrightarrow{\beta^{+}}.\alpha^{-}$ implies $\Gamma \vdash [\sigma_2]\gamma^{-} \simeq^{\leqslant} \alpha^{-}$.
  (ii) The proof is symmetric: We swap P and Q, σ_1 and σ_2, and exploit $\Gamma \vdash [\sigma_1][\sigma_2]\alpha^{\pm} \simeq^{\leqslant} \alpha^{\pm}$ instead of $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^{\pm} \simeq^{\leqslant} \alpha^{\pm}$.
− The proof is symmetric to the positive case.

□

Lemma 32 (Equivalent substitution act equivalently). *Suppose that* $\Gamma' \vdash \sigma_1 : \Gamma$ *and* $\Gamma' \vdash \sigma_2 : \Gamma$ *are substitutions equivalent on their domain, that is* $\Gamma' \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \Gamma$. *Then*

+ *for any* $\Gamma \vdash P$, $\Gamma' \vdash [\sigma_1]P \simeq^{\leqslant} [\sigma_2]P$;
− *for any* $\Gamma \vdash N$, $\Gamma' \vdash [\sigma_1]N \simeq^{\leqslant} [\sigma_2]N$.

Proof. We prove it by induction on P (and mutually on N).

Case 1. $N = \alpha^-$
Then since by inversion, $\alpha^- \in \Gamma, \Gamma' \vdash [\sigma_1]\alpha^- \simeq^{\leqslant} [\sigma_2]\alpha^-$ holds by definition of $\Gamma' \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \Gamma$.

Case 2. $N = \uparrow P$
Then by inversion, $\Gamma \vdash P$. By the induction hypothesis, $\Gamma' \vdash [\sigma_1]P \simeq^{\leqslant} [\sigma_2]P$, Then by Rule $(\uparrow^{\leqslant})$, $\Gamma' \vdash \uparrow[\sigma_1]P \leqslant \uparrow[\sigma_2]P$, and symmetrically, $\Gamma' \vdash \uparrow[\sigma_2]P \leqslant \uparrow[\sigma_1]P$, together meaning that $\Gamma' \vdash \uparrow[\sigma_1]P \simeq^{\leqslant} \uparrow[\sigma_2]P$, or equivalently, $\Gamma' \vdash [\sigma_1]\uparrow P \simeq^{\leqslant} [\sigma_2]\uparrow P$.

Case 3. $N = P \to M$
Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. By the induction hypothesis, $\Gamma' \vdash [\sigma_1]P \simeq^{\leqslant} [\sigma_2]P$ and $\Gamma' \vdash [\sigma_1]M \simeq^{\leqslant} [\sigma_2]M$, that is $\Gamma' \vdash [\sigma_1]P \geqslant [\sigma_2]P$, $\Gamma' \vdash [\sigma_2]P \geqslant [\sigma_1]P$, $\Gamma' \vdash [\sigma_1]M \leqslant [\sigma_2]M$, and $\Gamma' \vdash [\sigma_2]M \leqslant [\sigma_1]M$. Then by Rule $(\to^{\leqslant})$, $\Gamma' \vdash [\sigma_1]P \to [\sigma_1]M \leqslant [\sigma_2]P \to [\sigma_2]M$, and again by Rule $(\to^{\leqslant})$, $\Gamma' \vdash [\sigma_2]P \to [\sigma_2]M \leqslant [\sigma_1]P \to [\sigma_1]M$. This way, $\Gamma' \vdash [\sigma_1]P \to [\sigma_1]M \simeq^{\leqslant} [\sigma_2]P \to [\sigma_2]M$, or equivalently, $\Gamma' \vdash [\sigma_1](P \to M) \simeq^{\leqslant} [\sigma_2](P \to M)$.

Case 4. $N = \forall \overrightarrow{\alpha^+}.M$ We can assume that $\overrightarrow{\alpha^+}$ is disjoint from Γ and Γ'. By inversion, $\Gamma \vdash \forall \overrightarrow{\alpha^+}.M$ implies $\Gamma, \overrightarrow{\alpha^+} \vdash M$. Notice that $\Gamma' \vdash \sigma_i : \Gamma$ and $\Gamma' \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \Gamma$ can be extended to $\Gamma', \overrightarrow{\alpha^+} \vdash \sigma_i : \Gamma, \overrightarrow{\alpha^+}$ and $\Gamma', \overrightarrow{\alpha^+} \vdash \sigma_1 \simeq^{\leqslant} \sigma_2 : \Gamma, \overrightarrow{\alpha^+}$ by lemma 14. Then by the induction hypothesis, $\Gamma', \overrightarrow{\alpha^+} \vdash [\sigma_1]M \simeq^{\leqslant} [\sigma_2]M$, meaning by inversion that $\Gamma', \overrightarrow{\alpha^+} \vdash [\sigma_1]M \leqslant [\sigma_2]M$ and $\Gamma', \overrightarrow{\alpha^+} \vdash [\sigma_2]M \leqslant [\sigma_1]M$.
To infer $\Gamma' \vdash \forall \overrightarrow{\alpha^+}.[\sigma_1]M \leqslant \forall \overrightarrow{\alpha^+}.[\sigma_2]M$, we apply Rule $(\forall^{\leqslant})$ with the substitution $\Gamma', \overrightarrow{\alpha^+} \vdash \mathrm{id} : \overrightarrow{\alpha^+}$, noting that $\Gamma', \overrightarrow{\alpha^+} \vdash [\mathrm{id}][\sigma_1]M \leqslant [\sigma_2]M$ holds since $\Gamma', \overrightarrow{\alpha^+} \vdash [\sigma_1]M \leqslant [\sigma_2]M$, as noted above.
Symmetrically, we infer $\Gamma' \vdash \forall \overrightarrow{\alpha^+}.[\sigma_2]M \leqslant \forall \overrightarrow{\alpha^+}.[\sigma_1]M$, which together with $\Gamma' \vdash \forall \overrightarrow{\alpha^+}.[\sigma_1]M \leqslant \forall \overrightarrow{\alpha^+}.[\sigma_2]M$ means $\Gamma' \vdash \forall \overrightarrow{\alpha^+}.[\sigma_1]M \simeq^{\leqslant} \forall \overrightarrow{\alpha^+}.[\sigma_2]M$, or equivalently, $\Gamma' \vdash [\sigma_1]\forall \overrightarrow{\alpha^+}.M \simeq^{\leqslant} [\sigma_2]\forall \overrightarrow{\alpha^+}.M$.

Case 5. The positive cases are proved symmetrically.

□

Lemma 33 (Equivalence of polymorphic types).

- *For* $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N$ *and* $\Gamma \vdash \forall \overrightarrow{\beta^+}.M$,
  if $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \simeq^{\leqslant} \forall \overrightarrow{\beta^+}.M$ *then there exists a bijection* $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv}\, M \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$ *such that* $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq^{\leqslant} [\mu]M$,
- *For* $\Gamma \vdash \exists \overrightarrow{\alpha^-}.P$ *and* $\Gamma \vdash \exists \overrightarrow{\beta^-}.Q$,
  if $\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \simeq^{\leqslant} \exists \overrightarrow{\beta^-}.Q$ *then there exists a bijection* $\mu : \overrightarrow{\beta^-} \cap \mathbf{fv}\, Q \leftrightarrow \overrightarrow{\alpha^-} \cap \mathbf{fv}\, P$ *such that* $\Gamma, \overrightarrow{\beta^-} \vdash P \simeq^{\leqslant} [\mu]Q$.

Proof.

- First, by α-conversion, we ensure $\overrightarrow{\alpha^+} \cap \mathbf{fv}\, M = \varnothing$ and $\overrightarrow{\beta^+} \cap \mathbf{fv}\, N = \varnothing$. By inversion, $\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \simeq^{\leqslant} \forall \overrightarrow{\beta^+}.M$ implies
  (1) $\Gamma, \overrightarrow{\beta^+} \vdash [\sigma_1]N \leqslant M$ for $\Gamma, \overrightarrow{\beta^+} \vdash \sigma_1 : \overrightarrow{\alpha^+}$ and
  (2) $\Gamma, \overrightarrow{\alpha^+} \vdash [\sigma_2]M \leqslant N$ for $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 : \overrightarrow{\beta^+}$.
  To apply lemma 31, we weaken and rearrange the contexts, and extend the substitutions to act as identity outside of their initial domain:
  (1) $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \leqslant M$ for $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 : \Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$ and
  (2) $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \leqslant N$ for $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 : \Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$.

Then from lemma 31, there exists a bijection $\mu : \mathbf{fv}\, M \leftrightarrow \mathbf{fv}\, N$ such that $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq^{\leqslant} \mu : \mathbf{fv}\, M$ and $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 \simeq^{\leqslant} \mu^{-1} : \mathbf{fv}\, N$.
Let us show that $\mu|_{\overrightarrow{\beta^+}}$ is the appropriate candidate.

First, we show that if we restrict the domain of μ to $\overrightarrow{\beta^+}$, its range will be contained in $\overrightarrow{\alpha^+}$.
Let us take $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv}\, M$ and assume $[\mu]\gamma^+ \notin \overrightarrow{\alpha^+}$. Then since $\Gamma, \overrightarrow{\beta^+} \vdash \sigma_1 : \overrightarrow{\alpha^+}$, σ_1 acts as identity outside of $\overrightarrow{\alpha^+}$, i.e. $[\sigma_1][\mu]\gamma^+ = [\mu]\gamma^+$ (notice that γ^+ is in the domain of μ). Since $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_1 \simeq^{\leqslant} \mu^{-1} : \mathbf{fv}\, N$, application of σ_1 to $[\mu]\gamma^+ \in \mathbf{fv}\, N$ is equivalent to application of μ^{-1}, then $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}][\mu]\gamma^+ \simeq^{\leqslant} [\mu]\gamma^+$, i.e. $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \gamma^+ \simeq^{\leqslant} [\mu]\gamma^+$, which means $\gamma^+ \in \mathbf{fv}\, [\mu]\gamma^+ \subseteq \mathbf{fv}\, N$. By assumption, $\gamma^+ \in \overrightarrow{\beta^+} \cap \mathbf{fv}\, M$, i.e. $\overrightarrow{\beta^+} \cap \mathbf{fv}\, N \neq \varnothing$, hence contradiction.
Second, we will show $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$.
Since $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 : \overrightarrow{\beta^+}$ and $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq^{\leqslant} \mu : \mathbf{fv}\, M$, we have $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq^{\leqslant} \mu|_{\overrightarrow{\beta^+}} : \mathbf{fv}\, M$: for any $\alpha^{\pm} \in \mathbf{fv}\, M \backslash \overrightarrow{\beta^+}$, $[\sigma_2]\alpha^{\pm} = \alpha^{\pm}$ since $\Gamma, \overrightarrow{\alpha^+} \vdash \sigma_2 : \overrightarrow{\beta^+}$, and $[\mu|_{\overrightarrow{\beta^+}}]\alpha^{\pm} = \alpha^{\pm}$ by definition of substitution restriction; for $\beta^+ \in \overrightarrow{\beta^+}$, $[\mu|_{\overrightarrow{\beta^+}}]\beta^+ = [\mu]\beta^+$, and thus, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]\beta^+ \simeq^{\leqslant} [\mu]\beta^+$ can be rewritten to $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]\beta^+ \simeq^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]\beta^+$.
By lemma 32, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash \sigma_2 \simeq^{\leqslant} \mu|_{\overrightarrow{\beta^+}} : \mathbf{fv}\, M$ implies $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_2]M \simeq^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$. By similar reasoning, $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\sigma_1]N \simeq^{\leqslant} [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N$.
This way, by transitivity of subtyping (lemma 24),

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu^{-1}|_{\overrightarrow{\alpha^+}}]N \leqslant M \tag{5}$$

$$\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu|_{\overrightarrow{\beta^+}}]M \leqslant N \tag{6}$$

By applying $\mu|_{\overrightarrow{\beta^+}}$ to both sides of 5 (lemma 23), we have $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash [\mu|_{\overrightarrow{\beta^+}}][\mu^{-1}|_{\overrightarrow{\alpha^+}}]N \leqslant [\mu|_{\overrightarrow{\beta^+}}]M$. By contracting $\mu^{-1}|_{\overrightarrow{\alpha^+}} \circ \mu|_{\overrightarrow{\beta^+}} = \mu|^{-1}_{\overrightarrow{\beta^+}} \circ \mu|_{\overrightarrow{\beta^+}}$ (notice that $\mathbf{fv}\, N \cap \overrightarrow{\beta^+} = \varnothing$), we have $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \leqslant [\mu|_{\overrightarrow{\beta^+}}]M$, which together with 6 means $\Gamma, \overrightarrow{\alpha^+}, \overrightarrow{\beta^+} \vdash N \simeq^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$, and by strengthening, $\Gamma, \overrightarrow{\alpha^+} \vdash N \simeq^{\leqslant} [\mu|_{\overrightarrow{\beta^+}}]M$.

\+ The proof is symmetric to the proof of the negative case.

□

Lemma 34 (Completeness of equivalence). *Mutual subtyping implies declarative equivalence. Assuming all the types below are well-formed in* Γ*:*

\+ *if* $\Gamma \vdash P \simeq^{\leqslant} Q$ *then* $P \simeq^{D} Q$,
\− *if* $\Gamma \vdash N \simeq^{\leqslant} M$ *then* $N \simeq^{D} M$.

Proof. \− Induction on the sum of sizes of N and M. By inversion, $\Gamma \vdash N \simeq^{\leqslant} M$ means $\Gamma \vdash N \leqslant M$ and $\Gamma \vdash M \leqslant N$. Let us consider the last rule that forms $\Gamma \vdash N \leqslant M$:

Case 1. Rule ($\text{Var}^{\leqslant}_{-}$) i.e. $\Gamma \vdash N \leqslant M$ is of the form $\Gamma \vdash \alpha^- \leqslant \alpha^-$
Then $N \simeq^{D} M$ (i.e. $\alpha^- \simeq^{D} \alpha^-$) holds immediately by Rule ($\text{Var}^{\simeq^D}_{-}$).

Case 2. Rule ($\uparrow^{\leqslant}$) i.e. $\Gamma \vdash N \leqslant M$ is of the form $\Gamma \vdash \uparrow P \leqslant \uparrow Q$
Then by inversion, $\Gamma \vdash P \simeq^{\leqslant} Q$, and by induction hypothesis, $P \simeq^{D} Q$. Then $N \simeq^{D} M$ (i.e. $\uparrow P \simeq^{D} \uparrow Q$) holds by Rule ($\uparrow^{\simeq^D}$).

Case 3. Rule $(\to^{\leqslant})$ i.e. $\Gamma \vdash N \leqslant M$ is of the form $\Gamma \vdash P \to N' \leqslant Q \to M'$
Then by inversion, $\Gamma \vdash P \geqslant Q$ and $\Gamma \vdash N' \leqslant M'$. Notice that $\Gamma \vdash M \leqslant N$ is of the form $\Gamma \vdash Q \to M' \leqslant P \to N'$, which by inversion means $\Gamma \vdash Q \geqslant P$ and $\Gamma \vdash M' \leqslant N'$.
This way, $\Gamma \vdash Q \simeq^{\leqslant} P$ and $\Gamma \vdash M' \simeq^{\leqslant} N'$. Then by induction hypothesis, $Q \simeq^{D} P$ and $M' \simeq^{D} N'$. Then $N \simeq^{D} M$ (i.e. $P \to N' \simeq^{D} Q \to M'$) holds by Rule $(\to^{\simeq^{D}})$.

Case 4. Rule $(\forall^{\leqslant})$ i.e. $\Gamma \vdash N \leqslant M$ is of the form $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N' \leqslant \forall\overrightarrow{\beta^{+}}.M'$
Then by lemma 33, $\Gamma \vdash \forall\overrightarrow{\alpha^{+}}.N' \simeq^{\leqslant} \forall\overrightarrow{\beta^{+}}.M'$ means that there exists a bijection $\mu : \overrightarrow{\beta^{+}} \cap \mathbf{fv}\, M' \leftrightarrow \overrightarrow{\alpha^{+}} \cap \mathbf{fv}\, N'$ such that $\Gamma, \overrightarrow{\alpha^{+}} \vdash [\mu]M' \simeq^{\leqslant} N'$.
Notice that the application of bijection μ to M' does not change its size (which is less than the size of M), hence the induction hypothesis applies. This way, $[\mu]M' \simeq^{D} N'$ (and by symmetry, $N' \simeq^{D} [\mu]M'$) holds by induction. Then we apply Rule $(\forall^{\simeq^{D}})$ to get $\forall\overrightarrow{\alpha^{+}}.N' \simeq^{D} \forall\overrightarrow{\beta^{+}}.M'$, i.e. $N \simeq^{D} M$.

\+ The proof is symmetric to the proof of the negative case.

□

Corollary 16 (Normalization is complete w.r.t. subtyping-induced equivalence). *Assuming all the types below are well-formed in* Γ*:*

\+ *if* $\Gamma \vdash P \simeq^{\leqslant} Q$ *then* $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$,
− *if* $\Gamma \vdash N \simeq^{\leqslant} M$ *then* $\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$.

Proof. Immediately from lemmas 34 and 46. □

Lemma 35 (Algorithmization of subtyping-induced equivalence). *Mutual subtyping is the equality of normal forms. Assuming all the types below are well-formed in* Γ*:*

\+ $\Gamma \vdash P \simeq^{\leqslant} Q \iff \mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$,
− $\Gamma \vdash N \simeq^{\leqslant} M \iff \mathbf{nf}\,(N) = \mathbf{nf}\,(M)$.

Proof. Let us prove the positive case, the negative case is symmetric. We prove both directions of $\iff$ separately:

$\Rightarrow$ exactly corollary 16;
$\Leftarrow$ by lemmas 29 and 57.

□

Lemma 36 (Substitution preserves declarative equivalence). *Suppose that* σ *is a substitution. Then*

\+ $P \simeq^{D} Q$ *implies* $[\sigma]P \simeq^{D} [\sigma]Q$
− $N \simeq^{D} M$ *implies* $[\sigma]N \simeq^{D} [\sigma]M$

Proof.
$P \simeq^{D} Q \Rightarrow \mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$ by lemma 35
$\Rightarrow [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(Q)$
$\Rightarrow \mathbf{nf}\,([\sigma]P) = \mathbf{nf}\,([\sigma]Q)$ by lemma 45
$\Rightarrow [\sigma]P \simeq^{D} [\sigma]Q$ by lemma 35
□

5.5 Variable Ordering

Observation 1 (Ordering is deterministic). *If* **ord** *vars* **in** $N = \vec{\alpha}_1$ *and* **ord** *vars* **in** $N = \vec{\alpha}_2$ *then* $\vec{\alpha}_1 = \vec{\alpha}_2$. *If* **ord** *vars* **in** $P = \vec{\alpha}_1$ *and* **ord** *vars* **in** $P = \vec{\alpha}_2$ *then* $\vec{\alpha}_1 = \vec{\alpha}_2$. *This way, we can use* **ord** *vars* **in** N *and as a function on* N*, and* **ord** *vars* **in** P *as a function on* P.

Proof. By mutual structural induction on N and P. Notice that the shape of the term N or P uniquely determines the last used inference rule, and all the premises are deterministic on the input. □

Lemma 37 (Soundness of variable ordering). *Variable ordering extracts used free variables.*

- $-$ $\mathbf{ord}\ vars\ \mathbf{in}\ N = vars \cap \mathbf{fv}\ N$ *(as sets)*
- $+$ $\mathbf{ord}\ vars\ \mathbf{in}\ P = vars \cap \mathbf{fv}\ P$ *(as sets)*

Proof. We prove it by mutual induction on $\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}$ and $\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}$. The only non-trivial cases are Rule ($\to^{\text{Ord}}$) and Rule ($\forall^{\text{Ord}}$).

Case 1. Rule ($\to^{\text{Ord}}$) Then the inferred ordering judgement has shape $\mathbf{ord}\ vars\ \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)$ and by inversion, $\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}_1$ and $\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}_2$.
By definition of free variables, $vars \cap \mathbf{fv}\ P \to N = vars \cap \mathbf{fv}\ P \cup vars \cap \mathbf{fv}\ N$, and since by the induction hypothesis $vars \cap \mathbf{fv}\ P = \vec{\alpha}_1$ and $vars \cap \mathbf{fv}\ N = \vec{\alpha}_2$, we have $vars \cap \mathbf{fv}\ P \to N = \vec{\alpha}_1 \cup \vec{\alpha}_2$.
On the other hand, As a set, $\vec{\alpha}_1 \cup \vec{\alpha}_2$ is equal to $\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)$.

Case 2. Rule ($\forall^{\text{Ord}}$). Then the inferred ordering judgement has shape $\mathbf{ord}\ vars\ \mathbf{in}\ \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}$, and by inversion, $vars \cap \overrightarrow{\alpha^+} = \varnothing$ $\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}$. The latter implies that $vars \cap \mathbf{fv}\ N = \vec{\alpha}$.
We need to show that $vars \cap \mathbf{fv}\ \forall \overrightarrow{\alpha^+}.N = \vec{\alpha}$, or equivalently, that $vars \cap (\mathbf{fv}\ N \backslash \overrightarrow{\alpha^+}) = vars \cap \mathbf{fv}\ N$, which holds since $vars \cap \overrightarrow{\alpha^+} = \varnothing$.

□

Corollary 17 (Additivity of ordering). *Variable ordering is additive (in terms of set union) with respect to its first argument.*

- $-$ $\mathbf{ord}\ (vars_1 \cup vars_2)\ \mathbf{in}\ N = \mathbf{ord}\ vars_1\ \mathbf{in}\ N \cup \mathbf{ord}\ vars_2\ \mathbf{in}\ N$ *(as sets)*
- $+$ $\mathbf{ord}\ (vars_1 \cup vars_2)\ \mathbf{in}\ P = \mathbf{ord}\ vars_1\ \mathbf{in}\ P \cup \mathbf{ord}\ vars_2\ \mathbf{in}\ P$ *(as sets)*

Lemma 38 (Weakening of ordering). *Only used variables matter in the first argument of the ordering,*

- $-$ $\mathbf{ord}\ (vars \cap \mathbf{fv}\ N)\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ N$
- $+$ $\mathbf{ord}\ (vars \cap \mathbf{fv}\ P)\ \mathbf{in}\ P = \mathbf{ord}\ vars\ \mathbf{in}\ P$

Proof. Mutual structural induction on N and P.

Case 1. If N is a variable α^-, we notice that $\alpha^- \in vars$ is equivalent to $\alpha^- \in vars \cap \alpha^-$.

Case 2. If N has shape $\uparrow P$, then the required property holds immediately by the induction hypothesis, since $\mathbf{fv}\ (\uparrow P) = \mathbf{fv}\ (P)$.

Case 3. If the term has shape $P \to N$ then Rule ($\to^{\text{Ord}}$) was applied to infer $\mathbf{ord}\ (vars \cap (\mathbf{fv}\ P \cup \mathbf{fv}\ N))\ \mathbf{in}\ P \to N$ and $\mathbf{ord}\ vars\ \mathbf{in}\ P \to N$. By inversion, the result of $\mathbf{ord}\ (vars \cap (\mathbf{fv}\ P \cup \mathbf{fv}\ N))\ \mathbf{in}\ P \to N$ depends on $A = \mathbf{ord}\ (vars \cap (\mathbf{fv}\ P \cup \mathbf{fv}\ N))\ \mathbf{in}\ P$ and $B = \mathbf{ord}\ (vars \cap (\mathbf{fv}\ P \cup \mathbf{fv}\ N))\ \mathbf{in}\ N$. The result of $\mathbf{ord}\ vars\ \mathbf{in}\ P \to N$ depends on $X = \mathbf{ord}\ vars\ \mathbf{in}\ P$ and $Y = \mathbf{ord}\ vars\ \mathbf{in}\ N$.
Let us show that that $A = B$ and $X = Y$, so the results are equal. By the induction hypothesis and set properties, $\mathbf{ord}\ (vars \cap (\mathbf{fv}\ P \cup \mathbf{fv}\ N))\ \mathbf{in}\ P = \mathbf{ord}\ (vars \cap (\mathbf{fv}\ P \cup \mathbf{fv}\ N)) \cap \mathbf{fv}\ (P)\ \mathbf{in}\ P = \mathbf{ord}\ vars \cap \mathbf{fv}\ (P)\ \mathbf{in}\ P = \mathbf{ord}\ vars\ \mathbf{in}\ P$. Analogously, $\mathbf{ord}\ (vars \cap (\mathbf{fv}\ P \cup \mathbf{fv}\ N))\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ N$.

Case 4. If the term has shape $\forall \overrightarrow{\alpha^+}.N$, we can assume that $\overrightarrow{\alpha^+}$ is disjoint from $vars$, since we operate on alpha-equivalence classes. Then using the induction hypothesis, set properties and Rule ($\forall^{\text{Ord}}$): $\mathbf{ord}\ vars \cap (\mathbf{fv}\ (\forall \overrightarrow{\alpha^+}.N))\ \mathbf{in}\ \forall \overrightarrow{\alpha^+}.N = \mathbf{ord}\ vars \cap (\mathbf{fv}\ (N) \backslash \overrightarrow{\alpha^+})\ \mathbf{in}\ N = \mathbf{ord}\ vars \cap (\mathbf{fv}\ (N) \backslash \overrightarrow{\alpha^+}) \cap \mathbf{fv}\ (N)\ \mathbf{in}\ N = \mathbf{ord}\ vars \cap \mathbf{fv}\ (N)\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ N$.

□

Corollary 18 (Idempotency of ordering).

– *If* $\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}$ *then* $\mathbf{ord}\ \vec{\alpha}\ \mathbf{in}\ N = \vec{\alpha}$,
+ *If* $\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}$ *then* $\mathbf{ord}\ \vec{\alpha}\ \mathbf{in}\ P = \vec{\alpha}$;

Proof. By lemmas 37 and 38. □

Next we make a set-theoretical observation that will be useful further. In general, any injective function (its image) distributes over set intersection. However, for convenience we allow the bijections on variables to be applied *outside of their domains* (as identities), which may violate the injectivity. To deal with these cases, we define a special notion of bijections collision-free on certain sets in such a way that a bijection that is collision-free on P and Q, distributes over intersection of P and Q.

Definition 29 (Collision free bijection). *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is* ***collision free on sets*** *P and Q if and only if*

(1) $\mu(P \cap A) \cap Q = \varnothing$
(2) $\mu(Q \cap A) \cap P = \varnothing$

Observation 2. *Suppose that $\mu : A \leftrightarrow B$ is a bijection between two sets of variables, and μ is collision free on P and Q. Then $\mu(P \cap Q) = \mu(P) \cap \mu(Q)$.*

Lemma 39 (Distributivity of renaming over variable ordering). *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

– *If μ is collision free on vars and* $\mathbf{fv}\ N$ *then* $[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) = \mathbf{ord}\ ([\mu]vars)\ \mathbf{in}\ [\mu]N$
+ *If μ is collision free on vars and* $\mathbf{fv}\ P$ *then* $[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ P) = \mathbf{ord}\ ([\mu]vars)\ \mathbf{in}\ [\mu]P$

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$
Then
$$\begin{aligned}
[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) &= [\mu](\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^-) \\
&= [\mu]\alpha^- && \text{by Rule } (\textsc{Var}^{\textsc{Ord}}_{+\in}) \\
&= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars) \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ \beta^- && \text{by Rule } (\textsc{Var}^{\textsc{Ord}}_{+\in}), \text{ because } \beta^- \in [\mu]vars \\
&= \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]\alpha^-
\end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$
Notice that $[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) = [\mu](\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^-) = \cdot$ by Rule $(\textsc{Var}^{\textsc{Ord}}_{+\notin})$. On the other hand, $\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]\alpha^- = \mathbf{ord}\ [\mu]vars\ \mathbf{in}\ \alpha^- = \cdot$ The latter equality is from Rule $(\textsc{Var}^{\textsc{Ord}}_{+\notin})$, because μ is collision free on $vars$ and $\mathbf{fv}\ N$, so $\mathbf{fv}\ N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$
Then $[\mu](\mathbf{ord}\ vars\ \mathbf{in}\ N) = [\mu](\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^-) = \cdot$ by Rule $(\textsc{Var}^{\textsc{Ord}}_{+\notin})$. To prove that $\mathbf{ord}\ [\mu]vars\ \mathbf{in}\ [\mu]\alpha^- = \cdot$, we apply Rule $(\textsc{Var}^{\textsc{Ord}}_{+\notin})$. Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.
(i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ, which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.

(ii) Since μ is collision free on *vars* and $\mathbf{fv}\, N$, $\mu(A \cap \mathbf{fv}\, N) \ni \mu(\alpha^-) \notin vars$.

d. $\alpha^- \notin A$ but $\alpha^- \in vars$

$\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\alpha^- = \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\alpha^- = \alpha^-$. The latter is by Rule ($\textsc{Var}^{\textsc{Ord}}_{+\notin}$), because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) = [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$

$$\begin{aligned}
[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\uparrow P)\\
&= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P) && \text{by Rule } (\uparrow^{\textsc{Ord}})\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P && \text{by the induction hypothesis}\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\uparrow[\mu]P && \text{by Rule } (\uparrow^{\textsc{Ord}})\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\uparrow P && \text{by the definition of substitution}\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N
\end{aligned}$$

Case 3. $N = P \to M$

$$\begin{aligned}
[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) &= [\mu](\mathbf{ord}\,vars\,\mathbf{in}\,P \to M)\\
&= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)) && \text{where } \mathbf{ord}\,vars\,\mathbf{in}\,P = \vec{\alpha}_1 \text{ and } \mathbf{ord}\,vars\,\mathbf{in}\,M = \vec{\alpha}_2\\
&= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \backslash \vec{\alpha}_1)\\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1) && \text{by induction on } \vec{\alpha}_2\text{; the inductive step is similar to cas}\\
& && \text{collision free on } \vec{\alpha}_1 \text{ and } \vec{\alpha}_2 \text{ since } \vec{\alpha}_1 \subseteq vars \text{ and } \vec{\alpha}_2 \subseteq\\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)\\
(\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N) &= (\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P \to [\mu]M)\\
&= (\vec{\beta}_1, (\vec{\beta}_2 \backslash \vec{\beta}_1)) && \text{where } \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord}\,[\mu]va\\
& && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}\\
&= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \backslash [\mu]\vec{\alpha}_1)
\end{aligned}$$

Case 4. $N = \forall\overrightarrow{\alpha^+}.M$

$$\begin{aligned}
[\mu](\mathbf{ord}\,vars\,\mathbf{in}\,N) &= [\mu]\mathbf{ord}\,vars\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.M\\
&= [\mu]\mathbf{ord}\,vars\,\mathbf{in}\,M\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M && \text{by the induction hypothesis}\\
(\mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]N) &= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]\forall\overrightarrow{\alpha^+}.M\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,\forall\overrightarrow{\alpha^+}.[\mu]M\\
&= \mathbf{ord}\,[\mu]vars\,\mathbf{in}\,[\mu]M
\end{aligned}$$

□

Lemma 40 (Ordering is not affected by independent substitutions). *Suppose that* $\Gamma_2 \vdash \sigma : \Gamma_1$, *i.e.* σ *maps variables from* Γ_1 *into types taking free variables from* Γ_2, *and vars is a set of variables disjoint with both* Γ_1 *and* Γ_2, N *and* P *are types. Then*

$-$ $\mathbf{ord}\,vars\,\mathbf{in}\,[\sigma]N = \mathbf{ord}\,vars\,\mathbf{in}\,N$

$+$ $\mathbf{ord}\,vars\,\mathbf{in}\,[\sigma]P = \mathbf{ord}\,vars\,\mathbf{in}\,P$

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$

If $\alpha^- \notin \Gamma_1$ then $[\sigma]\alpha^- = \alpha^-$ and $\mathbf{ord}\,vars\,\mathbf{in}\,[\sigma]\alpha^- = \mathbf{ord}\,vars\,\mathbf{in}\,\alpha^-$, as requried. If $\alpha^- \in \Gamma_1$

then $\alpha^- \notin vars$, so $\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^- = \cdot$. Moreover, $\Gamma_2 \vdash \sigma : \Gamma_1$ means $\mathbf{fv}\,([\sigma]\alpha^-) \subseteq \Gamma_2$, and thus, as a set, $\mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]\alpha^- = vars \cap \mathbf{fv}\,([\sigma]\alpha^-) \subseteq vars \cap \Gamma_2 = \cdot$.

Case 2. $N = \forall\overrightarrow{\alpha^+}.M$

We can assume $\overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing$ and $\overrightarrow{\alpha^+} \cap vars = \varnothing$. Then

$$\begin{aligned}
\mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]N &= \mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]\forall\overrightarrow{\alpha^+}.M \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.[\sigma]M \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]M && \text{by the induction hypothesis} \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ M \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.M \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ N
\end{aligned}$$

Case 3. $N = \uparrow P$

$$\begin{aligned}
\mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]N &= \mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]\uparrow P \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ \uparrow[\sigma]P && \text{by the definition of substitution} \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]P && \text{by the induction hypothesis} \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ P && \text{by the definition of substitution} \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ \uparrow P && \text{by the definition of ordering} \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ N
\end{aligned}$$

Case 4. $N = P \to M$

$$\begin{aligned}
\mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]N &= \mathbf{ord}\ vars\ \mathbf{in}\ [\sigma](P \to M) \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ ([\sigma]P \to [\sigma]M) && \text{by the definition of substitut} \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]P, (\mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]M \backslash \mathbf{ord}\ vars\ \mathbf{in}\ [\sigma]P) && \text{by the definition of ordering} \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ P, (\mathbf{ord}\ vars\ \mathbf{in}\ M \backslash \mathbf{ord}\ vars\ \mathbf{in}\ P) && \text{by the induction hypothesis} \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ P \to M && \text{by the definition of ordering} \\
&= \mathbf{ord}\ vars\ \mathbf{in}\ N
\end{aligned}$$

Case 5. The proofs of the positive cases are symmetric.

□

Lemma 41 (Completeness of variable ordering). *Variable ordering is invariant under equivalence. For arbitrary vars,*

- *If* $N \simeq^D M$ *then* $\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ M$ *(as lists)*
- *If* $P \simeq^D Q$ *then* $\mathbf{ord}\ vars\ \mathbf{in}\ P = \mathbf{ord}\ vars\ \mathbf{in}\ Q$ *(as lists)*

Proof. Mutual induction on $N \simeq^D M$ and $P \simeq^D Q$. Let us consider the rule inferring $N \simeq^D M$.

Case 1. Rule ($\mathrm{Var}^{\simeq^D}_{-}$)

Case 2. Rule ($\uparrow^{\simeq^D}$)

Case 3. Rule ($\to^{\simeq^D}$) Then the equivalence has shape $P \to N \simeq^D Q \to M$, and by inversion, $P \simeq^D Q$ and $N \simeq^D M$. They by the induction hypothesis, $\mathbf{ord}\ vars\ \mathbf{in}\ P = \mathbf{ord}\ vars\ \mathbf{in}\ Q$ and $\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ M$. Since the resulting ordering for $P \to N$ and $Q \to M$ depend on the ordering of the corresponding components, which are equal, the results are equal.

Case 4. Rule ($\forall^{\simeq^D}$) Then the equivalence has shape $\forall\overrightarrow{\alpha^+}.N \simeq^D \forall\overrightarrow{\beta^+}.M$. and by inversion there exists $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)$ such that

- $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing$ and

- $N \simeq^D [\mu]M$

Let us assume that *vars* is disjoint from $\overrightarrow{\alpha^+}$ and $\overrightarrow{\beta^+}$ (we can always alpha-rename the bound variables). Then $\mathbf{ord}\ vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.N = \mathbf{ord}\ vars\ \mathbf{in}\ N$, $\mathbf{ord}\ vars\ \mathbf{in}\ \forall\overrightarrow{\alpha^+}.M = \mathbf{ord}\ vars\ \mathbf{in}\ M$ and by the induction hypothesis, $\mathbf{ord}\ vars\ \mathbf{in}\ N = \mathbf{ord}\ vars\ \mathbf{in}\ [\mu]M$. This way, it suffices tho show that $\mathbf{ord}\ vars\ \mathbf{in}\ [\mu]M = \mathbf{ord}\ vars\ \mathbf{in}\ M$. It holds by lemma 40 since *vars* is disjoint form the domain and the codomain of $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N)$ by assumption.

Case 5. The positive cases are proved symmetrically.

□

5.6 Normaliztaion

Observation 3 (Normalization is deterministic). *If* $\mathbf{nf}\,(N) = M$ *and* $\mathbf{nf}\,(N) = M'$ *then* $M = M'$. *If* $\mathbf{nf}\,(P) = Q$ *and* $\mathbf{nf}\,(P) = Q'$ *then* $Q = Q'$. *This way, we can use normalization as a function.*

Proof. By straightforward induction using observation 1. □

Lemma 42. *Set of free variables is invariant under equivalence.*

– *If* $N \simeq^D M$ *then* $\mathbf{fv}\, N = \mathbf{fv}\, M$ *(as sets)*
+ *If* $P \simeq^D Q$ *then* $\mathbf{fv}\, P = \mathbf{fv}\, Q$ *(as sets)*

Proof. Mutual induction on $N \simeq^D M$ and $P \simeq^D Q$ The base cases (Rule ($\text{Var}^{\simeq^D}_-$) and Rule ($\text{Var}^{\simeq^D}_+$)) are trivial. So are Rule ($\uparrow^{\simeq^D}$), Rule ($\downarrow^{\simeq^D}$), and Rule ($\to^{\simeq^D}$), where the required property follows from the induction hypothesis.

Let us consider the case when the equivalence is formed by Rule ($\forall^{\simeq^D}$), that is the equivalence has shape $\forall\overrightarrow{\alpha^+}.N \simeq^D \forall\overrightarrow{\beta^+}.M$, and by inversion, there is a bijection $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N)$ such that $N \simeq^D [\mu]M$, which by the induction hypothesis means $\mathbf{fv}\, N = \mathbf{fv}\,[\mu]M = [\mu]\mathbf{fv}\, M$.

Let us ensure by alpha-equivalence that $\overrightarrow{\alpha^+}$ is disjoint from $\mathbf{fv}\, M$. Then $(\mathbf{fv}\,\forall\overrightarrow{\beta^+}.M)\backslash\overrightarrow{\alpha^+} = \mathbf{fv}\,\forall\overrightarrow{\beta^+}.M$. Then we apply the following chain of equalities: $\mathbf{fv}\,\forall\overrightarrow{\alpha^+}.N = \mathbf{fv}\, N\backslash\overrightarrow{\alpha^+} = ([\mu]\mathbf{fv}\, M)\backslash\overrightarrow{\alpha^+} = [\mu](\mathbf{fv}\,\forall\overrightarrow{\beta^+}.M \cup (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M))\backslash\overrightarrow{\alpha^+} = ([\mu]\mathbf{fv}\,\forall\overrightarrow{\beta^+}.M \cup [\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\, M))\backslash\overrightarrow{\alpha^+} = ([\mu]\mathbf{fv}\,\forall\overrightarrow{\beta^+}.M)\backslash\overrightarrow{\alpha^+} = (\mathbf{fv}\,\forall\overrightarrow{\beta^+}.M)\backslash\overrightarrow{\alpha^+} = \mathbf{fv}\,\forall\overrightarrow{\beta^+}.M$.

Symmetrically, we prove the case when the equivalence is formed by Rule ($\exists^{\simeq^D}$). □

Lemma 43. *Free variables are not changed by the normalization*

– $\mathbf{fv}\, N = \mathbf{fv}\,\mathbf{nf}\,(N)$
+ $\mathbf{fv}\, P = \mathbf{fv}\,\mathbf{nf}\,(P)$

Proof. By mutual induction on N and P. The base cases (Rule (Var^{NF}_-) and Rule (Var^{NF}_-)) are trivial; the congruent cases (Rule ($\uparrow^{\text{NF}}$), Rule ($\downarrow^{\text{NF}}$), and Rule ($\to^{\text{NF}}$)) are proved by the induction hypothesis.

Let us consider the case when the term is formed by $\forall$, that is the normalization judgement has shape $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.N'$, where by inversion $\mathbf{nf}\,(N) = N'$ and $\mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,N' = \overrightarrow{\alpha^{+\prime}}$. By the induction hypothesis, $\mathbf{fv}\, N = \mathbf{fv}\, N'$. Since $\mathbf{fv}\,(\forall\overrightarrow{\alpha^+}.N) = \mathbf{fv}\, N\backslash\overrightarrow{\alpha^+}$, and $\mathbf{fv}\,(\forall\overrightarrow{\alpha^{+\prime}}.N') = \mathbf{fv}\, N'\backslash\overrightarrow{\alpha^{+\prime}}$, it is left to show that $\mathbf{fv}\, N\backslash\overrightarrow{\alpha^+} = \mathbf{fv}\, N\backslash\overrightarrow{\alpha^{+\prime}}$. By lemma 41, $\overrightarrow{\alpha^{+\prime}} = \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N' = \overrightarrow{\alpha^+} \cap \mathbf{fv}\, N$. Then $\mathbf{fv}\, N\backslash\overrightarrow{\alpha^+} = \mathbf{fv}\, N\backslash(\overrightarrow{\alpha^+} \cup \mathbf{fv}\, N)$ by set-theoretic properties, and thus, $\mathbf{fv}\, N\backslash\overrightarrow{\alpha^+} = \mathbf{fv}\, N\backslash\overrightarrow{\alpha^{+\prime}}$.

The case when the term is positive and formed by $\exists$ is symmetric. □

Lemma 44 (Soundness of normalization).

- $N \simeq^D \mathbf{nf}\,(N)$
+ $P \simeq^D \mathbf{nf}\,(P)$

Proof. Mutual induction on $\mathbf{nf}\,(N) = M$ and $\mathbf{nf}\,(P) = Q$. Let us consider how this judgment is formed:

Case 1. ($\text{Var}^{\text{NF}}_{-}$) and ($\text{Var}^{\text{NF}}_{-}$)
By the corresponding equivalence rules.

Case 2. ($\uparrow^{\text{NF}}$), ($\downarrow^{\text{NF}}$), and ($\to^{\text{NF}}$)
By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. ($\forall^{\text{NF}}$), i.e. $\mathbf{nf}\,(\forall\overrightarrow{\alpha^+}.N) = \forall\overrightarrow{\alpha^{+\prime}}.N'$
From the induction hypothesis, we know that $N \simeq^D N'$. In particular, by lemma 42, $\mathbf{fv}\,N \equiv \mathbf{fv}\,N'$. Then by lemma 37, $\overrightarrow{\alpha^{+\prime}} \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$, and thus, $\overrightarrow{\alpha^{+\prime}} \cap \mathbf{fv}\,N' \equiv \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$. To prove $\forall\overrightarrow{\alpha^+}.N \simeq^D \forall\overrightarrow{\alpha^{+\prime}}.N'$, it suffices to provide a bijection $\mu : \overrightarrow{\alpha^{+\prime}} \cap \mathbf{fv}\,N' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$ such that $N \simeq^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. ($\exists^{\text{NF}}$) Same as for case 3.

□

Corollary 19 (Normalization preserves ordering). *For any vars,*

- $\mathbf{ord}\ vars\ \mathbf{in}\ \mathbf{nf}\,(N) = \mathbf{ord}\ vars\ \mathbf{in}\ M$
+ $\mathbf{ord}\ vars\ \mathbf{in}\ \mathbf{nf}\,(P) = \mathbf{ord}\ vars\ \mathbf{in}\ Q$

Proof. Immediately from lemmas 41 and 44. □

Lemma 45 (Distributivity of normalization over substitution). *Normalization of a term distributes over substitution. Suppose that σ is a substitution, N and P are types. Then*

- $\mathbf{nf}\,([\sigma]N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)$
+ $\mathbf{nf}\,([\sigma]P) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P)$

where $\mathbf{nf}\,(\sigma)$ *means pointwise normalization:* $[\mathbf{nf}\,(\sigma)]\alpha^- = \mathbf{nf}\,([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P.

Case 1. $N = \alpha^-$
$\mathbf{nf}\,([\sigma]N) = \mathbf{nf}\,([\sigma]\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.
$[\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N) = [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(\alpha^-) = [\mathbf{nf}\,(\sigma)]\alpha^-$.

Case 2. $P = \alpha^+$
Similar to case 1.

Case 3. If the type is formed by $\to$, $\uparrow$, or $\downarrow$, the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \to M$ then

$$
\begin{aligned}
\mathbf{nf}\,([\sigma]N) &= \mathbf{nf}\,([\sigma](P \to M)) \\
&= \mathbf{nf}\,([\sigma]P \to [\sigma]M) && \text{By the congruence of substitution} \\
&= \mathbf{nf}\,([\sigma]P) \to \mathbf{nf}\,([\sigma]M) && \text{By the congruence of normalization, i.e. Rule } (\to^{\text{NF}}) \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P) \to [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(M) && \text{By the induction hypothesis} \\
&= [\mathbf{nf}\,(\sigma)](\mathbf{nf}\,(P) \to \mathbf{nf}\,(M)) && \text{By the congruence of substitution} \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(P \to M) && \text{By the congruence of normalization} \\
&= [\mathbf{nf}\,(\sigma)]\mathbf{nf}\,(N)
\end{aligned}
$$

Case 4. $N = \forall \overrightarrow{\alpha^+}.M$

$[\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (N) = [\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (\forall \overrightarrow{\alpha^+}.M)$

$= [\mathbf{nf}\ (\sigma)]\forall \overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\ (M)$ Where $\overrightarrow{\alpha^{+\prime}} = \mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ \mathbf{nf}\ (M) = \mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ M$ (the latter is l

$\mathbf{nf}\ ([\sigma]N) = \mathbf{nf}\ ([\sigma]\forall \overrightarrow{\alpha^+}.M)$

$= \mathbf{nf}\ (\forall \overrightarrow{\alpha^+}.[\sigma]M)$ Assuming $\overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing$ and $\overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$

$= \forall \overrightarrow{\beta^+}.\mathbf{nf}\ ([\sigma]M)$ Where $\overrightarrow{\beta^+} = \mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ \mathbf{nf}\ ([\sigma]M) = \mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ [\sigma]M$ (the latter is

$= \forall \overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\ ([\sigma]M)$ By lemma 40, $\overrightarrow{\beta^+} = \overrightarrow{\alpha^{+\prime}}$ since $\overrightarrow{\alpha^+}$ is disjoint with Γ_1 and Γ_2

$= \forall \overrightarrow{\alpha^{+\prime}}.[\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (M)$ By the induction hypothesis

To show alpha-equivalence of $[\mathbf{nf}\ (\sigma)]\forall \overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\ (M)$ and $\forall \overrightarrow{\alpha^{+\prime}}.[\mathbf{nf}\ (\sigma)]\mathbf{nf}\ (M)$, we can assume that $\overrightarrow{\alpha^{+\prime}} \cap \Gamma_1 = \varnothing$, and $\overrightarrow{\alpha^{+\prime}} \cap \Gamma_2 = \varnothing$.

Case 5. $P = \exists \overrightarrow{\alpha^-}.Q$

Same as for case 4.

□

Corollary 20 (Commutativity of normalization and renaming). *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

- $\mathbf{nf}\ ([\mu]N) = [\mu]\mathbf{nf}\ (N)$
- $\mathbf{nf}\ ([\mu]P) = [\mu]\mathbf{nf}\ (P)$

Proof. Immediately from lemma 45, after noticing that $\mathbf{nf}\ (\mu) = \mu$. □

Lemma 46 (Completeness of quantified normalization). *Normalization returns the same representative for equivalent types.*

- *If* $N \simeq^D M$ *then* $\mathbf{nf}\ (N) = \mathbf{nf}\ (M)$
- *If* $P \simeq^D Q$ *then* $\mathbf{nf}\ (P) = \mathbf{nf}\ (Q)$

Proof. Mutual induction on $N \simeq^D M$ and $P \simeq^D Q$.

Case 1. $(\forall^{\simeq^D})$

From the definition of the normalization,

- $\mathbf{nf}\ (\forall \overrightarrow{\alpha^+}.N) = \forall \overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\ (N)$ where $\overrightarrow{\alpha^{+\prime}}$ is $\mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ \mathbf{nf}\ (N)$
- $\mathbf{nf}\ (\forall \overrightarrow{\beta^+}.M) = \forall \overrightarrow{\beta^{+\prime}}.\mathbf{nf}\ (M)$ where $\overrightarrow{\beta^{+\prime}}$ is $\mathbf{ord}\ \overrightarrow{\beta^+}\ \mathbf{in}\ \mathbf{nf}\ (M)$

Let us take $\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\ M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\ N)$ from the inversion of the equivalence judgment. Notice that from lemmas 37 and 43, the domain and the codomain of μ can be written as $\mu : \overrightarrow{\beta^{+\prime}} \leftrightarrow \overrightarrow{\alpha^{+\prime}}$.

To show the alpha-equivalence of $\forall \overrightarrow{\alpha^{+\prime}}.\mathbf{nf}\ (N)$ and $\forall \overrightarrow{\beta^{+\prime}}.\mathbf{nf}\ (M)$, it suffices to prove that
(i) $[\mu]\mathbf{nf}\ (M) = \mathbf{nf}\ (N)$ and
(ii) $[\mu]\overrightarrow{\beta^{+\prime}} = \overrightarrow{\alpha^{+\prime}}$.

(i) $[\mu]\mathbf{nf}\ (M) = \mathbf{nf}\ ([\mu]M) = \mathbf{nf}\ (N)$. The first equality holds by corollary 20, the second— by the induction hypothesis.

(ii) $[\mu]\overrightarrow{\beta^+}' = [\mu]\mathbf{ord}\,\overrightarrow{\beta^+}\,\mathbf{in}\,\mathbf{nf}\,(M)$ by the definition of $\overrightarrow{\beta^+}'$

$= [\mu]\mathbf{ord}\,(\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(M)$ from lemmas 38 and 43

$= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,[\mu]\mathbf{nf}\,(M)$ by lemma 39, because $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap \mathbf{fv}\,\mathbf{nf}\,(M) \subseteq \overrightarrow{\alpha^+}$ and $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N \cap (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \subseteq \overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing$

$= \mathbf{ord}\,[\mu](\overrightarrow{\beta^+} \cap \mathbf{fv}\,M)\,\mathbf{in}\,\mathbf{nf}\,(N)$ since $[\mu]\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$ is proved

$= \mathbf{ord}\,(\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N)\,\mathbf{in}\,\mathbf{nf}\,(N)$ because μ is a bijection between $\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N$ and $\overrightarrow{\beta^+}$

$= \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,\mathbf{nf}\,(N)$ from lemmas 38 and 43

$= \overrightarrow{\alpha^+}'$ by the definition of $\overrightarrow{\alpha^+}'$

Case 2. ($\exists^{\simeq^D}$) Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□

Lemma 47 (Idempotence of normalization). *Normalization is idempotent*

$-\ \mathbf{nf}\,(\mathbf{nf}\,(N)) = \mathbf{nf}\,(N)$
$+\ \mathbf{nf}\,(\mathbf{nf}\,(P)) = \mathbf{nf}\,(P)$

Proof. By applying lemma 46 to lemma 44. □

Lemma 48. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized.*

Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then

$$+\ [\sigma]P \text{ is normal} \iff \begin{cases} \sigma|_{\mathbf{fv}\,(P)} & \text{is normal} \\ P & \text{is normal} \end{cases}$$

$$-\ [\sigma]N \text{ is normal} \iff \begin{cases} \sigma|_{\mathbf{fv}\,(N)} & \text{is normal} \\ N & \text{is normal} \end{cases}$$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$
Then N is always normal, and the normality of $\sigma|_{\alpha^-}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \to M$

$$[\sigma](P \to M) \text{ is normal} \iff [\sigma]P \to [\sigma]M \text{ is normal} \qquad \text{by the substitution congruence}$$

$$\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases}$$

$$\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(M)} & \text{is normal} \end{cases} \qquad \text{by the induction hypothesis}$$

$$\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P)\cup\mathbf{fv}\,(M)} & \text{is normal} \end{cases}$$

$$\iff \begin{cases} P \to M & \text{is normal} \\ \sigma|_{\mathbf{fv}\,(P\to M)} & \text{is normal} \end{cases}$$

Case 3. $N = \uparrow P$

By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall \overrightarrow{\alpha^+}.M$

$$[\sigma](\forall \overrightarrow{\alpha^+}.M) \text{ is normal} \iff (\forall \overrightarrow{\alpha^+}.[\sigma]M) \text{ is normal} \qquad \text{assuming } \overrightarrow{\alpha^+} \cap \Gamma_1 = \varnothing \text{ and } \overrightarrow{\alpha^+} \cap \Gamma_2 = \varnothing$$

$$\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,[\sigma]M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by the definition of normalization}$$

$$\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by lemma 40}$$

$$\iff \begin{cases} \sigma|_{\mathbf{fv}\,(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord}\,\overrightarrow{\alpha^+}\,\mathbf{in}\,M = \overrightarrow{\alpha^+} \end{cases} \qquad \text{by the induction hypothesis}$$

$$\iff \begin{cases} \sigma|_{\mathbf{fv}\,(\forall \overrightarrow{\alpha^+}.M)} \text{ is normal} & \text{since } \mathbf{fv}\,(\forall \overrightarrow{\alpha^+}.M) = \mathbf{fv}\,(M); \\ \forall \overrightarrow{\alpha^+}.M \text{ is normal} & \text{by the definition of normalization} \end{cases}$$

Case 5. $P = \ldots$

The positive cases are done in the same way as the negative ones.

□

6 PROPERTIES OF THE ALGORITHMIC TYPE SYSTEM

6.1 Algorithmic Type Well-formedness

Lemma 49 (Soundness of algorithmic type well-formedness).

+ *if* $\Gamma; \Xi \vdash P$ *then* $\mathbf{fv}\,(P) \subseteq \Gamma$ *and* $\mathbf{uv}\,(P) \subseteq \Xi$;
− *if* $\Gamma; \Xi \vdash N$ *then* $\mathbf{fv}\,(N) \subseteq \Gamma$ *and* $\mathbf{uv}\,(N) \subseteq \Xi$.

Proof. The proof is analogous to lemma 3. The additional base case is when $\Gamma; \Xi \vdash P$ is derived by Rule ($\text{UVAR}^{\text{WF}}_{+}$), and the symmetric negative case. In this case, $P = \widehat{\alpha}^+$, and $\mathbf{uv}\,(P) = \widehat{\alpha}^+ \subseteq \Xi$ by inversion; $\mathbf{fv}\,(P) = \varnothing \subseteq \Gamma$ vacuously. □

Lemma 50 (Completeness of algorithmic type well-formedness). *In the well-formedness judgment, only used variables matter:*

+ *if* $\Gamma_1 \cap \mathbf{fv}\, P = \Gamma_2 \cap \mathbf{fv}\, P$ *and* $\Xi_1 \cap \mathbf{uv}\, P = \Xi_2 \cap \mathbf{uv}\, P$ *then* $\Gamma_1; \Xi_1 \vdash P \iff \Gamma_2; \Xi_2 \vdash P$, *and*
− *if* $\Gamma_1 \cap \mathbf{fv}\, N = \Gamma_2 \cap \mathbf{fv}\, N$ *and* $\Xi_1 \cap \mathbf{uv}\, N = \Xi_2 \cap \mathbf{uv}\, N$ *then* $\Gamma_1; \Xi_1 \vdash N \iff \Gamma_2; \Xi_2 \vdash N$.

Proof. By mutual structural induction on P and N. □

Lemma 51 (Variable algorithmization agrees with well-formedness).

+ $\Gamma, \overrightarrow{\alpha^-} \vdash P$ *implies* $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P$;
− $\Gamma, \overrightarrow{\alpha^-} \vdash N$ *implies* $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]N$.

Proof. The proof is a structural induction on $\Gamma, \overrightarrow{\alpha^-} \vdash P$ and mutually, on $\Gamma, \overrightarrow{\alpha^-} \vdash N$. Notice that the substitutions commute with all the constructors, providing the step of the induction. □

Lemma 52 (Variable dealgorithmization agrees with well-formedness).

+ $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash P$ *implies* $\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P$;
− $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash N$ *implies* $\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]N$.

Proof. As for lemma 51, the proof is a structural induction on $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash P$ and mutually, on $\Gamma; \overrightarrow{\widehat{\alpha}^-} \vdash N$. □

Corollary 21 (Well-formedness Algorithmic Context Weakening). *Suppose that* $\Gamma_1 \subseteq \Gamma_2$, *and* $\Xi_1 \subseteq \Xi_2$. *Then*

+ *if* $\Gamma_1; \Xi_1 \vdash P$ *implies* $\Gamma_2; \Xi_2 \vdash P$,
− *if* $\Gamma_1; \Xi_1 \vdash N$ *implies* $\Gamma_2; \Xi_2 \vdash N$.

Proof. By lemma 49, $\Gamma_1; \Xi_1 \vdash P$ implies $\mathbf{fv}(P) \subseteq \Gamma_1 \subseteq \Gamma_2$ and $\mathbf{uv}(P) \subseteq \Xi_1 \subseteq \Xi_2$, and thus, $\mathbf{fv}(P) = \mathbf{fv}(P) \cap \Gamma_1 = \mathbf{fv}(P) \cap \Gamma_2$, and $\mathbf{uv}(P) = \mathbf{uv}(P) \cap \Xi_1 = \mathbf{uv}(P) \cap \Xi_2$. Then by lemma 50, $\Gamma_2; \Xi_2 \vdash P$. The negative case is symmetric. □

6.2 Substitution

Lemma 53 (Algorithmic Substitution Strengthening). *Restricting the substitution to the algorithmic variables of the substitution subject does not affect the result. Suppose that* $\widehat{\sigma}$ *is an algorithmic substitution,* P *and* N *are algorithmic types. Then*

+ $[\widehat{\sigma}]P = [\widehat{\sigma}|_{\mathbf{uv}\, P}]P$,
− $[\widehat{\sigma}]N = [\widehat{\sigma}|_{\mathbf{uv}\, N}]N$

Proof. The proof is analogous to the proof of lemma 6. □

Lemma 54 (Substitutions equal on the algorithmic variables). *Suppose that* $\widehat{\sigma}_1$ *and* $\widehat{\sigma}_2$ *are normalized substitutions of signature* $\Theta \vdash \widehat{\sigma}_i : \Xi$. *Then*

+ *for a normalized type* $\Gamma; \Xi \vdash P$, *if* $[\widehat{\sigma}_1]P = [\widehat{\sigma}_2]P$ *then* $\widehat{\sigma}_1|_{(\mathbf{uv}\, P)} = \widehat{\sigma}_2|_{(\mathbf{uv}\, P)}$;
− *for a normalized type* $\Gamma; \Xi \vdash N$, *if* $[\widehat{\sigma}_1]N = [\widehat{\sigma}_2]N$ *then* $\widehat{\sigma}_1|_{(\mathbf{uv}\, N)} = \widehat{\sigma}_2|_{(\mathbf{uv}\, N)}$.

Proof. The proof is a simple structural induction on $\Gamma; \Xi \vdash P$ and mutually, on $\Gamma; \Xi \vdash N$. Let us consider the shape of N (the cases of P are symmetric).

Case 1. $\Gamma; \Xi \vdash \widehat{\alpha}^-$. Then $[\widehat{\sigma}_1]\widehat{\alpha}^- = [\widehat{\sigma}_2]\widehat{\alpha}^-$ implies $\widehat{\sigma}_1|_{(\mathbf{uv}\, \widehat{\alpha}^-)} = \widehat{\sigma}_2|_{(\mathbf{uv}\, \widehat{\alpha}^-)}$ immediately.
Case 2. $\Gamma; \Xi \vdash \alpha^-$. Then $\mathbf{uv}\, \widehat{\alpha}^- = \varnothing$, and $\widehat{\sigma}_1|_{(\mathbf{uv}\, \widehat{\alpha}^-)} = \widehat{\sigma}_2|_{(\mathbf{uv}\, \widehat{\alpha}^-)}$ holds vacuously.
Case 3. $\Gamma; \Xi \vdash \forall \overrightarrow{\alpha^+}.N$. Then we are proving that $[\widehat{\sigma}_1]\forall \overrightarrow{\alpha^+}.N = [\widehat{\sigma}_2]\forall \overrightarrow{\alpha^+}.N$ implies $\widehat{\sigma}_1|_{(\mathbf{uv}\, \forall \overrightarrow{\alpha^+}.N)} = \widehat{\sigma}_2|_{(\mathbf{uv}\, \forall \overrightarrow{\alpha^+}.N)}$. By definition of substitution and Rule ($\forall^{\simeq^D}$), $[\widehat{\sigma}_1]N = [\widehat{\sigma}_2]N$ implies $\widehat{\sigma}_1|_{\mathbf{uv}\, N} = \widehat{\sigma}_2|_{\mathbf{uv}\, N}$. Since $\forall \overrightarrow{\alpha^+}.N$ is normalized, so is $\Gamma, \overrightarrow{\alpha^+}; \Xi \vdash N$, hence, the induction hypothesis is applicable and implies $\widehat{\sigma}_1|_{\mathbf{uv}\, N} = \widehat{\sigma}_2|_{\mathbf{uv}\, N}$, as required.

Case 4. $\Gamma;\Xi \vdash P \to N$. Then we are proving that $[\widehat{\sigma}_1](P \to N) = [\widehat{\sigma}_2](P \to N)$ implies $\widehat{\sigma}_1|_{(\mathbf{uv}\ P \to N)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ P \to N)}$. By definition of substitution and congruence of equality, $[\widehat{\sigma}_1](P \to N) = [\widehat{\sigma}_2](P \to N)$ means $[\widehat{\sigma}_1]P = [\widehat{\sigma}_2]P$ and $[\widehat{\sigma}_1]N = [\widehat{\sigma}_2]N$. Notice that P and N are normalized since $P \to N$ is normalized, and well-formed in the same contexts. This way, by the induction hypothesis, $\widehat{\sigma}_1|_{(\mathbf{uv}\ P)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ P)}$ and $\widehat{\sigma}_1|_{(\mathbf{uv}\ N)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ N)}$, which since $\mathbf{uv}\ (P \to N) = \mathbf{uv}\ P \cup \mathbf{uv}\ N$ implies $\widehat{\sigma}_1|_{(\mathbf{uv}\ P \to N)} = \widehat{\sigma}_2|_{(\mathbf{uv}\ P \to N)}$.

Case 5. $\Gamma;\Xi \vdash \uparrow P$. The proof is similar to the previous case: we apply congruence of substitution, equality, and normalization, then the induction hypothesis, and then the fact that $\mathbf{uv}\ (\uparrow P) = \mathbf{uv}\ P$.

□

Corollary 22 (Substitutions equivalent on the algorithmic variables). *Suppose that* $\widehat{\sigma}_1$ *and* $\widehat{\sigma}_2$ *are substitutions of signature* $\Theta \vdash \widehat{\sigma}_i : \Xi$. *Then*

+ *for a type* $\Gamma;\Xi \vdash P$, *if* $\Theta \vdash \quad [\widehat{\sigma}_1]P \simeq^D [\widehat{\sigma}_2]P$ *then* $\Theta \vdash \widehat{\sigma}_1 \simeq^{\leqslant} \widehat{\sigma}_2 : \mathbf{uv}\ P$;
− *for a type* $\Gamma;\Xi \vdash N$, *if* $\Theta \vdash \quad [\widehat{\sigma}_1]N \simeq^D [\widehat{\sigma}_2]N$ *then* $\Theta \vdash \widehat{\sigma}_1 \simeq^{\leqslant} \widehat{\sigma}_2 : \mathbf{uv}\ N$.

Proof. First, let us normalize the types and the substitutions, and show that the given equivalences and well-formedness properties are preserved. $\Gamma;\Xi \vdash P$ implies $\Gamma;\Xi \vdash \mathbf{nf}\ (P)$ by corollary 23. $\Theta \vdash \quad [\widehat{\sigma}_1]P \simeq^D [\widehat{\sigma}_2]P$ implies $\mathbf{nf}\ ([\widehat{\sigma}_1]P) = \mathbf{nf}\ ([\widehat{\sigma}_2]P)$ by lemma 35. Then $\mathbf{nf}\ ([\widehat{\sigma}_1]P) = \mathbf{nf}\ ([\widehat{\sigma}_2]P)$ implies $[\mathbf{nf}\ (\widehat{\sigma}_1)]\mathbf{nf}\ (P) = [\mathbf{nf}\ (\widehat{\sigma}_2)]\mathbf{nf}\ (P)$ by lemma 45. Notice that by corollary 24 $\Theta \vdash \widehat{\sigma}_i : \Xi$ implies $\Theta \vdash \mathbf{nf}\ (\widehat{\sigma}_i) : \Xi$.

This way, by lemma 54, $\Theta \vdash \quad [\widehat{\sigma}_1]P \simeq^D [\widehat{\sigma}_2]P$ implies $\mathbf{nf}\ (\widehat{\sigma}_1)|_{(\mathbf{uv}\,\mathbf{nf}\,(P))} = \mathbf{nf}\ (\widehat{\sigma}_2)|_{(\mathbf{uv}\,\mathbf{nf}\,(P))}$. Then by lemma 55, $\mathbf{nf}\ (\widehat{\sigma}_1)|_{(\mathbf{uv}\ P)} = \mathbf{nf}\ (\widehat{\sigma}_2)|_{(\mathbf{uv}\ P)}$, and by *corollary* : *subst* − *subt* − *equiv* − *algorithmization*, $\Theta \vdash \widehat{\sigma}_1 \simeq^{\leqslant} \widehat{\sigma}_2 : \mathbf{uv}\ P$.

Symmetrically, $\Theta \vdash \quad [\widehat{\sigma}_1]N \simeq^D [\widehat{\sigma}_2]N$ implies $\Theta \vdash \widehat{\sigma}_1 \simeq^{\leqslant} \widehat{\sigma}_2 : \mathbf{uv}\ N$. □

6.3 Normalization

Lemma 55. *Algorithmic variables are not changed by the normalization*

− $\mathbf{uv}\ N \equiv \mathbf{uv}\ \mathbf{nf}\ (N)$
+ $\mathbf{uv}\ P \equiv \mathbf{uv}\ \mathbf{nf}\ (P)$

Proof. By straightforward induction on N and mutually on P, similar to the proof of lemma 43. □

Lemma 56 (Soundness of normalization of algorithmic types).

− $N \simeq^D \mathbf{nf}\ (N)$
+ $P \simeq^D \mathbf{nf}\ (P)$

Proof. The proof coincides with the proof of lemma 44. □

6.4 Equivalence

Lemma 57 (Algorithmization of declarative equivalence). *Declarative equivalence is the equality of normal forms.*

+ $P \simeq^D Q \iff \mathbf{nf}\ (P) = \mathbf{nf}\ (Q)$,
− $N \simeq^D M \iff \mathbf{nf}\ (N) = \mathbf{nf}\ (M)$.

Proof.

+ Let us prove both directions separately.
  ⇒ exactly by lemma 46,

$\Leftarrow$ from lemma 44, we know $P \simeq^D \mathbf{nf}\,(P) = \mathbf{nf}\,(Q) \simeq^D Q$, then by transitivity (lemma 26), $P \simeq^D Q$.

– For the negative case, the proof is the same.

□

Lemma 58 (Algorithmic type well-formedness is invariant under equivalence). *Mutual subtyping implies declarative equivalence.*

+ *if* $P \simeq^D Q$ *then* $\Gamma;\Xi \vdash P \iff \Gamma;\Xi \vdash Q$,
– *if* $N \simeq^D M$ *then* $\Gamma;\Xi \vdash N \iff \Gamma;\Xi \vdash M$

Proof. The proof coincides with the proof of lemma 27, and adds two cases for equating two positive or two negative algorithmic variables, which must be equal by inversion, and thus, $\Gamma;\Xi \vdash \widehat{\alpha}^{\pm} \iff \Gamma;\Xi \vdash \widehat{\alpha}^{\pm}$ holds trivially. □

Corollary 23 (Normalization preserves well-formedness of algorithmic types). +
$\Gamma;\Xi \vdash P \iff \Gamma;\Xi \vdash \mathbf{nf}\,(P)$,
– $\Gamma;\Xi \vdash N \iff \Gamma;\Xi \vdash \mathbf{nf}\,(N)$

Proof. Immediately from lemmas 56 and 58. □

Corollary 24 (Normalization preserves the signature of the algorithmic substitution).

$$\Theta \vdash \widehat{\sigma} : \Xi \iff \Theta \vdash \mathbf{nf}\,(\widehat{\sigma}) : \Xi,\ \Gamma \vdash \widehat{\sigma} : \Xi \iff \Gamma \vdash \mathbf{nf}\,(\widehat{\sigma}) : \Xi.$$

Proof. The proof is analogous to corollary 12. □

Lemma 59 (Algorithmic substitution equivalence becomes equality after normalization). *Suppose that* $\Theta \vdash \widehat{\sigma}_1 : \Xi'$ *and* $\Theta \vdash \widehat{\sigma}_2 : \Xi'$ *are algorithmic substitutions and* $\Xi \subseteq \Xi'$. *Then* $\Theta \vdash \widehat{\sigma}_1 \simeq^{\leqslant} \widehat{\sigma}_2 : \Xi \iff \mathbf{nf}\,(\widehat{\sigma}_1)|_{\Xi} = \mathbf{nf}\,(\widehat{\sigma}_2)|_{\Xi}$.

Proof. Follows immediately from lemma 35:

$\Rightarrow$ If $\widehat{\alpha}^{\pm} \notin \Xi$, then $[\mathbf{nf}\,(\widehat{\sigma}_1)|_{\Xi}]\widehat{\alpha}^{\pm} = [\mathbf{nf}\,(\widehat{\sigma}_2)|_{\Xi}]\widehat{\alpha}^{\pm} = \widehat{\alpha}^{\pm}$ by definition. For any $\widehat{\alpha}^{\pm} \in \Xi$, $[\mathbf{nf}\,(\widehat{\sigma}_1)|_{\Xi}]\widehat{\alpha}^{\pm} = \mathbf{nf}\,([\widehat{\sigma}_1]\widehat{\alpha}^{\pm})$ and $[\mathbf{nf}\,(\widehat{\sigma}_2)|_{\Xi}]\widehat{\alpha}^{\pm} = \mathbf{nf}\,([\widehat{\sigma}_2]\widehat{\alpha}^{\pm})$; $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}_1]\widehat{\alpha}^{\pm} \simeq^{\leqslant} [\widehat{\sigma}_2]\widehat{\alpha}^{\pm}$ implies $\mathbf{nf}\,([\widehat{\sigma}_1]\widehat{\alpha}^{\pm}) = \mathbf{nf}\,([\widehat{\sigma}_2]\widehat{\alpha}^{\pm})$ by lemma 57.

$\Leftarrow$ If $\widehat{\alpha}^{\pm} \in \Xi$, then $\mathbf{nf}\,(\widehat{\sigma}_1)|_{\Xi} = \mathbf{nf}\,(\widehat{\sigma}_2)|_{\Xi}$ implies $\mathbf{nf}\,([\widehat{\sigma}_1]\widehat{\alpha}^{\pm}) = \mathbf{nf}\,([\widehat{\sigma}_2]\widehat{\alpha}^{\pm})$ by definition of substitution restriction and normalization. In turn, $\mathbf{nf}\,([\widehat{\sigma}_1]\widehat{\alpha}^{\pm}) = \mathbf{nf}\,([\widehat{\sigma}_2]\widehat{\alpha}^{\pm})$ means $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}_1]\widehat{\alpha}^{\pm} \simeq^{\leqslant} [\widehat{\sigma}_2]\widehat{\alpha}^{\pm}$ by lemma 57.

□

6.5 Unification Constraint Merge

Observation 4 (Unification Constraint Merge is Deterministic). *Suppose that* $\Theta \vdash UC_1$ *and* $\Theta \vdash UC_2$ *If* $\Theta \vdash UC_1 \& UC_2 = UC$ *and* $\Theta \vdash UC_1 \& UC_2 = UC'$ *are defined then* $UC = UC'$.

Proof. UC and UC' both consists of three parts: Entries of UC_1 that do not have matching entries in UC_2, entries of UC_2 that do not have matching entries in UC_1, and the merge of matching entries. The parts corresponding to unmatched entries of UC_1 and UC_2 coincide, since UC_1 and UC_2 are fixed. To show that the merge of matching entries coincide, let us take any pair of matching $e_1 \in UC_1$ and $e_2 \in UC_2$ and consider their shape.

Case 1. e_1 is $\widehat{\alpha}^{+} :\simeq Q_1$ and e_2 is $\widehat{\alpha}^{+} :\simeq Q_2$ then the result, if it exists, is always e_1, by inversion of Rule $(\simeq \&^{+} \simeq)$.

Case 2. e_1 is $\widehat{\alpha}^- :\simeq N_1$ and e_2 is $\widehat{\alpha}^- :\simeq N_2$ then analogously, the result, if it exists, is always e_1, by inversion of Rule $(\simeq \&^- \simeq)$.

This way, the third group of entries coincide as well. □

Lemma 60 (Soundness of Unification Constraint Merge). *Suppose that* $\Theta \vdash UC_1$ *and* $\Theta \vdash UC_2$ *are normalized unification constraints. If* $\Theta \vdash UC_1 \& UC_2 = UC$ *is defined then* $UC = UC_1 \cup UC_2$.

Proof.

- $UC_1 \,\&\, UC_2 \subseteq UC_1 \cup UC_2$
  By definition, $UC_1 \,\&\, UC_2$ consists of three parts: entries of UC_1 that do not have matching entries of UC_2, entries of UC_2 that do not have matching entries of UC_1, and the merge of matching entries.
  If e is from the first or the second part, then $e \in UC_1 \cup UC_2$ holds immediately. If e is from the third part, then e is the merge of two matching entries $e_1 \in UC_1$ and $e_2 \in UC_2$. Since UC_1 and UC_2 are normalized unification , e_1 and e_2 have one of the following forms:
  - $\widehat{\alpha}^+ :\simeq P_1$ and $\widehat{\alpha}^+ :\simeq P_2$, where P_1 and P_2 are normalized, and then since $\Theta(\widehat{\alpha}^+) \vdash e_1 \& e_2 = e$ exists, Rule $(\simeq \&^+ \simeq)$ was applied to infer it. It means that $e = e_1 = e_2$;
  - $\widehat{\alpha}^- :\simeq N_1$ and $\widehat{\alpha}^- :\simeq N_2$, then symmetrically, $\Theta(\widehat{\alpha}^-) \vdash e_1 \& e_2 = e = e_1 = e_2$

  In both cases, $e \in UC_1 \cup UC_2$.
- $UC_1 \cup UC_2 \subseteq UC_1 \,\&\, UC_2$
  Let us take an arbitrary $e_1 \in UC_1$. Then since UC_1 is a unification constraint, e_1 has one of the following forms:
  - $\widehat{\alpha}^+ :\simeq P$ where P is normalized. If $\widehat{\alpha}^+ \notin \mathbf{dom}\,(UC_2)$, then $e_1 \in UC_1 \,\&\, UC_2$. Otherwise, there is a normalized matching $e_2 = (\widehat{\alpha}^+ :\simeq P') \in UC_2$ and then since $UC_1 \,\&\, UC_2$ exists, Rule $(\simeq \&^+ \simeq)$ was applied to construct $e_1 \,\&\, e_2 \in UC_1 \,\&\, UC_2$. By inversion of Rule $(\simeq \&^+ \simeq)$, $e_1 \,\&\, e_2 = e_1$, and $\mathbf{nf}\,(P) = \mathbf{nf}\,(P')$, which since P and P' are normalized, implies that $P = P'$, that is $e_1 = e_2 \in UC_1 \,\&\, UC_2$.
  - $\widehat{\alpha}^- :\simeq N$ where N is normalized. Then symmetrically, $e_1 = e_2 \in UC_1 \,\&\, UC_2$.

  Similarly, if we take an arbitrary $e_2 \in UC_2$, then $e_1 = e_2 \in UC_1 \,\&\, UC_2$.

□

Corollary 25. *Suppose that* $\Theta \vdash UC_1$ *and* $\Theta \vdash UC_2$ *are normalized unification constraints. If* $\Theta \vdash UC_1 \& UC_2 = UC$ *is defined then*

(1) $\Theta \vdash UC$ *is normalized unification constraint,*
(2) for any substitution $\Theta \vdash \widehat{\sigma} : \mathbf{dom}\,(UC)$, $\Theta \vdash \widehat{\sigma} : UC$ *implies* $\Theta \vdash \widehat{\sigma} : UC_1$ *and* $\Theta \vdash \widehat{\sigma} : UC_2$.

Proof. It is clear that since $UC = UC_1 \cup UC_2$ (by lemma 60), and being normalized means that all entries are normalized, UC is a normalized unification constraint. Analogously, $\Theta \vdash UC = UC_1 \cup UC_2$ holds immediately, since $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$.

Let us take an arbitrary substitution $\Theta \vdash \widehat{\sigma} : \mathbf{dom}\,(UC)$ and assume that $\Theta \vdash \widehat{\sigma} : UC$. Then $\Theta \vdash \widehat{\sigma} : UC_i$ holds by definition: If $e \in UC_i \subseteq UC_1 \cup UC_2 = UC$ then $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$ (where e restricts $\widehat{\alpha}^{\pm}$) holds since $\Theta \vdash \widehat{\sigma} : \mathbf{dom}\,(UC)$. □

Lemma 61 (Completeness of Unification Constraint Entry Merge). *For a fixed context* Γ, *suppose that* $\Gamma \vdash e_1$ *and* $\Gamma \vdash e_2$ *are matching constraint entries.*

- *for a type* P *such that* $\Gamma \vdash P : e_1$ *and* $\Gamma \vdash P : e_2$, $\Gamma \vdash e_1 \& e_2 = e$ *is defined and* $\Gamma \vdash P : e$.
- *for a type* N *such that* $\Gamma \vdash N : e_1$ *and* $\Gamma \vdash N : e_2$, $\Gamma \vdash e_1 \& e_2 = e$ *is defined and* $\Gamma \vdash N : e$.

Proof. Let us consider the shape of e_1 and e_2.

Case 1. e_1 is $\widehat{\alpha}^+ :\simeq Q_1$ and e_2 is $\widehat{\alpha}^+ :\simeq Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \simeq^{\leqslant} Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \simeq^{\leqslant} Q_2$. Then by transitivity of equivalence (corollary 10), $\Gamma \vdash Q_1 \simeq^{\leqslant} Q_2$, which means $\mathbf{nf}(Q_1) = \mathbf{nf}(Q_2)$ by lemma 35. Hence, Rule $(\simeq \&^+ \simeq)$ applies to infer $\Gamma \vdash e_1 \& e_2 = e_2$, and $\Gamma \vdash P : e_2$ holds by assumption.

Case 2. e_1 is $\widehat{\alpha}^- :\simeq N_1$ and e_2 is $\widehat{\alpha}^- :\simeq M_2$. The proof is symmetric.

□

Lemma 62 (Completeness of Unification Constraint Merge). *Suppose that* $\Theta \vdash UC_1$ *and* $\Theta \vdash UC_2$. *Then for any* $\Xi \supseteq \mathbf{dom}(UC_1) \cup \mathbf{dom}(UC_2)$ *and substitution* $\Theta \vdash \widehat{\sigma} : \Xi$ *such that* $\Theta \vdash \widehat{\sigma} : UC_1$ *and* $\Theta \vdash \widehat{\sigma} : UC_2$,

(1) $\Theta \vdash UC_1 \& UC_2 = UC$ *is defined and*
(2) $\Theta \vdash \widehat{\sigma} : UC$.

Proof. The proof repeats the proof of lemma 82 for cases uses lemma 61 instead of lemma 81. □

6.6 Unification

Observation 5 (Unification is deterministic).

+ *If* $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC$ *and* $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC'$ *then* $UC = UC'$.
− *If* $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv UC$ *and* $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv UC'$ *then* $UC = UC'$.

Proof. We prove it by mutual structural induction on $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC$ and $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv UC'$. Let us consider the positive case, since the negative case is symmetric.

First, notice that the rule applied the last is uniquely determined by the shape of P and Q. Second, the premises of each rule are deterministic on the input either by the induction hypothesis or by observation 4. □

Lemma 63 (Soundness of Unification).

+ *For normalized* P *and* Q *such that* $\Gamma; \mathbf{dom}(\Theta) \vdash P$ *and* $\Gamma \vdash Q$,
if $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC$ *then* $\Theta \vdash UC : \mathbf{uv}\, P$ *and for any normalized* $\widehat{\sigma}$ *such that* $\Theta \vdash \widehat{\sigma} : UC$, $[\widehat{\sigma}]P = Q$.
− *For normalized* N *and* M *such that* $\Gamma; \mathbf{dom}(\Theta) \vdash N$ *and* $\Gamma \vdash M$,
if $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv UC$ *then* $\Theta \vdash UC : \mathbf{uv}\, N$ *and for any normalized* $\widehat{\sigma}$ *such that* $\Theta \vdash \widehat{\sigma} : UC$, $[\widehat{\sigma}]N = M$.

Proof. We prove by induction on the derivation of $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv UC$ and mutually $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC$. Let us consider the last rule forming this derivation.

Case 1. Rule $(\mathrm{VAR}^{u}_{\simeq})$, then $N = \alpha^- = M$. The resulting unification constraint is empty: $UC = \cdot$. It satisfies $\Theta \vdash UC : \cdot$ vacuously, and $[\widehat{\sigma}]\alpha^- = \alpha^-$, that is $[\widehat{\sigma}]N = M$.

Case 2. Rule $(\uparrow^{u}_{\simeq})$, then $N = \uparrow P$ and $M = \uparrow Q$. The algorithm makes a recursive call to $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC$ returning UC. By induction hypothesis, $\Theta \vdash UC : \mathbf{uv}\, P$ and thus, $\Theta \vdash UC : \mathbf{uv} \uparrow P$, and for any $\Theta \vdash \widehat{\sigma} : UC$, $[\widehat{\sigma}]N = [\widehat{\sigma}]\uparrow P = \uparrow[\widehat{\sigma}]P = \uparrow Q = M$, as required.

Case 3. Rule $(\rightarrow^{u}_{\simeq})$, then $N = P \to N'$ and $M = Q \to M'$. The algorithm makes two recursive calls to $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC_1$ and $\Gamma; \Theta \vDash N' \overset{u}{\simeq} M' \dashv UC_2$ returning $\Theta \vdash UC_1 \& UC_2 = UC$ as the result.
It is clear that P, N', Q, and M' are normalized, and that $\Gamma; \mathbf{dom}(\Theta) \vdash P, \Gamma; \mathbf{dom}(\Theta) \vdash N'$, $\Gamma \vdash Q$, and $\Gamma \vdash M'$. This way, the induction hypothesis is applicable to both recursive calls.
By applying the induction hypothesis to $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC_1$, we have:

- $\Theta \vdash UC_1 : \mathbf{uv}\, P$,
- for any $\Theta \vdash \widehat{\sigma}' : UC_1$, $[\widehat{\sigma}']P = Q$.

By applying it to $\Gamma; \Theta \vDash N' \overset{u}{\simeq} M' \dashv UC_2$, we have:

- $\Theta \vdash UC_2 : \mathbf{uv}\, N'$,
- for any $\Theta \vdash \widehat{\sigma}' : UC_2$, $[\widehat{\sigma}']N' = M'$.

Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : UC$. By the soundness of the constraint merge (lemma 80), $\Theta \vdash UC_1 \& UC_2 = UC$ implies $\Theta \vdash \widehat{\sigma} : UC_1$ and $\Theta \vdash \widehat{\sigma} : UC_2$.
Applying the induction hypothesis to $\Theta \vdash \widehat{\sigma} : UC_1$, we have $[\widehat{\sigma}]P = Q$; applying it to $\Theta \vdash \widehat{\sigma} : UC_2$, we have $[\widehat{\sigma}]N' = M'$. This way, $[\widehat{\sigma}]N = [\widehat{\sigma}]P \to [\widehat{\sigma}]N' = Q \to M' = M$.

Case 4. Rule $(\forall^{\overset{u}{\simeq}})$, then $N = \forall\overrightarrow{\alpha^+}.N'$ and $M = \forall\overrightarrow{\alpha^+}.M'$. The algorithm makes a recursive call to $\Gamma, \overrightarrow{\alpha^+}; \Theta \vDash N' \overset{u}{\simeq} M' \dashv UC$ returning UC as the result.
The induction hypothesis is applicable: $\Gamma, \overrightarrow{\alpha^+}; \mathbf{dom}\,(\Theta) \vdash N'$ and $\Gamma, \overrightarrow{\alpha^+} \vdash M'$ hold by inversion, and N' and M' are normalized, since N and M are. Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : UC$. By the induction hypothesis, $[\widehat{\sigma}]N' = M'$. Then $[\widehat{\sigma}]N = [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N' = \forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N' = \forall\overrightarrow{\alpha^+}.M' = M$.

Case 5. Rule $(\text{UVAR}^{\overset{u}{\simeq}}_{-})$, then $N = \widehat{\alpha}^-$, $\widehat{\alpha}^-\{\Delta\} \in \Theta$, and $\Delta \vdash M$. As the result, the algorithm returns $UC = (\widehat{\alpha}^- :\simeq M)$.
It is clear that $\widehat{\alpha}^-\{\Delta\} \vdash (\widehat{\alpha}^- :\simeq M)$, since $\Delta \vdash M$, meaning that $\Theta \vdash UC$.
Let us take an arbitrary $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : UC$. Since $UC = (\widehat{\alpha}^- :\simeq M)$, $\Theta \vdash \widehat{\sigma} : UC$ implies $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]\widehat{\alpha}^- : (\widehat{\alpha}^- :\simeq M)$. By inversion of Rule $(:\simeq^{\text{SAT}}_{-})$, it means $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]\widehat{\alpha}^- \simeq^{\leqslant} M$. This way, $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M$. Notice that $\widehat{\sigma}$ and N are normalized, and by lemma 45, so is $[\widehat{\sigma}]N$. Since both sides of $\Theta(\widehat{\alpha}^-) \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M$ are normalized, by lemma 35, we have $[\widehat{\sigma}]N = M$.

Case 6. The positive cases are proved symmetrically.

□

Lemma 64 (Completeness of Unification).

+ *For normalized P and Q such that $\Gamma; \mathbf{dom}\,(\Theta) \vdash P$ and $\Gamma \vdash Q$, suppose that there exists $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P)$ such that $[\widehat{\sigma}]P = Q$, then $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC$ for some UC.*
− *For normalized N and M such that $\Gamma; \mathbf{dom}\,(\Theta) \vdash N$ and $\Gamma \vdash M$, suppose that there exists $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(N)$ such that $[\widehat{\sigma}]N = M$, then $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv UC$ for some UC.*

Proof. We prove it by induction on the structure of P and mutually, N.

Case 1. $N = \widehat{\alpha}^-$
$\Gamma; \mathbf{dom}\,(\Theta) \vdash \widehat{\alpha}^-$ means that $\widehat{\alpha}^-\{\Delta\} \in \Theta$ for some Δ.
Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : \widehat{\alpha}^-$ such that $[\widehat{\sigma}]\widehat{\alpha}^- = M$. $\Theta \vdash \widehat{\sigma} : \widehat{\alpha}^-$ means that $\Delta \vdash M$.
This way, Rule $(\text{UVAR}^{\overset{u}{\simeq}}_{-})$ is applicable to infer $\Gamma; \Theta \vDash \widehat{\alpha}^- \overset{u}{\simeq} M \dashv (\widehat{\alpha}^- :\simeq M)$.

Case 2. $N = \alpha^-$
Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(\alpha^-)$ such that $[\widehat{\sigma}]\alpha^- = M$. Since $\mathbf{uv}\,(\alpha^-) = \varnothing$, $[\widehat{\sigma}]\alpha^- = M$ means $M = \alpha^-$.
This way, Rule $(\text{VAR}^{\overset{u}{\simeq}}_{-})$ infers $\Gamma; \Theta \vDash \alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot$, which is rewritten as $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv \cdot$.

Case 3. $N = \uparrow P$
Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P)$ such that $[\widehat{\sigma}]\uparrow P = M$. The latter means $\uparrow[\widehat{\sigma}]P = M$, i.e. $M = \uparrow Q$ for some Q and $[\widehat{\sigma}]P = Q$.

Let us show that the induction hypothesis is applicable to $[\widehat{\sigma}]P = Q$. Notice that P is normalized, since $N = \uparrow P$ is normalized, $\Gamma; \mathbf{dom}(\Theta) \vdash P$ holds by inversion of $\Gamma; \mathbf{dom}(\Theta) \vdash \uparrow P$, and $\Gamma \vdash Q$ holds by inversion of $\Gamma \vdash \uparrow Q$.

This way, by the induction hypothesis there exists UC such that $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC$.

Case 4. $N = P \to N'$

Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(P \to N')$ such that $[\widehat{\sigma}](P \to N') = M$. The latter means $[\widehat{\sigma}]P \to [\widehat{\sigma}]N' = M$, i.e. $M = Q \to M'$ for some Q and M', such that $[\widehat{\sigma}]P = Q$ and $[\widehat{\sigma}]N' = M'$.

Let us show that the induction hypothesis is applicable to $\Theta \vdash \widehat{\sigma}|_{\mathbf{uv}(P)} : \mathbf{uv}(P)$ and $[\widehat{\sigma}|_{\mathbf{uv}(P)}]P = Q$ (the latter holds since $[\widehat{\sigma}|_{\mathbf{uv}(P)}]P = [\widehat{\sigma}]P$ by lemma 53),

- P is normalized, since $N = P \to N'$ is normalized
- $\Gamma; \mathbf{dom}(\Theta) \vdash P$ follows from the inversion of $\Gamma; \mathbf{dom}(\Theta) \vdash P \to N'$,
- $\Gamma \vdash Q$.

Then by the induction hypothesis, $\Gamma; \Theta \vDash P \overset{u}{\simeq} Q \dashv UC_1$. Analogously, the induction hypothesis is applicable to $[\widehat{\sigma}|_{\mathbf{uv}\,N'}]N' = M'$, and thus, $\Gamma; \Theta \vDash N' \overset{u}{\simeq} M' \dashv UC_2$.

To apply Rule $(\to\overset{u}{\simeq})$ and infer the required $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv UC$, it is left to show that $\Theta \vdash UC_1 \& UC_2 = UC$. It holds by completeness of the unification constraint merge (lemma 62) for $\Theta \vdash UC_1 : \mathbf{uv}\,P$, $\Theta \vdash UC_2 : \mathbf{uv}\,N'$ (which hold by soundness), and $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(P) \cup \mathbf{uv}(N')$, which holds since $\mathbf{uv}(P) \cup \mathbf{uv}(N') = \mathbf{uv}(P \to N')$. Notice that by soundness, $\Theta \vdash \widehat{\sigma}|_{\mathbf{uv}(P)} : UC_1$, which implies $\Theta \vdash \widehat{\sigma} : UC_1$. Analogously, $\Theta \vdash \widehat{\sigma} : UC_2$.

Case 5. $N = \forall \overrightarrow{\alpha^+}.N'$

Let us take an arbitrary $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(N')$ such that $[\widehat{\sigma}]\forall \overrightarrow{\alpha^+}.N' = M$. The latter means $\forall \overrightarrow{\alpha^+}.[\widehat{\sigma}]N' = M$, i.e. $M = \forall \overrightarrow{\alpha^+}.M'$ for some M' such that $[\widehat{\sigma}]N' = M'$.

Let us show that the induction hypothesis is applicable to $[\widehat{\sigma}]N' = M'$. Notice that N' is normalized, since $N = \forall \overrightarrow{\alpha^+}.N'$ is normalized, $\Gamma, \overrightarrow{\alpha^+}; \mathbf{dom}(\Theta) \vdash N'$ follows from inversion of $\Gamma; \mathbf{dom}(\Theta) \vdash \forall \overrightarrow{\alpha^+}.N'$, $\Gamma, \overrightarrow{\alpha^+} \vdash M'$ follows from inversion of $\Gamma \vdash \forall \overrightarrow{\alpha^+}.M'$, and $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(N')$ by assumption.

This way, by the induction hypothesis, $\Gamma, \overrightarrow{\alpha^+}; \Theta \vDash N' \overset{u}{\simeq} M' \dashv UC$ exists and moreover, $\Theta \vdash \widehat{\sigma} : UC$. Hence, Rule $(\forall\overset{u}{\simeq})$ is applicable to infer $\Gamma; \Theta \vDash \forall \overrightarrow{\alpha^+}.N' \overset{u}{\simeq} \forall \overrightarrow{\alpha^+}.M' \dashv UC$, that is $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv UC$.

Case 6. The positive cases are proved symmetrically.

□

6.7 Anti-unification

Observation 6 (Anti-unification algorithm is deterministic).

- $+$ *If* $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ *and* $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$, *then* $\Xi = \Xi'$, $Q = Q'$, $\widehat{\tau}_1 = \widehat{\tau}'_1$, *and* $\widehat{\tau}_2 = \widehat{\tau}'_2$.
- $-$ *If* $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ *and* $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi', M', \widehat{\tau}'_1, \widehat{\tau}'_2)$, *then* $\Xi = \Xi'$, $M = M'$, $\widehat{\tau}_1 = \widehat{\tau}'_1$, *and* $\widehat{\tau}_2 = \widehat{\tau}'_2$.

Proof. By trivial induction on $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$. □

Observation 7. *Names of the anti-unification variables are uniquely defined by the types they are mapped to by the resulting substitutions.*

- + *Assuming P_1 and P_2 are normalized, if $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ then for any $\widehat{\beta}^- \in \Xi$, $\widehat{\beta}^- = \widehat{\alpha}^-_{\{[\widehat{\tau}_1]\widehat{\beta}^-, [\widehat{\tau}_2]\widehat{\beta}^-\}}$*
- − *Assuming N_1 and N_2 are normalized, if $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ then for any $\widehat{\beta}^- \in \Xi$, $\widehat{\beta}^- = \widehat{\alpha}^-_{\{[\widehat{\tau}_1]\widehat{\beta}^-, [\widehat{\tau}_2]\widehat{\beta}^-\}}$*

Proof. By simple induction on $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$. Let us consider tha last rule applied to infer this judgment.

Case 1. Rule (Var$^{\overset{a}{\simeq}}_+$) or Rule (Var$^{\overset{a}{\simeq}}_-$), then $\Xi = \cdot$, and the property holds vacuously.

Case 2. Rule (AU) Then $\Xi = \widehat{\alpha}^-_{\{N_1,N_2\}}$, $\widehat{\tau}_1 = \widehat{\alpha}^-_{\{N_1,N_2\}} \mapsto N_1$, and $\widehat{\tau}_2 = \widehat{\alpha}^-_{\{N_1,N_2\}} \mapsto N_2$. So the property holds trivially.

Case 3. Rule ($\to^{\overset{a}{\simeq}}$) In this case, $\Xi = \Xi' \cup \Xi''$, $\widehat{\tau}_1 = \widehat{\tau}'_1 \cup \widehat{\tau}''_1$, and $\widehat{\tau}_2 = \widehat{\tau}'_2 \cup \widehat{\tau}''_2$, where the property holds for $(\Xi', \widehat{\tau}'_1, \widehat{\tau}'_2)$ and $(\Xi'', \widehat{\tau}''_1, \widehat{\tau}''_2)$ by the induction hypothesis. Then since the union of solutions does not change the types the variables are mapped to, the required property holds for Ξ, $\widehat{\tau}_1$, and $\widehat{\tau}_2$.

Case 4. For the other rules, the resulting Ξ is taken from the recursive call and the required property holds immediately by the induction hypothesis.

□

Lemma 65 (Soundness of Anti-Unification).

- + *Assuming P_1 and P_2 are normalized, if $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ then*
  (1) $\Gamma; \Xi \vdash Q$,
  (2) $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ for $i \in \{1, 2\}$ are anti-unification substitutions, and
  (3) $[\widehat{\tau}_i]Q = P_i$ for $i \in \{1, 2\}$.
- − *Assuming N_1 and N_2 are normalized, if $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ then*
  (1) $\Gamma; \Xi \vdash M$,
  (2) $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ for $i \in \{1, 2\}$ are anti-unification substitutions, and
  (3) $[\widehat{\tau}_i]M = N_i$ for $i \in \{1, 2\}$.

Proof. We prove it by induction on $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually, $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$. Let us consider the last rule applied to infer this judgement.

Case 1. Rule (Var$^{\overset{a}{\simeq}}_-$), then $N_1 = \alpha^- = N_2$, $\Xi = \cdot$, $M = \alpha^-$, and $\widehat{\tau}_1 = \widehat{\tau}_2 = \cdot$.
(1) $\Gamma; \cdot \vdash \alpha^-$ follows from the assumption $\Gamma \vdash \alpha^-$,
(2) $\Gamma; \cdot \vdash \cdot : \cdot$ holds trivially, and
(3) $[\cdot]\alpha^- = \alpha^-$ holds trivially.

Case 2. Rule ($\uparrow^{\overset{a}{\simeq}}$), then $N_1 = \uparrow P_1$, $N_2 = \uparrow P_2$, and the algorithm makes the recursive call: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, returning $(\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ as the result.
Since $N_1 = \uparrow P_1$ and $N_2 = \uparrow P_2$ are normalized, so are P_1 and P_2, and thus, the induction hypothesis is applicable to $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$:
(1) $\Gamma; \Xi \vdash Q$, and hence, $\Gamma; \Xi \vdash \uparrow Q$,
(2) $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ for $i \in \{1, 2\}$, and
(3) $[\widehat{\tau}_i]Q = P_i$ for $i \in \{1, 2\}$, and then by the definition of the substitution, $[\widehat{\tau}_i]\uparrow Q = \uparrow P_i$ for $i \in \{1, 2\}$.

Case 3. Rule ($\rightarrow^{\stackrel{a}{\simeq}}$), then $N_1 = P_1 \rightarrow N_1'$, $N_2 = P_2 \rightarrow N_2'$, and the algorithm makes two recursive calls: $\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and $\Gamma \vDash N_1' \stackrel{a}{\simeq} N_2' \dashv (\Xi', M, \widehat{\tau}_1', \widehat{\tau}_2')$ and and returns $(\Xi \cup \Xi', Q \rightarrow M, \widehat{\tau}_1 \cup \widehat{\tau}_1', \widehat{\tau}_2 \cup \widehat{\tau}_2')$ as the result.

Notice that the induction hypothesis is applicable to $\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$: P_1 and P_2 are normalized, since $N_1 = P_1 \rightarrow N_1'$ and $N_2 = P_2 \rightarrow N_2'$ are normalized. Similarly, the induction hypothesis is applicable to $\Gamma \vDash N_1' \stackrel{a}{\simeq} N_2' \dashv (\Xi', M, \widehat{\tau}_1', \widehat{\tau}_2')$.

This way, by the induction hypothesis:

(1) $\Gamma; \Xi \vdash Q$ and $\Gamma; \Xi' \vdash M$. Then by weakening (corollary 21), $\Gamma; \Xi \cup \Xi' \vdash Q$ and $\Gamma; \Xi \cup \Xi' \vdash M$, which implies $\Gamma; \Xi \cup \Xi' \vdash Q \rightarrow M$;

(2) $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$ Then $\Gamma; \cdot \vdash \widehat{\tau}_i \cup \widehat{\tau}_i' : \Xi \cup \Xi'$ are well-defined anti-unification substitutions. Let us take an arbitrary $\widehat{\beta}^- \in \Xi \cup \Xi'$. If $\widehat{\beta}^- \in \Xi$. then $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ implies that $\widehat{\tau}_i$, and hence, $\widehat{\tau}_i \cup \widehat{\tau}_i'$ contains an entry well-formed in Γ. If $\widehat{\beta}^- \in \Xi'$, the reasoning is symmetric.
$\widehat{\tau}_i \cup \widehat{\tau}_i'$ is a well-defined anti-unification substitution: any anti-unification variable occurs uniquely $\widehat{\tau}_i \cup \widehat{\tau}_i'$, since by observation 7, the name of the variable is in one-to-one correspondence with the pair of types it is mapped to by $\widehat{\tau}_1$ and $\widehat{\tau}_2$, an is in one-to-one correspondence with the pair of types it is mapped to by $\widehat{\tau}_1'$ and $\widehat{\tau}_2'$ i.e. if $\widehat{\beta}^- \in \Xi \cap \Xi'$ then $[\widehat{\tau}_1]\widehat{\beta}^- = [\widehat{\tau}_1']\widehat{\beta}^-$, and $[\widehat{\tau}_2]\widehat{\beta}^- = [\widehat{\tau}_2']\widehat{\beta}^-$.

(3) $[\widehat{\tau}_i]Q = P_i$ and $[\widehat{\tau}_i']M = N_i'$. Since $\widehat{\tau}_i \cup \widehat{\tau}_i'$ restricted to Ξ is $\widehat{\tau}_i$, and $\widehat{\tau}_i \cup \widehat{\tau}_i'$ restricted to Ξ' is $\widehat{\tau}_i'$, we have $[\widehat{\tau}_i \cup \widehat{\tau}_i']Q = P_i$ and $[\widehat{\tau}_i \cup \widehat{\tau}_i']M = N_i'$, and thus, $[\widehat{\tau}_i \cup \widehat{\tau}_i']Q \rightarrow M = P_1 \rightarrow N_1'$

Case 4. Rule ($\forall^{\stackrel{a}{\simeq}}$), then $N_1 = \forall\overrightarrow{\alpha^+}.N_1'$, $N_2 = \forall\overrightarrow{\alpha^+}.N_2'$, and the algorithm makes a recursive call: $\Gamma \vDash N_1' \stackrel{a}{\simeq} N_2' \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ and returns $(\Xi, \forall\overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)$ as the result.

Similarly to case 2, we apply the induction hypothesis to $\Gamma \vDash N_1' \stackrel{a}{\simeq} N_2' \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ to obtain:

(1) $\Gamma; \Xi \vdash M$, and hence, $\Gamma; \Xi \vdash \forall\overrightarrow{\alpha^+}.M$;

(2) $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ for $i \in \{1, 2\}$, and

(3) $[\widehat{\tau}_i]M = N_i'$ for $i \in \{1, 2\}$, and then by the definition of the substitution, $[\widehat{\tau}_i]\forall\overrightarrow{\alpha^+}.M = \forall\overrightarrow{\alpha^+}.N_i'$ for $i \in \{1, 2\}$.

Case 5. Rule (AU), which applies when other rules do not, and $\Gamma \vdash N_i$, returning as the result $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2) = (\widehat{\alpha^-}_{\{N_1,N_2\}}, \widehat{\alpha^-}_{\{N_1,N_2\}}, (\widehat{\alpha^-}_{\{N_1,N_2\}} \mapsto N_1), (\widehat{\alpha^-}_{\{N_1,N_2\}} \mapsto N_2))$.

(1) $\Gamma; \Xi \vdash M$ is rewritten as $\Gamma; \widehat{\alpha^-}_{\{N_1,N_2\}} \vdash \widehat{\alpha^-}_{\{N_1,N_2\}}$, which holds trivially;

(2) $\Gamma; \cdot \vdash \widehat{\tau}_i : \Xi$ is rewritten as $\Gamma; \cdot \vdash (\widehat{\alpha^-}_{\{N_1,N_2\}} \mapsto N_i) : \widehat{\alpha^-}_{\{N_1,N_2\}}$, which holds since $\Gamma \vdash N_i$ by the premise of the rule;

(3) $[\widehat{\tau}_i]M = N_i$ is rewritten as $[\widehat{\alpha^-}_{\{N_1,N_2\}} \mapsto N_i]\widehat{\alpha^-}_{\{N_1,N_2\}} = N_i$, which holds trivially by the definition of substitution.

Case 6. Positive cases are proved symmetrically.

□

Lemma 66 (Completeness of Anti-Unification).

\+ *Assume that P_1 and P_2 are normalized, and there exists $(\Xi', Q', \widehat{\tau}_1', \widehat{\tau}_2')$ such that*
(1) $\Gamma; \Xi' \vdash Q'$,
(2) $\Gamma; \cdot \vdash \widehat{\tau}_i' : \Xi'$ for $i \in \{1, 2\}$ are anti-unification substitutions, and
(3) $[\widehat{\tau}_i']Q' = P_i$ for $i \in \{1, 2\}$.

Then the anti-unification algorithm terminates, that is there exists $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ *such that* $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$

– *Assume that* N_1 *and* N_2 *are normalized, and there exists* $(\Xi', M', \widehat{\tau}'_1, \widehat{\tau}'_2)$ *such that*
(1) $\Gamma; \Xi' \vdash M'$,
(2) $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ *for* $i \in \{1, 2\}$, *are anti-unification substitutions, and*
(3) $[\widehat{\tau}'_i]M' = N_i$ *for* $i \in \{1, 2\}$.
Then the anti-unification algorithm succeeds, that is there exists $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ *such that* $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$.

Proof. We prove it by the induction on M' and mutually on Q'.

Case 1. $M' = \widehat{\alpha}^-$ Then since $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$, $\Gamma \vdash [\widehat{\tau}'_i]M' = N_i$. This way, Rule (AU) is always applicable if other rules are not.

Case 2. $M' = \alpha^-$ Then $\alpha^- = [\widehat{\tau}'_i]\alpha^- = N_i$, which means that Rule (Var$^{\overset{a}{-}}_{-}$) is applicable.

Case 3. $M' = \uparrow Q'$ Then $\uparrow[\widehat{\tau}'_i]Q' = [\widehat{\tau}'_i]\uparrow Q' = N_i$, that is N_1 and N_2 have form $\uparrow P_1$ and $\uparrow P_2$ respectively.
Moreover, $[\widehat{\tau}'_i]Q' = P_i$, which means that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2. Then by the induction hypothesis, there exists $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, and hence, $\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ by Rule ($\uparrow^{\overset{a}{\simeq}}$).

Case 4. $M' = \forall\overrightarrow{\alpha^+}.M''$ This case is similar to the previous one: we consider $\forall\overrightarrow{\alpha^+}$ as a constructor. Notice that $\forall\overrightarrow{\alpha^+}.[\widehat{\tau}'_i]M'' = [\widehat{\tau}'_i]\forall\overrightarrow{\alpha^+}.M'' = N_i$, that is N_1 and N_2 have form $\forall\overrightarrow{\alpha^+}.N''_1$ and $\forall\overrightarrow{\alpha^+}.N''_2$ respectively.
Moreover, $[\widehat{\tau}'_i]M'' = N''_i$, which means that $(\Xi', M'', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of N''_1 and N''_2. Then by the induction hypothesis, there exists $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ such that $\Gamma \vDash N''_1 \overset{a}{\simeq} N''_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$, and hence, $\Gamma \vDash \forall\overrightarrow{\alpha^+}.N''_1 \overset{a}{\simeq} \forall\overrightarrow{\alpha^+}.N''_2 \dashv (\Xi, \forall\overrightarrow{\alpha^+}.M, \widehat{\tau}_1, \widehat{\tau}_2)$ by Rule ($\forall^{\overset{a}{\simeq}}$).

Case 5. $M' = Q' \to M''$ Then $[\widehat{\tau}'_i]Q' \to [\widehat{\tau}'_i]M'' = [\widehat{\tau}'_i](Q' \to M'') = N_i$, that is N_1 and N_2 have form $P_1 \to N'_1$ and $P_2 \to N'_2$ respectively.
Moreover, $[\widehat{\tau}'_i]Q' = P_i$ and $[\widehat{\tau}'_i]M'' = N''_i$, which means that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2, and $(\Xi', M'', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of N''_1 and N''_2. Then by the induction hypothesis, $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and $\Gamma \vDash N''_1 \overset{a}{\simeq} N''_2 \dashv (\Xi_2, M, \widehat{\tau}_3, \widehat{\tau}_4)$ succeed. The result of the algorithm is $(\Xi_1 \cup \Xi_2, Q \to M, \widehat{\tau}_1 \cup \widehat{\tau}_3, \widehat{\tau}_2 \cup \widehat{\tau}_4)$.

Case 6. $Q' = \widehat{\alpha}^+$ This case if not possible, since $\Gamma; \Xi' \vdash Q'$ means $\widehat{\alpha}^+ \in \Xi'$, but Ξ' can only contain negative variables.

Case 7. Other positive cases are proved symmetrically to the corresponding negative ones.

□

Lemma 67 (Initiality of Anti-Unification).

\+ *Assume that* P_1 *and* P_2 *are normalized, and* $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, *then* $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ *is more specific than any other sound anti-unifier* $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$, *i.e. if*
(1) $\Gamma; \Xi' \vdash Q'$,
(2) $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ *for* $i \in \{1, 2\}$, *and*
(3) $[\widehat{\tau}'_i]Q' = P_i$ *for* $i \in \{1, 2\}$
then there exists $\widehat{\rho}$ *such that* $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\ Q'})$ *and* $[\widehat{\rho}]Q' = Q$. *Moreover,* $[\widehat{\rho}]\widehat{\beta}^-$ *can be uniquely determined by* $[\widehat{\tau}'_1]\widehat{\beta}^-$, $[\widehat{\tau}'_2]\widehat{\beta}^-$, *and* Γ.

– *Assume that* N_1 *and* N_2 *are normalized, and* $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$, *then* $(\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$ *is more specific than any other sound anti-unifier* $(\Xi', M', \widehat{\tau}'_1, \widehat{\tau}'_2)$, *i.e. if*

(1) $\Gamma; \Xi' \vdash M'$,
(2) $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ *for* $i \in \{1, 2\}$, *and*
(3) $[\widehat{\tau}'_i]M' = N_i$ *for* $i \in \{1, 2\}$
then there exists $\widehat{\rho}$ *such that* $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\, M'})$ *and* $[\widehat{\rho}]M' = M$. *Moreover,* $[\widehat{\rho}]\widehat{\beta}^-$ *can be uniquely determined by* $[\widehat{\tau}'_1]\widehat{\beta}^-$, $[\widehat{\tau}'_2]\widehat{\beta}^-$, *and* Γ.

Proof. First, let us assume that M' is a algorithmic variable $\widehat{\alpha}^-$. Then we can take $\widehat{\rho} = \widehat{\alpha}^- \mapsto M$, which satisfies the required properties:

- $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\, M'})$ holds since $\Xi'|_{\mathbf{uv}\, M'} = \widehat{\alpha}^-$ and $\Gamma; \Xi \vdash M$ by the soundness of anti-unification (lemma 65);
- $[\widehat{\rho}]M' = M$ holds by construction
- $[\widehat{\rho}]\widehat{\alpha}^- = M$ is the anti-unifier of $N_1 = [\widehat{\tau}'_1]\widehat{\alpha}^-$ and $N_2 = [\widehat{\tau}'_2]\widehat{\alpha}^-$ in context Γ, and hence, it is uniquely determined by them (observation 6).

Now, we can assume that M' is not a algorithmic variable. We prove by induction on the derivation of $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ and mutually on the derivation of $\Gamma \vDash N_1 \overset{a}{\simeq} N_2 \dashv (\Xi, M, \widehat{\tau}_1, \widehat{\tau}_2)$.

Since M' is not a algorithmic variable, the substitution acting on M' preserves its outer constructor. In other words, $[\widehat{\tau}'_i]M' = N_i$ means that M', N_1 and N_2 have the same outer constructor. Let us consider the algorithmic anti-unification rule corresponding to this constructor, and show that it was successfully applied to anti-unify N_1 and N_2 (or P_1 and P_2).

Case 1. Rule $(\text{Var}^{\overset{a}{\simeq}}_{-})$, i.e. $N_1 = \alpha^- = N_2$. This rule is applicable since it has no premises.
Then $\Xi = \cdot$, $M = \alpha^-$, and $\widehat{\tau}_1 = \widehat{\tau}_2 = \cdot$. Since $[\widehat{\tau}'_i]M' = N_i = \alpha^-$ and M' is not a algorithmic variable, $M' = \alpha^-$. Then we can take $\widehat{\rho} = \cdot$, which satisfies the required properties:
- $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\, M'})$ holds vacuously since $\Xi'|_{\mathbf{uv}\, M'} = \varnothing$;
- $[\widehat{\rho}]M' = M$, that is $[\cdot]\alpha^- = \alpha^-$ holds by substitution properties;
- the unique determination of $[\widehat{\rho}]\widehat{\alpha}^-$ for $\widehat{\alpha}^- \in \Xi'|_{\mathbf{uv}\, M'} = \varnothing$ holds vacuously.

Case 2. Rule $(\uparrow^{\overset{a}{\simeq}})$, i.e. $N_1 = \uparrow P_1$ and $N_2 = \uparrow P_2$.
Then since $[\widehat{\tau}'_i]M' = N_i = \uparrow P_i$ and M' is not a algorithmic variable, $M' = \uparrow Q'$, where $[\widehat{\tau}'_i]Q' = P_i$. Let us show that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2.
(1) $\Gamma; \Xi' \vdash Q'$ holds by inversion of $\Gamma; \Xi' \vdash \uparrow Q'$;
(2) $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ holds by assumption;
(3) $[\widehat{\tau}'_i]Q' = P_i$ holds by assumption.
This way, by the completeness of anti-unification (lemma 66), the anti-unification algorithm succeeds on P_1 and P_2: $\Gamma \vDash P_1 \overset{a}{\simeq} P_2 \dashv (\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$, which means that Rule $(\uparrow^{\overset{a}{\simeq}})$ is applicable to infer $\Gamma \vDash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 \dashv (\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$.
Moreover, by the induction hypothesis, $(\Xi, Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$, which immediately implies that $(\Xi, \uparrow Q, \widehat{\tau}_1, \widehat{\tau}_2)$ is more specific than $(\Xi', \uparrow Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ (we keep the same $\widehat{\rho}$).

Case 3. Rule $(\forall^{\overset{a}{\simeq}})$, i.e. $N_1 = \forall \overrightarrow{\alpha^+}.N'_1$ and $N_2 = \forall \overrightarrow{\alpha^+}.N'_2$. The proof is symmetric to the previous case. Notice that the context Γ is not changed in Rule $(\forall^{\overset{a}{\simeq}})$, as it represents the context in which the anti-unification variables must be instantiated, rather than the context forming the types that are being anti-unified.

Case 4. Rule $(\to^{\overset{a}{\simeq}})$, i.e. $N_1 = P_1 \to N'_1$ and $N_2 = P_2 \to N'_2$.
Then since $[\widehat{\tau}'_i]M' = N_i = P_i \to N'_i$ and M' is not a algorithmic variable, $M' = Q' \to M''$, where $[\widehat{\tau}'_i]Q' = P_i$ and $[\widehat{\tau}'_i]M'' = N''_i$.
Let us show that $(\Xi', Q', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of P_1 and P_2.

(1) $\Gamma; \Xi' \vdash Q'$ holds by inversion of $\Gamma; \Xi' \vdash Q' \to M''$;
(2) $\Gamma; \cdot \vdash \widehat{\tau}'_i : \Xi'$ holds by assumption;
(3) $[\widehat{\tau}'_i]Q' = P_i$ holds by assumption.

Similarly, $(\Xi', M'', \widehat{\tau}'_1, \widehat{\tau}'_2)$ is an anti-unifier of N''_1 and N''_2.

Then by the completeness of anti-unification (lemma 66), the anti-unification algorithm succeeds on P_1 and P_2: $\Gamma \vDash P_1 \stackrel{a}{\simeq} P_2 \dashv (\Xi_1, Q, \widehat{\tau}_1, \widehat{\tau}_2)$; and on N'_1 and N'_2: $\Gamma \vDash N''_1 \stackrel{a}{\simeq} N''_2 \dashv (\Xi_2, M''', \widehat{\tau}_3, \widehat{\tau}_4)$. Notice that $\widehat{\tau}_1$ & $\widehat{\tau}_3$ and $\widehat{\tau}_2$ & $\widehat{\tau}_4$ are defined, in other words, for any $\widehat{\beta}^- \in \Xi_1 \cap \Xi_2$, $[\widehat{\tau}_1]\widehat{\beta}^- = [\widehat{\tau}_2]\widehat{\beta}^-$ and $[\widehat{\tau}_3]\widehat{\beta}^- = [\widehat{\tau}_4]\widehat{\beta}^-$, which follows immediately from observation 7. This way, the algorithm proceeds by applying Rule $(\to\stackrel{a}{\simeq})$ and returns $(\Xi_1 \cup \Xi_2, Q \to M''', \widehat{\tau}_1 \cup \widehat{\tau}_3, \widehat{\tau}_2 \cup \widehat{\tau}_4)$.

It is left to construct $\widehat{\rho}$ such that $\Gamma; \Xi \vdash \widehat{\rho} : (\Xi'|_{\mathbf{uv}\, M'})$ and $[\widehat{\rho}]M' = M$. By the induction hypothesis, there exist $\widehat{\rho}_1$ and $\widehat{\rho}_2$ such that $\Gamma; \Xi_1 \vdash \widehat{\rho}_1 : (\Xi'|_{\mathbf{uv}\, Q'})$, $\Gamma; \Xi_2 \vdash \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\, M''})$, $[\widehat{\rho}_1]Q' = Q$, and $[\widehat{\rho}_2]M'' = M'''$.

Let us show that $\widehat{\rho} = \widehat{\rho}_1 \cup \widehat{\rho}_2$ satisfies the required properties:

- $\Gamma; \Xi_1 \cup \Xi_2 \vdash \widehat{\rho}_1 \cup \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\, M'})$ holds since $\Xi'|_{\mathbf{uv}\, M'} = \Xi'|_{\mathbf{uv}\, Q' \to M''} = (\Xi'|_{\mathbf{uv}\, Q'}) \cup (\Xi'|_{\mathbf{uv}\, M''})$, $\Gamma; \Xi_1 \vdash \widehat{\rho}_1 : (\Xi'|_{\mathbf{uv}\, Q'})$ and $\Gamma; \Xi_2 \vdash \widehat{\rho}_2 : (\Xi'|_{\mathbf{uv}\, M''})$;
- $[\widehat{\rho}]M' = [\widehat{\rho}](Q' \to M'') = [\widehat{\rho}|_{\mathbf{uv}\, Q'}]Q' \to [\widehat{\rho}|_{\mathbf{uv}\, M''}]M'' = [\widehat{\rho}_1]Q' \to [\widehat{\rho}_2]M'' = Q \to M''' = M$;
- Since $[\widehat{\rho}]\widehat{\beta}^-$ is either equal to $[\widehat{\rho}_1]\widehat{\beta}^-$ or $[\widehat{\rho}_2]\widehat{\beta}^-$, it inherits their property that it is uniquely determined by $[\widehat{\tau}'_1]\widehat{\beta}^-$, $[\widehat{\tau}'_2]\widehat{\beta}^-$, and Γ.

Case 5. $P_1 = P_2 = \alpha^+$. This case is symmetric to case 1.

Case 6. $P_1 = \downarrow N_1$ and $P_2 = \downarrow N_2$. This case is symmetric to case 2

Case 7. $P_1 = \exists \overrightarrow{\alpha^-}.P'_1$ and $P_2 = \exists \overrightarrow{\alpha^-}.P'_2$. This case is symmetric to case 3

□

6.8 Upper Bounds

Observation 8 (Least Upper Bound algorithm is deterministic). *For types* $\Gamma \vdash P_1$, *and* $\Gamma \vdash P_2$, *if* $\Gamma \vDash P_1 \vee P_2 = Q$ *and* $\Gamma \vDash P_1 \vee P_2 = Q'$ *then* $Q = Q'$.

Proof. The shape of P_1 and P_2 uniquely determines the rule applied to infer the upper bound. By looking at the inference rules, it is easy to see that the result of the least upper bound algorithm depends on

- the inputs of the algorithm (that is P_1, P_2, and Γ), which are fixed;
- the result of the anti-unification algorithm applied to normalized input, which is deterministic by observation 6;
- the result of the recursive call, which is deterministic by the induction hypothesis.

□

Lemma 68 (Characterization of the Supertypes). *Let us define the set of upper bounds of a positive type* $\mathsf{UB}(P)$ *in the following way:*

$\Gamma \vdash P$	$\mathsf{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists \overrightarrow{\alpha^-}.\beta^+ \mid \textit{for } \overrightarrow{\alpha^-}\}$
$\Gamma \vdash \exists \overrightarrow{\beta^-}.Q$	$\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash Q)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \textit{for } \overrightarrow{\alpha^-}, M', \textit{and } \vec{N} \textit{ s.t.} \\ \Gamma \vdash N_i, \Gamma, \overrightarrow{\alpha^-} \vdash M', \textit{and } [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq^D \downarrow M \end{array} \right\}$

Then $\mathsf{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geqslant P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$
Immediately from lemma 19

Case 2. $P = \exists \overrightarrow{\beta^-}.P'$
Then if $\Gamma \vdash Q \geqslant \exists \overrightarrow{\beta^-}.P'$, then by lemma 18, $\Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant P'$, and $\mathbf{fv}\, Q \cap \overrightarrow{\beta^-} = \varnothing$ by the convention. The other direction holds by Rule ($\exists^{\geqslant}$). This way, $\{Q \mid \Gamma \vdash Q \geqslant \exists \overrightarrow{\beta^-}.P'\} = \{Q \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant P' \text{ s.t. } \mathbf{fv}\,(Q) \cap \overrightarrow{\beta^-} = \varnothing\}$. From the induction hypothesis, the latter is equal to $\mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$, i.e. $\mathsf{UB}(\Gamma \vdash \exists \overrightarrow{\beta^-}.P')$.

Case 3. $P = \downarrow M$
Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|_{\nexists}$) and upper bounds with outer quantifiers ($|_{\exists}$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists \overrightarrow{\beta^-}.Q'$
Then the last applied rule to infer $\Gamma \vdash Q \geqslant \downarrow M$ must be Rule ($\downarrow^{\geqslant}$), which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq^{\leqslant} M$, then by lemma 34 and Rule ($\downarrow^{\simeq^D}$), $\downarrow M' \simeq^D \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq^D \downarrow M\} = \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\nexists}$.
In the other direction, $\downarrow M' \simeq^D \downarrow M \Rightarrow \Gamma \vdash \downarrow M' \simeq^{\leqslant} \downarrow M$ by lemma 29, since $\Gamma \vdash \downarrow M'$ by lemma 2
$\Rightarrow \Gamma \vdash \downarrow M' \geqslant \downarrow M$ by inversion

b. $Q = \exists \overrightarrow{\beta^-}.Q'$ (for non-empty $\overrightarrow{\beta^-}$)
Then the last rule applied to infer $\Gamma \vdash \exists \overrightarrow{\beta^-}.Q' \geqslant \downarrow M$ must be Rule ($\exists^{\geqslant}$). Inversion of this rule gives us $\Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]Q' \geqslant \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\vec{N}/\overrightarrow{\beta^-}]Q'$ has no outer quantifiers. Thus from case 3.*a*, $[\vec{N}/\overrightarrow{\beta^-}]Q' \simeq^D \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists \overrightarrow{\beta^-}.\downarrow M' \in \mathsf{UB}(\Gamma \vdash \downarrow M)|_{\exists}$ (notice that $\overrightarrow{\beta^-}$ is not empty).
In the other direction, $[\vec{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq^D \downarrow M \Rightarrow \Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]\downarrow M' \simeq^{\leqslant} \downarrow M$ by lemma 29, since $\Gamma \vdash [$
$\Rightarrow \Gamma \vdash [\vec{N}/\overrightarrow{\beta^-}]\downarrow M' \geqslant \downarrow M$ by inversion
$\Rightarrow \Gamma \vdash \exists \overrightarrow{\beta^-}.\downarrow M' \geqslant \downarrow M$ by Rule ($\exists^{\geqslant}$)

□

Lemma 69 (Characterization of the Normalized Supertypes). *For a normalized positive type* $P = \mathbf{nf}\,(P)$*, let us define the set of normalized upper bounds in the following way:*

$\Gamma \vdash P$	$\mathsf{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists\overrightarrow{\beta^-}.P$	$\mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P)$ *not using* $\overrightarrow{\beta^-}$
$\Gamma \vdash \downarrow M$	$\left\{ \exists\overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i, \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow M \end{array} \right\}$

Then $\mathsf{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Then from lemma 68, $\{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant \beta^+\} = \{\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\beta^+) \mid \text{for some } \overrightarrow{\alpha^-}\} = \{\beta^+\}$

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$

$$\begin{aligned}
\mathsf{NFUB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P') &= \mathsf{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ not using } \overrightarrow{\beta^-} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant P'\} \text{ not using } \overrightarrow{\beta^-} && \text{by the induction hypothe} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant P' \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{because } \mathbf{fv}\,\mathbf{nf}\,(Q) = \mathbf{fv}\,Q \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv}\,Q \cap \overrightarrow{\beta^-} = \varnothing\} && \text{by lemma 68} \\
&= \{\mathbf{nf}\,(Q) \mid Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}.P')\} && \text{by the definition of UB} \\
&= \{\mathbf{nf}\,(Q) \mid \Gamma \vdash Q \geqslant \exists\overrightarrow{\beta^-}.P'\} && \text{by lemma 68}
\end{aligned}$$

Case 3. $P = \downarrow M$ Let us prove the set equality by two inclusions.

$\subseteq$ Suppose that $\Gamma \vdash Q \geqslant \downarrow M$ and M is normalized.

By lemma 68, $Q \in \mathsf{UB}(\Gamma \vdash \downarrow M)$. Then by definition of UB, $Q = \exists\overrightarrow{\alpha^-}.\downarrow M'$ for some $\overrightarrow{\alpha^-}$, M', and $\Gamma \vdash \sigma : \overrightarrow{\alpha^-}$ s.t. $[\sigma]\downarrow M' \simeq^D \downarrow M$.

We need to show that $\mathbf{nf}\,(Q) \in \mathsf{NFUB}(\Gamma \vdash \downarrow M)$. Notice that $\mathbf{nf}\,(Q) = \mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.\downarrow M') = \exists\overrightarrow{\alpha^-}_0.\downarrow M_0$, where $\mathbf{nf}\,(M') = M_0$ and $\mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M_0 = \overrightarrow{\alpha^-}_0$.

The belonging of $\exists\overrightarrow{\alpha^-}_0.\downarrow M_0$ to $\mathsf{NFUB}(\Gamma \vdash \downarrow M)$ means that

(1) $\mathbf{ord}\,\overrightarrow{\alpha^-}_0\,\mathbf{in}\,M_0 = \overrightarrow{\alpha^-}_0$ and

(2) that there exists $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^-}_0$ such that $[\sigma_0]\downarrow M_0 = \downarrow M$.

The first requirement holds by corollary 18. To show the second requirement, we construct σ_0 as $\mathbf{nf}\,(\sigma|_{\mathbf{fv}\,M'})$. Let us show the required properties of σ_0:

(1) $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^-}_0$. Notice that by lemma 7, $\Gamma \vdash \sigma|_{\mathbf{fv}\,(M')} : \overrightarrow{\alpha^-} \cap \mathbf{fv}\,(M')$, which we rewrite as $\Gamma \vdash \sigma|_{\mathbf{fv}\,(M')} : \overrightarrow{\alpha^-}_0$ (since by lemma 37 $\overrightarrow{\alpha^-}_0 = \overrightarrow{\alpha^-} \cap \mathbf{fv}\,M_0$ as sets, and $\mathbf{fv}\,(M_0) = \mathbf{fv}\,(M')$ by lemma 43). Then by lemma 28, $\Gamma \vdash \mathbf{nf}\,(\sigma|_{\mathbf{fv}\,(M')}) : \overrightarrow{\alpha^-}_0$, that is $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^-}_0$.

(2) $[\sigma_0]\downarrow M_0 = \downarrow M$. $[\sigma]\downarrow M' \simeq^D \downarrow M$ means $[\sigma|_{\mathbf{fv}\,(M')}]\downarrow M' \simeq^D \downarrow M$ by lemma 6. Then by lemma 57, $\mathbf{nf}\,([\sigma|_{\mathbf{fv}\,(M')}]\downarrow M') = \mathbf{nf}\,(\downarrow M)$, implying $[\sigma_0]\downarrow M_0 = \mathbf{nf}\,(\downarrow M)$ by lemma 45, and further $[\sigma_0]\downarrow M_0 = \downarrow M$ by lemma 47 (since $\downarrow M$ is normal by assumption).

$\supseteq$ Suppose that a type belongs to $\mathsf{NFUB}(\Gamma \vdash \downarrow M)$ for a normalized $\downarrow M$. Then it must have shape $\exists\overrightarrow{\alpha^-}_0.\downarrow M_0$ for some $\overrightarrow{\alpha^-}_0$, M_0, and $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^-}_0$ such that $\mathbf{ord}\,\overrightarrow{\alpha^-}_0\,\mathbf{in}\,M_0 = \overrightarrow{\alpha^-}_0$

and $[\sigma_0]\downarrow M_0 = \downarrow M$. It is suffices to show that (1) $\exists\overrightarrow{\alpha^-}_0.\downarrow M_0$ is normalized itself, and (2) $\Gamma \vdash \exists\overrightarrow{\alpha^-}_0.\downarrow M_0 \geqslant \downarrow M$.

(1) By definition, $\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}_0.\downarrow M_0) = \exists\overrightarrow{\alpha^-}_1.\downarrow M_1$, where $M_1 = \mathbf{nf}\,(M_0)$ and $\mathbf{ord}\,\overrightarrow{\alpha^-}_0\,\mathbf{in}\,M_1 = \overrightarrow{\alpha^-}_1$. First, notice that by lemmas 41 and 44, $\mathbf{ord}\,\overrightarrow{\alpha^-}_0\,\mathbf{in}\,M_1 = \mathbf{ord}\,\overrightarrow{\alpha^-}_0\,\mathbf{in}\,M_0 = \overrightarrow{\alpha^-}_0$. This way, $\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}_0.\downarrow M_0) = \exists\overrightarrow{\alpha^-}_0.\downarrow\mathbf{nf}\,(M_0)$. Second, M_0 is normalized by lemma 48, since $[\sigma_0]\downarrow M_0 = \downarrow M$ is normal. As such, $\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}_0.\downarrow M_0) = \exists\overrightarrow{\alpha^-}_0.\downarrow M_0$, in other words, $\exists\overrightarrow{\alpha^-}_0.\downarrow M_0$ is normalized.

(2) $\Gamma \vdash \exists\overrightarrow{\alpha^-}_0.\downarrow M_0 \geqslant \downarrow M$ holds immediately by Rule ($\exists^{\geqslant}$) with the substitution σ_0. Notice that $\Gamma \vdash [\sigma_0]\downarrow M_0 \geqslant \downarrow M$ follows from $[\sigma_0]\downarrow M_0 = \downarrow M$ by reflexivity of subtyping (lemma 22).

□

Lemma 70. *Upper bounds of a type do not depend on the context as soon as the type is well-formed in it.*

If $\Gamma_1 \vdash P$ *and* $\Gamma_2 \vdash P$ *then* $\mathsf{UB}(\Gamma_1 \vdash P) = \mathsf{UB}(\Gamma_2 \vdash P)$ *and* $\mathsf{NFUB}(\Gamma_1 \vdash P) = \mathsf{NFUB}(\Gamma_2 \vdash P)$

Proof. We prove both inclusions by structural induction on P.

Case 1. $P = \beta^+$ Then $\mathsf{UB}(\Gamma_1 \vdash \beta^+) = \mathsf{UB}(\Gamma_2 \vdash \beta^+) = \{\exists\overrightarrow{\alpha^-}.\beta^+ \mid \text{for some } \overrightarrow{\alpha^-}\}$. $\mathsf{NFUB}(\Gamma_1 \vdash \beta^+) = \mathsf{NFUB}(\Gamma_2 \vdash \beta^+) = \{\beta^+\}$.

Case 2. $P = \exists\overrightarrow{\beta^-}.P'$. Then $\mathsf{UB}(\Gamma_1 \vdash \exists\overrightarrow{\beta^-}.P') = \mathsf{UB}(\Gamma_1, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$. $\mathsf{UB}(\Gamma_2 \vdash \exists\overrightarrow{\beta^-}.P') = \mathsf{UB}(\Gamma_2, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$. By the induction hypothesis, $\mathsf{UB}(\Gamma_1, \overrightarrow{\beta^-} \vdash P') = \mathsf{UB}(\Gamma_2, \overrightarrow{\beta^-} \vdash P')$, and if we restrict these sets to the same domain, they stay equal. Analogously, $\mathsf{NFUB}(\Gamma_1 \vdash \exists\overrightarrow{\beta^-}.P') = \mathsf{NFUB}(\Gamma_2 \vdash \exists\overrightarrow{\beta^-}.P')$.

Case 3. $P = \downarrow M$. Suppose that $\exists\overrightarrow{\alpha^-}.\downarrow M' \in \mathsf{UB}(\Gamma_1 \vdash \downarrow M)$. It means that $\Gamma_1, \overrightarrow{\alpha^-} \vdash M'$ and there exist $\Gamma_1 \vdash \overrightarrow{N}$ s.t. $[\overrightarrow{N}/\overrightarrow{\alpha^-}]\downarrow M' \simeq^D \downarrow M$, or in other terms, there exists $\Gamma_1 \vdash \sigma : \overrightarrow{\alpha^-}$ such that $[\sigma]\downarrow M' \simeq^D \downarrow M$.

We need to show that $\exists\overrightarrow{\alpha^-}.\downarrow M' \in UB(\Gamma_2 \vdash \downarrow M)$, in other words, $\Gamma_2, \overrightarrow{\alpha^-} \vdash M'$ and there exists $\Gamma_2 \vdash \sigma_0 : \overrightarrow{\alpha^-}$ such that $[\sigma_0]\downarrow M' \simeq^D \downarrow M$.

First, let us show $\Gamma_2, \overrightarrow{\alpha^-} \vdash M'$. Notice that $[\sigma]\downarrow M' \simeq^D \downarrow M$ implies $\mathbf{fv}\,([\sigma]M') = \mathbf{fv}\,(\downarrow M)$ by lemma 42. By lemma 15, $\mathbf{fv}\,(M')\backslash\overrightarrow{\alpha^-} \subseteq \mathbf{fv}\,([\sigma]M')$. This way, $\mathbf{fv}\,(M')\backslash\overrightarrow{\alpha^-} \subseteq \mathbf{fv}\,(M)$, implying $\mathbf{fv}\,(M') \subseteq \mathbf{fv}\,(M) \cup \overrightarrow{\alpha^-}$. By lemma 3, $\Gamma_2 \vdash \downarrow M$ implies $\mathbf{fv}\,M \subseteq \Gamma_2$, hence, $\mathbf{fv}\,M' \subseteq (\Gamma_2, \overrightarrow{\alpha^-})$, which by corollary 1 means $\Gamma_2, \overrightarrow{\alpha^-} \vdash M'$.

Second, let us construct the required σ_0 in the following way:

$$\begin{cases} [\sigma_0]\alpha_i^- = [\sigma]\alpha_i^- & \text{for } \alpha_i^- \in \overrightarrow{\alpha^-} \cap \mathbf{fv}\,(M') \\ [\sigma_0]\alpha_i^- = \forall\gamma^+.\uparrow\gamma^+ & \text{for } \alpha_i^- \in \overrightarrow{\alpha^-}\backslash\mathbf{fv}\,(M') \\ [\sigma_0]\gamma^\pm = \gamma^\pm & \text{for any other } \gamma^\pm \end{cases}$$

This construction of a substitution coincides with the one from the proof of lemma 20. This way, for σ_0, hold the same properties:

(1) $[\sigma_0]M' = [\sigma]M'$, which in particular, implies $[\sigma_0]\downarrow M = [\sigma]\downarrow M$, and thus, $[\sigma]\downarrow M' \simeq^D \downarrow M$ can be rewritten to $[\sigma_0]\downarrow M' \simeq^D \downarrow M$; and

(2) $\mathbf{fv}\,([\sigma]M') \vdash \sigma_0 : \overrightarrow{\alpha^-}$, which, as noted above, can be rewritten to $\mathbf{fv}\,(M) \vdash \sigma_0 : \overrightarrow{\alpha^-}$, and since $\mathbf{fv}\,M \subseteq \Gamma_2$, weakened to $\Gamma_2 \vdash \sigma_0 : \overrightarrow{\alpha^-}$.

The proof of $\mathsf{NFUB}(\Gamma_1 \vdash \downarrow M) \subseteq \mathsf{NFUB}(\Gamma_2 \vdash \downarrow M)$ is analogous. The differences are:

(1) $\mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}$ holds by assumption,
(2) $[\sigma]{\downarrow}M' = {\downarrow}M$ implies $\mathbf{fv}\,([\sigma]M') = \mathbf{fv}\,({\downarrow}M)$ by rewriting,
(3) $[\sigma]{\downarrow}M' = {\downarrow}M$ and $[\sigma_0]{\downarrow}M = [\sigma]{\downarrow}M$ imply $[\sigma_0]{\downarrow}M' = {\downarrow}M$ by rewriting.

□

Lemma 71 (Soundness of the Least Upper Bound). *For types* $\Gamma \vdash P_1$, *and* $\Gamma \vdash P_2$, *if* $\Gamma \vDash P_1 \vee P_2 = Q$ *then*

(i) $\Gamma \vdash Q$
(ii) $\Gamma \vdash Q \geqslant P_1$ *and* $\Gamma \vdash Q \geqslant P_2$

Proof. Induction on $\Gamma \vDash P_1 \vee P_2 = Q$.

Case 1. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$
Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant \alpha^+$ by Rule ($\textsc{Var}^{\geqslant}_{+}$).

Case 2. $\Gamma \vDash \exists\overrightarrow{\alpha^-}.P_1 \vee \exists\overrightarrow{\beta^-}.P_2 = Q$
Then by inversion of $\Gamma \vdash \exists\overrightarrow{\alpha^-}.P_i$ and weakening, $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash P_i$, hence, the induction hypothesis applies to $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vDash P_1 \vee P_2 = Q$. Then
(i) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q$,
(ii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant P_1$,
(iii) $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant P_2$.
To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}\,(Q) \cap (\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}) = \mathbf{fv}\,(Q) \cap \Gamma$ (and then apply lemma 4). The inclusion right-to-left is self-evident. To show $\mathbf{fv}\,(Q) \cap (\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-}) \subseteq \mathbf{fv}\,(Q) \cap \Gamma$, we prove that $\mathbf{fv}\,(Q) \subseteq \Gamma$

$$\begin{aligned}
\mathbf{fv}\,(Q) &\subseteq \mathbf{fv}\,P_1 \cap \mathbf{fv}\,P_2 && \text{by lemma 17}\\
&\subseteq ((\Gamma, \overrightarrow{\alpha^-})\backslash\overrightarrow{\beta^-}) \cap ((\Gamma, \overrightarrow{\beta^-})\backslash\overrightarrow{\alpha^-}) && \text{since } \Gamma \vdash \exists\overrightarrow{\alpha^-}.P_1,\ \mathbf{fv}\,(P_1) \subseteq (\Gamma, \overrightarrow{\alpha^-}) = (\Gamma, \overrightarrow{\alpha^-})\backslash\overrightarrow{\beta^-}\\
& && \text{(the latter is because by the Barendregt's convention,}\\
& && (\Gamma, \overrightarrow{\alpha^-}) \cap \overrightarrow{\beta^-} = \varnothing\text{); similarly, } \mathbf{fv}\,(P_2) \subseteq (\Gamma, \overrightarrow{\beta^-})\backslash\overrightarrow{\alpha^-}\\
&\subseteq \Gamma
\end{aligned}$$

To show $\Gamma \vdash Q \geqslant \exists\overrightarrow{\alpha^-}.P_1$, we apply Rule ($\exists^{\geqslant}$). Then $\Gamma, \overrightarrow{\alpha^-} \vdash Q \geqslant P_1$ holds since $\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \vdash Q \geqslant P_1$ (by the induction hypothesis), $\Gamma, \overrightarrow{\alpha^-} \vdash Q$ (by weakening), and $\Gamma, \overrightarrow{\alpha^-} \vdash P_1$.
Judgment $\Gamma \vdash Q \geqslant \exists\overrightarrow{\beta^-}.P_2$ is proved symmetrically.

Case 3. $\Gamma \vDash {\downarrow}N \vee {\downarrow}M = \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$. By the inversion, $\Gamma, \cdot \vDash \mathbf{nf}\,({\downarrow}N) \stackrel{a}{\simeq} \mathbf{nf}\,({\downarrow}M) \dashv (\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$. Then by the soundness of anti-unification (lemma 65),
(i) $\Gamma; \Xi \vdash P$, then by lemma 52,

$$\Gamma, \overrightarrow{\alpha^-} \vdash [\overrightarrow{\alpha^-}/\Xi]P \tag{7}$$

(ii) $\Gamma; \cdot \vdash \widehat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \widehat{\tau}_2 : \Xi$. Assuming that $\Xi = \widehat{\beta_1^-}, .., \widehat{\beta_n^-}$, the antiunification solutions $\widehat{\tau}_1$ and $\widehat{\tau}_2$ can be put explicitly as $\widehat{\tau}_1 = (\widehat{\beta_1^-} :\simeq N_1, .., \widehat{\beta_n^-} :\simeq N_n)$, and $\widehat{\tau}_2 = (\widehat{\beta_1^-} :\simeq M_1, .., \widehat{\beta_n^-} :\simeq M_n)$. Then

$$\widehat{\tau}_1 = (\overrightarrow{N}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \tag{8}$$

$$\widehat{\tau}_2 = (\overrightarrow{M}/\overrightarrow{\alpha^-}) \circ (\overrightarrow{\alpha^-}/\Xi) \tag{9}$$

(iii) $[\widehat{\tau_1}]Q = P_1$ and $[\widehat{\tau_2}]Q = P_1$, which, by 8 and 9, means

$$[\vec{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P = \mathbf{nf}\,(\downarrow N) \tag{10}$$

$$[\vec{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P = \mathbf{nf}\,(\downarrow M) \tag{11}$$

Then $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P$ follows directly from 7.
To show $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant \downarrow N$, we apply Rule ($\exists^{\geqslant}$), instantiating $\overrightarrow{\alpha^-}$ with $\vec{N}$. Then $\Gamma \vdash [\vec{N}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P \geqslant \downarrow N$ follows from 10 and since $\Gamma \vdash \mathbf{nf}\,(\downarrow N) \geqslant \downarrow N$ (by corollary 13). Analogously, instantiating $\overrightarrow{\alpha^-}$ with $\vec{M}$, gives us $\Gamma \vdash [\vec{M}/\overrightarrow{\alpha^-}][\overrightarrow{\alpha^-}/\Xi]P \geqslant \downarrow M$ (from 11), and hence, $\Gamma \vdash \exists\overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\Xi]P \geqslant \downarrow M$.

□

Lemma 72 (Completeness and Initiality of the Least Upper Bound). *For types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q$ such that $\Gamma \vdash Q \geqslant P_1$ and $\Gamma \vdash Q \geqslant P_2$, there exists Q' s.t. $\Gamma \vDash P_1 \vee P_2 = Q'$ and $\Gamma \vdash Q \geqslant Q'$.*

Proof. Induction on the pair (P_1, P_2). From lemma 69, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$. Let us consider the cases of what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_i$).

Case 1. $P_1 = \exists\overrightarrow{\beta^-}_1.Q_1$, $P_2 = \exists\overrightarrow{\beta^-}_2.Q_2$, where either $\overrightarrow{\beta^-}_1$ or $\overrightarrow{\beta^-}_2$ is not empty

Then $Q \in \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_1.Q_1) \cap \mathsf{UB}(\Gamma \vdash \exists\overrightarrow{\beta^-}_2.Q_2)$

$\subseteq \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_2 \vdash Q_2)$ from the definition of U

$= \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_1) \cap \mathsf{UB}(\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q_2)$ by lemma 70, weakening

$= \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant Q_1\} \cap \{Q' \mid \Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q' \geqslant Q_2\}$ by lemma 68,

meaning that $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant Q_1$ and $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant Q_2$. Then the next step of the algorithm—the recursive call $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vDash Q_1 \vee Q_2 = Q'$ terminates by the induction hypothesis, and moreover, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant Q'$. This way, the result of the algorithm is Q', i.e. $\Gamma \vDash P_1 \vee P_2 = Q'$.
Since both Q and Q' are sound upper bounds, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \overrightarrow{\beta^-}_1, \overrightarrow{\beta^-}_2 \vdash Q \geqslant Q'$ can be strengthened to $\Gamma \vdash Q \geqslant Q'$ by lemma 20.

Case 2. $P_1 = \alpha^+$ and $P_2 = \downarrow N$
Then the set of common upper bounds of $\downarrow N$ and α^+ is empty, and thus, $Q \in \mathsf{UB}(\Gamma \vdash P_1) \cap \mathsf{UB}(\Gamma \vdash P_2)$ gives a contradiction:
$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \downarrow N)$

$= \{\exists\overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists\overrightarrow{\beta^-}.\downarrow M' \mid \cdots\}$ by the definition of UB

$= \varnothing$ since $\alpha^+ \neq \downarrow M'$ for any M'

Case 3. $P_1 = \downarrow N$ and $P_2 = \alpha^+$
Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)
Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$$Q \in \mathsf{UB}(\Gamma \vdash \alpha^+) \cap \mathsf{UB}(\Gamma \vdash \beta^+)$$
$$= \{\exists \overrightarrow{\alpha^-}.\alpha^+ \mid \cdots\} \cap \{\exists \overrightarrow{\beta^-}.\beta^+ \mid \cdots\} \quad \text{by the definition of UB}$$
$$= \varnothing \quad \text{since } \alpha^+ \neq \beta^+$$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Then the algorithm terminates in one step (Rule ($\textsc{Var}^\vee$)) and the result is α^+, i.e. $\Gamma \vDash \alpha^+ \vee \alpha^+ = \alpha^+$.

Since $Q \in \mathsf{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists \overrightarrow{\alpha^-}.\alpha^+$. Then $\Gamma \vdash \exists \overrightarrow{\alpha^-}.\alpha^+ \geqslant \alpha^+$ by Rule ($\exists^{\geqslant}$): $\overrightarrow{\alpha^-}$ can be instantiated with arbitrary negative types (for example $\forall\beta^+.\uparrow\beta^+$), since the substitution for unused variables does not change the term $[\vec{N}/\overrightarrow{\alpha^-}]\alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geqslant \alpha^+$ by Rule ($\textsc{Var}^{\geqslant}_+$).

Case 6. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$

Then on the next step, the algorithm tries to anti-unify $\mathbf{nf}\,(\downarrow M_1)$ and $\mathbf{nf}\,(\downarrow M_2)$. By lemma 66, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$$\mathbf{nf}\,(Q) \in \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,(\downarrow M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \mathbf{nf}\,(\downarrow M_2))$$
$$= \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}\,(M_1)) \cap \mathsf{NFUB}(\Gamma \vdash \downarrow\mathbf{nf}\,(M_2))$$
$$\left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash N_i, \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1) \end{array} \right\}$$
$$= \cap$$
$$\left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \vec{N}_1, \Gamma \vdash \vec{N}_2, \Gamma, \overrightarrow{\alpha^-} \vdash M', \text{ and } [\vec{N}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2) \end{array} \right\}$$
$$= \left\{ \exists \overrightarrow{\alpha^-}.\downarrow M' \;\middle|\; \begin{array}{l} \text{for } \overrightarrow{\alpha^-}, M', \vec{N}_1 \text{ and } \vec{N}_2 \text{ s.t. } \mathbf{ord}\,\overrightarrow{\alpha^-}\,\mathbf{in}\,M' = \overrightarrow{\alpha^-}, \\ \Gamma \vdash \vec{N}_1, \Gamma \vdash \vec{N}_2, \Gamma, \overrightarrow{\alpha^-} \vdash M', [\vec{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1), \text{ and } [\vec{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \end{array} \right\}$$

The fact that the latter set is non-empty means that there exist $\overrightarrow{\alpha^-}, M', \vec{N}_1$ and $\vec{N}_2$ such that

(i) $\Gamma, \overrightarrow{\alpha^-} \vdash M'$ (notice that M' is normal)
(ii) $\Gamma \vdash \vec{N}_1$ and $\Gamma \vdash \vec{N}_1$,
(iii) $[\vec{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1)$ and $[\vec{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2)$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative anti-unification variable $\widehat{\alpha^-}$, and denote the list of these variables as $\overrightarrow{\widehat{\alpha^-}}$. Let us show that $(\overrightarrow{\widehat{\alpha^-}}, [\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]\downarrow M', \vec{N}_1/\overrightarrow{\widehat{\alpha^-}}, \vec{N}_2/\overrightarrow{\widehat{\alpha^-}}$ is a sound anti-unifier of $\mathbf{nf}\,(\downarrow M_1)$ and $\mathbf{nf}\,(\downarrow M_2)$ in context Γ:

- $\overrightarrow{\widehat{\alpha^-}}$ is negative by construction,
- $\Gamma; \overrightarrow{\widehat{\alpha^-}} \vdash [\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]\downarrow M'$ because $\Gamma, \overrightarrow{\alpha^-} \vdash \downarrow M'$ (lemma 51),
- $\Gamma; \cdot \vdash (\vec{N}_1/\overrightarrow{\widehat{\alpha^-}}) : \overrightarrow{\widehat{\alpha^-}}$ because $\Gamma \vdash \vec{N}_1$ and $\Gamma; \cdot \vdash (\vec{N}_2/\overrightarrow{\widehat{\alpha^-}}) : \overrightarrow{\widehat{\alpha^-}}$ because $\Gamma \vdash \vec{N}_2$,
- $[\vec{N}_1/\overrightarrow{\widehat{\alpha^-}}][\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]\downarrow M' = [\vec{N}_1/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_1) = \mathbf{nf}\,(\downarrow M_1)$.
- $[\vec{N}_2/\overrightarrow{\widehat{\alpha^-}}][\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]\downarrow M' = [\vec{N}_2/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow\mathbf{nf}\,(M_2) = \mathbf{nf}\,(\downarrow M_2)$.

Then by the completeness of the anti-unification (lemma 66), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it, i.e. $Q' = \exists \overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, where $(\Xi, P, \widehat{\tau}_1, \widehat{\tau}_2)$ is the result of the anti-unification of $\mathbf{nf}\,(\downarrow M_1)$ and $\mathbf{nf}\,(\downarrow M_2)$ in context Γ.

Moreover, lemma 66 also says that the found anti-unification solution is initial, i.e. there exists $\widehat{\tau}$ such that $\Gamma; \Xi \vdash \widehat{\tau} : \overrightarrow{\widehat{\alpha^-}}$ and $[\widehat{\tau}][\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]\downarrow M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}$, (ii) $\widehat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\Xi$. Notice that $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi][\widehat{\tau}][\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}]{\downarrow}M' = [\overrightarrow{\beta^-}/\Xi]P$. In particular, from the reflexivity of subtyping: $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]{\downarrow}M' \geqslant [\overrightarrow{\beta^-}/\Xi]P$.
It allows us to show $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant Q'$, i.e. $\Gamma \vdash \exists\overrightarrow{\alpha^-}.{\downarrow}M' \geqslant \exists\overrightarrow{\beta^-}.[\overrightarrow{\beta^-}/\Xi]P$, by applying Rule ($\exists^{\geqslant}$), instantiating $\overrightarrow{\alpha^-}$ with respect to σ. Finally, $\Gamma \vdash Q \geqslant Q'$ by transitively combining $\Gamma \vdash \mathbf{nf}\,(Q) \geqslant Q'$ and $\Gamma \vdash Q \geqslant \mathbf{nf}\,(Q)$ (holds by corollary 13 and inversion).

□

6.9 Upgrade

Let us consider a type P well-formed in Γ. Some of its Γ-supertypes are also well-formed in a smaller context $\Delta \subseteq \Gamma$. The upgrade is the operation that returns the least of such supertypes.

Observation 9 (Upgrade is deterministic). *Assuming P is well-formed in $\Gamma \subseteq \Delta$, if* $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ *and* $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q'$ *are defined then $Q = Q'$.*

Proof. It follows directly from observation 8, and the convention that the fresh variables are chosen by a fixed deterministic algorithm (section 2.2). □

Lemma 73 (Soundness of Upgrade). *Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^\pm}$, if* $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ *then*

(1) $\Delta \vdash Q$
(2) $\Gamma \vdash Q \geqslant P$

Proof. By inversion, $\mathbf{upgrade}\,\Gamma \vdash P\,\mathbf{to}\,\Delta = Q$ means that for fresh $\overrightarrow{\beta^\pm}$ and $\overrightarrow{\gamma^\pm}$, $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \models [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q$. Then by the soundness of the least upper bound (lemma 71),

(1) $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q$,
(2) $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$, and
(3) $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P$.

$$
\begin{aligned}
\mathbf{fv}\,Q &\subseteq \mathbf{fv}\,[\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \cap \mathbf{fv}\,[\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P && \text{since by lemma 17, } \mathbf{fv}\,Q \subseteq \mathbf{fv}\,[\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \text{ and } \mathbf{fv}\,Q \subseteq \mathbf{fv}\,[\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P \\
&\subseteq ((\mathbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) \cup \overrightarrow{\beta^\pm}) \cap ((\mathbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) \cup \overrightarrow{\gamma^\pm}) \\
&= (\mathbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) \cap (\mathbf{fv}\,P \backslash \overrightarrow{\alpha^\pm}) && \text{since } \overrightarrow{\beta^\pm} \text{ and } \overrightarrow{\gamma^\pm} \text{ are fresh} \\
&= \mathbf{fv}\,P \backslash \overrightarrow{\alpha^\pm} \\
&\subseteq \Gamma \backslash \overrightarrow{\alpha^\pm} && \text{since } P \text{ is well-formed in } \Gamma \\
&\subseteq \Delta
\end{aligned}
$$

This way, by lemma 4, $\Delta \vdash Q$.

Let us apply $\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}$—the inverse of the substitution $\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}$ to both sides of $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$ and by lemma 23 (since $\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}$ can be specified as $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash \overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm} : \Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm}$ by lemma 14) obtain $\Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm} \vdash [\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}]Q \geqslant P$. Notice that $\Delta \vdash Q$ implies that $\mathbf{fv}\,Q \cap \overrightarrow{\beta^\pm} = \varnothing$, then by corollary 4, $[\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}]Q = Q$, and thus $\Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geqslant P$. By context strengthening, $\Delta, \overrightarrow{\alpha^\pm} \vdash Q \geqslant P$. □

Lemma 74 (Completeness and Initiality of Upgrade). *The upgrade returns the least Γ-supertype of P well-formed in Δ. Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^\pm}$,*
For any Q' such that

(1) $\Delta \vdash Q'$ *and*
(2) $\Gamma \vdash Q' \geqslant P$,

The result of the upgrade algorithm Q *exists (*$\textbf{upgrade}\,\Gamma \vdash P\,\textbf{to}\,\Delta = Q$*) and satisfies* $\Delta \vdash Q' \geqslant Q$.

Proof. Let us consider fresh (not intersecting with Γ) $\overrightarrow{\beta^{\pm}}$ and $\overrightarrow{\gamma^{\pm}}$.

If we apply substitution $\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q' \geqslant P$, we have $\Delta, \overrightarrow{\beta^{\pm}} \vdash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]Q' \geqslant [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$, which by corollary 4, since $\overrightarrow{\alpha^{\pm}}$ is disjoint from $\textbf{fv}\,(Q')$ (because $\Delta \vdash Q'$), simplifies to $\Delta, \overrightarrow{\beta^{\pm}} \vdash Q' \geqslant [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$.

Analogously, if we apply substitution $\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}$ to both sides of $\Delta, \overrightarrow{\alpha^{\pm}} \vdash Q' \geqslant P$, we have $\Delta, \overrightarrow{\gamma^{\pm}} \vdash Q' \geqslant [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$.

This way, Q' is a common supertype of $[\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$ and $[\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P$ in context $\Delta, \overrightarrow{\beta^{\pm}}, \overrightarrow{\gamma^{\pm}}$. It means that we can apply the completeness of the least upper bound (lemma 72):

(1) there exists Q s.t. $\Gamma \vDash [\overrightarrow{\beta^{\pm}}/\overrightarrow{\alpha^{\pm}}]P \vee [\overrightarrow{\gamma^{\pm}}/\overrightarrow{\alpha^{\pm}}]P = Q$
(2) $\Gamma \vdash Q' \geqslant Q$.

The former means that the upgrade algorithm terminates and returns Q. The latter means that since both Q' and Q are well-formed in Δ and Γ, by lemma 20, $\Delta \vdash Q' \geqslant Q$. □

6.10 Constraint Satisfaction

Lemma 75 (Any constraint is satisfiable). *Suppose that* $\Theta \vdash SC$ *and* Ξ *is a set such that* $\textbf{dom}\,(SC) \subseteq \Xi \subseteq \textbf{dom}\,(\Theta)$. *Then there exists* $\widehat{\sigma}$ *such that* $\Theta \vdash \widehat{\sigma} : \Xi$ *and* $\Theta \vdash \widehat{\sigma} : SC$.

Proof. Let us define $\widehat{\sigma}$ on $\textbf{dom}\,(SC)$ in the following way:

$$[\widehat{\sigma}]\widehat{\alpha}^{\pm} = \begin{cases} P & \text{if } (\widehat{\alpha}^{\pm} :\simeq P) \in SC \\ P & \text{if } (\widehat{\alpha}^{\pm} :\geqslant P) \in SC \\ N & \text{if } (\widehat{\alpha}^{\pm} :\simeq N) \in SC \\ \exists \beta^{-}.{\downarrow}\beta^{-} & \text{if } \widehat{\alpha}^{\pm} = \widehat{\alpha}^{+} \in \Xi \backslash \textbf{dom}\,(SC) \\ \forall \beta^{+}.{\uparrow}\beta^{+} & \text{if } \widehat{\alpha}^{\pm} = \widehat{\alpha}^{-} \in \Xi \backslash \textbf{dom}\,(SC) \end{cases}$$

Then $\Theta \vdash \widehat{\sigma} : SC$ follows immediately from the reflexivity of equivalence and subtyping (lemma 22) and the corresponding rules Rule $(:\simeq^{\text{SAT}}_{+})$, Rule $(:\simeq^{\text{SAT}}_{-})$, and Rule $(:\geqslant^{\text{SAT}}_{+})$. □

Lemma 76 (Constraint Entry Satiisfaction is Stable under Equivalence). − *If* $\Gamma \vdash N_1 : e$ *and* $\Gamma \vdash N_1 \simeq^{\leqslant} N_2$ *then* $\Gamma \vdash N_2 : e$.
\+ *If* $\Gamma \vdash P_1 : e$ *and* $\Gamma \vdash P_1 \simeq^{\leqslant} P_2$ *then* $\Gamma \vdash P_2 : e$.

Proof. − Then e has form $(\widehat{\alpha}^{-} :\simeq M)$, and by inversion, $\Gamma \vdash N_1 \simeq^{\leqslant} M$. Then by transitivity, $\Gamma \vdash N_2 \simeq^{\leqslant} M$, meaning $\Gamma \vdash N_2 : e$.
\+ Let us consider what form e has.
Case 1. $e = (\widehat{\alpha}^{+} :\simeq Q)$. Then $\Gamma \vdash P_1 \simeq^{\leqslant} Q$, and hence, $\Gamma \vdash P_2 \simeq^{\leqslant} Q$ by transitivity. Then $\Gamma \vdash P_2 : e$.
Case 2. $e = (\widehat{\alpha}^{+} :\geqslant Q)$. Then $\Gamma \vdash P_1 \geqslant Q$, and hence, $\Gamma \vdash P_2 \geqslant Q$ by transitivity. Then $\Gamma \vdash P_2 : e$.

□

Corollary 26 (Constraint Satisfaction is stable under Equivalence).
If $\Theta \vdash \widehat{\sigma}_1 : SC$ *and* $\Theta \vdash \widehat{\sigma}_1 \simeq^{\leqslant} \widehat{\sigma}_2 : \textbf{dom}\,(SC)$ *then* $\Theta \vdash \widehat{\sigma}_2 : SC$;
if $\Theta \vdash \widehat{\sigma}_1 : UC$ *and* $\Theta \vdash \widehat{\sigma}_1 \simeq^{\leqslant} \widehat{\sigma}_2 : \textbf{dom}\,(SC)$ *then* $\Theta \vdash \widehat{\sigma}_2 : UC$.

Corollary 27 (Normalization preserves Constraint Satisfaction).
If $\Theta \vdash \widehat{\sigma} : SC$ *then* $\Theta \vdash \mathbf{nf}\,(\widehat{\sigma}) : SC$;
if $\Theta \vdash \widehat{\sigma} : UC$ *then* $\Theta \vdash \mathbf{nf}\,(\widehat{\sigma}) : UC$.

6.11 Positive Subtyping

Observation 10 (Positive Subtyping is Deterministic). *For fixed* Γ, Θ, P, *and* Q, *if* $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ *and* $\Gamma; \Theta \vDash P \geqslant Q \dashv SC'$ *then* $SC = SC'$.

Proof. We prove it by induction on $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$. First, it is easy to see that the rule applied to infer $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ uniquely depends on the input, and those, it is the same rule that is inferring $\Gamma; \Theta \vDash P \geqslant Q \dashv SC'$. Second, the premises of each rule are deterministic on the input: unification is deterministic by observation 5, upgrade is deterministic by observation 9, the choice of the fresh algorithmic variables is deterministic by convention, as discussed in section 2.2, positive subtyping by the induction hypothesis. □

Lemma 77 (Soundness of the Positive Subtyping). *If* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash Q$, $\Gamma; \mathbf{dom}\,(\Theta) \vdash P$, *and* $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$, *then* $\Theta \vdash SC : \mathbf{uv}\,P$ *and for any normalized* $\widehat{\sigma}$ *such that* $\Theta \vdash \widehat{\sigma} : SC$, $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$.

Proof. We prove it by induction on $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$. Let us consider the last rule to infer this judgment.

Case 1. Rule (UVar$^{\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \vDash \widehat{\alpha}^+ \geqslant P' \dashv (\widehat{\alpha}^+ :\geqslant Q')$ where $\widehat{\alpha}^+\{\Delta\} \in \Theta$ and **upgrade** $\Gamma \vdash P'$ **to** $\Delta = Q'$.
Notice that $\widehat{\alpha}^+\{\Delta\} \in \Theta$ and $\Gamma \vdash^{\supseteq} \Theta$ implies $\Gamma = \Delta, \overrightarrow{\alpha^{\pm}}$ for some $\overrightarrow{\alpha^{\pm}}$, hence, the soundness of upgrade (lemma 73) is applicable:
(1) $\Delta \vdash Q'$ and
(2) $\Gamma \vdash Q' \geqslant P$.
Since $\widehat{\alpha}^+\{\Delta\} \in \Theta$ and $\Delta \vdash Q'$, it is clear that $\Theta \vdash (\widehat{\alpha}^+ :\geqslant Q') : \widehat{\alpha}^+$.
It is left to show that $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant P'$ for any normalized $\widehat{\sigma}$ s.t. $\Theta \vdash \widehat{\sigma} : (\widehat{\alpha}^+ :\geqslant Q')$. The latter means that $\Theta(\widehat{\alpha}^+) \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant Q'$, i.e. $\Delta \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant Q'$. By weakening the context to Γ and combining this judgment transitively with $\Gamma \vdash Q' \geqslant P$, we have $\Gamma \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant P$, as required.

Case 2. Rule (Var$^{\geqslant}_{+}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot$. Then $\mathbf{uv}\,\alpha^+ = \varnothing$, and $SC = \cdot$ satisfies $\Theta \vdash SC : \cdot$. Since $\mathbf{uv}\,\alpha^+ = \varnothing$, application of any substitution $\widehat{\sigma}$ does not change α^+, i.e. $[\widehat{\sigma}]\alpha^+ = \alpha^+$. Therefore, $\Gamma \vdash [\widehat{\sigma}]\alpha^+ \geqslant \alpha^+$ holds by Rule (Var$^{\leqslant}_{-}$).

Case 3. Rule ($\downarrow^{\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \vDash \downarrow N \geqslant \downarrow M \dashv SC$.
Then the next step of the algorithm is the unification of $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$, and it returns the resulting unification constraint $UC = SC$ as the result. By the soundness of unification (lemma 63), $\Theta \vdash SC : \mathbf{uv}\,(N)$ and for any normalized $\widehat{\sigma}$, $\Theta \vdash \widehat{\sigma} : SC$ implies $[\widehat{\sigma}]\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$, then we rewrite the left-hand side by lemma 45: $\mathbf{nf}\,([\widehat{\sigma}]N) = \mathbf{nf}\,(M)$ and apply lemma 35: $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M$, then by Rule ($\uparrow^{\leqslant}$), $\Gamma \vdash \downarrow[\widehat{\sigma}]N \geqslant \downarrow M$.

Case 4. Rule ($\exists^{\geqslant}$) then $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ has shape $\Gamma; \Theta \vDash \exists\overrightarrow{\alpha^-}.P' \geqslant \exists\overrightarrow{\beta^-}.Q' \dashv SC$ s.t. either $\overrightarrow{\alpha^-}$ or $\overrightarrow{\beta^-}$ is not empty.
Then the algorithm creates fresh unification variables $\overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\}$, substitutes the old $\overrightarrow{\alpha^-}$ with them in P', and makes the recursive call: $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vDash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P' \geqslant Q' \dashv SC'$, returning as the result $SC = SC' \backslash \overrightarrow{\widehat{\alpha}^-}$.

Let us take an arbitrary normalized $\widehat{\sigma}$ s.t. $\Theta \vdash \widehat{\sigma} : SC' \backslash \overrightarrow{\widehat{\alpha}^-}$. We wish to show $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$, i.e. $\Gamma \vdash \exists \overrightarrow{\alpha^-}.[\widehat{\sigma}]P' \geqslant \exists \overrightarrow{\beta^-}.Q'$. To do that, we apply Rule ($\exists^{\geqslant}$), and what is left to show is $\Gamma, \overrightarrow{\beta^-} \vdash [\vec{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]P' \geqslant Q'$ for some $\vec{N}$. If we construct a normalized $\widehat{\sigma}'$ such that $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}' : SC'$ and for some $\vec{N}$, $[\vec{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]P' = [\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$, we can apply the induction hypothesis to $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vDash [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P \geqslant Q \dashv SC'$ and infer the required subtyping.

Let us construct such $\widehat{\sigma}'$ by extending $\widehat{\sigma}$ with $\overrightarrow{\widehat{\alpha}^-}$ mapped to the corresponding types in SC':

$$[\widehat{\sigma}']\widehat{\beta}^{\pm} = \begin{cases} [\widehat{\sigma}]\widehat{\beta}^{\pm} & \text{if } \widehat{\beta}^{\pm} \in \mathbf{dom}\,(SC') \backslash \overrightarrow{\widehat{\alpha}^-} \\ \mathbf{nf}\,(N) & \text{if } \widehat{\beta}^{\pm} \in \overrightarrow{\widehat{\alpha}^-} \text{ and } (\widehat{\beta}^{\pm} :\simeq N) \in SC' \end{cases}$$

It is easy to see that $\widehat{\sigma}'$ is normalized: it inherits this property from $\widehat{\sigma}$. Let us show that $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}' : SC'$. Let us take an arbitrary entry e from SC' restricting a variable $\widehat{\beta}^{\pm}$. Suppose $\widehat{\beta}^{\pm} \in \mathbf{dom}\,(SC') \backslash \overrightarrow{\widehat{\alpha}^-}$. Then $(\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\})(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}']\widehat{\beta}^{\pm} : e$ is rewritten as $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\beta}^{\pm} : e$, which holds since $\Theta \vdash \widehat{\sigma} : SC'$. Suppose $\widehat{\beta}^{\pm} = \widehat{\alpha}_i^- \in \overrightarrow{\widehat{\alpha}^-}$. Then $e = (\widehat{\alpha}_i^- :\simeq N)$ for some N, $[\widehat{\sigma}']\widehat{\alpha}_i^- = \mathbf{nf}\,(N)$ by the definition, and $\Gamma, \overrightarrow{\beta^-} \vdash \mathbf{nf}\,(N) : (\widehat{\alpha}_i^- :\simeq N)$ by Rule ($:\simeq^{\mathrm{SAT}}_-$), since $\Gamma \vdash \mathbf{nf}\,(N) \simeq^{\leqslant} N$ by lemma 35.

Finally, let us show that $[\vec{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]P' = [\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$. For N_i, we take the *normalized* type restricting $\widehat{\alpha}_i^-$ in SC'. Let us take an arbitrary variable from P.

(1) If this variable is a unification variable $\widehat{\beta}^{\pm}$, then $[\vec{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$, since $\Theta \vdash \widehat{\sigma} : SC' \backslash \overrightarrow{\widehat{\alpha}^-}$ and $\mathbf{dom}\,(\Theta) \cap \overrightarrow{\alpha^-} = \varnothing$.
Notice that $\widehat{\beta}^{\pm} \in \mathbf{dom}\,(\Theta)$, which is disjoint from $\overrightarrow{\widehat{\alpha}^-}$, that is $\widehat{\beta}^{\pm} \in \mathbf{dom}\,(SC') \backslash \overrightarrow{\widehat{\alpha}^-}$. This way, $[\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\widehat{\beta}^{\pm} = [\widehat{\sigma}']\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$ by the definition of $\widehat{\sigma}'$,

(2) If this variable is a regular variable $\beta^{\pm} \notin \overrightarrow{\alpha^-}$, then $[\vec{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]\beta^{\pm} = \beta^{\pm}$ and $[\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\beta^{\pm} = \beta^{\pm}$.

(3) If this variable is a regular variable $\alpha_i^- \in \overrightarrow{\alpha^-}$, then $[\vec{N}/\overrightarrow{\alpha^-}][\widehat{\sigma}]\alpha_i^- = N_i = \mathbf{nf}\,(N_i)$ (the latter equality holds since N_i is normalized) and $[\widehat{\sigma}'][\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]\alpha_i^- = [\widehat{\sigma}']\widehat{\alpha}_i^- = \mathbf{nf}\,(N_i)$.

□

Lemma 78 (Completeness of the Positive Subtyping). *Suppose that* $\Gamma \vdash^{\supseteq} \Theta$, $\Gamma \vdash Q$ *and* $\Gamma; \mathbf{dom}\,(\Theta) \vdash P$. *Then for any* $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P)$ *such that* $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$, *there exists* $\Gamma; \Theta \vDash P \geqslant Q \dashv SC$ *and moreover,* $\Theta \vdash \widehat{\sigma} : SC$.

Proof. Let us prove this lemma by induction on $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$. Let us consider the last rule used in the derivation, but first, consider the base case for the substitution $[\widehat{\sigma}]P$:

Case 1. $P = \exists \overrightarrow{\beta^-}.\widehat{\alpha}^+$ (for potentially empty $\overrightarrow{\beta^-}$)
Then by assumption, $\Gamma \vdash \exists \overrightarrow{\beta^-}.[\widehat{\sigma}]\widehat{\alpha}^+ \geqslant Q$ (where $\overrightarrow{\beta^-} \cap \mathbf{fv}\,[\widehat{\sigma}]\widehat{\alpha}^+ = \varnothing$). Let us decompose Q as $Q = \exists \overrightarrow{\gamma^-}.Q_0$, where Q_0 does not start with $\exists$.
By inversion, $\Gamma; \mathbf{dom}\,(\Theta) \vdash \exists \overrightarrow{\beta^-}.\widehat{\alpha}^+$ implies $\widehat{\alpha}^+\{\Delta\} \in \Theta$ for some $\Delta \subseteq \Gamma$.
By lemma 18 applied twice, $\Gamma \vdash \exists \overrightarrow{\beta^-}.[\widehat{\sigma}]\widehat{\alpha}^+ \geqslant \exists \overrightarrow{\gamma^-}.Q_0$ implies $\Gamma, \overrightarrow{\gamma^-} \vdash [\vec{N}/\overrightarrow{\beta^-}][\widehat{\sigma}]\widehat{\alpha}^+ \geqslant Q_0$ for some N, and since $\overrightarrow{\beta^-} \cap \mathbf{fv}\,([\widehat{\sigma}]\widehat{\alpha}^+) \subseteq \overrightarrow{\beta^-} \cap \Theta(\widehat{\alpha}^+) \subseteq \overrightarrow{\beta^-} \cap \Gamma = \varnothing$, $[\vec{N}/\overrightarrow{\beta^-}][\widehat{\sigma}]\widehat{\alpha}^+ = [\widehat{\sigma}]\widehat{\alpha}^+$, that is $\Gamma, \overrightarrow{\gamma^-} \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant Q_0$.

When algorithm tires to infer the subtyping $\Gamma; \Theta \vDash \exists\overrightarrow{\beta^-}.\widehat{\alpha}^+ \geqslant \exists\overrightarrow{\gamma^-}.Q_0 \dashv SC$, it applies Rule ($\exists^{\geqslant}$), which reduces the problem to $\Gamma, \overrightarrow{\gamma^-}; \Theta, \overrightarrow{\widehat{\beta}^-}\{\Gamma, \overrightarrow{\gamma^-}\} \vDash [\overrightarrow{\widehat{\beta}^-}/\overrightarrow{\beta^-}]\widehat{\alpha}^+ \geqslant Q_0 \dashv SC$, which is equivalent to $\Gamma, \overrightarrow{\gamma^-}; \Theta, \overrightarrow{\widehat{\beta}^-}\{\Gamma, \overrightarrow{\gamma^-}\} \vDash \widehat{\alpha}^+ \geqslant Q_0 \dashv SC$.
Next, the algorithm tries to apply Rule (UVar$^{\geqslant}$) and the resulting restriction is $SC = (\widehat{\alpha}^+ :\geqslant Q_0')$ where **upgrade** $\Gamma, \overrightarrow{\gamma^-} \vdash Q_0$ **to** $\Delta = Q_0'$.
Why does the upgrade procedure terminate? Because $[\widehat{\sigma}]\widehat{\alpha}^+$ satisfies the pre-conditions of the completeness of the upgrade (lemma 74):
(1) $\Delta \vdash [\widehat{\sigma}]\widehat{\alpha}^+$ because $\Theta \vdash \widehat{\sigma} : \widehat{\alpha}^+$ and $\widehat{\alpha}^+\{\Delta\} \in \Theta$,
(2) $\Gamma, \overrightarrow{\gamma^-} \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant Q_0$ as noted above
Moreover, the completeness of upgrade also says that Q_0' is *the least* supertype of Q_0 among types well-formed in Δ, that is $\Delta \vdash [\widehat{\sigma}]\widehat{\alpha}^+ \geqslant Q_0'$, which means $\Theta \vdash \widehat{\sigma} : (\widehat{\alpha}^+ :\geqslant Q_0')$, that is $\Theta \vdash \widehat{\sigma} : SC$.

Case 2. $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$ is derived by Rule (Var$^{\geqslant}_+$)
Then $P = [\widehat{\sigma}]P = \alpha^+ = Q$, where the first equality holds because P is not a unification variable: it has been covered by case 1; and the second equality hold because Rule (Var$^{\geqslant}_+$) was applied.
The algorithm applies Rule (Var$^{\geqslant}_+$) and infers $SC = \cdot$, i.e. $\Gamma; \Theta \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot$. Then $\Theta \vdash \widehat{\sigma} : \cdot$ holds trivially.

Case 3. $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$ is derived by Rule ($\downarrow^{\geqslant}$),
Then $P = \downarrow N$, since the substitution $[\widehat{\sigma}]P$ must preserve the top-level constructor of $P \neq \widehat{\alpha}^+$ (the case $P = \widehat{\alpha}^+$ has been covered by case 1), and $Q = \downarrow M$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M$.
Since both types start with $\downarrow$, the algorithm tries to apply Rule ($\downarrow^{\geqslant}$): $\Gamma; \Theta \vDash \downarrow N \geqslant \downarrow M \dashv SC$. The premise of this rule is the unification of $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$: $\Gamma; \Theta \vDash \mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv UC$. And the algorithm returns it as a subtyping constraint $SC = UC$.
To demonstrate that the unification terminates ant $\widehat{\sigma}$ satisfies the resulting constraints, we apply the completeness of the unification algorithm (lemma 64). In order to do that, we need to provide a substitution unifying $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$. Let us show that $\mathbf{nf}\,(\widehat{\sigma})$ is such a substitution.

- $\mathbf{nf}\,(N)$ and $\mathbf{nf}\,(M)$ are normalized
- $\Gamma; \mathbf{dom}\,(\Theta) \vdash \mathbf{nf}\,(N)$ because $\Gamma; \mathbf{dom}\,(\Theta) \vdash N$ (corollary 23)
- $\Gamma \vdash \mathbf{nf}\,(M)$ because $\Gamma \vdash M$ (corollary 11)
- $\Theta \vdash \mathbf{nf}\,(\widehat{\sigma}) : \mathbf{uv}\,(P)$ because $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P)$ (corollary 24)
- $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M \Rightarrow [\widehat{\sigma}]N \simeq^{D} M$ by lemma 34
  $\Rightarrow \mathbf{nf}\,([\widehat{\sigma}]N) = \mathbf{nf}\,(M)$ by lemma 46
  $\Rightarrow [\mathbf{nf}\,(\widehat{\sigma})]\mathbf{nf}\,(N) = \mathbf{nf}\,(M)$ by lemma 45

By the completeness of the unification, $\Gamma; \Theta \vDash N \overset{u}{\simeq} M \dashv UC$ exists, and $\Theta \vdash \mathbf{nf}\,(\widehat{\sigma}) : UC$, and by corollary 26, $\Theta \vdash \widehat{\sigma} : UC$.

Case 4. $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$ is derived by Rule ($\exists^{\geqslant}$).
We should only consider the case when the substitution $[\widehat{\sigma}]P$ results in the existential type $\exists\overrightarrow{\alpha^-}.P''$ (for $P'' \neq \exists\ldots$) by congruence, i.e. $P = \exists\overrightarrow{\alpha^-}.P'$ (for $P' \neq \exists\ldots$) and $[\widehat{\sigma}]P' = P''$. This is because the case when $P = \exists\overrightarrow{\beta^-}.\widehat{\alpha}^+$ has been covered (case 1), and thus, the substitution $\widehat{\sigma}$ must preserve all the outer quantifiers of P and does not generate any new ones.

This way, $P = \exists\overrightarrow{\alpha^-}.P'$, $[\widehat{\sigma}]P = \exists\overrightarrow{\alpha^-}.[\widehat{\sigma}]P'$ (assuming $\overrightarrow{\alpha^-}$ does not intersect with the range of $\widehat{\sigma}$) and $Q = \exists\overrightarrow{\beta^-}.Q'$, where either $\overrightarrow{\alpha^-}$ or $\overrightarrow{\beta^-}$ is not empty.
By inversion, $\Gamma \vdash [\sigma][\widehat{\sigma}]P' \geqslant Q'$ for some $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$. Since σ and $\widehat{\sigma}$ have disjoint domains, and the range of one does not intersect with the domain of the other, they commute, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}][\sigma]P' \geqslant Q'$ (notice that the tree inferring this judgement is a proper subtree of the tree inferring $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$).
At the next step, the algorithm creates fresh (disjoint with $\mathbf{uv}\,P'$) unification variables $\overrightarrow{\widehat{\alpha}^-}$, replaces $\overrightarrow{\alpha^-}$ with them in P', and makes the recursive call: $\Gamma, \overrightarrow{\beta^-}; \Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vDash P_0 \geqslant Q' \dashv SC_1$, (where $P_0 = [\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P'$), returning $SC_1 \backslash \overrightarrow{\widehat{\alpha}^-}$ as the result.
To show that the recursive call terminates and that $\Theta \vdash \widehat{\sigma} : SC_1 \backslash \overrightarrow{\widehat{\alpha}^-}$, it suffices to build $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}_0 : \mathbf{uv}\,(P_0)$—an extension of $\widehat{\sigma}$ with $\overrightarrow{\widehat{\alpha}^-} \cap \mathbf{uv}\,(P_0)$ such that $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant Q$. Then by the induction hypothesis, $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}_0 : SC_1$, and hence, $\Theta \vdash \widehat{\sigma} : SC_1 \backslash \overrightarrow{\widehat{\alpha}^-}$, as required.
Let us construct such a substitution $\widehat{\sigma}_0$:

$$[\widehat{\sigma}_0]\widehat{\beta}^{\pm} = \begin{cases} [\sigma]\alpha_i^- & \text{if } \widehat{\beta}^{\pm} = \widehat{\alpha}_i^- \in \overrightarrow{\widehat{\alpha}^-} \cap \mathbf{uv}\,(P_0) \\ [\widehat{\sigma}]\widehat{\beta}^{\pm} & \text{if } \widehat{\beta}^{\pm} \in \mathbf{uv}\,(P') \end{cases}$$

It is easy to see $\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\} \vdash \widehat{\sigma}_0 : \mathbf{uv}\,(P_0) : \mathbf{uv}\,(P_0) = \mathbf{uv}\,([\overrightarrow{\widehat{\alpha}^-}/\overrightarrow{\alpha^-}]P') = \overrightarrow{\widehat{\alpha}^-} \cap \mathbf{uv}\,(P_0) \cup \mathbf{uv}\,(P')$. Then

(1) for $\widehat{\alpha}_i^- \in \overrightarrow{\widehat{\alpha}^-} \cap \mathbf{uv}\,(P_0)$, $(\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\})(\widehat{\alpha}_i^-) \vdash [\widehat{\sigma}_0]\widehat{\alpha}_i^-$, i.e. $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]\alpha_i^-$ holds since $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$,
(2) for $\widehat{\beta}^{\pm} \in \mathbf{uv}\,(P') \subseteq \mathbf{dom}\,(\Theta)$, $(\Theta, \overrightarrow{\widehat{\alpha}^-}\{\Gamma, \overrightarrow{\beta^-}\})(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}_0]\widehat{\beta}^{\pm}$, i.e. $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\beta}^{\pm}$ holds since $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P)$ and $\widehat{\beta}^{\pm} \in \mathbf{uv}\,(P') = \mathbf{uv}\,(P)$.

Now, let us show that $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant Q$. To do that, we notice that $[\widehat{\sigma}_0]P_0 = [\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P_0$: let us consider an arbitrary variable appearing freely in P_0:

(1) if this variable is a algorithmic variable $\widehat{\alpha}_i^- \in \overrightarrow{\widehat{\alpha}^-}$, then $[\widehat{\sigma}_0]\widehat{\alpha}_i^- = [\sigma]\alpha_i^-$ and $[\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]\widehat{\alpha}_i^- = [\widehat{\sigma}][\sigma]\alpha_i^- = [\sigma]\alpha_i^-$,
(2) if this variable is a algorithmic variable $\widehat{\beta}^{\pm} \in \mathbf{uv}\,(P_0) \backslash \overrightarrow{\widehat{\alpha}^-} = \mathbf{uv}\,(P')$, then $[\widehat{\sigma}_0]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$ and $[\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]\widehat{\beta}^{\pm} = [\widehat{\sigma}][\sigma]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$,
(3) if this variable is a regular variable from $\mathbf{fv}\,(P_0)$, both substitutions do not change it: $\widehat{\sigma}_0$, $\widehat{\sigma}$ and $\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}$ act on algorithmic variables, and σ is defined on $\overrightarrow{\alpha^-}$, however, $\overrightarrow{\alpha^-} \cap \mathbf{fv}\,(P_0) = \varnothing$.

This way, $[\widehat{\sigma}_0]P_0 = [\widehat{\sigma}][\sigma][\overrightarrow{\alpha^-}/\overrightarrow{\widehat{\alpha}^-}]P_0 = [\widehat{\sigma}][\sigma]P'$, and thus, $\Gamma, \overrightarrow{\beta^-} \vdash [\widehat{\sigma}_0]P_0 \geqslant Q'$.

□

6.12 Subtyping Constraint Merge

Observation 11 (Constraint Entry Merge is Deterministic). *For fixed* Γ, e_1, e_2, *if* $\Gamma \vdash e_1 \,\&\, e_2 = e$ *and* $\Gamma \vdash e_1 \,\&\, e_2 = e'$ *then* $e = e'$.

Proof. First, notice that the shape of e_1 and e_2 uniquely determines the rule applied to infer $\Gamma \vdash e_1 \,\&\, e_2 = e$, which is consequently, the same rule used to infer $\Gamma \vdash e_1 \,\&\, e_2 = e'$. Second, notice that the premises of each rule are deterministic on the input: the positive subtyping is deterministic by observation 10, and the least upper bound is deterministic by observation 8. □

Observation 12 (Subgyping Constraint Merge is Deterministic). *Suppose that* $\Theta \vdash SC_1$ *and* $\Theta \vdash SC_2$ *If* $\Theta \vdash SC_1 \;\&\; SC_2 \quad = UC$ *and* $\Theta \vdash SC_1 \;\&\; SC_2 \quad = UC'$ *are defined then* $SC = SC'$.

Proof. The proof is analogous to the proof of observation 4 but uses observation 11 to show that the merge of the matching constraint entries is fixed. □

Lemma 79 (Soundness of Constraint Entry Merge). *For a fixed context* Γ, *suppose that* $\Gamma \vdash e_1$ *and* $\Gamma \vdash e_2$. *If* $\Gamma \vdash e_1 \;\&\; e_2 = e$ *is defined then*

(1) $\Gamma \vdash e$
(2) For any $\Gamma \vdash P$, $\Gamma \vdash P : e$ *implies* $\Gamma \vdash P : e_1$ *and* $\Gamma \vdash P : e_2$

Proof. Let us consider the rule forming $\Gamma \vdash e_1 \;\&\; e_2 = e$.

Case 1. Rule $(\simeq \;\&^+\; \simeq)$, i.e. $\Gamma \vdash e_1 \;\&\; e_2 = e$ has form $\Gamma \vdash (\widehat{\alpha}^+ :\simeq Q) \;\&\; (\widehat{\alpha}^+ :\simeq Q') = (\widehat{\alpha}^+ :\simeq Q)$ and $\mathbf{nf}\,(Q) = \mathbf{nf}\,(Q')$. The latter implies $\Gamma \vdash Q \simeq^{\leqslant} Q'$ by lemma 35. Then
(1) $\Gamma \vdash e$, i.e. $\Gamma \vdash \widehat{\alpha}^+ :\simeq Q$ holds by assumption;
(2) by inversion, $\Gamma \vdash P : (\widehat{\alpha}^+ :\simeq Q)$ means $\Gamma \vdash P \simeq^{\leqslant} Q$, and by transitivity of equivalence (corollary 10), $\Gamma \vdash P \simeq^{\leqslant} Q'$. Thus, $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$ hold by Rule $(:\simeq^{\mathrm{SAT}}_{+})$.

Case 2. Rule $(\simeq \;\&^-\; \simeq)$ the negative case is proved in exactly the same way as the positive one.

Case 3. Rule $(\geqslant \;\&^+\; \geqslant)$ Then e_1 is $\widehat{\alpha}^+ :\geqslant Q_1$, e_2 is $\widehat{\alpha}^+ :\geqslant Q_2$, and $e_1 \;\&\; e_2 = e$ is $\widehat{\alpha}^+ :\geqslant Q$ where Q is the least upper bound of Q_1 and Q_2. Then by lemma 71,
- $\Gamma \vdash Q$,
- $\Gamma \vdash Q \geqslant Q_1$,
- $\Gamma \vdash Q \geqslant Q_2$.

Let us show the required properties.
- $\Gamma \vdash e$ holds from $\Gamma \vdash Q$,
- Assuming $\Gamma \vdash P : e$, by inversion, we have $\Gamma \vdash P \geqslant Q$. Combining it transitively with $\Gamma \vdash Q \geqslant Q_1$, we have $\Gamma \vdash P \geqslant Q_1$. Analogously, $\Gamma \vdash P \geqslant Q_2$. Then $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$ hold by Rule $(:\geqslant^{\mathrm{SAT}}_{+})$.

Case 4. Rule $(\geqslant \;\&^+\; \simeq)$ Then e_1 is $\widehat{\alpha}^+ :\geqslant Q_1$, e_2 is $\widehat{\alpha}^+ :\simeq Q_2$, where $\Gamma; \cdot \vDash Q_2 \geqslant Q_1 \dashv \cdot$, and the resulting $e_1 \;\&\; e_2 = e$ is equal to e_2, that is $\widehat{\alpha}^+ :\simeq Q_2$.
Let us show the required properties.
- By assumption, $\Gamma \vdash Q$, and hence $\Gamma \vdash e$.
- Since $\mathbf{uv}\,(Q_2) = \varnothing$, $\Gamma; \cdot \vDash Q_2 \geqslant Q_1 \dashv \cdot$ implies $\Gamma \vdash Q_2 \geqslant Q_1$ by the soundness of positive subtyping (lemma 77). Then let us take an arbitrary $\Gamma \vdash P$ such that $\Gamma \vdash P : e$. Since $e_2 = e$, $\Gamma \vdash P : e_2$ holds immediately.
  By inversion, $\Gamma \vdash P : (\widehat{\alpha}^+ :\simeq Q_2)$ means $\Gamma \vdash P \simeq^{\leqslant} Q_2$, and then by transitivity of subtyping (lemma 24), $\Gamma \vdash P \geqslant Q_1$. Then $\Gamma \vdash P : e_1$ holds by Rule $(:\geqslant^{\mathrm{SAT}}_{+})$.

Case 5. Rule $(\simeq \;\&^+\; \geqslant)$ Thee proof is analogous to the previous case.

□

Lemma 80 (Soundness of Constraint Merge). *Suppose that* $\Theta \vdash SC_1 : \Xi_1$ *and* $\Theta \vdash SC_2 : \Xi_2$ *and* $\Theta \vdash SC_1 \& SC_2 = SC$ *is defined. Then*

(1) $\Theta \vdash SC : \Xi_1 \cup \Xi_2$,
(2) for any substitution $\Theta \vdash \widehat{\sigma} : \Xi_1 \cup \Xi_2$, $\Theta \vdash \widehat{\sigma} : SC$ *implies* $\Theta \vdash \widehat{\sigma} : SC_1$ *and* $\Theta \vdash \widehat{\sigma} : SC_2$.

Proof. By definition, $\Theta \vdash SC_1 \& SC_2 = SC$ consists of three parts: entries of SC_1 that do not have matching entries of SC_2, entries of SC_2 that do not have matching entries of SC_1, and the merge of matching entries.

Notice that $\widehat{\alpha}^{\pm} \in \Xi_1 \backslash \Xi_2$ if and only if there is an entry e in SC_1 restricting $\widehat{\alpha}^{\pm}$, but there is no such entry in SC_2. Therefore, for any $\widehat{\alpha}^{\pm} \in \Xi_1 \backslash \Xi_2$, there is an entry e in SC restricting $\widehat{\alpha}^{\pm}$. Notice that $\Theta(\widehat{\alpha}^{\pm}) \vdash e$ holds since $\Theta \vdash SC_1 : \Xi_1$.

Analogously, for any $\widehat{\beta}^{\pm} \in \Xi_2 \backslash \Xi_1$, there is an entry e in SC restricting $\widehat{\beta}^{\pm}$. Notice that $\Theta(\widehat{\beta}^{\pm}) \vdash e$ holds since $\Theta \vdash SC_2 : \Xi_2$.

Finally, for any $\widehat{\gamma}^{\pm} \in \Xi_1 \cap \Xi_2$, there is an entry e_1 in SC_1 restricting $\widehat{\gamma}^{\pm}$ and an entry e_2 in SC_2 restricting $\widehat{\gamma}^{\pm}$. Since $\Theta \vdash SC_1 \& SC_2 = SC$ is defined, $\Theta(\widehat{\gamma}^{\pm}) \vdash e_1 \,\&\, e_2 = e$ restricting $\widehat{\gamma}^{\pm}$ is defined and belongs to SC, moreover, $\Theta(\widehat{\gamma}^{\pm}) \vdash e$ by lemma 79. This way, $\Theta \vdash SC : \Xi_1 \cup \Xi_2$.

Let us show the second property. We take an arbitrary $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : \Xi_1 \cup \Xi_2$ and $\Theta \vdash \widehat{\sigma} : SC$. To prove $\Theta \vdash \widehat{\sigma} : SC_1$, we need to show that for any $e_1 \in SC_1$, restricting $\widehat{\alpha}^{\pm}$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$ holds.

Let us assume that $\widehat{\alpha}^{\pm} \notin \mathbf{dom}\,(SC_2)$. It means that $SC \ni e_1$, and then since $\Theta \vdash \widehat{\sigma} : SC$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$.

Otherwise, SC_2 contains an entry e_2 restricting $\widehat{\alpha}^{\pm}$, and $SC \ni e$ where $\Theta(\widehat{\alpha}^{\pm}) \vdash e_1 \,\&\, e_2 = e$. Then since $\Theta \vdash \widehat{\sigma} : SC$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$, and by lemma 79, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$.

The proof of $\Theta \vdash \widehat{\sigma} : SC_2$ is symmetric. □

Lemma 81 (Completeness of Constraint Entry Merge). *For a fixed context* Γ, *suppose that* $\Gamma \vdash e_1$ *and* $\Gamma \vdash e_2$ *are matching constraint entries.*

- *for a type* P *such that* $\Gamma \vdash P : e_1$ *and* $\Gamma \vdash P : e_2$, $\Gamma \vdash e_1 \,\&\, e_2 = e$ *is defined and* $\Gamma \vdash P : e$.
- *for a type* N *such that* $\Gamma \vdash N : e_1$ *and* $\Gamma \vdash N : e_2$, $\Gamma \vdash e_1 \,\&\, e_2 = e$ *is defined and* $\Gamma \vdash N : e$.

Proof. Let us consider the shape of e_1 and e_2.

Case 1. e_1 is $\widehat{\alpha}^{+} :\simeq Q_1$ and e_2 is $\widehat{\alpha}^{+} :\simeq Q_2$. The proof repeats the corresponding case of lemma 61

Case 2. e_1 is $\widehat{\alpha}^{+} :\simeq Q_1$ and e_2 is $\widehat{\alpha}^{+} :\geqslant Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \simeq^{\leqslant} Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \geqslant Q_2$. Then by transitivity of subtyping, $\Gamma \vdash Q_1 \geqslant Q_2$, which means $\Gamma; \cdot \vDash Q_1 \geqslant Q_2 \dashv \cdot$ by lemma 78. This way, Rule $(\simeq \,\&^{+}\, \geqslant)$ applies to infer $\Gamma \vdash e_1 \,\&\, e_2 = e_1$, and $\Gamma \vdash P : e_1$ holds by assumption.

Case 3. e_1 is $\widehat{\alpha}^{+} :\geqslant Q_1$ and e_2 is $\widehat{\alpha}^{+} :\geqslant Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \geqslant Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \geqslant Q_2$. By the completeness of the least upper bound (lemma 72), $\Gamma \vDash Q_1 \vee Q_2 = Q$, and $\Gamma \vdash P \geqslant Q$. This way, Rule $(\geqslant \,\&^{+}\, \geqslant)$ applies to infer $\Gamma \vdash e_1 \,\&\, e_2 = (\widehat{\alpha}^{+} :\geqslant Q)$, and $\Gamma \vdash P : (\widehat{\alpha}^{+} :\geqslant Q)$ holds by Rule $(:\geqslant^{\mathrm{SAT}}_{+})$.

Case 4. The negative cases are proved symmetrically.

□

Lemma 82 (Completeness of Constraint Merge). *Suppose that* $\Theta \vdash SC_1 : \Xi_1$ *and* $\Theta \vdash SC_2 : \Xi_2$. *If there exists a substitution* $\Theta \vdash \widehat{\sigma} : \Xi_1 \cup \Xi_2$ *such that* $\Theta \vdash \widehat{\sigma} : SC_1$ *and* $\Theta \vdash \widehat{\sigma} : SC_2$ *then* $\Theta \vdash SC_1 \& SC_2 = SC$ *is defined.*

Proof. By definition, $SC_1 \,\&\, SC_2$ is a union of

(1) entries of SC_1, which do not have matching entries in SC_2,
(2) entries of SC_2, which do not have matching entries in SC_1, and
(3) the merge of matching entries.

This way, to show that $\Theta \vdash SC_1 \& SC_2 = SC$ is defined, we need to demonstrate that each of these components is defined and satisfies the required property (that the result of $\widehat{\sigma}$ satisfies the corresponding constraint entry).

It is clear that the first two components of this union exist. Moreover, if e is an entry of SC_i restricting $\widehat{\alpha}^{\pm} \notin \mathbf{dom}\,(SC_2)$, then $\Theta \vdash \widehat{\sigma} : SC_i$ implies $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$,

Let us show that the third component exists. Let us take two entries $e_1 \in SC_1$ and $e_2 \in SC_2$ restricting the same variable $\widehat{\alpha}^{\pm}$. $\Theta \vdash \widehat{\sigma} : SC_1$ means that $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$ means $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e_2$. Then by lemma 81, $\Theta(\widehat{\alpha}^{\pm}) \vdash e_1 \,\&\, e_2 = e$ is defined and $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\alpha}^{\pm} : e$.

□

6.13 Negative Subtyping

Observation 13 (Negative Algorithmic Subtyping is Deterministic). *For fixed* $\Gamma, \Theta, M,$ *and* N, *if* $\Gamma; \Theta \vDash N \leqslant M \dashv SC$ *and* $\Gamma; \Theta \vDash N \leqslant M \dashv SC'$ *then* $SC = SC'$.

Proof. First, notice that the shape of the input uniquely determines the rule applied to infer $\Gamma; \Theta \vDash N \leqslant M \dashv SC$, which is consequently, the same rule used to infer $\Gamma; \Theta \vDash N \leqslant M \dashv SC'$.

Second, notice that for each of the inference rules, the premises are deterministic on the input. Specifically,

- Rule ($\uparrow^{\leqslant}$) relies on unification, which is deterministic by observation 5;
- Rule ($\forall^{\leqslant}$) relies on the choice of fresh algorithmic variables, which is deterministic as discussed in section 2.2, and on the negative subtyping, which is deterministic by the induction hypothesis;
- Rule ($\rightarrow^{\leqslant}$) uses the negative subtyping (deterministic by the induction hypothesis), the positive subtyping (observation 10), and the merge of subtyping constraints (observation 12);

□

Lemma 83 (Soundness of Negative Subtyping). *If* $\Gamma \vdash^{\supseteq} \Theta, \Gamma \vdash M, \Gamma; \mathbf{dom}(\Theta) \vdash N$ *and* $\Gamma; \Theta \vDash N \leqslant M \dashv SC$, *then* $\Theta \vdash SC : \mathbf{uv}(N)$ *and for any normalized* $\widehat{\sigma}$ *such that* $\Theta \vdash \widehat{\sigma} : SC$, $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$.

Proof. We prove it by induction on $\Gamma; \Theta \vDash N \leqslant M \dashv SC$.

Suppose that $\widehat{\sigma}$ is normalized and $\Theta \vdash \widehat{\sigma} : SC$, Let us consider the last rule to infer this judgment.

Case 1. Rule ($\rightarrow^{\leqslant}$). Then $\Gamma; \Theta \vDash N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \vDash P \rightarrow N' \leqslant Q \rightarrow M' \dashv SC$
On the next step, the the algorithm makes two recursive calls: $\Gamma; \Theta \vDash P \geqslant Q \dashv SC_1$ and $\Gamma; \Theta \vDash N' \leqslant M' \dashv SC_2$ and returns $\Theta \vdash SC_1 \& SC_2 = SC$ as the result.
By the soundness of constraint merge (lemma 80), $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$. Then by the soundness of positive subtyping (lemma 77), $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$; and by the induction hypothesis, $\Gamma \vdash [\widehat{\sigma}]N' \leqslant M'$. This way, by Rule ($\rightarrow^{\leqslant}$), $\Gamma \vdash [\widehat{\sigma}](P \rightarrow N') \leqslant Q \rightarrow M'$.

Case 2. Rule ($\mathrm{Var}^{\leqslant}_{-}$), and then $\Gamma; \Theta \vDash N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \vDash \alpha^{-} \leqslant \alpha^{-} \dashv \cdot$
This case is symmetric to case 2 of lemma 77.

Case 3. Rule ($\uparrow^{\leqslant}$), and then $\Gamma; \Theta \vDash N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \vDash \uparrow P \leqslant \uparrow Q \dashv SC$
This case is symmetric to case 3 of lemma 77.

Case 4. Rule ($\forall^{\leqslant}$), and then $\Gamma; \Theta \vDash N \leqslant M \dashv SC$ has shape $\Gamma; \Theta \vDash \forall \overrightarrow{\alpha^{+}}.N' \leqslant \forall \overrightarrow{\beta^{+}}.M' \dashv SC$ s.t. either $\overrightarrow{\alpha^{+}}$ or $\overrightarrow{\beta^{+}}$ is not empty
This case is symmetric to case 4 of lemma 77.

□

Lemma 84 (Completeness of the Negative Subtyping). *Suppose that* $\Gamma \vdash^{\supseteq} \Theta, \Gamma \vdash M, \Gamma; \mathbf{dom}(\Theta) \vdash N$, *and* N *does not contain negative unification variables* ($\widehat{\alpha}^{-} \notin \mathbf{uv}\, N$). *Then for any* $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(N)$ *such that* $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$, *there exists* $\Gamma; \Theta \vDash N \leqslant M \dashv SC$ *and moreover,* $\Theta \vdash \widehat{\sigma} : SC$.

Proof. We prove it by induction on $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$. Let us consider the last rule used in the derivation of $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$.

Case 1. $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$ is derived by Rule $(\uparrow^{\leqslant})$
Then $N = \uparrow P$, since the substitution $[\widehat{\sigma}]N$ must preserve the top-level constructor of $N \neq \widehat{\alpha}^-$ (since by assumption, $\widehat{\alpha}^- \notin \mathbf{uv}\, N$), and $Q = \downarrow M$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} M$.
The rest of the proof is symmetric to case 3 of lemma 78: notice that the algorithm does not make a recursive call, and the difference in the induction statement for the positive and the negative case here does not matter.

Case 2. $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$ is derived by Rule $(\rightarrow^{\leqslant})$, i.e. $[\widehat{\sigma}]N = [\widehat{\sigma}]P \rightarrow [\widehat{\sigma}]N'$ and $M = Q \rightarrow M'$, and by inversion, $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$ and $\Gamma \vdash [\widehat{\sigma}]N' \leqslant M'$.
The algorithm makes two recursive calls: $\Gamma; \Theta \vDash P \geqslant Q \dashv SC_1$ and $\Gamma; \Theta \vDash N' \leqslant M' \dashv SC_2$, and then returns $\Theta \vdash SC_1 \& SC_2 = SC$ as the result. Let us show that these recursive calls are successful and the returning constraints are fulfilled by $\widehat{\sigma}$.
Notice that from the inversion of $\Gamma \vdash M$, we have: $\Gamma \vdash Q$ and $\Gamma \vdash M'$; from the inversion of $\Gamma; \mathbf{dom}\,(\Theta) \vdash N$, we have: $\Gamma; \mathbf{dom}\,(\Theta) \vdash P$ and $\Gamma; \mathbf{dom}\,(\Theta) \vdash N'$; and since N does not contain negative unification variables, N' does not contain negative unification variables either.
This way, we can apply the induction hypothesis to $\Gamma \vdash [\widehat{\sigma}]N' \leqslant M'$ to obtain $\Gamma; \Theta \vDash N' \leqslant M' \dashv SC_2$ such that $\Theta \vdash SC_2 : \mathbf{uv}\,(N')$ and $\Theta \vdash \widehat{\sigma} : SC_2$. Also, we can apply the completeness of the positive subtyping (lemma 78) to $\Gamma \vdash [\widehat{\sigma}]P \geqslant Q$ to obtain $\Gamma; \Theta \vDash P \geqslant Q \dashv SC_1$ such that $\Theta \vdash SC_1 : \mathbf{uv}\,(P)$ and $\Theta \vdash \widehat{\sigma} : SC_1$.
Finally, we need to show that the merge of SC_1 and SC_2 is successful and satisfies the required properties. To do so, we apply the completeness of subtyping constraint merge (lemma 82) (notice that $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P \rightarrow N')$ means $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(P) \cup \mathbf{uv}\,(N')$). This way, $\Theta \vdash SC_1 \& SC_2 = SC$ is defined and $\Theta \vdash \widehat{\sigma} : SC$ holds.

Case 3. $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$ is derived by Rule $(\forall^{\leqslant})$. Since N does not contain negative unification variables, N must be of the form $\forall\overrightarrow{\alpha^+}.N'$, such that $[\widehat{\sigma}]N = \forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N'$ and $[\widehat{\sigma}]N' \neq \forall \ldots$ (assuming $\overrightarrow{\alpha^+}$ does not intersect with the range of $\widehat{\sigma}$). Also, $M = \forall\overrightarrow{\beta^+}.M'$ and either $\overrightarrow{\alpha^+}$ or $\overrightarrow{\beta^+}$ is non-empty.
The rest of the proof is symmetric to case 4 of lemma 78. To apply the induction hypothesis, we need to show additionally that there are no negative unification variables in $N_0 = [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N'$. This is because $\mathbf{uv}\, N_0 \subseteq \mathbf{uv}\, N \cup \overrightarrow{\widehat{\alpha}^+}$, and N is free of negative unification variables by assumption.

Case 4. $\Gamma \vdash [\widehat{\sigma}]N \leqslant M$ is derived by Rule $(\textsc{Var}^{\leqslant}_{-})$.
Then $N = [\widehat{\sigma}]N = \alpha^- = M$. Here the first equality holds because N is not a unification variable: by assumption, N is free of negative unification variables. The second and the third equations hold because Rule $(\textsc{Var}^{\leqslant}_{-})$ was applied.
The rest of the proof is symmetric to case 2 of lemma 78.

□

7 PROPERTIES OF THE DECLARATIVE TYPING

Lemma 85. *If* $\Gamma; \Phi \vdash N_1 \bullet \vec{v} \Longrightarrow\!\!> M$ *and* $\Gamma \vdash N_1 \simeq^{\leqslant} N_2$ *then* $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Longrightarrow\!\!> M$.

Proof. By lemma 34, $\Gamma \vdash N_1 \simeq^{\leqslant} N_2$ implies $N_1 \simeq^{D} N_2$. Let us prove the required judgement by induction on $N_1 \simeq^{D} N_2$. Let us consider the last rule used in the derivation.

Case 1. Rule $(\textsc{Var}^{\simeq^D}_{-})$. It means that N_1 is α^- and N_2 is α^-. Then the required property coincides with the assumption.

Case 2. Rule $(\uparrow^{\simeq^D})$. It means that N_1 is $\uparrow P_1$ and N_2 is $\uparrow P_2$. where $P_1 \simeq^{D} P_2$.

Then the only rule applicable to infer $\Gamma;\Phi \vdash \uparrow P_1 \bullet \vec{v} \Rrightarrow M$ is Rule $(\varnothing^{\text{INF}}_{\bullet\Rrightarrow})$, meaning that $\vec{v} = \cdot$ and $\Gamma \vdash \uparrow P_1 \simeq^{\leqslant} M$. Then by transitivity of equivalence corollary 10, $\Gamma \vdash \uparrow P_2 \simeq^{\leqslant} M$, and then Rule $(\varnothing^{\text{INF}}_{\bullet\Rrightarrow})$ is applicable to infer $\Gamma;\Phi \vdash \uparrow P_2 \bullet \cdot \Rrightarrow M$.

Case 3. Rule $(\to^{\simeq^D})$. Then we are proving that $\Gamma;\Phi \vdash (Q_1 \to N_1) \bullet v, \vec{v} \Rrightarrow M$ and $Q_1 \to N_1 \simeq^D Q_2 \to N_2$ imply $\Gamma;\Phi \vdash (Q_2 \to N_2) \bullet v, \vec{v} \Rrightarrow M$.
By inversion, $(Q_1 \to N_1) \simeq^D (Q_2 \to N_2)$ means $Q_1 \simeq^D Q_2$ and $N_1 \simeq^D N_2$.
By inversion of $\Gamma;\Phi \vdash (Q_1 \to N_1) \bullet v, \vec{v} \Rrightarrow M$:
(1) $\Gamma;\Phi \vdash v\colon P$
(2) $\Gamma \vdash Q_1 \geqslant P$, and then by transitivity lemma 24, $\Gamma \vdash Q_2 \geqslant P$;
(3) $\Gamma;\Phi \vdash N_1 \bullet \vec{v} \Rrightarrow M$, and then by induction hypothesis, $\Gamma;\Phi \vdash N_2 \bullet \vec{v} \Rrightarrow M$.
Since we have $\Gamma;\Phi \vdash v\colon P$, $\Gamma \vdash Q_2 \geqslant P$ and $\Gamma;\Phi \vdash N_2 \bullet \vec{v} \Rrightarrow M$, we can apply Rule $(\to^{\text{INF}}_{\bullet\Rrightarrow})$ to infer $\Gamma;\Phi \vdash (Q_2 \to N_2) \bullet v, \vec{v} \Rrightarrow M$.

Case 4. Rule $(\forall^{\simeq^D})$ Then we are proving that $\Gamma;\Phi \vdash \forall\overrightarrow{\alpha^+}_1.N_1' \bullet \vec{v} \Rrightarrow M$ and $\forall\overrightarrow{\alpha^+}_1.N_1' \simeq^D \forall\overrightarrow{\alpha^+}_2.N_2'$ imply $\Gamma;\Phi \vdash \forall\overrightarrow{\alpha^+}_2.N_2' \bullet \vec{v} \Rrightarrow M$.
By inversion of $\forall\overrightarrow{\alpha^+}_1.N_1' \simeq^D \forall\overrightarrow{\alpha^+}_2.N_2'$:
(1) $\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, N_1 = \varnothing$,
(2) there exists a bijection $\mu : (\overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, N_2') \leftrightarrow (\overrightarrow{\alpha^+}_1 \cap \mathbf{fv}\, N_1')$ such that $N_1' \simeq^D [\mu]N_2'$.
By inversion of $\Gamma;\Phi \vdash \forall\overrightarrow{\alpha^+}_1.N_1' \bullet \vec{v} \Rrightarrow M$:
(1) $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}_1$
(2) $\Gamma;\Phi \vdash [\sigma]N_1' \bullet \vec{v} \Rrightarrow M$
(3) $\vec{v} \neq \cdot$
Let us construct $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^+}_2$ in the following way:

$$\begin{cases} [\sigma_0]\alpha^+ = [\sigma][\mu]\alpha^+ & \text{if } \alpha^+ \in \overrightarrow{\alpha^+}_2 \cap \mathbf{fv}\, N_2' \\ [\sigma_0]\alpha^+ = \exists\beta^-.{\downarrow}\beta^- & \text{otherwise (the type does not matter here)} \end{cases}$$

Then to infer $\Gamma;\Phi \vdash N_2 \bullet \vec{v} \Rrightarrow M$, we apply Rule $(\to^{\text{INF}}_{\bullet\Rrightarrow})$ with σ_0. Let us show the required premises:
(1) $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^+}_2$ by construction;
(2) $\vec{v} \neq \cdot$ as noted above;
(3) To show $\Gamma;\Phi \vdash [\sigma_0]N_2' \bullet \vec{v} \Rrightarrow M$, Notice that $[\sigma_0]N_2' = [\sigma][\mu]N_2'$ and since $[\mu]N_2' \simeq^D N_1'$, $[\sigma][\mu]N_2' \simeq^D [\sigma]N_1'$. This way, by lemma 29, $\Gamma \vdash [\sigma]N_1' \simeq^{\leqslant} [\sigma_0]N_2'$. Then the required judgement holds by the induction hypothesis applied to $\Gamma;\Phi \vdash [\sigma]N_1' \bullet \vec{v} \Rrightarrow > M$.

□

Definition 30 (Number of prenex quantifiers). *Let us define* $\mathsf{npq}(N)$ *and* $\mathsf{npq}(P)$ *as the number of prenex quantifiers in these types, i.e.*

$+$ $\mathsf{npq}(\exists\overrightarrow{\alpha^-}.P) = |nas|$, *if* $P \neq \exists\overrightarrow{\beta^-}.P'$,
$-$ $\mathsf{npq}(\forall\overrightarrow{\alpha^+}.N) = |pas|$, *if* $N \neq \forall\overrightarrow{\beta^+}.N'$.

Definition 31 (Size of a Declarative Judgement). *For a declarative typing judgement* J *let us define a metrics* $\mathsf{size}(J)$ *as a pair of numbers in the following way:*

$+$ $\mathsf{size}(\Gamma;\Phi \vdash v\colon P) = (\mathsf{size}(v), 0)$;
$-$ $\mathsf{size}(\Gamma;\Phi \vdash c\colon N) = (\mathsf{size}(c), 0)$;
$\bullet$ $\mathsf{size}(\Gamma;\Phi \vdash N \bullet \vec{v} \Rrightarrow M) = (\mathsf{size}(\vec{v}), \mathsf{npq}(N)))$

where $\mathsf{size}(v)$ *or* $\mathsf{size}(c)$ *is the size of the syntax tree of the term* v *or* c *and* $\mathsf{size}(\vec{v})$ *is the sum of sizes of the terms in* $\vec{v}$.

Definition 32 (Number of Equivalence Nodes). *For a tree* T *inferring a declarative typing judgement, let us a function* $\mathsf{eq\_nodes}(T)$ *as the number of nodes in* T *labeled with Rule* $(\simeq^{INF}_{+})$ *or Rule* $(\simeq^{INF}_{-})$.

Definition 33 (Metric). *For a tree* T *inferring a declarative typing judgement* J*, let us define a metric* $\mathsf{metric}(T)$ *as a pair* $(\mathsf{size}(J), \mathsf{eq\_nodes}(T))$.

Lemma 86 (Declarative typing is preserved under context equivalence). *Assuming* $\Gamma \vdash \Phi_1$, $\Gamma \vdash \Phi_2$, *and* $\Gamma \vdash \Phi_1 \simeq^{\leqslant} \Phi_2$:

- $+$ *for any tree* T_1 *inferring* $\Gamma; \Phi_1 \vdash v \colon P$, *there exists a tree* T_2 *inferring* $\Gamma; \Phi_2 \vdash v \colon P$.
- $-$ *for any tree* T_1 *inferring* $\Gamma; \Phi_1 \vdash c \colon N$, *there exists a tree* T_2 *inferring* $\Gamma; \Phi_2 \vdash c \colon N$.
- $\bullet$ *for any tree* T_1 *inferring* $\Gamma; \Phi_1 \vdash N \bullet \vec{v} \Rrightarrow M$, *there exists a tree* T_2 *inferring* $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rightarrow > M$.

Proof. Let us prove it by induction on the $\mathsf{metric}(T_1)$. Let us consider the last rule applied in T_1 (i.e., its root node).

Case 1. Rule $(\textsc{Var}^{\textsc{inf}})$
Then we are proving that $\Gamma; \Phi_1 \vdash x \colon P$ implies $\Gamma; \Phi_2 \vdash x \colon P$. By inversion, $x : P \in \Phi_1$, and since $\Gamma \vdash \Phi_1 \simeq^{\leqslant} \Phi_2$, $x : P' \in \Phi_2$ for some P' such that $\Gamma \vdash P \simeq^{\leqslant} P'$. Then we infer $\Gamma; \Phi_2 \vdash x \colon P'$ by Rule $(\textsc{Var}^{\textsc{inf}})$, and next, $\Gamma; \Phi_2 \vdash x \colon P$ by Rule $(\simeq^{\textsc{inf}}_{+})$.

Case 2. For Rule $(\{\}^{\textsc{inf}})$, Rule $(\textsc{ann}^{\textsc{inf}}_{+})$, Rule $(\Lambda^{\textsc{inf}})$, Rule $(\textsc{ret}^{\textsc{inf}})$, and Rule $(\textsc{ann}^{\textsc{inf}}_{-})$ the proof is analogous. We apply the induction hypothesis to the premise of the rule to substitute Φ_1 for Φ_2 in it. The induction is applicable because the metric of the premises is less than the metric of the conclusion: the term in the premise is a syntactic subterm of the term in the conclusion.
And after that, we apply the same rule to infer the required judgement.

Case 3. Rule $(\simeq^{\textsc{inf}}_{+})$ and Rule $(\simeq^{\textsc{inf}}_{-})$ In these cases, the induction hypothesis is also applicable to the premise: although the first component of the metric is the same for the premise and the conclusion: $\mathsf{size}(\Gamma; \Phi \vdash c \colon N') = \mathsf{size}(\Gamma; \Phi \vdash c \colon N) = \mathsf{size}(c)$, the second component of the metric is less for the premise by one, since the equivalence rule was applied to turn the premise tree into $T1$. Having made this note, we continue the proof in the same way as in the previous case.

Case 4. Rule $(\lambda^{\textsc{inf}})$ Then we are proving that $\Gamma; \Phi_1 \vdash \lambda x : P.c \colon P \to N$ implies $\Gamma; \Phi_2 \vdash \lambda x : P.c \colon P \to N$. Analogously to the previous cases, we apply the induction hypothesis to the equivalent contexts $\Gamma \vdash \Phi_1, x : P \simeq^{\leqslant} \Phi_2, x : P$ and the premise $\Gamma; \Phi_1, x : P \vdash c \colon N$ to obtain $\Gamma; \Phi_2, x : P \vdash c \colon N$. Notice that c is a subterm of $\lambda x : P.c$, i.e., the metric of the premise tree is less than the metric of the conclusion, and the induction hypothesis is applicable. Then we infer $\Gamma; \Phi_2 \vdash \lambda x : P.c \colon P \to N$ by Rule $(\lambda^{\textsc{inf}})$.

Case 5. Rule $(\textsc{let}^{\textsc{inf}})$ Then we are proving that $\Gamma; \Phi_1 \vdash \mathbf{let}\, x = v; c \colon N$ implies $\Gamma; \Phi_2 \vdash \mathbf{let}\, x = v; c \colon N$. First, we apply the induction hypothesis to $\Gamma; \Phi_1 \vdash v \colon P$ to obtain $\Gamma; \Phi_2 \vdash v \colon P$ of the same pure size.
Then we apply the induction hypothesis to the equivalent contexts $\Gamma \vdash \Phi_1, x : P \simeq^{\leqslant} \Phi_2, x : P$ and the premise $\Gamma; \Phi_1, x : P \vdash c \colon N$ to obtain $\Gamma; \Phi_2, x : P \vdash c \colon N$. Then we infer $\Gamma; \Phi_2 \vdash \mathbf{let}\, x = v; c \colon N$ by Rule $(\textsc{let}^{\textsc{inf}})$.

Case 6. Rule $(\textsc{let}^{\textsc{inf}}_{@})$ Then we are proving that $\Gamma; \Phi_1 \vdash \mathbf{let}\, x = v(\vec{v}); c \colon N$ implies $\Gamma; \Phi_2 \vdash \mathbf{let}\, x = v(\vec{v}); c \colon N$.
We apply the induction hypothesis to each of the premises. to rewrite:

- $\Gamma; \Phi_1 \vdash v\colon \downarrow M$ into $\Gamma; \Phi_2 \vdash v\colon \downarrow M$,
- $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q$ into $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q$.
- $\Gamma; \Phi_1, x : Q \vdash c\colon N$ into $\Gamma; \Phi_2, x : Q \vdash c\colon N$ (notice that $\Gamma \vdash \Phi_1, x : Q \simeq^{\leqslant} \Phi_2, x : Q$).

It is left to show the uniqueness of $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q$. Let us assume that this judgement holds for other Q', i.e. there exists a tree T_0 inferring $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q'$. Then notice that the induction hypothesis is applicable to T_0: the first component of the first component of $\mathsf{metric}(T_0)$ is $S = \sum_{v \in \vec{v}} \mathsf{size}(v)$, and it is less than the corresponding component of $\mathsf{metric}(T_1)$, which is $\mathsf{size}(\mathbf{let}\, x = v(\vec{v}); c) = 1 + \mathsf{size}(v) + \mathsf{size}(c) + S$. This way, $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q'$ holds by the induction hypothesis, but since $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q$ unique, we have $\Gamma \vdash Q' \simeq^{\leqslant} Q$.

Then we infer $\Gamma; \Phi_2 \vdash \mathbf{let}\, x = v(\vec{v}); c\colon N$ by Rule $(\textsc{let}^{\textsc{inf}}_{@})$.

Case 7. Rule $(\textsc{let}^{\textsc{inf}}_{:@})$ Then we are proving that $\Gamma; \Phi_1 \vdash \mathbf{let}\, x : P = v(\vec{v}); c\colon N$ implies $\Gamma; \Phi_2 \vdash \mathbf{let}\, x : P = v(\vec{v}); c\colon N$.

As in the previous case, we apply the induction hypothesis to each of the premises and rewrite:

- $\Gamma; \Phi_1 \vdash v\colon \downarrow M$ into $\Gamma; \Phi_2 \vdash v\colon \downarrow M$,
- $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q$ into $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rrightarrow \uparrow Q$, and
- $\Gamma; \Phi_1, x : P \vdash c\colon N$ into $\Gamma; \Phi_2, x : P \vdash c\colon N$ (notice that $\Gamma \vdash \Phi_1, x : P \simeq^{\leqslant} \Phi_2, x : P$).

Notice that $\Gamma \vdash P$ and $\Gamma \vdash \uparrow Q \leqslant \uparrow P$ do not depend on the variable context, and hold by assumption. Then we infer $\Gamma; \Phi_2 \vdash \mathbf{let}\, x : P = v(\vec{v}); c\colon N$ by Rule $(\textsc{let}^{\textsc{inf}}_{:@})$.

Case 8. Rule $(\textsc{let}^{\textsc{inf}}_{\exists})$, and Rule $(\textsc{ann}^{\textsc{inf}}_{-})$ are proved in the same way.

Case 9. Rule $(\varnothing^{\textsc{inf}}_{\bullet\Rrightarrow})$ Then we are proving that $\Gamma; \Phi_1 \vdash N \bullet \cdot \Rrightarrow N'$ (inferred by Rule $(\varnothing^{\textsc{inf}}_{\bullet\Rrightarrow})$) implies $\Gamma; \Phi_2 \vdash N \bullet \cdot \Rrightarrow N'$.

To infer $\Gamma; \Phi_2 \vdash N \bullet \cdot \Rrightarrow N'$, we apply Rule $(\varnothing^{\textsc{inf}}_{\bullet\Rrightarrow})$, noting that $\Gamma \vdash N \simeq^{\leqslant} N'$ holds by assumption.

Case 10. Rule $(\rightarrow^{\textsc{inf}}_{\bullet\Rrightarrow})$ Then we are proving that $\Gamma; \Phi_1 \vdash Q \to N \bullet v, \vec{v} \Rrightarrow M$ (inferred by Rule $(\rightarrow^{\textsc{inf}}_{\bullet\Rrightarrow})$) implies $\Gamma; \Phi_2 \vdash Q \to N \bullet v, \vec{v} \Rrightarrow M$. And uniqueness of the M in the first case implies uniqueness in the second case.

By induction, we rewrite $\Gamma; \Phi_1 \vdash v\colon P$ into $\Gamma; \Phi_2 \vdash v\colon P$, and $\Gamma; \Phi_1 \vdash N \bullet \vec{v} \Rrightarrow M$ into $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rrightarrow M$. Then we infer $\Gamma; \Phi_2 \vdash Q \to N \bullet v, \vec{v} \Rrightarrow M$ by Rule $(\rightarrow^{\textsc{inf}}_{\bullet\Rrightarrow})$.

Now, let us show the uniqueness. The only rule that can infer $\Gamma; \Phi_1 \vdash Q \to N \bullet v, \vec{v} \Rrightarrow M$ is Rule $(\rightarrow^{\textsc{inf}}_{\bullet\Rrightarrow})$. Then by inversion, uniqueness of $\Gamma; \Phi_1 \vdash Q \to N \bullet v, \vec{v} \Rrightarrow M$ implies uniqueness of $\Gamma; \Phi_1 \vdash N \bullet \vec{v} \Rrightarrow M$. By the induction hypothesis, it implies the uniqueness of $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rrightarrow M$.

Suppose that $\Gamma; \Phi_2 \vdash Q \to N \bullet v, \vec{v} \Rrightarrow M'$. By inversion, $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rrightarrow M'$, which by uniqueness of $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rrightarrow M$ implies $\Gamma \vdash M \simeq^{\leqslant} M'$.

Case 11. Rule $(\forall^{\textsc{inf}}_{\bullet\Rrightarrow})$ Then we are proving that $\Gamma; \Phi_1 \vdash \forall \overrightarrow{\alpha^+}.N \bullet \vec{v} \Rrightarrow M$ (inferred by Rule $(\forall^{\textsc{inf}}_{\bullet\Rrightarrow})$) implies $\Gamma; \Phi_2 \vdash \forall \overrightarrow{\alpha^+}.N \bullet \vec{v} \Rrightarrow M$.

By inversion, we have σ such that $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}$ and $\Gamma; \Phi_1 \vdash [\sigma]N \bullet \vec{v} \Rrightarrow M$ is inferred. Let us denote the inference tree as T_1'. Notice that the induction hypothesis is applicable to T_1': $\mathsf{metric}(T_1') = ((\mathsf{size}(\vec{v}), 0), x)$ is less than $\mathsf{metric}(T_1) = ((\mathsf{size}(\vec{v}), |\overrightarrow{\alpha^+}|), y)$ for any x and y, since $|\overrightarrow{\alpha^+}| > 0$ by inversion.

This way, by the induction hypothesis, there exists a tree T_2' inferring $\Gamma; \Phi_2 \vdash [\sigma]N \bullet \vec{v} \Rrightarrow M$. Notice that the premises $\vec{v} \neq \cdot$, $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}$, and $\overrightarrow{\alpha^+} \neq \cdot$ do not depend on the variable context, and hold by inversion. Then we infer $\Gamma; \Phi_2 \vdash \forall \overrightarrow{\alpha^+}.N \bullet \vec{v} \Rrightarrow M$ by Rule $(\forall^{\textsc{inf}}_{\bullet\Rrightarrow})$.

□

8 PROPERTIES OF THE ALGORITHMIC TYPING

8.1 Singularity

Lemma 87 (Soundness of Entry Singularity). + *Suppose* e **singular with** P *for* P *well-formed in* Γ. *Then* $\Gamma \vdash P : e$ *and for any* $\Gamma \vdash P' : e$ *such that* $\Gamma \vdash P' \simeq^{\leqslant} P$;

− *Suppose* e **singular with** N *for* N *well-formed in* Γ. *Then* $\Gamma \vdash N : e$ *and for any* $\Gamma \vdash N' : e$ *such that* $\Gamma \vdash N' : e$, $\Gamma \vdash N' \simeq^{\leqslant} N$.

Proof. Let us consider how e **singular with** P or e **singular with** N is formed.

Case 1. Rule $(\simeq^{\text{SING}}_{-})$, that is $e = \widehat{\alpha}^{-} :\simeq N_0$. and N is $\mathbf{nf}\,(N_0)$. Then $\Gamma \vdash N' : e$ means $\Gamma \vdash N' \simeq^{\leqslant} N_0$, (by inversion of Rule $(:\simeq^{\text{SAT}}_{-})$), which by transitivity, using corollary 13, means $\Gamma \vdash N' \simeq^{\leqslant} \mathbf{nf}\,(N_0)$, as required.

Case 2. Rule $(\simeq^{\text{SING}}_{+})$. This case is symmetric to the previous one.

Case 3. Rule $(:\geqslant \alpha^{\text{SING}})$, that is $e = \widehat{\alpha}^{+} :\geqslant \exists\overrightarrow{\alpha^{-}}.\beta^{+}$, and $P = \beta^{+}$.

Since $\Gamma \vdash \beta^{+} \geqslant \exists\overrightarrow{\alpha^{-}}.\beta^{+}$, we have $\Gamma \vdash \beta^{+} : e$, as required.

Notice that $\Gamma \vdash P' : e$ means $\Gamma \vdash P' \geqslant \exists\overrightarrow{\alpha^{-}}.\beta^{+}$. Let us show that it implies $\Gamma \vdash P' \simeq^{\leqslant} \beta^{+}$.

By applying lemma 68 once, we have $\Gamma, \overrightarrow{\alpha^{-}} \vdash P' \geqslant \beta^{+}$. By applying it again, we notice that $\Gamma, \overrightarrow{\alpha^{-}} \vdash P' \geqslant \beta^{+}$ implies $P_i = \exists\overrightarrow{\alpha^{-}}'.\beta^{+}$. Finally, it is easy to see that $\Gamma \vdash \exists\overrightarrow{\alpha^{-}}'.\beta^{+} \simeq^{\leqslant} \beta^{+}$

Case 4. Rule $(:\geqslant \downarrow^{\text{SING}})$, that is $e = \widehat{\alpha}^{+} :\geqslant \exists\overrightarrow{\beta^{-}}.\downarrow N_1$, where $N_1 \simeq^{D} \beta_j^{-}$, and $P = \exists\alpha^{-}.\downarrow\alpha^{-}$.

Since $\Gamma \vdash \exists\alpha^{-}.\downarrow\alpha^{-} \geqslant \exists\overrightarrow{\beta^{-}}.\downarrow N_1$ (by Rule $(\exists^{\geqslant})$, with substitution N_1/α^{-}), we have $\Gamma \vdash \exists\alpha^{-}.\downarrow\alpha^{-} : e$, as required.

Notice $\Gamma \vdash P' : e$ means $\Gamma \vdash P' \geqslant \exists\overrightarrow{\beta^{-}}.\downarrow N_1$. Let us show that it implies $\Gamma \vdash P' \simeq^{\leqslant} \exists\alpha^{-}.\downarrow\alpha^{-}$.

$[h]\Gamma \vdash P' \geqslant \exists\overrightarrow{\beta^{-}}.\downarrow N_1 \Rightarrow \Gamma \vdash \mathbf{nf}\,(P') \geqslant \exists\overrightarrow{\beta^{-}}'.\downarrow\mathbf{nf}\,(N_1)$ where $\mathbf{ord}\,\overrightarrow{\beta^{-}}\,\mathbf{in}\,N' = \overrightarrow{\beta^{-}}'$ by corollary 14

$\Rightarrow \Gamma \vdash \mathbf{nf}\,(P') \geqslant \exists\overrightarrow{\beta^{-}}'.\downarrow\mathbf{nf}\,(\beta_j^{-})$ by lemma 46

$\Rightarrow \Gamma \vdash \mathbf{nf}\,(P') \geqslant \exists\overrightarrow{\beta^{-}}'.\downarrow\beta_n^{-}$ by definition of n

$\Rightarrow \Gamma \vdash \mathbf{nf}\,(P') \geqslant \exists\beta_j^{-}.\downarrow\beta_j^{-}$ since $\mathbf{ord}\,\overrightarrow{\beta^{-}}\,\mathbf{in}$ nf

$\Rightarrow \Gamma, \beta_j^{-} \vdash \mathbf{nf}\,(P') \geqslant \downarrow\beta_j^{-}$ and $\beta_j^{-} \notin \mathbf{fv}\,(\mathbf{nf}\,(P'))$ by lemma 69

By lemma 69, the last subtyping means that $\mathbf{nf}\,(P') = \exists\overrightarrow{\alpha^{-}}.\downarrow N'$, such that

(1) $\Gamma, \beta_j^{-}, \overrightarrow{\alpha^{-}} \vdash N'$

(2) $\mathbf{ord}\,\overrightarrow{\alpha^{-}}\,\mathbf{in}\,N' = \overrightarrow{\alpha^{-}}$

(3) for some substitution $\Gamma, \beta_j^{-} \vdash \sigma : \overrightarrow{\alpha^{-}}$, $[\sigma]N' = \beta_j^{-}$.

Since $\beta_j^{-} \notin \mathbf{fv}\,(\mathbf{nf}\,(P'))$, the latter means that $N' = \alpha^{-}$, and then $\mathbf{nf}\,(P') = \exists\alpha^{-}.\downarrow\alpha^{-}$ for some α^{-}. Finally, notice that all the types of shape $\exists\alpha^{-}.\downarrow\alpha^{-}$ are equal.

□

Lemma 88 (Completeness of Entry Singularity).

− *Suppose that there exists* N *well-formed in* Γ *such that for any* N' *well-formed in* Γ, $\Gamma \vdash N' : e$ *implies* $\Gamma \vdash N' \simeq^{\leqslant} N$. *Then* e **singular with nf** (N).

+ *Suppose that there exists* P *well-formed in* Γ *such that for any* P' *well-formed in* Γ, $\Gamma \vdash P' : e$ *implies* $\Gamma \vdash P' \simeq^{\leqslant} P$. *Then* e **singular with nf** (P) .

Proof.

− By lemma 75, there exists $\Gamma \vdash N' : e$. Since N' is negative, by inversion of $\Gamma \vdash N' : e$, e has shape $\widehat{\alpha}^- :\simeq M$, where $\Gamma \vdash N' \simeq^{\leqslant} M$, and transitively, $\Gamma \vdash N \simeq^{\leqslant} M$. Then $\mathbf{nf}\,(M) = \mathbf{nf}\,(N)$, and e **singular with nf** (M) (by Rule $(\simeq^{\text{SING}}_{-})$) is rewritten as e **singular with nf** (N).

+ By lemma 75, there exists $\Gamma \vdash P' : e$, then by assumption, $\Gamma \vdash P' \simeq^{\leqslant} P$, which by lemma 76 implies $\Gamma \vdash P : e$.

Let us consider the shape of e:

Case 1. $e = (\widehat{\alpha}^+ :\simeq Q)$ then inversion of $\Gamma \vdash P : e$ implies $\Gamma \vdash P \simeq^{\leqslant} Q$, and hence, $\mathbf{nf}\,(P) = \mathbf{nf}\,(Q)$ (by lemma 35). Then e **singular with nf** (Q), which holds by Rule $(\simeq^{\text{SING}}_{+})$, is rewritten as e **singular with nf** (P).

Case 2. $e = (\widehat{\alpha}^+ :\geqslant Q)$. Then the inversion of $\Gamma \vdash P : e$ implies $\Gamma \vdash P \geqslant Q$. Let us consider the shape of Q:

a. $Q = \exists\overrightarrow{\beta^-}.\beta^+$ (for potentially empty $\overrightarrow{\beta^-}$). Then $\Gamma \vdash P \geqslant \exists\overrightarrow{\beta^-}.\beta^+$ implies $\Gamma \vdash P \simeq^{\leqslant} \beta^+$ by lemma 68, as was noted in the proof of lemma 87, and hence, $\mathbf{nf}\,(P) = \beta^+$.
Then e **singular with** β^+, which holds by Rule $(:\geqslant \alpha^{\text{SING}})$, can be rewritten as e **singular with nf** (P).

b. $Q = \exists\overrightarrow{\beta^-}.{\downarrow}N$ (for potentially empty $\overrightarrow{\beta^-}$). Notice that $\Gamma \vdash \exists\gamma^-.{\downarrow}\gamma^- \geqslant \exists\overrightarrow{\beta^-}.{\downarrow}N$ (by Rule $(\exists^{\geqslant})$, with substitution N/γ^-), and thus, $\Gamma \vdash \exists\gamma^-.{\downarrow}\gamma^- : e$ by Rule $(:\geqslant^{\text{SAT}}_{+})$.
Then by assumption, $\Gamma \vdash \exists\gamma^-.{\downarrow}\gamma^- \simeq^{\leqslant} P$, that is $\mathbf{nf}\,(P) = \exists\gamma^-.{\downarrow}\gamma^-$. To apply Rule $(:\geqslant {\downarrow}^{\text{SING}})$ to infer $(\widehat{\alpha}^+ :\geqslant \exists\overrightarrow{\beta^-}.{\downarrow}N)$ **singular with** $\exists\gamma^-.{\downarrow}\gamma^-$, it is left to show that $N \simeq^{D} \beta_i^-$ for some i.
Since $\Gamma \vdash Q : e$, by assumption, $\Gamma \vdash Q \simeq^{\leqslant} P$, and by transitivity, $\Gamma \vdash Q \simeq^{\leqslant} \exists\gamma^-.{\downarrow}\gamma^-$. It implies $\mathbf{nf}\,(\exists\overrightarrow{\beta^-}.{\downarrow}N) = \exists\gamma^-.{\downarrow}\gamma^-$ (by lemma 35), which by definition of normalization means $\exists\overrightarrow{\beta^-}'.{\downarrow}\mathbf{nf}\,(N) = \exists\gamma^-.{\downarrow}\gamma^-$, where $\mathbf{ord}\,\overrightarrow{\beta^-}\,\mathbf{in}\,N' = \overrightarrow{\beta^-}'$. This way, $\overrightarrow{\beta^-}'$ is a variable β^-, and $\mathbf{nf}\,(N) = \beta^-$. Notice that $\beta^- \in \overrightarrow{\beta^-}' \subseteq \overrightarrow{\beta^-}$ by lemma 37. This way, $N \simeq^{D} \beta^-$ for $\beta^- \in \overrightarrow{\beta^-}$ (by lemma 35),

□

Lemma 89 (Soundness of Singularity). *Suppose* $\Theta \vdash SC : \Xi$, *and* SC **singular with** $\widehat{\sigma}$. *Then* $\Theta \vdash \widehat{\sigma} : \Xi$, $\Theta \vdash \widehat{\sigma} : SC$ *and for any* $\widehat{\sigma}'$ *such that* $\Theta \vdash \widehat{\sigma} : SC$, $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma} : \Xi$.

Proof. Suppose that $\Theta \vdash \widehat{\sigma}' : SC$. It means that for every $e \in SC$ restricting $\widehat{\alpha}^{\pm}$, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}']\widehat{\alpha}^{\pm} : e$ holds. SC **singular with** $\widehat{\sigma}$ means e **singular with** $[\widehat{\sigma}]\widehat{\alpha}^{\pm}$, and hence, by lemma 88, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}']\widehat{\alpha}^{\pm} \simeq^{\leqslant} [\widehat{\sigma}]\widehat{\alpha}^{\pm}$ holds.
Since the uniqueness holds for every variable from $\mathbf{dom}\,(SC)$, $\widehat{\sigma}$ is equivalent to $\widehat{\sigma}'$ on this set. □

Observation 14 (Singularity is Deterministic). *For a fixed* SC *such that* $\Theta \vdash SC : \Xi$, *if* SC **singular with** $\widehat{\sigma}$ *and* SC **singular with** $\widehat{\sigma}'$, *then* $\widehat{\sigma} = \widehat{\sigma}'$.

Proof. By lemma 89, $\Theta \vdash \widehat{\sigma} : \Xi$ and $\Theta \vdash \widehat{\sigma}' : \Xi$. It means that both $\widehat{\sigma}$ and $\widehat{\sigma}'$ act as identity outside of Ξ.
Moreover, for any $\widehat{\alpha}^{\pm} \in \Xi$, $\Theta \vdash SC : \Xi$ means that there is a unique $e \in SC$ restricting $\widehat{\alpha}^{\pm}$. Then SC **singular with** $\widehat{\sigma}$ means that e **singular with** $[\widehat{\sigma}]\widehat{\alpha}^{\pm}$. By looking at the inference rules, it is easy to see that $[\widehat{\sigma}]\widehat{\alpha}^{\pm}$ is uniquely determined by e, which, Similarly, $[\widehat{\sigma}']\widehat{\alpha}^{\pm}$ is also uniquely determined by e, in the same way, and hence, $[\widehat{\sigma}]\widehat{\alpha}^{\pm} = [\widehat{\sigma}']\widehat{\alpha}^{\pm}$. □

Lemma 90 (Completeness of Singularity). *For a given $\Theta \vdash SC$, suppose that all the substitutions satisfying SC are equivalent on $\Xi \supseteq \mathbf{dom}\,(SC)$. In other words, suppose that there exists $\Theta \vdash \widehat{\sigma}_1 : \Xi$ such that for any $\Theta \vdash \widehat{\sigma} : \Xi$, $\Theta \vdash \widehat{\sigma} : SC$ implies $\Theta \vdash \widehat{\sigma} \simeq^{\leqslant} \widehat{\sigma}_1 : \Xi$. Then*

- *SC* **singular with** *$\widehat{\sigma}_0$ for some $\widehat{\sigma}_0$ and*
- *$\Xi = \mathbf{dom}\,(SC)$.*

Proof. First, let us assume $\Xi \neq \mathbf{dom}\,(SC)$. Then there exists $\widehat{\alpha}^{\pm} \in \Xi \backslash \mathbf{dom}\,(SC)$. Let us take $\Theta \vdash \widehat{\sigma}_1 : \Xi$ such that any other substitution $\Theta \vdash \widehat{\sigma} : \Xi$ satisfying SC is equivalent to $\widehat{\sigma}_1$ on Ξ.

Notice that $\Theta \vdash \widehat{\sigma}_1 : SC$: by lemma 75, there exists $\widehat{\sigma}'$ such that $\Theta \vdash \widehat{\sigma}' : \Xi$ and $\Theta \vdash \widehat{\sigma}' : SC$, and by assumption, $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma}_1 : \Xi$, implying $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma}_1 : \mathbf{dom}\,(SC)$.

Let us construct $\widehat{\sigma}_2$ such that $\Theta \vdash \widehat{\sigma}_2 : \Xi$ as follows:

$$\begin{cases} [\widehat{\sigma}_2]\widehat{\beta}^{\pm} = [\widehat{\sigma}_1]\widehat{\beta}^{\pm} & \text{if } \widehat{\beta}^{\pm} \neq \widehat{\alpha}^{\pm} \\ [\widehat{\sigma}_2]\widehat{\alpha}^{\pm} = T & \text{where } T \text{ is any closed type not equivalent to } [\widehat{\sigma}_1]\widehat{\alpha}^{\pm} \end{cases}$$

It is easy to see that $\Theta \vdash \widehat{\sigma}_2 : SC$ since $\widehat{\sigma}_1|_{\mathbf{dom}\,(SC)} = \widehat{\sigma}_2|_{\mathbf{dom}\,(SC)}$, and $\Theta \vdash \widehat{\sigma}_1 : SC$. However, $\Theta \vdash \widehat{\sigma}_2 \simeq^{\leqslant} \widehat{\sigma}_1 : \Xi$ does not hold because by construction, $\Theta(\widehat{\alpha}^{\pm}) \vdash [\widehat{\sigma}_2]\widehat{\alpha}^{\pm} \simeq^{\leqslant} [\widehat{\sigma}_1]\widehat{\alpha}^{\pm}$ does not hold. This way, we have a contradiction.

Second, let us show SC **singular with** $\widehat{\sigma}_0$. Let us take arbitrary $e \in SC$ restricting $\widehat{\beta}^{\pm}$. We need to show that e is singular. Notice that $\Theta \vdash \widehat{\sigma}_1 : SC$ implies $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}_1]\widehat{\beta}^{\pm}$ and $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}_1]\widehat{\beta}^{\pm} : e$. We will show that any other type satisfying e is equivalent to $[\widehat{\sigma}_1]\widehat{\beta}^{\pm}$, then by lemma 88, e **singular with** $[\widehat{\sigma}_1]\widehat{\beta}^{\pm}$.

- if $\widehat{\beta}^{\pm}$ is positive, let us take any type $\Theta(\widehat{\beta}^{\pm}) \vdash P'$ and assume $\Theta(\widehat{\beta}^{\pm}) \vdash P' : e$. We will show that $\Theta(\widehat{\beta}^{\pm}) \vdash P' \simeq^{\leqslant} [\widehat{\sigma}_1]\widehat{\beta}^{\pm}$, which by lemma 35 will imply e **singular with nf** $([\widehat{\sigma}_1]\widehat{\beta}^{\pm})$.
  Let us construct $\widehat{\sigma}_2$ such that $\Theta \vdash \widehat{\sigma}_2 : \Xi$ as follows:
  $$\begin{cases} [\widehat{\sigma}_2]\widehat{\gamma}^{\pm} = [\widehat{\sigma}_1]\widehat{\gamma}^{\pm} & \text{if } \widehat{\gamma}^{\pm} \neq \widehat{\beta}^{\pm} \\ [\widehat{\sigma}_2]\widehat{\beta}^{\pm} = P' \end{cases}$$
  It is easy to see that $\Theta \vdash \widehat{\sigma}_2 : SC$: for e, $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}_2]\widehat{\beta}^{\pm} : e$ by construction, since $\Theta(\widehat{\beta}^{\pm}) \vdash P' : e$; for any other $e' \in SC$ restricting $\widehat{\gamma}^{\pm}$, $[\widehat{\sigma}_2]\widehat{\gamma}^{\pm} = [\widehat{\sigma}_1]\widehat{\gamma}^{\pm}$, and $\Theta(\widehat{\gamma}^{\pm}) \vdash [\widehat{\sigma}_1]\widehat{\gamma}^{\pm} : e'$ since $\Theta \vdash \widehat{\sigma}_1 : SC$.
  Then by assumption, $\Theta \vdash \widehat{\sigma}_2 \simeq^{\leqslant} \widehat{\sigma}_1 : \Xi$, which in particular means $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}_2]\widehat{\beta}^{\pm} \simeq^{\leqslant} [\widehat{\sigma}_1]\widehat{\beta}^{\pm}$, that is $\Theta(\widehat{\beta}^{\pm}) \vdash P' \simeq^{\leqslant} [\widehat{\sigma}_1]\widehat{\beta}^{\pm}$.
- if $\widehat{\beta}^{\pm}$ is negative, the proof is analogous.

□

8.2 Correctness of the Typing Algorithm

Lemma 91 (Determinicity of typing algorithm). *Suppose that $\Gamma \vdash \Phi$ and $\Gamma \vdash^{\supseteq} \Theta$. Then*

- *If $\Gamma; \Phi \vDash v : P$ and $\Gamma; \Phi \vDash v : P'$ then $P = P'$.*
- *If $\Gamma; \Phi \vDash c : N$ and $\Gamma; \Phi \vDash c : N'$ then $N = N'$.*
- *If $\Gamma; \Phi; \Theta \vDash N \bullet \vec{v} \Rrightarrow M \dashv \Theta'; SC$ and $\Gamma; \Phi; \Theta \vDash N \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC'$ then $M = M'$, $\Theta = \Theta'$, and $SC = SC'$.*

Proof. We show it by structural induction on the inference tree. Notice that the last rule used to infer the judgement is uniquely determined by the input, and that each premise of each inference rule is deterministic by the corresponding observation. □

Let us extend the declarative typing metric (definition 33) to the algorithmic typing.

Definition 34 (Size of an Algorithmic Judgement). *For an algorithmic typing judgement J let us define a metrics* $\mathsf{size}(J)$ *as a pair of numbers in the following way:*

- $+$ $\mathsf{size}(\Gamma;\Phi \vDash v\colon P) = (\mathsf{size}(v), 0)$;
- $-$ $\mathsf{size}(\Gamma;\Phi \vDash c\colon N) = (\mathsf{size}(c), 0)$;
- $\bullet$ $\mathsf{size}(\Gamma;\Phi;\Theta \vDash N \bullet \vec{v} \gg M \dashv \Theta';SC) = (\mathsf{size}(\vec{v}), \mathsf{npq}(N)))$

Definition 35 (Metric). *We extend the metric from definition 33 to the algorithmic typing in the following way. For a tree T inferring an algorithmic typing judgement J, we define* $\mathsf{size}(T)$ *as* $(\mathsf{size}(J), 0)$.

Soundness and the completeness are proved by mutual induction on the metric of the inference tree.

Lemma 92 (Soundness of typing). *Suppose that $\Gamma \vdash \Phi$. For an inference tree T_1,*

- $+$ *If T_1 infers $\Gamma;\Phi \vDash v\colon P$ then $\Gamma \vdash P$ and $\Gamma;\Phi \vdash v\colon P$*
- $-$ *If T_1 infers $\Gamma;\Phi \vDash c\colon N$ then $\Gamma \vdash N$ and $\Gamma;\Phi \vdash c\colon N$*
- $\bullet$ *If T_1 infers $\Gamma;\Phi;\Theta \vDash N \bullet \vec{v} \gg M \dashv \Theta';SC$ for $\Gamma \vdash^{\supseteq} \Theta$ and $\Gamma;\mathbf{dom}\,(\Theta) \vdash N$ free from negative algorithmic variables, then*
  - *(1)* $\Gamma \vdash^{\supseteq} \Theta'$
  - *(2)* $\Theta \subseteq \Theta'$
  - *(3)* $\Gamma;\mathbf{dom}\,(\Theta') \vdash M$
  - *(4)* $\mathbf{dom}\,(\Theta) \cap \mathbf{uv}\,(M) \subseteq \mathbf{uv}\,N$
  - *(5)* M *is normalized and free from negative algorithmic variables*
  - *(6)* $\Theta'|_{\mathbf{uv}\,N \cup \mathbf{uv}\,M} \vdash SC$
  - *(7) for any* $\Theta' \vdash \widehat{\sigma} : \mathbf{uv}\,N \cup \mathbf{uv}\,M$, $\Theta' \vdash \widehat{\sigma} : SC$ *implies* $\Gamma;\Phi \vdash [\widehat{\sigma}]N \bullet \vec{v} \gg [\widehat{\sigma}]M$

Proof. We prove it by induction on metric(T_1), mutually with the completeness of typing (lemma 92). Let us consider the last rule used to infer the derivation.

Case 1. Rule ($\textsc{Var}^{\textsc{inf}}$) We are proving that if $\Gamma;\Phi \vDash x\colon \mathbf{nf}\,(P)$ then $\Gamma \vdash \mathbf{nf}\,(P)$ and $\Gamma;\Phi \vdash x\colon \mathbf{nf}\,(P)$.
By inversion, $x : P \in \Phi$. Since $\Gamma \vdash \Phi$, we have $\Gamma \vdash P$, and by corollary 11, $\Gamma \vdash \mathbf{nf}\,(P)$.
By applying Rule ($\textsc{Var}^{\textsc{inf}}$) to $x : P \in \Phi$, we infer $\Gamma;\Phi \vdash x\colon P$. Finally, by Rule ($\simeq^{\textsc{inf}}_{+}$), since $\Gamma \vdash P \simeq^{\leqslant} \mathbf{nf}\,(P)$ (corollary 13), we have $\Gamma;\Phi \vdash x\colon \mathbf{nf}\,(P)$.

Case 2. Rule ($\{\}^{\textsc{inf}}$)
We are proving that if $\Gamma;\Phi \vDash \{c\}\colon \downarrow N$ then $\Gamma \vdash \downarrow N$ and $\Gamma;\Phi \vdash \{c\}\colon \downarrow N$.
By inversion of $\Gamma;\Phi \vDash \{c\}\colon \downarrow N$, we have $\Gamma;\Phi \vDash c\colon N$. By the induction hypothesis applied to $\Gamma;\Phi \vDash c\colon N$, we have
(1) $\Gamma \vdash N$, and hence, $\Gamma \vdash \downarrow N$;
(2) $\Gamma;\Phi \vdash c\colon N$, which by Rule ($\{\}^{\textsc{inf}}$) implies $\Gamma;\Phi \vdash \{c\}\colon \downarrow N$.

Case 3. Rule ($\textsc{ret}^{\textsc{inf}}$) The proof is symmetric to the previous case (case 2).

Case 4. Rule ($\textsc{ann}^{\textsc{inf}}_{+}$) We are proving that if $\Gamma;\Phi \vDash (v : Q)\colon \mathbf{nf}\,(Q)$ then $\Gamma \vdash \mathbf{nf}\,(Q)$ and $\Gamma;\Phi \vdash (v : Q)\colon \mathbf{nf}\,(Q)$.
By inversion of $\Gamma;\Phi \vDash (v : Q)\colon \mathbf{nf}\,(Q)$, we have:
(1) $\Gamma \vdash (v : Q)$, hence, $\Gamma \vdash Q$, and by corollary 11, $\Gamma \vdash \mathbf{nf}\,(Q)$;
(2) $\Gamma;\Phi \vDash v\colon P$, which by the induction hypothesis implies $\Gamma \vdash P$ and $\Gamma;\Phi \vdash v\colon P$;
(3) $\Gamma;\cdot \vDash Q \geqslant P \dashv \cdot$, which by lemma 77 implies $\Gamma \vdash [\cdot]Q \geqslant P$, that is $\Gamma \vdash Q \geqslant P$.
To infer $\Gamma;\Phi \vdash (v : Q)\colon Q$, we apply Rule ($\textsc{ann}^{\textsc{inf}}_{+}$) to $\Gamma;\Phi \vdash v\colon P$ and $\Gamma \vdash Q \geqslant P$. Then by Rule ($\simeq^{\textsc{inf}}_{+}$), $\Gamma;\Phi \vdash (v : Q)\colon \mathbf{nf}\,(Q)$.

Case 5. Rule ($\textsc{ann}^{\textsc{inf}}_{-}$) The proof is symmetric to the previous case (case 4).

Case 6. Rule (λ^{INF}) We are proving that if $\Gamma;\Phi \vDash \lambda x : P.c\colon \mathbf{nf}\,(P \to N)$ then $\Gamma \vdash \mathbf{nf}\,(P \to N)$ and $\Gamma;\Phi \vdash \lambda x : P.c\colon \mathbf{nf}\,(P \to N)$.
By inversion of $\Gamma;\Phi \vDash \lambda x : P.c\colon \mathbf{nf}\,(P \to N)$, we have $\Gamma \vdash \lambda x : P.c$, which implies $\Gamma \vdash P$.
Also by inversion of $\Gamma;\Phi \vDash \lambda x : P.c\colon \mathbf{nf}\,(P \to N)$, we have $\Gamma;\Phi, x : P \vDash c\colon N$, applying induction hypothesis to which gives us:
(1) $\Gamma \vdash N$, thus $\Gamma \vdash P \to N$, and by corollary 11, $\Gamma \vdash \mathbf{nf}\,(P \to N)$;
(2) $\Gamma;\Phi, x : P \vdash c\colon N$, which by Rule (λ^{INF}) implies $\Gamma;\Phi \vdash \lambda x : P.c\colon P \to N$, and by Rule $(\simeq^{\text{INF}}_{+})$, $\Gamma;\Phi \vdash \lambda x : P.c\colon \mathbf{nf}\,(P \to N)$.

Case 7. Rule (Λ^{INF}) We are proving that if $\Gamma;\Phi \vDash \Lambda\alpha^{+}.c\colon \mathbf{nf}\,(\forall\alpha^{+}.N)$ then $\Gamma;\Phi \vdash \Lambda\alpha^{+}.c\colon \mathbf{nf}\,(\forall\alpha^{+}.N)$ and $\Gamma \vdash \mathbf{nf}\,(\forall\alpha^{+}.N)$.
By inversion of $\Gamma, \alpha^{+};\Phi \vDash c\colon N$, we have $\Gamma \vdash \Lambda\alpha^{+}.c$, which implies $\Gamma, \alpha^{+} \vdash c$.
Also by inversion of $\Gamma, \alpha^{+};\Phi \vDash c\colon N$, we have $\Gamma, \alpha^{+};\Phi \vDash c\colon N$. Obtaining the induction hypothesis to $\Gamma, \alpha^{+};\Phi \vDash c\colon N$, we have:
(1) $\Gamma, \alpha^{+} \vdash N$, thus $\Gamma \vdash \forall\alpha^{+}.N$, and by corollary 11, $\Gamma \vdash \mathbf{nf}\,(\forall\alpha^{+}.N)$;
(2) $\Gamma, \alpha^{+};\Phi \vdash c\colon N$, which by Rule (Λ^{INF}) implies $\Gamma;\Phi \vdash \Lambda\alpha^{+}.c\colon \forall\alpha^{+}.N$, and by Rule $(\simeq^{\text{INF}}_{+})$, $\Gamma;\Phi \vdash \Lambda\alpha^{+}.c\colon \mathbf{nf}\,(\forall\alpha^{+}.N)$.

Case 8. Rule $(\text{LET}^{\text{INF}})$ We are proving that if $\Gamma;\Phi \vDash \mathbf{let}\, x = v; c\colon N$ then $\Gamma;\Phi \vdash \mathbf{let}\, x = v; c\colon N$ and $\Gamma \vdash N$.
By inversion of $\Gamma;\Phi \vDash \mathbf{let}\, x = v; c\colon N$, we have:
(1) $\Gamma \vdash \mathbf{let}\, x = v; c$, which gives us $\Gamma \vdash v$ and $\Gamma \vdash c$.
(2) $\Gamma;\Phi \vDash v\colon P$, which by the induction hypothesis implies $\Gamma \vdash P$ (and thus, $\Gamma \vdash \Phi, x : P$) and $\Gamma;\Phi \vdash v\colon P$;
(3) $\Gamma;\Phi, x : P \vDash c\colon N$, which by the induction hypothesis implies $\Gamma \vdash N$ and $\Gamma;\Phi, x : P \vdash c\colon N$.
This way, $\Gamma;\Phi \vdash \mathbf{let}\, x = v; c\colon N$ holds by Rule $(\text{LET}^{\text{INF}})$.

Case 9. Rule $(\text{LET}^{\text{INF}}_{:@})$ We are proving that if $\Gamma;\Phi \vDash \mathbf{let}\, x : P = v(\vec{v}); c'\colon N$ then $\Gamma;\Phi \vdash \mathbf{let}\, x : P = v(\vec{v}); c'\colon N$ and $\Gamma \vdash N$.
By inversion, we have:
(1) $\Gamma \vdash P$, hence, $\Gamma \vdash \Phi, x : P$
(2) $\Gamma;\Phi \vDash v\colon \downarrow M$
(3) $\Gamma;\Phi;\cdot \vDash M \bullet \vec{v} \Rrightarrow \uparrow Q \dashv \Theta; SC_1$
(4) $\Gamma;\ \Theta \vDash \uparrow Q \leqslant \uparrow P \dashv SC_2$
(5) $\Theta \vdash SC_1 \& SC_2 = SC$
(6) $\Gamma;\Phi, x : P \vDash c'\colon N$
By the induction hypothesis applied to $\Gamma;\Phi \vDash v\colon \downarrow M$, we have $\Gamma;\Phi \vdash v\colon \downarrow M$ and $\Gamma \vdash \downarrow M$ (and hence, $\Gamma;\mathbf{dom}\,(\Theta) \vdash M$).
By the induction hypothesis applied to $\Gamma;\Phi, x : P \vDash c'\colon N$, we have $\Gamma;\Phi, x : P \vdash c'\colon N$ and $\Gamma \vdash N$.
By the induction hypothesis applied to $\Gamma;\Phi;\cdot \vDash M \bullet \vec{v} \Rrightarrow \uparrow Q \dashv \Theta; SC_1$, we have:
(1) $\Gamma \vdash^{\supseteq} \Theta$,
(2) $\Gamma;\mathbf{dom}\,(\Theta) \vdash \uparrow Q$,
(3) $\Theta'|_{\mathbf{uv}\,M \cup \mathbf{uv}\,Q} \vdash SC_1$, and thus, $\mathbf{dom}\,(SC_1) \subseteq \mathbf{uv}\,M \cup \mathbf{uv}\,Q$.
(4) for any $\Theta' \vdash \widehat{\sigma} : SC_1$, we have $\Gamma;\Phi \vdash [\widehat{\sigma}]M \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]\uparrow Q$.
By soundness of negative subtyping (lemma 83) applied to $\Gamma;\ \Theta \vDash \uparrow Q \leqslant \uparrow P \dashv SC_2$, we have $\Theta \vdash SC_2 : \mathbf{uv}\,(\uparrow Q)$, and thus, $\mathbf{uv}\,(\uparrow Q) = \mathbf{dom}\,(SC_2)$.
By soundness of constraint merge (lemma 80), $\mathbf{dom}\,(SC) = \mathbf{dom}\,(SC_1) \cup \mathbf{dom}\,(SC_2) \subseteq \mathbf{uv}\,M \cup \mathbf{uv}\,Q$ Then by lemma 75, let us take $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(M) \cup \mathbf{uv}\,(Q)$ and

$\Theta \vdash \widehat{\sigma} : SC$. By the soundness of constraint merge, $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash \widehat{\sigma} : SC_2$, and by weakening, $\Theta' \vdash \widehat{\sigma} : SC_1$ and $\Theta' \vdash \widehat{\sigma} : SC_2$.
Then as noted above (4), $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]\uparrow Q$ And again, by soundness of negative subtyping (lemma 83) applied to $\Gamma;\ \Theta \vDash \uparrow Q \leqslant \uparrow P \dashv SC_2$, we have $\Gamma \vdash [\widehat{\sigma}]\uparrow Q \leqslant \uparrow P$.
To infer $\Gamma; \Phi \vdash \mathbf{let}\, x : P = v(\vec{v}); c' : N$, we apply the corresponding declarative rule Rule ($\textsc{let}^{\textsc{inf}}_{:@}$), where Q is $[\widehat{\sigma}]\, Q$. Notice that all the premises were already shown to hold above:
(1) $\Gamma \vdash P$ and $\Gamma; \Phi \vdash v : \downarrow M$ from the assumption,
(2) $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow \uparrow[\widehat{\sigma}]\, Q$ holds since $[\widehat{\sigma}]\uparrow Q = \uparrow[\widehat{\sigma}]\, Q$,
(3) $\Gamma \vdash \uparrow[\widehat{\sigma}]\, Q \leqslant \uparrow P$ by soundness of negative subtyping,
(4) $\Gamma; \Phi, x : P \vdash c' : N$ from the the induction hypothesis.

Case 10. Rule ($\textsc{let}^{\textsc{inf}}_{@}$) We are proving that if $\Gamma; \Phi \vDash \mathbf{let}\, x = v(\vec{v}); c' : N$ then $\Gamma; \Phi \vdash \mathbf{let}\, x = v(\vec{v}); c' : N$ and $\Gamma \vdash N$.
By the inversion, we have:
(1) $\Gamma; \Phi \vDash v : \downarrow M$
(2) $\Gamma; \Phi; \cdot \vDash M \bullet \vec{v} \Rrightarrow \uparrow Q \dashv \Theta; SC$
(3) $\mathbf{uv}\, Q \subseteq \mathbf{dom}\,(SC)$
(4) $SC|_{\mathbf{uv}\,(Q)}$ **singular with** $\widehat{\sigma}_3$
(5) $\Gamma; \Phi, x : [\widehat{\sigma}_3]\, Q \vDash c' : N$
By the induction hypothesis applied to $\Gamma; \Phi \vDash v : \downarrow M$, we have $\Gamma; \Phi \vdash v : \downarrow M$ and $\Gamma \vdash \downarrow M$ (and thus, $\Gamma; \mathbf{dom}\,(\Theta) \vdash M$).
By the induction hypothesis applied to $\Gamma; \Phi, x : [\widehat{\sigma}_3]\, Q \vDash c' : N$, we have $\Gamma \vdash N$ and $\Gamma; \Phi, x : [\widehat{\sigma}_3]\, Q \vdash c' : N$.
By the induction hypothesis applied to $\Gamma; \Phi; \cdot \vDash M \bullet \vec{v} \Rrightarrow \uparrow Q \dashv \Theta; SC$, we have:
(1) $\Gamma \vdash^{\supseteq} \Theta$
(2) $\Gamma; \mathbf{dom}\,(\Theta) \vdash \uparrow Q$
(3) $\Theta|_{\mathbf{uv}\, M \cup \mathbf{uv}\, Q} \vdash SC$ (and thus, $\mathbf{dom}\,(SC) \subseteq \mathbf{uv}\, M \cup \mathbf{uv}\, Q$)
(4) for any $\Theta \vdash \widehat{\sigma} : SC$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}]\, M \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]\uparrow Q$, which, since M is ground means $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow \uparrow[\widehat{\sigma}]\, Q$.
To infer $\Gamma; \Phi \vdash \mathbf{let}\, x = v(\vec{v}); c' : N$, we apply the corresponding declarative rule Rule ($\textsc{let}^{\textsc{inf}}_{@}$). Let us show that the premises hold:
- $\Gamma; \Phi \vdash v : \downarrow M$ holds by the induction hypothesis;
- $\Gamma; \Phi, x : [\widehat{\sigma}_3]\, Q \vdash c' : N$ also holds by the induction hypothesis, as noted above;
- Let us take an arbitrary substitution $\widehat{\sigma}\ \Theta \vdash \widehat{\sigma} : \mathbf{uv}\, M \cup \mathbf{uv}\, Q$ satisfying $\Theta \vdash \widehat{\sigma} : SC$ (it exists by lemma 75). Then $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow \uparrow[\widehat{\sigma}]\, Q$ holds, as noted above;
- To show the uniqueness of $\uparrow[\widehat{\sigma}]\, Q$, we assume that for some other type K holds $\Gamma; \Phi \vdash M \bullet \vec{v} \Rrightarrow K$, that is $\Gamma; \Phi \vdash [\cdot] M \bullet \vec{v} \Rrightarrow K$. Then by the completeness of typing (lemma 93), there exist N', Θ', and SC' such that
  (1) $\Gamma; \Phi; \cdot \vDash M \bullet \vec{v} \Rrightarrow N' \dashv \Theta'; SC'$ and
  (2) there exists a substitution $\Theta' \vdash \widehat{\sigma}' : SC'$ such that $\Gamma \vdash [\widehat{\sigma}']\, N' \simeq^{\leqslant} K$.
  By determinicity of the typing algorithm (lemma 91), $\Gamma; \Phi; \cdot \vDash M \bullet \vec{v} \Rrightarrow N' \dashv \Theta'; SC'$, means that SC' is SC, Θ' is Θ, and N' is $\uparrow Q$. This way, $\Gamma \vdash [\widehat{\sigma}']\uparrow Q \simeq^{\leqslant} K$ for substitution $\Theta \vdash \widehat{\sigma}' : SC$.
  It is left to show that $\Gamma \vdash [\widehat{\sigma}']\uparrow Q \simeq^{\leqslant} [\widehat{\sigma}]\uparrow Q$, then by transitivity of equivalence, we will have $\Gamma \vdash [\widehat{\sigma}]\uparrow Q \simeq^{\leqslant} K$. Since $\Theta \vdash \widehat{\sigma} : SC|_{\mathbf{uv}\,(Q)}$ and $\Theta \vdash \widehat{\sigma}' : SC|_{\mathbf{uv}\,(Q)}$, and $SC|_{\mathbf{uv}\,(Q)}$ **singular with** $\widehat{\sigma}_3$, we have $\Theta \vdash \widehat{\sigma} \simeq^{\leqslant} \widehat{\sigma}_3 : \mathbf{dom}\,(SC) \cap \mathbf{uv}\,(Q)$ and $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma}_3 : \mathbf{dom}\,(SC) \cap \mathbf{uv}\,(Q)$. Then since $\mathbf{uv}\,(Q) \subseteq \mathbf{dom}\,(SC)$, we have $\mathbf{dom}\,(SC) \cap$

$\mathbf{uv}\,(Q) = \mathbf{uv}\,(Q)$. This way, by transitivity and symmetry of the equivalence, $\Theta \vdash \widehat{\sigma} \simeq^{\leqslant} \widehat{\sigma}' : \mathbf{uv}\,(Q)$, which implies $\Gamma \vdash [\widehat{\sigma}']\uparrow Q \simeq^{\leqslant} [\widehat{\sigma}]\uparrow Q$.

Case 11. Rule ($\textsc{let}^{\textsc{inf}}_{\exists}$) We are proving that if $\Gamma;\Phi \vDash \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c' : N$ then $\Gamma;\Phi \vdash \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c' : N$ and $\Gamma \vdash N$. By the inversion, we have:

(1) $\Gamma;\Phi \vDash v : \exists\overrightarrow{\alpha^-}.P$
(2) $\Gamma, \overrightarrow{\alpha^-};\Phi, x : P \vDash c' : N$
(3) $\Gamma \vdash N$

By the induction hypothesis applied to $\Gamma;\Phi \vDash v : \exists\overrightarrow{\alpha^-}.P$, we have $\Gamma;\Phi \vdash v : \exists\overrightarrow{\alpha^-}.P$ and $\exists\overrightarrow{\alpha^-}.P$ is normalized. By the induction hypothesis applied to $\Gamma, \overrightarrow{\alpha^-};\Phi, x : P \vDash c' : N$, we have $\Gamma, \overrightarrow{\alpha^-};\Phi, x : P \vdash c' : N$.

To show $\Gamma;\Phi \vdash \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c' : N$, we apply the corresponding declarative rule Rule ($\textsc{let}^{\textsc{inf}}_{\exists}$). Let us show that the premises hold:

(1) $\Gamma;\Phi \vdash v : \exists\overrightarrow{\alpha^-}.P$ holds by the induction hypothesis, as noted above,
(2) $\mathbf{nf}\,(\exists\overrightarrow{\alpha^-}.P) = \exists\overrightarrow{\alpha^-}.P$ holds since $\exists\overrightarrow{\alpha^-}.P$ is normalized,
(3) $\Gamma, \overrightarrow{\alpha^-};\Phi, x : P \vdash c' : N$ also holds by the induction hypothesis,
(4) $\Gamma \vdash N$ holds by the inversion, as noted above.

Case 12. Rule ($\varnothing^{\textsc{inf}}_{\bullet\Rightarrow}$) Then by assumption:

- $\Gamma \vdash^{\supseteq} \Theta$,
- $\Gamma;\mathbf{dom}\,(\Theta) \vdash N$ is free from negative algorithmic variables,
- $\Gamma;\Phi;\Theta \vDash N \bullet \cdot \Rightarrow\!\!> \mathbf{nf}\,(N) \dashv \Theta;\cdot$.

Let us show the required properties:

(1) $\Gamma \vdash^{\supseteq} \Theta$ holds by assumption,
(2) $\Theta \subseteq \Theta$ holds trivially,
(3) $\mathbf{nf}\,(N)$ is evidently normalized, $\Gamma;\mathbf{dom}\,(\Theta) \vdash N$ implies $\Gamma;\mathbf{dom}\,(\Theta) \vdash \mathbf{nf}\,(N)$ by corollary 23, and lemma 43 means that $\mathbf{nf}\,(N)$ is inherently free from negative algorithmic variables,
(4) $\mathbf{dom}\,(\Theta) \cap \mathbf{uv}\,(\mathbf{nf}\,(N)) \subseteq \mathbf{uv}\,N$ holds since $\mathbf{uv}\,(\mathbf{nf}\,(N)) = \mathbf{uv}\,(N)$,
(5) $\Theta|_{\mathbf{uv}\,N \cup \mathbf{uv}\,\mathbf{nf}\,(N)} \vdash \cdot$ holds trivially,
(6) suppose that $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,N \cup \mathbf{uv}\,\mathbf{nf}\,(N)$. To show $\Gamma;\Phi \vdash [\widehat{\sigma}]N \bullet \cdot \Rightarrow\!\!> [\widehat{\sigma}]\mathbf{nf}\,(N)$, we apply the corresponding declarative rule Rule ($\varnothing^{\textsc{inf}}_{\bullet\Rightarrow}$). To show $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} [\widehat{\sigma}]\mathbf{nf}\,(N)$, we apply the following sequence: $N \simeq^{D} \mathbf{nf}\,(N)$ by lemma 44, then $[\widehat{\sigma}]N \simeq^{D} [\widehat{\sigma}]\mathbf{nf}\,(N)$ by lemma 36, then $\Gamma \vdash [\widehat{\sigma}]N \simeq^{\leqslant} [\widehat{\sigma}]\mathbf{nf}\,(N)$ by lemma 29.

Case 13. Rule ($\rightarrow^{\textsc{inf}}_{\bullet\Rightarrow}$) By assumption:

(1) $\Gamma \vdash^{\supseteq} \Theta$,
(2) $\Gamma;\mathbf{dom}\,(\Theta) \vdash Q \to N$ is free from negative algorithmic variables, and hence, so are Q and N,
(3) $\Gamma;\Phi;\Theta \vDash Q \to N \bullet v, \vec{v} \Rightarrow\!\!> M \dashv \Theta'; SC$, and by inversion:
   (a) $\Gamma;\Phi \vDash v : P$, and by the induction hypothesis applied to this judgment, we have $\Gamma;\Phi \vdash v : P$, and $\Gamma \vdash P$;
   (b) $\Gamma;\Theta \vDash Q \geqslant P \dashv SC_1$, and by the soundness of subtyping: $\Theta \vdash SC_1 : \mathbf{uv}\,Q$ (and thus, $\mathbf{dom}\,(SC_1) = \mathbf{uv}\,Q$), and for any $\Theta \vdash \widehat{\sigma} : SC_1$, we have $\Gamma \vdash [\widehat{\sigma}]Q \geqslant P$;
   (c) $\Gamma;\Phi;\Theta \vDash N \bullet \vec{v} \Rightarrow\!\!> M \dashv \Theta'; SC_2$, and by the induction hypothesis applied to this judgment,
      (i) $\Gamma \vdash^{\supseteq} \Theta'$,
      (ii) $\Theta \subseteq \Theta'$,

(iii) $\Gamma; \mathbf{dom}\,(\Theta') \vdash M$ is normalized and free from negative algorithmic variables,
(iv) $\mathbf{dom}\,(\Theta) \cap \mathbf{uv}\,(M) \subseteq \mathbf{uv}\,N$,
(v) $\Theta'|_{\mathbf{uv}\,(M) \cup \mathbf{uv}\,(N)} \vdash SC_2$, and thus, $\mathbf{dom}\,(SC_2) \subseteq \mathbf{uv}\,(M) \cup \mathbf{uv}\,(N)$,
(vi) for any $\widehat{\sigma}$ such that $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(M) \cup \mathbf{uv}\,(N)$ and $\Theta' \vdash \widehat{\sigma} : SC_2$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]M$;

(d) $\Theta \vdash SC_1 \& SC_2 = SC$, which by lemma 80 implies $\mathbf{dom}\,(SC) = \mathbf{dom}\,(SC_1) \cup \mathbf{dom}\,(SC_2) \subseteq \mathbf{uv}\,Q \cup \mathbf{uv}\,M \cup \mathbf{uv}\,N$.

Let us show the required properties:

(1) $\Gamma \vdash^{\supseteq} \Theta'$ is shown above,
(2) $\Theta \subseteq \Theta'$ is shown above,
(3) $\Gamma; \mathbf{dom}\,(\Theta') \vdash M$ is normalized and free from negative algorithmic variables, as shown above,
(4) $\mathbf{dom}\,(\Theta) \cap \mathbf{uv}\,(M) \subseteq \mathbf{uv}\,N \subseteq \mathbf{uv}\,(Q \to N)$ (the first inclusion is shown above, the second one is by definition),
(5) To show $\Theta'|_{\mathbf{uv}\,(Q) \cup \mathbf{uv}\,(N) \cup \mathbf{uv}\,(M)} \vdash SC$, first let us notice that $\mathbf{uv}\,(Q) \cup \mathbf{uv}\,(N) \cup \mathbf{uv}\,(M) \subseteq \mathbf{dom}\,(SC)$, as mentioned above. Then we demonstrate $\Theta' \vdash SC$: $\Theta \vdash SC_1$ and $\Theta \subseteq \Theta'$ imply $\Theta' \vdash SC_1$, by the soundness of constraint merge (lemma 80) applied to $\Theta' \vdash SC_1 \& SC_2 = SC$:
   (a) $\Theta' \vdash SC$,
   (b) for any $\Theta' \vdash \widehat{\sigma} : SC$, $\Theta' \vdash \widehat{\sigma} : SC_i$ holds;
(6) Suppose that $\Theta' \vdash \widehat{\sigma} : \mathbf{uv}\,(Q) \cup \mathbf{uv}\,(N) \cup \mathbf{uv}\,(M)$ and $\Theta' \vdash \widehat{\sigma} : SC$. To show $\Gamma; \Phi \vdash [\widehat{\sigma}](Q \to N) \bullet v, \vec{v} \Rrightarrow [\widehat{\sigma}]M$, that is $\Gamma; \Phi \vdash [\widehat{\sigma}]Q \to [\widehat{\sigma}]N \bullet v, \vec{v} \Rrightarrow [\widehat{\sigma}]M$, we apply the corresponding declarative rule Rule $(\to^{\mathrm{INF}}_{\bullet\Rrightarrow})$. Let us show the required premises:
   (a) $\Gamma; \Phi \vdash v : P$ holds as shown above,
   (b) $\Gamma \vdash [\widehat{\sigma}]Q \geqslant P$ holds since $\Gamma \vdash [\widehat{\sigma}|_{\mathbf{uv}\,(Q)}]Q \geqslant P$ by the soundness of subtyping as noted above: since $\Theta' \vdash \widehat{\sigma} : SC$ implies $\Theta' \vdash \widehat{\sigma}|_{\mathbf{uv}\,(Q)} : SC_1$, which we strengthen to $\Theta \vdash \widehat{\sigma}|_{\mathbf{uv}\,(Q)} : SC_1$,
   (c) $\Gamma; \Phi \vdash [\widehat{\sigma}]N \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]M$ holds by the induction hypothesis as shown above, since $\Theta' \vdash \widehat{\sigma} : SC$ implies $\Theta' \vdash \widehat{\sigma} : SC_2$, and then $\Theta' \vdash \widehat{\sigma}|_{\mathbf{uv}\,(N) \cup \mathbf{uv}\,(M)} : SC_2$ and $\Theta \vdash \widehat{\sigma}|_{\mathbf{uv}\,(N) \cup \mathbf{uv}\,(M)} : \mathbf{uv}\,(N) \cup \mathbf{uv}\,(M)$.

Case 14. Rule $(\forall^{\mathrm{INF}}_{\bullet\Rrightarrow})$

By assumption:

(1) $\Gamma \vdash^{\supseteq} \Theta$,
(2) $\Gamma; \mathbf{dom}\,(\Theta) \vdash \forall \overrightarrow{\alpha^+}.N$ is free from negative algorithmic variables,
(3) $\Gamma; \Phi; \Theta \vDash \forall \overrightarrow{\alpha^+}.N \bullet \vec{v} \Rrightarrow M \dashv \Theta'; SC$, which by inversion means $\vec{v} \neq \cdot$, $\overrightarrow{\alpha^+} \neq \cdot$, and $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \vec{v} \Rrightarrow M \dashv \Theta'; SC$. It is easy to see that the induction hypothesis is applicable to the latter judgment:
   - $\Gamma \vdash^{\supseteq} \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\}$ holds by $\Gamma \vdash^{\supseteq} \Theta$,
   - $\Gamma; \mathbf{dom}\,(\Theta), \overrightarrow{\widehat{\alpha}^+} \vdash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N$ holds since $\Gamma; \mathbf{dom}\,(\Theta) \vdash \forall \overrightarrow{\alpha^+}.N\ [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N$ is normalized and free from negative algorithmic variables since so is N;

   This way, by the inductive hypothesis applied to $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \vec{v} \Rrightarrow > M \dashv \Theta'; SC$, we have:
   (a) $\Gamma \vdash^{\supseteq} \Theta'$,
   (b) $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \subseteq \Theta'$,

(c) $\Gamma; \mathbf{dom}(\Theta') \vdash M$ is normalized and free from negative algorithmic variables,
(d) $\mathbf{dom}(\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\}) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}([\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N)$,
(e) $\Theta'|_{\Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)} \vdash SC$, where Ξ denotes $\mathbf{uv}([\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N) \cap \overrightarrow{\widehat{\alpha}^+}$, that is the algorithmization of the $\forall$-variables that are actually used in N.
(f) for any $\widehat{\sigma}$ such that $\Theta' \vdash \widehat{\sigma} : \Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$ and $\Theta' \vdash \widehat{\sigma} : SC$, we have $\Gamma; \Phi \vdash [\widehat{\sigma}][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]M$.

Let us show the required properties:

(1) $\Gamma \vdash^{\supseteq} \Theta'$ is shown above;
(2) $\Theta \subseteq \Theta'$ since $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \subseteq \Theta'$;
(3) $\Gamma; \mathbf{dom}(\Theta') \vdash M$ is normalized and free from negative algorithmic variables, as shown above;
(4) $\mathbf{dom}(\Theta) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}(N)$ since $\mathbf{dom}(\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\}) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}([\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N)$ implies $(\mathbf{dom}(\Theta) \cup \overrightarrow{\widehat{\alpha}^+}) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}(N) \cup \overrightarrow{\widehat{\alpha}^+}$, thus, $\mathbf{dom}(\Theta) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}(N) \cup \overrightarrow{\widehat{\alpha}^+}$, and since $\mathbf{dom}(\Theta)$ is disjoint with $\overrightarrow{\widehat{\alpha}^+}$, $\mathbf{dom}(\Theta) \cap \mathbf{uv}(M) \subseteq \mathbf{uv}(N)$;
(5) $\Theta'|_{\mathbf{uv}(N) \cup \mathbf{uv}(M)} \vdash SC|_{\mathbf{uv}(N) \cup \mathbf{uv}(M)}$ follows from $\Theta'|_{\Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)} \vdash SC$ if we restrict both sides to $\mathbf{uv}(N) \cup \mathbf{uv}(M)$.
(6) Let us assume $\Theta' \vdash \widehat{\sigma} : \mathbf{uv}(N) \cup \mathbf{uv}(M)$ and $\Theta' \vdash \widehat{\sigma} : SC|_{\mathbf{uv}(N) \cup \mathbf{uv}(M)}$. Then to show $\Gamma; \Phi \vdash [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]M$, that is $\Gamma; \Phi \vdash \forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]M$, we apply the corresponding declarative rule Rule ($\forall^{\mathrm{INF}}_{\bullet\Rrightarrow}$). To do so, we need to provide a substitution for $\overrightarrow{\alpha^+}$, i.e. $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^+}$ such that $\Gamma; \Phi \vdash [\sigma_0][\widehat{\sigma}]N \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]M$.
By lemma 75, we construct $\widehat{\sigma}_0$ such that $\Theta' \vdash \widehat{\sigma}_0 : \overrightarrow{\widehat{\alpha}^+}$ and $\Theta' \vdash \widehat{\sigma}_0 : SC|_{\overrightarrow{\widehat{\alpha}^+}}$.
Then σ_0 is defined as $\widehat{\sigma}_0 \circ \widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^+}} \circ \overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}$.
Let us show that the premises of Rule ($\forall^{\mathrm{INF}}_{\bullet\Rrightarrow}$) hold:

- To show $\Gamma \vdash \sigma_0 : \overrightarrow{\alpha^+}$, let us take $\alpha_i^+ \in \overrightarrow{\alpha^+}$. If $\widehat{\alpha}_i^+ \in \mathbf{uv}(M)$ then $[\sigma_0]\alpha_i^+ = [\widehat{\sigma}]\widehat{\alpha}_i^+$, and $\Theta' \vdash \widehat{\sigma} : \mathbf{uv}(N) \cup \mathbf{uv}(M)$ implies $\Theta'(\widehat{\alpha}^+) \vdash [\widehat{\sigma}]\widehat{\alpha}^+$. Analogously, if $\widehat{\alpha}_i^+ \in \overrightarrow{\widehat{\alpha}^+} \backslash \mathbf{uv}(M)$ then $[\sigma_0]\alpha_i^+ = [\widehat{\sigma}_0]\widehat{\alpha}_i^+$, and $\Theta' \vdash \widehat{\sigma}_0 : \overrightarrow{\widehat{\alpha}^+}$ implies $\Theta'(\widehat{\alpha}_i^+) \vdash [\widehat{\sigma}_0]\widehat{\alpha}_i^+$. In any case, $\Theta'(\widehat{\alpha}_i^+) \vdash [\sigma]\alpha_i^+$ can be weakened to $\Gamma \vdash [\sigma_0]\alpha_i^+$, since $\Gamma \vdash^{\supseteq} \Theta'$.
- Let us show $\Gamma; \Phi \vdash [\sigma_0][\widehat{\sigma}]N \bullet \vec{v} \Rrightarrow [\widehat{\sigma}]M$. It suffices to construct $\widehat{\sigma}_1$ such that
  (a) $\Theta' \vdash \widehat{\sigma}_1 : \Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$,
  (b) $\Theta' \vdash \widehat{\sigma}_1 : SC$,
  (c) $[\sigma_0][\widehat{\sigma}]N = [\widehat{\sigma}_1][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N$, and
  (d) $[\widehat{\sigma}]M = [\widehat{\sigma}_1]M$,
  because then we can apply the induction hypothesis (3f) to $\widehat{\sigma}_1$, rewrite the conclusion by $[\widehat{\sigma}_1][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N = [\sigma_0][\widehat{\sigma}]N$ and $[\widehat{\sigma}_1]M = [\widehat{\sigma}]M$, and infer the required judgement.
  Let us take $\widehat{\sigma}_1 = (\widehat{\sigma}_0 \circ \widehat{\sigma})|_{\Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)}$, then
  (a) $\Theta' \vdash \widehat{\sigma}_1 : \Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$, since $\Theta' \vdash \widehat{\sigma}_0 : \overrightarrow{\widehat{\alpha}^+}$ and $\Theta' \vdash \widehat{\sigma} : \mathbf{uv}(N) \cup \mathbf{uv}(M)$, we have $\Theta' \vdash \widehat{\sigma}_0 \circ \widehat{\sigma} : \overrightarrow{\widehat{\alpha}^+} \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$, which we restrict to $\Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$.
  (b) $\Theta' \vdash \widehat{\sigma}_1 : SC$, Let us take any constraint $e \in SC$ restricting variable $\widehat{\beta}^{\pm}$. $\Theta'|_{\Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)} \vdash SC$ implies that $\widehat{\beta}^{\pm} \in \Xi \cup \mathbf{uv}(N) \cup \mathbf{uv}(M)$.

If $\widehat{\beta}^{\pm} \in \mathbf{uv}\,(N) \cup \mathbf{uv}\,(M)$ then $[\widehat{\sigma}_1]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$. Additionally, $e \in SC|_{\mathbf{uv}\,(N)\cup\mathbf{uv}\,(M)}$, which, since $\Theta' \vdash \widehat{\sigma} : SC|_{\mathbf{uv}\,(N)\cup\mathbf{uv}\,(M)}$, means $\Theta'(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}]\widehat{\beta}^{\pm} : e$.
If $\widehat{\beta}^{\pm} \in \Xi\backslash(\mathbf{uv}\,(N) \cup \mathbf{uv}\,(M))$ then $[\widehat{\sigma}_1]\widehat{\beta}^{\pm} = [\widehat{\sigma}_0]\widehat{\beta}^{\pm}$. Additionally, $e \in SC|_{\overrightarrow{\widehat{\alpha}^{+}}}$, which, since $\Theta' \vdash \widehat{\sigma}_0 : SC|_{\overrightarrow{\widehat{\alpha}^{+}}}$, means $\Theta'(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}_0]\widehat{\beta}^{\pm} : e$.

(c) Let us prove $[\sigma_0][\widehat{\sigma}]N = [\widehat{\sigma}_1][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N$ by the following reasoning

$$
\begin{aligned}
[\sigma_0][\widehat{\sigma}]N &= [\widehat{\sigma}_0][\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^{+}}}][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}][\widehat{\sigma}]N && \text{by definition of } \sigma_0\\
&= [\widehat{\sigma}_0][\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^{+}}}][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}][\widehat{\sigma}|_{\mathbf{uv}\,(N)}]N && \text{by lemma 53}\\
&= [\widehat{\sigma}_0][\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^{+}}}][\widehat{\sigma}|_{\mathbf{uv}\,(N)}][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N && \mathbf{uv}\,(N) \cap \overrightarrow{\widehat{\alpha}^{+}} = \varnothing \text{ and } \overrightarrow{\alpha^{+}} \cap \Gamma = \varnothing\\
&= [\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^{+}}}][\widehat{\sigma}|_{\mathbf{uv}\,(N)}][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N && [\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^{+}}}][\widehat{\sigma}|_{\mathbf{uv}\,(N)}][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N \text{ is ground}\\
&= [\widehat{\sigma}|_{\overrightarrow{\widehat{\alpha}^{+}}\cup\mathbf{uv}\,(N)}][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N\\
&= [\widehat{\sigma}|_{\Xi\cup\mathbf{uv}\,(N)}][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N && \text{by lemma 53: } \mathbf{uv}\,([\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N) = \Xi \cup \mathbf{uv}\,(N)\\
&= [\widehat{\sigma}|_{\Xi\cup\mathbf{uv}\,(N)\cup\mathbf{uv}\,(M)}][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N && \text{also by lemma 53}\\
&= [(\widehat{\sigma}_0 \circ \widehat{\sigma})|_{\Xi\cup\mathbf{uv}\,(N)\cup\mathbf{uv}\,(M)}][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N && [\widehat{\sigma}|_{\Xi\cup\mathbf{uv}\,(N)\cup\mathbf{uv}\,(M)}][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N \text{ is ground}\\
&= [\widehat{\sigma}_1][\overrightarrow{\widehat{\alpha}^{+}}/\overrightarrow{\alpha^{+}}]N && \text{by definition of } \widehat{\sigma}_1
\end{aligned}
$$

(d) $[\widehat{\sigma}]M = [\widehat{\sigma}_1]M$ By definition of $\widehat{\sigma}_1$, $[\widehat{\sigma}_1]M$ is equal to $[(\widehat{\sigma}_0 \circ \widehat{\sigma})|_{\Xi\cup\mathbf{uv}\,(N)\cup\mathbf{uv}\,(M)}]M$, which by lemma 53 is equal to $[\widehat{\sigma}_0 \circ \widehat{\sigma}]M$, that is $[\widehat{\sigma}_0][\widehat{\sigma}]M$, and since $[\widehat{\sigma}]M$ is ground, $[\widehat{\sigma}_0][\widehat{\sigma}]M = [\widehat{\sigma}]M$.

- $\overrightarrow{\alpha^{+}} \neq \cdot$ and $\vec{v} \neq \cdot$ hold by assumption.

□

Lemma 93 (Completeness of Typing). *Suppose that* $\Gamma \vdash \Phi$. *For an inference tree* T_1,

+ *If* T_1 *infers* $\Gamma;\Phi \vdash v\colon P$ *then* $\Gamma;\Phi \vDash v\colon \mathbf{nf}\,(P)$
− *If* T_1 *infers* $\Gamma;\Phi \vdash c\colon N$ *then* $\Gamma;\Phi \vDash c\colon \mathbf{nf}\,(N)$
• *If* T_1 *infers* $\Gamma;\Phi \vdash [\widehat{\sigma}]N \bullet \vec{v} \gg\!\gg M$ *and*
  (1) $\Gamma \vdash^{\supseteq} \Theta$,
  (2) $\Gamma \vdash M$,
  (3) $\Gamma;\mathbf{dom}\,(\Theta) \vdash N$ *(free from negative algorithmic variables, that is* $\widehat{\alpha}^{-} \notin \mathbf{uv}\,N$*), and*
  (4) $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(N)$,
  then there exist M', Θ', *and* SC *such that*
  (1) $\Gamma;\Phi;\Theta \vDash N \bullet \vec{v} \gg\!\gg M' \dashv \Theta'; SC$ *and*
  (2) for any $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(N)$ *and* $\Gamma \vdash M$ *such that* $\Gamma;\Phi \vdash [\widehat{\sigma}]N \bullet \vec{v} \gg\!\gg M$, *there exists* $\widehat{\sigma}'$ *such that*
    (a) $\Theta' \vdash \widehat{\sigma}' : \mathbf{uv}\,N \cup \mathbf{uv}\,M$ *and* $\Theta' \vdash \widehat{\sigma}' : SC$,
    (b) $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}\,N$, *and*
    (c) $\Gamma \vdash [\widehat{\sigma}']M' \simeq^{\leqslant} M$.

Proof. We prove it by induction on $\mathrm{metric}(T_1)$, mutually with the soundness of typing (lemma 92). Let us consider the last rule applied to infer the derivation.

Case 1. Rule $(\{\}^{\text{INF}})$

Then we are proving that if $\Gamma;\Phi \vdash \{c\}\colon \downarrow N$ (inferred by Rule $(\{\}^{\text{INF}})$) then $\Gamma;\Phi \vDash \{c\}\colon \mathbf{nf}\,(\downarrow N)$. By inversion of $\Gamma;\Phi \vdash \{c\}\colon \downarrow N$, we have $\Gamma;\Phi \vdash c\colon N$, which we apply

the induction hypothesis to to obtain $\Gamma;\Phi \vDash c\colon \mathbf{nf}\,(N)$. Then by Rule $(\{\}^{\text{INF}})$, we have $\Gamma;\Phi \vDash \{c\}\colon \downarrow\mathbf{nf}\,(N)$. It is left to notice that $\downarrow\mathbf{nf}\,(N) = \mathbf{nf}\,(\downarrow N)$.

Case 2. Rule $(\text{RET}^{\text{INF}})$

The proof is symmetric to the previous case (case 1).

Case 3. Rule $(\text{ANN}_{+}^{\text{INF}})$

Then we are proving that if $\Gamma;\Phi \vdash (v : Q)\colon Q$ is inferred by Rule $(\text{ANN}_{+}^{\text{INF}})$ then $\Gamma;\Phi \vDash (v : Q)\colon \mathbf{nf}\,(Q)$. By inversion, we have:

(1) $\Gamma \vdash Q$;
(2) $\Gamma;\Phi \vdash v\colon P$, which by the induction hypothesis implies $\Gamma;\Phi \vDash v\colon \mathbf{nf}\,(P)$;
(3) $\Gamma \vdash Q \geqslant P$, and by transitivity, $\Gamma \vdash Q \geqslant \mathbf{nf}\,(P)$; Since Q is ground, we have $\Gamma;\cdot \vdash Q$ and $\Gamma \vdash [\cdot]Q \geqslant \mathbf{nf}\,(P)$. Then by the completeness of subtyping (lemma 78), we have $\Gamma;\cdot \vDash Q \geqslant \mathbf{nf}\,(P) \dashv SC$, where $\cdot \vdash SC$ (implying $SC = \cdot$). This way, $\Gamma;\cdot \vDash Q \geqslant \mathbf{nf}\,(P) \dashv \cdot$.

Then we can apply Rule $(\text{ANN}_{+}^{\text{INF}})$ to $\Gamma \vdash Q$, $\Gamma;\Phi \vDash v\colon \mathbf{nf}\,(P)$ and $\Gamma;\cdot \vDash Q \geqslant \mathbf{nf}\,(P) \dashv \cdot$ to infer $\Gamma;\Phi \vDash (v : Q)\colon \mathbf{nf}\,(Q)$.

Case 4. Rule $(\text{ANN}_{-}^{\text{INF}})$

The proof is symmetric to the previous case (case 3).

Case 5. Rule (λ^{INF})

Then we are proving that if $\Gamma;\Phi \vdash \lambda x : P.c\colon P \to N$ is inferred by Rule (λ^{INF}), then $\Gamma;\Phi \vDash \lambda x : P.c\colon \mathbf{nf}\,(P \to N)$.

By inversion of $\Gamma;\Phi \vdash \lambda x : P.c\colon P \to N$, we have $\Gamma \vdash P$ and $\Gamma;\Phi, x : P \vdash c\colon N$. Then by the induction hypothesis, $\Gamma;\Phi, x : P \vDash c\colon \mathbf{nf}\,(N)$. By Rule (λ^{INF}), we infer $\Gamma;\Phi \vDash \lambda x : P.c\colon \mathbf{nf}\,(P \to \mathbf{nf}\,(N))$. By idempotence of normalization (lemma 47), $\mathbf{nf}\,(P \to \mathbf{nf}\,(N)) = \mathbf{nf}\,(P \to N)$, which concludes the proof for this case.

Case 6. Rule (Λ^{INF})

Then we are proving that if $\Gamma;\Phi \vdash \Lambda\alpha^{+}.c\colon \forall\alpha^{+}.N$ is inferred by Rule (Λ^{INF}), then $\Gamma;\Phi \vDash \Lambda\alpha^{+}.c\colon \mathbf{nf}\,(\forall\alpha^{+}.N)$. Similar to the previous case, by inversion of $\Gamma;\Phi \vdash \Lambda\alpha^{+}.c\colon \forall\alpha^{+}.N$, we have $\Gamma, \alpha^{+};\Phi \vdash c\colon N$, and then by the induction hypothesis, $\Gamma, \alpha^{+};\Phi \vDash c\colon \mathbf{nf}\,(N)$. After that, application of Rule (Λ^{INF}), gives as $\Gamma;\Phi \vDash \Lambda\alpha^{+}.c\colon \mathbf{nf}\,(\forall\alpha^{+}.\mathbf{nf}\,(N))$.

It is left to show that $\mathbf{nf}\,(\forall\alpha^{+}.\mathbf{nf}\,(N)) = \mathbf{nf}\,(\forall\alpha^{+}.N)$. Assume $N = \forall\overrightarrow{\beta^{+}}.M$ (where M does not start with $\forall$).

- Then by definition, $\mathbf{nf}\,(\forall\alpha^{+}.N) = \mathbf{nf}\,(\forall\alpha^{+}, \overrightarrow{\beta^{+}}.M) = \forall\overrightarrow{\gamma^{+}}.\mathbf{nf}\,(M)$, where $\mathbf{ord}\,\alpha^{+}, \overrightarrow{\beta^{+}}\ \mathbf{in}\ \mathbf{nf}\,(M) = \overrightarrow{\gamma^{+}}$.
- On the other hand, $\mathbf{nf}\,(N) = \forall\overrightarrow{\gamma^{+\prime}}.\mathbf{nf}\,(M)$, where $\mathbf{ord}\,\overrightarrow{\beta^{+}}\ \mathbf{in}\ \mathbf{nf}\,(M) = \overrightarrow{\gamma^{+\prime}}$, and thus, $\mathbf{nf}\,(\forall\alpha^{+}.\mathbf{nf}\,(N)) = \mathbf{nf}\,(\forall\alpha^{+}, \overrightarrow{\gamma^{+\prime}}.\mathbf{nf}\,(M)) = \forall\overrightarrow{\gamma^{+\prime\prime}}.\mathbf{nf}\,(\mathbf{nf}\,(M)) = \forall\overrightarrow{\gamma^{+\prime\prime}}.\mathbf{nf}\,(M)$, where $\mathbf{ord}\,\alpha^{+}, \overrightarrow{\gamma^{+\prime}}\ \mathbf{in}\ \mathbf{nf}\,(\mathbf{nf}\,(M)) = \overrightarrow{\gamma^{+\prime\prime}}$.

It is left to show that $\overrightarrow{\gamma^{+}}'' = \overrightarrow{\gamma^{+}}$.

$$
\begin{aligned}
\overrightarrow{\gamma^{+}}'' &= \mathbf{ord}\,\alpha^{+}, \overrightarrow{\gamma^{+}}' \,\mathbf{in}\,\mathbf{nf}\,(\mathbf{nf}\,(M)) \\
&= \mathbf{ord}\,\alpha^{+}, \overrightarrow{\gamma^{+}}' \,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by idempotence (lemma 47)} \\
&= \mathbf{ord}\,\alpha^{+} \cup \overrightarrow{\beta^{+}} \cap \mathbf{fv}\,\mathbf{nf}\,(M) \,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by definition of } \overrightarrow{\gamma^{+}}' \text{ and lemma 37} \\
&= \mathbf{ord}\,(\alpha^{+} \cup \overrightarrow{\beta^{+}} \cap \mathbf{fv}\,\mathbf{nf}\,(M)) \cap \mathbf{fv}\,\mathbf{nf}\,(M) \,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by lemma 38} \\
&= \mathbf{ord}\,(\alpha^{+} \cup \overrightarrow{\beta^{+}}) \cap \mathbf{fv}\,\mathbf{nf}\,(M) \,\mathbf{in}\,\mathbf{nf}\,(M) && \text{by set properties} \\
&= \mathbf{ord}\,\alpha^{+}, \overrightarrow{\beta^{+}} \,\mathbf{in}\,\mathbf{nf}\,(M) \\
&= \overrightarrow{\gamma^{+}}
\end{aligned}
$$

Case 7. Rule ($\textsc{let}^{\textsc{inf}}_{\exists}$)

Then we are proving that if $\Gamma; \Phi \vdash \mathbf{let}^{\exists}(\overrightarrow{\alpha^{-}}, x) = v; c\colon N$ is inferred by Rule ($\textsc{let}^{\textsc{inf}}_{\exists}$), then $\Gamma; \Phi \vDash \mathbf{let}^{\exists}(\overrightarrow{\alpha^{-}}, x) = v; c\colon \mathbf{nf}\,(N)$.

By inversion of $\Gamma; \Phi \vdash \mathbf{let}^{\exists}(\overrightarrow{\alpha^{-}}, x) = v; c\colon N$, we have

(1) $\mathbf{nf}\,(\exists\overrightarrow{\alpha^{-}}.P) = \exists\overrightarrow{\alpha^{-}}.P$,
(2) $\Gamma; \Phi \vdash v\colon \exists\overrightarrow{\alpha^{-}}.P$, which by the induction hypothesis implies $\Gamma; \Phi \vDash v\colon \mathbf{nf}\,(\exists\overrightarrow{\alpha^{-}}.P)$, and hence, $\Gamma; \Phi \vDash v\colon \exists\overrightarrow{\alpha^{-}}.P$.
(3) $\Gamma, \overrightarrow{\alpha^{-}}; \Phi, x : P \vdash c\colon N$, and by the induction hypothesis, $\Gamma, \overrightarrow{\alpha^{-}}; \Phi, x : P \vDash c\colon \mathbf{nf}\,(N)$.
(4) $\Gamma \vdash N$.

This way, we can apply Rule ($\textsc{let}^{\textsc{inf}}_{\exists}$) to infer $\Gamma; \Phi \vDash \mathbf{let}^{\exists}(\overrightarrow{\alpha^{-}}, x) = v; c\colon \mathbf{nf}\,(N)$.

Case 8. Rule ($\simeq^{\textsc{inf}}_{+}$)

Then we are proving that if $\Gamma; \Phi \vdash v\colon P'$ is inferred by Rule ($\simeq^{\textsc{inf}}_{+}$), then $\Gamma; \Phi \vDash v\colon \mathbf{nf}\,(P')$. By inversion, $\Gamma; \Phi \vdash v\colon P$ and $\Gamma \vdash P \simeq^{\leqslant} P'$, and the metric of the tree inferring $\Gamma; \Phi \vdash v\colon P$ is less than the one inferring $\Gamma; \Phi \vdash v\colon P'$. Then by the induction hypothesis, $\Gamma; \Phi \vDash v\colon \mathbf{nf}\,(P)$. By lemma 35 $\Gamma \vdash P \simeq^{\leqslant} P'$ implies $\mathbf{nf}\,(P) = \mathbf{nf}\,(P')$, and thus, $\Gamma; \Phi \vDash v\colon \mathbf{nf}\,(P)$ can be rewritten to $\Gamma; \Phi \vDash v\colon \mathbf{nf}\,(P')$.

Case 9. Rule ($\textsc{Var}^{\textsc{inf}}$)

Then we are proving that $\Gamma; \Phi \vdash x\colon P$ implies $\Gamma; \Phi \vDash x\colon \mathbf{nf}\,(P)$. By inversion of $\Gamma; \Phi \vdash x\colon P$, we have $x : P \in \Phi$. Then Rule ($\textsc{Var}^{\textsc{inf}}$) applies to infer $\Gamma; \Phi \vDash x\colon \mathbf{nf}\,(P)$.

Case 10. Rule ($\textsc{let}^{\textsc{inf}}$)

Then we are proving that $\Gamma; \Phi \vdash \mathbf{let}\,x = v(\vec{v}); c\colon N$ implies $\Gamma; \Phi \vDash \mathbf{let}\,x = v(\vec{v}); c\colon \mathbf{nf}\,(N)$.

By inversion of $\Gamma; \Phi \vdash \mathbf{let}\,x = v(\vec{v}); c\colon N$, we have

(1) $\Gamma; \Phi \vdash v\colon P$, and by the induction hypothesis, $\Gamma; \Phi \vDash v\colon \mathbf{nf}\,(P)$.
(2) $\Gamma; \Phi, x : P \vdash c\colon N$, and by lemma 86, since $\Gamma \vdash P \simeq^{\leqslant} \mathbf{nf}\,(P)$, we have $\Gamma; \Phi, x : \mathbf{nf}\,(P) \vdash c\colon N$. Then by the induction hypothesis, $\Gamma; \Phi, x : \mathbf{nf}\,(P) \vDash c\colon \mathbf{nf}\,(N)$.

Together, $\Gamma; \Phi \vDash v\colon \mathbf{nf}\,(P)$ and $\Gamma; \Phi, x : \mathbf{nf}\,(P) \vDash c\colon \mathbf{nf}\,(N)$ imply $\Gamma; \Phi \vDash \mathbf{let}\,x = v(\vec{v}); c\colon \mathbf{nf}\,(N)$ by Rule ($\textsc{let}^{\textsc{inf}}$).

Case 11. Rule ($\textsc{let}^{\textsc{inf}}_{:@}$)

Then we are proving that $\Gamma; \Phi \vdash \mathbf{let}\,x : P = v(\vec{v}); c\colon N$ implies $\Gamma; \Phi \vDash \mathbf{let}\,x : P = v(\vec{v}); c\colon \mathbf{nf}\,(N)$.

By inversion of $\Gamma; \Phi \vdash \mathbf{let}\,x : P = v(\vec{v}); c\colon N$, we have

(1) $\Gamma \vdash P$

(2) $\Gamma;\Phi \vdash v\colon \downarrow M$ for some ground M, which by the induction hypothesis means $\Gamma;\Phi \models v\colon \downarrow\mathbf{nf}\,(M)$

(3) $\Gamma;\Phi \vdash M \bullet \vec{v} \Rightarrow\!\!> \uparrow Q$. By lemma 85, since $\Gamma \vdash M \simeq^{\leqslant} \mathbf{nf}\,(M)$, we have $\Gamma;\Phi \vdash [\cdot]\mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> \uparrow Q$, which by the induction hypothesis means that there exist normalized M', Θ, and SC_1 such that (noting that M is ground):

(a) $\Gamma;\Phi;\cdot \models \mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> M' \dashv \Theta; SC_1$, where by the soundness, $\Gamma; \mathbf{dom}\,(\Theta) \vdash M'$ and $\Theta \vdash SC_1$.

(b) for any $\Gamma \vdash M''$ such that $\Gamma;\Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> M''$ there exists $\widehat{\sigma}$ such that

(i) $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,M', \Theta \vdash \widehat{\sigma} : SC_1$, and

(ii) $\Gamma \vdash [\widehat{\sigma}]M' \simeq^{\leqslant} M''$,

In particular, there exists $\widehat{\sigma}_0$ such that $\Theta \vdash \widehat{\sigma}_0 : \mathbf{uv}\,M', \Theta \vdash \widehat{\sigma}_0 : SC_1, \Gamma \vdash [\widehat{\sigma}_0]M' \simeq^{\leqslant} \uparrow Q$. Since M' is normalized and free of negative algorithmic variables, the latter equivalence means $M' = \uparrow Q_0$ for some Q_0, and $\Gamma \vdash [\widehat{\sigma}_0]Q_0 \simeq^{\leqslant} Q$.

(4) $\Gamma \vdash \uparrow Q \leqslant \uparrow P$, and by transitivity, since $\Gamma \vdash [\widehat{\sigma}_0]\uparrow Q_0 \simeq^{\leqslant} \uparrow Q$, we have $\Gamma \vdash [\widehat{\sigma}_0]\uparrow Q_0 \leqslant \uparrow P$.

Let us apply lemma 84 to $\Gamma \vdash [\widehat{\sigma}_0]\uparrow Q_0 \leqslant \uparrow P$ and obtain $\Theta \vdash SC_2$ such that

(a) $\Gamma;\Theta \models \uparrow Q_0 \leqslant \uparrow P \dashv SC_2$ and

(b) $\Theta \vdash \widehat{\sigma}_0 : SC_2$.

(5) $\Gamma;\Phi, x : P \vdash c\colon N$, and by the induction hypothesis, $\Gamma;\Phi, x : P \models c\colon \mathbf{nf}\,(N)$.

To infer $\Gamma;\Phi \models \mathbf{let}\, x : P = v(\vec{v}); c\colon \mathbf{nf}\,(N)$, we apply the corresponding algorithmic rule Rule ($\textsc{let}^{\textsc{inf}}_{:@}$). Let us show that the premises hold:

(1) $\Gamma \vdash P$,

(2) $\Gamma;\Phi \models v\colon \downarrow\mathbf{nf}\,(M)$,

(3) $\Gamma;\Phi;\cdot \models \mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> \uparrow Q_0 \dashv \Theta; SC_1$,

(4) $\Gamma;\Theta \models \uparrow Q_0 \leqslant \uparrow P \dashv SC_2$, and

(5) $\Gamma;\Phi, x : P \models c\colon \mathbf{nf}\,(N)$ hold as noted above;

(6) $\Theta \vdash SC_1 \& SC_2 = SC$ is defined by lemma 82, since $\Theta \vdash \widehat{\sigma}_0 : SC_1$ and $\Theta \vdash \widehat{\sigma}_0 : SC_2$.

Case 12. Rule ($\textsc{let}^{\textsc{inf}}_{@}$)

By assumption, c is $\mathbf{let}\, x = v(\vec{v}); c'$. Then by inversion of $\Gamma;\Phi \vdash \mathbf{let}\, x = v(\vec{v}); c' : N$:

- $\Gamma;\Phi \vdash v\colon \downarrow M$, which by the induction hypothesis means $\Gamma;\Phi \models v\colon \downarrow\mathbf{nf}\,(M)$;
- $\Gamma;\Phi \vdash M \bullet \vec{v} \Rightarrow\!\!> \uparrow Q$ unique. Then by lemma 85, since $\Gamma \vdash M \simeq^{\leqslant} \mathbf{nf}\,(M)$, we have $\Gamma;\Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> \uparrow Q$ and moreover, $\Gamma;\Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> \uparrow Q$ unique (since symmetrically, $\mathbf{nf}\,(M)$ can be replaced back by M). Then the induction hypothesis applied to $\Gamma;\Phi \vdash [\cdot]\mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> \uparrow Q$ implies that there exist M', Θ, and SC such that (considering M is ground):

(1) $\Gamma;\Phi;\cdot \models \mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> M' \dashv \Theta; SC$, which, by the soundness, implies, in particular that

(a) $\Gamma; \mathbf{dom}\,(\Theta) \vdash M'$ is normalized and free of negative algorithmic variables,

(b) $\Theta|_{\mathbf{uv}\,(M')} \vdash SC$, which means $\mathbf{dom}\,(SC) \subseteq \mathbf{uv}\,(M')$,

(c) for any $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,M'$ such that $\Theta \vdash \widehat{\sigma} : SC$, we have $\Gamma;\Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> [\widehat{\sigma}]M'$, which, since $\Gamma;\Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> \uparrow Q$ unique, means $\Gamma \vdash [\widehat{\sigma}]M' \simeq^{\leqslant} \uparrow Q$.

and

(2) for any $\Gamma \vdash M''$ such that $\Gamma;\Phi \vdash \mathbf{nf}\,(M) \bullet \vec{v} \Rightarrow\!\!> M''$, (and in particular, for $\Gamma \vdash \uparrow Q$) there exists $\widehat{\sigma}_1$ such that

(a) $\Theta \vdash \widehat{\sigma}_1 : \mathbf{uv}\,M', \Theta \vdash \widehat{\sigma}_1 : SC$, and

(b) $\Gamma \vdash [\widehat{\sigma}_1]M' \simeq^{\leqslant} M''$, and in particular, $\Gamma \vdash [\widehat{\sigma}_1]M' \simeq^{\leqslant} \uparrow Q$. Since M' is normalized and free of negative algorithmic variables, it means that $M' = \uparrow P$ for some P ($\Gamma; \mathbf{dom}(\Theta) \vdash P$) that is $\Gamma \vdash [\widehat{\sigma}_1]P \simeq^{\leqslant} Q$.

- $\Gamma; \Phi, x : Q \vdash c' : N$

To infer $\Gamma; \Phi \vDash \mathbf{let}\, x = v(\overrightarrow{v}); c' : \mathbf{nf}(N)$, let us apply the corresponding algorithmic rule (Rule ($\textsc{let}^{\textsc{inf}}_{@}$)):

(1) $\Gamma; \Phi \vDash v : \downarrow\mathbf{nf}(M)$ holds as noted above;
(2) $\Gamma; \Phi; \cdot \vDash \mathbf{nf}(M) \bullet \overrightarrow{v} \Rrightarrow \uparrow P \dashv \Theta; SC$ holds as noted above;
(3) To show $\mathbf{uv}\, P = \mathbf{dom}(SC)$ and SC **singular with** $\widehat{\sigma}_0$ for some $\widehat{\sigma}_0$, we apply lemma 90 with $\Xi = \mathbf{uv}\, P = \mathbf{uv}(M')$ (as noted above, $\mathbf{dom}(SC) \subseteq \mathbf{uv}(M') = \Xi$).
Now we will show that any substitution satisfying SC is equivalent to $\widehat{\sigma}_1$. As noted in 1c, for any substitution $\Theta \vdash \widehat{\sigma} : \Xi$, $\Theta \vdash \widehat{\sigma} : SC$ implies $\Gamma \vdash [\widehat{\sigma}]M' \simeq^{\leqslant} \uparrow Q$, which is rewritten as $\Gamma \vdash [\widehat{\sigma}]P \simeq^{\leqslant} Q$. And since $\Gamma \vdash [\widehat{\sigma}_1]P \simeq^{\leqslant} Q$, we have $\Gamma \vdash [\widehat{\sigma}]P \simeq^{\leqslant} [\widehat{\sigma}_1]P$, which implies $\Theta \vdash \widehat{\sigma} \simeq^{\leqslant} \widehat{\sigma}_1 : \Xi$ by corollary 22.
(4) Let us show $\Gamma; \Phi, x : [\widehat{\sigma}_0]P \vDash c' : \mathbf{nf}(N)$. By the soundness of singularity (lemma 89), we have $\Theta \vdash \widehat{\sigma}_0 : SC$, which by 1c means $\Gamma \vdash [\widehat{\sigma}_0]M' \simeq^{\leqslant} \uparrow Q$, that is $\Gamma \vdash [\widehat{\sigma}_0]P \simeq^{\leqslant} Q$, and thus, $\Gamma \vdash \Phi, x : Q \simeq^{\leqslant} \Phi, x : [\widehat{\sigma}_0]P$.
Then by lemma 86, $\Gamma; \Phi, x : Q \vdash c' : N$ can be rewritten as $\Gamma; \Phi, x : [\widehat{\sigma}_0]P \vdash c' : N$.
Then by the induction hypothesis applied to it, $\Gamma; \Phi, x : [\widehat{\sigma}_0]P \vDash c' : \mathbf{nf}(N)$ holds.

Case 13. Rule ($\forall^{\textsc{inf}}_{\bullet\Rrightarrow}$)

Since N cannot be a algorithmic variable, if $[\widehat{\sigma}]N$ starts with $\forall$, so does N. This way, $N = \forall\overrightarrow{\alpha^+}.N_1$. Then by assumption:

(1) $\Gamma \vdash^{\supseteq} \Theta$
(2) $\Gamma; \mathbf{dom}(\Theta) \vdash \forall\overrightarrow{\alpha^+}.N_1$ is free from negative algorithmic variables, and then $\Gamma, \overrightarrow{\alpha^+}; \mathbf{dom}(\Theta) \vdash N_1$ is free from negative algorithmic variables too;
(3) $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\, N_1$;
(4) $\Gamma \vdash M$;
(5) $\Gamma; \Phi \vdash [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N_1 \bullet \overrightarrow{v} \Rrightarrow M$, that is $\Gamma; \Phi \vdash (\forall\overrightarrow{\alpha^+}.[\widehat{\sigma}]N_1) \bullet \overrightarrow{v} \Rrightarrow M$. Then by inversion there exists σ such that
   (a) $\Gamma \vdash \sigma : \overrightarrow{\alpha^+}$;
   (b) $\overrightarrow{v} \neq \cdot$ and $\overrightarrow{\alpha^+} \neq \cdot$; and
   (c) $\Gamma; \Phi \vdash [\sigma][\widehat{\sigma}]N_1 \bullet \overrightarrow{v} \Rrightarrow M$. Notice that σ and $\widehat{\sigma}$ commute because the codomain of σ does not contain algorithmic variables (and thus, does not intersect with the domain of $\widehat{\sigma}$), and the codomain of $\widehat{\sigma}$ is Γ and does not intersect with $\overrightarrow{\alpha^+}$—the domain of σ.
   Let us take fresh $\overrightarrow{\widehat{\alpha}^+}$ and construct $N_0 = [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N_1$ and $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}_0 : \mathbf{uv}(N_0)$ defined as

$$\begin{cases} [\widehat{\sigma}_0]\widehat{\alpha}_i^{\,+} = [\sigma]\alpha_i^+ & \text{for } \widehat{\alpha}_i^{\,+} \in \overrightarrow{\widehat{\alpha}^+} \cap \mathbf{uv}\, N_0 \\ [\widehat{\sigma}_0]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm} & \text{for } \widehat{\beta}^{\pm} \in \mathbf{uv}\, N_1 \end{cases}$$

   Then it is easy to see that $[\widehat{\sigma}_0][\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N_1 = [\sigma][\widehat{\sigma}]N_1$ because this substitution compositions coincide on $\mathbf{uv}(N_1) \cup \mathbf{fv}(N_1)$. In other words, $[\widehat{\sigma}_0]N_0 = [\sigma][\widehat{\sigma}]N_1$.
   Then let us apply the induction hypothesis to $\Gamma; \Phi \vdash [\widehat{\sigma}_0]N_0 \bullet \overrightarrow{v} \Rrightarrow M$ and obtain M', Θ', and SC such that
   - $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash N_0 \bullet \overrightarrow{v} \Rrightarrow M' \dashv \Theta'; SC$ and

- for any $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}_0 : \mathbf{uv}\,(N_0)$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\widehat{\sigma}_0]N_0 \bullet \vec{v} \Rightarrow > M$, there exists $\widehat{\sigma}'_0$ such that
  (i) $\Theta' \vdash \widehat{\sigma}'_0 : \mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M')$, $\Theta' \vdash \widehat{\sigma}'_0 : SC$,
  (ii) $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}'_0 \simeq^{\leqslant} \widehat{\sigma}_0 : \mathbf{uv}\,N_0$, and
  (iii) $\Gamma \vdash [\widehat{\sigma}'_0]M' \simeq^{\leqslant} M$.

Let us take M', Θ', and SC from the induction hypothesis (5c) (from SC we subtract entries restricting $\overrightarrow{\widehat{\alpha}^+}$) and show they satisfy the required properties

(1) To infer $\Gamma; \Phi; \Theta \vDash \forall\overrightarrow{\alpha^+}.N_1 \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC\backslash\overrightarrow{\widehat{\alpha}^+}$ we apply the corresponding algorithmic rule Rule $(\forall^{\mathrm{INF}}_{\bullet\Rrightarrow})$. As noted above, the required premises hold:
  (a) $\vec{v} \neq \cdot$, $\overrightarrow{\alpha^+} \neq \cdot$; and
  (b) $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash [\overrightarrow{\widehat{\alpha}^+}/\overrightarrow{\alpha^+}]N_1 \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC$ is obtained by unfolding the definition of N_0 in $\Gamma; \Phi; \Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vDash N_0 \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC$ (5c).

(2) Let us take and arbitrary $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,N_1$ and $\Gamma \vdash M$ and assume $\Gamma; \Phi \vdash [\widehat{\sigma}]\forall\overrightarrow{\alpha^+}.N_1 \bullet \vec{v} \Rrightarrow M$. Then the same reasoning as in 5c applies. In particular, we construct $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}_0 : \mathbf{uv}\,(N_0)$ as an extension of $\widehat{\sigma}$ and obtain $\Gamma; \Phi \vdash [\widehat{\sigma}_0]N_0 \bullet \vec{v} \Rrightarrow M$. It means we can apply the property inferred from the induction hypothesis (5c) to obtain $\widehat{\sigma}'_0$ such that
  (a) $\Theta' \vdash \widehat{\sigma}'_0 : \mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M')$ and $\Theta' \vdash \widehat{\sigma}'_0 : SC$,
  (b) $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}'_0 \simeq^{\leqslant} \widehat{\sigma}_0 : \mathbf{uv}\,N_0$, and
  (c) $\Gamma \vdash [\widehat{\sigma}'_0]M' \simeq^{\leqslant} M$.

  Let us show that $\widehat{\sigma}'_0|_{(\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))}$ satisfies the required properties.
  (a) $\Theta' \vdash \widehat{\sigma}'_0|_{(\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))} : (\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))$ holds since $\Theta' \vdash \widehat{\sigma}'_0 : \mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M')$ and $\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M') \subseteq \mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M')$; $\Theta' \vdash \widehat{\sigma}'_0|_{(\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))} : SC\backslash\overrightarrow{\widehat{\alpha}^+}$ holds since $\Theta' \vdash \widehat{\sigma}'_0 : SC$, $\Theta' \vdash \widehat{\sigma}'_0 : \mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M')$, and $(\mathbf{uv}\,(N_0) \cup \mathbf{uv}\,(M'))\backslash\overrightarrow{\widehat{\alpha}^+} = \mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M')$.
  (b) $\Gamma \vdash [\widehat{\sigma}'_0]M' \simeq^{\leqslant} M$ holds as shown, and hence it holds for $\widehat{\sigma}'_0|_{(\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))}$;
  (c) We show $\Theta \vdash \widehat{\sigma}'_0 \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}\,N_1$, from which it follows that it holds for $\widehat{\sigma}'_0|_{(\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M'))}$. Let us take an arbitrary $\widehat{\beta}^{\pm} \in \mathbf{dom}\,(\Theta) \subseteq \mathbf{dom}\,(\Theta) \cup \overrightarrow{\widehat{\alpha}^+}$. Then since $\Theta, \overrightarrow{\widehat{\alpha}^+}\{\Gamma\} \vdash \widehat{\sigma}'_0 \simeq^{\leqslant} \widehat{\sigma}_0 : \mathbf{uv}\,N_0$, we have $\Theta(\widehat{\beta}^{\pm}) \vdash [\widehat{\sigma}'_0]\widehat{\beta}^{\pm} \simeq^{\leqslant} [\widehat{\sigma}_0]\widehat{\beta}^{\pm}$ and by definition of $\widehat{\sigma}_0$, $[\widehat{\sigma}_0]\widehat{\beta}^{\pm} = [\widehat{\sigma}]\widehat{\beta}^{\pm}$.

Case 14. Rule $(\rightarrow^{\mathrm{INF}}_{\bullet\Rrightarrow})$

Since N cannot be a algorithmic variable, if the shape of $[\widehat{\sigma}]N$ is an arrow, so is the shape of N. This way, $N = Q \to N_1$. Then by assumption:

(1) $\Gamma \vdash^{\supseteq} \Theta$;
(2) $\Gamma; \mathbf{dom}\,(\Theta) \vdash Q \to N_1$ is free from negative algorithmic variables;
(3) $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,Q \cup \mathbf{uv}\,N_1$;
(4) $\Gamma \vdash M$;
(5) $\Gamma; \Phi \vdash [\widehat{\sigma}](Q \to N_1) \bullet v, \vec{v} \Rrightarrow M$, that is $\Gamma; \Phi \vdash ([\widehat{\sigma}]Q \to [\widehat{\sigma}]N_1) \bullet v, \vec{v} \Rrightarrow M$, and by inversion:
  (a) $\Gamma; \Phi \vdash v\colon P$, and by the induction hypothesis, $\Gamma; \Phi \vDash v\colon \mathbf{nf}\,(P)$;
  (b) $\Gamma \vdash [\widehat{\sigma}]Q \geqslant P$, which by transitivity (lemma 24) means $\Gamma \vdash [\widehat{\sigma}]Q \geqslant \mathbf{nf}\,(P)$, and then by completeness of subtyping (lemma 78), $\Gamma;\ \Theta \vDash Q \geqslant \mathbf{nf}\,(P) \dashv SC_1$, for some $\Theta \vdash SC_1 : \mathbf{uv}\,(Q)$, and moreover, $\Theta \vdash \widehat{\sigma} : SC_1$;

(c) $\Gamma;\Phi \vdash [\widehat{\sigma}]N_1 \bullet \vec{v} \Rrightarrow M$. Notice that the induction hypothesis is applicable to this case: $\Gamma;\mathbf{dom}(\Theta) \vdash N_1$ is free from negative algorithmic variables because so is $Q \to N_1$. This way, there exist M', Θ', and SC_2 such that
  (i) $\Gamma;\Phi;\Theta \vDash N_1 \bullet \vec{v} \Rrightarrow M' \dashv \Theta';SC_2$ and then by soundness of typing (i.e. the induction hypothesis),
    (A) $\Theta \subseteq \Theta'$
    (B) $\Gamma;\mathbf{dom}(\Theta') \vdash M'$
    (C) $\mathbf{dom}(\Theta) \cap \mathbf{uv}(M') \subseteq \mathbf{uv}\,N_1$
    (D) $\Theta'|_{\mathbf{uv}\,N_1 \cup \mathbf{uv}\,M'} \vdash SC_2$
  (ii) for any $\Theta \vdash \widehat{\sigma} : \mathbf{uv}(N_1)$ and $\Gamma \vdash M$ such that $\Gamma;\Phi \vdash [\widehat{\sigma}]N_1 \bullet \vec{v} \Rrightarrow M$, there exists $\widehat{\sigma}'$ such that
    (A) $\Theta' \vdash \widehat{\sigma}' : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ and $\Theta' \vdash \widehat{\sigma}' : SC_2$,
    (B) $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(N_1)$, and
    (C) $\Gamma \vdash [\widehat{\sigma}']M' \simeq^{\leqslant} M$.

We need to show that there exist M', Θ', and SC such that $\Gamma;\Phi;\Theta \vDash Q \to N_1 \bullet v, \vec{v} \Rightarrow > M' \dashv \Theta';SC$ and the initiality property holds. We take M' and Θ' from the induction hypothesis (5c), and SC as a merge of SC_1 and SC_2. To show that $\Theta' \vdash SC_1 \& SC_2 = SC$ exists, we apply lemma 82. To do so, we need to provide a substitution satisfying both SC_1 and SC_2. Notice that $\mathbf{dom}(SC_1) = \mathbf{uv}(Q)$ and $\mathbf{dom}(SC_2) \subseteq \mathbf{uv}\,N_1 \cup \mathbf{uv}\,M'$. This way, it suffices to construct $\Theta' \vdash \widehat{\sigma}'' : \mathbf{uv}(Q) \cup \mathbf{uv}\,N_1 \cup \mathbf{uv}\,M'$ such that $\Theta' \vdash \widehat{\sigma}'' : SC_1$ and $\Theta' \vdash \widehat{\sigma}'' : SC_2$. By the induction hypothesis (5(c)ii), $\widehat{\sigma}|_{\mathbf{uv}(N_1)}$ can be extended to $\widehat{\sigma}'$ such that

(1) $\Theta' \vdash \widehat{\sigma}' : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ and $\Theta' \vdash \widehat{\sigma}' : SC_2$,
(2) $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(N_1)$, and
(3) $\Gamma \vdash [\widehat{\sigma}']M' \simeq^{\leqslant} M$.

Let us extend $\widehat{\sigma}'$ to $\widehat{\sigma}''$ defined on $\mathbf{uv}(Q) \cup \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ with values of $\widehat{\sigma}$ as follows:

$$\begin{cases} [\widehat{\sigma}'']\widehat{\beta}^{\pm} = [\widehat{\sigma}']\widehat{\beta}^{\pm} & \text{for } \widehat{\beta}^{\pm} \in \mathbf{uv}(N_1) \cup \mathbf{uv}(M') \\ [\widehat{\sigma}'']\widehat{\gamma}^{\pm} = [\widehat{\sigma}]\widehat{\gamma}^{\pm} & \text{for } \widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \backslash (\mathbf{uv}(N_1) \cup \mathbf{uv}(M')) \end{cases}$$

First, notice that $\Theta' \vdash \widehat{\sigma}'' \simeq^{\leqslant} \widehat{\sigma}' : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$ by definition. Then since $\Theta' \vdash \widehat{\sigma}' : SC_2$ and $\Theta' \vdash SC_2 : \mathbf{uv}(N_1) \cup \mathbf{uv}(M')$, we have $\Theta' \vdash \widehat{\sigma}'' : SC_2$.
Second, notice that $\Theta \vdash \widehat{\sigma}'' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(N_1) \cup \mathbf{uv}(Q)$:

- if $\widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \backslash (\mathbf{uv}(N_1) \cup \mathbf{uv}(M'))$ then $[\widehat{\sigma}'']\widehat{\gamma}^{\pm} = [\widehat{\sigma}]\widehat{\gamma}^{\pm}$ by definition of $\widehat{\sigma}''$;
- if $\widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \cap \mathbf{uv}(N_1)$ then $[\widehat{\sigma}'']\widehat{\gamma}^{\pm} = [\widehat{\sigma}']\widehat{\gamma}^{\pm}$, and $\Theta \vdash \widehat{\sigma}' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(N_1)$, as noted above;
- if $\widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \cap \mathbf{uv}(M')$ then since $\Gamma;\mathbf{dom}(\Theta) \vdash Q$, we have $\mathbf{uv}(Q) \subseteq \mathbf{dom}(\Theta)$, implying $\widehat{\gamma}^{\pm} \in \mathbf{dom}(\Theta) \cap \mathbf{uv}(M') \subseteq \mathbf{uv}(N_1)$. This way, $\widehat{\gamma}^{\pm} \in \mathbf{uv}(Q) \cap \mathbf{uv}(N_1)$, and this case is covered by the previous one.

In particular, $\Theta \vdash \widehat{\sigma}'' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}(Q)$. Then since $\Theta \vdash \widehat{\sigma} : SC_1$ and $\Theta \vdash SC_1 : \mathbf{uv}(Q)$, we have $\Theta \vdash \widehat{\sigma}'' : SC_1$.
This way, $\widehat{\sigma}'$ satisfies both SC_1 and SC_2, and by the completeness of constraint merge (lemma 82), $\Theta' \vdash SC_1 \& SC_2 = SC$ exists.
Finally, to show the required properties, we take M' and Θ' from the induction hypothesis (5(c)ii), and SC defined above. Then

(1) $\Gamma;\Phi;\Theta \vDash Q \to N_1 \bullet v, \vec{v} \Rrightarrow M' \dashv \Theta';SC$ is inferred by Rule $(\to^{\mathrm{INF}}_{\bullet\Rrightarrow})$. As noted above:
  (a) $\Gamma;\Phi \vDash v\colon \mathbf{nf}(P)$,
  (b) $\Gamma;\Theta \vDash Q \geqslant \mathbf{nf}(P) \dashv SC_1$,

(c) $\Gamma; \Phi; \Theta \vDash N_1 \bullet \vec{v} \Rrightarrow M' \dashv \Theta'; SC_2$, and
(d) $\Theta' \vdash SC_1 \& SC_2 = SC$.

(2) let us take an arbitrary $\Theta \vdash \widehat{\sigma}_0 : \mathbf{uv}\, Q \cup \mathbf{uv}\, N_1$; and $\Gamma \vdash M_0$; such that $\Gamma; \Phi \vdash [\widehat{\sigma}_0](Q \to N_1) \bullet v, \vec{v} \Rrightarrow M_0$. Then by inversion of $\Gamma; \Phi \vdash [\widehat{\sigma}_0] Q \to [\widehat{\sigma}_0] N_1 \bullet v, \vec{v} \Rrightarrow M_0$, we have the same properties as in 5. In particular,
- $\Gamma \vdash [\widehat{\sigma}_0] Q \geqslant \mathbf{nf}\,(P)$ and by the completeness of subtyping (lemma 78), $\Theta \vdash \widehat{\sigma}_0 : SC_1$.
- $\Gamma; \Phi \vdash [\widehat{\sigma}_0] N_1 \bullet \vec{v} \Rrightarrow M_0$. Then by 5(c)ii, there exists $\widehat{\sigma}'_0$ such that
  (a) $\Theta' \vdash \widehat{\sigma}'_0 : \mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M')$ and $\Theta' \vdash \widehat{\sigma}'_0 : SC_2$,
  (b) $\Theta \vdash \widehat{\sigma}'_0 \simeq^{\leqslant} \widehat{\sigma}_0 : \mathbf{uv}\,(N_1)$, and
  (c) $\Gamma \vdash [\widehat{\sigma}'_0] M' \simeq^{\leqslant} M_0$.

Let us extend $\widehat{\sigma}'_0$ to be defined on $\mathbf{uv}\,(Q) \cup \mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M')$ with the values of $\widehat{\sigma}_0$. We define $\widehat{\sigma}''_0$ as follows:

$$\begin{cases} [\widehat{\sigma}''_0]\widehat{\gamma}^{\pm} = [\widehat{\sigma}'_0]\widehat{\gamma}^{\pm} & \text{for } \widehat{\gamma}^{\pm} \in \mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M') \\ [\widehat{\sigma}''_0]\widehat{\gamma}^{\pm} = [\widehat{\sigma}_0]\widehat{\gamma}^{\pm} & \text{for } \widehat{\gamma}^{\pm} \in \mathbf{uv}\,(Q) \backslash (\mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M')) \end{cases}$$

This way,
- $\Theta' \vdash \widehat{\sigma}''_0 : \mathbf{uv}\,(Q) \cup \mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M')$,
- $\Theta' \vdash \widehat{\sigma}''_0 : SC$, since $\Theta' \vdash \widehat{\sigma}''_0 : SC_1$ and $\Theta' \vdash \widehat{\sigma}''_0 : SC_2$, which is proved similarly to $\Theta' \vdash \widehat{\sigma}'' : SC_1$ and $\Theta' \vdash \widehat{\sigma}'' : SC_2$ above;
- $\Theta \vdash \widehat{\sigma}''_0 \simeq^{\leqslant} \widehat{\sigma}_0 : \mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(Q)$: the proof is analogous to $\Theta \vdash \widehat{\sigma}'' \simeq^{\leqslant} \widehat{\sigma} : \mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(Q)$ above.
- $\Gamma \vdash [\widehat{\sigma}''_0] M' \simeq^{\leqslant} M_0$ Notice that $\Theta' \vdash \widehat{\sigma}''_0 \simeq^{\leqslant} \widehat{\sigma}'_0 : \mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M')$, which is proved analogously to $\Theta' \vdash \widehat{\sigma}'' \simeq^{\leqslant} \widehat{\sigma}' : \mathbf{uv}\,(N_1) \cup \mathbf{uv}\,(M')$ above. Then $\Gamma \vdash [\widehat{\sigma}'_0] M' \simeq^{\leqslant} M_0$ can be rewritten to $\Gamma \vdash [\widehat{\sigma}''_0] M' \simeq^{\leqslant} M_0$.

Case 15. Rule $(\varnothing^{\mathrm{INF}}_{\bullet\Rrightarrow})$

By assumption:
(1) $\Gamma \vdash^{\supseteq} \Theta$,
(2) $\Gamma \vdash N'$,
(3) $\Gamma; \mathbf{dom}\,(\Theta) \vdash N$ and N is free from negative variables,
(4) $\Theta \vdash \widehat{\sigma} : \mathbf{uv}\,(N)$,
(5) $\Gamma; \Phi \vdash [\widehat{\sigma}] N \bullet \cdot \Rrightarrow N'$, and by inversion, $\Gamma \vdash [\widehat{\sigma}] N \simeq^{\leqslant} N'$.

Then we can apply the corresponding algorithmic rule Rule $(\varnothing^{\mathrm{INF}}_{\bullet\Rrightarrow})$ to infer $\Gamma; \Phi; \Theta \vDash N \bullet \cdot \Rrightarrow \mathbf{nf}\,(N) \dashv \Theta; \cdot$. Let us show the required properties. Let us take an arbitrary $\Theta \vdash \widehat{\sigma}_0 : \mathbf{uv}\,(N)$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\widehat{\sigma}_1] N \bullet \cdot \Rrightarrow M$. Then we can take $\widehat{\sigma}_0$ as the required substitution:
(1) $\Theta \vdash \widehat{\sigma}_0 : \mathbf{uv}\,(N) \cup \mathbf{uv}\,(\mathbf{nf}\,(N))$, since $\mathbf{uv}\,(\mathbf{nf}\,(N)) = \mathbf{uv}\,(N)$, and thus, $\mathbf{uv}\,(N) \cup \mathbf{uv}\,(\mathbf{nf}\,(N)) = \mathbf{uv}\,(N)$;
(2) $\Theta \vdash \widehat{\sigma}_0 : \cdot$ vacuously;
(3) $\Theta \vdash \widehat{\sigma}_0 \simeq^{\leqslant} \widehat{\sigma}_0 : \mathbf{uv}\,(N)$ by reflexivity;
(4) Let us show $\Gamma \vdash [\widehat{\sigma}_0]\mathbf{nf}\,(N) \simeq^{\leqslant} M$. Notice that $\Gamma; \Phi \vdash [\widehat{\sigma}_0] N \bullet \cdot \Rrightarrow M$ can only be inferred by Rule $(\varnothing^{\mathrm{INF}}_{\bullet\Rrightarrow})$, and thus, $\Gamma \vdash [\widehat{\sigma}_0] N \simeq^{\leqslant} M$. By corollary 14, $\Gamma \vdash [\widehat{\sigma}_0] N \simeq^{\leqslant} [\widehat{\sigma}_0]\mathbf{nf}\,(N)$, and then by transitivity, $\Gamma \vdash [\widehat{\sigma}_0]\mathbf{nf}\,(N) \simeq^{\leqslant} M$, that is $\Gamma \vdash [\widehat{\sigma}_0]\mathbf{nf}\,(N) \simeq^{\leqslant} M$.

□

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