

$\alpha, \beta, \alpha, \beta, \gamma, \delta$	type variables
n, m, i, j	index variables
x, y, z	term variables

		$\hat{\alpha}^- : \approx N$		
		(e)	S	
		$e_1 \ \& \ e_2$	M	
UC	$::=$			unification constraint
		\cdot		
		e		
		$UC \backslash vars$		
		$UC vars$		
		$UC_1 \cup UC_2$		
		$\overline{UC_i}^i$		concatenate
		(UC)	S	
		$\mathbf{UC} vars$	M	
		$UC_1 \ \& \ UC_2$	M	
		$UC_1 \cup UC_2$	M	
		$ SC $	M	
SC	$::=$			subtyping constraint
		\cdot		
		e		
		$SC \backslash vars$		
		$SC vars$		
		$SC_1 \cup SC_2$		
		UC		
		$\overline{SC_i}^i$		concatenate
		(SC)	S	
		$\mathbf{SC} vars$	M	
		$SC_1 \ \& \ SC_2$	M	
$\hat{\sigma}$	$::=$			unification substitution
		\cdot		
		$P/\hat{\alpha}^+$		
		$N/\hat{\alpha}^-$		
		$\vec{P}/\vec{\alpha}^+$		
		$\vec{N}/\vec{\alpha}^-$		
		$(\hat{\sigma})$	S	
		$\overline{\hat{\sigma}_i}^i$		concatenate
		$\mathbf{nf}(\hat{\sigma}')$	M	
		$\hat{\sigma}' vars$	M	
$\hat{\tau}, \hat{\rho}$	$::=$			anti-unification substitution
		\cdot		
		$\hat{\alpha}^- : \approx N$		
		$\hat{\alpha}^- : \approx N$		
		$\vec{\alpha}^- / \vec{\alpha}^-$		
		$\vec{N} / \vec{\alpha}^-$		
		$\hat{\tau}_1 \cup \hat{\tau}_2$		
		$\overline{\hat{\tau}_i}^i$		concatenate
		$(\hat{\tau})$	S	
		$\hat{\tau}' vars$	M	

		$\hat{\tau}_1 \ \& \ \hat{\tau}_2$	M	
P, Q, R	::=			positive types
		α^+		
		$\downarrow N$		
		$\exists \alpha^-. P$		
		$[\sigma]P$	M	
N, M, K	::=			negative types
		α^-		
		$\uparrow P$		
		$\forall \alpha^+. N$		
		$P \rightarrow N$		
		$[\sigma]N$	M	
$\overrightarrow{\alpha^+}, \overrightarrow{\beta^+}, \overrightarrow{\gamma^+}, \overrightarrow{\delta^+}$::=			positive variable list
		\cdot		empty list
		α^+		a variable
		$\overrightarrow{\alpha^+}$		a variable
		$\overrightarrow{\overrightarrow{\alpha^+}}^i$		concatenate lists
$\overrightarrow{\alpha^-}, \overrightarrow{\beta^-}, \overrightarrow{\gamma^-}, \overrightarrow{\delta^-}$::=			negative variables
		\cdot		empty list
		α^-		a variable
		$\overrightarrow{\alpha^-}$		variables
		$\overrightarrow{\overrightarrow{\alpha^-}}^i$		concatenate lists
$\overrightarrow{\alpha^\pm}, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm}, \overrightarrow{\delta^\pm}$::=			positive or negative variable list
		\cdot		empty list
		α^\pm		a variable
		$\overrightarrow{\mathbf{pa}}$		variables
		$\overrightarrow{\overrightarrow{\alpha^\pm}}^i$		concatenate lists
P, Q, R	::=			multi-quantified positive types
		α^+		
		$\downarrow N$		
		$\exists \overrightarrow{\alpha^-}. P$		$P \neq \exists \dots$
		$[\sigma]P$	M	
		$[\hat{\tau}]P$	M	
		$[\hat{\sigma}]P$	M	
		$[\mu]P$	M	
		(P)	S	
		$P_1 \vee P_2$	M	
		$\mathbf{nf}(P')$	M	
N, M, K	::=			multi-quantified negative types
		α^-		
		$\uparrow P$		
		$P \rightarrow N$		

		$\forall \vec{\alpha}^+. N$		$N \neq \forall \dots$
		$[\sigma] N$	M	
		$[\hat{\tau}] N$	M	
		$[\mu] N$	M	
		$[\hat{\sigma}] N$	M	
		(N)	S	
		$\mathbf{nf}(N')$	M	
\vec{P}, \vec{Q}	::=			list of positive types
		\cdot		empty list
		P		a singel type
		$[\sigma] \vec{P}$	M	
		\vec{P}_i		concatenate lists
		(\vec{P})	S	
		$\mathbf{nf}(\vec{P}')$	M	
\vec{N}, \vec{M}	::=			list of negative types
		\cdot		empty list
		N		a singel type
		$[\sigma] \vec{N}$	M	
		\vec{N}_i		concatenate lists
		(\vec{N})	S	
		$\mathbf{nf}(\vec{N}')$	M	
Δ, Γ	::=			declarative type context
		\cdot		empty context
		$\vec{\alpha}^+$		list of variables
		$\vec{\alpha}^-$		list of variables
		$\vec{\alpha}^\pm$		list of variables
		$vars$		
		$\vec{\Gamma}_i$		concatenate contexts
		(Γ)	S	
		$\Theta(\hat{\alpha}^+)$	M	
		$\Theta(\hat{\alpha}^-)$	M	
		$\Gamma_1 \cup \Gamma_2$	M	
Θ	::=			unification type variable context
		\cdot		empty context
		$\vec{\alpha}\{\Delta\}$		from an ordered list of variables
		$\hat{\alpha}^+\{\Delta\}$		from a variable to a list
		$\vec{\Theta}_i$		concatenate contexts
		(Θ)	S	
		$\Theta _{vars}$		leave only those variables that are in the set
		$\Theta_1 \cup \Theta_2$		
Ξ	::=			anti-unification type variable context
		\cdot		empty context
		$\vec{\alpha}^-$		list of variables
		$\vec{\Xi}_i$		concatenate contexts
		(Ξ)	S	

		$\Xi_1 \cup \Xi_2$		
		$\Xi_1 \cap \Xi_2$		
		$\Xi'_{ vars}$	M	
$\vec{\alpha}, \vec{\beta}$::=			ordered positive or negative variables
		\cdot		empty list
		$\vec{\alpha}^+$		list of variables
		$\vec{\alpha}^-$		list of variables
		$\vec{\alpha}^\pm$		list of variables
		$\vec{\alpha}^+$		list of variables
		$\vec{\alpha}^-$		list of variables
		$\vec{\alpha}_1 \setminus vars$		setminus
		Γ		context
		$vars$		
		$\vec{\alpha}_i^i$		concatenate contexts
		$(\vec{\alpha})$	S	parenthesis
		$[\mu]\vec{\alpha}$	M	apply moving to list
		ord $vars$ in P	M	
		ord $vars$ in N	M	
		ord $vars$ in P	M	
		ord $vars$ in N	M	
$vars$::=			set of variables
		\emptyset		empty set
		fv P		free variables
		fv N		free variables
		fv imP		free variables
		fv imN		free variables
		$vars_1 \cap vars_2$		set intersection
		$vars_1 \cup vars_2$		set union
		$vars_1 \setminus vars_2$		set complement
		mv imP		movable variables
		mv imN		movable variables
		uv N		unification variables
		uv P		unification variables
		fv N		free variables
		fv P		free variables
		$(vars)$	S	parenthesis
		$\vec{\alpha}$		ordered list of variables
		$[\mu]vars$	M	apply moving to varset
		dom (UC)	M	
		dom (SC)	M	
		dom $(\hat{\sigma})$	M	
		dom $(\hat{\tau})$	M	
		dom (Θ)	M	
μ	::=			
		\cdot		empty moving
		$pma1 \mapsto pma2$		Positive unit substitution
		$nma1 \mapsto nma2$		Positive unit substitution

		$\mu_1 \cup \mu_2$	M	Set-like union of movings
		$\mu_1 \circ \mu_2$	M	Composition
		$\overline{\mu_i}^i$		concatenate movings
		$\mu _{vars}$	M	restriction on a set
		μ^{-1}	M	inversion
		$\mathbf{nf}(\mu')$	M	
$\hat{\alpha}^\pm$::=			positive/negative unification variable
		$\hat{\alpha}^\pm$		
$\hat{\alpha}^+$::=			positive unification variable
		$\hat{\alpha}^+$		
		$\hat{\alpha}^+\{\Delta\}$		
		$\hat{\alpha}^\pm$		
$\hat{\alpha}^-, \hat{\beta}^-$::=			negative unification variable
		$\hat{\alpha}^-$		
		$\hat{\alpha}^-_{\{N,M\}}$		
		$\hat{\alpha}^-\{\Delta\}$		
		$\hat{\alpha}^\pm$		
$\overrightarrow{\alpha}^+, \overrightarrow{\beta}^+$::=			positive unification variable list
		\cdot		empty list
		$\hat{\alpha}^+$		a variable
		$\overrightarrow{\hat{\alpha}^+}$		from a normal variable, context unspecified
		$\overrightarrow{\overrightarrow{\alpha}^+}^i$		concatenate lists
$\overrightarrow{\alpha}^-, \overrightarrow{\beta}^-$::=			negative unification variable list
		\cdot		empty list
		$\hat{\alpha}^-$		a variable
		$\overrightarrow{\Xi}$		from an antiunification context
		$\overrightarrow{\hat{\alpha}^-\{\Delta\}}$		from a normal variable
		$\overrightarrow{\hat{\alpha}^-}$		from a normal variable, context unspecified
		$\overrightarrow{\overrightarrow{\alpha}^-}^i$		concatenate lists
\boxed{P}, \boxed{Q}	::=			a positive algorithmic type (potentially with metavariables)
		α^+		
		\mathbf{pma}		
		$\hat{\alpha}^+$		
		$\downarrow \overrightarrow{N}$		
		$\exists \overrightarrow{\alpha}^-. \boxed{P}$		
		$[\sigma] \boxed{P}$	M	
		$[\hat{\tau}] \boxed{P}$	M	
		$[\mu] \boxed{P}$	M	
		(\boxed{P})	S	
		$\mathbf{nf}(\boxed{P}')$	M	
\boxed{N}, \boxed{M}	::=			a negative algorithmic type (potentially with metavariables)
		α^-		

		$\hat{\alpha}^-$	
		$\uparrow P$	
		$P \rightarrow N$	
		$\forall \vec{\alpha}^+. N$	
		$[\sigma] N$	M
		$[\hat{\tau}] N$	M
		$[\mu] N$	M
		(N)	S
		$\mathbf{nf}(N')$	M
$auSol$	$::=$		
		$(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$	
		$(\Xi, N, \hat{\tau}_1, \hat{\tau}_2)$	
$terminals$	$::=$		
		\exists	
		\forall	
		\uparrow	
		\downarrow	
		\rightarrow	
		\leftrightarrow	
		\in	
		\notin	
		\cdot	
		\top	
		\leq	
		\geq	
		\approx	
		\subset	
		\supset	
		\setminus	
		\sqsubseteq	
		\mapsto	
		\approx^u	
		\approx^a	
		\emptyset	
		\circ	
		\Rightarrow	
		\models	
		\models	
		\neq	
		\equiv_n	
		\vee	
		\Downarrow	
		$:\geq$	
		$:\approx$	
		Λ	
		λ	
		\mathbf{let}^\exists	
		\bullet	

	\Rightarrow \Leftarrow	
v, w	$::=$ x $\{c\}$ $(v : P)$ (v)	value terms M
\vec{v}	$::=$ $.$ v \vec{v}_i^i	list of arguments concatenate
c, d	$::=$ $(c : N)$ $\lambda x : P. c$ $\Lambda \alpha^+. c$ return v let $x = v; c$ let $x : P = v(\vec{v}); c$ let $x = v(\vec{v}); c$ let ³ $(\alpha^-, x) = v; c$	computation terms
$vctx, \Phi$	$::=$ $.$ $x : P$ Φ_i^i	variable context concatenate contexts
<i>formula</i>	$::=$ <i>judgement</i> <i>judgement</i> unique <i>formula</i> ₁ .. <i>formula</i> _n $\mu : vars_1 \leftrightarrow vars_2$ μ is bijective $x : P \in \Phi$ $UC_1 \subseteq UC_2$ $UC_1 = UC_2$ $SC_1 \subseteq SC_2$ $vars_1 \subseteq vars_2$ $vars_1 = vars_2$ vars is fresh $\alpha^- \notin vars$ $\alpha^+ \notin vars$ $\alpha^- \in vars$ $\alpha^+ \in vars$ $\hat{\alpha}^+ \in vars$ $\hat{\alpha}^- \in vars$ $\hat{\alpha}^- \in \Theta$ $\hat{\alpha}^+ \in \Theta$	

	$ \begin{array}{ l} \text{if any other rule is not applicable} \\ \vec{\alpha}_1 = \vec{\alpha}_2 \\ e_1 = e_2 \\ e_1 = e_2 \\ \hat{\sigma}_1 = \hat{\sigma}_2 \\ N = M \\ \Theta \subseteq \Theta' \\ \vec{v}_1 = \vec{v}_2 \\ N \neq M \\ P \neq Q \\ N \neq M \\ P \neq Q \\ P \neq Q \\ N \neq M \\ \vec{v}_1 \neq \vec{v}_2 \\ \alpha_1^+ \neq \alpha_2^+ \end{array} $	
A	$ \begin{array}{ l} \Gamma; \Theta \models N \leq M \Rightarrow SC \\ \Gamma; \Theta \models P \geq Q \Rightarrow SC \end{array} $	Negative subtyping Positive supertyping
AT	$ \begin{array}{ l} \Gamma; \Phi \models v : P \\ \Gamma; \Phi \models c : N \\ \Gamma; \Phi; \Theta_1 \models N \bullet \vec{v} \Rightarrow M \Rightarrow \Theta_2; SC \end{array} $	Positive type inference Negative type inference Application type inference
AU	$ \begin{array}{ l} \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \\ \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, M, \hat{\tau}_1, \hat{\tau}_2) \end{array} $	
SCM	$ \begin{array}{ l} \Gamma \vdash e_1 \& e_2 = e_3 \\ \Theta \vdash SC_1 \& SC_2 = SC_3 \end{array} $	Subtyping Constraint Entry Merge Merge of subtyping constraints
UCM	$ \begin{array}{ l} \Gamma \vdash e_1 \& e_2 = e_3 \\ \Theta \vdash UC_1 \& UC_2 = UC_3 \end{array} $	
$SATSCE$	$ \begin{array}{ l} \Gamma \vdash P : e \\ \Gamma \vdash N : e \end{array} $	Positive type satisfies with the subtyping constraint Negative type satisfies with the subtyping constraint
$SING$	$ \begin{array}{ l} e_1 \text{ singular with } P \\ e_1 \text{ singular with } N \\ SC \text{ singular with } \hat{\sigma} \end{array} $	Positive Subtyping Constraint Entry Is Singular Negative Subtyping Constraint Entry Is Singular Subtyping Constraint Is Singular
$E1$	$ \begin{array}{ l} N \simeq_1^D M \end{array} $	Negative multi-quantified type equivalence

		$P \simeq_1^D Q$	Positive multi-quantified type equivalence
		$\boxed{P} \simeq_1^D \boxed{Q}$	Positive unification type equivalence
		$N \simeq_1^D M$	Positive unification type equivalence
$D1$	$::=$		
		$\Gamma \vdash N \simeq_1^{\leq} M$	Negative equivalence on MQ types
		$\Gamma \vdash P \simeq_1^{\leq} Q$	Positive equivalence on MQ types
		$\Gamma \vdash N \leq_1 M$	Negative subtyping
		$\Gamma \vdash P \geq_1 Q$	Positive supertyping
		$\Gamma_2 \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : \Gamma_1$	Equivalence of substitutions
		$\Gamma \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : vars$	Equivalence of substitutions
		$\Theta \vdash \hat{\sigma}_1 \simeq_1^{\leq} \hat{\sigma}_2 : vars$	Equivalence of unification substitutions
		$\Gamma \vdash \Phi_1 \simeq_1^{\leq} \Phi_2$	Equivalence of contexts
$D0$	$::=$		
		$\Gamma \vdash N \simeq_0^{\leq} M$	Negative equivalence
		$\Gamma \vdash P \simeq_0^{\leq} Q$	Positive equivalence
		$\Gamma \vdash N \leq_0 M$	Negative subtyping
		$\Gamma \vdash P \geq_0 Q$	Positive supertyping
DT	$::=$		
		$\Gamma; \Phi \vdash v : P$	Positive type inference
		$\Gamma; \Phi \vdash c : N$	Negative type inference
		$\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M$	Application type inference
EQ	$::=$		
		$N = M$	Negative type equality (alpha-equivalence)
		$P = Q$	Positive type equality (alpha-equivalence)
		$\boxed{P} = \boxed{Q}$	
$LUBF$	$::=$		
		$P_1 \vee P_2 === Q$	
		ord $vars$ in $P === \vec{\alpha}$	
		ord $vars$ in $N === \vec{\alpha}$	
		ord $vars$ in $P === \vec{\alpha}$	
		ord $vars$ in $N === \vec{\alpha}$	
		nf $(N') === N$	
		nf $(P') === P$	
		nf $(N') === \boxed{N}$	
		nf $(P') === \boxed{P}$	
		nf $(\vec{N}') === \vec{N}$	
		nf $(\vec{P}') === \vec{P}$	
		nf $(\sigma') === \sigma$	
		nf $(\hat{\sigma}') === \hat{\sigma}$	
		nf $(\mu') === \mu$	
		$\sigma' _{vars}$	
		$\hat{\sigma}' _{vars}$	
		$\hat{\tau}' _{vars}$	
		$\Xi' _{vars}$	
		$SC _{vars}$	

		$UC _{vars}$	
		$e_1 \& e_2$	
		$e_1 \& e_2$	
		$UC_1 \& UC_2$	
		$UC_1 \cup UC_2$	
		$\Gamma_1 \cup \Gamma_2$	
		$SC_1 \& SC_2$	
		$\hat{\tau}_1 \& \hat{\tau}_2$	
		$\mathbf{dom}(UC) === vars$	
		$\mathbf{dom}(SC) === vars$	
		$\mathbf{dom}(\hat{\sigma}) === vars$	
		$\mathbf{dom}(\hat{\tau}) === vars$	
		$\mathbf{dom}(\Theta) === vars$	
		$ SC === UC$	
LUB	::=		
		$\Gamma \models P_1 \vee P_2 = Q$	Least Upper Bound (Least Common Supertype)
		$\mathbf{upgrade} \Gamma \vdash P \mathbf{to} \Delta = Q$	
Nrm	::=		
		$\mathbf{nf}(N) = M$	
		$\mathbf{nf}(P) = Q$	
		$\mathbf{nf}(N) = \textcolor{gray}{M}$	
		$\mathbf{nf}(P) = \textcolor{gray}{Q}$	
$Order$::=		
		$\mathbf{ord} \mathit{vars} \mathbf{in} N = \vec{\alpha}$	
		$\mathbf{ord} \mathit{vars} \mathbf{in} P = \vec{\alpha}$	
		$\mathbf{ord} \mathit{vars} \mathbf{in} \textcolor{gray}{N} = \vec{\alpha}$	
		$\mathbf{ord} \mathit{vars} \mathbf{in} \textcolor{gray}{P} = \vec{\alpha}$	
U	::=		
		$\Gamma; \Theta \models N \overset{u}{\simeq} M \Rightarrow UC$	Negative unification
		$\Gamma; \Theta \models \textcolor{gray}{P} \overset{u}{\simeq} Q \Rightarrow UC$	Positive unification
WF	::=		
		$\Gamma \vdash N$	Negative type well-formedness
		$\Gamma \vdash P$	Positive type well-formedness
		$\Gamma \vdash N$	Negative type well-formedness
		$\Gamma \vdash P$	Positive type well-formedness
		$\Gamma \vdash \vec{N}$	Negative type list well-formedness
		$\Gamma \vdash \vec{P}$	Positive type list well-formedness
		$\Gamma; \Theta \vdash \textcolor{gray}{N}$	Negative unification type well-formedness
		$\Gamma; \Theta \vdash \textcolor{gray}{P}$	Positive unification type well-formedness
		$\Gamma; \Xi \vdash \textcolor{gray}{N}$	Negative anti-unification type well-formedness
		$\Gamma; \Xi \vdash \textcolor{gray}{P}$	Positive anti-unification type well-formedness
		$\Gamma; \Xi_2 \vdash \hat{\tau} : \Xi_1$	Antiunification substitution well-formedness
		$\Gamma \vdash^{\supseteq} \Theta$	Unification context well-formedness
		$\Gamma_1 \vdash \sigma : \Gamma_2$	Substitution well-formedness
		$\Theta \vdash \hat{\sigma}$	Unification substitution well-formedness

		$\Theta \vdash \hat{\sigma} : UC$	Unification substitution satisfies unification constraint
		$\Theta \vdash \hat{\sigma} : SC$	Unification substitution satisfies subtyping constraint
		$\Gamma \vdash e$	Unification constraint entry well-formedness
		$\Gamma \vdash e$	Subtyping constraint entry well-formedness
		$\Gamma \vdash P : e$	Positive type satisfies unification constraint
		$\Gamma \vdash N : e$	Negative type satisfies unification constraint
		$\Gamma \vdash P : e$	Positive type satisfies subtyping constraint
		$\Gamma \vdash N : e$	Negative type satisfies subtyping constraint
		$\Theta \vdash UC$	Unification constraint well-formedness
		$\Theta \vdash SC$	Subtyping constraint well-formedness
		$\Gamma \vdash \vec{v}$	Argument List well-formedness
		$\Gamma \vdash \Phi$	Context well-formedness
		$\Gamma \vdash v$	Value well-formedness
		$\Gamma \vdash c$	Computation well-formedness
<i>judgement</i>	::=		
		A	
		AT	
		AU	
		SCM	
		UCM	
		$SATSCE$	
		$SING$	
		$E1$	
		$D1$	
		$D0$	
		DT	
		EQ	
		LUB	
		Nrm	
		$Order$	
		U	
		WF	
<i>user_syntax</i>	::=		
		α	
		n	
		x	
		n	
		α^+	
		α^-	
		α^\pm	
		σ	
		e	
		e	
		UC	
		SC	
		$\hat{\sigma}$	
		$\hat{\tau}$	
		P	

N
 $\overrightarrow{\alpha^+}$
 $\overrightarrow{\alpha^-}$
 $\overrightarrow{\alpha^\pm}$
 P
 N
 \vec{P}
 \vec{N}
 Γ
 Θ
 Ξ
 $\vec{\alpha}$
 $vars$
 μ
 $\hat{\alpha}^\pm$
 $\hat{\alpha}^+$
 $\hat{\alpha}^-$
 $\widetilde{\alpha^+}$
 $\widetilde{\alpha^-}$
 P
 N
 $auSol$
 $terminals$
 v
 \vec{v}
 c
 $vctx$
 $formula$

$\boxed{\Gamma; \Theta \models N \leq M \Rightarrow SC}$ Negative subtyping

$$\begin{array}{c}
\overline{\Gamma; \Theta \models \alpha^- \leq \alpha^- \Rightarrow} \quad \text{ANVAR} \\
\\
\frac{\Gamma; \Theta \models \mathbf{nf}(P) \stackrel{u}{\simeq} \mathbf{nf}(Q) \Rightarrow UC}{\Gamma; \Theta \models \uparrow P \leq \uparrow Q \Rightarrow UC} \quad \text{AShiftU} \\
\\
\frac{\Gamma; \Theta \models P \geq Q \Rightarrow SC_1 \quad \Gamma; \Theta \models N \leq M \Rightarrow SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC}{\Gamma; \Theta \models P \rightarrow N \leq Q \rightarrow M \Rightarrow SC} \quad \text{AArrow} \\
\\
\frac{\text{<<multiple parses>>}}{\Gamma; \Theta \models \forall \alpha^+. N \leq \forall \beta^+. M \Rightarrow SC \setminus \hat{\alpha}^+} \quad \text{AForall}
\end{array}$$

$\boxed{\Gamma; \Theta \models P \geq Q \Rightarrow SC}$ Positive supertyping

$$\begin{array}{c}
\overline{\Gamma; \Theta \models \alpha^+ \geq \alpha^+ \Rightarrow} \quad \text{APVar} \\
\\
\frac{\Gamma; \Theta \models \mathbf{nf}(\vec{N}) \stackrel{u}{\simeq} \mathbf{nf}(\vec{M}) \Rightarrow UC}{\Gamma; \Theta \models \downarrow \vec{N} \geq \downarrow \vec{M} \Rightarrow UC} \quad \text{AShiftD} \\
\\
\frac{\Gamma, \vec{\beta}^-; \Theta, \vec{\alpha}^- \{ \Gamma, \vec{\beta}^- \} \models [\vec{\alpha}^- / \alpha^-] P \geq Q \Rightarrow SC}{\Gamma; \Theta \models \exists \alpha^-. P \geq \exists \beta^-. Q \Rightarrow SC \setminus \hat{\alpha}^-} \quad \text{AExists}
\end{array}$$

$$\begin{array}{c}
\frac{\hat{\alpha}^+\{\Delta\} \in \Theta \quad \mathbf{upgrade} \Gamma \vdash P \text{ to } \Delta = Q}{\Gamma; \Theta \models \hat{\alpha}^+ \succcurlyeq P \Rightarrow (\hat{\alpha}^+ : \geq Q)} \quad \text{APUVAR} \\
\\
\boxed{\Gamma; \Phi \models v : P} \quad \text{Positive type inference} \\
\\
\frac{x : P \in \Phi}{\Gamma; \Phi \models x : \mathbf{nf}(P)} \quad \text{ATVAR} \\
\\
\frac{\Gamma; \Phi \models c : N}{\Gamma; \Phi \models \{c\} : \downarrow N} \quad \text{ATT HUNK} \\
\\
\frac{\Gamma \vdash Q \quad \Gamma; \Phi \models v : P \quad \Gamma; \cdot \models Q \succcurlyeq P \Rightarrow \cdot}{\Gamma; \Phi \models (v : Q) : \mathbf{nf}(Q)} \quad \text{ATPANNOT} \\
\\
\boxed{\Gamma; \Phi \models c : N} \quad \text{Negative type inference} \\
\\
\frac{\Gamma \vdash M \quad \Gamma; \Phi \models c : N \quad \Gamma; \cdot \models N \preccurlyeq M \Rightarrow \cdot}{\Gamma; \Phi \models (c : M) : \mathbf{nf}(M)} \quad \text{ATNANNOT} \\
\\
\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \lambda x : P. c : \mathbf{nf}(P \rightarrow N)} \quad \text{ATTLAM} \\
\\
\frac{\Gamma, \alpha^+; \Phi \models c : N}{\Gamma; \Phi \models \Lambda \alpha^+. c : \mathbf{nf}(\forall \alpha^+. N)} \quad \text{ATTLAM} \\
\\
\frac{\Gamma; \Phi \models v : P}{\Gamma; \Phi \models \mathbf{return} v : \uparrow P} \quad \text{ATRETURN} \\
\\
\frac{\Gamma; \Phi \models v : P \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \mathbf{let} x = v; c : N} \quad \text{ATVARLET} \\
\\
\frac{\begin{array}{l} \Gamma \vdash P \quad \Gamma; \Phi \models v : \downarrow M \\ \Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow \uparrow Q \Rightarrow \Theta; SC_1 \quad \Gamma; \Theta \models \uparrow Q \preccurlyeq \uparrow P \Rightarrow SC_2 \\ \Theta \vdash SC_1 \& SC_2 = SC \quad \Gamma; \Phi, x : P \models c : N \end{array}}{\Gamma; \Phi \models \mathbf{let} x : P = v(\vec{v}); c : N} \quad \text{ATAPPLETANN} \\
\\
\frac{\begin{array}{l} \Gamma; \Phi \models v : \downarrow M \quad \Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow \uparrow Q \Rightarrow \Theta; SC \\ \text{<<multiple parses>>} \\ \Gamma; \Phi, x : [\hat{\sigma}] Q \models c : N \end{array}}{\Gamma; \Phi \models \mathbf{let} x = v(\vec{v}); c : N} \quad \text{ATAPPLET} \\
\\
\frac{\begin{array}{l} \Gamma; \Phi \models v : \exists \alpha^-. P \quad \Gamma, \alpha^+; \Phi, x : P \models c : N \quad \Gamma \vdash N \\ \vec{v} \neq \cdot \quad \alpha^+ \neq \cdot \end{array}}{\Gamma; \Phi \models \mathbf{let}^3(\alpha^-, x) = v; c : N} \quad \text{ATUNPACK} \\
\\
\boxed{\Gamma; \Phi; \Theta_1 \models N \bullet \vec{v} \Rightarrow M \Rightarrow \Theta_2; SC} \quad \text{Application type inference} \\
\\
\frac{}{\Gamma; \Phi; \Theta \models N \bullet \cdot \Rightarrow \mathbf{nf}(N) \Rightarrow \Theta; \cdot} \quad \text{ATEMPTYAPP} \\
\\
\frac{\begin{array}{l} \Gamma; \Phi \models v : P \quad \Gamma; \Theta \models Q \succcurlyeq P \Rightarrow SC_1 \quad \Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rightarrow M \Rightarrow \Theta'; SC_2 \\ \Theta \vdash SC_1 \& SC_2 = SC \end{array}}{\Gamma; \Phi; \Theta \models Q \rightarrow N \bullet v, \vec{v} \Rightarrow M \Rightarrow \Theta'; SC} \quad \text{ATARROWAPP} \\
\\
\frac{\begin{array}{l} \text{<<multiple parses>>} \\ \vec{v} \neq \cdot \quad \alpha^+ \neq \cdot \end{array}}{\Gamma; \Phi; \Theta \models \forall \alpha^+. N \bullet \vec{v} \Rightarrow M \Rightarrow \Theta'; SC} \quad \text{ATFORALLAPP} \\
\\
\boxed{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}
\end{array}$$

$$\begin{array}{c}
\frac{}{\Gamma \vdash \alpha^+ \overset{a}{\simeq} \alpha^+ = (\cdot, \alpha^+, \cdot, \cdot)} \text{AUPVAR} \\
\frac{\Gamma \vdash N_1 \overset{a}{\simeq} N_2 = (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vdash \downarrow N_1 \overset{a}{\simeq} \downarrow N_2 = (\Xi, \downarrow M, \hat{\tau}_1, \hat{\tau}_2)} \text{AUSHIFTD} \\
\frac{\overrightarrow{\alpha^-} \cap \Gamma = \emptyset \quad \Gamma \vdash P_1 \overset{a}{\simeq} P_2 = (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vdash \overrightarrow{\exists \alpha^-}. P_1 \overset{a}{\simeq} \overrightarrow{\exists \alpha^-}. P_2 = (\Xi, \overrightarrow{\exists \alpha^-}. Q, \hat{\tau}_1, \hat{\tau}_2)} \text{AUEXISTS} \\
\boxed{\Gamma \vdash N_1 \overset{a}{\simeq} N_2 = (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)} \\
\frac{}{\Gamma \vdash \alpha^- \overset{a}{\simeq} \alpha^- = (\cdot, \alpha^-, \cdot, \cdot)} \text{AUNVAR} \\
\frac{\Gamma \vdash P_1 \overset{a}{\simeq} P_2 = (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vdash \uparrow P_1 \overset{a}{\simeq} \uparrow P_2 = (\Xi, \uparrow Q, \hat{\tau}_1, \hat{\tau}_2)} \text{AUSHIFTU} \\
\frac{\overrightarrow{\alpha^+} \cap \Gamma = \emptyset \quad \Gamma \vdash N_1 \overset{a}{\simeq} N_2 = (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \vdash \overrightarrow{\forall \alpha^+}. N_1 \overset{a}{\simeq} \overrightarrow{\forall \alpha^+}. N_2 = (\Xi, \overrightarrow{\forall \alpha^+}. M, \hat{\tau}_1, \hat{\tau}_2)} \text{AUFORALL} \\
\frac{\Gamma \vdash P_1 \overset{a}{\simeq} P_2 = (\Xi_1, Q, \hat{\tau}_1, \hat{\tau}_2) \quad \Gamma \vdash N_1 \overset{a}{\simeq} N_2 = (\Xi_2, M, \hat{\tau}'_1, \hat{\tau}'_2)}{\Gamma \vdash P_1 \rightarrow N_1 \overset{a}{\simeq} P_2 \rightarrow N_2 = (\Xi_1 \cup \Xi_2, Q \rightarrow M, \hat{\tau}_1 \cup \hat{\tau}'_1, \hat{\tau}_2 \cup \hat{\tau}'_2)} \text{AUARROW} \\
\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \vdash N \overset{a}{\simeq} M = (\hat{\alpha}_{\{N,M\}}^-, \hat{\alpha}_{\{N,M\}}^-, (\hat{\alpha}_{\{N,M\}}^- : \approx N), (\hat{\alpha}_{\{N,M\}}^- : \approx M))} \text{AUAU} \\
\boxed{\Gamma \vdash e_1 \& e_2 = e_3} \quad \text{Subtyping Constraint Entry Merge} \\
\frac{\Gamma \vdash P_1 \vee P_2 = Q}{\Gamma \vdash (\hat{\alpha}^+ : \geq P_1) \& (\hat{\alpha}^+ : \geq P_2) = (\hat{\alpha}^+ : \geq Q)} \text{SCMESUPSUP} \\
\frac{\Gamma; \cdot \vdash P \geq Q = \cdot}{\Gamma \vdash (\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \geq Q) = (\hat{\alpha}^+ : \approx P)} \text{SCMEEQSUP} \\
\frac{\Gamma; \cdot \vdash Q \geq P = \cdot}{\Gamma \vdash (\hat{\alpha}^+ : \geq P) \& (\hat{\alpha}^+ : \approx Q) = (\hat{\alpha}^+ : \approx Q)} \text{SCMESUPEQ} \\
\frac{\text{<<multiple parses>>}}{\Gamma \vdash (\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \approx P') = (\hat{\alpha}^+ : \approx P)} \text{SCMEPEQEQ} \\
\frac{\text{<<multiple parses>>}}{\Gamma \vdash (\hat{\alpha}^- : \approx N_1) \& (\hat{\alpha}^- : \approx N') = (\hat{\alpha}^- : \approx N)} \text{SCMENEQEQ} \\
\boxed{\Theta \vdash SC_1 \& SC_2 = SC_3} \quad \text{Merge of subtyping constraints} \\
\boxed{\Gamma \vdash e_1 \& e_2 = e_3} \\
\frac{\text{<<multiple parses>>}}{\Gamma \vdash (\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \approx P') = (\hat{\alpha}^+ : \approx P)} \text{UCMEPEQEQ} \\
\frac{\text{<<multiple parses>>}}{\Gamma \vdash (\hat{\alpha}^- : \approx N_1) \& (\hat{\alpha}^- : \approx N') = (\hat{\alpha}^- : \approx N)} \text{UCMENEQEQ} \\
\boxed{\Theta \vdash UC_1 \& UC_2 = UC_3} \\
\boxed{\Gamma \vdash P : e} \quad \text{Positive type satisfies with the subtyping constraint entry}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash P \geq_1 Q}{\Gamma \vdash P : (\hat{\alpha}^+ : \geq Q)} \text{ SATSCESUP} \\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\Gamma \vdash P : (\hat{\alpha}^+ : \approx Q)} \text{ SATSCEPEQ} \\
\boxed{\Gamma \vdash N : e} \quad \text{Negative type satisfies with the subtyping constraint entry} \\
\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\Gamma \vdash N : (\hat{\alpha}^- : \approx M)} \text{ SATSCENEQ} \\
\boxed{e_1 \text{ singular with } P} \quad \text{Positive Subtyping Constraint Entry Is Singular} \\
\frac{}{\hat{\alpha}^+ : \approx P \text{ singular with nf } (P)} \text{ SINGPEQ} \\
\frac{}{\hat{\alpha}^+ : \geq \exists \alpha^- . \alpha^+ \text{ singular with } \alpha^+} \text{ SINGSUPVAR} \\
\frac{N \simeq_1^D \alpha_i^-}{\hat{\alpha}^+ : \geq \exists \alpha^- . \downarrow N \text{ singular with } \exists \alpha^- . \downarrow \alpha^-} \text{ SINGSUPSHIFT} \\
\boxed{e_1 \text{ singular with } N} \quad \text{Negative Subtyping Constraint Entry Is Singular} \\
\frac{}{\hat{\alpha}^- : \approx N \text{ singular with nf } (N)} \text{ SINGNEQ} \\
\boxed{SC \text{ singular with } \hat{\sigma}} \quad \text{Subtyping Constraint Is Singular} \\
\boxed{N \simeq_1^D M} \quad \text{Negative multi-quantified type equivalence} \\
\frac{}{\alpha^- \simeq_1^D \alpha^-} \text{ E1NVAR} \\
\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \text{ E1SHIFTU} \\
\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \rightarrow N \simeq_1^D Q \rightarrow M} \text{ E1ARROW} \\
\frac{\vec{\alpha}^+ \cap \mathbf{fv} M = \emptyset \quad \mu : (\vec{\beta}^+ \cap \mathbf{fv} M) \leftrightarrow (\vec{\alpha}^+ \cap \mathbf{fv} N) \quad N \simeq_1^D [\mu]M}{\forall \alpha^+ . N \simeq_1^D \forall \beta^+ . M} \text{ E1FORALL} \\
\boxed{P \simeq_1^D Q} \quad \text{Positive multi-quantified type equivalence} \\
\frac{}{\alpha^+ \simeq_1^D \alpha^+} \text{ E1PVAR} \\
\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \text{ E1SHIFTD} \\
\frac{\vec{\alpha}^+ \cap \mathbf{fv} Q = \emptyset \quad \mu : (\vec{\beta}^- \cap \mathbf{fv} Q) \leftrightarrow (\vec{\alpha}^+ \cap \mathbf{fv} P) \quad P \simeq_1^D [\mu]Q}{\exists \alpha^- . P \simeq_1^D \exists \beta^- . Q} \text{ E1EXISTS} \\
\boxed{P \simeq_1^D Q} \quad \text{Positive unification type equivalence} \\
\boxed{N \simeq_1^D M} \quad \text{Positive unification type equivalence} \\
\boxed{\Gamma \vdash N \simeq_1^{\leq} M} \quad \text{Negative equivalence on MQ types}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \quad \text{D1NDEF} \\
\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q} \quad \text{Positive equivalence on MQ types} \\
\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\geqslant} Q} \quad \text{D1PDEF} \\
\boxed{\Gamma \vdash N \leqslant_1 M} \quad \text{Negative subtyping} \\
\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \quad \text{D1NVAR} \\
\frac{\text{<<multiple parses>>}}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q} \quad \text{D1SHIFTU} \\
\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \rightarrow N \leqslant_1 Q \rightarrow M} \quad \text{D1ARROW} \\
\frac{\mathbf{fv} N \cap \vec{\beta}^+ = \emptyset \quad \Gamma, \vec{\beta}^+ \vdash P_i \quad \Gamma, \vec{\beta}^+ \vdash [\vec{P}/\alpha^+] N \leqslant_1 M}{\Gamma \vdash \forall \alpha^+. N \leqslant_1 \forall \vec{\beta}^+. M} \quad \text{D1FORALL} \\
\boxed{\Gamma \vdash P \geqslant_1 Q} \quad \text{Positive supertyping} \\
\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+} \quad \text{D1PVAR} \\
\frac{\text{<<multiple parses>>}}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M} \quad \text{D1SHIFTD} \\
\frac{\mathbf{fv} P \cap \vec{\beta}^- = \emptyset \quad \Gamma, \vec{\beta}^- \vdash N_i \quad \Gamma, \vec{\beta}^- \vdash [\vec{N}/\alpha^-] P \geqslant_1 Q}{\Gamma \vdash \exists \alpha^-. P \geqslant_1 \exists \vec{\beta}^-. Q} \quad \text{D1EXISTS} \\
\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : \Gamma_1} \quad \text{Equivalence of substitutions} \\
\boxed{\Gamma \vdash \sigma_1 \simeq_1^{\leqslant} \sigma_2 : vars} \quad \text{Equivalence of substitutions} \\
\boxed{\Theta \vdash \hat{\sigma}_1 \simeq_1^{\leqslant} \hat{\sigma}_2 : vars} \quad \text{Equivalence of unification substitutions} \\
\boxed{\Gamma \vdash \Phi_1 \simeq_1^{\leqslant} \Phi_2} \quad \text{Equivalence of contexts} \\
\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M} \quad \text{Negative equivalence} \\
\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M} \quad \text{D0NDEF} \\
\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q} \quad \text{Positive equivalence} \\
\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\geqslant} Q} \quad \text{D0PDEF} \\
\boxed{\Gamma \vdash N \leqslant_0 M} \quad \text{Negative subtyping} \\
\frac{}{\Gamma \vdash \alpha^- \leqslant_0 \alpha^-} \quad \text{D0NVAR} \\
\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \quad \text{D0SHIFTU} \\
\frac{\Gamma \vdash P \quad \Gamma \vdash [P/\alpha^+] N \leqslant_0 M \quad M \neq \forall \beta^+. M'}{\Gamma \vdash \forall \alpha^+. N \leqslant_0 M} \quad \text{D0FORALLL}
\end{array}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leq_0 M}{\Gamma \vdash N \leq_0 \forall \alpha^+. M} \quad \text{D0FORALLR}$$

$$\frac{\Gamma \vdash P \geq_0 Q \quad \Gamma \vdash N \leq_0 M}{\Gamma \vdash P \rightarrow N \leq_0 Q \rightarrow M} \quad \text{D0ARROW}$$

$\boxed{\Gamma \vdash P \geq_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geq_0 \alpha^+} \quad \text{D0PVAR}$$

$$\frac{\Gamma \vdash N \simeq_0^s M}{\Gamma \vdash \downarrow N \geq_0 \downarrow M} \quad \text{D0SHIFTD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/\alpha^-]P \geq_0 Q \quad Q \neq \exists \alpha^-. Q'}{\Gamma \vdash \exists \alpha^-. P \geq_0 Q} \quad \text{D0EXISTS L}$$

$$\frac{\Gamma, \alpha^- \vdash P \geq_0 Q}{\Gamma \vdash P \geq_0 \exists \alpha^-. Q} \quad \text{D0EXISTS R}$$

$\boxed{\Gamma; \Phi \vdash v : P}$ Positive type inference

$$\frac{x : P \in \Phi}{\Gamma; \Phi \vdash x : P} \quad \text{DTVAR}$$

$$\frac{\Gamma; \Phi \vdash c : N}{\Gamma; \Phi \vdash \{c\} : \downarrow N} \quad \text{DTTHUNK}$$

$$\frac{\Gamma \vdash Q \quad \Gamma; \Phi \vdash v : P \quad \Gamma \vdash Q \geq_1 P}{\Gamma; \Phi \vdash (v : Q) : Q} \quad \text{DTPANNOT}$$

$$\frac{\text{<<multiple parses>>}}{\Gamma; \Phi \vdash v : P'} \quad \text{DTPEQUIV}$$

$\boxed{\Gamma; \Phi \vdash c : N}$ Negative type inference

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \lambda x : P. c : P \rightarrow N} \quad \text{DTTLAM}$$

$$\frac{\Gamma, \alpha^+; \Phi \vdash c : N}{\Gamma; \Phi \vdash \Lambda \alpha^+. c : \forall \alpha^+. N} \quad \text{DTTLAM}$$

$$\frac{\Gamma; \Phi \vdash v : P}{\Gamma; \Phi \vdash \mathbf{return} v : \uparrow P} \quad \text{DTRETURN}$$

$$\frac{\Gamma; \Phi \vdash v : P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x = v; c : N} \quad \text{DTVARLET}$$

$$\frac{\Gamma; \Phi \vdash v : \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q \text{ unique} \quad \Gamma; \Phi, x : Q \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x = v(\vec{v}); c : N} \quad \text{DTAPPLET}$$

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi \vdash v : \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q \quad \Gamma \vdash \uparrow Q \leq_1 \uparrow P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x : P = v(\vec{v}); c : N} \quad \text{DTAPPLETANN}$$

$$\frac{\text{<<multiple parses>>}}{\Gamma, \alpha^-; \Phi, x : P \vdash c : N \quad \Gamma \vdash N} \quad \text{DTUNPACK}$$

$$\frac{}{\Gamma; \Phi \vdash \mathbf{let}^3(\alpha^-, x) = v; c : N} \quad \text{DTUNPACK}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\Gamma; \Phi \vdash (c : M) : M} \quad \text{DTNANNOT}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\Gamma; \Phi \vdash c : N'} \quad \text{DTNEQUIV}$$

$$\boxed{\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M} \quad \text{Application type inference}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\Gamma; \Phi \vdash N \bullet \cdot \Rightarrow N'} \quad \text{DTEMTYAPP}$$

$$\frac{\Gamma; \Phi \vdash v : P \quad \Gamma \vdash Q \geqslant_1 P \quad \Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M}{\Gamma; \Phi \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M} \quad \text{DTARROWAPP}$$

$$\frac{\begin{array}{c} \Gamma \vdash \sigma : \vec{\alpha}^+ \quad \Gamma; \Phi \vdash [\sigma]N \bullet \vec{v} \Rightarrow M \\ \vec{v} \neq \cdot \quad \vec{\alpha}^+ \neq \cdot \end{array}}{\Gamma; \Phi \vdash \vec{\forall \alpha^+}. N \bullet \vec{v} \Rightarrow M} \quad \text{DTFORALLAPP}$$

$$\boxed{N = M} \quad \text{Negative type equality (alpha-equivalence)}$$

$$\boxed{P = Q} \quad \text{Positive type equality (alphha-equivalence)}$$

$$\boxed{P = Q}$$

$$\boxed{P_1 \vee P_2}$$

$$\boxed{\text{ord vars in } P}$$

$$\boxed{\text{ord vars in } N}$$

$$\boxed{\text{ord vars in } P}$$

$$\boxed{\text{ord vars in } N}$$

$$\boxed{\text{nf}(N')}$$

$$\boxed{\text{nf}(P')}$$

$$\boxed{\text{nf}(N')}$$

$$\boxed{\text{nf}(P')}$$

$$\boxed{\text{nf}(\vec{N'})}$$

$$\mathbf{nf}\left(\vec{P}'\right)$$

$$\mathbf{nf}\left(\sigma'\right)$$

$$\mathbf{nf}\left(\hat{\sigma}'\right)$$

$$\mathbf{nf}\left(\mu'\right)$$

$$\sigma'|_{vars}$$

$$\hat{\sigma}'|_{vars}$$

$$\hat{\tau}'|_{vars}$$

$$\Xi'|_{vars}$$

$$\mathbf{SC}|_{vars}$$

$$\mathbf{UC}|_{vars}$$

$$e_1 \ \& \ e_2$$

$$e_1 \ \& \ e_2$$

$$UC_1 \ \& \ UC_2$$

$$UC_1 \cup UC_2$$

$$\Gamma_1 \cup \Gamma_2$$

$$\boxed{SC_1 \ \& \ SC_2}$$

$$\boxed{\hat{\tau}_1 \ \& \ \hat{\tau}_2}$$

$$\boxed{\mathbf{dom} \, (UC)}$$

$$\boxed{\mathbf{dom} \, (SC)}$$

$$\boxed{\mathbf{dom} \, (\hat{\sigma})}$$

$$\boxed{\mathbf{dom} \, (\hat{\tau})}$$

$$\boxed{\mathbf{dom} \, (\Theta)}$$

$$\boxed{\|SC\|}$$

$$\boxed{\Gamma \models P_1 \vee P_2 = Q} \quad \text{Least Upper Bound (Least Common Supertype)}$$

$$\frac{\overline{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVAR}}{\Gamma, \cdot \models \mathbf{nf} \, (\downarrow N) \stackrel{a}{\simeq} \mathbf{nf} \, (\downarrow M) \Rightarrow (\Xi, P, \hat{\tau}_1, \hat{\tau}_2) \quad \text{LUBSHIFT}} \quad \frac{\Gamma \models \downarrow N \vee \downarrow M = \exists \alpha^-. [\alpha^- / \Xi] P}{\Gamma \models \downarrow N \vee \downarrow M = \exists \alpha^-. [\alpha^- / \Xi] P} \quad \text{LUBEXISTS}$$

$$\boxed{\mathbf{upgrade} \Gamma \vdash P \text{ to } \Delta = Q}$$

$$\frac{\begin{array}{l} \Gamma = \Delta, \overrightarrow{\alpha^\pm} \quad \overrightarrow{\beta^\pm} \text{ is fresh} \quad \overrightarrow{\gamma^\pm} \text{ is fresh} \\ \Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \models [\overrightarrow{\beta^\pm} / \overrightarrow{\alpha^\pm}] P \vee [\overrightarrow{\gamma^\pm} / \overrightarrow{\alpha^\pm}] P = Q \end{array}}{\mathbf{upgrade} \Gamma \vdash P \text{ to } \Delta = Q} \quad \text{LUBUPGRADE}$$

$$\boxed{\mathbf{nf} \, (N) = M}$$

$$\frac{\overline{\mathbf{nf} \, (\alpha^-) = \alpha^-} \quad \text{NRMNVAR}}{\mathbf{nf} \, (\uparrow P) = \uparrow Q \quad \text{NRMSHIFTU}} \quad \text{NRMSHIFTU}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(P \rightarrow N) = Q \rightarrow M} \quad \text{NRMARROW}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(\overrightarrow{\forall \alpha^+}.N) = \overrightarrow{\forall \alpha^{+'}}.N'} \quad \text{NRMFORALL}$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\overline{\mathbf{nf}(\alpha^+) = \alpha^+} \quad \text{NRMPVAR}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(\downarrow N) = \downarrow M} \quad \text{NRMSHIFTD}$$

$$\frac{\langle\langle \text{multiple parses} \rangle\rangle}{\mathbf{nf}(\overrightarrow{\exists \alpha^-}.P) = \overrightarrow{\exists \alpha^{-'}}.P'} \quad \text{NRME EXISTS}$$

$$\boxed{\mathbf{nf}(N) = M}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^-) = \hat{\alpha}^-} \quad \text{NRMNUVAR}$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^+) = \hat{\alpha}^+} \quad \text{NRMPUVAR}$$

$$\boxed{\mathbf{ord vars in } N = \vec{\alpha}}$$

$$\frac{\alpha^- \in vars}{\mathbf{ord vars in } \alpha^- = \alpha^-} \quad \text{ONVARIN}$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord vars in } \alpha^- = \cdot} \quad \text{ONVARININ}$$

$$\frac{\mathbf{ord vars in } P = \vec{\alpha}}{\mathbf{ord vars in } \uparrow P = \vec{\alpha}} \quad \text{OSHIFTU}$$

$$\frac{\mathbf{ord vars in } P = \vec{\alpha}_1 \quad \mathbf{ord vars in } N = \vec{\alpha}_2}{\mathbf{ord vars in } P \rightarrow N = \vec{\alpha}_1, (\vec{\alpha}_2 \setminus \vec{\alpha}_1)} \quad \text{OARROW}$$

$$\frac{vars \cap \vec{\alpha}^+ = \emptyset \quad \mathbf{ord vars in } N = \vec{\alpha}}{\mathbf{ord vars in } \overrightarrow{\forall \alpha^+}.N = \vec{\alpha}} \quad \text{OFORALL}$$

$$\boxed{\mathbf{ord vars in } P = \vec{\alpha}}$$

$$\frac{\alpha^+ \in vars}{\mathbf{ord vars in } \alpha^+ = \alpha^+} \quad \text{OPVARIN}$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord vars in } \alpha^+ = \cdot} \quad \text{OPVARININ}$$

$$\frac{\mathbf{ord vars in } N = \vec{\alpha}}{\mathbf{ord vars in } \downarrow N = \vec{\alpha}} \quad \text{OSHIFTD}$$

$$\frac{vars \cap \vec{\alpha}^- = \emptyset \quad \mathbf{ord vars in } P = \vec{\alpha}}{\mathbf{ord vars in } \overrightarrow{\exists \alpha^-}.P = \vec{\alpha}} \quad \text{OEXISTS}$$

$$\boxed{\mathbf{ord vars in } N = \vec{\alpha}}$$

$$\overline{\mathbf{ord vars in } \hat{\alpha}^- = \cdot} \quad \text{ONUVAR}$$

$$\boxed{\text{ord vars in } P = \vec{\alpha}}$$

$$\frac{}{\text{ord vars in } \hat{\alpha}^+ = \cdot} \text{OPUVAR}$$

$$\boxed{\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC} \quad \text{Negative unification}$$

$$\frac{}{\Gamma; \Theta \models \alpha^- \stackrel{u}{\simeq} \alpha^- \Rightarrow \cdot} \text{UNVAR}$$

$$\frac{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC}{\Gamma; \Theta \models \uparrow P \stackrel{u}{\simeq} \uparrow Q \Rightarrow UC} \text{USHIFTU}$$

$$\frac{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC_1 \quad \Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC_2}{\Gamma; \Theta \models P \rightarrow N \stackrel{u}{\simeq} Q \rightarrow M \Rightarrow UC_1 \ \& \ UC_2} \text{UARROW}$$

$$\frac{\Gamma, \vec{\alpha}^+; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC}{\Gamma; \Theta \models \forall \alpha^+. N \stackrel{u}{\simeq} \forall \alpha^+. M \Rightarrow UC} \text{Uforall}$$

$$\frac{\hat{\alpha}^-\{\Delta\} \in \Theta \quad \Delta \vdash N}{\Gamma; \Theta \models \hat{\alpha}^- \stackrel{u}{\simeq} N \Rightarrow (\hat{\alpha}^- : \approx N)} \text{UNUVAR}$$

$$\boxed{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC} \quad \text{Positive unification}$$

$$\frac{}{\Gamma; \Theta \models \alpha^+ \stackrel{u}{\simeq} \alpha^+ \Rightarrow \cdot} \text{UPVAR}$$

$$\frac{\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC}{\Gamma; \Theta \models \downarrow N \stackrel{u}{\simeq} \downarrow M \Rightarrow UC} \text{USHIFTD}$$

$$\frac{\Gamma, \vec{\alpha}^-; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC}{\Gamma; \Theta \models \exists \alpha^-. P \stackrel{u}{\simeq} \exists \alpha^-. Q \Rightarrow UC} \text{UEXISTS}$$

$$\frac{\hat{\alpha}^+\{\Delta\} \in \Theta \quad \Delta \vdash P}{\Gamma; \Theta \models \hat{\alpha}^+ \stackrel{u}{\simeq} P \Rightarrow (\hat{\alpha}^+ : \approx P)} \text{UPUVAR}$$

$$\boxed{\Gamma \vdash N} \quad \text{Negative type well-formedness}$$

$$\boxed{\Gamma \vdash P} \quad \text{Positive type well-formedness}$$

$$\boxed{\Gamma \vdash N} \quad \text{Negative type well-formedness}$$

$$\boxed{\Gamma \vdash P} \quad \text{Positive type well-formedness}$$

$$\boxed{\Gamma \vdash \vec{N}} \quad \text{Negative type list well-formedness}$$

$$\boxed{\Gamma \vdash \vec{P}} \quad \text{Positive type list well-formedness}$$

$$\boxed{\Gamma; \Theta \vdash N} \quad \text{Negative unification type well-formedness}$$

$$\boxed{\Gamma; \Theta \vdash P} \quad \text{Positive unification type well-formedness}$$

$$\boxed{\Gamma; \Xi \vdash N} \quad \text{Negative anti-unification type well-formedness}$$

$$\boxed{\Gamma; \Xi \vdash P} \quad \text{Positive anti-unification type well-formedness}$$

$$\boxed{\Gamma; \Xi_2 \vdash \hat{\tau} : \Xi_1} \quad \text{Antiunification substitution well-formedness}$$

$$\boxed{\Gamma \vdash \supset \Theta} \quad \text{Unification context well-formedness}$$

$$\boxed{\Gamma_1 \vdash \sigma : \Gamma_2} \quad \text{Substitution well-formedness}$$

$$\boxed{\Theta \vdash \hat{\sigma}} \quad \text{Unification substitution well-formedness}$$

$$\boxed{\Theta \vdash \hat{\sigma} : UC} \quad \text{Unification substitution satisfies unification constraint}$$

$$\boxed{\Theta \vdash \hat{\sigma} : SC} \quad \text{Unification substitution satisfies subtyping constraint}$$

$$\boxed{\Gamma \vdash e} \quad \text{Unification constraint entry well-formedness}$$

$$\boxed{\Gamma \vdash e} \quad \text{Subtyping constraint entry well-formedness}$$

$$\boxed{\Gamma \vdash P : e} \quad \text{Positive type satisfies unification constraint}$$

$\boxed{\Gamma \vdash N : e}$	Negative type satisfies unification constraint
$\boxed{\Gamma \vdash P : e}$	Positive type satisfies subtyping constraint
$\boxed{\Gamma \vdash N : e}$	Negative type satisfies subtyping constraint
$\boxed{\Theta \vdash UC}$	Unification constraint well-formedness
$\boxed{\Theta \vdash SC}$	Subtyping constraint well-formedness
$\boxed{\Gamma \vdash \vec{v}}$	Argument List well-formedness
$\boxed{\Gamma \vdash \Phi}$	Context well-formedness
$\boxed{\Gamma \vdash v}$	Value well-formedness

$$\frac{}{\Gamma \vdash x} \text{WFVAR}$$

$\boxed{\Gamma \vdash c}$ Computation well-formedness

$$\frac{\Gamma \vdash v \quad \Gamma \vdash c \quad \Gamma \vdash \vec{v}}{\Gamma \vdash \mathbf{let} \, x = v(\vec{v}); c} \text{WFAPPLET}$$

Definition rules: 101 good 21 bad
Definition rule clauses: 209 good 21 bad