α, β	type variables
n, m, i, j	index variables

$\alpha^+,\ \beta^+$	::=			positive variable
	|	α^+		
$\alpha^-,\ \beta^-$	::=			negative variable
	|	α^-		
σ	::=			substitution
	|	$\cdot$		
	|	$P/a+$		
	|	$N/a-$		
	|	$\vec{P}/\overrightarrow{\alpha^+}$		
	|	$\vec{N}/\overrightarrow{\alpha^-}$		
	|	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	|	$\overrightarrow{\widetilde{\alpha^-}}/\overrightarrow{\alpha^-}$		
	|	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	|	$\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}$		
	|	$vars_1/vars_2$		
	|	$\overline{\sigma_i}^{\,i}$		concatenate
e	::=			entry of a unification solution
	|	$\widehat{\alpha}^+ :\approx P$		
	|	$\widehat{\alpha}^- :\approx N$		
	|	$\widehat{\alpha}^+ :\geqslant P$		
$\widehat{\sigma}$	::=			unification solution (substitution)
	|	$\cdot$		
	|	e		
	|	$\widehat{\sigma}\backslash\overrightarrow{\widetilde{\alpha^+}}$		
	|	$\widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha^-}}$		
	|	$\widehat{\sigma}\backslash\widehat{\alpha}^+$		
	|	$\widehat{\sigma}\backslash\widehat{\alpha}^-$		
	|	$\widehat{\sigma}_1 \cup \widehat{\sigma}_2$		
	|	$\overline{\widehat{\sigma}_i}^{\,i}$		concatenate
	|	$(\widehat{\sigma})$	S	
	|	$\widehat{\sigma}_1 \& \widehat{\sigma}_2$	M	
$P,\ Q$	::=			positive types
	|	$a+$		
	|	$\downarrow N$		
	|	$\exists\alpha^-.P$		
	|	$[\sigma]P$	M	
$N,\ M$	::=			negative types
	|	$a-$		
	|	$\uparrow P$		
	|	$\forall\alpha^+.N$		
	|	$P \to N$		
	|	$[\sigma]N$	M	

$\overrightarrow{\alpha^+},\ \overrightarrow{\beta^+}$	$::=$			positive variable list
	$\mid$	$\cdot$		empty list
	$\mid$	α^+		a variable
	$\mid$	$\overline{\overrightarrow{\alpha^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^-},\ \overrightarrow{\beta^-}$	$::=$			negative variables
	$\mid$	$\cdot$		empty list
	$\mid$	α^-		a variable
	$\mid$	$\overline{\overrightarrow{\alpha^-}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$			multi-quantified positive types
	$\mid$	α^+		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists \overrightarrow{\alpha^-}.P$		$P \neq \exists \ldots$
	$\mid$	$[\sigma]P$	M	
	$\mid$	$[\mu]P$	M	
	$\mid$	$P_1 \vee P_2$	M	
$N,\ M$	$::=$			multi-quantified negative types
	$\mid$	α^-		
	$\mid$	$\uparrow P$		
	$\mid$	$P \rightarrow N$		
	$\mid$	$\forall \overrightarrow{\alpha^+}.N$		$N \neq \forall \ldots$
	$\mid$	$[\sigma]N$	M	
	$\mid$	$[\mu]N$	M	
$\vec{P}$	$::=$			list of positive types
	$\mid$	$\cdot$		empty list
	$\mid$	P		a singel type
	$\mid$	$\overline{\vec{P}_i}^{\,i}$		concatenate lists
$\vec{N}$	$::=$			list of negative types
	$\mid$	$\cdot$		empty list
	$\mid$	N		a singel type
	$\mid$	$\overline{\vec{N}_i}^{\,i}$		concatenate lists
Γ	$::=$			declarative type context
	$\mid$	$\cdot$		empty context
	$\mid$	$vars$		
	$\mid$	$\overrightarrow{\alpha^+}$		list of variables
	$\mid$	$\overrightarrow{\alpha^-}$		list of variables
	$\mid$	$\overline{\Gamma_i}^{\,i}$		concatenate contexts
	$\mid$	(Γ)	S	
	$\mid$	$\Gamma_1 \cup \Gamma_2$		
$\vec{\alpha}$	$::=$			ordered positive or negative variables
	$\mid$	$\cdot$		empty set
	$\mid$	$\overrightarrow{\alpha^+}$		list of variables
	$\mid$	$\overrightarrow{\alpha^-}$		list of variables

	|	$\vec{\alpha}_1 \backslash \vec{\alpha}_2$		setminus
	|	$\overline{\vec{\alpha}_i}^{\,i}$		concatenate contexts
	|	$(\vec{\alpha})$	S	parenthesis
$vars$	::=			set of variables
	|	$\varnothing$		empty set
	|	$\mathbf{fv}\, P$		free variables
	|	$\mathbf{fv}\, N$		free variables
	|	$\mathbf{fv}\, P$		free variables
	|	$\mathbf{fv}\, N$		free variables
	|	$vars_1 \cap vars_2$		set intersection
	|	$vars_1 \cup vars_2$		set union
	|	$vars_1 \backslash vars_2$		set complement
	|	$\mathbf{mv}\, P$		movable variables
	|	$\mathbf{mv}\, N$		movable variables
	|	$\mathbf{uv}\, N$		unification variables
	|	$\mathbf{uv}\, P$		unification variables
	|	$(vars)$	S	parenthesis
	|	Γ		context
	|	$\vec{\alpha}$		ordered list of variables
μ	::=			
	|	$\cdot$		empty moving
	|	$\tilde{\alpha}_1^+ \mapsto \tilde{\alpha}_2^+$		Positive unit substitution
	|	$\tilde{\alpha}_1^- \mapsto \tilde{\alpha}_2^-$		Positive unit substitution
	|	$\mu_1 \cup \mu_2$	M	Set-like union of movings
	|	$\overline{\mu_i}^{\,i}$		concatenate movings
	|	$\mu\vert_{vars}$	M	restriction on a set
n	::=			cohort index
	|	0		
	|	$n+1$		
$\tilde{\alpha}^+$	::=			positive movable variable
	|	$\tilde{\alpha}^{+n}$		
$\tilde{\alpha}^-$	::=			negative movable variable
	|	$\tilde{\alpha}^{-n}$		
$\overrightarrow{\tilde{\alpha}^+},\ \overrightarrow{\tilde{\beta}^+}$	::=			positive movable variable list
	|	$\cdot$		empty list
	|	$\tilde{\alpha}^+$		a variable
	|	$\overrightarrow{\widetilde{\alpha^{+n}}}$		from a non-movable variable
	|	$\overline{\overrightarrow{\tilde{\alpha}^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\tilde{\alpha}^-},\ \overrightarrow{\tilde{\beta}^-}$	::=			negatiive movable variable list
	|	$\cdot$		empty list
	|	$\tilde{\alpha}^-$		a variable
	|	$\overrightarrow{\widetilde{\alpha^{-n}}}$		from a non-movable variable

	$\mid$	$\overline{\overrightarrow{\widetilde{\alpha^-}}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$			multi-quantified positive types with movable variables
	$\mid$	α^+		
	$\mid$	$\widetilde{\alpha}^+$		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists\overrightarrow{\alpha^-}.P$		
	$\mid$	$[\sigma]P$	M	
	$\mid$	$[\mu]P$	M	
$N,\ M$	$::=$			multi-quantified negative types with movable variables
	$\mid$	α^-		
	$\mid$	$\widetilde{\alpha}^-$		
	$\mid$	$\uparrow P$		
	$\mid$	$P \to N$		
	$\mid$	$\forall\overrightarrow{\alpha^+}.N$		
	$\mid$	$[\sigma]N$	M	
	$\mid$	$[\mu]N$	M	
$\widehat{\alpha}^+$	$::=$			positive unification variable
	$\mid$	$\widehat{\alpha}^+$		
$\widehat{\alpha}^-$	$::=$			negative unification variable
	$\mid$	$\widehat{\alpha}^-$		
	$\mid$	$\widehat{\alpha}^-_{\{N,M\}}$		
$\overrightarrow{\widetilde{\alpha^+}},\ \overrightarrow{\widetilde{\beta^+}}$	$::=$			positive unification variable list
	$\mid$	$\cdot$		empty list
	$\mid$	$\widehat{\alpha}^+$		a variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^+\{vars\}}$		from a normal variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^+}$		from a normal variable, context unspecified
	$\mid$	$\overline{\overrightarrow{\widetilde{\alpha^+}}_i}^{\,i}$		concatenate lists
$\overrightarrow{\widehat{\alpha^-}},\ \overrightarrow{\widehat{\beta^-}}$	$::=$			negative unification variable list
	$\mid$	$\cdot$		empty list
	$\mid$	$\widehat{\alpha}^-$		a variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^-\{vars\}}$		from a normal variable
	$\mid$	$\overrightarrow{\widehat{\alpha}^-}$		from a normal variable, context unspecified
	$\mid$	$\overline{\overrightarrow{\widehat{\alpha^-}}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$			a positive algorithmic type (potentially with metavariables)
	$\mid$	α^+		
	$\mid$	$\widetilde{\alpha}^+$		
	$\mid$	$\widehat{\alpha}^+\{vars\}$		
	$\mid$	$\downarrow N$		
	$\mid$	$\exists\overrightarrow{\alpha^-}.P$		
	$\mid$	$[\sigma]P$	M	
	$\mid$	$[\mu]P$	M	

N, M	::=			a negative algorithmic type (potentially with metavariables)
	|	α^-		
	|	$\widetilde{\alpha}^-$		
	|	$\widehat{\alpha}^-\{vars\}$		
	|	$\widehat{\alpha}^-$		
	|	$\uparrow P$		
	|	$P \to N$		
	|	$\forall \overrightarrow{\alpha^+}. N$		
	|	$[\sigma] N$	M	
	|	$[\mu] N$	M	
terminals	::=			
	|	$\exists$		
	|	$\forall$		
	|	$\uparrow$		
	|	$\downarrow$		
	|	$\to$		
	|	$\leftrightarrow$		
	|	$\in$		
	|	$\notin$		
	|	$\cdot$		
	|	$\vdash$		
	|	$\leqslant$		
	|	$\geqslant$		
	|	$\simeq$		
	|	$\cup$		
	|	$\cap$		
	|	$\setminus$		
	|	$\subseteq$		
	|	$\mapsto$		
	|	$\stackrel{u}{\simeq}$		
	|	$\stackrel{a}{\simeq}$		
	|	$\varnothing$		
	|	$\vDash$		
	|	$\dashv$		
	|	$\neq$		
	|	$\equiv_n$		
	|	$\vee$		
	|	$\Downarrow$		
formula	::=			
	|	*judgement*		
	|	$formula_1 \quad .. \quad formula_n$		
	|	$\mu : vars_1 \leftrightarrow vars_2$		
	|	μ **is bijective**		
	|	$\widehat{\sigma}$ **is functional**		
	|	$\widehat{\sigma}_1 \in \widehat{\sigma}_2$		
	|	$vars_1 \subseteq vars_2$		
	|	$vars_1 = vars_2$		
	|	$vars$ **is fresh**		

	$\mid$	$\alpha^- \notin vars$	
	$\mid$	$\alpha^+ \notin vars$	
	$\mid$	$\alpha^- \in vars$	
	$\mid$	$\alpha^+ \in vars$	
	$\mid$	$N \neq M$	
	$\mid$	$P \neq Q$	
$E1A$	$::=$		
	$\mid$	$n \models N \simeq_1^A M \dashv \mu$	Negative multi-quantified type equivalence (algorithmic)
	$\mid$	$n \models P \simeq_1^A Q \dashv \mu$	Positive multi-quantified type equivalence (algorithmic)
A	$::=$		
	$\mid$	$\Gamma \models \boxed{N} \leqslant M \dashv \widehat{\sigma}$	Negative subtyping
	$\mid$	$\Gamma \models \boxed{P} \geqslant Q \dashv \widehat{\sigma}$	Positive supertyping
$E1$	$::=$		
	$\mid$	$N \simeq_1^D M$	Negative multi-quantified type equivalence
	$\mid$	$P \simeq_1^D Q$	Positive multi-quantified type equivalence
$D1$	$::=$		
	$\mid$	$\Gamma \vdash N \simeq_1^{\leqslant} M$	Negative equivalence on MQ types
	$\mid$	$\Gamma \vdash P \simeq_1^{\leqslant} Q$	Positive equivalence on MQ types
	$\mid$	$\Gamma \vdash N \leqslant_1 M$	Negative subtyping
	$\mid$	$\Gamma \vdash P \geqslant_1 Q$	Positive supertyping
$D0$	$::=$		
	$\mid$	$\Gamma \vdash N \simeq_0^{\leqslant} M$	Negative equivalence
	$\mid$	$\Gamma \vdash P \simeq_0^{\leqslant} Q$	Positive equivalence
	$\mid$	$\Gamma \vdash N \leqslant_0 M$	Negative subtyping
	$\mid$	$\Gamma \vdash P \geqslant_0 Q$	Positive supertyping
$LUBF$	$::=$		
	$\mid$	$P_1 \vee P_2 === Q$	
	$\mid$	$\widehat{\sigma}_1 \& \widehat{\sigma}_2 === \widehat{\sigma}$	
LUB	$::=$		
	$\mid$	$P_1 \vee P_2 = Q$	Least Upper Bound (Least Common Supertype)
AU	$::=$		
	$\mid$	$\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (\boxed{Q}, \widehat{\sigma}_1, \widehat{\sigma}_2)$	
	$\mid$	$\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (\boxed{M}, \widehat{\sigma}_1, \widehat{\sigma}_2)$	
$Order$	$::=$		
	$\mid$	$\mathbf{ord}\ vars\ \mathbf{in}\ \boxed{N} = vars'$	
	$\mid$	$\mathbf{ord}\ vars\ \mathbf{in}\ \boxed{P} = vars'$	
Nrm	$::=$		
	$\mid$	$N \Downarrow M$	
	$\mid$	$P \Downarrow Q$	

	$\vert$	$N \Downarrow M$	
	$\vert$	$P \Downarrow Q$	
SM	$::=$		
	$\vert$	$e_1 \& e_2 = e_3$	Unification Solution Entry Merge
	$\vert$	$\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3$	Merge unification solutions
U	$::=$		
	$\vert$	$N \overset{u}{\simeq} M \dashv \widehat{\sigma}$	Negative unification
	$\vert$	$P \overset{u}{\simeq} Q \dashv \widehat{\sigma}$	Positive unification
WF	$::=$		
	$\vert$	$\Gamma \vdash N$	Negative type well-formedness
	$\vert$	$\Gamma \vdash P$	Positive type well-formedness
	$\vert$	$\Gamma \vdash N$	Negative type well-formedness
	$\vert$	$\Gamma \vdash P$	Positive type well-formedness
judgement	$::=$		
	$\vert$	*E1A*	
	$\vert$	*A*	
	$\vert$	*E1*	
	$\vert$	*D1*	
	$\vert$	*D0*	
	$\vert$	*LUB*	
	$\vert$	*AU*	
	$\vert$	*Order*	
	$\vert$	*Nrm*	
	$\vert$	*SM*	
	$\vert$	*U*	
	$\vert$	*WF*	
user_syntax	$::=$		
	$\vert$	α	
	$\vert$	n	
	$\vert$	α^+	
	$\vert$	α^-	
	$\vert$	σ	
	$\vert$	e	
	$\vert$	$\widehat{\sigma}$	
	$\vert$	P	
	$\vert$	N	
	$\vert$	$\overrightarrow{\alpha^+}$	
	$\vert$	$\overrightarrow{\alpha^-}$	
	$\vert$	P	
	$\vert$	N	
	$\vert$	$\vec{P}$	
	$\vert$	$\vec{N}$	
	$\vert$	Γ	
	$\vert$	$\vec{\alpha}$	
	$\vert$	*vars*	

| μ
| n
| $\widetilde{\alpha}^+$
| $\widetilde{\alpha}^-$
| $\overrightarrow{\widetilde{\alpha}^+}$
| $\overrightarrow{\widetilde{\alpha}^-}$
| P
| N
| $\widehat{\alpha}^+$
| $\widehat{\alpha}^-$
| $\overrightarrow{\widehat{\alpha}^+}$
| $\overrightarrow{\widehat{\alpha}^-}$
| P
| N
| *terminals*
| *formula*

$\boxed{n \models N \simeq_1^A M \dashv \mu}$ Negative multi-quantified type equivalence (algorithmic)

$$\frac{}{n \models \alpha^- \simeq_1^A \alpha^- \dashv \cdot} \quad \text{E1ANVAR}$$

$$\frac{n \models P \simeq_1^A Q \dashv \mu}{n \models \uparrow P \simeq_1^A \uparrow Q \dashv \mu} \quad \text{E1ASHIFTU}$$

$$\frac{n \models P \simeq_1^A Q \dashv \mu_1 \quad n \models N \simeq_1^A M \dashv \mu_2 \quad \mu_1 \cup \mu_2 \textbf{ is bijective}}{n \models P \to N \simeq_1^A Q \to M \dashv \mu_1 \cup \mu_2} \quad \text{E1AARROW}$$

$$\frac{n+1 \models [\overrightarrow{\widetilde{\alpha}^{+n}}/\overrightarrow{\alpha^+}]N \simeq_1^A [\overrightarrow{\widetilde{\beta}^{+n}}/\overrightarrow{\beta^+}]M \dashv \mu}{n \models \forall\overrightarrow{\alpha^+}.N \simeq_1^A \forall\overrightarrow{\beta^+}.M \dashv \mu|_{\mathbf{mv}\,M}} \quad \text{E1AFORALL}$$

$$\frac{}{n \models \widetilde{\alpha}^{-n} \simeq_1^A \widetilde{\beta}^{-n} \dashv \widetilde{\beta}^{-n} \mapsto \widetilde{\alpha}^{-n}} \quad \text{E1ANMVAR}$$

$\boxed{n \models P \simeq_1^A Q \dashv \mu}$ Positive multi-quantified type equivalence (algorithmic)

$$\frac{}{n \models \alpha^+ \simeq_1^A \alpha^+ \dashv \cdot} \quad \text{E1APVAR}$$

$$\frac{n \models N \simeq_1^A M \dashv \mu}{n \models \downarrow N \simeq_1^A \downarrow M \dashv \mu} \quad \text{E1ASHIFTD}$$

$$\frac{n+1 \models [\overrightarrow{\widetilde{\alpha}^{-n}}/\overrightarrow{\alpha^-}]P \simeq_1^A [\overrightarrow{\widetilde{\beta}^{-n}}/\overrightarrow{\beta^-}]Q \dashv \mu}{n \models \exists\overrightarrow{\alpha^-}.P \simeq_1^A \exists\overrightarrow{\beta^-}.Q \dashv \mu|_{\mathbf{mv}\,Q}} \quad \text{E1AEXISTS}$$

$$\frac{}{n \models \widetilde{\alpha}^{+n} \simeq_1^A \widetilde{\beta}^{+n} \dashv \widetilde{\beta}^{+n} \mapsto \widetilde{\alpha}^{+n}} \quad \text{E1APMVAR}$$

$\boxed{\Gamma \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma \models \alpha^- \leqslant \alpha^- \dashv \cdot} \quad \text{ANVAR}$$

$$\frac{P \Downarrow P' \quad Q \Downarrow Q' \quad P' \overset{u}{\simeq} Q' \dashv \widehat{\sigma}}{\Gamma \models \uparrow P \leqslant \uparrow Q \dashv \widehat{\sigma}} \quad \text{ASHIFTU}$$

$$\frac{\Gamma \vDash P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma \vDash N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma \vDash P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2} \textsc{ AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vDash [\overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}/\alpha^+}] N \leqslant M \dashv \widehat{\sigma}}{\Gamma \vDash \forall \overrightarrow{\alpha^+}. N \leqslant \forall \overrightarrow{\beta^+}. M \dashv \widehat{\sigma} \backslash \overrightarrow{\widehat{\alpha}^+}} \textsc{ AForall}$$

$\boxed{\Gamma \vDash P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma \vDash \alpha^+ \geqslant \alpha^+ \dashv \cdot} \textsc{ APVar}$$

$$\frac{N \Downarrow N' \quad M \Downarrow M' \quad N' \stackrel{u}{\simeq} M' \dashv \widehat{\sigma}}{\Gamma \vDash \downarrow N \geqslant \downarrow M \dashv \widehat{\sigma}} \textsc{ AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vDash [\overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}/\alpha^-}] P \geqslant Q \dashv \widehat{\sigma}}{\Gamma \vDash \exists \overrightarrow{\alpha^-}. P \geqslant \exists \overrightarrow{\beta^-}. Q \dashv \widehat{\sigma}} \textsc{ AExists}$$

$$\frac{\mathit{vars}_1 = \mathbf{fv}\, P \backslash \mathit{vars} \quad \mathit{vars}_2 \textbf{ is fresh}}{\Gamma \vDash \widehat{\alpha}^+\{\mathit{vars}\} \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant P \vee [\mathit{vars}_2/\mathit{vars}_1]P)} \textsc{ APUVar}$$

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-} \textsc{ E1NVar}$$

$$\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \textsc{ E1ShiftU}$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M} \textsc{ E1Arrow}$$

$$\frac{\mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\, M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\, N) \quad N \simeq_1^D [\mu] M}{\forall \overrightarrow{\alpha^+}. N \simeq_1^D \forall \overrightarrow{\beta^+}. M} \textsc{ E1Forall}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+} \textsc{ E1PVar}$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \textsc{ E1ShiftD}$$

$$\frac{\mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu] Q}{\exists \overrightarrow{\alpha^-}. P \simeq_1^D \exists \overrightarrow{\beta^-}. Q} \textsc{ E1Exists}$$

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M} \textsc{ D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q} \textsc{ D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-} \textsc{ D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash {\uparrow}P \leqslant_1 {\uparrow}Q}\ \textsc{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M}\ \textsc{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M}\ \textsc{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+}\ \textsc{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash {\downarrow}N \geqslant_1 {\downarrow}M}\ \textsc{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q'}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q}\ \textsc{D1ExistsL}$$

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M}\ \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q}\ \textsc{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash a- \leqslant_0 a-}\ \textsc{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash {\uparrow}P \leqslant_0 {\uparrow}Q}\ \textsc{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/a+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M}\ \textsc{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M}\ \textsc{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M}\ \textsc{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash a+ \geqslant_0 a+}\ \textsc{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash {\downarrow}N \geqslant_0 {\downarrow}M}\ \textsc{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/a-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q}\ \textsc{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q}\ \textsc{D0ExistsR}$$

$\boxed{P_1 \vee P_2}$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2}$

$\boxed{P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\alpha^+ \vee \alpha^+ = \alpha^+}\ \textsc{LUBVar}$$

$$\frac{N \Downarrow N' \quad M \Downarrow M' \quad (\mathbf{fv}\, N' \cup \mathbf{fv}\, M') \models {\downarrow} N' \overset{a}{\simeq} {\downarrow} M' \Dashv (P, \widehat{\sigma}_1, \widehat{\sigma}_2)}{{\downarrow} N \vee {\downarrow} M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\mathbf{uv}\ P]P}\ \textsc{LUBShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing}{\exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = P_1 \vee P_2}\ \textsc{LUBExists}$$

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \Dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \Dashv (\alpha^+, \cdot, \cdot)}\ \textsc{AUPVar}$$

$$\frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \Dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models {\downarrow} N_1 \overset{a}{\simeq} {\downarrow} N_2 \Dashv ({\downarrow} M, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \textsc{AUPShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \Dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \Dashv (\exists \overrightarrow{\alpha^-}.\, Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \textsc{AUPExists}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \Dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \Dashv (\alpha^-, \cdot, \cdot)}\ \textsc{AUNVar}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \Dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models {\uparrow} P_1 \overset{a}{\simeq} {\uparrow} P_2 \Dashv ({\uparrow} Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}\ \textsc{AUNShift}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \Dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \Dashv (M, \widehat{\sigma}'_1, \widehat{\sigma}'_2)}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \Dashv (Q \to M, \widehat{\sigma}_1 \cup \widehat{\sigma}'_1, \widehat{\sigma}_2 \cup \widehat{\sigma}'_2)}\ \textsc{AUNArrow}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \Dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \Dashv (M, \widehat{\sigma}'_1, \widehat{\sigma}'_2)}{\Gamma \models N \overset{a}{\simeq} M \Dashv (\widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))}\ \textsc{AUNAU}$$

$\boxed{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ N = \mathit{vars}'}$

$$\frac{\alpha^- \in \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^- = \alpha^-}\ \textsc{ONVarIn}$$

$$\frac{\alpha^- \notin \mathit{vars}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \alpha^- = \cdot}\ \textsc{ONVarNIn}$$

$$\frac{}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ \widehat{\alpha}^-\{\mathit{vars}'\} = \cdot}\ \textsc{ONUVar}$$

$$\frac{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ \mathit{vars}\ \mathbf{in}\ {\uparrow} P = \vec{\alpha}}\ \textsc{OShiftU}$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ P = \vec{\alpha}_1 \quad \mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}_2}{\mathbf{ord}\ vars\ \mathbf{in}\ P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)} \quad \textsc{OArrow}$$

$$\frac{\mathbf{ord}\ (vars \backslash \overrightarrow{\alpha^+})\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \forall \overrightarrow{\alpha^+}. N = \vec{\alpha}} \quad \textsc{OForall}$$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P = vars'}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \alpha^+} \quad \textsc{OPVarIn}$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\ vars\ \mathbf{in}\ \alpha^+ = \cdot} \quad \textsc{OPVarNIn}$$

$$\frac{}{\mathbf{ord}\ vars\ \mathbf{in}\ \widehat{\alpha}^+\{vars'\} = \cdot} \quad \textsc{OPUVar}$$

$$\frac{\mathbf{ord}\ vars\ \mathbf{in}\ N = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ {\downarrow} N = \vec{\alpha}} \quad \textsc{OShiftD}$$

$$\frac{\mathbf{ord}\ (vars \backslash \overrightarrow{\alpha^-})\ \mathbf{in}\ P = \vec{\alpha}}{\mathbf{ord}\ vars\ \mathbf{in}\ \exists \overrightarrow{\alpha^-}. P = \vec{\alpha}} \quad \textsc{OExists}$$

$\boxed{N \Downarrow M}$
$\boxed{P \Downarrow Q}$
$\boxed{N \Downarrow M}$

$$\frac{}{\alpha^- \Downarrow \alpha^-} \quad \textsc{NrmNVar}$$

$$\frac{}{\widehat{\alpha}^-\{vars\} \Downarrow \widehat{\alpha}^-\{vars\}} \quad \textsc{NrmNUVar}$$

$$\frac{P \Downarrow Q}{{\uparrow} P \Downarrow {\uparrow} Q} \quad \textsc{NrmShiftU}$$

$$\frac{P \Downarrow Q \quad N \Downarrow M}{P \to N \Downarrow Q \to M} \quad \textsc{NrmArrow}$$

$$\frac{N \Downarrow N' \quad \mathbf{ord}\ \overrightarrow{\alpha^+}\ \mathbf{in}\ N' = \overrightarrow{\alpha^+}'}{\forall \overrightarrow{\alpha^+}. N \Downarrow \forall \overrightarrow{\alpha^+}'. N'} \quad \textsc{NrmForall}$$

$\boxed{P \Downarrow Q}$

$$\frac{}{\alpha^+ \Downarrow \alpha^+} \quad \textsc{NrmPVar}$$

$$\frac{}{\widehat{\alpha}^+\{vars\} \Downarrow \widehat{\alpha}^+\{vars\}} \quad \textsc{NrmPUVar}$$

$$\frac{N \Downarrow M}{{\downarrow} N \Downarrow {\downarrow} M} \quad \textsc{NrmShiftD}$$

$$\frac{P \Downarrow P' \quad \mathbf{ord}\ \overrightarrow{\alpha^-}\ \mathbf{in}\ P' = \overrightarrow{\alpha^-}'}{\exists \overrightarrow{\alpha^-}. P \Downarrow \exists \overrightarrow{\alpha^-}'. P'} \quad \textsc{NrmExists}$$

$\boxed{e_1 \& e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{}{\widehat{\alpha}^+ :\geqslant P \& \widehat{\alpha}^+ :\geqslant Q = \widehat{\alpha}^+ :\geqslant P \vee Q} \quad \textsc{SMEPSupSup}$$

$$\frac{\mathbf{fv}\, P \cup \mathbf{fv}\, Q \vDash \boxed{P} \geqslant Q \dashv \widehat{\sigma}'}{\widehat{\alpha}^+ :\approx P \& \widehat{\alpha}^+ :\geqslant Q = \widehat{\alpha}^+ :\approx P} \quad \text{SMEPEqSup}$$

$$\frac{\mathbf{fv}\, P \cup \mathbf{fv}\, Q \vDash \boxed{Q} \geqslant P \dashv \widehat{\sigma}'}{\widehat{\alpha}^+ :\geqslant P \& \widehat{\alpha}^+ :\approx Q = \widehat{\alpha}^+ :\approx Q} \quad \text{SMEPSupEq}$$

$$\frac{0 \vDash P \simeq_1^A Q \dashv \mu}{\widehat{\alpha}^+ :\approx P \& \widehat{\alpha}^+ :\approx Q = \widehat{\alpha}^+ :\approx Q} \quad \text{SMEPEqEq}$$

$$\frac{0 \vDash N \simeq_1^A M \dashv \mu}{\widehat{\alpha}^- :\approx N \& \widehat{\alpha}^- :\approx M = \widehat{\alpha}^+ :\approx Q} \quad \text{SMENEqEq}$$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$$\frac{}{\cdot \& \widehat{\sigma} = \widehat{\sigma}} \quad \text{SMEmpty}$$

$$\frac{(\widehat{\alpha}^+ :\approx Q) \in \widehat{\sigma}_2 \quad 0 \vDash P \simeq_1^A Q \dashv \mu \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)} \quad \text{SMPEqEq}$$

$$\frac{(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\geqslant P \vee Q, \widehat{\sigma}_3)} \quad \text{SMPSupSup}$$

$$\frac{(\widehat{\alpha}^+ :\approx Q) \in \widehat{\sigma}_2 \quad \mathbf{fv}\, Q \cup \mathbf{fv}\, P \vDash \boxed{Q} \geqslant P \dashv \widehat{\sigma}' \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx Q, \widehat{\sigma}_3)} \quad \text{SMPSupEq}$$

$$\frac{(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \quad \mathbf{fv}\, Q \cup \mathbf{fv}\, P \vDash \boxed{P} \geqslant Q \dashv \widehat{\sigma}' \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)} \quad \text{SMPEqSup}$$

$$\frac{(\widehat{\alpha}^- :\approx M) \in \widehat{\sigma}_2 \quad 0 \vDash N \simeq_1^A M \dashv \mu \quad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^-) = \widehat{\sigma}_3}{(\widehat{\alpha}^- :\approx N, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^- :\approx N, \widehat{\sigma}_3)} \quad \text{SMNEqEq}$$

$\boxed{N \stackrel{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\alpha^- \stackrel{u}{\simeq} \alpha^- \dashv \cdot} \quad \text{UNVar}$$

$$\frac{\boxed{P} \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}{\uparrow \boxed{P} \stackrel{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}} \quad \text{UShiftU}$$

$$\frac{\boxed{P} \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}_1 \quad \boxed{N} \stackrel{u}{\simeq} M \dashv \widehat{\sigma}_2}{\boxed{P} \to \boxed{N} \stackrel{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2} \quad \text{UArrow}$$

$$\frac{[\overrightarrow{\widetilde{\alpha}^{+n}} / \overrightarrow{\alpha^+}]\boxed{N} \stackrel{u}{\simeq} [\overrightarrow{\widetilde{\beta}^{+n}} / \overrightarrow{\beta^+}] M \dashv \widehat{\sigma}}{\forall \overrightarrow{\alpha^+}. \boxed{N} \stackrel{u}{\simeq} \forall \overrightarrow{\beta^+}. M \dashv \widehat{\sigma}} \quad \text{UForall}$$

$$\frac{\mathbf{fv}\, N \subseteq \mathit{vars}}{\widehat{\alpha}^-\{\mathit{vars}\} \stackrel{u}{\simeq} N \dashv \widehat{\alpha}^- :\approx N} \quad \text{UNUVar}$$

$\boxed{P \stackrel{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\alpha^+ \stackrel{u}{\simeq} \alpha^+ \dashv \cdot} \quad \text{UPVar}$$

$$\frac{\boxed{N} \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\downarrow \boxed{N} \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}} \quad \textsc{UShiftD}$$

$$\frac{[\overrightarrow{\widetilde{\alpha^{-n}}}/\overrightarrow{\alpha^{-}}]\, P \overset{u}{\simeq} [\overrightarrow{\widetilde{\beta^{-n}}}/\overrightarrow{\beta^{-}}]Q \dashv \widehat{\sigma}}{\exists \overrightarrow{\alpha^{-}}.\, P \overset{u}{\simeq} \exists \overrightarrow{\beta^{-}}.Q \dashv \widehat{\sigma}} \quad \textsc{UExists}$$

$$\frac{\mathbf{fv}\, P \subseteq \mathit{vars}}{\widehat{\alpha}^{+}\{\mathit{vars}\} \overset{u}{\simeq} P \dashv \widehat{\alpha}^{+} :\approx P} \quad \textsc{UPUVar}$$

$\boxed{\Gamma \vdash N}$ Negative type well-formedness

$\boxed{\Gamma \vdash P}$ Positive type well-formedness

$\boxed{\Gamma \vdash N}$ Negative type well-formedness

$\boxed{\Gamma \vdash P}$ Positive type well-formedness

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Definition rules:        94 good    0 bad
Definition rule clauses: 167 good    0 bad
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