

Contents

1	Declarative System	2
1.1	Grammar	2
1.2	Declarative Typing	2
1.3	Declarative Subtyping	3
1.4	Declarative Equivalence	4
2	Algorithm	4
2.1	Normalization	4
2.1.1	Ordering	4
2.1.2	Quantifier Normalization	5
2.2	Singularity	5
2.3	Unification	6
2.4	Algorithmic Subtyping	6
2.5	Constraints	7
2.6	Constraint Satisfaction	8
2.7	Least Upper Bound	8
2.8	Antiunification	8
2.9	Typing	9
3	Proofs	9
3.1	Substitution	9
3.2	Declarative Subtyping	11
3.3	Type well-formedness	16
3.4	Overview	17
3.5	Variable Ordering	17
3.6	Normaliztaion	19
3.7	Equivalence	22
3.8	Unification Constraint Merge	27
3.9	Unification	29
3.10	Anti-unification	30
3.11	Upper Bounds	35
3.12	Upgrade	40
3.13	Positive Subtyping	41
3.14	Subtyping Constraint Merge	44
3.15	Constraint Satisfaction	45
3.16	Negative Subtyping	46
3.17	Singularity	47
3.18	Declarative Typing	48
3.19	Algorithmic Typing	51

1 Declarative System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

P, Q, R	$::=$	multi-quantified positive types
	$\mid \alpha^+$	
	$\mid \downarrow N$	
	$\mid \exists \alpha^{\rightarrow}.P$	$P \neq \exists \dots$
N, M, K	$::=$	multi-quantified negative types
	$\mid \alpha^-$	
	$\mid \uparrow P$	
	$\mid P \rightarrow N$	
	$\mid \forall \alpha^+.N$	$N \neq \forall \dots$
v, w	$::=$	value terms
	$\mid x$	
	$\mid \{c\}$	
	$\mid (v : P)$	
c, d	$::=$	computation terms
	$\mid (c : N)$	
	$\mid \lambda x : P. c$	
	$\mid \Lambda \alpha^+. c$	
	$\mid \mathbf{return} \ v$	
	$\mid \mathbf{let} \ x = v ; c$	
	$\mid \mathbf{let} \ x : P = v(\vec{v}) ; c$	
	$\mid \mathbf{let} \ x = v(\vec{v}) ; c$	
	$\mid \mathbf{let}^{\exists}(\alpha^-, x) = v ; c$	

1.2 Declarative Typing

$\boxed{\Gamma; \Phi \vdash v : P}$ Positive type inference

$$\begin{array}{c}
\frac{x : P \in \Phi}{\Gamma; \Phi \vdash x : P} \quad \text{DTVAR} \\
\\
\frac{\Gamma; \Phi \vdash c : N}{\Gamma; \Phi \vdash \{c\} : \downarrow N} \quad \text{DTTHUNK} \\
\\
\frac{\Gamma \vdash Q \quad \Gamma; \Phi \vdash v : P \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma; \Phi \vdash (v : Q) : Q} \quad \text{DTPANNOT} \\
\\
\frac{\Gamma; \Phi \vdash v : P \quad \Gamma \vdash P \simeq_1^{\leq} P'}{\Gamma; \Phi \vdash v : P'} \quad \text{DTPEQUIV}
\end{array}$$

$\boxed{\Gamma; \Phi \vdash c : N}$ Negative type inference

$$\begin{array}{c}
\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \lambda x : P. c : P \rightarrow N} \quad \text{DTTLAM} \\
\\
\frac{\Gamma, \alpha^+; \Phi \vdash c : N}{\Gamma; \Phi \vdash \Lambda \alpha^+. c : \forall \alpha^+. N} \quad \text{DTTLAM} \\
\\
\frac{\Gamma; \Phi \vdash v : P}{\Gamma; \Phi \vdash \mathbf{return} \ v : \uparrow P} \quad \text{DTReturn} \\
\\
\frac{\Gamma; \Phi \vdash v : P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} \ x = v ; c : N} \quad \text{DTVARLET}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma; \Phi \vdash v : \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q \text{ unique} \quad \Gamma; \Phi, x : Q \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x = v(\vec{v}); c : N} \quad \text{DTAPPLET} \\
\\
\frac{\Gamma \vdash P \quad \Gamma; \Phi \vdash v : \downarrow M \quad \Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q \quad \Gamma \vdash \uparrow Q \leq_1 \uparrow P \quad \Gamma; \Phi, x : P \vdash c : N}{\Gamma; \Phi \vdash \mathbf{let} x : P = v(\vec{v}); c : N} \quad \text{DTAPPLETANN} \\
\\
\frac{\Gamma; \Phi \vdash v : \exists \alpha^-. P \quad \mathbf{nf}(\exists \alpha^-. P) = \exists \alpha^-. P \quad \Gamma, \alpha^-; \Phi, x : P \vdash c : N \quad \Gamma \vdash N}{\Gamma; \Phi \vdash \mathbf{let}^{\exists}(\alpha^-, x) = v; c : N} \quad \text{DTUNPACK} \\
\\
\frac{\Gamma \vdash M \quad \Gamma; \Phi \vdash c : N \quad \Gamma \vdash N \simeq_1^{\leq} M}{\Gamma; \Phi \vdash (c : M) : M} \quad \text{DTNANNOT} \\
\\
\frac{\Gamma; \Phi \vdash c : N \quad \Gamma \vdash N \simeq_1^{\leq} N'}{\Gamma; \Phi \vdash c : N'} \quad \text{DTNEQUIV}
\end{array}$$

$\boxed{\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M}$ Application type inference

$$\begin{array}{c}
\frac{\Gamma \vdash N \simeq_1^{\leq} N'}{\Gamma; \Phi \vdash N \bullet \cdot \Rightarrow N'} \quad \text{DTEEMPTYAPP} \\
\\
\frac{\Gamma; \Phi \vdash v : P \quad \Gamma \vdash Q \geq_1 P \quad \Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M}{\Gamma; \Phi \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M} \quad \text{DTARROWAPP} \\
\\
\frac{\Gamma \vdash \sigma : \alpha^+ \quad \Gamma; \Phi \vdash [\sigma]N \bullet \vec{v} \Rightarrow M \quad \vec{v} \neq \cdot \quad \alpha^+ \neq \cdot}{\Gamma; \Phi \vdash \forall \alpha^+. N \bullet \vec{v} \Rightarrow M} \quad \text{DTFORALLAPP}
\end{array}$$

1.3 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leq} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leq_1 M \quad \Gamma \vdash M \leq_1 N}{\Gamma \vdash N \simeq_1^{\leq} M} \quad (\simeq_1^{\leq} -)$$

$\boxed{\Gamma \vdash P \simeq_1^{\leq} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash Q \geq_1 P}{\Gamma \vdash P \simeq_1^{\leq} Q} \quad (\simeq_1^{\leq} +)$$

$\boxed{\Gamma \vdash N \leq_1 M}$ Negative subtyping

$$\begin{array}{c}
\overline{\Gamma \vdash \alpha^- \leq_1 \alpha^-} \quad (\text{VAR}^{-\leq_1}) \\
\\
\frac{\Gamma \vdash P \simeq_1^{\leq} Q}{\Gamma \vdash \uparrow P \leq_1 \uparrow Q} \quad (\uparrow^{\leq_1}) \\
\\
\frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash N \leq_1 M}{\Gamma \vdash P \rightarrow N \leq_1 Q \rightarrow M} \quad (\rightarrow^{\leq_1}) \\
\\
\frac{\mathbf{fv} N \cap \vec{\beta}^+ = \emptyset \quad \Gamma, \vec{\beta}^+ \vdash P_i \quad \Gamma, \vec{\beta}^+ \vdash [\vec{P}/\alpha^+]N \leq_1 M}{\Gamma \vdash \forall \alpha^+. N \leq_1 \forall \vec{\beta}^+. M} \quad (\forall^{\leq_1})
\end{array}$$

$\boxed{\Gamma \vdash P \geq_1 Q}$ Positive supertyping

$$\begin{array}{c}
\overline{\Gamma \vdash \alpha^+ \geq_1 \alpha^+} \quad (\text{VAR}^{+\geq_1}) \\
\\
\frac{\Gamma \vdash N \simeq_1^{\leq} M}{\Gamma \vdash \downarrow N \geq_1 \downarrow M} \quad (\downarrow^{\geq_1}) \\
\\
\frac{\mathbf{fv} P \cap \vec{\beta}^- = \emptyset \quad \Gamma, \vec{\beta}^- \vdash N_i \quad \Gamma, \vec{\beta}^- \vdash [\vec{N}/\alpha^-]P \geq_1 Q}{\Gamma \vdash \exists \alpha^-. P \geq_1 \exists \vec{\beta}^-. Q} \quad (\exists^{\geq_1})
\end{array}$$

$\boxed{\Gamma_2 \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : \Gamma_1}$	Equivalence of substitutions
$\boxed{\Gamma \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : vars}$	Equivalence of substitutions
$\boxed{\Theta \vdash \hat{\sigma}_1 \simeq_1^{\leq} \hat{\sigma}_2 : vars}$	Equivalence of unification substitutions
$\boxed{\Gamma \vdash \Phi_1 \simeq_1^{\leq} \Phi_2}$	Equivalence of contexts

1.4 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\begin{array}{c}
\frac{}{\alpha^- \simeq_1^D \alpha^-} \quad (\text{VAR}^- \simeq_1^D) \\
\frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad (\uparrow \simeq_1^D) \\
\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \rightarrow N \simeq_1^D Q \rightarrow M} \quad (\rightarrow \simeq_1^D) \\
\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv} M = \emptyset \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv} M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv} N) \quad N \simeq_1^D [\mu]M}{\forall \overrightarrow{\alpha^+}. N \simeq_1^D \forall \overrightarrow{\beta^+}. M} \quad (\forall \simeq_1^D)
\end{array}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\begin{array}{c}
\frac{}{\alpha^+ \simeq_1^D \alpha^+} \quad (\text{VAR}^+ \simeq_1^D) \\
\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow \simeq_1^D) \\
\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv} Q = \emptyset \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv} Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv} P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}. P \simeq_1^D \exists \overrightarrow{\beta^-}. Q} \quad (\exists \simeq_1^D)
\end{array}$$

$\boxed{P \simeq_1^D Q}$ Positive unification type equivalence

$\boxed{N \simeq_1^D M}$ Positive unification type equivalence

2 Algorithm

2.1 Normalization

2.1.1 Ordering

$\boxed{\text{ord vars in } N = \vec{\alpha}}$

$$\begin{array}{c}
\frac{\alpha^- \in vars}{\text{ord vars in } \alpha^- = \alpha^-} \quad (\text{VAR}_{\in}^-) \\
\frac{\alpha^- \notin vars}{\text{ord vars in } \alpha^- = \cdot} \quad (\text{VAR}_{\notin}^-) \\
\frac{\text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \uparrow P = \vec{\alpha}} \quad (\uparrow) \\
\frac{\text{ord vars in } P = \vec{\alpha}_1 \quad \text{ord vars in } N = \vec{\alpha}_2}{\text{ord vars in } P \rightarrow N = \vec{\alpha}_1, (\vec{\alpha}_2 \setminus \vec{\alpha}_1)} \quad (\rightarrow) \\
\frac{vars \cap \overrightarrow{\alpha^+} = \emptyset \quad \text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \forall \overrightarrow{\alpha^+}. N = \vec{\alpha}} \quad (\forall)
\end{array}$$

$\boxed{\text{ord vars in } P = \vec{\alpha}}$

$$\frac{\alpha^+ \in vars}{\text{ord vars in } \alpha^+ = \alpha^+} \quad (\text{VAR}_{\in}^+)$$

$$\begin{array}{c}
\frac{\alpha^+ \notin \text{vars}}{\text{ord vars in } \alpha^+ = .} \quad (\text{VAR}_{\neq}^+) \\
\frac{\text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \downarrow N = \vec{\alpha}} \quad (\downarrow) \\
\frac{\text{vars} \cap \vec{\alpha}^- = \emptyset \quad \text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \exists \alpha^-. P = \vec{\alpha}} \quad (\exists)
\end{array}$$

$$\boxed{\text{ord vars in } N = \vec{\alpha}}$$

$$\frac{}{\text{ord vars in } \hat{\alpha}^- = .} \quad (\text{UVar}^-)$$

$$\boxed{\text{ord vars in } P = \vec{\alpha}}$$

$$\frac{}{\text{ord vars in } \hat{\alpha}^+ = .} \quad (\text{UVar}^+)$$

2.1.2 Quantifier Normalization

$$\boxed{\text{nf}(N) = M}$$

$$\begin{array}{c}
\frac{}{\text{nf}(\alpha^-) = \alpha^-} \quad (\text{VAR}^-) \\
\frac{\text{nf}(P) = Q}{\text{nf}(\uparrow P) = \uparrow Q} \quad (\uparrow) \\
\frac{\text{nf}(P) = Q \quad \text{nf}(N) = M}{\text{nf}(P \rightarrow N) = Q \rightarrow M} \quad (\rightarrow) \\
\frac{\text{nf}(N) = N' \quad \text{ord } \vec{\alpha}^+ \text{ in } N' = \vec{\alpha}^{+'}}{\text{nf}(\vec{\forall} \alpha^+. N) = \vec{\forall} \alpha^{+'}. N'} \quad (\forall)
\end{array}$$

$$\boxed{\text{nf}(P) = Q}$$

$$\begin{array}{c}
\frac{}{\text{nf}(\alpha^+) = \alpha^+} \quad (\text{VAR}^+) \\
\frac{\text{nf}(N) = M}{\text{nf}(\downarrow N) = \downarrow M} \quad (\downarrow) \\
\frac{\text{nf}(P) = P' \quad \text{ord } \vec{\alpha}^- \text{ in } P' = \vec{\alpha}^{-'}}{\text{nf}(\vec{\exists} \alpha^-. P) = \vec{\exists} \alpha^{-'}. P'} \quad (\exists)
\end{array}$$

$$\boxed{\text{nf}(N) = M}$$

$$\frac{}{\text{nf}(\hat{\alpha}^-) = \hat{\alpha}^-} \quad (\text{UVar}^-)$$

$$\boxed{\text{nf}(P) = Q}$$

$$\frac{}{\text{nf}(\hat{\alpha}^+) = \hat{\alpha}^+} \quad (\text{UVar}^+)$$

2.2 Singularity

e_1 **singular with** P Positive Subtyping Constraint Entry Is Singular

$$\begin{array}{c}
\frac{}{\hat{\alpha}^+ : \approx P \text{ singular with } \text{nf}(P)} \quad \text{SINGPEQ} \\
\frac{}{\hat{\alpha}^+ : \geq \vec{\exists} \alpha^-. \alpha^+ \text{ singular with } \alpha^+} \quad \text{SINGSUPVAR} \\
\frac{N \simeq_1^D \alpha_i^-}{\hat{\alpha}^+ : \geq \vec{\exists} \alpha^-. \downarrow N \text{ singular with } \vec{\exists} \alpha^-. \downarrow \alpha^-} \quad \text{SINGSUPSHIFT}
\end{array}$$

e_1 **singular with** N Negative Subtyping Constraint Entry Is Singular

$$\overline{\hat{\alpha}^- : \approx N \text{ singular with nf } (N)} \quad \text{SINGNEQ}$$

$SC \text{ singular with } \hat{\sigma}$

Subtyping Constraint Is Singular

2.3 Unification

$\Gamma; \Theta \models N \stackrel{u}{\simeq} M \models UC$

Negative unification

$$\begin{array}{c} \overline{\Gamma; \Theta \models \alpha^- \stackrel{u}{\simeq} \alpha^- \models \cdot} \quad (\text{VAR}^{-\stackrel{u}{\simeq}}) \\ \overline{\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \models UC} \quad (\uparrow^{\stackrel{u}{\simeq}}) \\ \Gamma; \Theta \models \uparrow P \stackrel{u}{\simeq} \uparrow Q \models UC \\ \hline \Gamma; \Theta \models P \stackrel{u}{\simeq} Q \models UC_1 \quad \Gamma; \Theta \models N \stackrel{u}{\simeq} M \models UC_2 \quad (\rightarrow^{\stackrel{u}{\simeq}}) \\ \Gamma; \Theta \models P \rightarrow N \stackrel{u}{\simeq} Q \rightarrow M \models UC_1 \ \& \ UC_2 \\ \hline \Gamma, \overrightarrow{\alpha^+}; \Theta \models N \stackrel{u}{\simeq} M \models UC \quad (\forall^{\stackrel{u}{\simeq}}) \\ \Gamma; \Theta \models \forall \overrightarrow{\alpha^+}. N \stackrel{u}{\simeq} \forall \overrightarrow{\alpha^+}. M \models UC \\ \hline \hat{\alpha}^- \{\Delta\} \in \Theta \quad \Delta \vdash N \quad (\text{UVAR}^{-\stackrel{u}{\simeq}}) \\ \Gamma; \Theta \models \hat{\alpha}^- \stackrel{u}{\simeq} N \models (\hat{\alpha}^- : \approx N) \end{array}$$

$\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \models UC$

Positive unification

$$\begin{array}{c} \overline{\Gamma; \Theta \models \alpha^+ \stackrel{u}{\simeq} \alpha^+ \models \cdot} \quad (\text{VAR}^{+\stackrel{u}{\simeq}}) \\ \overline{\Gamma; \Theta \models N \stackrel{u}{\simeq} M \models UC} \quad (\downarrow^{\stackrel{u}{\simeq}}) \\ \Gamma; \Theta \models \downarrow N \stackrel{u}{\simeq} \downarrow M \models UC \\ \hline \Gamma, \overrightarrow{\alpha^-}; \Theta \models P \stackrel{u}{\simeq} Q \models UC \quad (\exists^{\stackrel{u}{\simeq}}) \\ \Gamma; \Theta \models \exists \overrightarrow{\alpha^-}. P \stackrel{u}{\simeq} \exists \overrightarrow{\alpha^-}. Q \models UC \\ \hline \hat{\alpha}^+ \{\Delta\} \in \Theta \quad \Delta \vdash P \quad (\text{UVAR}^{+\stackrel{u}{\simeq}}) \\ \Gamma; \Theta \models \hat{\alpha}^+ \stackrel{u}{\simeq} P \models (\hat{\alpha}^+ : \approx P) \end{array}$$

2.4 Algorithmic Subtyping

$\Gamma; \Theta \models N \leq M \models SC$

Negative subtyping

$$\begin{array}{c} \overline{\Gamma; \Theta \models \alpha^- \leq \alpha^- \models \cdot} \quad (\text{VAR}^{-\leq}) \\ \overline{\Gamma; \Theta \models \text{nf}(P) \stackrel{u}{\simeq} \text{nf}(Q) \models UC} \quad (\uparrow^{\leq}) \\ \Gamma; \Theta \models \uparrow P \leq \uparrow Q \models UC \\ \hline \Gamma; \Theta \models P \geq Q \models SC_1 \quad \Gamma; \Theta \models N \leq M \models SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC \quad (\rightarrow^{\leq}) \\ \Gamma; \Theta \models P \rightarrow N \leq Q \rightarrow M \models SC \\ \hline \Gamma, \overrightarrow{\beta^+}; \Theta, \hat{\alpha}^+ \{\Gamma, \overrightarrow{\beta^+}\} \models [\hat{\alpha}^+ / \alpha^+] N \leq M \models SC \quad (\forall^{\leq}) \\ \Gamma; \Theta \models \forall \overrightarrow{\alpha^+}. N \leq \forall \overrightarrow{\beta^+}. M \models SC \setminus \hat{\alpha}^+ \end{array}$$

$\Gamma; \Theta \models P \geq Q \models SC$

Positive supertyping

$$\begin{array}{c} \overline{\Gamma; \Theta \models \alpha^+ \geq \alpha^+ \models \cdot} \quad (\text{VAR}^{+\geq}) \\ \overline{\Gamma; \Theta \models \text{nf}(N) \stackrel{u}{\simeq} \text{nf}(M) \models UC} \quad (\downarrow^{\geq}) \\ \Gamma; \Theta \models \downarrow N \geq \downarrow M \models UC \\ \hline \Gamma, \overrightarrow{\beta^-}; \Theta, \hat{\alpha}^- \{\Gamma, \overrightarrow{\beta^-}\} \models [\hat{\alpha}^- / \alpha^-] P \geq Q \models SC \quad (\exists^{\geq}) \\ \Gamma; \Theta \models \exists \overrightarrow{\alpha^-}. P \geq \exists \overrightarrow{\beta^-}. Q \models SC \setminus \hat{\alpha}^- \\ \hline \hat{\alpha}^+ \{\Delta\} \in \Theta \quad \text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q \quad (\text{UVAR}^{\geq}) \\ \Gamma; \Theta \models \hat{\alpha}^+ \geq P \models (\hat{\alpha}^+ : \geq Q) \end{array}$$

2.5 Constraints

Unification and subtyping algorithms are based on the constraint generation. The constraints are represented by set of constraint entries.

Definition 1 (Unification Constraint).

- *Unification entry (denoted as e) is an expression of shape $\hat{\alpha}^+ : \approx P$ or $\hat{\alpha}^- : \approx N$;*
- *unification constraint (denoted as UC) is a set of unification constraint entries.*

Definition 2 (Subtyping Constraint).

- *Subtyping entry (denoted as e) is an expression of shape $\hat{\alpha}^+ : \geq P$, $\hat{\alpha}^- : \approx N$, or $\hat{\alpha}^+ : \approx P$;*
- *subtyping constraint (denoted as SC) is a set of subtyping constraint entries.*

Definition 3 (Well-formed Constraint Entry). *We say that a constraint entry is well-formed in a context Γ if the type it restricts the unification variable to is well-formed in Γ .*

- $\Gamma \vdash \hat{\alpha}^+ : \geq P$ iff $\Gamma \vdash P$;
- $\Gamma \vdash \hat{\alpha}^+ : \approx P$ iff $\Gamma \vdash P$;
- $\Gamma \vdash \hat{\alpha}^- : \approx N$ iff $\Gamma \vdash N$.

Definition 4 (Matching Entries). *We call two unification constraint entries or two subtyping constraint entries matching if they are restricting the same unification variable.*

Two matching entries formed in the same context Γ can be merged in the following way:

Definition 5 (Merge of Matching Constraint Entries).

$\boxed{\Gamma \vdash e_1 \ \& \ e_2 = e_3}$ *Subtyping Constraint Entry Merge*

$$\begin{array}{c}
\frac{\Gamma \models P_1 \vee P_2 = Q}{\Gamma \vdash (\hat{\alpha}^+ : \geq P_1) \ \& \ (\hat{\alpha}^+ : \geq P_2) = (\hat{\alpha}^+ : \geq Q)} \quad (\geq \ \&^+ \ \geq) \\
\\
\frac{\Gamma; \cdot \models P \geq Q = \cdot}{\Gamma \vdash (\hat{\alpha}^+ : \approx P) \ \& \ (\hat{\alpha}^+ : \geq Q) = (\hat{\alpha}^+ : \approx P)} \quad (\approx \ \&^+ \ \geq) \\
\\
\frac{\Gamma; \cdot \models Q \geq P = \cdot}{\Gamma \vdash (\hat{\alpha}^+ : \geq P) \ \& \ (\hat{\alpha}^+ : \approx Q) = (\hat{\alpha}^+ : \approx Q)} \quad (\geq \ \&^+ \ \approx) \\
\\
\frac{\mathbf{nf}(P) = \mathbf{nf}(P')}{\Gamma \vdash (\hat{\alpha}^+ : \approx P) \ \& \ (\hat{\alpha}^+ : \approx P') = (\hat{\alpha}^+ : \approx P)} \quad (\approx \ \&^+ \ \approx) \\
\\
\frac{\mathbf{nf}(N) = \mathbf{nf}(N')}{\Gamma \vdash (\hat{\alpha}^- : \approx N_1) \ \& \ (\hat{\alpha}^- : \approx N') = (\hat{\alpha}^- : \approx N)} \quad (\approx \ \&^- \ \approx)
\end{array}$$

Definition 6 (Constraint Context). *Constraint context (denoted as Θ) is a set of entries of shape $\hat{\alpha}^+ \{\Gamma\}$ and $\hat{\alpha}^- \{\Gamma\}$, specifying that the unification variable ($\hat{\alpha}^+$ or $\hat{\alpha}^-$) must be instantiated in a type well-formed in context Γ . We assume that for each $\hat{\alpha}^\pm$, at most one entry of shape $\hat{\alpha}^\pm \{\Gamma\}$ is in Θ . If $\hat{\alpha}^\pm \{\Gamma\} \in \Theta$, we denote Γ as $\Theta(\hat{\alpha}^\pm)$.*

Definition 7 (Well-formed Constraint). *We say that a constraint is well-formed in a constraint context Θ if all its entries are well-formed in the corresponding elements of Θ . More formally, $\Theta \vdash UC$ iff for every $e \in UC$, such that e restricts $\hat{\alpha}^\pm$, there exists $\hat{\alpha}^\pm \{\Gamma\} \in \Theta$ and $\Gamma \vdash e$.*

Definition 8 (Merge of Subtyping Constraints). *Suppose that $\Theta \vdash SC_1$ and $\Theta \vdash SC_2$. Then $\Theta \vdash SC_1 \ \& \ SC_2 = SC$ defines a set such that $e \in SC$ iff either*

- $e \in SC_1$ and there is no matching $e' \in SC_2$; or
- $e \in SC_2$ and there is no matching $e' \in SC_1$; or
- $\Theta(\hat{\alpha}^\pm) \vdash e_1 \ \& \ e_2 = e$ for some $e_1 \in SC_1$ and $e_2 \in SC_2$ such that e_1 matches with e_2 restricting variable $\hat{\alpha}^\pm$.

2.6 Constraint Satisfaction

$\boxed{\Gamma \vdash P : e}$ Positive type satisfies with the subtyping constraint entry

$$\frac{\Gamma \vdash P \geqslant_1 Q}{\Gamma \vdash P : (\hat{\alpha}^+ : \geqslant Q)} \quad \text{SATSCESUP}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leq} Q}{\Gamma \vdash P : (\hat{\alpha}^+ : \approx Q)} \quad \text{SATSCEPEQ}$$

$\boxed{\Gamma \vdash N : e}$ Negative type satisfies with the subtyping constraint entry

$$\frac{\Gamma \vdash N \simeq_1^{\leq} M}{\Gamma \vdash N : (\hat{\alpha}^- : \approx M)} \quad \text{SATSCENEQ}$$

2.7 Least Upper Bound

$\boxed{\Gamma \models P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\begin{aligned} & \overline{\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+} \quad (\text{VAR}^\vee) \\ & \frac{\Gamma, \cdot \models \mathbf{nf}(\downarrow N) \stackrel{a}{\simeq} \mathbf{nf}(\downarrow M) \Rightarrow (\Xi, \mathbf{P}, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N \vee \downarrow M = \exists \alpha^-. [\overrightarrow{\alpha^-} / \Xi] \mathbf{P}} \quad (\downarrow^\vee) \\ & \frac{\Gamma, \overrightarrow{\alpha^-}, \overrightarrow{\beta^-} \models P_1 \vee P_2 = Q}{\Gamma \models \exists \alpha^-. P_1 \vee \exists \beta^-. P_2 = Q} \quad (\exists^\vee) \end{aligned}$$

$\boxed{\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q}$

$$\frac{\begin{array}{l} \Gamma = \Delta, \overrightarrow{\alpha^\pm} \quad \overrightarrow{\beta^\pm} \text{ is fresh} \quad \overrightarrow{\gamma^\pm} \text{ is fresh} \\ \Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \models [\overrightarrow{\beta^\pm} / \overrightarrow{\alpha^\pm}] P \vee [\overrightarrow{\gamma^\pm} / \overrightarrow{\alpha^\pm}] P = Q \end{array}}{\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q} \quad (\text{UPG})$$

2.8 Antiunification

$\boxed{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)}$

$$\begin{aligned} & \overline{\Gamma \models \alpha^+ \stackrel{a}{\simeq} \alpha^+ \Rightarrow (\cdot, \alpha^+, \cdot, \cdot)} \quad (\text{VAR}^{+\stackrel{a}{\simeq}}) \\ & \frac{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \mathbf{M}, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \downarrow N_1 \stackrel{a}{\simeq} \downarrow N_2 \Rightarrow (\Xi, \downarrow \mathbf{M}, \hat{\tau}_1, \hat{\tau}_2)} \quad (\downarrow^{\stackrel{a}{\simeq}}) \\ & \frac{\overrightarrow{\alpha^+} \cap \Gamma = \emptyset \quad \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \exists \overrightarrow{\alpha^+}. P_1 \stackrel{a}{\simeq} \exists \overrightarrow{\alpha^+}. P_2 \Rightarrow (\Xi, \exists \overrightarrow{\alpha^+}. \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)} \quad (\exists^{\stackrel{a}{\simeq}}) \end{aligned}$$

$\boxed{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \mathbf{M}, \hat{\tau}_1, \hat{\tau}_2)}$

$$\begin{aligned} & \overline{\Gamma \models \alpha^- \stackrel{a}{\simeq} \alpha^- \Rightarrow (\cdot, \alpha^-, \cdot, \cdot)} \quad (\text{VAR}^{-\stackrel{a}{\simeq}}) \\ & \frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \uparrow P_1 \stackrel{a}{\simeq} \uparrow P_2 \Rightarrow (\Xi, \uparrow \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)} \quad (\uparrow^{\stackrel{a}{\simeq}}) \\ & \frac{\overrightarrow{\alpha^+} \cap \Gamma = \emptyset \quad \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \mathbf{M}, \hat{\tau}_1, \hat{\tau}_2)}{\Gamma \models \forall \overrightarrow{\alpha^+}. N_1 \stackrel{a}{\simeq} \forall \overrightarrow{\alpha^+}. N_2 \Rightarrow (\Xi, \forall \overrightarrow{\alpha^+}. \mathbf{M}, \hat{\tau}_1, \hat{\tau}_2)} \quad (\forall^{\stackrel{a}{\simeq}}) \\ & \frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi_1, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2) \quad \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi_2, \mathbf{M}, \hat{\tau}'_1, \hat{\tau}'_2)}{\Gamma \models P_1 \rightarrow N_1 \stackrel{a}{\simeq} P_2 \rightarrow N_2 \Rightarrow (\Xi_1 \cup \Xi_2, \mathbf{Q} \rightarrow \mathbf{M}, \hat{\tau}_1 \cup \hat{\tau}'_1, \hat{\tau}_2 \cup \hat{\tau}'_2)} \quad (\rightarrow^{\stackrel{a}{\simeq}}) \\ & \frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \stackrel{a}{\simeq} M \Rightarrow (\hat{\alpha}_{\{N,M\}}^-, \hat{\alpha}_{\{N,M\}}^-, (\hat{\alpha}_{\{N,M\}}^- : \approx N), (\hat{\alpha}_{\{N,M\}}^- : \approx M))} \quad (\text{AU}^-) \end{aligned}$$

2.9 Typing

$\boxed{\Gamma; \Phi \models v : P}$ Positive type inference

$$\frac{x : P \in \Phi}{\Gamma; \Phi \models x : \mathbf{nf}(P)} \text{ ATVAR}$$

$$\frac{\Gamma; \Phi \models c : N}{\Gamma; \Phi \models \{c\} : \downarrow N} \text{ ATTHUNK}$$

$$\frac{\Gamma \vdash Q \quad \Gamma; \Phi \models v : P \quad \Gamma; \cdot \models Q \geq P \Rightarrow \cdot}{\Gamma; \Phi \models (v : Q) : \mathbf{nf}(Q)} \text{ ATPANNOT}$$

$\boxed{\Gamma; \Phi \models c : N}$ Negative type inference

$$\frac{\Gamma \vdash M \quad \Gamma; \Phi \models c : N \quad \Gamma; \cdot \models N \leq M \Rightarrow \cdot}{\Gamma; \Phi \models (c : M) : \mathbf{nf}(M)} \text{ ATNANNOT}$$

$$\frac{\Gamma \vdash P \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \lambda x : P. c : \mathbf{nf}(P \rightarrow N)} \text{ ATTLAM}$$

$$\frac{\Gamma, \alpha^+; \Phi \models c : N}{\Gamma; \Phi \models \Lambda \alpha^+. c : \mathbf{nf}(\forall \alpha^+. N)} \text{ ATTLAM}$$

$$\frac{\Gamma; \Phi \models v : P}{\Gamma; \Phi \models \mathbf{return} v : \uparrow P} \text{ ATRETURN}$$

$$\frac{\Gamma; \Phi \models v : P \quad \Gamma; \Phi, x : P \models c : N}{\Gamma; \Phi \models \mathbf{let} x = v; c : N} \text{ ATVARLET}$$

$$\frac{\begin{array}{l} \Gamma \vdash P \quad \Gamma; \Phi \models v : \downarrow M \\ \Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow \uparrow Q \Rightarrow \Theta; SC_1 \quad \Gamma; \Theta \models \uparrow Q \leq \uparrow P \Rightarrow SC_2 \\ \Theta \vdash SC_1 \& SC_2 = SC \quad \Gamma; \Phi, x : P \models c : N \end{array}}{\Gamma; \Phi \models \mathbf{let} x : P = v(\vec{v}); c : N} \text{ ATAPPLETANN}$$

$$\frac{\begin{array}{l} \Gamma; \Phi \models v : \downarrow M \quad \Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow \uparrow Q \Rightarrow \Theta; SC \\ \mathbf{uv} Q \subseteq \mathbf{dom}(SC) \quad \mathbf{SC}|_{\mathbf{uv} Q} \text{ singular with } \hat{\sigma} \\ \Gamma; \Phi, x : [\hat{\sigma}] Q \models c : N \end{array}}{\Gamma; \Phi \models \mathbf{let} x = v(\vec{v}); c : N} \text{ ATAPPLET}$$

$$\frac{\Gamma; \Phi \models v : \exists \alpha^-. P \quad \Gamma, \alpha^-; \Phi, x : P \models c : N \quad \Gamma \vdash N}{\Gamma; \Phi \models \mathbf{let}^{\exists}(\alpha^-, x) = v; c : N} \text{ ATUNPACK}$$

$\boxed{\Gamma; \Phi; \Theta_1 \models N \bullet \vec{v} \Rightarrow M \Rightarrow \Theta_2; SC}$ Application type inference

$$\frac{}{\Gamma; \Phi; \Theta \models N \bullet \cdot \Rightarrow \mathbf{nf}(N) \Rightarrow \Theta; \cdot} \text{ AEMPTYAPP}$$

$$\frac{\Gamma; \Phi \models v : P \quad \Gamma; \Theta \models Q \geq P \Rightarrow SC_1 \quad \Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rightarrow M \Rightarrow \Theta'; SC_2 \quad \Theta \vdash SC_1 \& SC_2 = SC}{\Gamma; \Phi; \Theta \models Q \rightarrow N \bullet v, \vec{v} \Rightarrow M \Rightarrow \Theta'; SC} \text{ ATARROWAPP}$$

$$\frac{\begin{array}{l} \Gamma; \Phi; \Theta, \hat{\alpha}^+ \{ \Gamma \} \models [\hat{\alpha}^+ / \alpha^+] N \bullet \vec{v} \Rightarrow M \Rightarrow \Theta'; SC \\ \vec{v} \neq \cdot \quad \hat{\alpha}^+ \neq \cdot \end{array}}{\Gamma; \Phi; \Theta \models \forall \alpha^+. N \bullet \vec{v} \Rightarrow M \Rightarrow \Theta'; SC} \text{ ATFORALLAPP}$$

3 Proofs

3.1 Substitution

Lemma 1 (Substitution strengthening). *Restricting the substitution to the free variables of the substitution subject does not affect the result. Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

- + if $\Gamma_1 \vdash P$ then $[\sigma]P = [\sigma|_{\mathbf{fv} P}]P$,
- if $\Gamma_1 \vdash N$ then $[\sigma]N = [\sigma|_{\mathbf{fv} N}]N$

Proof. Ilya: todo □

Lemma 2. Suppose that $\Gamma' \subseteq \Gamma$, σ_1 and σ_2 are substitutions of signature $\Gamma \vdash \sigma_i : \Gamma'$. Then

- + for a type $\Gamma \vdash P$, if $\Gamma \vdash [\sigma_1]P \simeq_1^{\leq} [\sigma_2]P$ then $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : \mathbf{fv} P \cap \Gamma'$;
- for a type $\Gamma \vdash N$, if $\Gamma \vdash [\sigma_1]N \simeq_1^{\leq} [\sigma_2]N$ then $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : \mathbf{fv} N \cap \Gamma'$.

Proof. Let us make an additional assumption that σ_1 , σ_2 , and the mentioned types are normalized. If they are not, we normalize them first.

Notice that the normalization preserves the set of free variables (lemma 21), well-formedness (corollary 14), and equivalence (lemma 35), and distributes over substitution (lemma 23). This way, the assumed and desired properties are equivalent to their normalized versions.

We prove it by induction on the structure of P and mutually, N . Let us consider the shape of this type.

Case 1. $P = \alpha^+ \in \Gamma'$. Then $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : \mathbf{fv} P \cap \Gamma'$ means $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : \alpha^+$, i.e. $\Gamma \vdash [\sigma_1]\alpha^+ \simeq_1^{\leq} [\sigma_2]\alpha^+$, which holds by assumption.

Case 2. $P = \alpha^+ \in \Gamma \setminus \Gamma'$. Then $\mathbf{fv} P \cap \Gamma' = \emptyset$, so $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : \mathbf{fv} P \cap \Gamma'$ holds vacuously.

Case 3. $P = \downarrow N$. Then the induction hypothesis is applicable to type N :

1. N is normalized,
2. $\Gamma \vdash N$ by inversion of $\Gamma \vdash \downarrow N$,
3. $\Gamma \vdash [\sigma_1]N \simeq_1^{\leq} [\sigma_2]N$ holds by inversion of $\Gamma \vdash [\sigma_1]\downarrow N \simeq_1^{\leq} [\sigma_2]\downarrow N$, i.e. $\Gamma \vdash \downarrow[\sigma_1]N \simeq_1^{\leq} \downarrow[\sigma_2]N$.

This way, we obtain $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \sigma_2 : \mathbf{fv} N \cap \Gamma'$, which implies the required equivalence since $\mathbf{fv} P \cap \Gamma' = \mathbf{fv} \downarrow N \cap \Gamma' = \mathbf{fv} N \cap \Gamma'$.

Case 4. $P = \exists \alpha^{\rightarrow}.Q$. Then the induction hypothesis is applicable to type Q well-formed in context $\Gamma, \alpha^{\rightarrow}$:

1. $\Gamma' \subseteq \Gamma, \alpha^{\rightarrow}$ since $\Gamma' \subseteq \Gamma$,
2. $\Gamma, \alpha^{\rightarrow} \vdash \sigma_i : \Gamma'$ by weakening,
3. Q is normalized,
4. $\Gamma, \alpha^{\rightarrow} \vdash Q$ by inversion of $\Gamma \vdash \exists \alpha^{\rightarrow}.Q$,
5. Notice that $[\sigma_i]\exists \alpha^{\rightarrow}.Q$ is normalized, and thus, $[\sigma_1]\exists \alpha^{\rightarrow}.Q \simeq_1^D [\sigma_2]\exists \alpha^{\rightarrow}.Q$ implies $[\sigma_1]\exists \alpha^{\rightarrow}.Q = [\sigma_2]\exists \alpha^{\rightarrow}.Q$ (by lemma 35). This equality means $[\sigma_1]Q = [\sigma_2]Q$, which implies $\Gamma \vdash [\sigma_1]Q \simeq_1^{\leq} [\sigma_2]Q$.

Case 5. $N = P \rightarrow M$

□

Lemma 3 (Substitutions equivalent on the metavariables). Suppose that $\Gamma \vdash^{\exists} \Theta$, $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are substitutions of signature $\Theta \vdash \hat{\sigma}_i$. Then

- + for a type $\Gamma; \Theta \vdash P$, if $\Gamma \vdash [\hat{\sigma}_1]P \simeq_1^{\leq} [\hat{\sigma}_2]P$ then $\Theta \vdash \hat{\sigma}_1 \simeq_1^{\leq} \hat{\sigma}_2 : \mathbf{uv} P$;
- for a type $\Gamma; \Theta \vdash N$, if $\Gamma \vdash [\hat{\sigma}_1]N \simeq_1^{\leq} [\hat{\sigma}_2]N$ then $\Theta \vdash \hat{\sigma}_1 \simeq_1^{\leq} \hat{\sigma}_2 : \mathbf{uv} N$.

Proof. The proof is a trivial structural induction on $\Gamma; \Theta \vdash P$ and mutually, on $\Gamma; \Theta \vdash N$. □

Lemma 4 (Substitution composition well-formedness). If $\Gamma'_1 \vdash \sigma_1 : \Gamma_1$ and $\Gamma'_2 \vdash \sigma_2 : \Gamma_2$, then $\Gamma'_1, \Gamma'_2 \vdash \sigma_2 \circ \sigma_1 : \Gamma_1, \Gamma_2$.

Lemma 5 (Substitution monadic composition well-formedness). If $\Gamma'_1 \vdash \sigma_1 : \Gamma_1$ and $\Gamma'_2 \vdash \sigma_2 : \Gamma_2$, then $\Gamma'_2 \vdash \sigma_2 \ll \sigma_1 : \Gamma_1$.

Lemma 6 (Substitution composition). If $\Gamma'_1 \vdash \sigma_1 : \Gamma_1$, $\Gamma'_2 \vdash \sigma_2 : \Gamma_2$, $\Gamma_1 \cap \Gamma'_2 = \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$ then $\sigma_2 \circ \sigma_1 = (\sigma_2 \ll \sigma_1) \circ \sigma_2$.

Corollary 1 (Substitution composition commutativity). If $\Gamma'_1 \vdash \sigma_1 : \Gamma_1$, $\Gamma'_2 \vdash \sigma_2 : \Gamma_2$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \cap \Gamma'_2 = \emptyset$, and $\Gamma'_1 \cap \Gamma_2 = \emptyset$ then $\sigma_2 \circ \sigma_1 = \sigma_1 \circ \sigma_2$.

Proof. by lemma 6, $\sigma_2 \circ \sigma_1 = (\sigma_2 \ll \sigma_1) \circ \sigma_2$. Since the codomain of σ_1 is Γ'_1 , and it is disjoint with the domain of σ_2 , $\sigma_2 \ll \sigma_1 = \sigma_1$. □

Lemma 7 (Substitution domain weakening). If $\Gamma_2 \vdash \sigma : \Gamma_1$ then $\Gamma_2, \Gamma' \vdash \sigma : \Gamma_1, \Gamma'$

Proof. If the variable α^{\pm} is in Γ_1 then $\Gamma_2 \vdash [\sigma]\alpha^{\pm}$ by assumption, and then $\Gamma_2, \Gamma' \vdash [\sigma]\alpha^{\pm}$ by weakening. If the variable α^{\pm} is in $\Gamma' \setminus \Gamma_1$ then $[\sigma]\alpha^{\pm} = \alpha^{\pm} \in \Gamma' \Gamma_2, \Gamma'$, and thus, $\Gamma_2, \Gamma' \vdash \alpha^{\pm}$. □

3.2 Declarative Subtyping

Lemma 8 (Free Variable Propagation). *In the judgments of negative subtyping or positive supertyping, free variables propagate left-to-right. For a context Γ ,*

- $-$ if $\Gamma \vdash N \leq_1 M$ then $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$
- $+$ if $\Gamma \vdash P \geq_1 Q$ then $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$

Proof. Mutual induction on $\Gamma \vdash N \leq_1 M$ and $\Gamma \vdash P \geq_1 Q$.

Case 1. $\Gamma \vdash \alpha^- \leq_1 \alpha^-$

It is self-evident that $\alpha^- \subseteq \alpha^-$.

Case 2. $\Gamma \vdash \uparrow P \leq_1 \uparrow Q$ From the inversion (and unfolding $\Gamma \vdash P \simeq_1^< Q$), we have $\Gamma \vdash P \geq_1 Q$. Then by the induction hypothesis, $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$. The desired inclusion holds, since $\mathbf{fv}(\uparrow P) = \mathbf{fv}(P)$ and $\mathbf{fv}(\uparrow Q) = \mathbf{fv}(Q)$.

Case 3. $\Gamma \vdash P \rightarrow N \leq_1 Q \rightarrow M$ The induction hypothesis applied to the premises gives: $\mathbf{fv}(P) \subseteq \mathbf{fv}(Q)$ and $\mathbf{fv}(N) \subseteq \mathbf{fv}(M)$. Then $\mathbf{fv}(P \rightarrow N) = \mathbf{fv}(P) \cup \mathbf{fv}(N) \subseteq \mathbf{fv}(Q) \cup \mathbf{fv}(M) = \mathbf{fv}(Q \rightarrow M)$.

Case 4. $\Gamma \vdash \forall \alpha^+. N \leq_1 \forall \beta^+. M$

$$\begin{aligned} \mathbf{fv} \forall \alpha^+. N &\subseteq \mathbf{fv}([\vec{P}/\alpha^+]N) \setminus \vec{\beta}^+ && \text{here } \vec{\beta}^+ \text{ is excluded by the premise } \mathbf{fv} N \cap \vec{\beta}^+ = \emptyset \\ &\subseteq \mathbf{fv} M \setminus \vec{\beta}^+ && \text{by the induction hypothesis, } \mathbf{fv}([\vec{P}/\alpha^+]N) \subseteq \mathbf{fv} M \\ &\subseteq \mathbf{fv} \forall \beta^+. M \end{aligned}$$

Case 5. The positive cases are symmetric. □

Corollary 2 (Free Variables of mutual subtypes).

- $-$ If $\Gamma \vdash N \simeq_1^< M$ then $\mathbf{fv} N = \mathbf{fv} M$,
- $+$ If $\Gamma \vdash P \simeq_1^< Q$ then $\mathbf{fv} P = \mathbf{fv} Q$

Lemma 9 (Decomposition of quantifier rules). *Assuming that $\vec{\alpha}^+$, $\vec{\beta}^+$, $\vec{\alpha}^-$, and $\vec{\alpha}^-$ are disjoint from Γ ,*

$-_R$ $\Gamma \vdash N \leq_1 \forall \vec{\beta}^+. M$ holds if and only if $\Gamma, \vec{\beta}^+ \vdash N \leq_1 M$;

$+_R$ $\Gamma \vdash P \geq_1 \exists \vec{\beta}^+. Q$ holds if and only if $\Gamma, \vec{\beta}^+ \vdash P \geq_1 Q$;

$-_L$ suppose $M \neq \forall \dots$ then $\Gamma \vdash \forall \vec{\alpha}^+. N \leq_1 M$ holds if and only if $\Gamma \vdash [\vec{P}/\vec{\alpha}^+]N \leq_1 M$ for some $\Gamma \vdash \vec{P}$;

$+_L$ suppose $Q \neq \exists \dots$ then $\Gamma \vdash \exists \vec{\alpha}^+. P \geq_1 Q$ holds if and only if $\Gamma \vdash [\vec{N}/\vec{\alpha}^+]P \geq_1 Q$ for some $\Gamma \vdash \vec{N}$.

Proof.

$-_R$ Let us prove both directions.

\Rightarrow Let us assume $\Gamma \vdash N \leq_1 \forall \vec{\beta}^+. M$. $\Gamma \vdash N \leq_1 \forall \vec{\beta}^+. M$. Let us decompose M as $\forall \vec{\beta}^+. M'$ where M' does not start with \forall , and decompose N as $\forall \vec{\alpha}^+. N'$ where N' does not start with \forall . If $\vec{\beta}^+$ is empty, then $\Gamma, \vec{\beta}^+ \vdash N \leq_1 M$ holds by assumption. Otherwise, $\Gamma \vdash \forall \vec{\alpha}^+. N' \leq_1 \forall \vec{\beta}^+. \forall \vec{\beta}^+. M'$ is inferred by Rule (\forall^{\leq_1}) , and by inversion: $\Gamma, \vec{\beta}^+, \vec{\beta}^+ \vdash [\vec{P}/\vec{\alpha}^+]N' \leq_1 M'$ for some $\Gamma, \vec{\beta}^+, \vec{\beta}^+ \vdash \vec{P}$. Then again by Rule (\forall^{\leq_1}) with the same \vec{P} , $\Gamma, \vec{\beta}^+ \vdash \forall \vec{\alpha}^+. N' \leq_1 \forall \vec{\beta}^+. M'$, that is $\Gamma, \vec{\beta}^+ \vdash N \leq_1 M$.

\Leftarrow let us assume $\Gamma, \vec{\beta}^+ \vdash N \leq_1 M$, and let us decompose N as $\forall \vec{\alpha}^+. N'$ where N' does not start with \forall , and M as $\forall \vec{\beta}^+. M'$ where M' does not start with \forall . if $\vec{\alpha}^+$ and $\vec{\beta}^+$ are empty then $\Gamma, \vec{\beta}^+ \vdash N \leq_1 M$ is turned into $\Gamma \vdash N \leq_1 \forall \vec{\beta}^+. M$ by Rule (\forall^{\leq_1}) . Otherwise, $\Gamma, \vec{\beta}^+ \vdash \forall \vec{\alpha}^+. N' \leq_1 \forall \vec{\beta}^+. M'$ is inferred by Rule (\forall^{\leq_1}) , that is $\Gamma, \vec{\beta}^+, \vec{\beta}^+ \vdash [\vec{P}/\vec{\alpha}^+]N' \leq_1 M'$ for some $\Gamma, \vec{\beta}^+, \vec{\beta}^+ \vdash \vec{P}$. Then by Rule (\forall^{\leq_1}) again, $\Gamma \vdash \forall \vec{\alpha}^+. N' \leq_1 \forall \vec{\beta}^+, \vec{\beta}^+. M'$, in other words, $\Gamma \vdash \forall \vec{\alpha}^+. N' \leq_1 \forall \vec{\beta}^+. \forall \vec{\beta}^+. M'$, that is $\Gamma \vdash N \leq_1 \forall \vec{\beta}^+. M$.

$-_L$ Suppose $M \neq \forall \dots$. Let us prove both directions.

\Rightarrow Let us assume $\Gamma \vdash \forall \vec{\alpha}^+. N \leq_1 M$. then if $\vec{\alpha}^+ = \cdot$, $\Gamma \vdash N \leq_1 M$ holds immediately. Otherwise, let us decompose iN as $\forall \vec{\alpha}^+. N'$ where N' does not start with \forall . Then $\Gamma \vdash \forall \vec{\alpha}^+. \forall \vec{\alpha}^+. N' \leq_1 M'$ is inferred by Rule (\forall^{\leq_1}) , and by inversion, there exist $\Gamma \vdash \vec{P}, \vec{P}'$ such that $\Gamma \vdash [\vec{P}/\vec{\alpha}^+][\vec{P}'/\vec{\alpha}^+]N' \leq_1 M'$ (the decomposition of substitutions is possible since $\vec{\alpha}^+ \cap \Gamma = \emptyset$). Then by Rule (\forall^{\leq_1}) again, $\Gamma \vdash \forall \vec{\alpha}^+. [\vec{P}'/\vec{\alpha}^+]N' \leq_1 M'$ (notice that $[\vec{P}'/\vec{\alpha}^+]N'$ cannot start with \forall).

\Leftarrow Let us assume $\Gamma \vdash [\vec{P}/\vec{\alpha}^+]N \leq_1 M$ for some $\Gamma \vdash \vec{P}$. let us decompose iN as $\forall \vec{\alpha}^{+'}.N'$ where N' does not start with \forall . Then $\Gamma \vdash [\vec{P}/\vec{\alpha}^+]\forall \vec{\alpha}^{+'}.N' \leq_1 M'$ or, equivalently, $\Gamma \vdash \forall \vec{\alpha}^{+'}. [\vec{P}/\vec{\alpha}^+]N' \leq_1 M'$ is inferred by Rule (\forall^{\leq_1}) (notice that $[\vec{P}/\vec{\alpha}^+]N'$ cannot start with \forall). By inversion, there exist $\Gamma \vdash \vec{P}'$ such that $\Gamma \vdash [\vec{P}'/\vec{\alpha}^+][\vec{P}/\vec{\alpha}^+]N' \leq_1 M'$. Since $\vec{\alpha}^{+'}$ is disjoint from the free variables of \vec{P} and from $\vec{\alpha}^+$, the composition of $\vec{P}'/\vec{\alpha}^{+'}$ and $\vec{P}/\vec{\alpha}^+$ can be joined into a single substitution well-formed in Γ . Then by Rule (\forall^{\leq_1}) again, $\Gamma \vdash \forall \vec{\alpha}^+.N \leq_1 M$.

+ The positive cases are proved symmetrically. □

Corollary 3 (Redundant quantifier elimination).

$-_L$ Suppose that $\vec{\alpha}^+ \cap \mathbf{fv}(N) = \emptyset$ then $\Gamma \vdash \forall \vec{\alpha}^+.N \leq_1 M$ holds if and only if $\Gamma \vdash N \leq_1 M$;

$-_R$ Suppose that $\vec{\alpha}^+ \cap \mathbf{fv}(M) = \emptyset$ then $\Gamma \vdash N \leq_1 \forall \vec{\alpha}^+.M$ holds if and only if $\Gamma \vdash N \leq_1 M$;

$+_L$ Suppose that $\vec{\alpha}^- \cap \mathbf{fv}(P) = \emptyset$ then $\Gamma \vdash \exists \vec{\alpha}^-.P \geq_1 Q$ holds if and only if $\Gamma \vdash P \geq_1 Q$.

$+_R$ Suppose that $\vec{\alpha}^- \cap \mathbf{fv}(Q) = \emptyset$ then $\Gamma \vdash P \geq_1 \exists \vec{\alpha}^-.Q$ holds if and only if $\Gamma \vdash P \geq_1 Q$.

Proof. $\rightarrow -_R$ Suppose that $\vec{\alpha}^+ \cap \mathbf{fv}(M) = \emptyset$ then by lemma 9, $\Gamma \vdash N \leq_1 \forall \vec{\alpha}^+.M$ is equivalent to $\Gamma, \vec{\alpha}^+ \vdash N \leq_1 M$. By , since $\vec{\alpha}^+ \cap \mathbf{fv}(N) = \emptyset$ and $\vec{\alpha}^+ \cap \mathbf{fv}(M) = \emptyset$, $\Gamma, \vec{\alpha}^+ \vdash N \leq_1 M$ is equivalent to $\Gamma \vdash N \leq_1 M$.

$-_L$ Suppose that $\vec{\alpha}^+ \cap \mathbf{fv}(N) = \emptyset$. Let us decompose M as $\forall \vec{\beta}^+.M'$ where M' does not start with \forall . By lemma 9, $\Gamma \vdash \forall \vec{\alpha}^+.N \leq_1 \forall \vec{\beta}^+.M'$ is equivalent to $\Gamma, \vec{\beta}^+ \vdash \forall \vec{\alpha}^+.N \leq_1 M'$, which is equivalent to existence of $\Gamma, \vec{\beta}^+ \vdash \vec{P}$ such that $\Gamma, \vec{\beta}^+ \vdash [\vec{P}/\vec{\alpha}^+]N \leq_1 M'$. Since $[\vec{P}/\vec{\alpha}^+]N = N$, the latter is equivalent to $\Gamma, \vec{\beta}^+ \vdash N \leq_1 M'$, which is equivalent to $\Gamma \vdash N \leq_1 \forall \vec{\beta}^+.M'$. $\Gamma, \vec{\beta}^+ \vdash \vec{P}$ can be chosen arbitrary, for example, $\vec{P}_i = \exists \alpha^-. \downarrow \alpha^-$.

+ The positive cases are proved symmetrically. □

Lemma 10 (Subtypes and supertypes of a variable). Assuming $\Gamma \vdash \alpha^-$, $\Gamma \vdash \alpha^+$, $\Gamma \vdash N$, and $\Gamma \vdash P$,

+ if $\Gamma \vdash P \geq_1 \exists \vec{\alpha}^-. \alpha^+$ or $\Gamma \vdash \exists \vec{\alpha}^-. \alpha^+ \geq_1 P$ then $P = \exists \vec{\beta}^-. \alpha^+$ (for some potentially empty $\vec{\beta}^-$)

- if $\Gamma \vdash N \leq_1 \forall \vec{\alpha}^+. \alpha^-$ or $\Gamma \vdash \forall \vec{\alpha}^+. \alpha^- \leq_1 N$ then $N = \forall \vec{\beta}^+. \alpha^-$ (for some potentially empty $\vec{\beta}^+$)

Proof. We prove by induction on the tree inferring $\Gamma \vdash P \geq_1 \exists \vec{\alpha}^-. \alpha^+$ or $\Gamma \vdash \exists \vec{\alpha}^-. \alpha^+ \geq_1 P$ or $\Gamma \vdash N \leq_1 \forall \vec{\alpha}^+. \alpha^-$ or $\Gamma \vdash \forall \vec{\alpha}^+. \alpha^- \leq_1 N$.

Let us consider which one of these judgments is inferred.

Case 1. $\Gamma \vdash P \geq_1 \exists \vec{\alpha}^-. \alpha^+$

If the size of the inference tree is 1 then the only rule that can infer it is Rule $(\text{Var}^{+ \geq_1})$, which implies that $\vec{\alpha}^-$ is empty and $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule (\exists^{\geq_1}) . By inverting this rule, $P = \exists \vec{\beta}^-. P'$ where P' does not start with \exists and $\Gamma, \vec{\alpha}^- \vdash [\vec{N}/\vec{\beta}^-]P' \geq_1 \alpha^+$ for some $\Gamma, \vec{\alpha}^- \vdash N_i$.

By the induction hypothesis, $[\vec{N}/\vec{\beta}^-]P' = \exists \vec{\gamma}^-. \alpha^+$. What shape can P' have? As mentioned, it does not start with \exists , and it cannot start with \uparrow (otherwise, $[\vec{N}/\vec{\alpha}^-]P'$ would also start with \uparrow and would not be equal to $\exists \vec{\beta}^-. \alpha^+$). This way, P' is a *positive* variable. As such, $[\vec{N}/\vec{\alpha}^-]P' = P'$, and then $P' = \exists \vec{\gamma}^-. \alpha^+$ meaning that $\vec{\gamma}^-$ is empty and $P' = \alpha^+$. This way, $P = \exists \vec{\beta}^-. P' = \exists \vec{\beta}^-. \alpha^+$, as required.

Case 2. $\Gamma \vdash \exists \vec{\alpha}^-. \alpha^+ \geq_1 P$

If the size of the inference tree is 1 then the only rule that can infer it is Rule $(\text{Var}^{+ \geq_1})$, which implies that $\vec{\alpha}^-$ is empty and $P = \alpha^+$.

If the size of the inference tree is > 1 then the last rule inferring it must be Rule (\exists^{\geq_1}) . By inverting this rule, $P = \exists \vec{\beta}^-. Q$ where $\Gamma, \vec{\beta}^- \vdash [\vec{N}/\vec{\alpha}^-] \alpha^+ \geq_1 Q$ and Q does not start with \exists . Notice that since α^+ is positive, $[\vec{N}/\vec{\alpha}^-] \alpha^+ = \alpha^+$, i.e. $\Gamma, \vec{\beta}^- \vdash \alpha^+ \geq_1 Q$.

By the induction hypothesis, $Q = \exists \vec{\beta}'^-. \alpha^+$, and since Q does not start with \exists , $\vec{\beta}'^-$ is empty. This way, $P = \exists \vec{\beta}^-. Q = \exists \vec{\beta}^-. \alpha^+$, as required.

Case 3. The negative cases ($\Gamma \vdash N \leq_1 \forall \vec{\alpha}^+. \alpha^-$ and $\Gamma \vdash \forall \vec{\alpha}^+. \alpha^- \leq_1 N$) are proved analogously.

□

Corollary 4 (Variables have no proper subtypes and supertypes). *Assuming that all mentioned types are well-formed in Γ ,*

$$\begin{aligned}
\Gamma \vdash P \geq_1 \alpha^+ &\iff P = \exists \vec{\beta}^-. \alpha^+ \iff \Gamma \vdash P \simeq_1^{\leq} \alpha^+ \iff P \simeq_1^D \alpha^+ \\
\Gamma \vdash \alpha^+ \geq_1 P &\iff P = \exists \vec{\beta}^-. \alpha^+ \iff \Gamma \vdash P \simeq_1^{\leq} \alpha^+ \iff P \simeq_1^D \alpha^+ \\
\Gamma \vdash N \leq_1 \alpha^- &\iff N = \forall \vec{\beta}^+. \alpha^- \iff \Gamma \vdash N \simeq_1^{\leq} \alpha^- \iff N \simeq_1^D \alpha^- \\
\Gamma \vdash \alpha^- \leq_1 N &\iff N = \forall \vec{\beta}^+. \alpha^- \iff \Gamma \vdash N \simeq_1^{\leq} \alpha^- \iff N \simeq_1^D \alpha^-
\end{aligned}$$

Proof. Notice that $\Gamma \vdash \exists \vec{\beta}^-. \alpha^+ \simeq_1^{\leq} \alpha^+$ and $\exists \vec{\beta}^-. \alpha^+ \simeq_1^D \alpha^+$ and apply lemma 10. □

Lemma 11 (Reflexivity of subtyping). *Assuming all the types are well-formed in Γ ,*

$$- \Gamma \vdash N \leq_1 N$$

$$+ \Gamma \vdash P \geq_1 P$$

Proof. Let us prove it by the size of N and mutually, P .

Case 1. $N = \alpha^-$

Then $\Gamma \vdash \alpha^- \leq_1 \alpha^-$ is inferred immediately by Rule (Var^{-≤₁}).

Case 2. $N = \forall \vec{\alpha}^+. N'$ where $\vec{\alpha}^+$ is not empty

First, we rename $\vec{\alpha}^+$ to fresh $\vec{\beta}^+$ in $\forall \vec{\alpha}^+. N'$ to avoid name clashes: $\forall \vec{\alpha}^+. N' = \forall \vec{\beta}^+. [\vec{\alpha}^+ / \vec{\beta}^+] N'$. Then to infer $\Gamma \vdash \forall \vec{\alpha}^+. N' \leq_1 \forall \vec{\beta}^+. [\vec{\alpha}^+ / \vec{\beta}^+] N'$ we can apply Rule (V^{≤₁}), instantiating $\vec{\alpha}^+$ with $\vec{\beta}^+$:

- $\text{fv } N \cap \vec{\beta}^+ = \emptyset$ by choice of $\vec{\beta}^+$,
- $\Gamma, \vec{\beta}^+ \vdash \beta_i^+$,
- $\Gamma, \vec{\beta}^+ \vdash [\vec{\beta}^+ / \vec{\alpha}^+] N' \leq_1 [\vec{\beta}^+ / \vec{\alpha}^+] N'$ by the induction hypothesis, since the size of $[\vec{\beta}^+ / \vec{\alpha}^+] N'$ is equal to the size of N' , which is smaller than the size of $N = \forall \vec{\alpha}^+. N'$.

Case 3. $N = P \rightarrow M$

Then $\Gamma \vdash P \rightarrow M \leq_1 P \rightarrow M$ is inferred by Rule (\rightarrow^{\leq_1}), since $\Gamma \vdash P \geq_1 P$ and $\Gamma \vdash M \leq_1 M$ hold the induction hypothesis.

Case 4. $N = \uparrow P$

Then $\Gamma \vdash \uparrow P \leq_1 \uparrow P$ is inferred by Rule (\uparrow^{\leq_1}), since $\Gamma \vdash P \geq_1 P$ holds by the induction hypothesis.

Case 5. The positive cases are symmetric to the negative ones.

□

Lemma 12 (Substitution preserves subtyping). *Suppose that all mentioned types are well-formed in Γ_1 , and σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$.*

- - If $\Gamma_1 \vdash N \leq_1 M$ then $\Gamma_2 \vdash [\sigma]N \leq_1 [\sigma]M$
- + If $\Gamma_1 \vdash P \geq_1 Q$ then $\Gamma_2 \vdash [\sigma]P \geq_1 [\sigma]Q$

Proof. We prove it by induction on the size of the derivation of $\Gamma_1 \vdash N \leq_1 M$ and mutually, $\Gamma_1 \vdash P \geq_1 Q$. Let us consider the last rule used in the derivation:

Case 1. Rule (Var^{-≤₁}). Then by inversion, $N = \alpha^-$ and $M = \alpha^-$. By reflexivity of subtyping (lemma 11), we have $\Gamma_2 \vdash [\sigma]\alpha^- \leq_1 [\sigma]\alpha^-$, i.e. $\Gamma_2 \vdash [\sigma]N \leq_1 [\sigma]M$, as required.

Case 2. Rule (V^{≤₁}). Then by inversion, $N = \forall \vec{\alpha}^+. N'$, $M = \forall \vec{\beta}^+. M'$, where $\vec{\alpha}^+$ or $\vec{\beta}^+$ is not empty. Moreover, $\Gamma_1, \vec{\beta}^+ \vdash [\vec{P} / \vec{\alpha}^+] N' \leq_1 M'$ for some $\Gamma_1, \vec{\beta}^+ \vdash \vec{P}$, and $\text{fv } N \cap \vec{\beta}^+ = \emptyset$.

Notice that since the derivation of $\Gamma_1, \vec{\beta}^+ \vdash [\vec{P} / \vec{\alpha}^+] N' \leq_1 M'$ is a subderivation of the derivation of $\Gamma \vdash N \leq_1 M$, its size is smaller, and hence, the induction hypothesis applies ($\Gamma_1, \vec{\beta}^+ \vdash \sigma : \Gamma_1, \vec{\beta}^+$ by) : $\Gamma_2, \vec{\beta}^+ \vdash [\sigma][\vec{P} / \vec{\alpha}^+] N' \leq_1 [\sigma]M'$.

Notice that by convention, $\vec{\alpha}^+$ and $\vec{\beta}^+$ are fresh, and thus, $[\sigma]\forall \vec{\alpha}^+. N' = \forall \vec{\alpha}^+. [\sigma]N'$ and $[\sigma]\forall \vec{\beta}^+. M' = \forall \vec{\beta}^+. [\sigma]M'$, which means that the required $\Gamma_2, \Gamma \vdash [\sigma]\forall \vec{\alpha}^+. N' \leq_1 [\sigma]\forall \vec{\beta}^+. M'$ is rewritten as $\Gamma_2, \Gamma \vdash \forall \vec{\alpha}^+. [\sigma]N' \leq_1 \forall \vec{\beta}^+. [\sigma]M'$.

To infer it, we apply Rule (V^{≤₁}), instantiating α_i^+ with $[\sigma]P_i$:

- $\mathbf{fv}[\sigma]N \cap \overrightarrow{\beta^+} = \emptyset$;
- $\Gamma_2, \Gamma, \overrightarrow{\beta^+} \vdash [\sigma]P_i$, by corollary 8 since from the inversion, $\Gamma_1, \Gamma, \overrightarrow{\beta^+} \vdash P_i$;
- $\Gamma, \overrightarrow{\beta^+} \vdash [[\sigma]\overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N' \leq_1 [\sigma]M'$ holds by lemma 6: Since $\overrightarrow{\alpha^+}$ is fresh, it is disjoint with the domain and the codomain of σ (Γ_1 and Γ_2), and thus, $[\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' = [\sigma \ll \overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N' = [[\sigma]\overrightarrow{P}/\overrightarrow{\alpha^+}][\sigma]N'$. Then $\Gamma_2, \Gamma, \overrightarrow{\beta^+} \vdash [\sigma][\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leq_1 [\sigma]M'$ holds by the induction hypothesis.

Case 3. Rule (\rightarrow^{\leq_1}) . Then by inversion, $N = P \rightarrow N_1$, $M = Q \rightarrow M_1$, $\Gamma \vdash P \geq_1 Q$, and $\Gamma \vdash N_1 \leq_1 M_1$. And by the induction hypothesis, $\Gamma' \vdash [\sigma]P \geq_1 [\sigma]Q$ and $\Gamma' \vdash [\sigma]N_1 \leq_1 [\sigma]M_1$. Then $\Gamma' \vdash [\sigma]N \leq_1 [\sigma]M$, i.e. $\Gamma' \vdash [\sigma]P \rightarrow [\sigma]N_1 \leq_1 [\sigma]Q \rightarrow [\sigma]M_1$, is inferred by Rule (\rightarrow^{\leq_1}) .

Case 4. Rule (\uparrow^{\leq_1}) . Then by inversion, $N = \uparrow P$, $M = \uparrow Q$, and $\Gamma \vdash P \leq_1^{\leq} Q$, which by inversion means that $\Gamma \vdash P \geq_1 Q$ and $\Gamma \vdash Q \geq_1 P$. Then the induction hypothesis applies, and we have $\Gamma' \vdash [\sigma]P \geq_1 [\sigma]Q$ and $\Gamma' \vdash [\sigma]Q \geq_1 [\sigma]P$. Then by sequential application of Rule (\simeq_1^-) and Rule (\uparrow^{\leq_1}) to these judgments, we have $\Gamma' \vdash \uparrow[\sigma]P \leq_1 \uparrow[\sigma]Q$, i.e. $\Gamma' \vdash [\sigma]N \leq_1 [\sigma]M$, as required.

Case 5. The positive cases are proved symmetrically. □

Corollary 5 (Substitution preserves subtyping induced equivalence). *Suppose that $\Gamma \vdash \sigma : \Gamma_1$. Then*

- + if $\Gamma_1 \vdash P$, $\Gamma_1 \vdash Q$, and $\Gamma_1 \vdash P \simeq_1^{\leq} Q$ then $\Gamma \vdash [\sigma]P \simeq_1^{\leq} [\sigma]Q$
- if $\Gamma_1 \vdash N$, $\Gamma_1 \vdash M$, and $\Gamma_1 \vdash N \simeq_1^{\leq} M$ then $\Gamma \vdash [\sigma]N \simeq_1^{\leq} [\sigma]M$

Lemma 13 (Strong transitivity of subtyping). *Assuming all the types are well-formed in Γ ,*

- - if $\Gamma \vdash N \leq_1 M_1$, $\Gamma \vdash M_2 \leq_1 K$, and for $\Gamma' \vdash \sigma : \Gamma$, $[\sigma]M_1 = [\sigma]M_2$ then $\Gamma' \vdash [\sigma]N \leq_1 [\sigma]K$
- + if $\Gamma \vdash P \geq_1 Q_1$, $\Gamma \vdash Q_2 \geq_1 R$, and for $\Gamma' \vdash \sigma : \Gamma$, $[\sigma]Q_1 = [\sigma]Q_2$ then $\Gamma' \vdash [\sigma]P \geq_1 [\sigma]R$

Proof. We prove it by induction on $\text{depth}(\Gamma \vdash N \leq_1 M_1) + \text{depth}(\Gamma \vdash M_2 \leq_1 K)$ and mutually, on $\text{depth}(\Gamma \vdash P \geq_1 Q_1) + \text{depth}(\Gamma \vdash Q_2 \geq_1 R)$.

First, let us consider the 3 important cases.

Case 1. Let us consider the case when $M_1 = \forall \overrightarrow{\beta^+}_1. \alpha^-$. Then by lemma 10, $\Gamma \vdash N \leq_1 M_1$ means that $N = \forall \overrightarrow{\alpha^+}. \alpha^-$. $[\sigma]M_1 = [\sigma]M_2$ means that $\forall \overrightarrow{\beta^+}_1. [\sigma]\alpha^- = [\sigma]M_2$. Applying σ to both sides of $\Gamma \vdash M_2 \leq_1 K$ (by lemma 12), we obtain $\Gamma' \vdash [\sigma]M_2 \leq_1 [\sigma]K$, that is $\Gamma' \vdash \forall \overrightarrow{\beta^+}_1. [\sigma]\alpha^- \leq_1 [\sigma]K$. Since $\mathbf{fv}([\sigma]\alpha^-) \subseteq \Gamma, \alpha^-$, it is disjoint from $\overrightarrow{\alpha^+}$ and $\overrightarrow{\beta^+}_1$. This way, by corollary 3, $\Gamma' \vdash \forall \overrightarrow{\beta^+}_1. [\sigma]\alpha^- \leq_1 [\sigma]K$ is equivalent to $\Gamma' \vdash [\sigma]\alpha^- \leq_1 [\sigma]K$, which is equivalent to $\Gamma' \vdash \forall \overrightarrow{\alpha^+}. [\sigma]\alpha^- \leq_1 [\sigma]K$, that is $\Gamma' \vdash [\sigma]N \leq_1 [\sigma]K$.

Case 2. Let us consider the case when $M_2 = \forall \overrightarrow{\beta^+}_2. \alpha^-$. This case is symmetric to the previous one. Notice that lemma 10 and corollary 3 are agnostic to the side on which the the quantifiers occur, and thus, the proof stays the same.

Case 3. Let us decompose the types, by extracting the outer quantifiers:

- $N = \forall \overrightarrow{\alpha^+}. N'$, where $N' \neq \forall \dots$,
- $M_1 = \forall \overrightarrow{\beta^+}_1. M'_1$, where $M'_1 \neq \forall \dots$,
- $M_2 = \forall \overrightarrow{\beta^+}_2. M'_2$, where $M'_2 \neq \forall \dots$,
- $K = \forall \overrightarrow{\gamma^+}. K'$, where $K' \neq \forall \dots$.

and assume that at least one of $\overrightarrow{\alpha^+}$, $\overrightarrow{\beta^+}_1$, $\overrightarrow{\beta^+}_2$, and $\overrightarrow{\gamma^+}$ is not empty. Since $[\sigma]M_1 = [\sigma]M_2$, we have $\forall \overrightarrow{\beta^+}_1. [\sigma]M'_1 = \forall \overrightarrow{\beta^+}_2. [\sigma]M'_2$, and since M'_i are not variables (which was covered by the previous cases) and do not start with \forall , $[\sigma]M'_i$ do not start with \forall either, which means $\overrightarrow{\beta^+}_1 = \overrightarrow{\beta^+}_2$ and $[\sigma]M'_1 = [\sigma]M'_2$. Let us rename $\overrightarrow{\beta^+}_1$ and $\overrightarrow{\beta^+}_2$ to $\overrightarrow{\beta^+}$. Then $M_1 = \forall \overrightarrow{\beta^+}. M'_1$ and $M_2 = \forall \overrightarrow{\beta^+}. M'_2$.

By lemma 9 applied twice to $\Gamma \vdash \forall \overrightarrow{\alpha^+}. N' \leq_1 \forall \overrightarrow{\beta^+}. M'_1$ and to $\Gamma \vdash \forall \overrightarrow{\beta^+}. M'_2 \leq_1 \forall \overrightarrow{\gamma^+}. K'$, we have the following:

1. $\Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N' \leq_1 M'_1$ for some $\Gamma, \overrightarrow{\beta^+} \vdash \overrightarrow{P}$;
2. $\Gamma, \overrightarrow{\gamma^+} \vdash [\overrightarrow{Q}/\overrightarrow{\beta^+}]M'_2 \leq_1 K'$ for some $\Gamma, \overrightarrow{\gamma^+} \vdash \overrightarrow{Q}$.

And since at least one of $\vec{\alpha}^+$, $\vec{\beta}^+$, and $\vec{\gamma}^+$ is not empty, either $\Gamma \vdash N \leq_1 M_1$ or $\Gamma \vdash M_2 \leq_1 K$ is inferred by Rule (\forall^{\leq_1}), meaning that either $\Gamma, \vec{\beta}^+ \vdash [\vec{P}/\vec{\alpha}^+]N' \leq_1 M'_1$ is a proper subderivation of $\Gamma \vdash N \leq_1 M_1$ or $\Gamma, \vec{\gamma}^+ \vdash [\vec{Q}/\vec{\beta}^+]M'_2 \leq_1 K'$ is a proper subderivation of $\Gamma \vdash M_2 \leq_1 K$.

Notice that we can weaken and rearrange the contexts without changing the depths of the derivations: $\Gamma, \vec{\beta}^+, \vec{\gamma}^+ \vdash [\vec{P}/\vec{\alpha}^+]N' \leq_1 M'_1$ and $\Gamma, \vec{\beta}^+, \vec{\gamma}^+ \vdash [\vec{Q}/\vec{\beta}^+]M'_2 \leq_1 K'$. This way, the sum of the depths of these derivations is smaller than the sum of the depths of $\Gamma \vdash N \leq_1 M_1$ and $\Gamma \vdash M_2 \leq_1 K$. Let us apply the induction hypothesis to these derivations, with the substitution $\Gamma', \vec{\gamma}^+ \vdash \sigma \circ (\vec{Q}/\vec{\beta}^+) : \Gamma, \vec{\beta}^+, \vec{\gamma}^+$ (section 3.12). To apply the induction hypothesis, it is left to show that $\sigma \circ (\vec{Q}/\vec{\beta}^+)$ unifies M'_1 and $[\vec{Q}/\vec{\beta}^+]M'_2$:

$$\begin{aligned} [\sigma \circ \vec{Q}/\vec{\beta}^+]M'_1 &= [\sigma][\vec{Q}/\vec{\beta}^+]M'_1 \\ &= [[\sigma]\vec{Q}/\vec{\beta}^+][\sigma]M'_2 && \text{by lemma 6} \\ &= [[\sigma]\vec{Q}/\vec{\beta}^+][\sigma]M'_2 && \text{Since } [\sigma]M'_1 = [\sigma]M'_2 \\ &= [\sigma][\vec{Q}/\vec{\beta}^+]M'_2 && \text{by lemma 6} \\ &= [\sigma][\vec{Q}/\vec{\beta}^+][\vec{Q}/\vec{\beta}^+]M'_2 && \text{Since } \Gamma, \vec{\gamma}^+ \vdash \vec{Q}, \text{ and } (\Gamma, \vec{\gamma}^+) \cap \vec{\beta}^+ = \emptyset \\ &= [\sigma \circ \vec{Q}/\vec{\beta}^+][\vec{Q}/\vec{\beta}^+]M'_2 \end{aligned}$$

This way the induction hypothesis gives us $\Gamma', \vec{\gamma}^+ \vdash [\sigma][\vec{Q}/\vec{\beta}^+][\vec{P}/\vec{\alpha}^+]N' \leq_1 [\sigma][\vec{Q}/\vec{\beta}^+]K'$, and since $\Gamma, \vec{\gamma}^+ \vdash K'$, $[\vec{Q}/\vec{\beta}^+]K' = K'$, that is $\Gamma', \vec{\gamma}^+ \vdash [\sigma][\vec{Q}/\vec{\beta}^+][\vec{P}/\vec{\alpha}^+]N' \leq_1 [\sigma]K'$. Let us rewrite the substitution that we apply to N' :

$$\begin{aligned} [\sigma \circ \vec{Q}/\vec{\beta}^+ \circ \vec{P}/\vec{\alpha}^+]N' &= [(\sigma \ll \vec{Q}/\vec{\beta}^+) \circ \sigma \circ \vec{P}/\vec{\alpha}^+]N' && \text{by lemma 6} \\ &= [(\sigma \ll \vec{Q}/\vec{\beta}^+) \circ (\sigma \ll \vec{P}/\vec{\alpha}^+) \circ \sigma]N' && \text{by lemma 6} \\ &= [(((\sigma \ll \vec{Q}/\vec{\beta}^+) \circ \sigma) \ll \vec{P}/\vec{\alpha}^+) \circ \sigma]N' && \text{Since } \mathbf{fv}([\sigma]N') \cap \vec{\beta}^+ = \emptyset \\ &= [((\sigma \circ \vec{Q}/\vec{\beta}^+) \ll \vec{P}/\vec{\alpha}^+) \circ \sigma]N' && \text{by lemma 6} \\ &= [(\sigma \circ \vec{Q}/\vec{\beta}^+) \ll \vec{P}/\vec{\alpha}^+][\sigma]N' \end{aligned}$$

Notice that $(\sigma \circ \vec{Q}/\vec{\beta}^+) \ll \vec{P}/\vec{\alpha}^+$ is a substitution that turns α_i^+ into $[\sigma \circ \vec{Q}/\vec{\beta}^+]P_i$, where $\Gamma', \vec{\gamma}^+ \vdash [\sigma \circ \vec{Q}/\vec{\beta}^+]P_i$. This way, $\Gamma', \vec{\gamma}^+ \vdash [(\sigma \circ \vec{Q}/\vec{\beta}^+) \ll \vec{P}/\vec{\alpha}^+][\sigma]N' \leq_1 [\sigma]K'$ means $\Gamma \vdash \forall \alpha^+. [\sigma]N' \leq_1 \forall \gamma^+. [\sigma]K'$ by lemma 9, that is $\Gamma \vdash [\sigma]N \leq_1 [\sigma]K$, as required.

Now, we can assume that neither $\Gamma \vdash N \leq_1 M_1$ nor $\Gamma \vdash M_2 \leq_1 K$ is inferred by Rule (\forall^{\leq_1}), and that neither M_1 nor M_2 is equivalent to a variable. Because of that, $[\sigma]M_1 = [\sigma]M_2$ means that M_1 and M_2 have the same outer constructor. Let us consider the shape of M_1 .

Case 1. $M_1 = \alpha^-$ this case has been considered;

Case 2. $M_1 = \forall \vec{\beta}^+. M'_1$ this case has been considered;

Case 3. $M_1 = \uparrow Q_1$. Then as noted above, $[\sigma]M_1 = [\sigma]M_2$ means that $M_2 = \uparrow Q_2$ and $[\sigma]Q_1 = [\sigma]Q_2$. Moreover, $\Gamma \vdash N \leq_1 \uparrow Q_1$ can only be inferred by Rule (\uparrow^{\leq_1}), and thus, $N = \uparrow P$, and by inversion, $\Gamma \vdash P \geq_1 Q_1$ and $\Gamma \vdash Q_1 \geq_1 P$. Analogously, $\Gamma \vdash \uparrow Q_2 \leq_1 K$ means that $K = \uparrow R$, $\Gamma \vdash Q_2 \geq_1 R$, and $\Gamma \vdash R \geq_1 Q_2$.

Notice that the derivations of $\Gamma \vdash P \geq_1 Q_1$ and $\Gamma \vdash Q_1 \geq_1 P$ are proper sub-derivations of $\Gamma \vdash N \leq_1 M_1$, and the derivations of $\Gamma \vdash Q_2 \geq_1 R$ and $\Gamma \vdash R \geq_1 Q_2$ are proper sub-derivations of $\Gamma \vdash M_2 \leq_1 K$. This way, the induction hypothesis is applicable:

- applying the induction hypothesis to $\Gamma \vdash P \geq_1 Q_1$ and $\Gamma \vdash Q_2 \geq_1 R$ with $\Gamma' \vdash \sigma : \Gamma$ unifying Q_1 and Q_2 , we obtain $\Gamma' \vdash [\sigma]P \geq_1 [\sigma]R$;
- applying the induction hypothesis to $\Gamma \vdash R \geq_1 Q_2$ and $\Gamma \vdash Q_1 \geq_1 P$ with $\Gamma' \vdash \sigma : \Gamma$ unifying Q_2 and Q_1 , we obtain $\Gamma' \vdash [\sigma]R \geq_1 [\sigma]P$.

This way, by Rule (\uparrow^{\leq_1}), $\Gamma' \vdash [\sigma]N \leq_1 [\sigma]K$, as required.

Case 4. $M_1 = Q_1 \rightarrow M'_1$. Then as noted above, $[\sigma]M_1 = [\sigma]M_2$ means that $M_2 = Q_2 \rightarrow M'_2$, $[\sigma]Q_1 = [\sigma]Q_2$, and $[\sigma]M'_1 = [\sigma]M'_2$. Moreover, $\Gamma \vdash N \leq_1 Q_1 \rightarrow M'_1$ can only be inferred by Rule (\rightarrow^{\leq_1}), and thus, $N = P \rightarrow N'$, and by inversion, $\Gamma \vdash P \geq_1 Q_1$ and $\Gamma \vdash N' \leq_1 M'_1$. Analogously, $\Gamma \vdash Q_2 \rightarrow M'_2 \leq_1 K$ means that $K = R \rightarrow K'$, $\Gamma \vdash Q_2 \geq_1 R$, and $\Gamma \vdash M'_2 \leq_1 K'$.

Notice that the derivations of $\Gamma \vdash P \geq_1 Q_1$ and $\Gamma \vdash N' \leq_1 M'_1$ are proper sub-derivations of $\Gamma \vdash P \rightarrow N' \leq_1 Q_1 \rightarrow M'_1$, and the derivations of $\Gamma \vdash Q_2 \geq_1 R$ and $\Gamma \vdash M'_2 \leq_1 K'$ are proper sub-derivations of $\Gamma \vdash Q_2 \rightarrow M'_2 \leq_1 R \rightarrow K'$. This way, the induction hypothesis is applicable:

- applying the induction hypothesis to $\Gamma \vdash P \geq_1 Q_1$ and $\Gamma \vdash Q_2 \geq_1 R$ with $\Gamma' \vdash \sigma : \Gamma$ unifying Q_1 and Q_2 , we obtain $\Gamma' \vdash [\sigma]P \geq_1 [\sigma]R$;
- applying the induction hypothesis to $\Gamma \vdash N' \leq_1 M'_1$ and $\Gamma \vdash M'_2 \leq_1 K'$ with $\Gamma' \vdash \sigma : \Gamma$ unifying M'_1 and M'_2 , we obtain $\Gamma' \vdash [\sigma]N' \leq_1 [\sigma]K'$.

This way, by Rule (\rightarrow^{\leq_1}), $\Gamma' \vdash [\sigma]P \rightarrow [\sigma]N' \leq_1 [\sigma]R \rightarrow [\sigma]K'$, that is $\Gamma' \vdash [\sigma]N \leq_1 [\sigma]K$, as required.

After that we consider all the analogous positive cases, and prove them symmetrically. □

Corollary 6 (Transitivity of subtyping). *Assuming the types are well-formed in Γ ,*

- if $\Gamma \vdash N_1 \leq_1 N_2$ and $\Gamma \vdash N_2 \leq_1 N_3$ then $\Gamma \vdash N_1 \leq_1 N_3$,
- + if $\Gamma \vdash P_1 \geq_1 P_2$ and $\Gamma \vdash P_2 \geq_1 P_3$ then $\Gamma \vdash P_1 \geq_1 P_3$.

Corollary 7 (Transitivity of equivalence). *Assuming the types are well-formed in Γ ,*

- if $\Gamma \vdash N_1 \simeq_1^{\leq} N_2$ and $\Gamma \vdash N_2 \simeq_1^{\leq} N_3$ then $\Gamma \vdash N_1 \simeq_1^{\leq} N_3$,
- + if $\Gamma \vdash P_1 \simeq_1^{\leq} P_2$ and $\Gamma \vdash P_2 \simeq_1^{\leq} P_3$ then $\Gamma \vdash P_1 \simeq_1^{\leq} P_3$.

3.3 Type well-formedness

Lemma 14 (Well-formedness agrees with substitution). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

- + $\Gamma, \Gamma_1 \vdash P \Leftrightarrow \Gamma, \Gamma_2 \vdash [\sigma]P$
- $\Gamma, \Gamma_1 \vdash N \Leftrightarrow \Gamma, \Gamma_2 \vdash [\sigma]N$

Proof. **Ilya:** todo □

Corollary 8. *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$. Then*

- + $\Gamma_1, \Gamma_2 \vdash P \Leftrightarrow \Gamma_2 \vdash [\sigma]P$
- $\Gamma_1, \Gamma_2 \vdash N \Leftrightarrow \Gamma_2 \vdash [\sigma]N$

Lemma 15 (Equivalent Contexts). *In the well-formedness judgment, only used variables matter:*

- + if $\Gamma_1 \cap \mathbf{fv} P = \Gamma_2 \cap \mathbf{fv} P$ then $\Gamma_1 \vdash P \iff \Gamma_2 \vdash P$,
- if $\Gamma_1 \cap \mathbf{fv} N = \Gamma_2 \cap \mathbf{fv} N$ then $\Gamma_1 \vdash N \iff \Gamma_2 \vdash N$.

Proof. By simple mutual induction on P and Q . □

Corollary 9. *Suppose that all the types below are well-formed in Γ and $\Gamma' \subseteq \Gamma$. Then*

- + $\Gamma \vdash P \simeq_1^{\leq} Q$ implies $\Gamma' \vdash P \iff \Gamma' \vdash Q$
- $\Gamma \vdash N \simeq_1^{\leq} M$ implies $\Gamma' \vdash N \iff \Gamma' \vdash M$

Proof. From lemma 15 and corollary 2. □

3.4 Overview

Algorithm	Soundness	Completeness	Initiality
Ordering	$\text{ord vars in } N \equiv \text{vars} \cap \text{fv } N$	$\frac{N \simeq_1^D M}{\text{ord vars in } N = \text{ord vars in } M}$	—
Normalization	$\frac{}{N \simeq_1^D \text{nf}(N)}$	$\frac{N \simeq_1^D M}{\text{nf}(N) = \text{nf}(M)}$	—
Equivalence	$\frac{\Gamma \vdash P \quad \Gamma \vdash Q \quad P \simeq_1^D Q}{\Gamma \vdash P \simeq_1^{\leq} Q}$	$\frac{\Gamma \vdash P \simeq_1^{\leq} Q}{P \simeq_1^D Q}$	—
Uppgrade	$\frac{\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q}{Q \text{ is sound } \left\{ \begin{array}{l} \Delta \vdash Q \\ \Gamma \vdash Q \geq_1 P \end{array} \right.}$	$\frac{\exists \text{ sound } Q' \quad \exists Q \text{ s.t. } \text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q}{\text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q}$	$\frac{Q' \text{ is sound} \quad \text{upgrade } \Gamma \vdash P \text{ to } \Delta = Q}{\Delta \vdash Q' \geq_1 Q}$
LUB	$\frac{\Gamma \models P_1 \vee P_2 = Q}{Q \text{ is sound } \left\{ \begin{array}{l} \Gamma \vdash Q \\ \Gamma \vdash Q \geq_1 P_1 \\ \Gamma \vdash Q \geq_1 P_2 \end{array} \right.}$	$\frac{\exists \text{ sound } Q' \quad \exists Q \text{ s.t. } \Gamma \models P_1 \vee P_2 = Q}{\Gamma \models P_1 \vee P_2 = Q}$	$\frac{Q' \text{ is sound} \quad \Gamma \models P_1 \vee P_2 = Q}{\Delta \vdash Q' \geq_1 Q}$
Anti-unification	$\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \left\{ \begin{array}{l} \Xi \text{ is negative} \\ \Gamma; \Xi \vdash Q \\ \Gamma; \cdot \vdash \hat{\tau}_i : \Xi \\ [\hat{\tau}_i] Q = P_i \end{array} \right. \text{ is sound}}$	$\frac{\exists \text{ sound } (\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2) \quad \exists (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2) \text{ s.t.}}{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}$	$\frac{(\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2) \text{ is sound} \quad \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)}{\exists \Gamma; \Xi \vdash \hat{\tau} : \Xi' \text{ s.t. } [\hat{\tau}] Q' = Q}$
Unification (matching)			—
Subtyping			—

3.5 Variable Ordering

Definition 9 (Collision free bijection). *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is **collision free on sets** P and Q if and only if*

1. $\mu(P \cap A) \cap Q = \emptyset$
2. $\mu(Q \cap A) \cap P = \emptyset$

Lemma 16 (Soundness of variable ordering). *Variable ordering extracts precisely used free variables.*

- $\text{ord vars in } N \equiv \text{vars} \cap \text{fv } N$ (as sets)
- + $\text{ord vars in } P \equiv \text{vars} \cap \text{fv } P$ (as sets)

Proof. Straightforward mutual induction on $\text{ord vars in } N = \vec{\alpha}$ and $\text{ord vars in } P = \vec{\alpha}$ □

Corollary 10 (Additivity of ordering). *Variable ordering is additive (in terms of set union) with respect to its first argument.*

- $\text{ord}(\text{vars}_1 \cup \text{vars}_2) \text{ in } N \equiv \text{ord vars}_1 \text{ in } N \cup \text{ord vars}_2 \text{ in } N$ (as sets)
- + $\text{ord}(\text{vars}_1 \cup \text{vars}_2) \text{ in } P \equiv \text{ord vars}_1 \text{ in } P \cup \text{ord vars}_2 \text{ in } P$ (as sets)

Corollary 11 (Weakening of ordering). *Extending the first argument of the ordering with unused variables does not change the result.*

- $\text{ord}(\text{vars} \cap \text{fv } N) \text{ in } N = \text{ord vars in } N$
- + $\text{ord}(\text{vars} \cap \text{fv } P) \text{ in } P = \text{ord vars in } P$

Lemma 17 (Distributivity of renaming over variable ordering). *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

- *If μ is collision free on vars and $\text{fv } N$ then $[\mu](\text{ord vars in } N) = \text{ord}([\mu]\text{vars}) \text{ in } [\mu]N$*

+ If μ is collision free on $vars$ and $\mathbf{fv} P$ then $[\mu](\mathbf{ord} \text{ vars in } P) = \mathbf{ord}([\mu]vars) \mathbf{in} [\mu]P$

Proof. Mutual induction on N and P .

Case 1. $N = \alpha^-$

let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$

$$\begin{aligned} \text{Then } [\mu](\mathbf{ord} \text{ vars in } N) &= [\mu](\mathbf{ord} \text{ vars in } \alpha^-) \\ &= [\mu]\alpha^- && \text{by Rule (Var}_{\epsilon}^+) \\ &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars) \\ &= \mathbf{ord} [\mu]vars \mathbf{in} \beta^- && \text{by Rule (Var}_{\epsilon}^+), \text{ because } \beta^- \in [\mu]vars \\ &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^- \end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$

Notice that $[\mu](\mathbf{ord} \text{ vars in } N) = [\mu](\mathbf{ord} \text{ vars in } \alpha^-) = \cdot$ by Rule (Var $_{\epsilon}^+$). On the other hand, $\mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^- = \mathbf{ord} [\mu]vars \mathbf{in} \alpha^- = \cdot$. The latter equality is from Rule (Var $_{\epsilon}^+$), because μ is collision free on $vars$ and $\mathbf{fv} N$, so $\mathbf{fv} N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$

Then $[\mu](\mathbf{ord} \text{ vars in } N) = [\mu](\mathbf{ord} \text{ vars in } \alpha^-) = \cdot$ by Rule (Var $_{\epsilon}^+$). To prove that $\mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^- = \cdot$, we apply Rule (Var $_{\epsilon}^+$). Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

(i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ , which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.

(ii) Since μ is collision free on $vars$ and $\mathbf{fv} N$, $\mu(A \cap \mathbf{fv} N) \ni \mu(\alpha^-) \notin vars$.

d. $\alpha^- \notin A$ but $\alpha^- \in vars$

$\mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^- = \mathbf{ord} [\mu]vars \mathbf{in} \alpha^- = \alpha^-$. The latter is by Rule (Var $_{\epsilon}^+$), because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord} \text{ vars in } N) = [\mu](\mathbf{ord} \text{ vars in } \alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$

$$\begin{aligned} [\mu](\mathbf{ord} \text{ vars in } N) &= [\mu](\mathbf{ord} \text{ vars in } \uparrow P) \\ &= [\mu](\mathbf{ord} \text{ vars in } P) && \text{by Rule } (\uparrow) \\ &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]P && \text{by the induction hypothesis} \\ &= \mathbf{ord} [\mu]vars \mathbf{in} \uparrow[\mu]P && \text{by Rule } (\uparrow) \\ &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]\uparrow P && \text{by the definition of substitution} \\ &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]N \end{aligned}$$

Case 3. $N = P \rightarrow M$

$$\begin{aligned} [\mu](\mathbf{ord} \text{ vars in } N) &= [\mu](\mathbf{ord} \text{ vars in } P \rightarrow M) \\ &= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \setminus \vec{\alpha}_1)) && \text{where } \mathbf{ord} \text{ vars in } P = \vec{\alpha}_1 \text{ and } \mathbf{ord} \text{ vars in } M = \vec{\alpha}_2 \\ &= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \setminus \vec{\alpha}_1) \\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus [\mu]\vec{\alpha}_1) && \text{by induction on } \vec{\alpha}_2; \text{ the inductive step is similar to case 1. Notice that } \mu \text{ is} \\ & && \text{collision free on } \vec{\alpha}_1 \text{ and } \vec{\alpha}_2 \text{ since } \vec{\alpha}_1 \subseteq vars \text{ and } \vec{\alpha}_2 \subseteq \mathbf{fv} N \\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus [\mu]\vec{\alpha}_1) \\ (\mathbf{ord} [\mu]vars \mathbf{in} [\mu]N) &= (\mathbf{ord} [\mu]vars \mathbf{in} [\mu]P \rightarrow [\mu]M) \\ &= (\vec{\beta}_1, (\vec{\beta}_2 \setminus \vec{\beta}_1)) && \text{where } \mathbf{ord} [\mu]vars \mathbf{in} [\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord} [\mu]vars \mathbf{in} [\mu]M = \vec{\beta}_2 \\ & && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1, \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\ &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus [\mu]\vec{\alpha}_1) \end{aligned}$$

Case 4. $N = \forall \alpha^+. M$

$$\begin{aligned} [\mu](\mathbf{ord} \text{ vars in } N) &= [\mu]\mathbf{ord} \text{ vars in } \forall \alpha^+. M \\ &= [\mu]\mathbf{ord} \text{ vars in } M \\ &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]M && \text{by the induction hypothesis} \\ (\mathbf{ord} [\mu]vars \mathbf{in} [\mu]N) &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]\forall \alpha^+. M \\ &= \mathbf{ord} [\mu]vars \mathbf{in} \forall \alpha^+. [\mu]M \\ &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]M \end{aligned}$$

□

Lemma 18 (Ordering is not affected by independent substitutions). *Suppose that $\Gamma_2 \vdash \sigma : \Gamma_1$, i.e. σ maps variables from Γ_1 into types taking free variables from Γ_2 , and vars is a set of variables disjoint with both Γ_1 and Γ_2 . Then*

- $\text{ord vars in } [\sigma]N = \text{ord vars in } N$
- + $\text{ord vars in } [\sigma]P = \text{ord vars in } P$

Proof. Ilya: Should be easy

□

Lemma 19 (Completeness of variable ordering). *Variable ordering is invariant under equivalence. For arbitrary vars,*

- *If $N \simeq_1^D M$ then $\text{ord vars in } N = \text{ord vars in } M$ (as lists)*
- + *If $P \simeq_1^D Q$ then $\text{ord vars in } P = \text{ord vars in } Q$ (as lists)*

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

□

3.6 Normaliztaion

Lemma 20. *Set of free variables is invariant under equivalence.*

- *If $N \simeq_1^D M$ then $\text{fv } N \equiv \text{fv } M$ (as sets)*
- + *If $P \simeq_1^D Q$ then $\text{fv } P \equiv \text{fv } Q$ (as sets)*

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$

□

Lemma 21. *Free variables are not changed by the normalization*

- $\text{fv } N \equiv \text{fv nf } (N)$
- + $\text{fv } P \equiv \text{fv nf } (P)$

Proof. By straightforward induction on $\text{nf } (N) = M$.

□

Lemma 22 (Soundness of quantifier normalization).

- $N \simeq_1^D \text{nf } (N)$
- + $P \simeq_1^D \text{nf } (P)$

Proof. Mutual induction on $\text{nf } (N) = M$ and $\text{nf } (P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var^-) and (Var^+)

By the corresponding equivalence rules.

Case 2. (\uparrow), (\downarrow), and (\rightarrow)

By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. (\forall), i.e. $\text{nf } (\forall \alpha^+. N) = \forall \alpha^+. \text{nf } N$

From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 20, $\text{fv } N \equiv \text{fv } N'$. Then by lemma 16, $\overrightarrow{\alpha^+}' \equiv \overrightarrow{\alpha^+} \cap \text{fv } N' \equiv \overrightarrow{\alpha^+} \cap \text{fv } N$, and thus, $\overrightarrow{\alpha^+}' \cap \text{fv } N' \equiv \overrightarrow{\alpha^+} \cap \text{fv } N$.

To prove $\forall \alpha^+. N \simeq_1^D \forall \alpha^+. \text{nf } N'$, it suffices to provide a bijection $\mu : \overrightarrow{\alpha^+}' \cap \text{fv } N' \leftrightarrow \overrightarrow{\alpha^+} \cap \text{fv } N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = \text{id}$.

Case 4. (\exists) Same as for case 3.

□

Corollary 12 (Normalization preserves ordering). *For any vars,*

- $\text{ord vars in nf } (N) = \text{ord vars in } M$
- + $\text{ord vars in nf } (P) = \text{ord vars in } Q$

Proof. Immediately from lemmas 19 and 22.

□

Lemma 23 (Distributivity of normalization over substitution). *Normalization of a term distributes over substitution. Suppose that σ is a substitution, N and P are types. Then*

$$\begin{aligned} - \text{nf}([\sigma]N) &= [\text{nf}(\sigma)]\text{nf}(N) \\ + \text{nf}([\sigma]P) &= [\text{nf}(\sigma)]\text{nf}(P) \end{aligned}$$

where $\text{nf}(\sigma)$ means pointwise normalization: $[\text{nf}(\sigma)]\alpha^- = \text{nf}([\sigma]\alpha^-)$.

Proof. Mutual induction on N and P .

Case 1. $N = \alpha^-$

$$\begin{aligned} \text{nf}([\sigma]N) &= \text{nf}([\sigma]\alpha^-) = [\text{nf}(\sigma)]\alpha^- \\ [\text{nf}(\sigma)]\text{nf}(N) &= [\text{nf}(\sigma)]\text{nf}(\alpha^-) = [\text{nf}(\sigma)]\alpha^- \end{aligned}$$

Case 2. $P = \alpha^+$

Similar to case 1.

Case 3. If the type is formed by \rightarrow , \uparrow , or \downarrow , the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \rightarrow M$ then

$$\begin{aligned} \text{nf}([\sigma]N) &= \text{nf}([\sigma](P \rightarrow M)) \\ &= \text{nf}([\sigma]P \rightarrow [\sigma]M) && \text{By the congruence of substitution} \\ &= \text{nf}([\sigma]P) \rightarrow \text{nf}([\sigma]M) && \text{By the congruence of normalization, i.e. Rule } (\rightarrow) \\ &= [\text{nf}(\sigma)]\text{nf}(P) \rightarrow [\text{nf}(\sigma)]\text{nf}(M) && \text{By the induction hypothesis} \\ &= [\text{nf}(\sigma)](\text{nf}(P) \rightarrow \text{nf}(M)) && \text{By the congruence of substitution} \\ &= [\text{nf}(\sigma)]\text{nf}(P \rightarrow M) && \text{By the congruence of normalization} \\ &= [\text{nf}(\sigma)]\text{nf}(N) \end{aligned}$$

Case 4. $N = \forall \alpha^+. M$

$$\begin{aligned} [\text{nf}(\sigma)]\text{nf}(N) &= [\text{nf}(\sigma)]\text{nf}(\forall \alpha^+. M) \\ &= [\text{nf}(\sigma)]\forall \alpha^{+\prime}. \text{nf}(M) \quad \text{Where } \alpha^{+\prime} = \text{ord } \alpha^+ \text{ in } \text{nf}(M) = \text{ord } \alpha^+ \text{ in } M \text{ (the latter is by corollary 12)} \\ \text{nf}([\sigma]N) &= \text{nf}([\sigma]\forall \alpha^+. M) \\ &= \text{nf}(\forall \alpha^+. [\sigma]M) && \text{Assuming } \alpha^+ \cap \Gamma_1 = \emptyset \text{ and } \alpha^+ \cap \Gamma_2 = \emptyset \\ &= \forall \beta^+. \text{nf}([\sigma]M) && \text{Where } \beta^+ = \text{ord } \alpha^+ \text{ in } \text{nf}([\sigma]M) = \text{ord } \alpha^+ \text{ in } [\sigma]M \text{ (the latter is by corollary 12)} \\ &= \forall \alpha^{+\prime}. \text{nf}([\sigma]M) && \text{By lemma 18, } \beta^+ = \alpha^{+\prime} \text{ since } \alpha^+ \text{ is disjoint with } \Gamma_1 \text{ and } \Gamma_2 \\ &= \forall \alpha^{+\prime}. [\text{nf}(\sigma)]\text{nf}(M) && \text{By the induction hypothesis} \end{aligned}$$

To show alpha-equivalence of $[\text{nf}(\sigma)]\forall \alpha^{+\prime}. \text{nf}(M)$ and $\forall \alpha^{+\prime}. [\text{nf}(\sigma)]\text{nf}(M)$, we can assume that $\alpha^{+\prime} \cap \Gamma_1 = \emptyset$, and $\alpha^{+\prime} \cap \Gamma_2 = \emptyset$.

Case 5. $P = \exists \alpha^+. Q$

Same as for case 4.

□

Corollary 13 (Commutativity of normalization and renaming). *Normalization of a term commutes with renaming. Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then*

$$\begin{aligned} - \text{nf}([\mu]N) &= [\mu]\text{nf}(N) \\ + \text{nf}([\mu]P) &= [\mu]\text{nf}(P) \end{aligned}$$

Proof. Immediately from lemma 23, after noticing that $\text{nf}(\mu) = \mu$.

□

Lemma 24 (Completeness of quantified normalization). *Normalization returns the same representative for equivalent types.*

$$\begin{aligned} - \text{If } N \simeq_1^P M \text{ then } \text{nf}(N) &= \text{nf}(M) \\ + \text{If } P \simeq_1^P Q \text{ then } \text{nf}(P) &= \text{nf}(Q) \end{aligned}$$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. ($\forall \simeq_1^D$)

From the definition of the normalization,

- $\mathbf{nf}(\forall \alpha^+ . N) = \forall \alpha^{+'} . \mathbf{nf}(N)$ where $\alpha^{+'}$ is **ord** α^+ **in** $\mathbf{nf}(N)$
- $\mathbf{nf}(\forall \beta^+ . M) = \forall \beta^{+'} . \mathbf{nf}(M)$ where $\beta^{+'}$ is **ord** β^+ **in** $\mathbf{nf}(M)$

Let us take $\mu : (\beta^+ \cap \mathbf{fv} M) \leftrightarrow (\alpha^+ \cap \mathbf{fv} N)$ from the inversion of the equivalence judgment. Notice that from lemmas 16 and 21, the domain and the codomain of μ can be written as $\mu : \beta^{+'} \leftrightarrow \alpha^{+'}$.

To show the alpha-equivalence of $\forall \alpha^{+'} . \mathbf{nf}(N)$ and $\forall \beta^{+'} . \mathbf{nf}(M)$, it suffices to prove that (i) $[\mu]\mathbf{nf}(M) = \mathbf{nf}(N)$ and (ii) $[\mu]\beta^{+'} = \alpha^{+'}$.

(i) $[\mu]\mathbf{nf}(M) = \mathbf{nf}([\mu]M) = \mathbf{nf}(N)$. The first equality holds by corollary 13, the second—by the induction hypothesis.

$$\begin{aligned}
\text{(ii) } [\mu]\beta^{+'} &= [\mu]\mathbf{ord} \beta^+ \mathbf{in} \mathbf{nf}(M) && \text{by the definition of } \beta^{+'} \\
&= [\mu]\mathbf{ord} (\beta^+ \cap \mathbf{fv} M) \mathbf{in} \mathbf{nf}(M) && \text{from lemma 21 and corollary 11} \\
&= \mathbf{ord} [\mu](\beta^+ \cap \mathbf{fv} M) \mathbf{in} [\mu]\mathbf{nf}(M) && \text{by lemma 17, because } \alpha^+ \cap \mathbf{fv} N \cap \mathbf{fv} \mathbf{nf}(M) \subseteq \alpha^+ \cap \mathbf{fv} M = \emptyset \\
&&& \text{and } \alpha^+ \cap \mathbf{fv} N \cap (\beta^+ \cap \mathbf{fv} M) \subseteq \alpha^+ \cap \mathbf{fv} M = \emptyset \\
&= \mathbf{ord} [\mu](\beta^+ \cap \mathbf{fv} M) \mathbf{in} \mathbf{nf}(N) && \text{since } [\mu]\mathbf{nf}(M) = \mathbf{nf}(N) \text{ is proved} \\
&= \mathbf{ord} (\alpha^+ \cap \mathbf{fv} N) \mathbf{in} \mathbf{nf}(N) && \text{because } \mu \text{ is a bijection between } \alpha^+ \cap \mathbf{fv} N \text{ and } \beta^+ \cap \mathbf{fv} M \\
&= \mathbf{ord} \alpha^+ \mathbf{in} \mathbf{nf}(N) && \text{from lemma 21 and corollary 11} \\
&= \alpha^{+'} && \text{by the definition of } \alpha^{+'}
\end{aligned}$$

Case 2. ($\exists \simeq_1^D$) Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis. □

Lemma 25 (Idempotence of normalization). *Normalization is idempotent*

- $\mathbf{nf}(\mathbf{nf}(N)) = \mathbf{nf}(N)$
- + $\mathbf{nf}(\mathbf{nf}(P)) = \mathbf{nf}(P)$

Proof. By applying lemma 24 to lemma 22. □

Lemma 26. *The result of a substitution is normalized if and only if the initial type and the substitution are normalized.*

Suppose that σ is a substitution $\Gamma_2 \vdash \sigma : \Gamma_1$, P is a positive type ($\Gamma_1 \vdash P$), N is a negative type ($\Gamma_1 \vdash N$). Then

$$\begin{aligned}
+ \quad [\sigma]P \text{ is normal} &\iff \begin{cases} \sigma|_{\mathbf{fv}(P)} & \text{is normal} \\ P & \text{is normal} \end{cases} \\
- \quad [\sigma]N \text{ is normal} &\iff \begin{cases} \sigma|_{\mathbf{fv}(N)} & \text{is normal} \\ N & \text{is normal} \end{cases}
\end{aligned}$$

Proof. Mutual induction on $\Gamma_1 \vdash P$ and $\Gamma_1 \vdash N$.

Case 1. $N = \alpha^-$

Then N is always normal, and the normality of $\sigma|_{\alpha^-}$ by the definition means $[\sigma]\alpha^-$ is normal.

Case 2. $N = P \rightarrow M$

$$\begin{aligned}
[\sigma](P \rightarrow M) \text{ is normal} &\iff [\sigma]P \rightarrow [\sigma]M \text{ is normal} && \text{by the substitution congruence} \\
&\iff \begin{cases} [\sigma]P & \text{is normal} \\ [\sigma]M & \text{is normal} \end{cases} && \text{by congruence of normality Ilya: lemma?} \\
&\iff \begin{cases} P & \text{is normal} \\ \sigma|_{\mathbf{fv}(P)} & \text{is normal} \\ M & \text{is normal} \\ \sigma|_{\mathbf{fv}(M)} & \text{is normal} \end{cases} && \text{by the induction hypothesis} \\
&\iff \begin{cases} P \rightarrow M & \text{is normal} \\ \sigma|_{\mathbf{fv}(P) \cup \mathbf{fv}(M)} & \text{is normal} \end{cases} \iff \begin{cases} P \rightarrow M & \text{is normal} \\ \sigma|_{\mathbf{fv}(P \rightarrow M)} & \text{is normal} \end{cases}
\end{aligned}$$

Case 3. $N = \uparrow P$

By congruence and the inductive hypothesis, similar to case 2

Case 4. $N = \forall \alpha^+ . M$

$$\begin{aligned}
[\sigma](\forall \alpha^+ . M) \text{ is normal} &\iff (\forall \alpha^+ . [\sigma]M) \text{ is normal} && \text{assuming } \vec{\alpha}^+ \cap \Gamma_1 = \emptyset \text{ and } \vec{\alpha}^+ \cap \Gamma_2 = \emptyset \\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord} \vec{\alpha}^+ \text{ in } [\sigma]M = \vec{\alpha}^+ \end{cases} && \text{by the definition of normalization} \\
&\iff \begin{cases} [\sigma]M \text{ is normal} \\ \mathbf{ord} \vec{\alpha}^+ \text{ in } M = \vec{\alpha}^+ \end{cases} && \text{by lemma 18} \\
&\iff \begin{cases} \sigma|_{\mathbf{fv}(M)} \text{ is normal} \\ M \text{ is normal} \\ \mathbf{ord} \vec{\alpha}^+ \text{ in } M = \vec{\alpha}^+ \end{cases} && \text{by the induction hypothesis} \\
&\iff \begin{cases} \sigma|_{\mathbf{fv}(\forall \alpha^+ . M)} \text{ is normal} \\ \forall \alpha^+ . M \text{ is normal} \end{cases} && \begin{array}{l} \text{since } \mathbf{fv}(\forall \alpha^+ . M) = \mathbf{fv}(M); \\ \text{by the definition of normalization} \end{array}
\end{aligned}$$

Case 5. $P = \dots$

The positive cases are done in the same way as the negative ones.

□

3.7 Equivalence

Lemma 27 (Declarative equivalence is transitive).

- + if $P_1 \simeq_1^D P_2$ and $P_2 \simeq_1^D P_3$ then $P_1 \simeq_1^D P_3$,
- if $N_1 \simeq_1^D N_2$ and $N_2 \simeq_1^D N_3$ then $N_1 \simeq_1^D N_3$.

Proof. Ilya: should be easy to do by induction since the types are getting smaller

□

Lemma 28 (Algorithmization of declarative equivalence). *Declarative equivalence is equality of normal forms.*

- + $P \simeq_1^D Q \iff \mathbf{nf}(P) = \mathbf{nf}(Q)$,
- $N \simeq_1^D M \iff \mathbf{nf}(N) = \mathbf{nf}(M)$.

Proof.

- + Let us prove both directions separately.
 - \Rightarrow exactly by lemma 24,
 - \Leftarrow from lemma 22, we know $P \simeq_1^D \mathbf{nf}(P) = \mathbf{nf}(Q) \simeq_1^D Q$, then by transitivity (lemma 27), $P \simeq_1^D Q$.
- The proof is exactly the same.

□

Lemma 29 (Type well-formedness is invariant under equivalence). *Mutual subtyping implies declarative equivalence.*

- + if $P \simeq_1^D Q$ then $\Gamma \vdash P \iff \Gamma \vdash Q$,
- if $N \simeq_1^D M$ then $\Gamma \vdash N \iff \Gamma \vdash M$

Proof. Ilya: todo □

Corollary 14 (Normalization preserves well-formedness).

- + $\Gamma \vdash P \iff \Gamma \vdash \mathbf{nf}(P)$,
- $\Gamma \vdash N \iff \Gamma \vdash \mathbf{nf}(N)$

Proof. Immediately from lemmas 22 and 29. □

Corollary 15 (Normalization preserves well-formedness of substitution).

$$\Gamma_2 \vdash \sigma : \Gamma_1 \iff \Gamma_2 \vdash \mathbf{nf}(\sigma) : \Gamma_1$$

Lemma 30 (Soundness of equivalence). *Declarative equivalence implies mutual subtyping.*

- + if $\Gamma \vdash P, \Gamma \vdash Q$, and $P \simeq_1^D Q$ then $\Gamma \vdash P \simeq_1^\leq Q$,
- if $\Gamma \vdash N, \Gamma \vdash M$, and $N \simeq_1^D M$ then $\Gamma \vdash N \simeq_1^\leq M$.

Proof. We prove it by mutual induction on $P \simeq_1^D Q$ and $N \simeq_1^D M$.

Case 1. $\alpha^- \simeq_1^D \alpha^-$

Then $\Gamma \vdash \alpha^- \leq_1 \alpha^-$ by Rule ($\text{Var}^{-\leq_1}$), which immediately implies $\Gamma \vdash \alpha^- \simeq_1^\leq \alpha^-$ by Rule ($\simeq_1^\leq -$).

Case 2. $\uparrow P \simeq_1^D \uparrow Q$

Then by inversion of Rule (\uparrow^{\leq_1}), $P \simeq_1^D Q$, and from the induction hypothesis, $\Gamma \vdash P \simeq_1^\leq Q$, and (by symmetry) $\Gamma \vdash Q \simeq_1^\leq P$.

When Rule (\uparrow^{\leq_1}) is applied to $\Gamma \vdash P \simeq_1^\leq Q$, it gives us $\Gamma \vdash \uparrow P \leq_1 \uparrow Q$; when it is applied to $\Gamma \vdash Q \simeq_1^\leq P$, we obtain $\Gamma \vdash \uparrow Q \leq_1 \uparrow P$. Together, it implies $\Gamma \vdash \uparrow P \simeq_1^\leq \uparrow Q$.

Case 3. $P \rightarrow N \simeq_1^D Q \rightarrow M$

Then by inversion of Rule (\rightarrow^{\leq_1}), $P \simeq_1^D Q$ and $N \simeq_1^D M$. By the induction hypothesis, $\Gamma \vdash P \simeq_1^\leq Q$ and $\Gamma \vdash N \simeq_1^\leq M$, which means by inversion: (i) $\Gamma \vdash P \geq_1 Q$, (ii) $\Gamma \vdash Q \geq_1 P$, (iii) $\Gamma \vdash N \leq_1 M$, (iv) $\Gamma \vdash M \leq_1 N$. Applying Rule (\rightarrow^{\leq_1}) to (i) and (iii), we obtain $\Gamma \vdash P \rightarrow N \leq_1 Q \rightarrow M$; applying it to (ii) and (iv), we have $\Gamma \vdash Q \rightarrow M \leq_1 P \rightarrow N$. Together, it implies $\Gamma \vdash P \rightarrow N \simeq_1^\leq Q \rightarrow M$.

Case 4. $\forall \alpha^+. N \simeq_1^D \forall \beta^+. M$

Then by inversion, there exists bijection $\mu : (\vec{\beta}^+ \cap \mathbf{fv} M) \leftrightarrow (\vec{\alpha}^+ \cap \mathbf{fv} N)$, such that $N \simeq_1^D [\mu]M$. By the induction hypothesis, $\Gamma, \vec{\alpha}^+ \vdash N \simeq_1^\leq [\mu]M$. From lemma 36 and the fact that μ is bijective, we also have $\Gamma, \vec{\beta}^+ \vdash [\mu^{-1}]N \simeq_1^\leq M$.

Let us construct a substitution $\vec{\alpha}^+ \vdash \vec{P}/\vec{\beta}^+ : \vec{\beta}^+$ by extending μ with arbitrary positive types on $\vec{\beta}^+ \setminus \mathbf{fv} M$.

Notice that $[\mu]M = [\vec{P}/\vec{\beta}^+]M$, and therefore, $\Gamma, \vec{\alpha}^+ \vdash N \simeq_1^\leq [\mu]M$ implies $\Gamma, \vec{\alpha}^+ \vdash [\vec{P}/\vec{\beta}^+]M \leq_1 N$. Then by Rule (\forall^{\leq_1}), $\Gamma \vdash \forall \vec{\beta}^+. M \leq_1 \forall \vec{\alpha}^+. N$.

Analogously, we construct the substitution from μ^{-1} , and use it to instantiate $\vec{\alpha}^+$ in the application of Rule (\forall^{\leq_1}) to infer $\Gamma \vdash \forall \vec{\alpha}^+. N \leq_1 \forall \vec{\beta}^+. M$.

This way, $\Gamma \vdash \forall \vec{\beta}^+. M \leq_1 \forall \vec{\alpha}^+. N$ and $\Gamma \vdash \forall \vec{\alpha}^+. N \leq_1 \forall \vec{\beta}^+. M$ gives us $\Gamma \vdash \forall \vec{\beta}^+. M \simeq_1^\leq \forall \vec{\alpha}^+. N$.

Case 5. For the cases of the positive types, the proofs are symmetric. □

Corollary 16 (Normalization is sound w.r.t. subtyping-induced equivalence).

- + if $\Gamma \vdash P$ then $\Gamma \vdash P \simeq_1^\leq \mathbf{nf}(P)$,
- if $\Gamma \vdash N$ then $\Gamma \vdash N \simeq_1^\leq \mathbf{nf}(N)$.

Proof. Immediately from lemmas 22 and 30 and corollary 14. □

Corollary 17 (Normalization preserves subtyping). *Assuming all the types are well-formed in context Γ ,*

- + $\Gamma \vdash P \geq_1 Q \iff \Gamma \vdash \mathbf{nf}(P) \geq_1 \mathbf{nf}(Q)$,
- $\Gamma \vdash N \leq_1 M \iff \Gamma \vdash \mathbf{nf}(N) \leq_1 \mathbf{nf}(M)$.

Proof.

- + \Rightarrow Let us assume $\Gamma \vdash P \geq_1 Q$. By corollary 16, $\Gamma \vdash P \simeq_1^{\leq} \mathbf{nf}(P)$ and $\Gamma \vdash Q \simeq_1^{\leq} \mathbf{nf}(Q)$, in particular, by inversion, $\Gamma \vdash \mathbf{nf}(P) \geq_1 P$ and $\Gamma \vdash Q \geq_1 \mathbf{nf}(Q)$. Then by the transitivity of subtyping (corollary 6), $\Gamma \vdash \mathbf{nf}(P) \geq_1 \mathbf{nf}(Q)$.
- \Leftarrow Let us assume $\Gamma \vdash \mathbf{nf}(P) \geq_1 \mathbf{nf}(Q)$. Also by corollary 16 and inversion, $\Gamma \vdash P \geq_1 \mathbf{nf}(P)$ and $\Gamma \vdash \mathbf{nf}(Q) \geq_1 Q$. Then by the transitivity, $\Gamma \vdash P \geq_1 Q$.
- The negative case is proved symmetrically.

□

Lemma 31 (Subtyping induced by disjoint substitutions). *If two disjoint substitutions induce subtyping, they are degenerate (so is the subtyping). Suppose that $\Gamma \vdash \sigma_1 : \Gamma_1$ and $\Gamma \vdash \sigma_2 : \Gamma_2$, where $\Gamma_i \subseteq \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then*

- assuming $\Gamma \vdash N$, $\Gamma \vdash [\sigma_1]N \leq_1 [\sigma_2]N$ implies $\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \mathbf{fv} N$
- + assuming $\Gamma \vdash P$, $\Gamma \vdash [\sigma_1]P \geq_1 [\sigma_2]P$ implies $\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \mathbf{fv} P$

Proof. Proof by induction on $\Gamma \vdash N$ (and mutually on $\Gamma \vdash P$).

Case 1. $N = \alpha^-$

Then $\Gamma \vdash [\sigma_1]N \leq_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\alpha^- \leq_1 [\sigma_2]\alpha^-$. Let us consider the following cases:

- a. $\alpha^- \notin \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then $\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \alpha^-$ holds immediately, since $[\sigma_i]\alpha^- = [\text{id}]\alpha^- = \alpha^-$ and $\Gamma \vdash \alpha^- \simeq_1^{\leq} \alpha^-$.
- b. $\alpha^- \in \Gamma_1$ and $\alpha^- \in \Gamma_2$
This case is not possible by assumption: $\Gamma_1 \cap \Gamma_2 = \emptyset$.
- c. $\alpha^- \in \Gamma_1$ and $\alpha^- \notin \Gamma_2$
Then we have $\Gamma \vdash [\sigma_1]\alpha^- \leq_1 \alpha^-$, which by corollary 4 means $\Gamma \vdash [\sigma_1]\alpha^- \simeq_1^{\leq} \alpha^-$, and hence, $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \text{id} : \alpha^-$.
 $\Gamma \vdash \sigma_2 \simeq_1^{\leq} \text{id} : \alpha^-$ holds since $[\sigma_2]\alpha^- = \alpha^-$, similarly to case 1.a.
- d. $\alpha^- \notin \Gamma_1$ and $\alpha^- \in \Gamma_2$
Then we have $\Gamma \vdash \alpha^- \leq_1 [\sigma_2]\alpha^-$, which by corollary 4 means $\Gamma \vdash \alpha^- \simeq_1^{\leq} [\sigma_2]\alpha^-$, and hence, $\Gamma \vdash \sigma_2 \simeq_1^{\leq} \text{id} : \alpha^-$.
 $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \text{id} : \alpha^-$ holds since $[\sigma_1]\alpha^- = \alpha^-$, similarly to case 1.a.

Case 2. $N = \forall \alpha^+. M$

Then by inversion, $\Gamma, \alpha^+ \vdash M$. $\Gamma \vdash [\sigma_1]N \leq_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\forall \alpha^+. M \leq_1 [\sigma_2]\forall \alpha^+. M$. By the congruence of substitution and by the inversion of Rule (\forall^{\leq_1}) , $\Gamma, \alpha^+ \vdash [\vec{Q}/\alpha^+][\sigma_1]M \leq_1 [\sigma_2]M$, where $\Gamma, \alpha^+ \vdash Q_i$. Let us denote the (Kleisli) composition of σ_1 and \vec{Q}/α^+ as σ'_1 , noting that $\Gamma, \alpha^+ \vdash \sigma'_1 : \Gamma_1, \alpha^+$, and $\Gamma_1, \alpha^+ \cap \Gamma_2 = \emptyset$.

Let us apply the induction hypothesis to M and the substitutions σ'_1 and σ_2 with $\Gamma, \alpha^+ \vdash [\sigma'_1]M \leq_1 [\sigma_2]M$ to obtain:

$$\Gamma, \alpha^+ \vdash \sigma'_1 \simeq_1^{\leq} \text{id} : \mathbf{fv} M \quad (1)$$

$$\Gamma, \alpha^+ \vdash \sigma_2 \simeq_1^{\leq} \text{id} : \mathbf{fv} M \quad (2)$$

Then $\Gamma \vdash \sigma_2 \simeq_1^{\leq} \text{id} : \mathbf{fv} \forall \alpha^+. M$ holds by strengthening of 2: for any $\beta^\pm \in \mathbf{fv} \forall \alpha^+. M = \mathbf{fv} M \setminus \alpha^+$, $\Gamma, \alpha^+ \vdash [\sigma_2]\beta^\pm \simeq_1^{\leq} \beta^\pm$ is strengthened to $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leq} \beta^\pm$, because $\mathbf{fv} [\sigma_2]\beta^\pm = \mathbf{fv} \beta^\pm = \{\beta^\pm\} \subseteq \Gamma$.

To show that $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \text{id} : \mathbf{fv} \forall \alpha^+. M$, let us take an arbitrary $\beta^\pm \in \mathbf{fv} \forall \alpha^+. M = \mathbf{fv} M \setminus \alpha^+$.

$$\begin{aligned} \beta^\pm &= [\text{id}]\beta^\pm && \text{by definition of id} \\ &\simeq_1^{\leq} [\sigma'_1]\beta^\pm && \text{by 1} \\ &= [\vec{Q}/\alpha^+][\sigma_1]\beta^\pm && \text{by definition of } \sigma'_1 \\ &= [\sigma_1]\beta^\pm && \text{because } \alpha^+ \cap \mathbf{fv} [\sigma_1]\beta^\pm \subseteq \alpha^+ \cap \Gamma = \emptyset \end{aligned}$$

This way, $\Gamma \vdash [\sigma_1]\beta^\pm \simeq_1^{\leq} \beta^\pm$ for any $\beta^\pm \in \mathbf{fv} \forall \alpha^+. M$ and thus, $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \text{id} : \mathbf{fv} \forall \alpha^+. M$.

Case 3. $N = P \rightarrow M$

Then by inversion, $\Gamma \vdash P$ and $\Gamma \vdash M$. $\Gamma \vdash [\sigma_1]N \leq_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1](P \rightarrow M) \leq_1 [\sigma_2](P \rightarrow M)$, then by congruence of substitution, $\Gamma \vdash [\sigma_1]P \rightarrow [\sigma_1]M \leq_1 [\sigma_2]P \rightarrow [\sigma_2]M$, then by inversion $\Gamma \vdash [\sigma_1]P \geq_1 [\sigma_2]P$ and $\Gamma \vdash [\sigma_1]M \leq_1 [\sigma_2]M$.

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geq_1 [\sigma_2]P$ and to $\Gamma \vdash [\sigma_1]M \leq_1 [\sigma_2]M$, we obtain (respectively):

$$\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \mathbf{fv} P \quad (3)$$

$$\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \mathbf{fv} M \quad (4)$$

Noting that $\mathbf{fv}(P \rightarrow M) = \mathbf{fv} P \cup \mathbf{fv} M$, we combine eqs. (3) and (4) to conclude: $\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \mathbf{fv}(P \rightarrow M)$.

Case 4. $N = \uparrow P$

Then by inversion, $\Gamma \vdash P$. $\Gamma \vdash [\sigma_1]N \leq_1 [\sigma_2]N$ is rewritten as $\Gamma \vdash [\sigma_1]\uparrow P \leq_1 [\sigma_2]\uparrow P$, then by congruence of substitution and by inversion, $\Gamma \vdash [\sigma_1]P \geq_1 [\sigma_2]P$

Applying the induction hypothesis to $\Gamma \vdash [\sigma_1]P \geq_1 [\sigma_2]P$, we obtain $\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \mathbf{fv} P$. Since $\mathbf{fv} \uparrow P = \mathbf{fv} P$, we can conclude: $\Gamma \vdash \sigma_i \simeq_1^{\leq} \text{id} : \mathbf{fv} \uparrow P$.

Case 5. The positive cases are proved symmetrically. □

Corollary 18 (Substitution cannot induce proper subtypes or supertypes). *Assuming all mentioned types are well-formed in Γ and σ is a substitution $\Gamma \vdash \sigma : \Gamma$,*

$$\Gamma \vdash [\sigma]N \leq_1 N \Rightarrow \Gamma \vdash [\sigma]N \simeq_1^{\leq} N \text{ and } \Gamma \vdash \sigma \simeq_1^{\leq} \text{id} : \mathbf{fv} N$$

$$\Gamma \vdash N \leq_1 [\sigma]N \Rightarrow \Gamma \vdash N \simeq_1^{\leq} [\sigma]N \text{ and } \Gamma \vdash \sigma \simeq_1^{\leq} \text{id} : \mathbf{fv} N$$

$$\Gamma \vdash [\sigma]P \geq_1 P \Rightarrow \Gamma \vdash [\sigma]P \simeq_1^{\leq} P \text{ and } \Gamma \vdash \sigma \simeq_1^{\leq} \text{id} : \mathbf{fv} P$$

$$\Gamma \vdash P \geq_1 [\sigma]P \Rightarrow \Gamma \vdash P \simeq_1^{\leq} [\sigma]P \text{ and } \Gamma \vdash \sigma \simeq_1^{\leq} \text{id} : \mathbf{fv} P$$

Lemma 32. *Assuming that the mentioned types (P , Q , N , and M) are well-formed in Γ and that the substitutions (σ_1 and σ_2) have signature $\Gamma \vdash \sigma_i : \Gamma$,*

- + if $\Gamma \vdash [\sigma_1]P \geq_1 Q$ and $\Gamma \vdash [\sigma_2]Q \geq_1 P$
then there exists a bijection $\mu : \mathbf{fv} P \leftrightarrow \mathbf{fv} Q$ such that $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \mu : \mathbf{fv} P$ and $\Gamma \vdash \sigma_2 \simeq_1^{\leq} \mu^{-1} : \mathbf{fv} Q$;
- if $\Gamma \vdash [\sigma_1]N \leq_1 M$ and $\Gamma \vdash [\sigma_2]N \leq_1 M$
then there exists a bijection $\mu : \mathbf{fv} N \leftrightarrow \mathbf{fv} M$ such that $\Gamma \vdash \sigma_1 \simeq_1^{\leq} \mu : \mathbf{fv} N$ and $\Gamma \vdash \sigma_2 \simeq_1^{\leq} \mu^{-1} : \mathbf{fv} M$.

Proof.

- + Applying σ_2 to both sides of $\Gamma \vdash [\sigma_1]P \geq_1 Q$ (by ??), we have: $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geq_1 [\sigma_2]Q$. Composing it with $\Gamma \vdash [\sigma_2]Q \geq_1 P$ (by transitivity ??), we have $\Gamma \vdash [\sigma_2 \circ \sigma_1]P \geq_1 P$. Then by corollary 18, $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^{\leq} \text{id} : \mathbf{fv} P$.

By a symmetric argument, we also have: $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^{\leq} \text{id} : \mathbf{fv} Q$.

Now, we prove that $\Gamma \vdash \sigma_2 \circ \sigma_1 \simeq_1^{\leq} \text{id} : \mathbf{fv} P$ and $\Gamma \vdash \sigma_1 \circ \sigma_2 \simeq_1^{\leq} \text{id} : \mathbf{fv} Q$ implies that σ_1 and σ_1 are (equivalent to) mutually inverse bijections.

To do so, it suffices to prove that

- (i) for any $\alpha^\pm \in \mathbf{fv} P$ there exists $\beta^\pm \in \mathbf{fv} Q$ such that $\Gamma \vdash [\sigma_1]\alpha^\pm \simeq_1^{\leq} \beta^\pm$ and $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leq} \alpha^\pm$; and
- (ii) for any $\beta^\pm \in \mathbf{fv} Q$ there exists $\alpha^\pm \in \mathbf{fv} P$ such that $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leq} \alpha^\pm$ and $\Gamma \vdash [\sigma_1]\alpha^\pm \simeq_1^{\leq} \beta^\pm$.

Then these correspondences between $\mathbf{fv} P$ and $\mathbf{fv} Q$ are mutually inverse functions, since for any β^\pm there can be at most one α^\pm such that $\Gamma \vdash [\sigma_2]\beta^\pm \simeq_1^{\leq} \alpha^\pm$ (and vice versa).

- (i) Let us take $\alpha^\pm \in \mathbf{fv} P$.

- (a) if α^\pm is positive ($\alpha^\pm = \alpha^+$), from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^+ \simeq_1^{\leq} \alpha^+$, by corollary 4, we have $[\sigma_2][\sigma_1]\alpha^+ = \exists \overrightarrow{\beta^+}.\alpha^+$.
What shape can $[\sigma_1]\alpha^+$ have? It cannot be $\exists \overrightarrow{\alpha^-}.\downarrow N$ (for potentially empty α^-), because the outer constructor \downarrow would remain after the substitution σ_2 , whereas $\exists \overrightarrow{\beta^+}.\alpha^+$ does not have \downarrow . The only case left is $[\sigma_1]\alpha^+ = \exists \overrightarrow{\alpha^-}.\gamma^+$.
Notice that $\Gamma \vdash \exists \overrightarrow{\alpha^-}.\gamma^+ \simeq_1^{\leq} \gamma^+$, meaning that $\Gamma \vdash [\sigma_1]\alpha^+ \simeq_1^{\leq} \gamma^+$. Also notice that $[\sigma_2]\exists \overrightarrow{\alpha^-}.\gamma^+ = \exists \overrightarrow{\beta^+}.\alpha^+$ implies $\Gamma \vdash [\sigma_2]\gamma^+ \simeq_1^{\leq} \alpha^+$.
- (b) if α^\pm is negative ($\alpha^\pm = \alpha^-$) from $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^- \simeq_1^{\leq} \alpha^-$, by corollary 4, we have $[\sigma_2][\sigma_1]\alpha^- = \forall \overrightarrow{\beta^+}.\alpha^-$.
What shape can $[\sigma_1]\alpha^-$ have? It cannot be $\forall \overrightarrow{\alpha^+}.\uparrow P$ nor $\forall \overrightarrow{\alpha^+}.P \rightarrow M$ (for potentially empty α^+), because the outer constructor (\rightarrow or \uparrow), remaining after the substitution σ_2 , is however absent in the resulting $\forall \overrightarrow{\beta^+}.\alpha^-$. Hence, the only case left is $[\sigma_1]\alpha^- = \forall \overrightarrow{\alpha^+}.\gamma^-$. Notice that $\Gamma \vdash \gamma^- \simeq_1^{\leq} \forall \overrightarrow{\alpha^+}.\gamma^-$, meaning that $\Gamma \vdash [\sigma_1]\alpha^- \simeq_1^{\leq} \gamma^-$. Also notice that $[\sigma_2]\forall \overrightarrow{\alpha^+}.\gamma^- = \forall \overrightarrow{\beta^+}.\alpha^-$ implies $\Gamma \vdash [\sigma_2]\gamma^- \simeq_1^{\leq} \alpha^-$.

(ii) The proof is symmetric: We swap P and Q , σ_1 and σ_2 , and exploit $\Gamma \vdash [\sigma_1][\sigma_2]\alpha^\pm \simeq_1^\leq \alpha^\pm$ instead of $\Gamma \vdash [\sigma_2][\sigma_1]\alpha^\pm \simeq_1^\leq \alpha^\pm$.

– The proof is symmetric to the positive case.

□

Lemma 33 (Equivalence of polymorphic types).

- For $\Gamma \vdash \forall \vec{\alpha}^+. N$ and $\Gamma \vdash \forall \vec{\beta}^+. M$,
if $\Gamma \vdash \forall \vec{\alpha}^+. N \simeq_1^\leq \forall \vec{\beta}^+. M$ then there exists a bijection $\mu : \vec{\beta}^+ \cap \mathbf{fv} M \leftrightarrow \vec{\alpha}^+ \cap \mathbf{fv} N$ such that $\Gamma, \vec{\alpha}^+ \vdash N \simeq_1^\leq [\mu]N$,
- + For $\Gamma \vdash \exists \vec{\alpha}^-. P$ and $\Gamma \vdash \exists \vec{\beta}^-. Q$,
if $\Gamma \vdash \exists \vec{\alpha}^-. P \simeq_1^\leq \exists \vec{\beta}^-. Q$ then there exists a bijection $\mu : \vec{\beta}^- \cap \mathbf{fv} Q \leftrightarrow \vec{\alpha}^- \cap \mathbf{fv} P$ such that $\Gamma, \vec{\beta}^- \vdash P \simeq_1^\leq [\mu]Q$.

Proof.

- First, by α -conversion, we ensure $\vec{\alpha}^+ \cap \mathbf{fv} M = \emptyset$ and $\vec{\beta}^+ \cap \mathbf{fv} N = \emptyset$. By inversion, $\Gamma \vdash \forall \vec{\alpha}^+. N \simeq_1^\leq \forall \vec{\beta}^+. M$ implies
 1. $\Gamma, \vec{\beta}^+ \vdash [\sigma_1]N \leq_1 M$ for $\Gamma, \vec{\beta}^+ \vdash \sigma_1 : \vec{\alpha}^+$ and
 2. $\Gamma, \vec{\alpha}^+ \vdash [\sigma_2]M \leq_1 N$ for $\Gamma, \vec{\alpha}^+ \vdash \sigma_2 : \vec{\beta}^+$.

To apply lemma 32, we weaken and rearrange the contexts, and extend the substitutions to act as identity outside of their initial domain:

1. $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash [\sigma_1]N \leq_1 M$ for $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash \sigma_1 : \Gamma, \vec{\alpha}^+, \vec{\beta}^+$ and
2. $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash [\sigma_2]M \leq_1 N$ for $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash \sigma_2 : \Gamma, \vec{\alpha}^+, \vec{\beta}^+$.

Then from lemma 32, there exists a bijection $\mu : \mathbf{fv} M \leftrightarrow \mathbf{fv} N$ such that $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash \sigma_2 \simeq_1^\leq \mu : \mathbf{fv} M$ and $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash \sigma_1 \simeq_1^\leq \mu^{-1} : \mathbf{fv} N$.

Let us show that if we restrict the domain of μ to $\vec{\beta}^+$, its range will be contained in $\vec{\alpha}^+$. Let us take $\gamma^+ \in \vec{\beta}^+ \cap \mathbf{fv} M$ and assume $[\mu]\gamma^+ \notin \vec{\alpha}^+$. Then since $\Gamma, \vec{\beta}^+ \vdash \sigma_1 : \vec{\alpha}^+$, σ_1 acts as identity outside of $\vec{\alpha}^+$, i.e. $[\sigma_1][\mu]\gamma^+ = [\mu]\gamma^+$. Since $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash \sigma_1 \simeq_1^\leq \mu^{-1} : \mathbf{fv} N$, application of σ_1 is equivalent to application of μ^{-1} , then $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash [\mu^{-1}][\mu]\gamma^+ \simeq_1^\leq [\mu]\gamma^+$, i.e. $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash \gamma^+ \simeq_1^\leq [\mu]\gamma^+$, which means $\gamma^+ \in \mathbf{fv} [\mu]\gamma^+ \subseteq \mathbf{fv} N$. By assumption, $\gamma^+ \in \vec{\beta}^+ \cap \mathbf{fv} M$, i.e. $\vec{\beta}^+ \cap \mathbf{fv} N \neq \emptyset$, hence contradiction.

By ??, $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash \sigma_2 \simeq_1^\leq \mu|_{\vec{\beta}^+} : \mathbf{fv} M$ implies $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash [\sigma_2]M \simeq_1^\leq [\mu|_{\vec{\beta}^+}]M$. By similar reasoning, $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash [\sigma_1]N \simeq_1^\leq [\mu^{-1}|_{\vec{\alpha}^+}]N$.

This way,

$$\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash [\mu^{-1}|_{\vec{\alpha}^+}]N \leq_1 M \quad (5)$$

$$\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash [\mu|_{\vec{\beta}^+}]M \leq_1 N \quad (6)$$

By applying $\mu|_{\vec{\beta}^+}$ to both sides of 5 (??) and contracting $\mu^{-1}|_{\vec{\alpha}^+} \circ \mu|_{\vec{\beta}^+} = \mu|_{\vec{\beta}^+}^{-1} \circ \mu|_{\vec{\beta}^+} = \text{id}$, we have: $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash N \leq_1 [\mu|_{\vec{\beta}^+}]M$, which together with 6 means $\Gamma, \vec{\alpha}^+, \vec{\beta}^+ \vdash N \simeq_1^\leq [\mu|_{\vec{\beta}^+}]M$, and by strengthening, $\Gamma, \vec{\alpha}^+ \vdash N \simeq_1^\leq [\mu|_{\vec{\beta}^+}]M$. Symmetrically, $\Gamma, \vec{\beta}^+ \vdash M \simeq_1^\leq [\mu|_{\vec{\beta}^+}^{-1}]N$.

- + The proof is symmetric to the proof of the negative case.

□

Lemma 34 (Completeness of equivalence). *Mutual subtyping implies declarative equivalence. Assuming all the types below are well-formed in Γ :*

- + if $\Gamma \vdash P \simeq_1^\leq Q$ then $P \simeq_1^D Q$,
- if $\Gamma \vdash N \simeq_1^\leq M$ then $N \simeq_1^D M$.

Proof. – Induction on the sum of sizes of N and M . By inversion, $\Gamma \vdash N \simeq_1^\leq M$ means $\Gamma \vdash N \leq_1 M$ and $\Gamma \vdash M \leq_1 N$. Let us consider the last rule that forms $\Gamma \vdash N \leq_1 M$:

Case 1. Rule $(\text{Var}^{-\leq_1})$ i.e. $\Gamma \vdash N \leq_1 M$ is of the form $\Gamma \vdash \alpha^- \leq_1 \alpha^-$
Then $N \simeq_1^D M$ (i.e. $\alpha^- \simeq_1^D \alpha^-$) holds immediately by Rule $(\text{Var}^{-\simeq_1^D})$.

Case 2. Rule (\uparrow^{\leq_1}) i.e. $\Gamma \vdash N \leq_1 M$ is of the form $\Gamma \vdash \uparrow P \leq_1 \uparrow Q$
Then by inversion, $\Gamma \vdash P \simeq_1^{\leq} Q$, and by induction hypothesis, $P \simeq_1^D Q$. Then $N \simeq_1^D M$ (i.e. $\uparrow P \simeq_1^D \uparrow Q$) holds by Rule $(\uparrow^{\simeq_1^D})$.

Case 3. Rule (\rightarrow^{\leq_1}) i.e. $\Gamma \vdash N \leq_1 M$ is of the form $\Gamma \vdash P \rightarrow N' \leq_1 Q \rightarrow M'$
Then by inversion, $\Gamma \vdash P \geq_1 Q$ and $\Gamma \vdash N' \leq_1 M'$. Notice that $\Gamma \vdash M \leq_1 N$ is of the form $\Gamma \vdash Q \rightarrow M' \leq_1 P \rightarrow N'$, which by inversion means $\Gamma \vdash Q \geq_1 P$ and $\Gamma \vdash M' \leq_1 N'$.
This way, $\Gamma \vdash Q \simeq_1^{\leq} P$ and $\Gamma \vdash M' \simeq_1^{\leq} N'$. Then by induction hypothesis, $Q \simeq_1^D P$ and $M' \simeq_1^D N'$. Then $N \simeq_1^D M$ (i.e. $P \rightarrow N' \simeq_1^D Q \rightarrow M'$) holds by Rule $(\rightarrow^{\simeq_1^D})$.

Case 4. Rule (\forall^{\leq_1}) i.e. $\Gamma \vdash N \leq_1 M$ is of the form $\Gamma \vdash \forall \alpha^+. N' \leq_1 \forall \beta^+. M'$
Then by ??, $\Gamma \vdash \forall \alpha^+. N' \simeq_1^{\leq} \forall \beta^+. M'$ means that there exists a bijection $\mu : \overrightarrow{\beta^+} \cap \mathbf{fv} M' \leftrightarrow \overrightarrow{\alpha^+} \cap \mathbf{fv} N'$ such that $\Gamma, \overrightarrow{\alpha^+} \vdash [\mu]M' \simeq_1^{\leq} N'$.
Notice that the application of bijection μ to M' does not change its size (which is less than the size of M), hence the induction hypothesis applies. This way, $[\mu]M' \simeq_1^D N'$ (and by symmetry, $N' \simeq_1^D [\mu]M'$) holds by induction. Then we apply Rule $(\forall^{\simeq_1^D})$ to get $\forall \alpha^+. N' \simeq_1^D \forall \beta^+. M'$, i.e. $N \simeq_1^D M$.

+ The proof is symmetric to the proof of the negative case. □

Corollary 19 (Normalization is complete w.r.t. subtyping-induced equivalence). *Assuming all the types below are well-formed in Γ :*

- + if $\Gamma \vdash P \simeq_1^{\leq} Q$ then $\mathbf{nf}(P) = \mathbf{nf}(Q)$,
- if $\Gamma \vdash N \simeq_1^{\leq} M$ then $\mathbf{nf}(N) = \mathbf{nf}(M)$.

Proof. Immediately from lemmas 24 and 34. □

Lemma 35 (Algorithmization of subtyping-induced equivalence). *Mutual subtyping is equality of normal forms. Assuming all the types below are well-formed in Γ :*

- + $\Gamma \vdash P \simeq_1^{\leq} Q \iff \mathbf{nf}(P) = \mathbf{nf}(Q)$,
- $\Gamma \vdash N \simeq_1^{\leq} M \iff \mathbf{nf}(N) = \mathbf{nf}(M)$.

Proof. Let us prove the positive case, the negative case is symmetric. We prove both directions of \iff separately:

\Rightarrow exactly corollary 19;

\Leftarrow by lemmas 28 and 30. □

Lemma 36 (Substitution preserves declarative equivalence). *Suppose that σ is a substitution. Then*

- + $P \simeq_1^D Q$ implies $[\sigma]P \simeq_1^D [\sigma]Q$
- $N \simeq_1^D M$ implies $[\sigma]N \simeq_1^D [\sigma]M$

Proof. $P \simeq_1^D Q \Rightarrow \mathbf{nf}(P) = \mathbf{nf}(Q)$ by lemma 35
 $\Rightarrow [\mathbf{nf}(\sigma)]\mathbf{nf}(P) = [\mathbf{nf}(\sigma)]\mathbf{nf}(Q)$
 $\Rightarrow \mathbf{nf}([\sigma]P) = \mathbf{nf}([\sigma]Q)$ by lemma 23
 $\Rightarrow [\sigma]P \simeq_1^D [\sigma]Q$ by lemma 35 □

3.8 Unification Constraint Merge

Lemma 37 (Soundness of Unification Constraint Merge). *Suppose that $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$ are normalized unification constraints. If $\Theta \vdash UC_1 \& UC_2 = UC$ is defined then $UC = UC_1 \cup UC_2$.*

Proof.

- $UC_1 \& UC_2 \subseteq UC_1 \cup UC_2$

By definition, $UC_1 \& UC_2$ consists of three parts: entries of UC_1 that do not have matching entries of UC_2 , entries of UC_2 that do not have matching entries of UC_1 , and the merge of matching entries.

If e is from the first or the second part, then $e \in UC_1 \cup UC_2$ holds immediately. If e is from the third part, then e is the merge of two matching entries $e_1 \in UC_1$ and $e_2 \in UC_2$. Since UC_1 and UC_2 are normalized unification, e_1 and e_2 have one of the following forms:

- $\hat{\alpha}^+ : \approx P_1$ and $\hat{\alpha}^+ : \approx P_2$, where P_1 and P_2 are normalized, and then since $\Theta(\hat{\alpha}^+) \vdash e_1 \& e_2 = e$ exists, Rule ?? was applied to infer it. It means that $e = e_1 = e_2$;
- $\hat{\alpha}^- : \approx N_1$ and $\hat{\alpha}^- : \approx N_2$, then symmetrically, $\Theta(\hat{\alpha}^-) \vdash e_1 \& e_2 = e = e_1 = e_2$

In both cases, $e \in UC_1 \cup UC_2$.

- $UC_1 \cup UC_2 \subseteq UC_1 \& UC_2$

Let us take an arbitrary $e_1 \in UC_1$. Then since UC_1 is a unification constraint, e_1 has one of the following forms:

- $\hat{\alpha}^+ : \approx P$ where P is normalized. If $\hat{\alpha}^+ \notin \mathbf{dom}(UC_2)$, then $e_1 \in UC_1 \& UC_2$. Otherwise, there is a normalized matching $e_2 = (\hat{\alpha}^+ : \approx P') \in UC_2$ and then since $UC_1 \& UC_2$ exists, Rule ?? was applied to construct $e_1 \& e_2 \in UC_1 \& UC_2$. By inversion of Rule ??, $e_1 \& e_2 = e_1$, and $\mathbf{nf}(P) = \mathbf{nf}(P')$, which since P and P' are normalized, implies that $P = P'$, that is $e_1 = e_2 \in UC_1 \& UC_2$.
- $\hat{\alpha}^- : \approx N$ where N is normalized. Then symmetrically, $e_1 = e_2 \in UC_1 \& UC_2$.

Similarly, if we take an arbitrary $e_2 \in UC_2$, then $e_1 = e_2 \in UC_1 \& UC_2$.

□

Corollary 20. *Suppose that $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$ are normalized unification constraints. If $\Theta \vdash UC_1 \& UC_2 = UC$ is defined then*

1. $\Theta \vdash UC$ is normalized unification constraint,
2. for any substitution $\Theta \vdash \hat{\sigma}$, $\Theta \vdash \hat{\sigma} : UC$ implies $\Theta \vdash \hat{\sigma} : UC_1$ and $\Theta \vdash \hat{\sigma} : UC_2$.

Proof. It is clear that since $UC = UC_1 \cup UC_2$ (by lemma 37), and being normalized means that all entries are normalized, UC is a normalized unification constraint. Analogously, $\Theta \vdash UC = UC_1 \cup UC_2$ holds immediately, since $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$.

Let us take an arbitrary substitution $\Theta \vdash \hat{\sigma}$ and assume that $\Theta \vdash \hat{\sigma} : UC$. Then $\Theta \vdash \hat{\sigma} : UC_i$ holds by definition: If $e \in UC_i \subseteq UC_1 \cup UC_2 = UC$. So $\Theta(\hat{\alpha}^\pm) \vdash [\hat{\sigma}]\hat{\alpha}^\pm : e$ holds. □

Lemma 38 (Completeness of Unification Constraint Entry Merge). *For a fixed context Γ , suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$ are matching constraint entries.*

- for a type P such that $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$, $\Gamma \vdash e_1 \& e_2 = e$ is defined and $\Gamma \vdash P : e$.
- for a type N such that $\Gamma \vdash N : e_1$ and $\Gamma \vdash N : e_2$, $\Gamma \vdash e_1 \& e_2 = e$ is defined and $\Gamma \vdash N : e$.

Proof. The proof repeats the one of lemma 56 and is done by the case analysis on the shape of e_1 and e_2 . However, it only needs to consider two cases.

Case 1. e_1 is $\hat{\alpha}^+ : \approx Q_1$ and e_2 is $\hat{\alpha}^+ : \approx Q_2$.

Case 2. e_1 is $\hat{\alpha}^- : \approx N_1$ and e_2 is $\hat{\alpha}^- : \approx M_2$.

The proof of these cases is based only on lemma 35 and corollary 7, and does not require the properties of the least upper bound or subtyping. □

Lemma 39 (Completeness of Unification Constraint Merge). *Suppose that $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$. Then for any substitution $\Theta \vdash \hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} : UC_1$ and $\Theta \vdash \hat{\sigma} : UC_2$,*

1. $\Theta \vdash UC_1 \& UC_2 = UC$ is defined and
2. $\Theta \vdash \hat{\sigma} : UC$.

Proof. The proof repeats the proof of lemma 57 but uses lemma 38 instead of lemma 56. □

3.9 Unification

Lemma 40 (Soundness of Unification).

- + For normalized P and Q such that $\Gamma; \Theta \vdash P$ and $\Gamma \vdash Q$,
if $\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC$ then $\Theta \vdash UC$ and for any normalized $\hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} : UC$, $[\hat{\sigma}]P = Q$.
- For normalized N and M such that $\Gamma; \Theta \vdash N$ and $\Gamma \vdash M$,
if $\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC$ then $\Theta \vdash UC$ and for any normalized $\hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} : UC$, $[\hat{\sigma}]N = M$.

Proof. We prove by induction on the derivation of $\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC$ and mutually $\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC$. Let us consider the last rule forming this derivation.

Case 1. Rule $(\text{Var}^{-\stackrel{u}{\simeq}})$, then $N = \alpha^- = M$. The resulting unification constraint is empty: $UC = \cdot$. It satisfies $\Theta \vdash UC$ vacuously, and $[\hat{\sigma}]\alpha^- = \alpha^-$, that is $[\hat{\sigma}]N = M$.

Case 2. Rule $(\uparrow^{\stackrel{u}{\simeq}})$, then $N = \uparrow P$ and $M = \uparrow Q$. The algorithm makes a recursive call to $\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC$ returning UC . By induction hypothesis, $\Theta \vdash UC$ and for any $\Theta \vdash \hat{\sigma} : UC$, $[\hat{\sigma}]N = [\hat{\sigma}]\uparrow P = \uparrow[\hat{\sigma}]P = \uparrow Q = M$, as required.

Case 3. Rule $(\rightarrow^{\stackrel{u}{\simeq}})$, then $N = P \rightarrow N'$ and $M = Q \rightarrow M'$. The algorithm makes two recursive calls to $\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC_1$ and $\Gamma; \Theta \models N' \stackrel{u}{\simeq} M' \Rightarrow UC_2$ returning $\Theta \vdash UC_1 \& UC_2 = UC$ as the result.

It is clear that P , N' , Q , and M' are normalized, and that $\Gamma; \Theta \vdash P$, $\Gamma; \Theta \vdash N'$, $\Gamma \vdash Q$, and $\Gamma \vdash M'$. This way, the induction hypothesis is applicable to both recursive calls.

By applying the induction hypothesis to $\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC_1$, we have:

- $\Theta \vdash UC_1$,
- for any $\Theta \vdash \hat{\sigma}' : UC_1$, $[\hat{\sigma}']P = Q$.

By applying it to $\Gamma; \Theta \models N' \stackrel{u}{\simeq} M' \Rightarrow UC_2$, we have:

- $\Theta \vdash UC_2$,
- for any $\Theta \vdash \hat{\sigma}' : UC_2$, $[\hat{\sigma}']N' = M'$.

Let us take an arbitrary $\Theta \vdash \hat{\sigma} : UC$. By the soundness of the constraint merge (lemma 55), $\Theta \vdash UC_1 \& UC_2 = UC$ implies $\Theta \vdash \hat{\sigma} : UC_1$ and $\Theta \vdash \hat{\sigma} : UC_2$.

Applying the induction hypothesis to $\Theta \vdash \hat{\sigma} : UC_1$, we have $[\hat{\sigma}]P = Q$; applying it to $\Theta \vdash \hat{\sigma} : UC_2$, we have $[\hat{\sigma}]N' = M'$. This way, $[\hat{\sigma}]N = [\hat{\sigma}]P \rightarrow [\hat{\sigma}]N' = Q \rightarrow M' = M$.

Case 4. Rule $(\forall^{\stackrel{u}{\simeq}})$, then $N = \forall \alpha^+ . N'$ and $M = \forall \alpha^+ . M'$. The algorithm makes a recursive call to $\Gamma, \alpha^+; \Theta \models N' \stackrel{u}{\simeq} M' \Rightarrow UC$ returning UC as the result.

The induction hypothesis is applicable: $\Gamma, \alpha^+; \Theta \vdash N'$ and $\Gamma, \alpha^+ \vdash M'$ hold by inversion, and N' and M' are normalized, since N and M are. Let us take an arbitrary $\Theta \vdash \hat{\sigma} : UC$. By the induction hypothesis, $[\hat{\sigma}]N' = M'$. Then $[\hat{\sigma}]N = [\hat{\sigma}]\forall \alpha^+ . N' = \forall \alpha^+ . [\hat{\sigma}]N' = \forall \alpha^+ . M' = M$.

Case 5. Rule $(\text{UVar}^{-\stackrel{u}{\simeq}})$, then $N = \hat{\alpha}^-$, $\hat{\alpha}^- \{ \Delta \} \in \Theta$, and $\Delta \vdash M$. As the result, the algorithm returns $UC = (\hat{\alpha}^- : \approx M)$.

It is clear that $\hat{\alpha}^- \{ \Delta \} \vdash (\hat{\alpha}^- : \approx M)$, since $\Delta \vdash M$, meaning that $\Theta \vdash UC$.

Let us take an arbitrary $\hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} : UC$. Since $UC = (\hat{\alpha}^- : \approx M)$, $\Theta \vdash \hat{\sigma} : UC$ implies $\Theta(\hat{\alpha}^-) \vdash [\hat{\sigma}]\hat{\alpha}^- : (\hat{\alpha}^- : \approx M)$. By inversion of Rule SATSCENEq, it means $\Theta(\hat{\alpha}^-) \vdash [\hat{\sigma}]\hat{\alpha}^- \simeq_1^{\leq} M$. This way, $\Theta(\hat{\alpha}^-) \vdash [\hat{\sigma}]N \simeq_1^{\leq} M$. Notice that $\hat{\sigma}$ and N are normalized, and by ??, so is $[\hat{\sigma}]N$. Since both sides of $\Theta(\hat{\alpha}^-) \vdash [\hat{\sigma}]N \simeq_1^{\leq} M$ are normalized, by lemma 35, we have $[\hat{\sigma}]N = M$.

Case 6. The positive cases are proved symmetrically.

□

Lemma 41 (Completeness of Unification).

- + For normalized P and Q such that $\Gamma; \Theta \vdash P$ and $\Gamma \vdash Q$, for any $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]P = Q$, there exists $\Gamma; \Theta \models P \stackrel{u}{\simeq} Q \Rightarrow UC$, and $\Theta \vdash \hat{\sigma} : UC$.
- For normalized N and M such that $\Gamma; \Theta \vdash N$ and $\Gamma \vdash M$,
for any $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]N = M$, there exists $\Gamma; \Theta \models N \stackrel{u}{\simeq} M \Rightarrow UC$, and $\Theta \vdash \hat{\sigma} : UC$.

Proof. We prove it by induction on the structure of P and mutually, N .

Case 1. $N = \hat{\alpha}^-$

$\Gamma; \Theta \vdash \hat{\alpha}^-$ means that $\hat{\alpha}^- \{\Delta\} \in \Theta$ for some Δ .

Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]\hat{\alpha}^- = M$. $\Theta \vdash \hat{\sigma}$ means that $\Delta \vdash M$. This way, Rule (UVar $^{-\hat{\sigma}}$) is applicable to infer $\Gamma; \Theta \vdash \hat{\alpha}^- \stackrel{u}{\simeq} M \Rightarrow (\hat{\alpha}^- : \approx M)$. $\Theta \vdash \hat{\sigma} : (\hat{\alpha}^- : \approx M)$ holds by Rule SATSCENE $_q$.

Case 2. $N = \alpha^-$

Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]\alpha^- = M$. The latter means $M = \alpha^-$.

Then $[\hat{\sigma}]\alpha^- = M$ means $M = \alpha^-$. This way, Rule (Var $^{-\hat{\sigma}}$) infers $\Gamma; \Theta \vdash \alpha^- \stackrel{u}{\simeq} \alpha^- \Rightarrow \cdot$, which is rewritten as $\Gamma; \Theta \vdash N \stackrel{u}{\simeq} M \Rightarrow \cdot$, and $\Theta \vdash \hat{\sigma} : \cdot$ holds trivially.

Case 3. $N = \uparrow P$

Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]\uparrow P = M$. The latter means $\uparrow[\hat{\sigma}]P = M$, i.e. $M = \uparrow Q$ for some Q and $[\hat{\sigma}]P = Q$.

Let us show that the induction hypothesis is applicable to $[\hat{\sigma}]P = Q$. Notice that P is normalized, since $N = \uparrow P$ is normalized, $\Gamma; \Theta \vdash P$ holds by inversion of $\Gamma; \Theta \vdash \uparrow P$, and $\Gamma \vdash Q$ holds by inversion of $\Gamma \vdash \uparrow Q$.

This way, by the induction hypothesis there exists UC such that $\Gamma; \Theta \vdash P \stackrel{u}{\simeq} Q \Rightarrow UC$, and moreover, $\Theta \vdash \hat{\sigma} : UC$.

Case 4. $N = P \rightarrow N'$

Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}](P \rightarrow N') = M$. The latter means $[\hat{\sigma}]P \rightarrow [\hat{\sigma}]N' = M$, i.e. $M = Q \rightarrow M'$ for some Q and M' , such that $[\hat{\sigma}]P = Q$ and $[\hat{\sigma}]N' = M'$.

Let us show that the induction hypothesis is applicable to $[\hat{\sigma}]P = Q$ and to $[\hat{\sigma}]N' = M'$:

- P and N' are normalized, since $N = P \rightarrow N'$ is normalized
- $\Gamma; \Theta \vdash P$ and $\Gamma; \Theta \vdash N'$ follow from the inversion of $\Gamma; \Theta \vdash P \rightarrow N'$,
- $\Gamma \vdash Q$ and $\Gamma \vdash M'$ follow from inversion of $\Gamma \vdash Q \rightarrow M'$.

Then by the induction hypothesis, $\Gamma; \Theta \vdash P \stackrel{u}{\simeq} Q \Rightarrow UC_1$ and $\Theta \vdash \hat{\sigma} : UC_1$, $\Gamma; \Theta \vdash N' \stackrel{u}{\simeq} M' \Rightarrow UC_2$ and $\Theta \vdash \hat{\sigma} : UC_2$. To apply Rule ($\rightarrow^{\hat{\sigma}}$) and infer the required $\Gamma; \Theta \vdash N \stackrel{u}{\simeq} M \Rightarrow UC$, we need to show that $\Theta \vdash UC_1 \& UC_2 = UC$ is defined and $\Theta \vdash \hat{\sigma} : UC$. It holds by the completeness of the unification constraint merge (lemma 57):

- $\Theta \vdash UC_1$ and $\Theta \vdash UC_2$ holds by the soundness of unification (lemma 40)
- $\Theta \vdash \hat{\sigma} : UC_1$ and $\Theta \vdash \hat{\sigma} : UC_2$ holds as noted above

Case 5. $N = \forall \alpha^+. N'$

Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ such that $[\hat{\sigma}]\forall \alpha^+. N' = M$. The latter means $\forall \alpha^+. [\hat{\sigma}]N' = M$, i.e. $M = \forall \alpha^+. M'$ for some M' such that $[\hat{\sigma}]N' = M'$.

Let us show that the induction hypothesis is applicable to $[\hat{\sigma}]N' = M'$. Notice that N' is normalized, since $N = \forall \alpha^+. N'$ is normalized, $\Gamma, \alpha^+; \Theta \vdash N'$ follows from inversion of $\Gamma; \Theta \vdash \forall \alpha^+. N'$, $\Gamma, \alpha^+ \vdash M'$ follows from inversion of $\Gamma \vdash \forall \alpha^+. M'$, and $\Theta \vdash \hat{\sigma}$ by assumption.

This way, by the induction hypothesis, $\Gamma, \alpha^+; \Theta \vdash N' \stackrel{u}{\simeq} M' \Rightarrow UC$ exists and moreover, $\Theta \vdash \hat{\sigma} : UC$. Hence, Rule ($\forall^{\hat{\sigma}}$) is applicable to infer $\Gamma; \Theta \vdash \forall \alpha^+. N' \stackrel{u}{\simeq} \forall \alpha^+. M' \Rightarrow UC$, that is $\Gamma; \Theta \vdash N \stackrel{u}{\simeq} M \Rightarrow UC$.

Case 6. The positive cases are proved symmetrically.

□

3.10 Anti-unification

Observation 1 (Anti-unification algorithm is deterministic).

- + If $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$ and $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2)$, then $\Xi = \Xi'$, $Q = Q'$, $\hat{\tau}_1 = \hat{\tau}'_1$, and $\hat{\tau}_2 = \hat{\tau}'_2$.
- If $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$ and $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi', M', \hat{\tau}'_1, \hat{\tau}'_2)$, then $\Xi = \Xi'$, $M = M'$, $\hat{\tau}_1 = \hat{\tau}'_1$, and $\hat{\tau}_2 = \hat{\tau}'_2$.

Proof. By trivial induction on $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)$ and mutually on $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \mathbf{M}, \hat{\tau}_1, \hat{\tau}_2)$. \square

Observation 2. Names of the anti-unification variables are uniquely defined by the types they are mapped to by the resulting substitutions.

- + Assuming P_1 and P_2 are normalized, if $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)$ then for any $\hat{\beta}^- \in \Xi$, $\hat{\beta}^- = \hat{\alpha}^-_{\{[\hat{\tau}_1]\hat{\beta}^-, [\hat{\tau}_2]\hat{\beta}^-\}}$
- Assuming N_1 and N_2 are normalized, if $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \mathbf{M}, \hat{\tau}_1, \hat{\tau}_2)$ then for any $\hat{\beta}^- \in \Xi$, $\hat{\beta}^- = \hat{\alpha}^-_{\{[\hat{\tau}_1]\hat{\beta}^-, [\hat{\tau}_2]\hat{\beta}^-\}}$

Proof. By simple induction on $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)$ and mutually on $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \mathbf{M}, \hat{\tau}_1, \hat{\tau}_2)$. Let us consider the last rule applied to infer this judgment.

Case 1. Rule (Var⁺ _{\simeq}) or Rule (Var⁻ _{\simeq}), then $\Xi = \cdot$, and the property holds vacuously.

Case 2. Rule (AU⁻) Then $\Xi = \hat{\alpha}^-_{\{N_1, N_2\}}$, $\hat{\tau}_1 = \hat{\alpha}^-_{\{N_1, N_2\}} : \approx N_1$, and $\hat{\tau}_2 = \hat{\alpha}^-_{\{N_1, N_2\}} : \approx N_2$. So the property holds trivially.

Case 3. Rule ?? In this case, $\Xi = \Xi' \cup \Xi''$, $\hat{\tau}_1 = \hat{\tau}'_1 \cup \hat{\tau}''_1$, and $\hat{\tau}_2 = \hat{\tau}'_2 \cup \hat{\tau}''_2$, where the property holds for $(\Xi', \hat{\tau}'_1, \hat{\tau}'_2)$ and $(\Xi'', \hat{\tau}''_1, \hat{\tau}''_2)$ by the induction hypothesis. Then since the union of solutions does not change the types the variables are mapped to, the required property holds for Ξ , $\hat{\tau}_1$, and $\hat{\tau}_2$.

Case 4. For the other rules, the resulting Ξ is taken from the recursive call and the required property holds immediately by the induction hypothesis. \square

Lemma 42 (Soundness of Anti-Unification).

- + Assuming P_1 and P_2 are normalized, if $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)$ then
 1. $\Gamma; \Xi \vdash \mathbf{Q}$,
 2. $\Gamma; \cdot \vdash \hat{\tau}_i : \Xi$ for $i \in \{1, 2\}$ are anti-unification solutions, and
 3. $[\hat{\tau}_i]\mathbf{Q} = P_i$ for $i \in \{1, 2\}$.
- Assuming N_1 and N_2 are normalized, if $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \mathbf{M}, \hat{\tau}_1, \hat{\tau}_2)$ then
 1. $\Gamma; \Xi \vdash \mathbf{M}$,
 2. $\Gamma; \cdot \vdash \hat{\tau}_i : \Xi$ for $i \in \{1, 2\}$ are anti-unification solutions, and
 3. $[\hat{\tau}_i]\mathbf{M} = N_i$ for $i \in \{1, 2\}$.

Proof. We prove it by induction on $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, \mathbf{M}, \hat{\tau}_1, \hat{\tau}_2)$ and mutually, $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)$. Let us consider the last rule applied to infer this judgement.

Case 1. Rule (Var⁻ _{\simeq}), then $N_1 = \alpha^- = N_2$, $\Xi = \cdot$, $\mathbf{M} = \alpha^-$, and $\hat{\tau}_1 = \hat{\tau}_2 = \cdot$.

1. $\Gamma; \cdot \vdash \alpha^-$ follows from the assumption $\Gamma \vdash \alpha^-$,
2. $\Gamma; \cdot \vdash \cdot : \cdot$ holds trivially, and
3. $[\cdot]\alpha^- = \alpha^-$ holds trivially.

Case 2. Rule (\uparrow _{\simeq}), then $N_1 = \uparrow P_1$, $N_2 = \uparrow P_2$, and the algorithm makes the recursive call: $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)$, returning $(\Xi, \uparrow \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)$ as the result.

Since $N_1 = \uparrow P_1$ and $N_2 = \uparrow P_2$ are normalized, so are P_1 and P_2 , and thus, the induction hypothesis is applicable to $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)$:

1. $\Gamma; \Xi \vdash \mathbf{Q}$, and hence, $\Gamma; \Xi \vdash \uparrow \mathbf{Q}$,
2. $\Gamma; \cdot \vdash \hat{\tau}_i : \Xi$ for $i \in \{1, 2\}$, and
3. $[\hat{\tau}_i]\mathbf{Q} = P_i$ for $i \in \{1, 2\}$, and then by the definition of the substitution, $[\hat{\tau}_i]\uparrow \mathbf{Q} = \uparrow P_i$ for $i \in \{1, 2\}$.

Case 3. Rule (\rightarrow _{\simeq}), then $N_1 = P_1 \rightarrow N'_1$, $N_2 = P_2 \rightarrow N'_2$, and the algorithm makes two recursive calls: $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)$ and $\Gamma \models N'_1 \stackrel{a}{\simeq} N'_2 \Rightarrow (\Xi', \mathbf{M}, \hat{\tau}'_1, \hat{\tau}'_2)$ and returns $(\Xi \cup \Xi', \mathbf{Q} \rightarrow \mathbf{M}, \hat{\tau}_1 \cup \hat{\tau}'_1, \hat{\tau}_2 \cup \hat{\tau}'_2)$ as the result.

Notice that the induction hypothesis is applicable to $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, \mathbf{Q}, \hat{\tau}_1, \hat{\tau}_2)$: P_1 and P_2 are normalized, since $N_1 = P_1 \rightarrow N'_1$ and $N_2 = P_2 \rightarrow N'_2$ are normalized. Similarly, the induction hypothesis is applicable to $\Gamma \models N'_1 \stackrel{a}{\simeq} N'_2 \Rightarrow (\Xi', \mathbf{M}, \hat{\tau}'_1, \hat{\tau}'_2)$.

This way, by the induction hypothesis:

1. $\Gamma; \Xi \vdash Q$ and $\Gamma; \Xi' \vdash M$. Then by weakening (??), $\Gamma; \Xi \cup \Xi' \vdash Q$ and $\Gamma; \Xi \cup \Xi' \vdash M$, which implies $\Gamma; \Xi \cup \Xi' \vdash Q \rightarrow M$;
2. $\Gamma; \cdot \vdash \hat{\tau}_i : \Xi$ and $\Gamma; \cdot \vdash \hat{\tau}'_i : \Xi'$. Then $\Gamma; \cdot \vdash \hat{\tau}_i \cup \hat{\tau}'_i : \Xi \cup \Xi'$ are well-defined anti-unification solution. Let us take an arbitrary $\hat{\beta}^- \in \Xi \cup \Xi'$. If $\hat{\beta}^- \in \Xi$, then $\Gamma; \cdot \vdash \hat{\tau}_i : \Xi$ implies that $\hat{\tau}_i$, and hence, $\hat{\tau}_i \cup \hat{\tau}'_i$ contains an entry well-formed in Γ . If $\hat{\beta}^- \in \Xi'$, the reasoning is symmetric.
 $\hat{\tau}_i \cup \hat{\tau}'_i$ is a well-defined anti-unification solution: any anti-unification variable occurs uniquely $\hat{\tau}_i \cup \hat{\tau}'_i$, since by observation 2, the name of the variable is in one-to-one correspondence with the pair of types it is mapped to by $\hat{\tau}_1$ and $\hat{\tau}_2$, and is in one-to-one correspondence with the pair of types it is mapped to by $\hat{\tau}'_1$ and $\hat{\tau}'_2$ i.e. if $\hat{\beta}^- \in \Xi \cap \Xi'$ then $[\hat{\tau}_1]\hat{\beta}^- = [\hat{\tau}'_1]\hat{\beta}^-$, and $[\hat{\tau}_2]\hat{\beta}^- = [\hat{\tau}'_2]\hat{\beta}^-$.
3. $[\hat{\tau}_i]Q = P_i$ and $[\hat{\tau}'_i]M = N'_i$. Since $\hat{\tau}_i \cup \hat{\tau}'_i$ restricted to Ξ is $\hat{\tau}_i$, and $\hat{\tau}_i \cup \hat{\tau}'_i$ restricted to Ξ' is $\hat{\tau}'_i$, we have $[\hat{\tau}_i \cup \hat{\tau}'_i]Q = P_i$ and $[\hat{\tau}_i \cup \hat{\tau}'_i]M = N'_i$, and thus, $[\hat{\tau}_i \cup \hat{\tau}'_i]Q \rightarrow M = P_i \rightarrow N'_i$

Case 4. Rule $(\forall^{\hat{\alpha}})$, then $N_1 = \forall \alpha^+. N'_1$, $N_2 = \forall \alpha^+. N'_2$, and the algorithm makes a recursive call: $\Gamma \models N'_1 \stackrel{a}{\simeq} N'_2 \Rightarrow (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$ and returns $(\Xi, \forall \alpha^+. M, \hat{\tau}_1, \hat{\tau}_2)$ as the result.

Similarly to case 2, we apply the induction hypothesis to $\Gamma \models N'_1 \stackrel{a}{\simeq} N'_2 \Rightarrow (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$ to obtain:

1. $\Gamma; \Xi \vdash M$, and hence, $\Gamma; \Xi \vdash \forall \alpha^+. M$;
2. $\Gamma; \cdot \vdash \hat{\tau}_i : \Xi$ for $i \in \{1, 2\}$, and
3. $[\hat{\tau}_i]M = N'_i$ for $i \in \{1, 2\}$, and then by the definition of the substitution, $[\hat{\tau}_i]\forall \alpha^+. M = \forall \alpha^+. N'_i$ for $i \in \{1, 2\}$.

Case 5. Rule (AU^-) , which applies when other rules do not, and $\Gamma \vdash N_i$, returning as the result $(\Xi, M, \hat{\tau}_1, \hat{\tau}_2) = (\hat{\alpha}^-_{\{N_1, N_2\}}, \hat{\alpha}^-_{\{N_1, N_2\}}, N_1), (\hat{\alpha}^-_{\{N_1, N_2\}} : \approx N_2))$.

1. $\Gamma; \Xi \vdash M$ is rewritten as $\Gamma; \hat{\alpha}^-_{\{N_1, N_2\}} \vdash \hat{\alpha}^-_{\{N_1, N_2\}}$, which holds trivially;
2. $\Gamma; \cdot \vdash \hat{\tau}_i : \Xi$ is rewritten as $\Gamma; \cdot \vdash (\hat{\alpha}^-_{\{N_1, N_2\}} : \approx N_i) : \hat{\alpha}^-_{\{N_1, N_2\}}$, which holds since $\Gamma \vdash N_i$ by the premise of the rule;
3. $[\hat{\tau}_i]M = N_i$ is rewritten as $[\hat{\alpha}^-_{\{N_1, N_2\}} : \approx N_i]\hat{\alpha}^-_{\{N_1, N_2\}} = N_i$, which holds trivially by the definition of substitution.

Case 6. Positive cases are proved symmetrically. □

Lemma 43 (Completeness of Anti-Unification).

+ Assume that P_1 and P_2 are normalized, and there exists $(\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2)$ such that

1. $\Gamma; \Xi' \vdash Q'$,
2. $\Gamma; \cdot \vdash \hat{\tau}'_i : \Xi'$ for $i \in \{1, 2\}$ are anti-unification solutions, and
3. $[\hat{\tau}'_i]Q' = P_i$ for $i \in \{1, 2\}$.

Then the anti-unification algorithm terminates, that is there exists $(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$ such that $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$

– Assume that N_1 and N_2 are normalized, and there exists $(\Xi', M', \hat{\tau}'_1, \hat{\tau}'_2)$ such that

1. $\Gamma; \Xi' \vdash M'$,
2. $\Gamma; \cdot \vdash \hat{\tau}'_i : \Xi'$ for $i \in \{1, 2\}$, are anti-unification solutions, and
3. $[\hat{\tau}'_i]M' = N_i$ for $i \in \{1, 2\}$.

Then the anti-unification algorithm succeeds, that is there exists $(\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$ such that $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$.

Proof. We prove it by the induction on M' and mutually on Q' .

Case 1. $M' = \hat{\alpha}^-$ Then since $\Gamma; \cdot \vdash \hat{\tau}'_i : \Xi'$, $\Gamma \vdash [\hat{\tau}'_i]M' = N_i$. This way, Rule (AU^-) is always applicable if other rules are not.

Case 2. $M' = \alpha^-$ Then $\alpha^- = [\hat{\tau}'_i]\alpha^- = N_i$, which means that Rule $(Var^{-\hat{\alpha}})$ is applicable.

Case 3. $M' = \uparrow Q'$ Then $\uparrow[\hat{\tau}'_i]Q' = [\hat{\tau}'_i]\uparrow Q' = N_i$, that is N_1 and N_2 have form $\uparrow P_1$ and $\uparrow P_2$ respectively.

Moreover, $[\hat{\tau}'_i]Q' = P_i$, which means that $(\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2)$ is an anti-unifier of P_1 and P_2 . Then by the induction hypothesis, there exists $(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$ such that $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$, and hence, $\Gamma \models \uparrow P_1 \stackrel{a}{\simeq} \uparrow P_2 \Rightarrow (\Xi, \uparrow Q, \hat{\tau}_1, \hat{\tau}_2)$ by Rule $(\uparrow^{\hat{\alpha}})$.

Case 4. $M' = \forall \alpha^+ . M''$ This case is similar to the previous one: we consider $\forall \alpha^+$ as a constructor. Notice that $\forall \alpha^+ . [\hat{\tau}'_i] M'' = [\hat{\tau}'_i] \forall \alpha^+ . M'' = N_i$, that is N_1 and N_2 have form $\forall \alpha^+ . N_1''$ and $\forall \alpha^+ . N_2''$ respectively.

Moreover, $[\hat{\tau}'_i] M'' = N_i''$, which means that $(\Xi', M'', \hat{\tau}'_1, \hat{\tau}'_2)$ is an anti-unifier of N_1'' and N_2'' . Then by the induction hypothesis, there exists $(\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$ such that $\Gamma \models N_1'' \stackrel{a}{\simeq} N_2'' \Rightarrow (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$, and hence, $\Gamma \models \forall \alpha^+ . N_1'' \stackrel{a}{\simeq} \forall \alpha^+ . N_2'' \Rightarrow (\Xi, \forall \alpha^+ . M, \hat{\tau}_1, \hat{\tau}_2)$ by Rule (\forall^+) .

Case 5. $M' = Q' \rightarrow M''$ Then $[\hat{\tau}'_i] Q' \rightarrow [\hat{\tau}'_i] M'' = [\hat{\tau}'_i] (Q' \rightarrow M'') = N_i$, that is N_1 and N_2 have form $P_1 \rightarrow N_1'$ and $P_2 \rightarrow N_2'$ respectively.

Moreover, $[\hat{\tau}'_i] Q' = P_i$ and $[\hat{\tau}'_i] M'' = N_i''$, which means that $(\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2)$ is an anti-unifier of P_1 and P_2 , and $(\Xi', M'', \hat{\tau}'_1, \hat{\tau}'_2)$ is an anti-unifier of N_1'' and N_2'' . Then by the induction hypothesis, $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi_1, Q, \hat{\tau}_1, \hat{\tau}_2)$ and $\Gamma \models N_1'' \stackrel{a}{\simeq} N_2'' \Rightarrow (\Xi_2, M, \hat{\tau}_3, \hat{\tau}_4)$ succeed. The result of the algorithm is $(\Xi_1 \cup \Xi_2, Q \rightarrow M, \hat{\tau}_1 \cup \hat{\tau}_3, \hat{\tau}_2 \cup \hat{\tau}_4)$.

Case 6. $Q' = \hat{\alpha}^+$ This case is not possible, since $\Gamma; \Xi' \vdash Q'$ means $\hat{\alpha}^+ \in \Xi'$, but Ξ' can only contain negative variables.

Case 7. Other positive cases are proved symmetrically to the corresponding negative ones. □

Lemma 44 (Initiality of Anti-Unification).

+ Assume that P_1 and P_2 are normalized, and $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$, then $(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$ is more specific than any other sound anti-unifier $(\Xi', Q', \hat{\tau}'_1, \hat{\tau}'_2)$, i.e. if

1. $\Gamma; \Xi' \vdash Q'$,
2. $\Gamma; \cdot \vdash \hat{\tau}'_i : \Xi'$ for $i \in \{1, 2\}$ are anti-unification solutions, and
3. $[\hat{\tau}'_i] Q' = P_i$ for $i \in \{1, 2\}$

then there exists $\hat{\rho}$ such that $\Gamma; \Xi \vdash \hat{\rho} : (\Xi'|_{\mathbf{uv} Q'})$ and $[\hat{\rho}] Q' = Q$. Moreover, $[\hat{\rho}] \hat{\beta}^-$ can be uniquely determined by $[\hat{\tau}'_1] \hat{\beta}^-$, $[\hat{\tau}'_2] \hat{\beta}^-$, and Γ .

– Assume that N_1 and N_2 are normalized, and $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$, then $(\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$ is more specific than any other sound anti-unifier $(\Xi', M', \hat{\tau}'_1, \hat{\tau}'_2)$, i.e. if

1. $\Gamma; \Xi' \vdash M'$,
2. $\Gamma; \cdot \vdash \hat{\tau}'_i : \Xi'$ for $i \in \{1, 2\}$ are anti-unification solutions, and
3. $[\hat{\tau}'_i] M' = N_i$ for $i \in \{1, 2\}$

then there exists $\hat{\rho}$ such that $\Gamma; \Xi \vdash \hat{\rho} : (\Xi'|_{\mathbf{uv} M'})$ and $[\hat{\rho}] M' = M$. Moreover, $[\hat{\rho}] \hat{\beta}^-$ can be uniquely determined by $[\hat{\tau}'_1] \hat{\beta}^-$, $[\hat{\tau}'_2] \hat{\beta}^-$, and Γ .

Proof. First, let us assume that M' is a metavariable $\hat{\alpha}^-$. Then we can take $\hat{\rho} = \hat{\alpha}^- \mapsto M$, which satisfies the required properties:

- $\Gamma; \Xi \vdash \hat{\rho} : (\Xi'|_{\mathbf{uv} M'})$ holds since $\Xi'|_{\mathbf{uv} M'} = \hat{\alpha}^-$ and $\Gamma; \Xi \vdash M$ by the soundness of anti-unification (lemma 42);
- $[\hat{\rho}] M' = M$ holds by construction
- $[\hat{\rho}] \hat{\alpha}^- = M$ is the anti-unifier of $N_1 = [\hat{\tau}'_1] \hat{\alpha}^-$ and $N_2 = [\hat{\tau}'_2] \hat{\alpha}^-$ in context Γ , and hence, it is uniquely determined by them (observation 1).

Now, we can assume that M' is not a metavariable. We prove by induction on the derivation of $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$ and mutually on the derivation of $\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (\Xi, M, \hat{\tau}_1, \hat{\tau}_2)$.

Since M' is not a metavariable, the substitution acting on M' preserves its outer constructor. In other words, $[\hat{\tau}'_i] M' = N_i$ means that M' , N_1 and N_2 have the same outer constructor. Let us consider the algorithmic anti-unification rule corresponding to this constructor, and show that it was successfully applied to anti-unify N_1 and N_2 (or P_1 and P_2).

Case 1. Rule $(\text{Var}^{\hat{\alpha}^-})$, i.e. $N_1 = \alpha^- = N_2$. This rule is applicable since it has no premises.

Then $\Xi = \cdot$, $M = \alpha^-$, and $\hat{\tau}_1 = \hat{\tau}_2 = \cdot$. Since $[\hat{\tau}'_i] M' = N_i = \alpha^-$ and M' is not a metavariable, $M' = \alpha^-$. Then we can take $\hat{\rho} = \cdot$, which satisfies the required properties:

- $\Gamma; \Xi \vdash \hat{\rho} : (\Xi'|_{\mathbf{uv} M'})$ holds vacuously since $\Xi'|_{\mathbf{uv} M'} = \emptyset$;
- $[\hat{\rho}] M' = M$, that is $[\cdot] \alpha^- = \alpha^-$ holds by substitution properties;

- the unique determination of $[\hat{\rho}]\hat{\alpha}^-$ for $\hat{\alpha}^- \in \Xi'|_{\mathbf{uv} M'} = \emptyset$ holds vacuously.

Case 2. Rule $(\uparrow^{\hat{\alpha}})$, i.e. $N_1 = \uparrow P_1$ and $N_2 = \uparrow P_2$.

Then since $[\hat{\tau}_i]M' = N_i = \uparrow P_i$ and M' is not a metavariable, $M' = \uparrow Q'$, where $[\hat{\tau}_i]Q' = P_i$. Let us show that $(\Xi', Q', \hat{\tau}_1', \hat{\tau}_2')$ is an anti-unifier of P_1 and P_2 .

1. $\Gamma; \Xi' \vdash Q'$ holds by inversion of $\Gamma; \Xi' \vdash \uparrow Q'$;
2. $\Gamma; \cdot \vdash \hat{\tau}_i' : \Xi'$ holds by assumption;
3. $[\hat{\tau}_i']Q' = P_i$ holds by assumption.

This way, by the completeness of anti-unification (lemma 43), the anti-unification algorithm succeeds on P_1 and P_2 : $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$, which means that Rule $(\uparrow^{\hat{\alpha}})$ is applicable to infer $\Gamma \models \uparrow P_1 \stackrel{a}{\simeq} \uparrow P_2 \Rightarrow (\Xi, \uparrow Q, \hat{\tau}_1, \hat{\tau}_2)$.

Moreover, by the induction hypothesis, $(\Xi, Q, \hat{\tau}_1, \hat{\tau}_2)$ is more specific than $(\Xi', Q', \hat{\tau}_1', \hat{\tau}_2')$, which immediately implies that $(\Xi, \uparrow Q, \hat{\tau}_1, \hat{\tau}_2)$ is more specific than $(\Xi', \uparrow Q', \hat{\tau}_1', \hat{\tau}_2')$ (we keep the same $\hat{\rho}$).

Case 3. Rule $(\forall^{\hat{\alpha}})$, i.e. $N_1 = \forall \alpha^+. N_1'$ and $N_2 = \forall \alpha^+. N_2'$. The proof is symmetric to the previous case. Notice that the context Γ is not changed in Rule $(\forall^{\hat{\alpha}})$, as it represents the context in which the anti-unification variables must be instantiated, rather than the context forming the types that are being anti-unified.

Case 4. Rule $(\rightarrow^{\hat{\alpha}})$, i.e. $N_1 = P_1 \rightarrow N_1'$ and $N_2 = P_2 \rightarrow N_2'$.

Then since $[\hat{\tau}_i]M' = N_i = P_i \rightarrow N_i'$ and M' is not a metavariable, $M' = Q' \rightarrow M''$, where $[\hat{\tau}_i']Q' = P_i$ and $[\hat{\tau}_i']M'' = N_i'$.

Let us show that $(\Xi', Q', \hat{\tau}_1', \hat{\tau}_2')$ is an anti-unifier of P_1 and P_2 .

1. $\Gamma; \Xi' \vdash Q'$ holds by inversion of $\Gamma; \Xi' \vdash Q' \rightarrow M''$;
2. $\Gamma; \cdot \vdash \hat{\tau}_i' : \Xi'$ holds by assumption;
3. $[\hat{\tau}_i']Q' = P_i$ holds by assumption.

Similarly, $(\Xi', M'', \hat{\tau}_1', \hat{\tau}_2')$ is an anti-unifier of N_1' and N_2' .

Then by the completeness of anti-unification (lemma 43), the anti-unification algorithm succeeds on P_1 and P_2 : $\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (\Xi_1, Q, \hat{\tau}_1, \hat{\tau}_2)$; and on N_1' and N_2' : $\Gamma \models N_1' \stackrel{a}{\simeq} N_2' \Rightarrow (\Xi_2, M'', \hat{\tau}_3, \hat{\tau}_4)$. Notice that $\hat{\tau}_1$ & $\hat{\tau}_3$ and $\hat{\tau}_2$ & $\hat{\tau}_4$ are defined, in other words, for any $\hat{\beta}^- \in \Xi_1 \cap \Xi_2$, $[\hat{\tau}_1]\hat{\beta}^- = [\hat{\tau}_2]\hat{\beta}^-$ and $[\hat{\tau}_3]\hat{\beta}^- = [\hat{\tau}_4]\hat{\beta}^-$, which follows immediately from observation 2. This way, the algorithm proceeds by applying Rule $(\rightarrow^{\hat{\alpha}})$ and returns $(\Xi_1 \cup \Xi_2, Q \rightarrow M'', \hat{\tau}_1 \cup \hat{\tau}_3, \hat{\tau}_2 \cup \hat{\tau}_4)$.

It is left to construct $\hat{\rho}$ such that $\Gamma; \Xi \vdash \hat{\rho} : (\Xi'|_{\mathbf{uv} M'})$ and $[\hat{\rho}]M' = M$. By the induction hypothesis, there exist $\hat{\rho}_1$ and $\hat{\rho}_2$ such that $\Gamma; \Xi_1 \vdash \hat{\rho}_1 : (\Xi'|_{\mathbf{uv} Q'})$, $\Gamma; \Xi_2 \vdash \hat{\rho}_2 : (\Xi'|_{\mathbf{uv} M''})$, $[\hat{\rho}_1]Q' = Q$, and $[\hat{\rho}_2]M'' = M''$.

Let us show that $\hat{\rho} = \hat{\rho}_1 \cup \hat{\rho}_2$ satisfies the required properties:

- $\Gamma; \Xi_1 \cup \Xi_2 \vdash \hat{\rho}_1 \cup \hat{\rho}_2 : (\Xi'|_{\mathbf{uv} M'})$ holds since $\Xi'|_{\mathbf{uv} M'} = \Xi'|_{\mathbf{uv} Q' \rightarrow M''} = (\Xi'|_{\mathbf{uv} Q'}) \cup (\Xi'|_{\mathbf{uv} M''})$, $\Gamma; \Xi_1 \vdash \hat{\rho}_1 : (\Xi'|_{\mathbf{uv} Q'})$ and $\Gamma; \Xi_2 \vdash \hat{\rho}_2 : (\Xi'|_{\mathbf{uv} M''})$;
- $[\hat{\rho}]M' = [\hat{\rho}](Q' \rightarrow M'') = [\hat{\rho}|_{\mathbf{uv} Q'}]Q' \rightarrow [\hat{\rho}|_{\mathbf{uv} M''}]M'' = [\hat{\rho}_1]Q' \rightarrow [\hat{\rho}_2]M'' = Q \rightarrow M'' = M$;
- Since $[\hat{\rho}]\hat{\beta}^-$ is either equal to $[\hat{\rho}_1]\hat{\beta}^-$ or $[\hat{\rho}_2]\hat{\beta}^-$, it inherits their property that it is uniquely determined by $[\hat{\tau}_1']\hat{\beta}^-$, $[\hat{\tau}_2']\hat{\beta}^-$, and Γ .

Case 5. $P_1 = P_2 = \alpha^+$. This case is symmetric to case 1.

Case 6. $P_1 = \downarrow N_1$ and $P_2 = \downarrow N_2$. This case is symmetric to case 2

Case 7. $P_1 = \exists \alpha^+. P_1'$ and $P_2 = \exists \alpha^+. P_2'$. This case is symmetric to case 3

□

3.11 Upper Bounds

Lemma 45 (Decomposition of the quantifier rule). *Ilya: move somewhere* Whenever the quantifier rule (Rule $(\exists^{\geq 1})$ or Rule $(\forall^{\leq 1})$) is applied, one can assume that the rule adding quantifiers on the right-hand side was applied the last.

- If $\Gamma \vdash N \leq_1 \forall \vec{\beta}^+.M$ then $\Gamma, \vec{\beta}^+ \vdash N \leq_1 M$.
- + If $\Gamma \vdash P \geq_1 \exists \vec{\beta}^-.Q$ then $\Gamma, \vec{\beta}^- \vdash P \geq_1 Q$.

Lemma 46 (Characterization of the Supertypes). Let us define the set of upper bounds of a positive type $\text{UB}(P)$ in the following way:

$\Gamma \vdash P$	$\text{UB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\exists \vec{\alpha}^-. \beta^+ \mid \text{for } \vec{\alpha}^-\}$
$\Gamma \vdash \exists \vec{\beta}^-.Q$	$\text{UB}(\Gamma, \vec{\beta}^- \vdash Q)$ not using $\vec{\beta}^-$
$\Gamma \vdash \downarrow M$	$\left\{ \begin{array}{l} \exists \vec{\alpha}^-. \downarrow M' \mid \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t.} \\ \Gamma \vdash N_i, \Gamma, \vec{\alpha}^- \vdash M', \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' \simeq_1^D \downarrow M \end{array} \right\}$

Then $\text{UB}(\Gamma \vdash P) \equiv \{Q \mid \Gamma \vdash Q \geq_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Immediately from lemma 10

Case 2. $P = \exists \vec{\beta}^-.P'$

Then if $\Gamma \vdash Q \geq_1 \exists \vec{\beta}^-.P'$, then by lemma 45, $\Gamma, \vec{\beta}^- \vdash Q \geq_1 P'$, and $\mathbf{fv} Q \cap \vec{\beta}^- = \emptyset$ by the the Barendregt's convention. The other direction holds by Rule $(\exists^{\geq 1})$. This way, $\{Q \mid \Gamma \vdash Q \geq_1 \exists \vec{\beta}^-.P'\} = \{Q \mid \Gamma, \vec{\beta}^- \vdash Q \geq_1 P' \text{ s.t. } \mathbf{fv}(Q) \cap \vec{\beta}^- = \emptyset\}$. From the induction hypothesis, the latter is equal to $\text{UB}(\Gamma, \vec{\beta}^- \vdash P')$ not using $\vec{\beta}^-$, i.e. $\text{UB}(\Gamma \vdash \exists \vec{\beta}^-.P')$.

Case 3. $P = \downarrow M$

Then let us consider two subcases upper bounds without outer quantifiers (we denote the corresponding set restriction as $|\#$) and upper bounds with outer quantifiers ($|\exists$). We prove that for both of these groups, the restricted sets are equal.

a. $Q \neq \exists \vec{\beta}^-.Q'$

Then the last applied rule to infer $\Gamma \vdash Q \geq_1 \downarrow M$ must be Rule $(\downarrow^{\geq 1})$, which means $Q = \downarrow M'$, and by inversion, $\Gamma \vdash M' \simeq_1^< M$, then by lemma 34 and Rule $(\downarrow^{\simeq 1^D})$, $\downarrow M' \simeq_1^D \downarrow M$. This way, $Q = \downarrow M' \in \{\downarrow M' \mid \downarrow M' \simeq_1^D \downarrow M\} = \text{UB}(\Gamma \vdash \downarrow M)|\#$.

In the other direction, $\downarrow M' \simeq_1^D \downarrow M \Rightarrow \Gamma \vdash \downarrow M' \simeq_1^< \downarrow M$ by lemma 30, since $\Gamma \vdash \downarrow M'$ by lemma 29

$\Rightarrow \Gamma \vdash \downarrow M' \geq_1 \downarrow M$ by inversion

b. $Q = \exists \vec{\beta}^-.Q'$ (for non-empty $\vec{\beta}^-$)

Then the last rule applied to infer $\Gamma \vdash \exists \vec{\beta}^-.Q' \geq_1 \downarrow M$ must be Rule $(\exists^{\geq 1})$. Inversion of this rule gives us $\Gamma \vdash [\vec{N}/\vec{\beta}^-]Q' \geq_1 \downarrow M$ for some $\Gamma \vdash N_i$. Notice that $[\vec{N}/\vec{\beta}^-]Q'$ has no outer quantifiers. Thus from case 3.a, $[\vec{N}/\vec{\beta}^-]Q' \simeq_1^D \downarrow M$, which is only possible if $Q' = \downarrow M'$. This way, $Q = \exists \vec{\beta}^-. \downarrow M' \in \text{UB}(\Gamma \vdash \downarrow M)|\exists$ (notice that $\vec{\beta}^-$ is not empty).

In the other direction, $[\vec{N}/\vec{\beta}^-] \downarrow M' \simeq_1^D \downarrow M \Rightarrow \Gamma \vdash [\vec{N}/\vec{\beta}^-] \downarrow M' \simeq_1^< \downarrow M$ by lemma 30, since $\Gamma \vdash [\vec{N}/\vec{\beta}^-] \downarrow M'$ by lemma 29

$\Rightarrow \Gamma \vdash [\vec{N}/\vec{\beta}^-] \downarrow M' \geq_1 \downarrow M$ by inversion

$\Rightarrow \Gamma \vdash \exists \vec{\beta}^-. \downarrow M' \geq_1 \downarrow M$ by Rule $(\exists^{\geq 1})$

□

Lemma 47 (Characterization of the Normalized Supertypes). For a normalized positive type $P = \mathbf{nf}(P)$, let us define the set of normalized upper bounds in the following way:

$\Gamma \vdash P$	$\text{NFUB}(\Gamma \vdash P)$
$\Gamma \vdash \beta^+$	$\{\beta^+\}$
$\Gamma \vdash \exists \vec{\beta}^-.P$	$\text{NFUB}(\Gamma, \vec{\beta}^- \vdash P)$ not using $\vec{\beta}^-$
$\Gamma \vdash \downarrow M$	$\left\{ \begin{array}{l} \exists \vec{\alpha}^-. \downarrow M' \mid \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \Gamma \vdash N_i, \Gamma, \vec{\alpha}^- \vdash M', \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' = \downarrow M \end{array} \right\}$

Then $\text{NFUB}(\Gamma \vdash P) \equiv \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 P\}$.

Proof. By induction on $\Gamma \vdash P$.

Case 1. $P = \beta^+$

Then from lemma 46, $\{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 \beta^+\} = \{\mathbf{nf}(\overrightarrow{\exists \alpha^-}.\beta^+) \mid \text{for some } \overrightarrow{\alpha^-} = \{\beta^+\}\}$

Case 2. $P = \overrightarrow{\exists \beta^-}.P'$

$\text{NFUB}(\Gamma \vdash \overrightarrow{\exists \beta^-}.P') = \text{NFUB}(\Gamma, \overrightarrow{\beta^-} \vdash P')$ not using $\overrightarrow{\beta^-}$

$= \{\mathbf{nf}(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P'\}$ not using $\overrightarrow{\beta^-}$

by the induction hypothesis

$= \{\mathbf{nf}(Q) \mid \Gamma, \overrightarrow{\beta^-} \vdash Q \geqslant_1 P' \text{ s.t. } \mathbf{fv} Q \cap \overrightarrow{\beta^-} = \emptyset\}$

because $\mathbf{fv} \mathbf{nf}(Q) = \mathbf{fv} Q$ by lemma 21

$= \{\mathbf{nf}(Q) \mid Q \in \text{UB}(\Gamma, \overrightarrow{\beta^-} \vdash P') \text{ s.t. } \mathbf{fv} Q \cap \overrightarrow{\beta^-} = \emptyset\}$

by lemma 46

$= \{\mathbf{nf}(Q) \mid Q \in \text{UB}(\Gamma \vdash \overrightarrow{\exists \beta^-}.P')\}$

by the definition of UB

$= \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \geqslant_1 \overrightarrow{\exists \beta^-}.P'\}$

by lemma 46

Case 3. $P = \downarrow M$

In the following reasoning, we will use the following principle of variable replacement.

Observation 3. Suppose that $\nu : A \rightarrow A$ is an idempotent function, P is a predicate on A , $F : A \rightarrow B$ is a function. Then

$$\begin{aligned} & \{F(\nu x) \mid x \in A \text{ s.t. } P(\nu x)\} = \\ & = \{F(x) \mid x \in A \text{ s.t. } \nu x = x \text{ and } P(x)\}. \end{aligned}$$

In our case, the idempotent ν will be normalization, variable ordering, or domain restriction.

Another observation we will use is the following.

Observation 4. For functions F and ν , and predicates P and Q ,

$$\begin{aligned} & \{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } P(x)\} = \\ & = \{F(\nu x) \mid x \in A \text{ s.t. } Q(\nu x) \text{ and } (\exists x' \in A \text{ s.t. } P(x') \text{ and } \nu x' = \nu x)\}. \end{aligned}$$

Observation 5. There exist positive and negative types well-formed in empty context, hence, a type substitution can be extended to an arbitrary domain (if its values on the domain extension are irrelevant). Specifically, Suppose that $\text{vars}_1 \subseteq \text{vars}_2$. Then $\Gamma \vdash \sigma|_{\text{vars}_1} : \text{vars}_1$ implies $\exists \sigma' \text{ s.t. } \Gamma \vdash \sigma' : \text{vars}_2$ and $\sigma|_{\text{vars}_1} = \sigma'|_{\text{vars}_1}$.

$$\begin{aligned}
& \{\mathbf{nf}(Q) \mid \Gamma \vdash Q \gg_1 \downarrow M\} = \\
& = \{\mathbf{nf}(Q) \mid Q \in \mathbf{UB}(\Gamma \vdash \downarrow M)\} \\
& = \left\{ \mathbf{nf}(\exists \vec{\alpha}^-. \downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash N_i, \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \\
& = \left\{ \mathbf{nf}(\exists \vec{\alpha}^-. \downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } [\sigma] \downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \\
& = \left\{ \mathbf{nf}(\exists \vec{\alpha}^-. \downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } [\sigma|_{\mathbf{fv} M'}] \downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^-. \mathbf{nf}(\downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^-, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \text{and } [\sigma|_{\mathbf{fv} M'}] \downarrow M' \simeq_1^D \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^-. \mathbf{nf}(\downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^-, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \text{and } \mathbf{nf}([\sigma|_{\mathbf{fv} M'}] \downarrow M') = \mathbf{nf}(\downarrow M) \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^-. \mathbf{nf}(\downarrow M') \mid \begin{array}{l} \text{for } \vec{\alpha}^-, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \text{and } [\mathbf{nf}(\sigma|_{\mathbf{fv} M'})] \downarrow \mathbf{nf}(M') = \downarrow \mathbf{nf}(M) \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \text{and } [\sigma|_{\mathbf{fv} M'}] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ (\exists \sigma' \text{ s.t. } \Gamma \vdash \sigma' : \vec{\alpha}^- \text{ and } \sigma|_{\mathbf{fv}(\downarrow M')} = \sigma'|_{\mathbf{fv}(\downarrow M')}) \\ \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^- \text{ and } [\sigma|_{\mathbf{fv} M'}] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma|_{\mathbf{fv} M'} : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \text{and } [\sigma|_{\mathbf{fv} M'}] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \text{and } [\sigma] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \text{and } [\sigma] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \sigma \text{ s.t. } \Gamma, \vec{\alpha}^- \vdash M', \\ \Gamma \vdash \sigma : \vec{\alpha}^-, \text{ and } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \text{and } [\sigma] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \Gamma \vdash N_i, \Gamma, \vec{\alpha}^- \vdash M', \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' = \downarrow M \end{array} \right\} \\
& = \mathbf{NFUB}(\downarrow M) \\
& \text{hello}
\end{aligned}$$

by lemma 46

by the definition of UB

we reassigned the substitution $\vec{N}/\vec{\alpha}^-$ as σ

by lemma 1

by the definition of normalization

from lemmas 22 and 24, equivalence of types can be replaced with the equality of their normal forms

by congruence of normalization and lemma 23

by lemma 26, $\downarrow M'$ and $\sigma|_{\mathbf{fv} M'}$ are already normal, since the result of the substitution is normal; M is normal by assumption

We apply observation 4 (with $\nu\sigma = \sigma|_{\mathbf{fv} M'}$, and $P(\sigma) = \Gamma \vdash \sigma : \vec{\alpha}^-$)

Notice that

$\exists \sigma' \text{ s.t. } (\Gamma \vdash \sigma' : \vec{\alpha}^- \text{ and } \sigma|_{\mathbf{fv}(\downarrow M')} = \sigma'|_{\mathbf{fv}(\downarrow M')}) \downarrow \downarrow$ is equivalent to $\Gamma \vdash \sigma|_{\mathbf{fv}(\downarrow M')} : \vec{\alpha}^-$ (observation 5)

We apply observation 3 to the restriction of σ , and remove $\sigma|_{\mathbf{fv} M'} = \sigma$ as it follows from $\Gamma \vdash \sigma : \vec{\alpha}^-$

by lemma 15, since $\Gamma, \vec{\alpha}^- \cap \mathbf{fv} M' = \Gamma, \vec{\alpha}^- \cap \mathbf{fv} M'$

We apply observation 3 to the ordering of $\vec{\alpha}^-$

By reassigning σ explicitly as $\vec{N}/\vec{\alpha}^-$

by definition

□

Observation 6. Upper bounds of a type do not depend on the context as soon as the type are well-formed in it.

If $\Gamma_1 \vdash M$ and $\Gamma_2 \vdash M$ then $\mathbf{UB}(\Gamma_1 \vdash M) = \mathbf{UB}(\Gamma_2 \vdash M)$ and $\mathbf{NFUB}(\Gamma_1 \vdash M) = \mathbf{NFUB}(\Gamma_2 \vdash M)$

Proof. We prove both inclusions by induction on $\Gamma_1 \vdash M$. Notice that if $[\sigma]M' \simeq_1^D M$ and $\Gamma_2 \vdash M$ then the types from the range of $\sigma|_{\mathbf{fv} M'}$ are well-formed in 2 **Ilya: lemma**. □

Lemma 48 (Soundness of the Least Upper Bound). For types $\Gamma \vdash P_1$, and $\Gamma \vdash P_2$, if $\Gamma \models P_1 \vee P_2 = Q$ then

(i) $\Gamma \vdash Q$

(ii) $\Gamma \vdash Q \geqslant_1 P_1$ and $\Gamma \vdash Q \geqslant_1 P_2$

Proof. Induction on $\Gamma \models P_1 \vee P_2 = Q$.

Case 1. $\Gamma \models \alpha^+ \vee \alpha^+ = \alpha^+$

Then $\Gamma \vdash \alpha^+$ by assumption, and $\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+$ by Rule (Var⁺ \geqslant_1).

Case 2. $\Gamma \models \exists \vec{\alpha}^-. P_1 \vee \exists \vec{\beta}^-. P_2 = Q$

Then by inversion of $\Gamma \vdash \exists \vec{\alpha}^-. P_i$ and weakening, $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \vdash P_i$, hence, the induction hypothesis applies to $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \models P_1 \vee P_2 = Q$. Then

- (i) $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \vdash Q$,
- (ii) $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \vdash Q \geqslant_1 P_1$,
- (iii) $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \vdash Q \geqslant_1 P_2$.

To prove $\Gamma \vdash Q$, it suffices to show that $\mathbf{fv}(Q) \cap \Gamma, \vec{\alpha}^-, \vec{\beta}^- = \mathbf{fv}(Q) \cap \Gamma$ (and then apply lemma 15). The inclusion right-to-left is self-evident. To show $\mathbf{fv}(Q) \cap \Gamma, \vec{\alpha}^-, \vec{\beta}^- \subseteq \mathbf{fv}(Q) \cap \Gamma$, we prove that $\mathbf{fv}(Q) \subseteq \Gamma$

$$\begin{aligned} \mathbf{fv}(Q) &\subseteq \mathbf{fv} P_1 \cap \mathbf{fv} P_2 && \text{by lemma 8} \\ &\subseteq (\Gamma, \vec{\alpha}^- \setminus \vec{\beta}^-) \cap (\Gamma, \vec{\beta}^- \setminus \vec{\alpha}^-) && \begin{array}{l} \text{since } \Gamma \vdash \exists \vec{\alpha}^-. P_1, \mathbf{fv}(P_1) \subseteq \Gamma, \vec{\alpha}^- = \Gamma, \vec{\alpha}^- \setminus \vec{\beta}^- \\ \text{(the latter is because by the Barendregt's convention,} \\ \Gamma, \vec{\alpha}^- \cap \vec{\beta}^- = \emptyset); \text{ similarly, } \mathbf{fv}(P_2) \subseteq \Gamma, \vec{\beta}^- \setminus \vec{\alpha}^- \end{array} \\ &\subseteq \Gamma \end{aligned}$$

To show $\Gamma \vdash Q \geqslant_1 \exists \vec{\alpha}^-. P_1$, we apply Rule ($\exists \geqslant_1$). Then $\Gamma, \vec{\alpha}^- \vdash Q \geqslant_1 P_1$ holds since $\Gamma, \vec{\alpha}^-, \vec{\beta}^- \vdash Q \geqslant_1 P_1$ (by the induction hypothesis), $\Gamma, \vec{\alpha}^- \vdash Q$ (by weakening), and $\Gamma, \vec{\alpha}^- \vdash P_1$.

Judgment $\Gamma \vdash Q \geqslant_1 \exists \vec{\beta}^-. P_2$ is proved symmetrically.

Case 3. $\Gamma \models \downarrow N \vee \downarrow M = \exists \vec{\alpha}^-. [\vec{\alpha}^- / \Xi] P$. By the inversion, $\Gamma, \cdot \models \mathbf{nf}(\downarrow N) \stackrel{a}{\simeq} \mathbf{nf}(\downarrow M) = (\Xi, \vec{P}, \hat{\tau}_1, \hat{\tau}_2)$. Then by the soundness of anti-unification (??),

(i) $\Gamma; \Xi \vdash \vec{P}$, then by ??,

$$\Gamma, \vec{\alpha}^- \vdash [\vec{\alpha}^- / \Xi] P \quad (7)$$

(ii) $\Gamma; \cdot \vdash \hat{\tau}_1 : \Xi$ and $\Gamma; \cdot \vdash \hat{\tau}_2 : \Xi$. Assuming that $\Xi = \hat{\beta}_1^-, \dots, \hat{\beta}_n^-$, the antiunification solutions $\hat{\tau}_1$ and $\hat{\tau}_2$ can be put explicitly as $\hat{\tau}_1 = (\hat{\beta}_1^- : \approx N_1, \dots, \hat{\beta}_n^- : \approx N_n)$, and $\hat{\tau}_2 = (\hat{\beta}_1^- : \approx M_1, \dots, \hat{\beta}_n^- : \approx M_n)$. Then

$$\hat{\tau}_1 = (\vec{N} / \vec{\alpha}^-) \circ (\vec{\alpha}^- / \Xi) \quad (8)$$

$$\hat{\tau}_2 = (\vec{M} / \vec{\alpha}^-) \circ (\vec{\alpha}^- / \Xi) \quad (9)$$

(iii) $[\hat{\tau}_1] Q = P_1$ and $[\hat{\tau}_2] Q = P_1$, which, by 8 and 9, means

$$[\vec{N} / \vec{\alpha}^-][\vec{\alpha}^- / \Xi] P = \mathbf{nf}(\downarrow N) \quad (10)$$

$$[\vec{M} / \vec{\alpha}^-][\vec{\alpha}^- / \Xi] P = \mathbf{nf}(\downarrow M) \quad (11)$$

Then $\Gamma \vdash \exists \vec{\alpha}^-. [\vec{\alpha}^- / \Xi] P$ follows directly from 7.

To show $\Gamma \vdash \exists \vec{\alpha}^-. [\vec{\alpha}^- / \Xi] P \geqslant_1 \downarrow N$, we apply Rule ($\exists \geqslant_1$), instantiating $\vec{\alpha}^-$ with \vec{N} . Then $\Gamma \vdash [\vec{N} / \vec{\alpha}^-][\vec{\alpha}^- / \Xi] P \geqslant_1 \downarrow N$ follows from 10 and since $\Gamma \vdash \mathbf{nf}(\downarrow N) \geqslant_1 \downarrow N$ (by corollary 16).

Analogously, instantiating $\vec{\alpha}^-$ with \vec{M} , gives us $\Gamma \vdash [\vec{M} / \vec{\alpha}^-][\vec{\alpha}^- / \Xi] P \geqslant_1 \downarrow M$ (from 11), and hence, $\Gamma \vdash \exists \vec{\alpha}^-. [\vec{\alpha}^- / \Xi] P \geqslant_1 \downarrow M$. \square

Lemma 49 (Completeness and Initiality of the Least Upper Bound). *For types $\Gamma \vdash P_1$, $\Gamma \vdash P_2$, and $\Gamma \vdash Q$ such that $\Gamma \vdash Q \geqslant_1 P_1$ and $\Gamma \vdash Q \geqslant_1 P_2$, there exists Q' s.t. $\Gamma \models P_1 \vee P_2 = Q'$ and $\Gamma \vdash Q \geqslant_1 Q'$.*

Proof. Induction on the pair (P_1, P_2) . From lemma 47, $Q \in \mathbf{UB}(\Gamma \vdash P_1) \cap \mathbf{UB}(\Gamma \vdash P_2)$. Let us consider the cases of what P_1 and P_2 are (i.e. the last rules to infer $\Gamma \vdash P_i$).

Case 1. $P_1 = \exists \vec{\beta}^-_1. Q_1$, $P_2 = \exists \vec{\beta}^-_2. Q_2$, where either $\vec{\beta}^-_1$ or $\vec{\beta}^-_2$ is not empty

Then $Q \in \text{UB}(\Gamma \vdash \exists \vec{\beta}^-_1. Q_1) \cap \text{UB}(\Gamma \vdash \exists \vec{\beta}^-_2. Q_2)$
 $\subseteq \text{UB}(\Gamma, \vec{\beta}^-_1 \vdash Q_1) \cap \text{UB}(\Gamma, \vec{\beta}^-_2 \vdash Q_2)$ from the definition of UB
 $= \text{UB}(\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q_1) \cap \text{UB}(\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q_2)$ by observation 6, weakening and exchange
 $= \{Q' \mid \Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q' \geq_1 Q_1\} \cap \{Q' \mid \Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q' \geq_1 Q_2\}$ by lemma 46,
 meaning that $\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q \geq_1 Q_1$ and $\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q \geq_1 Q_2$. Then the next step of the algorithm—the recursive call $\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q_1 \vee Q_2 = Q'$ terminates by the induction hypothesis, and moreover, $\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q \geq_1 Q'$. This way, the result of the algorithm is Q' , i.e. $\Gamma \vdash P_1 \vee P_2 = Q'$.

Since both Q and Q' are sound, $\Gamma \vdash Q$ and $\Gamma \vdash Q'$, and therefore, $\Gamma, \vec{\beta}^-_1, \vec{\beta}^-_2 \vdash Q \geq_1 Q'$ can be strengthened to $\Gamma \vdash Q \geq_1 Q'$ by ??.

Case 2. $P_1 = \alpha^+$ and $P_2 = \downarrow N$

Then the set of common upper bounds of $\downarrow N$ and α^+ is empty, and thus, $Q \in \text{UB}(\Gamma \vdash P_1) \cap \text{UB}(\Gamma \vdash P_2)$ gives a contradiction:
 $Q \in \text{UB}(\Gamma \vdash \alpha^+) \cap \text{UB}(\Gamma \vdash \downarrow N)$
 $= \{\exists \vec{\alpha}^-. \alpha^+ \mid \dots\} \cap \{\exists \vec{\beta}^-. \downarrow M' \mid \dots\}$ by the definition of UB
 $= \emptyset$ since $\alpha^+ \neq \downarrow M'$ for any M'

Case 3. $P_1 = \downarrow N$ and $P_2 = \alpha^+$

Symmetric to case 2

Case 4. $P_1 = \alpha^+$ and $P_2 = \beta^+$ (where $\beta^+ \neq \alpha^+$)

Similarly to case 2, the set of common upper bounds is empty, which leads to the contradiction:

$Q \in \text{UB}(\Gamma \vdash \alpha^+) \cap \text{UB}(\Gamma \vdash \beta^+)$
 $= \{\exists \vec{\alpha}^-. \alpha^+ \mid \dots\} \cap \{\exists \vec{\beta}^-. \beta^+ \mid \dots\}$ by the definition of UB
 $= \emptyset$ since $\alpha^+ \neq \beta^+$

Case 5. $P_1 = \alpha^+$ and $P_2 = \alpha^+$

Then the algorithm terminates in one step (Rule (Var^v)) and the result is α^+ , i.e. $\Gamma \vdash \alpha^+ \vee \alpha^+ = \alpha^+$.

Since $Q \in \text{UB}(\Gamma \vdash \alpha^+)$, $Q = \exists \vec{\alpha}^-. \alpha^+$. Then $\Gamma \vdash \exists \vec{\alpha}^-. \alpha^+ \geq_1 \alpha^+$ by Rule ($\exists \geq_1$): $\vec{\alpha}^-$ can be instantiated with arbitrary negative types (for example $\forall \beta^+. \uparrow \beta^+$), since the substitution for unused variables does not change the term $[\vec{N}/\vec{\alpha}^-] \alpha^+ = \alpha^+$, and then $\Gamma \vdash \alpha^+ \geq_1 \alpha^+$ by Rule (Var⁺ \geq_1).

Case 6. $P_1 = \downarrow M_1$ and $P_2 = \downarrow M_2$

Then on the next step, the algorithm tries to anti-unify $\mathbf{nf}(\downarrow M_1)$ and $\mathbf{nf}(\downarrow M_2)$. By ??, to show that the anti-unification algorithm terminates, it suffices to demonstrate that a sound anti-unification solution exists.

Notice that

$$\begin{aligned} \mathbf{nf}(Q) &\in \text{NFUB}(\Gamma \vdash \mathbf{nf}(\downarrow M_1)) \cap \text{NFUB}(\Gamma \vdash \mathbf{nf}(\downarrow M_2)) \\ &= \text{NFUB}(\Gamma \vdash \downarrow \mathbf{nf}(M_1)) \cap \text{NFUB}(\Gamma \vdash \downarrow \mathbf{nf}(M_2)) \\ &= \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \Gamma \vdash N_i, \Gamma, \vec{\alpha}^- \vdash M', \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_1) \end{array} \right\} \\ &= \bigcap \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \text{ and } \vec{N} \text{ s.t. } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \Gamma \vdash \vec{N}_1, \Gamma \vdash \vec{N}_2, \Gamma, \vec{\alpha}^- \vdash M', \text{ and } [\vec{N}/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_2) \end{array} \right\} \\ &= \left\{ \exists \vec{\alpha}^-. \downarrow M' \mid \begin{array}{l} \text{for } \vec{\alpha}^-, M', \vec{N}_1 \text{ and } \vec{N}_2 \text{ s.t. } \mathbf{ord} \vec{\alpha}^- \text{ in } M' = \vec{\alpha}^-, \\ \Gamma \vdash \vec{N}_1, \Gamma \vdash \vec{N}_2, \Gamma, \vec{\alpha}^- \vdash M', [\vec{N}_1/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_1), \text{ and } [\vec{N}_2/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_2) \end{array} \right\} \end{aligned}$$

The fact that the latter set is non-empty means that there exist $\vec{\alpha}^-, M', \vec{N}_1$ and \vec{N}_2 such that

- (i) $\Gamma, \vec{\alpha}^- \vdash M'$ (notice that M' is normal)
- (ii) $\Gamma \vdash \vec{N}_1$ and $\Gamma \vdash \vec{N}_2$,
- (iii) $[\vec{N}_1/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_1)$ and $[\vec{N}_2/\vec{\alpha}^-] \downarrow M' = \downarrow \mathbf{nf}(M_2)$

For each negative variable α^- from $\overrightarrow{\alpha^-}$, let us choose a fresh negative anti-unification variable $\hat{\alpha}^-$, and denote the list of these variables as $\overrightarrow{\hat{\alpha}^-}$. Let us show that $(\overrightarrow{\hat{\alpha}^-}, [\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M', \overrightarrow{N_1}/\overrightarrow{\hat{\alpha}^-}, \overrightarrow{N_2}/\overrightarrow{\hat{\alpha}^-})$ is a sound anti-unifier of $\mathbf{nf}(\downarrow M_1)$ and $\mathbf{nf}(\downarrow M_2)$ in context Γ :

- $\overrightarrow{\hat{\alpha}^-}$ is negative by construction,
- $\Gamma; \overrightarrow{\hat{\alpha}^-} \vdash [\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M'$ because $\Gamma, \overrightarrow{\alpha^-} \vdash \downarrow M'$ **Ilya: lemma!**,
- $\Gamma; \cdot \vdash (\overrightarrow{N_1}/\overrightarrow{\hat{\alpha}^-}) : \overrightarrow{\hat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N_1}$ and $\Gamma; \cdot \vdash (\overrightarrow{N_2}/\overrightarrow{\hat{\alpha}^-}) : \overrightarrow{\hat{\alpha}^-}$ because $\Gamma \vdash \overrightarrow{N_2}$,
- $[\overrightarrow{N_1}/\overrightarrow{\hat{\alpha}^-}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{N_1}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow \mathbf{nf}(M_1) = \mathbf{nf}(\downarrow M_1)$.
- $[\overrightarrow{N_2}/\overrightarrow{\hat{\alpha}^-}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{N_2}/\overrightarrow{\alpha^-}]\downarrow M' = \downarrow \mathbf{nf}(M_2) = \mathbf{nf}(\downarrow M_2)$.

Then by the completeness of the anti-unification (??), the anti-unification algorithm terminates, so is the Least Upper Bound algorithm invoking it, i.e. $Q' = \exists \overrightarrow{\beta^-}. [\overrightarrow{\beta^-}/\overrightarrow{\Xi}]P$, where $(\overrightarrow{\Xi}, P, \hat{\tau}_1, \hat{\tau}_2)$ is the result of the anti-unification of $\mathbf{nf}(\downarrow M_1)$ and $\mathbf{nf}(\downarrow M_2)$ in context Γ .

Moreover, ?? also says that the found anti-unification solution is initial, i.e. there exists $\hat{\tau}$ such that $\Gamma; \overrightarrow{\Xi} \vdash \hat{\tau} : \overrightarrow{\hat{\alpha}^-}$ and $[\hat{\tau}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = P$.

Let σ be a sequential Kleisli composition of the following substitutions: (i) $\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}$, (ii) $\hat{\tau}$, and (iii) $\overrightarrow{\beta^-}/\overrightarrow{\Xi}$. Notice that $\Gamma, \overrightarrow{\beta^-} \vdash \sigma : \overrightarrow{\alpha^-}$ and $[\sigma]\downarrow M' = [\overrightarrow{\beta^-}/\overrightarrow{\Xi}][\hat{\tau}][\overrightarrow{\hat{\alpha}^-}/\overrightarrow{\alpha^-}]\downarrow M' = [\overrightarrow{\beta^-}/\overrightarrow{\Xi}]P$. In particular, from the reflexivity of subtyping: $\Gamma, \overrightarrow{\beta^-} \vdash [\sigma]\downarrow M' \geq_1 [\overrightarrow{\beta^-}/\overrightarrow{\Xi}]P$.

It allows us to show $\Gamma \vdash \mathbf{nf}(Q) \geq_1 Q'$, i.e. $\Gamma \vdash \exists \overrightarrow{\alpha^-}. \downarrow M' \geq_1 \exists \overrightarrow{\beta^-}. [\overrightarrow{\beta^-}/\overrightarrow{\Xi}]P$, by applying Rule $(\exists \geq_1)$, instantiating $\overrightarrow{\alpha^-}$ with respect to σ . Finally, $\Gamma \vdash Q \geq_1 Q'$ since $\Gamma \vdash \mathbf{nf}(Q) \simeq_1^{\leq} Q$, and equivalence implies subtyping by **Ilya: lemma**.

□

3.12 Upgrade

Let us consider a type P well-formed in Γ . Some of its Γ -supertypes are also well-formed in a smaller context $\Delta \subseteq \Gamma$. The upgrade is the operation that returns the least of such supertypes.

Lemma 50 (Soundness of Upgrade). *Assuming P is well-formed in $\Gamma = \Delta, \overrightarrow{\alpha^\pm}$, if $\mathbf{upgrade} \Gamma \vdash P$ to $\Delta = Q$ then*

1. $\Delta \vdash Q$
2. $\Gamma \vdash Q \geq_1 P$

Proof. By inversion, $\mathbf{upgrade} \Gamma \vdash P$ to $\Delta = Q$ means that for fresh $\overrightarrow{\beta^\pm}$ and $\overrightarrow{\gamma^\pm}$, $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \models [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \vee [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P = Q$. Then by the soundness of the least upper bound (lemma 48),

1. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q$,
2. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geq_1 [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$, and
3. $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geq_1 [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P$.

$$\begin{aligned} \mathbf{fv} Q &\subseteq \mathbf{fv} [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P \cap \mathbf{fv} [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P \\ &\subseteq ((\mathbf{fv} P \setminus \overrightarrow{\alpha^\pm}) \cup \overrightarrow{\beta^\pm}) \cap ((\mathbf{fv} P \setminus \overrightarrow{\alpha^\pm}) \cup \overrightarrow{\gamma^\pm}) \\ &= (\mathbf{fv} P \setminus \overrightarrow{\alpha^\pm}) \cap (\mathbf{fv} P \setminus \overrightarrow{\alpha^\pm}) \\ &= \mathbf{fv} P \setminus \overrightarrow{\alpha^\pm} \\ &\subseteq \Gamma \setminus \overrightarrow{\alpha^\pm} \\ &\subseteq \Delta \end{aligned}$$

Since by lemma 8, $\mathbf{fv} Q \subseteq \mathbf{fv} [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$ and $\mathbf{fv} Q \subseteq \mathbf{fv} [\overrightarrow{\gamma^\pm}/\overrightarrow{\alpha^\pm}]P$

since $\overrightarrow{\beta^\pm}$ and $\overrightarrow{\gamma^\pm}$ are fresh

since P is well-formed in Γ

This way, by lemma 15, $\Delta \vdash Q$.

Let us apply $\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}$ —the inverse of the substitution $\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}$ to both sides of $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geq_1 [\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}]P$ and by lemma 12 (since $\overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm}$ can be specified as $\Delta, \overrightarrow{\beta^\pm}, \overrightarrow{\gamma^\pm} \vdash \overrightarrow{\beta^\pm}/\overrightarrow{\alpha^\pm} : \Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm}$ by) obtain $\Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm} \vdash [\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}]Q \geq_1 P$. Notice that $\Delta \vdash Q$ implies that $\mathbf{fv} Q \cap \overrightarrow{\beta^\pm} = \emptyset$, then by ??, $[\overrightarrow{\alpha^\pm}/\overrightarrow{\beta^\pm}]Q = Q$, and thus $\Delta, \overrightarrow{\alpha^\pm}, \overrightarrow{\gamma^\pm} \vdash Q \geq_1 P$. By context strengthening, $\Delta, \overrightarrow{\alpha^\pm} \vdash Q \geq_1 P$. □

Lemma 51 (Completeness and Initiality of Upgrade). *The upgrade returns the least Γ -supertype of P well-formed in Δ . Assuming P is well-formed in $\Gamma = \Delta, \vec{\alpha}^\pm$, For any Q' such that*

1. $\Delta \vdash Q'$ and
2. $\Gamma \vdash Q' \geq_1 P$,

The result of the upgrade algorithm Q exists ($\mathbf{upgrade} \Gamma \vdash P \text{ to } \Delta = Q$) and satisfies $\Delta \vdash Q' \geq_1 Q$.

Proof. Let us consider fresh (not intersecting with Γ) $\vec{\beta}^\pm$ and $\vec{\gamma}^\pm$.

If we apply substitution $\vec{\beta}^\pm/\vec{\alpha}^\pm$ to both sides of $\Delta, \vec{\alpha}^\pm \vdash Q' \geq_1 P$, we have $\Delta, \vec{\beta}^\pm \vdash [\vec{\beta}^\pm/\vec{\alpha}^\pm]Q' \geq_1 [\vec{\beta}^\pm/\vec{\alpha}^\pm]P$, which by ??, since Q' is well-formed in Δ , simplifies to $\Delta, \vec{\beta}^\pm \vdash Q' \geq_1 [\vec{\beta}^\pm/\vec{\alpha}^\pm]P$.

Analogously, if we apply substitution $\vec{\gamma}^\pm/\vec{\alpha}^\pm$ to both sides of $\Delta, \vec{\alpha}^\pm \vdash Q' \geq_1 P$, we have $\Delta, \vec{\gamma}^\pm \vdash Q' \geq_1 [\vec{\gamma}^\pm/\vec{\alpha}^\pm]P$.

This way, Q' is a common supertype of $[\vec{\beta}^\pm/\vec{\alpha}^\pm]P$ and $[\vec{\gamma}^\pm/\vec{\alpha}^\pm]P$ in context $\Delta, \vec{\beta}^\pm, \vec{\gamma}^\pm$. It means that we can apply the completeness of the least upper bound (lemma 49):

1. there exists Q s.t. $\Gamma \models [\vec{\beta}^\pm/\vec{\alpha}^\pm]P \vee [\vec{\gamma}^\pm/\vec{\alpha}^\pm]P = Q$
2. $\Gamma \vdash Q' \geq_1 Q$.

The former means that the upgrade algorithm terminates and returns Q . The latter means that since both Q' and Q are well-formed in Δ , by ??, $\Delta \vdash Q' \geq_1 Q$. \square

3.13 Positive Subtyping

Lemma 52 (Soundness of the Positive Subtyping). *If $\Gamma \vdash^\pm \Theta$, $\Gamma \vdash Q$, $\Gamma; \Theta \vdash P$, and $\Gamma; \Theta \models P \geq Q \Rightarrow SC$, then $\Theta \vdash SC$ and for any normalized $\hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} : SC$, $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$.*

Proof. We prove it by induction on $\Gamma; \Theta \models P \geq Q \Rightarrow SC$. Let us consider the last rule to infer this judgment.

Case 1. Rule (UVar $^\geq$) then $\Gamma; \Theta \models P \geq Q \Rightarrow SC$ has shape $\Gamma; \Theta \models \hat{\alpha}^+ \geq P' \Rightarrow (\hat{\alpha}^+ \geq Q')$ where $\hat{\alpha}^+\{\Delta\} \in \Theta$ and $\mathbf{upgrade} \Gamma \vdash P' \text{ to } \Delta = Q'$.

Notice that $\hat{\alpha}^+\{\Delta\} \in \Theta$ and $\Gamma \vdash^\pm \Theta$ implies $\Gamma = \Delta, \vec{\alpha}^\pm$ for some $\vec{\alpha}^\pm$, hence, the soundness of upgrade (lemma 50) is applicable:

1. $\Delta \vdash Q'$ and
2. $\Gamma \vdash Q' \geq_1 P$.

Since $\hat{\alpha}^+\{\Delta\} \in \Theta$ and $\Delta \vdash Q'$, it is clear that $\Theta \vdash (\hat{\alpha}^+ \geq Q')$.

It is left to show that $\Gamma \vdash [\hat{\sigma}]\hat{\alpha}^+ \geq_1 P'$ for any normalized $\hat{\sigma}$ s.t. $\Theta \vdash \hat{\sigma} : (\hat{\alpha}^+ \geq Q')$. The latter means that $\Theta(\hat{\alpha}^+) \vdash [\hat{\sigma}]\hat{\alpha}^+ \geq_1 Q'$, i.e. $\Delta \vdash [\hat{\sigma}]\hat{\alpha}^+ \geq_1 Q'$. By weakening the context to Γ and combining this judgment transitively with $\Gamma \vdash Q' \geq_1 P$, we have $\Gamma \vdash [\hat{\sigma}]\hat{\alpha}^+ \geq_1 P$, as required.

Case 2. Rule (Var $^{+ \geq}$) then $\Gamma; \Theta \models P \geq Q \Rightarrow SC$ has shape $\Gamma; \Theta \models \alpha^+ \geq \alpha^+ \Rightarrow \cdot$. Then $\mathbf{uv} \alpha^+ = \emptyset$, and $SC = \cdot$ satisfies $\Theta \vdash \cdot$. Since $\mathbf{uv} \alpha^+ = \emptyset$, application of any substitution $\hat{\sigma}$ does not change α^+ , i.e. $[\hat{\sigma}]\alpha^+ = \alpha^+$. Therefore, $\Gamma \vdash [\hat{\sigma}]\alpha^+ \geq_1 \alpha^+$ holds by Rule (Var $^{+ \leq_1}$).

Case 3. Rule (\downarrow^\geq) then $\Gamma; \Theta \models P \geq Q \Rightarrow SC$ has shape $\Gamma; \Theta \models \downarrow N \geq \downarrow M \Rightarrow SC$.

Then the next step of the algorithm is the unification of $\mathbf{nf}(N)$ and $\mathbf{nf}(M)$, and it returns the resulting unification constraint $UC = SC$ as the result. By the soundness of unification (lemma 40), $\Theta \vdash SC$ and for any normalized $\hat{\sigma}$, $\Theta \vdash \hat{\sigma} : SC$ implies $[\hat{\sigma}]\mathbf{nf}(N) = \mathbf{nf}(M)$, then we rewrite the left-hand side by lemma 23: $\mathbf{nf}([\hat{\sigma}]N) = \mathbf{nf}(M)$ and apply lemma 35: $\Gamma \vdash [\hat{\sigma}]N \leq_1^{\leq} M$, then by Rule (\uparrow^{\leq_1}), $\Gamma \vdash \downarrow[\hat{\sigma}]N \geq_1 \downarrow M$.

Case 4. Rule (\exists^\geq) then $\Gamma; \Theta \models P \geq Q \Rightarrow SC$ has shape $\Gamma; \Theta \models \exists \vec{\alpha}^-. P' \geq \exists \vec{\beta}^-. Q' \Rightarrow SC$ s.t. either $\vec{\alpha}^-$ or $\vec{\beta}^-$ is not empty.

Then the algorithm creates fresh unification variables $\vec{\alpha}^-\{\Gamma, \vec{\beta}^-\}$, substitutes the old $\vec{\alpha}^-$ with them in P' , and makes the recursive call: $\Gamma, \vec{\beta}^-; \Theta, \vec{\alpha}^-\{\Gamma, \vec{\beta}^-\} \models [\vec{\alpha}^-/\vec{\alpha}^-]P' \geq Q' \Rightarrow SC'$, returning as the result $SC = SC' \setminus \vec{\alpha}^-$.

Let us take an arbitrary normalized $\hat{\sigma}$ s.t. $\Theta \vdash \hat{\sigma} : SC' \setminus \vec{\alpha}^-$. We wish to show $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$, i.e. $\Gamma \vdash \exists \vec{\alpha}^-. [\hat{\sigma}]P' \geq_1 \exists \vec{\beta}^-. Q'$. To do that, we apply Rule (\exists^{\geq_1}), and what is left to show is $\Gamma, \vec{\beta}^- \vdash [\vec{N}/\vec{\alpha}^-][\hat{\sigma}]P' \geq_1 Q'$ for some \vec{N} . If we construct a normalized $\hat{\sigma}'$ such that $\Theta, \vec{\alpha}^-\{\Gamma, \vec{\beta}^-\} \vdash \hat{\sigma}' : SC'$ and for some \vec{N} , $[\vec{N}/\vec{\alpha}^-][\hat{\sigma}]P' = [\hat{\sigma}']([\vec{\alpha}^-/\vec{\alpha}^-]P')$, we can apply the induction hypothesis to $\Gamma, \vec{\beta}^-; \Theta, \vec{\alpha}^-\{\Gamma, \vec{\beta}^-\} \models [\vec{\alpha}^-/\vec{\alpha}^-]P' \geq Q' \Rightarrow SC'$ and infer the required subtyping.

Let us construct such $\hat{\sigma}'$ by extending $\hat{\sigma}$ with $\hat{\alpha}^{\rightarrow}$ mapped to the corresponding types in SC' :

$$[\hat{\sigma}']\hat{\beta}^{\pm} = \begin{cases} [\hat{\sigma}]\hat{\beta}^{\pm} & \text{if } \hat{\beta}^{\pm} \in \mathbf{dom}(SC') \setminus \hat{\alpha}^{\rightarrow} \\ \mathbf{nf}(N) & \text{if } \hat{\beta}^{\pm} \in \hat{\alpha}^{\rightarrow} \text{ and } (\hat{\beta}^{\pm} : \approx N) \in SC' \end{cases}$$

It is easy to see that $\hat{\sigma}'$ is normalized. Let us show that $\Theta, \hat{\alpha}^{\rightarrow}\{\Gamma, \hat{\beta}^{\rightarrow}\} \vdash \hat{\sigma}' : SC'$. Let us take an arbitrary entry e from SC' restricting a variable $\hat{\beta}^{\pm}$. Suppose $\hat{\beta}^{\pm} \in \mathbf{dom}(SC') \setminus \hat{\alpha}^{\rightarrow}$. Then $(\Theta, \hat{\alpha}^{\rightarrow}\{\Gamma, \hat{\beta}^{\rightarrow}\})(\hat{\beta}^{\pm}) \vdash [\hat{\sigma}']\hat{\beta}^{\pm} : e$ is rewritten as $\Theta(\hat{\beta}^{\pm}) \vdash [\hat{\sigma}]\hat{\beta}^{\pm} : e$, which holds since $\Theta \vdash \hat{\sigma} : SC'$. Suppose $\hat{\beta}^{\pm} = \hat{\alpha}_i^{\rightarrow} \in \hat{\alpha}^{\rightarrow}$. Then $e = (\hat{\alpha}_i^{\rightarrow} : \approx N)$ for some N , $[\hat{\sigma}']\hat{\alpha}_i^{\rightarrow} = \mathbf{nf}(N)$ by the definition, and $\Gamma, \hat{\beta}^{\rightarrow} \vdash \mathbf{nf}(N) : (\hat{\alpha}_i^{\rightarrow} : \approx N)$ by Rule SATSCENEq, since $\Gamma \vdash \mathbf{nf}(N) \simeq_1^{\leq} N$ by lemma 35.

Finally, let us show that $[\vec{N}/\vec{\alpha}^{\rightarrow}][\hat{\sigma}]\vec{P}' = [\hat{\sigma}'][\vec{\alpha}^{\rightarrow}/\vec{\alpha}^{\rightarrow}]\vec{P}'$. For N_i , we take the *normalized* type restricting $\hat{\alpha}_i^{\rightarrow}$ in SC' . Let us take an arbitrary variable from \vec{P} .

1. If this variable is a unification variable $\hat{\beta}^{\pm}$, then $[\vec{N}/\vec{\alpha}^{\rightarrow}][\hat{\sigma}]\hat{\beta}^{\pm} = [\hat{\sigma}]\hat{\beta}^{\pm}$, since $\Theta \vdash \hat{\sigma} : SC' \setminus \hat{\alpha}^{\rightarrow}$ and $\mathbf{dom}(\Theta) \cap \hat{\alpha}^{\rightarrow} = \emptyset$. Notice that $\hat{\beta}^{\pm} \in \mathbf{dom}(\Theta)$, which is disjoint from $\hat{\alpha}^{\rightarrow}$, that is $\hat{\beta}^{\pm} \in \mathbf{dom}(SC') \setminus \hat{\alpha}^{\rightarrow}$. This way, $[\hat{\sigma}'][\vec{\alpha}^{\rightarrow}/\vec{\alpha}^{\rightarrow}]\hat{\beta}^{\pm} = [\hat{\sigma}']\hat{\beta}^{\pm} = [\hat{\sigma}]\hat{\beta}^{\pm}$ by the definition of $\hat{\sigma}'$,
2. If this variable is a regular variable $\beta^{\pm} \notin \hat{\alpha}^{\rightarrow}$, then $[\vec{N}/\vec{\alpha}^{\rightarrow}][\hat{\sigma}]\beta^{\pm} = \beta^{\pm}$ and $[\hat{\sigma}'][\vec{\alpha}^{\rightarrow}/\vec{\alpha}^{\rightarrow}]\beta^{\pm} = \beta^{\pm}$.
3. If this variable is a regular variable $\alpha_i^{\rightarrow} \in \hat{\alpha}^{\rightarrow}$, then $[\vec{N}/\vec{\alpha}^{\rightarrow}][\hat{\sigma}]\alpha_i^{\rightarrow} = N_i = \mathbf{nf}(N_i)$ (the latter equality holds since N_i is normalized) and $[\hat{\sigma}'][\vec{\alpha}^{\rightarrow}/\vec{\alpha}^{\rightarrow}]\alpha_i^{\rightarrow} = [\hat{\sigma}']\hat{\alpha}_i^{\rightarrow} = \mathbf{nf}(N_i)$.

□

Lemma 53 (Completeness of the Positive Subtyping). *Suppose that $\Gamma \vdash^{\exists} \Theta$, $\Gamma \vdash Q$ and $\Gamma; \Theta \vdash P$. Then for any $\Theta \vdash \hat{\sigma}$ such that $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$, there exists $\Gamma; \Theta \models P \geq Q \models SC$ and moreover, $\Theta \vdash \hat{\sigma} : SC$.*

Proof. Let us prove this lemma by induction on $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$. Let us consider the last rule used in the derivation, but first, consider the base case for the substitution $[\hat{\sigma}]P$:

Case 1. $P = \exists \vec{\beta}^{\rightarrow}. \hat{\alpha}^+$ (for potentially empty $\vec{\beta}^{\rightarrow}$)

Then by assumption, $\Gamma \vdash \exists \vec{\beta}^{\rightarrow}. [\hat{\sigma}]\hat{\alpha}^+ \geq_1 Q$ (where $\vec{\beta}^{\rightarrow} \cap \mathbf{fv}([\hat{\sigma}]\hat{\alpha}^+) = \emptyset$). Let us decompose Q as $Q = \exists \vec{\gamma}^{\rightarrow}. Q_0$, where Q_0 does not start with \exists .

By inversion, $\Gamma; \Theta \vdash \exists \vec{\beta}^{\rightarrow}. \hat{\alpha}^+$ implies $\hat{\alpha}^+\{\Delta\} \in \Theta$ for some $\Delta \subseteq \Gamma$.

By lemma 9 applied twice, $\Gamma \vdash \exists \vec{\beta}^{\rightarrow}. [\hat{\sigma}]\hat{\alpha}^+ \geq_1 \exists \vec{\gamma}^{\rightarrow}. Q_0$ implies $\Gamma, \vec{\gamma}^{\rightarrow} \vdash [\vec{N}/\vec{\beta}^{\rightarrow}][\hat{\sigma}]\hat{\alpha}^+ \geq_1 Q_0$ for some N , and since $\vec{\beta}^{\rightarrow} \cap \mathbf{fv}([\hat{\sigma}]\hat{\alpha}^+) \subseteq \vec{\beta}^{\rightarrow} \cap \Theta(\hat{\alpha}^+) \subseteq \vec{\beta}^{\rightarrow} \cap \Gamma = \emptyset$, $[\vec{N}/\vec{\beta}^{\rightarrow}][\hat{\sigma}]\hat{\alpha}^+ = [\hat{\sigma}]\hat{\alpha}^+$, that is $\Gamma, \vec{\gamma}^{\rightarrow} \vdash [\hat{\sigma}]\hat{\alpha}^+ \geq_1 Q_0$.

When algorithm tires to infer the subtyping $\Gamma; \Theta \models \exists \vec{\beta}^{\rightarrow}. \hat{\alpha}^+ \geq \exists \vec{\gamma}^{\rightarrow}. Q_0 \models SC$, it applies Rule (\exists^{\geq}) , which reduces the problem to $\Gamma, \vec{\gamma}^{\rightarrow}; \Theta, \vec{\beta}^{\rightarrow}\{\Gamma, \vec{\gamma}^{\rightarrow}\} \models [\vec{\beta}^{\rightarrow}/\vec{\beta}^{\rightarrow}]\hat{\alpha}^+ \geq Q_0 \models SC$, which is equivalent to $\Gamma, \vec{\gamma}^{\rightarrow}; \Theta, \vec{\beta}^{\rightarrow}\{\Gamma, \vec{\gamma}^{\rightarrow}\} \models \hat{\alpha}^+ \geq Q_0 \models SC$.

Next, the algorithm tries to apply Rule $(UVar^{\geq})$ and the resulting restriction is $SC = (\hat{\alpha}^+ : \geq Q'_0)$ where $\mathbf{upgrade} \Gamma, \vec{\gamma}^{\rightarrow} \vdash Q_0 \text{ to } \Delta = Q'_0$.

Why does the upgrade procedure terminate? Because $[\hat{\sigma}]\hat{\alpha}^+$ satisfies the pre-conditions of the completeness of the upgrade (lemma 51):

1. $\Delta \vdash [\hat{\sigma}]\hat{\alpha}^+$ because $\Theta \vdash \hat{\sigma}$ and $\hat{\alpha}^+\{\Delta\} \in \Theta$,
2. $\Gamma, \vec{\gamma}^{\rightarrow} \vdash [\hat{\sigma}]\hat{\alpha}^+ \geq_1 Q_0$ as noted above

Moreover, the completeness of upgrade also says that Q'_0 is the *least* supertype of Q_0 among types well-formed in Δ , that is $\Delta \vdash [\hat{\sigma}]\hat{\alpha}^+ \geq_1 Q'_0$, which means $\Theta \vdash \hat{\sigma} : (\hat{\alpha}^+ : \geq Q'_0)$, that is $\Theta \vdash \hat{\sigma} : SC$.

Case 2. $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$ is derived by Rule $(Var^{+ \geq_1})$

Then $P = [\hat{\sigma}]P = \alpha^+ = Q$, where the first equality holds because P is not a unification variable: it has been covered by case 1; and the second equality hold because Rule $(Var^{+ \geq_1})$ was applied.

The algorithm applies Rule $(Var^{+ \geq})$ and infers $SC = \cdot$, i.e. $\Gamma; \Theta \models \alpha^+ \geq \alpha^+ \models \cdot$. Then $\Theta \vdash \hat{\sigma} : \cdot$ holds trivially.

Case 3. $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$ is derived by Rule (\downarrow^{\geq_1}) ,

Then $P = \downarrow N$, since the substitution $[\hat{\sigma}]P$ must preserve the top-level constructor of $P \neq \hat{\alpha}^+$ (the case $P = \hat{\alpha}^+$ has been covered by case 1), and $Q = \downarrow M$, and by inversion, $\Gamma \vdash [\hat{\sigma}]N \simeq_1^{\leq} M$.

Since both types start with \downarrow , the algorithm tries to apply Rule ($\downarrow \triangleright$): $\Gamma; \Theta \models \downarrow N \triangleright \downarrow M \equiv SC$. The premise of this rule is the unification of $\mathbf{nf}(N)$ and $\mathbf{nf}(M)$: $\Gamma; \Theta \models \mathbf{nf}(N) \stackrel{u}{\simeq} \mathbf{nf}(M) \equiv UC$. And the algorithm returns it as a subtyping constraint $SC = UC$.

To demonstrate that the unification terminates and $\hat{\sigma}$ satisfies the resulting constraints, we apply the completeness of the unification algorithm (lemma 41). In order to do that, we need to provide a substitution unifying $\mathbf{nf}(N)$ and $\mathbf{nf}(M)$. Let us show that $\mathbf{nf}(\hat{\sigma})$ is such a substitution.

- $\mathbf{nf}(N)$ and $\mathbf{nf}(M)$ are normalized
- $\Gamma; \Theta \vdash \mathbf{nf}(N)$ because $\Gamma; \Theta \vdash N$ (??)
- $\Gamma \vdash \mathbf{nf}(M)$ because $\Gamma \vdash M$ (corollary 14)
- $\Theta \vdash \mathbf{nf}(\hat{\sigma})$ because $\Theta \vdash \hat{\sigma}$ (??)
- $\Gamma \vdash [\hat{\sigma}]N \simeq_1^{\leq} M \Rightarrow [\hat{\sigma}]N \simeq_1^D M$ by lemma 34
- $\Rightarrow \mathbf{nf}([\hat{\sigma}]N) = \mathbf{nf}(M)$ by lemma 24
- $\Rightarrow [\mathbf{nf}(\hat{\sigma})]\mathbf{nf}(N) = \mathbf{nf}(M)$ by lemma 23

Then by the completeness of the unification, $\Gamma; \Theta \models N \stackrel{u}{\simeq} M \equiv UC$ exists, and $\Theta \vdash \mathbf{nf}(\hat{\sigma}) : UC$. Then by ??, $\Theta \vdash \hat{\sigma} : UC$.

Case 4. $\Gamma \vdash [\hat{\sigma}]P \triangleright_1 Q$ is derived by Rule ($\exists \triangleright_1$).

We should only consider the case when the substitution $[\hat{\sigma}]P$ results in the existential type $\exists \alpha^-.P''$ (for $P'' \neq \exists \dots$) by congruence, i.e. $P = \exists \alpha^-.P'$ (for $P' \neq \exists \dots$) and $[\hat{\sigma}]P' = P''$. This is because the case when $P = \exists \beta^-. \hat{\alpha}^+$ has been covered (case 1), and thus, the substitution $\hat{\sigma}$ must preserve all the outer quantifiers of P and does not generate any new ones.

This way, $P = \exists \alpha^-.P'$, $[\hat{\sigma}]P = \exists \alpha^-.[\hat{\sigma}]P'$ (assuming α^- does not intersect with the range of $\hat{\sigma}$) and $Q = \exists \beta^-.Q'$, where either α^- or β^- is not empty.

By inversion, $\Gamma \vdash [\sigma][\hat{\sigma}]P' \triangleright_1 Q'$ for some $\Gamma, \beta^- \vdash \sigma : \alpha^-$. Since σ and $\hat{\sigma}$ have disjoint domains, and the range of one does not intersect with the domain of the other, they commute, i.e. $\Gamma, \beta^- \vdash [\hat{\sigma}][\sigma]P' \triangleright_1 Q'$ (notice that the tree inferring this judgement is a proper subtree of the tree inferring $\Gamma \vdash [\hat{\sigma}]P \triangleright_1 Q$).

At the next step, the algorithm creates fresh (disjoint with $\mathbf{uv}(P')$) unification variables $\hat{\alpha}^-$, replaces α^- with them in P' , and makes the recursive call: $\Gamma, \beta^-; \Theta, \hat{\alpha}^- \{ \Gamma, \beta^- \} \models P_0 \triangleright Q' \equiv SC_1$, (where $P_0 = [\hat{\alpha}^-/\alpha^-]P'$), returning $SC_1 \setminus \hat{\alpha}^-$ as the result.

To show that the recursive call terminates and that $\Theta \vdash \hat{\sigma} : SC_1 \setminus \hat{\alpha}^-$, it suffices to build $\Theta, \hat{\alpha}^- \{ \Gamma, \beta^- \} \vdash \hat{\sigma}_0$ —an extension of $\hat{\sigma}$ with $\hat{\alpha}^-$ such that $\Gamma, \beta^- \vdash [\hat{\sigma}_0]P_0 \triangleright_1 Q$. Then by the induction hypothesis, $\Theta, \hat{\alpha}^- \{ \Gamma, \beta^- \} \vdash \hat{\sigma}_0 : SC_1$, and hence, $\Theta \vdash \hat{\sigma} : SC_1 \setminus \hat{\alpha}^-$, as required.

Let us construct such a substitution $\hat{\sigma}_0$:

$$[\hat{\sigma}_0]\hat{\beta}^\pm = \begin{cases} [\sigma]\alpha_i^- & \text{if } \hat{\beta}^\pm = \hat{\alpha}_i^- \in \hat{\alpha}^- \\ [\hat{\sigma}]\hat{\beta}^\pm & \text{if } \hat{\beta}^\pm \in \mathbf{uv}(P') \end{cases}$$

It is easy to see $\Theta, \hat{\alpha}^- \{ \Gamma, \beta^- \} \vdash \hat{\sigma}_0$:

1. for $\hat{\alpha}_i^- \in \hat{\alpha}^-$, $(\Theta, \hat{\alpha}^- \{ \Gamma, \beta^- \})(\hat{\alpha}_i^-) \vdash [\hat{\sigma}_0]\hat{\alpha}_i^-$, i.e. $\Gamma, \beta^- \vdash [\sigma]\alpha_i^-$ holds since $\Gamma, \beta^- \vdash \sigma : \alpha^-$,
2. for $\hat{\beta}^\pm \in \mathbf{uv}(P') \subseteq \mathbf{dom}(\Theta)$, $(\Theta, \hat{\alpha}^- \{ \Gamma, \beta^- \})(\hat{\beta}^\pm) \vdash [\hat{\sigma}_0]\hat{\beta}^\pm$, i.e. $\Theta(\hat{\beta}^\pm) \vdash [\hat{\sigma}]\hat{\beta}^\pm$ holds since $\Theta \vdash \hat{\sigma}$.

Now, let us show that $\Gamma, \beta^- \vdash [\hat{\sigma}_0]P_0 \triangleright_1 Q$. To do that, we notice that $[\hat{\sigma}_0]P_0 = [\hat{\sigma}][\sigma][\hat{\alpha}^-/\alpha^-]P_0$: let us consider an arbitrary variable appearing freely in P_0 :

1. if this variable is a metavariable $\hat{\alpha}_i^- \in \hat{\alpha}^-$, then $[\hat{\sigma}_0]\hat{\alpha}_i^- = [\sigma]\alpha_i^-$ and $[\hat{\sigma}][\sigma][\hat{\alpha}^-/\alpha^-]\hat{\alpha}_i^- = [\hat{\sigma}][\sigma]\alpha_i^- = [\sigma]\alpha_i^-$,
2. if this variable is a metavariable $\hat{\beta}^\pm \in \mathbf{uv}(P_0) \setminus \hat{\alpha}^- = \mathbf{uv}(P')$, then $[\hat{\sigma}_0]\hat{\beta}^\pm = [\hat{\sigma}]\hat{\beta}^\pm$ and $[\hat{\sigma}][\sigma][\hat{\alpha}^-/\alpha^-]\hat{\beta}^\pm = [\hat{\sigma}][\sigma]\hat{\beta}^\pm = [\hat{\sigma}]\hat{\beta}^\pm$,
3. if this variable is a regular variable from $\mathbf{fv}(P_0)$, both substitutions do not change it: $\hat{\sigma}_0$, $\hat{\sigma}$ and $\hat{\alpha}^-/\hat{\alpha}^-$ act on metavariables, and σ is defined on α^- , however, $\alpha^- \cap \mathbf{fv}(P_0) = \emptyset$.

This way, $[\hat{\sigma}_0]P_0 = [\hat{\sigma}][\sigma][\hat{\alpha}^-/\alpha^-]P_0 = [\hat{\sigma}][\sigma]P'$, and thus, $\Gamma, \beta^- \vdash [\hat{\sigma}_0]P_0 \triangleright_1 Q'$.

□

3.14 Subtyping Constraint Merge

Lemma 54 (Soundness of Constraint Entry Merge). *For a fixed context Γ , suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$. If $\Gamma \vdash e_1 \& e_2 = e$ is defined then*

1. $\Gamma \vdash e$
2. For any $\Gamma \vdash P$, $\Gamma \vdash P : e$ implies $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$

Proof. Let us consider the rule forming $\Gamma \vdash e_1 \& e_2 = e$.

Case 1. Rule $(\simeq \&^+ \simeq)$, i.e. $\Gamma \vdash e_1 \& e_2 = e$ has form $\Gamma \vdash (\hat{\alpha}^+ : \approx Q) \& (\hat{\alpha}^+ : \approx Q') = (\hat{\alpha}^+ : \approx Q)$ and $\mathbf{nf}(Q) = \mathbf{nf}(Q')$. The latter implies $\Gamma \vdash Q \simeq_1^< Q'$ by lemma 35. Then

1. $\Gamma \vdash e$, i.e. $\Gamma \vdash \hat{\alpha}^+ : \approx Q$ holds by assumption;
2. by inversion, $\Gamma \vdash P : (\hat{\alpha}^+ : \approx Q)$ means $\Theta \vdash P \simeq_1^D Q$, and by transitivity of equivalence (corollary 7), $\Theta \vdash P \simeq_1^D Q'$. Thus, $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$ hold by Rule SATSCEPEq.

Case 2. Rule $(\simeq \&^- \simeq)$ the negative case is proved in exactly the same way as the positive one.

Case 3. Rule $(\geq \&^+ \geq)$ Then e_1 is $\hat{\alpha}^+ : \geq Q_1$, e_2 is $\hat{\alpha}^+ : \geq Q_2$, and $e_1 \& e_2 = e$ is $\hat{\alpha}^+ : \geq Q$ where Q is the least upper bound of Q_1 and Q_2 . Then by lemma 48,

- $\Gamma \vdash Q$,
- $\Gamma \vdash Q \geq_1 Q_1$,
- $\Gamma \vdash Q \geq_1 Q_2$.

Let us show the required properties.

- $\Gamma \vdash e$ holds from $\Gamma \vdash Q$,
- Assuming $\Gamma \vdash P : e$, by inversion, we have $\Gamma \vdash P \geq_1 Q$. Combining it transitively with $\Gamma \vdash Q \geq_1 Q_1$, we have $\Gamma \vdash P \geq_1 Q_1$. Analogously, $\Gamma \vdash P \geq_1 Q_2$. Then $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$ hold by Rule SATSCESup.

Case 4. Rule $(\geq \&^+ \simeq)$ Then e_1 is $\hat{\alpha}^+ : \geq Q_1$, e_2 is $\hat{\alpha}^+ : \approx Q_2$, where $\Gamma; \cdot \models Q_2 \geq Q_1 \Rightarrow \cdot$, and the resulting $e_1 \& e_2 = e$ is equal to e_2 , that is $\hat{\alpha}^+ : \approx Q_2$.

Let us show the required properties.

- By assumption, $\Gamma \vdash Q$, and hence $\Gamma \vdash e$.
- Since $\mathbf{uv}(Q_2) = \emptyset$, $\Gamma; \cdot \models Q_2 \geq Q_1 \Rightarrow \cdot$ implies $\Gamma \vdash Q_2 \geq_1 Q_1$ by the soundness of positive subtyping (lemma 52). Then let us take an arbitrary $\Gamma \vdash P$ such that $\Gamma \vdash P : e$. Since $e_2 = e$, $\Gamma \vdash P : e_2$ holds immediately. By inversion, $\Gamma \vdash P : (\hat{\alpha}^+ : \approx Q_2)$ means $\Gamma \vdash P \simeq_1^< Q_2$, and then by transitivity of subtyping (lemma 13), $\Gamma \vdash P \geq_1 Q_1$. Then $\Gamma \vdash P : e_1$ holds by Rule SATSCESup.

Case 5. Rule $(\simeq \&^+ \geq)$ The proof is analogous to the previous case.

□

Lemma 55 (Soundness of Constraint Merge). *Suppose that $\Theta \vdash SC_1$ and $\Theta \vdash SC_2$ and $\Theta \vdash SC_1 \& SC_2 = SC$ is defined. Then*

1. $\Theta \vdash SC$,
2. for any substitution $\Theta \vdash \hat{\sigma}$, $\Theta \vdash \hat{\sigma} : SC$ implies $\Theta \vdash \hat{\sigma} : SC_1$ and $\Theta \vdash \hat{\sigma} : SC_2$.

Proof. By definition, $SC_1 \& SC_2 = SC$ consists of three parts: entries of SC_1 that do not have matching entries of SC , entries of SC_2 that do not have matching entries of SC_1 , and the merge of matching entries.

Let us show $\Theta \vdash SC$. First, let us assume that an entry $e \in SC$ belongs to the first group, i.e. $e \in SC_1$. Let us denote the variable e as $\hat{\alpha}^\pm$. Then $\Theta(\hat{\alpha}^\pm) \vdash e$ holds since $\Theta \vdash SC_1 \ni e$. Analogously, if e belongs to the second group, then $\Theta(\hat{\alpha}^\pm) \vdash e$ holds since $\Theta \vdash SC_2 \ni e$. Finally, if e belongs to the third group, then e is a merge of two entries $\Theta(\hat{\alpha}^\pm) \vdash e_1$ and $\Theta(\hat{\alpha}^\pm) \vdash e_2$. Then $\Theta(\hat{\alpha}^\pm) \vdash e$ holds by lemma 54.

Let us show the second property. We take an arbitrary $\hat{\sigma}$ such that $\Theta \vdash \hat{\sigma}$ and $\Theta \vdash \hat{\sigma} : SC$. To prove $\Theta \vdash \hat{\sigma} : SC_1$, we need to show that for any $e_1 \in SC_1$, restricting $\hat{\alpha}^\pm$, $\Theta(\hat{\alpha}^\pm) \vdash [\hat{\sigma}]\hat{\alpha}^\pm : e_1$ holds.

Let us assume that $\hat{\alpha}^\pm \neq \mathbf{dom}(SC_2)$. It means that $SC \ni e_1$, and then since $\Theta \vdash \hat{\sigma} : SC$, $\Theta(\hat{\alpha}^\pm) \vdash [\hat{\sigma}]\hat{\alpha}^\pm : e_1$.

Otherwise, SC_2 contains an entry e_2 restricting $\hat{\alpha}^\pm$, and $SC \ni e$ where $\Theta(\hat{\alpha}^\pm) \vdash e_1 \& e_2 = e$. Then since $\Theta \vdash \hat{\sigma} : SC$, $\Theta(\hat{\alpha}^\pm) \vdash [\hat{\sigma}]\hat{\alpha}^\pm : e$, and by lemma 54, $\Theta(\hat{\alpha}^\pm) \vdash [\hat{\sigma}]\hat{\alpha}^\pm : e_1$.

The proof of $\Theta \vdash \hat{\sigma} : SC_2$ is symmetric.

□

Lemma 56 (Completeness of Constraint Entry Merge). *For a fixed context Γ , suppose that $\Gamma \vdash e_1$ and $\Gamma \vdash e_2$ are matching constraint entries.*

- for a type P such that $\Gamma \vdash P : e_1$ and $\Gamma \vdash P : e_2$, $\Gamma \vdash e_1 \& e_2 = e$ is defined and $\Gamma \vdash P : e$.
- for a type N such that $\Gamma \vdash N : e_1$ and $\Gamma \vdash N : e_2$, $\Gamma \vdash e_1 \& e_2 = e$ is defined and $\Gamma \vdash N : e$.

Proof. Let us consider the shape of e_1 and e_2 .

Case 1. e_1 is $\hat{\alpha}^+ : \approx Q_1$ and e_2 is $\hat{\alpha}^+ : \approx Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \simeq_1^< Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \simeq_1^< Q_2$. Then by transitivity of equivalence (corollary 7), $\Gamma \vdash Q_1 \simeq_1^< Q_2$, which means $\mathbf{nf}(Q_1) = \mathbf{nf}(Q_2)$ by lemma 35. Hence, Rule $(\simeq \&^+ \simeq)$ applies to infer $\Gamma \vdash e_1 \& e_2 = e_2$, and $\Gamma \vdash P : e_2$ holds by assumption.

Case 2. e_1 is $\hat{\alpha}^+ : \approx Q_1$ and e_2 is $\hat{\alpha}^+ : \geq Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \simeq_1^< Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \geq_1 Q_2$. Then by transitivity of subtyping, $\Gamma \vdash Q_1 \geq_1 Q_2$, which means $\Gamma; \cdot \models Q_1 \succcurlyeq Q_2 \Rightarrow \cdot$ by lemma 53. This way, Rule $(\simeq \&^+ \geq)$ applies to infer $\Gamma \vdash e_1 \& e_2 = e_1$, and $\Gamma \vdash P : e_1$ holds by assumption.

Case 3. e_1 is $\hat{\alpha}^+ : \geq Q_1$ and e_2 is $\hat{\alpha}^+ : \geq Q_2$. Then $\Gamma \vdash P : e_1$ means $\Gamma \vdash P \geq_1 Q_1$, and $\Gamma \vdash P : e_2$ means $\Gamma \vdash P \geq_1 Q_2$. By the completeness of the least upper bound (lemma 49), $\Gamma \models Q_1 \vee Q_2 = Q$, and $\Gamma \vdash P \geq_1 Q$. This way, Rule $(\geq \&^+ \geq)$ applies to infer $\Gamma \vdash e_1 \& e_2 = (\hat{\alpha}^+ : \geq Q)$, and $\Gamma \vdash P : (\hat{\alpha}^+ : \geq Q)$ holds by Rule SATSCESup.

Case 4. The negative cases are proved symmetrically. □

Lemma 57 (Completeness of Constraint Merge). *Suppose that $\Theta \vdash SC_1$ and $\Theta \vdash SC_2$. Then for any substitution $\Theta \vdash \hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} : SC_1$ and $\Theta \vdash \hat{\sigma} : SC_2$, $\Theta \vdash SC_1 \& SC_2 = SC$ is defined.*

Proof. By definition, $SC_1 \& SC_2$ is a union of

1. entries of SC_1 , which do not have matching entries in SC_2 ,
2. entries of SC_2 , which do not have matching entries in SC_1 , and
3. the merge of matching entries.

This way, to show that $\Theta \vdash SC_1 \& SC_2 = SC$ is defined, we need to demonstrate that each of these components is defined.

It is clear that the first two components of this union exist. Let us show that the third component exists. Let us take two entries $e_1 \in SC_1$ and $e_2 \in SC_2$ restricting the same variable $\hat{\alpha}^\pm$. $\Theta \vdash \hat{\sigma} : SC_1$ means that $\Theta(\hat{\alpha}^\pm) \vdash [\hat{\sigma}]\hat{\alpha}^\pm : e_1$ and $\Theta \vdash \hat{\sigma} : SC_2$ means $\Theta(\hat{\alpha}^\pm) \vdash [\hat{\sigma}]\hat{\alpha}^\pm : e_2$. Then by lemma 56, $\Theta(\hat{\alpha}^\pm) \vdash e_1 \& e_2 = e$ is defined and $\Theta(\hat{\alpha}^\pm) \vdash [\hat{\sigma}]\hat{\alpha}^\pm : e$. □

Lemma 58 (Substitution existence). *If $\Theta \vdash SC$ then there exists $\Theta \vdash \hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} : SC$.*

Proof. □

3.15 Constraint Satisfaction

Lemma 59. *Suppose that $\Theta \vdash SC$ then there exist $s \hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} : SC$.*

Proof. Let us define $\hat{\sigma}$ on $\mathbf{dom}(SC)$ in the following way:

$$[\hat{\sigma}]\hat{\alpha}^\pm = \begin{cases} P & \text{if } (\hat{\alpha}^\pm : \approx P) \in SC \\ P & \text{if } (\hat{\alpha}^\pm : \geq P) \in SC \\ N & \text{if } (\hat{\alpha}^\pm : \approx N) \in SC \end{cases}$$

Then $\Theta \vdash \hat{\sigma} : SC$ follows immediately from the reflexivity of equivalence and subtyping (lemma 11) and the corresponding rules Rule SATSCEPEq, Rule SATSCENEq, and Rule SATSCESup. □

Lemma 60 (Constraint Entry Satisfaction is Stable under Equivalence). *– If $\Gamma \vdash N_1 : e$ and $\Gamma \vdash N_1 \simeq_1^< N_2$ then $\Gamma \vdash N_2 : e$.*

+ If $\Gamma \vdash P_1 : e$ and $\Gamma \vdash P_1 \simeq_1^< P_2$ then $\Gamma \vdash P_2 : e$.

Proof. *–* Then e has form $(\hat{\alpha}^- : \approx M)$, and by inversion, $\Gamma \vdash N_1 \simeq_1^< M$. Then by transitivity, $\Gamma \vdash N_2 \simeq_1^< M$, meaning $\Gamma \vdash N_2 : e$.

+ Let us consider what form e has.

Case 1. $e = (\hat{\alpha}^+ : \approx Q)$. Then $\Gamma \vdash P_1 \simeq_1^< Q$, and hence, $\Gamma \vdash P_2 \simeq_1^< Q$ by transitivity. Then $\Gamma \vdash P_2 : e$.

Case 2. $e = (\hat{\alpha}^+ : \geq Q)$. Then $\Gamma \vdash P_1 \geq_1 Q$, and hence, $\Gamma \vdash P_2 \geq_1 Q$ by transitivity. Then $\Gamma \vdash P_2 : e$. □

3.16 Negative Subtyping

Lemma 61 (Soundness of Negative Subtyping). *If $\Gamma \vdash^\exists \Theta$, $\Gamma \vdash M$, $\Gamma; \Theta \vdash N$ and $\Gamma; \Theta \models N \leq M \Rightarrow SC$, then $\Theta \vdash SC$ and for any normalized $\hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} : SC$, $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$.*

Proof. We prove it by induction on $\Gamma; \Theta \models N \leq M \Rightarrow SC$.

Suppose that $\hat{\sigma}$ is normalized and $\Theta \vdash \hat{\sigma} : SC$. Let us consider the last rule to infer this judgment.

Case 1. Rule (\rightarrow^{\leq}) . Then $\Gamma; \Theta \models N \leq M \Rightarrow SC$ has shape $\Gamma; \Theta \models P \rightarrow N' \leq Q \rightarrow M' \Rightarrow SC$

On the next step, the algorithm makes two recursive calls: $\Gamma; \Theta \models P \geq Q \Rightarrow SC_1$ and $\Gamma; \Theta \models N' \leq M' \Rightarrow SC_2$ and returns $\Theta \vdash SC_1 \& SC_2 = SC$ as the result.

By the soundness of constraint merge (lemma 55), $\Theta \vdash \hat{\sigma} : SC_1$ and $\Theta \vdash \hat{\sigma} : SC_2$. Then by the soundness of positive subtyping (lemma 52), $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$; and by the induction hypothesis, $\Gamma \vdash [\hat{\sigma}]N' \leq_1 M'$. This way, by Rule (\rightarrow^{\leq_1}) , $\Gamma \vdash [\hat{\sigma}](P \rightarrow N') \leq_1 Q \rightarrow M'$.

Case 2. Rule (Var^{\leq}) , and then $\Gamma; \Theta \models N \leq M \Rightarrow SC$ has shape $\Gamma; \Theta \models \alpha^- \leq \alpha^- \Rightarrow \cdot$.

This case is symmetric to case 2 of lemma 52.

Case 3. Rule (\uparrow^{\leq}) , and then $\Gamma; \Theta \models N \leq M \Rightarrow SC$ has shape $\Gamma; \Theta \models \uparrow P \leq \uparrow Q \Rightarrow SC$

This case is symmetric to case 3 of lemma 52.

Case 4. Rule (\forall^{\leq}) , and then $\Gamma; \Theta \models N \leq M \Rightarrow SC$ has shape $\Gamma; \Theta \models \forall \alpha^+. N' \leq \forall \beta^+. M' \Rightarrow SC$ s.t. either α^+ or β^+ is not empty

This case is symmetric to case 4 of lemma 52.

□

Lemma 62 (Completeness of the Negative Subtyping). *Suppose that $\Gamma \vdash^\exists \Theta$, $\Gamma \vdash M$, $\Gamma; \Theta \vdash N$, and N does not contain negative unification variables ($\hat{\alpha}^- \notin \mathbf{uv} N$). Then for any $\Theta \vdash \hat{\sigma}$ such that $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$, there exists $\Gamma; \Theta \models N \leq M \Rightarrow SC$, such that $\Theta \vdash SC$ and moreover, $\Theta \vdash \hat{\sigma} : SC$.*

Proof. We prove it by induction on $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$. Let us consider the last rule used in the derivation of $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$.

Case 1. $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$ is derived by Rule (\uparrow^{\leq_1})

Then $N = \uparrow P$, since the substitution $[\hat{\sigma}]N$ must preserve the top-level constructor of $N \neq \hat{\alpha}^-$ (since by assumption, $\hat{\alpha}^- \notin \mathbf{uv} N$), and $Q = \downarrow M$, and by inversion, $\Gamma \vdash [\hat{\sigma}]N \simeq_1^{\leq} M$. The rest of the proof is symmetric to case 3 of lemma 53: notice that the algorithm does not make a recursive call, and the difference in the induction statement for the positive and the negative case here does not matter.

Case 2. $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$ is derived by Rule (\rightarrow^{\leq_1}) , i.e. $[\hat{\sigma}]N = [\hat{\sigma}]P \rightarrow [\hat{\sigma}]N'$ and $M = Q \rightarrow M'$, and by inversion, $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$ and $\Gamma \vdash [\hat{\sigma}]N' \leq_1 M'$.

The algorithm makes two recursive calls: $\Gamma; \Theta \models P \geq Q \Rightarrow SC_1$ and $\Gamma; \Theta \models N' \leq M' \Rightarrow SC_2$, and then returns $\Theta \vdash SC_1 \& SC_2 = SC$ as the result. Let us show that these recursive calls are successful and the returning constraints are fulfilled by $\hat{\sigma}$.

Notice that from the inversion of $\Gamma \vdash M$, we have: $\Gamma \vdash Q$ and $\Gamma \vdash M'$; from the inversion of $\Gamma; \Theta \vdash N$, we have: $\Gamma; \Theta \vdash P$ and $\Gamma; \Theta \vdash N'$; and since N does not contain negative unification variables, N' does not contain negative unification variables either.

This way, we can apply the induction hypothesis to $\Gamma \vdash [\hat{\sigma}]N' \leq_1 M'$ to obtain $\Gamma; \Theta \models N' \leq M' \Rightarrow SC_2$ such that $\Theta \vdash SC_2$ and $\Theta \vdash \hat{\sigma} : SC_2$. Also, we can apply the completeness of the positive subtyping (lemma 53) to $\Gamma \vdash [\hat{\sigma}]P \geq_1 Q$ to obtain $\Gamma; \Theta \models P \geq Q \Rightarrow SC_1$ such that $\Theta \vdash SC_1$ and $\Theta \vdash \hat{\sigma} : SC_1$.

Finally, we need to show that the merge of SC_1 and SC_2 is successful and satisfies the required properties. To do so, we apply the completeness of subtyping constraint merge (lemma 57). This way, $\Theta \vdash SC_1 \& SC_2 = SC$ is defined, $\Theta \vdash SC$, and $\Theta \vdash \hat{\sigma} : SC$, as required.

Case 3. $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$ is derived by Rule (\forall^{\leq_1}) . Since N does not contain negative unification variables, N must be of the form $\forall \alpha^+. N'$, such that $[\hat{\sigma}]N = \forall \alpha^+. [\hat{\sigma}]N'$ and $[\hat{\sigma}]N' \neq \forall \dots$ (assuming α^+ does not intersect with the range of $\hat{\sigma}$). Also, $M = \forall \beta^+. M'$ and either α^+ or β^+ is non-empty.

The rest of the proof is symmetric to ?? of lemma 53. To apply the induction hypothesis, we need to show additionally that there are no negative unification variables in $N_0 = [\hat{\alpha}^+ / \alpha^+]N'$. This is because $\mathbf{uv} N_0 \subseteq \mathbf{uv} N \cup \hat{\alpha}^+$, and N is free of negative unification variables by assumption.

Case 4. $\Gamma \vdash [\hat{\sigma}]N \leq_1 M$ is derived by Rule $(\text{Var}^{-\leq_1})$.

Then $N = [\hat{\sigma}]N = \alpha^- = M$. Here the first equality holds because N is not a unification variable: by assumption, N is free of negative unification variables. The second and the third equations hold because Rule $(\text{Var}^{-\leq_1})$ was applied.

The rest of the proof is symmetric to case 2 of lemma 53.

□

3.17 Singularity

Lemma 63 (Soundness of Entry Singularity). $\quad + \text{ Suppose } e \text{ singular with } P \text{ for } P \text{ well-formed in } \Gamma. \text{ Then } \Gamma \vdash P : e \text{ and for any } \Gamma \vdash P' \text{ such that } \Gamma \vdash P' : e, \Gamma \vdash P' \simeq_1^{\leq} P;$

$- \text{ Suppose } e \text{ singular with } N \text{ for } N \text{ well-formed in } \Gamma. \text{ Then } \Gamma \vdash N : e \text{ and for any } \Gamma \vdash N' \text{ such that } \Gamma \vdash N' : e, \Gamma \vdash N' \simeq_1^{\leq} N.$

Proof. Let us consider how e **singular with** P or e **singular with** N is formed.

Case 1. Rule SINGNEq, that is $e = \hat{\alpha}^- : \approx N_0$. and N is $\mathbf{nf}(N_0)$. Then $\Gamma \vdash N' : e$ means $\Gamma \vdash N' \simeq_1^{\leq} N_0$, (by inversion of Rule SATSCENEq), which by transitivity, using corollary 16, means $\Gamma \vdash N' \simeq_1^{\leq} \mathbf{nf}(N_0)$, as required.

Case 2. Rule SINGPEq. This case is symmetric to the previous one.

Case 3. Rule SINGSupVar, that is $e = \hat{\alpha}^+ : \geq \exists \alpha^- . \beta^+$, and $P = \beta^+$.

Since $\Gamma \vdash \beta^+ \geq_1 \exists \alpha^- . \beta^+$, we have $\Gamma \vdash \beta^+ : e$, as required.

Notice that $\Gamma \vdash P' : e$ means $\Gamma \vdash P' \geq_1 \exists \alpha^- . \beta^+$. Let us show that it implies $\Gamma \vdash P' \simeq_1^{\leq} \beta^+$. By applying lemma 46 once, we have $\Gamma, \alpha^- \vdash P' \geq_1 \beta^+$. By applying it again, we notice that $\Gamma, \alpha^- \vdash P' \geq_1 \beta^+$ implies $P_i = \exists \alpha^- . \beta^+$. Finally, it is easy to see that $\Gamma \vdash \exists \alpha^- . \beta^+ \simeq_1^{\leq} \beta^+$

Case 4. Rule SINGSupShift, that is $e = \hat{\alpha}^+ : \geq \exists \beta^- . \downarrow N_1$, where $N_1 \simeq_1^D \beta_j^-$, and $P = \exists \alpha^- . \downarrow \alpha^-$.

Since $\Gamma \vdash \exists \alpha^- . \downarrow \alpha^- \geq_1 \exists \beta^- . \downarrow N_1$ (by Rule $(\exists \geq_1)$, with substitution N_1/α^-), we have $\Gamma \vdash \exists \alpha^- . \downarrow \alpha^- : e$, as required.

Notice $\Gamma \vdash P' : e$ means $\Gamma \vdash P' \geq_1 \exists \beta^- . \downarrow N_1$. Let us show that it implies $\Gamma \vdash P' \simeq_1^{\leq} \exists \alpha^- . \downarrow \alpha^-$.

$$\begin{aligned}
[h]\Gamma \vdash P' \geq_1 \exists \beta^- . \downarrow N_1 &\Rightarrow \Gamma \vdash \mathbf{nf}(P') \geq_1 \exists \beta^- . \downarrow \mathbf{nf}(N_1) \text{ where } \mathbf{ord} \beta^- \text{ in } N' = \beta^- && \text{by corollary 17} \\
&\Rightarrow \Gamma \vdash \mathbf{nf}(P') \geq_1 \exists \beta^- . \downarrow \mathbf{nf}(\beta_j^-) && \text{by lemma 24} \\
&\Rightarrow \Gamma \vdash \mathbf{nf}(P') \geq_1 \exists \beta^- . \downarrow \beta_n^- && \text{by definition of normalization} \\
&\Rightarrow \Gamma \vdash \mathbf{nf}(P') \geq_1 \exists \beta_j^- . \downarrow \beta_j^- && \text{since } \mathbf{ord} \beta^- \text{ in } \mathbf{nf}(N_1) = \beta_j^- \\
&\Rightarrow \Gamma, \beta_j^- \vdash \mathbf{nf}(P') \geq_1 \downarrow \beta_j^- \text{ and } \beta_j^- \notin \mathbf{fv}(\mathbf{nf}(P')) && \text{by lemma 47}
\end{aligned}$$

By lemma 47, the last subtyping means that $\mathbf{nf}(P') = \exists \alpha^- . \downarrow N'$, such that

1. $\Gamma, \beta_j^-, \alpha^- \vdash N'$
2. $\mathbf{ord} \alpha^- \text{ in } N' = \alpha^-$
3. for some substitution $\Gamma, \beta_j^- \vdash \sigma : \alpha^-$, $[\sigma]N' = \beta_j^-$.

Since $\beta_j^- \notin \mathbf{fv}(\mathbf{nf}(P'))$, the latter means that $N' = \alpha^-$, and then $\mathbf{nf}(P') = \exists \alpha^- . \downarrow \alpha^-$ for some α^- . Finally, notice that all the types of shape $\exists \alpha^- . \downarrow \alpha^-$ are equal.

□

Lemma 64 (Completeness of Entry Singularity).

$- \text{ Suppose that there exists } N \text{ well-formed in } \Gamma \text{ such that for any } N' \text{ well-formed in } \Gamma, \Gamma \vdash N' : e \text{ implies } \Gamma \vdash N' \simeq_1^{\leq} N. \text{ Then } e \text{ singular with } \mathbf{nf}(N).$

$+ \text{ Suppose that there exists } P \text{ well-formed in } \Gamma \text{ such that for any } P' \text{ well-formed in } \Gamma, \Gamma \vdash P' : e \text{ implies } \Gamma \vdash P' \simeq_1^{\leq} P. \text{ Then } e \text{ singular with } \mathbf{nf}(P).$

Proof.

$-$ By lemma 59, there exists $\Gamma \vdash N' : e$. Since N' is negative, by inversion of $\Gamma \vdash N' : e$, e has shape $\hat{\alpha}^- : \approx M$, where $\Gamma \vdash N' \simeq_1^{\leq} M$, and transitively, $\Gamma \vdash N \simeq_1^{\leq} M$. Then $\mathbf{nf}(M) = \mathbf{nf}(N)$, and e **singular with** $\mathbf{nf}(M)$ (by Rule SINGNEq) is rewritten as e **singular with** $\mathbf{nf}(N)$.

+ By lemma 59, there exists $\Gamma \vdash P' : e$, then by assumption, $\Gamma \vdash P' \simeq_1^{\leq} P$, which by lemma 60 implies $\Gamma \vdash P : e$.

Let us consider the shape of e :

Case 1. $e = (\hat{\alpha}^+ : \approx Q)$ then inversion of $\Gamma \vdash P : e$ implies $\Gamma \vdash P \simeq_1^{\leq} Q$, and hence, $\mathbf{nf}(P) = \mathbf{nf}(Q)$ (by lemma 35). Then e **singular with $\mathbf{nf}(Q)$** , which holds by Rule SINGPEq, is rewritten as e **singular with $\mathbf{nf}(P)$** .

Case 2. $e = (\hat{\alpha}^+ : \geq Q)$. Then the inversion of $\Gamma \vdash P : e$ implies $\Gamma \vdash P \geq_1 Q$. Let us consider the shape of Q :

a. $Q = \exists \vec{\beta}^-. \beta^+$ (for potentially empty $\vec{\beta}^-$). Then $\Gamma \vdash P \geq_1 \exists \vec{\beta}^-. \beta^+$ implies $\Gamma \vdash P \simeq_1^{\leq} \beta^+$ by lemma 46, as was noted in the proof of lemma 63, and hence, $\mathbf{nf}(P) = \beta^+$.

Then e **singular with β^+** , which holds by Rule SINGSupVar, can be rewritten as e **singular with $\mathbf{nf}(P)$** .

b. $Q = \exists \vec{\beta}^-. \downarrow N$ (for potentially empty $\vec{\beta}^-$). Notice that $\Gamma \vdash \exists \gamma^-. \downarrow \gamma^- \geq_1 \exists \vec{\beta}^-. \downarrow N$ (by Rule $(\exists \geq_1)$), with substitution N/γ^- , and thus, $\Gamma \vdash \exists \gamma^-. \downarrow \gamma^- : e$ by Rule SATSCESup.

Then by assumption, $\Gamma \vdash \exists \gamma^-. \downarrow \gamma^- \simeq_1^{\leq} P$, that is $\mathbf{nf}(P) = \exists \gamma^-. \downarrow \gamma^-$. To apply Rule SINGSupShift to infer $(\hat{\alpha}^+ : \geq \exists \vec{\beta}^-. \downarrow N)$ **singular with $\exists \gamma^-. \downarrow \gamma^-$** , it is left to show that $N \simeq_1^D \beta_i^-$ for some i .

Since $\Gamma \vdash Q : e$, by assumption, $\Gamma \vdash Q \simeq_1^{\leq} P$, and by transitivity, $\Gamma \vdash Q \simeq_1^{\leq} \exists \gamma^-. \downarrow \gamma^-$. It implies $\mathbf{nf}(\exists \vec{\beta}^-. \downarrow N) = \exists \gamma^-. \downarrow \gamma^-$ (by lemma 35), which by definition of normalization means $\exists \vec{\beta}^-. \downarrow \mathbf{nf}(N) = \exists \gamma^-. \downarrow \gamma^-$, where **ord** $\vec{\beta}^-$ in $N' = \beta^{-'}$. This way, $\vec{\beta}^{-'}$ is a variable β^- , and $\mathbf{nf}(N) = \beta^-$. Notice that $\beta^- \in \beta^{-'} \subseteq \beta^-$ by lemma 16. This way, $N \simeq_1^D \beta^-$ for $\beta^- \in \beta^-$ (by lemma 35),

□

Lemma 65 (Soundness of Singularity). *Suppose $\Theta \vdash SC$, and SC **singular with $\hat{\sigma}$** . Then $\Theta \vdash \hat{\sigma} : SC$ and for any $\hat{\sigma}'$ such that $\Theta \vdash \hat{\sigma}' : SC$, $\Theta \vdash \hat{\sigma}' \simeq_1^{\leq} \hat{\sigma} : \mathbf{dom}(SC)$.*

Proof. Suppose that $\Theta \vdash \hat{\sigma}' : SC$. It means that for every $e \in SC$ restricting $\hat{\alpha}^\pm$, $\Theta(\hat{\alpha}^\pm) \vdash [\hat{\sigma}']\hat{\alpha}^\pm : e$ holds. SC **singular with $\hat{\sigma}$** means e **singular with $[\hat{\sigma}]\hat{\alpha}^\pm$** , and hence, by lemma 64, $\Theta(\hat{\alpha}^\pm) \vdash [\hat{\sigma}']\hat{\alpha}^\pm \simeq_1^{\leq} [\hat{\sigma}]\hat{\alpha}^\pm$ holds.

Since the uniqueness holds for every variable from $\mathbf{dom}(SC)$, $\hat{\sigma}$ is equivalent to $\hat{\sigma}'$ on this set. □

Lemma 66 (Completeness of Singularity).

Lemma 67 (Completeness of Singularity). *Suppose there exists $\Theta \vdash \hat{\sigma}_1$ such that for any $\Theta \vdash \hat{\sigma}$, $\Theta \vdash \hat{\sigma} : SC$ implies $\Theta \vdash \hat{\sigma} \simeq_1^{\leq} \hat{\sigma}_1 : \mathbf{vars}$. Then*

- $\mathbf{SC}|_{\mathbf{vars}}$ **singular with $\hat{\sigma}_0$** for some $\hat{\sigma}_0$, and
- $\mathbf{vars} \subseteq \mathbf{dom}(SC)$.

Proof. □

3.18 Declarative Typing

Lemma 68. *If $\Gamma; \Phi \vdash N_1 \bullet \vec{v} \Rightarrow M$ and $\Gamma \vdash N_1 \simeq_1^{\leq} N_2$ then $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Rightarrow M$.*

Proof. By lemma 34, $\Gamma \vdash N_1 \simeq_1^{\leq} N_2$ implies $N_1 \simeq_1^D N_2$. Let us prove the required judgement by induction on $N_1 \simeq_1^D N_2$. Let us consider the last rule used in the derivation.

Case 1. Rule $(\text{Var} \simeq_1^D)$. It means that N_1 is α^- and N_2 is α^- . Then the required property coincides with the assumption.

Case 2. Rule $(\uparrow \simeq_1^D)$. It means that N_1 is $\uparrow P_1$ and N_2 is $\uparrow P_2$. where $P_1 \simeq_1^D P_2$.

Then the only rule applicable to infer $\Gamma; \Phi \vdash \uparrow P_1 \bullet \vec{v} \Rightarrow M$ is Rule DTEmptyApp, meaning that $\vec{v} = \cdot$ and $\Gamma \vdash \uparrow P_1 \simeq_1^{\leq} M$. Then by transitivity of equivalence corollary 7, $\Gamma \vdash \uparrow P_2 \simeq_1^{\leq} M$, and then Rule DTEmptyApp is applicable to infer $\Gamma; \Phi \vdash \uparrow P_2 \bullet \cdot \Rightarrow M$.

Case 3. Rule $(\rightarrow \simeq_1^D)$. Then we are proving that $\Gamma; \Phi \vdash (Q_1 \rightarrow N_1) \bullet v, \vec{v} \Rightarrow M$ and $Q_1 \rightarrow N_1 \simeq_1^D Q_2 \rightarrow N_2$ imply $\Gamma; \Phi \vdash (Q_2 \rightarrow N_2) \bullet v, \vec{v} \Rightarrow M$.

By inversion, $(Q_1 \rightarrow N_1) \simeq_1^D (Q_2 \rightarrow N_2)$ means $Q_1 \simeq_1^D Q_2$ and $N_1 \simeq_1^D N_2$.

By inversion of $\Gamma; \Phi \vdash (Q_1 \rightarrow N_1) \bullet v, \vec{v} \Rightarrow M$:

1. $\Gamma; \Phi \vdash v : P$
2. $\Gamma \vdash Q_1 \geq_1 P$, and then by transitivity corollary 6, $\Gamma \vdash Q_2 \geq_1 P$;
3. $\Gamma; \Phi \vdash N_1 \bullet \vec{v} \Rightarrow M$, and then by induction hypothesis, $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Rightarrow M$.

Since we have $\Gamma; \Phi \vdash v : P$, $\Gamma \vdash Q_2 \geq_1 P$ and $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Rightarrow M$, we can apply Rule DTArrowApp to infer $\Gamma; \Phi \vdash (Q_2 \rightarrow N_2) \bullet v, \vec{v} \Rightarrow M$.

Case 4. Rule ($\forall \xrightarrow{P}$) Then we are proving that $\Gamma; \Phi \vdash \forall \alpha^+_{\cdot 1}. N'_1 \bullet \vec{v} \Rightarrow M$ and $\forall \alpha^+_{\cdot 1}. N'_1 \simeq_1^D \forall \alpha^+_{\cdot 2}. N'_2$ imply $\Gamma; \Phi \vdash \forall \alpha^+_{\cdot 2}. N'_2 \bullet \vec{v} \Rightarrow M$.

By inversion of $\forall \alpha^+_{\cdot 1}. N'_1 \simeq_1^D \forall \alpha^+_{\cdot 2}. N'_2$:

1. $\alpha^+_{\cdot 2} \cap \mathbf{fv} N_1 = \emptyset$,
2. there exists a bijection $\mu : (\alpha^+_{\cdot 2} \cap \mathbf{fv} N'_2) \leftrightarrow (\alpha^+_{\cdot 1} \cap \mathbf{fv} N'_1)$ such that $N'_1 \simeq_1^D [\mu]N'_2$.

By inversion of $\Gamma; \Phi \vdash \forall \alpha^+_{\cdot 1}. N'_1 \bullet \vec{v} \Rightarrow M$:

1. $\Gamma \vdash \sigma : \alpha^+_{\cdot 1}$
2. $\Gamma; \Phi \vdash [\sigma]N'_1 \bullet \vec{v} \Rightarrow M$
3. $\vec{v} \neq \cdot$.

Let us construct $\Gamma \vdash \sigma_0 : \alpha^+_{\cdot 2}$ in the following way:

$$\begin{cases} [\sigma_0]\alpha^+ = [\sigma][\mu]\alpha^+ & \text{if } \alpha^+ \in \alpha^+_{\cdot 2} \cap \mathbf{fv} N'_2 \\ [\sigma_0]\alpha^+ = \exists \beta^-. \downarrow \beta^- & \text{otherwise (the type does not matter here)} \end{cases}$$

Then to infer $\Gamma; \Phi \vdash N_2 \bullet \vec{v} \Rightarrow M$, we apply Rule DTArrowApp with σ_0 . Let us show the required premises:

1. $\Gamma \vdash \sigma_0 : \alpha^+_{\cdot 2}$ by construction;
2. $\vec{v} \neq \cdot$ as noted above;
3. To show $\Gamma; \Phi \vdash [\sigma_0]N'_2 \bullet \vec{v} \Rightarrow M$, Notice that $[\sigma_0]N'_2 = [\sigma][\mu]N'_2$ and since $[\mu]N'_2 \simeq_1^D N'_1$, $[\sigma][\mu]N'_2 \simeq_1^D [\sigma]N'_1$. This way, by lemma 30, $\Gamma \vdash [\sigma]N'_1 \simeq_1^{\leq} [\sigma_0]N'_2$. Then the required judgement holds by the induction hypothesis applied to $\Gamma; \Phi \vdash [\sigma]N'_1 \bullet \vec{v} \Rightarrow M$.

□

Definition 10 (Number of prenex quantifiers). *Let us define $\text{npq}(N)$ and $\text{npq}(P)$ as the number of prenex quantifiers in these types, i.e.*

- + $\text{npq}(\exists \alpha^-. P) = |\text{nas}|$, if $P \neq \exists \beta^-. P'$,
- $\text{npq}(\forall \alpha^+. N) = |\text{pas}|$, if $N \neq \forall \beta^+. N'$.

Definition 11 (Size of a Declarative Judgement). *For a declarative typing judgement J let us define a metrics $\text{size}(J)$ as a pair of numbers in the following way:*

- + $\text{size}(\Gamma; \Phi \vdash v : P) = (\text{size}(v), 0)$;
- $\text{size}(\Gamma; \Phi \vdash c : N) = (\text{size}(c), 0)$;
- $\text{size}(\Gamma; \Phi \vdash N \bullet \vec{v} \Rightarrow M) = (\text{size}(\vec{v}), \text{npq}(N))$

where $\text{size}(v)$ or $\text{size}(c)$ is the size of the syntax tree of the term v or c and $\text{size}(\vec{v})$ is the sum of sizes of the terms in \vec{v} .

Definition 12 (Number of Equivalence Nodes). *For a tree T inferring a declarative typing judgement, let us a function $\text{eq_nodes}(T)$ as the number of nodes in T labeled with Rule DTPEquiv or Rule DTNEquiv.*

Definition 13 (Metric). *For a tree T inferring a declarative typing judgement J , let us define a metric $\text{metric}(T)$ as a pair $(\text{size}(J), \text{eq_nodes}(T))$.*

Lemma 69 (Declarative typing is preserved under context equivalence). *Assuming $\Gamma \vdash \Phi_1$, $\Gamma \vdash \Phi_2$, and $\Gamma \vdash \Phi_1 \simeq_1^{\leq} \Phi_2$:*

- + for any tree T_1 inferring $\Gamma; \Phi_1 \vdash v : P$, there exists a tree T_2 inferring $\Gamma; \Phi_2 \vdash v : P$.
- for any tree T_1 inferring $\Gamma; \Phi_1 \vdash c : N$, there exists a tree T_2 inferring $\Gamma; \Phi_2 \vdash c : N$.
- for any tree T_1 inferring $\Gamma; \Phi_1 \vdash N \bullet \vec{v} \Rightarrow M$, there exists a tree T_2 inferring $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rightarrow M$.

Proof. Let us prove it by induction on the $\text{metric}(T_1)$. Let us consider the last rule applied in T_1 (i.e., its root node).

Case 1. Rule DTVar

Then we are proving that $\Gamma; \Phi_1 \vdash x : P$ implies $\Gamma; \Phi_2 \vdash x : P$. By inversion, $x : P \in \Phi_1$, and since $\Gamma \vdash \Phi_1 \simeq_1^{\leq} \Phi_2$, $x : P' \in \Phi_2$ for some P' such that $\Gamma \vdash P \simeq_1^{\leq} P'$. Then we infer $\Gamma; \Phi_2 \vdash x : P'$ by Rule DTVar, and next, $\Gamma; \Phi_2 \vdash x : P$ by Rule DTPEquiv.

Case 2. For Rule DTThunk, Rule DTPAnnot, Rule DTTLam, Rule DTReturn, and Rule DTNAnnot the proof is analogous. We apply the induction hypothesis to the premise of the rule to substitute Φ_1 for Φ_2 in it. The induction is applicable because the metric of the premises is less than the metric of the conclusion: the term in the premise is a syntactic subterm of the term in the conclusion.

And after that, we apply the same rule to infer the required judgement.

Case 3. Rule DTPEquiv and Rule DTNEquiv In these cases, the induction hypothesis is also applicable to the premise: although the first component of the metric is the same for the premise and the conclusion: $\text{size}(\Gamma; \Phi \vdash c : N') = \text{size}(\Gamma; \Phi \vdash c : N) = \text{size}(c)$, the second component of the metric is less for the premise by one, since the equivalence rule was applied to turn the premise tree into $T1$. Having made this note, we continue the proof in the same way as in the previous case.

Case 4. Rule DTtLam Then we are proving that $\Gamma; \Phi_1 \vdash \lambda x : P.c : P \rightarrow N$ implies $\Gamma; \Phi_2 \vdash \lambda x : P.c : P \rightarrow N$. Analogously to the previous cases, we apply the induction hypothesis to the equivalent contexts $\Gamma \vdash \Phi_1, x : P \simeq_1^{\leq} \Phi_2, x : P$ and the premise $\Gamma; \Phi_1, x : P \vdash c : N$ to obtain $\Gamma; \Phi_2, x : P \vdash c : N$. Notice that c is a subterm of $\lambda x : P.c$, i.e., the metric of the premise tree is less than the metric of the conclusion, and the induction hypothesis is applicable. Then we infer $\Gamma; \Phi_2 \vdash \lambda x : P.c : P \rightarrow N$ by Rule DTtLam.

Case 5. Rule DTVarLet Then we are proving that $\Gamma; \Phi_1 \vdash \text{let } x = v; c : N$ implies $\Gamma; \Phi_2 \vdash \text{let } x = v; c : N$. First, we apply the induction hypothesis to $\Gamma; \Phi_1 \vdash v : P$ to obtain $\Gamma; \Phi_2 \vdash v : P$ of the same pure size.

Then we apply the induction hypothesis to the equivalent contexts $\Gamma \vdash \Phi_1, x : P \simeq_1^{\leq} \Phi_2, x : P$ and the premise $\Gamma; \Phi_1, x : P \vdash c : N$ to obtain $\Gamma; \Phi_2, x : P \vdash c : N$. Then we infer $\Gamma; \Phi_2 \vdash \text{let } x = v; c : N$ by Rule DTVarLet.

Case 6. Rule DTAppLet Then we are proving that $\Gamma; \Phi_1 \vdash \text{let } x = v(\vec{v}); c : N$ implies $\Gamma; \Phi_2 \vdash \text{let } x = v(\vec{v}); c : N$.

We apply the induction hypothesis to each of the premises. to rewrite:

- $\Gamma; \Phi_1 \vdash v : \downarrow M$ into $\Gamma; \Phi_2 \vdash v : \downarrow M$,
- $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rightarrow \uparrow Q$ into $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rightarrow \uparrow Q$.
- $\Gamma; \Phi_1, x : Q \vdash c : N$ into $\Gamma; \Phi_2, x : Q \vdash c : N$ (notice that $\Gamma \vdash \Phi_1, x : Q \simeq_1^{\leq} \Phi_2, x : Q$).

It is left to show the uniqueness of $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rightarrow \uparrow Q$. Let us assume that this judgement holds for other Q' , i.e. there exists a tree T_0 inferring $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rightarrow \uparrow Q'$. Then notice that the induction hypothesis is applicable to T_0 : the first component of the first component of $\text{metric}(T_0)$ is $S = \sum_{v \in \vec{v}} \text{size}(v)$, and it is less than the corresponding component of $\text{metric}(T_1)$, which is $\text{size}(\text{let } x = v(\vec{v}); c) = 1 + \text{size}(v) + \text{size}(c) + S$. This way, $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rightarrow \uparrow Q'$ holds by the induction hypothesis, but since $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rightarrow \uparrow Q$ unique, we have $\Gamma \vdash Q' \simeq_1^{\leq} Q$.

Then we infer $\Gamma; \Phi_2 \vdash \text{let } x = v(\vec{v}); c : N$ by Rule DTAppLet.

Case 7. Rule DTAppLetAnn Then we are proving that $\Gamma; \Phi_1 \vdash \text{let } x : P = v(\vec{v}); c : N$ implies $\Gamma; \Phi_2 \vdash \text{let } x : P = v(\vec{v}); c : N$.

As in the previous case, we apply the induction hypothesis to each of the premises and rewrite:

- $\Gamma; \Phi_1 \vdash v : \downarrow M$ into $\Gamma; \Phi_2 \vdash v : \downarrow M$,
- $\Gamma; \Phi_1 \vdash M \bullet \vec{v} \Rightarrow \uparrow Q$ into $\Gamma; \Phi_2 \vdash M \bullet \vec{v} \Rightarrow \uparrow Q$, and
- $\Gamma; \Phi_1, x : P \vdash c : N$ into $\Gamma; \Phi_2, x : P \vdash c : N$ (notice that $\Gamma \vdash \Phi_1, x : P \simeq_1^{\leq} \Phi_2, x : P$).

Notice that $\Gamma \vdash P$ and $\Gamma \vdash \uparrow Q \leq_1 \uparrow P$ do not depend on the variable context, and hold by assumption. Then we infer $\Gamma; \Phi_2 \vdash \text{let } x : P = v(\vec{v}); c : N$ by Rule DTAppLetAnn.

Case 8. Rule DTUnpack, and Rule DTNAnnot are proved in the same way.

Case 9. Rule DTEmptyApp Then we are proving that $\Gamma; \Phi_1 \vdash N \bullet \cdot \Rightarrow N'$ (inferred by Rule DTEmptyApp) implies $\Gamma; \Phi_2 \vdash N \bullet \cdot \Rightarrow N'$.

To infer $\Gamma; \Phi_2 \vdash N \bullet \cdot \Rightarrow N'$, we apply Rule DTEmptyApp, noting that $\Gamma \vdash N \simeq_1^{\leq} N'$ holds by assumption.

Case 10. Rule DTArrowApp Then we are proving that $\Gamma; \Phi_1 \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M$ (inferred by Rule DTArrowApp) implies $\Gamma; \Phi_2 \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M$. And uniqueness of the M in the first case implies uniqueness in the second case.

By induction, we rewrite $\Gamma; \Phi_1 \vdash v : P$ into $\Gamma; \Phi_2 \vdash v : P$, and $\Gamma; \Phi_1 \vdash N \bullet \vec{v} \Rightarrow M$ into $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rightarrow M$. Then we infer $\Gamma; \Phi_2 \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M$ by Rule DTArrowApp.

Now, let us show the uniqueness. The only rule that can infer $\Gamma; \Phi_1 \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M$ is Rule DTArrowApp. Then by inversion, uniqueness of $\Gamma; \Phi_1 \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M$ implies uniqueness of $\Gamma; \Phi_1 \vdash N \bullet \vec{v} \Rightarrow M$. By the induction hypothesis, it implies the uniqueness of $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rightarrow M$.

Suppose that $\Gamma; \Phi_2 \vdash Q \rightarrow N \bullet v, \vec{v} \Rightarrow M'$. By inversion, $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rightarrow M'$, which by uniqueness of $\Gamma; \Phi_2 \vdash N \bullet \vec{v} \Rightarrow M$ implies $\Gamma \vdash M \simeq_1^{\leq} M'$.

Case 11. Rule DTForallApp Then we are proving that $\Gamma; \Phi_1 \vdash \forall \alpha^+ \vec{v}. N \bullet \vec{v} \Rightarrow M$ (inferred by Rule DTForallApp) implies $\Gamma; \Phi_2 \vdash \forall \alpha^+ \vec{v}. N \bullet \vec{v} \Rightarrow M$.

By inversion, we have σ such that $\Gamma \vdash \sigma : \alpha^+$ and $\Gamma; \Phi_1 \vdash [\sigma]N \bullet \vec{v} \Rightarrow M$ is inferred. Let us denote the inference tree as T'_1 . Notice that the induction hypothesis is applicable to T'_1 : $\text{metric}(T'_1) = ((\text{size}(\vec{v}), 0), x)$ is less than $\text{metric}(T_1) = ((\text{size}(\vec{v}), |\alpha^+|), y)$ for any x and y , since $|\alpha^+| > 0$ by inversion.

This way, by the induction hypothesis, there exists a tree T'_2 inferring $\Gamma; \Phi_2 \vdash [\sigma]N \bullet \vec{v} \Rightarrow M$. Notice that the premises $\vec{v} \neq \cdot$, $\Gamma \vdash \sigma : \alpha^+$, and $\alpha^+ \neq \cdot$ do not depend on the variable context, and hold by inversion. Then we infer $\Gamma; \Phi_2 \vdash \forall \alpha^+ \vec{v}. N \bullet \vec{v} \Rightarrow M$ by Rule DTForallApp.

□

3.19 Algorithmic Typing

Let us extend the declarative typing metric (definition 13) to the algorithmic typing.

Definition 14 (Size of an Algorithmic Judgement). *For an algorithmic typing judgement J let us define a metrics $\text{size}(J)$ as a pair of numbers in the following way:*

- + $\text{size}(\Gamma; \Phi \models v : P) = (\text{size}(v), 0);$
- $\text{size}(\Gamma; \Phi \models c : N) = (\text{size}(c), 0);$
- $\text{size}(\Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rightarrow M \Leftarrow \Theta'; SC) = (\text{size}(\vec{v}), \text{npq}(N))$

Definition 15 (Metric). *We extend the metric from definition 13 to the algorithmic typing in the following way. For a tree T inferring an algorithmic typing judgement J , we define $\text{size}(T)$ as $(\text{size}(J), 0)$.*

Soundness and the completeness are proved by mutual induction on the metric of the inference tree.

Lemma 70 (Soundness of typing). *Suppose that $\Gamma \vdash \Phi$. For an inference tree T_1 ,*

- + *If T_1 infers $\Gamma; \Phi \models v : P$ then $\Gamma \vdash P$ and $\Gamma; \Phi \vdash v : P$*
- *If T_1 infers $\Gamma; \Phi \models c : N$ then $\Gamma \vdash N$ and $\Gamma; \Phi \vdash c : N$*
- *For $\Gamma \vdash^\exists \Theta$ and $\Gamma; \Theta \vdash N$ free from negative metavariables, if T_1 infers $\Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rightarrow M \Leftarrow \Theta'; SC$ then*
 1. $\Gamma \vdash^\exists \Theta'$
 2. $\Theta \subseteq \Theta'$
 3. $\Gamma; \Theta' \vdash M$
 4. M is normalized and free from negative metavariables
 5. $\Theta' \vdash SC$
 6. *for any $\Theta' \vdash \hat{\sigma} : SC$, we have $\Gamma; \Phi \vdash [\hat{\sigma}]N \bullet \vec{v} \Rightarrow [\hat{\sigma}]M$*

Proof. We prove it by induction on $\text{metric}(T_1)$, mutually with the completeness of typing (lemma 70). Let us consider the last rule used to infer the derivation.

Case 1. Rule ATVar We are proving that if $\Gamma; \Phi \models x : \mathbf{nf}(P)$ then $\Gamma \vdash \mathbf{nf}(P)$ and $\Gamma; \Phi \vdash x : \mathbf{nf}(P)$.

By inversion, $x : P \in \Phi$. Since $\Gamma \vdash \Phi$, we have $\Gamma \vdash P$, and by corollary 14, $\Gamma \vdash \mathbf{nf}(P)$.

By applying Rule DTVar to $x : P \in \Phi$, we infer $\Gamma; \Phi \vdash x : P$. Finally, by Rule DTPEquiv, since $\Gamma \vdash P \simeq_1^{\leq} \mathbf{nf}(P)$ (corollary 16), we have $\Gamma; \Phi \vdash x : \mathbf{nf}(P)$.

Case 2. Rule ATThunk

We are proving that if $\Gamma; \Phi \models \{c\} : \downarrow N$ then $\Gamma \vdash \downarrow N$ and $\Gamma; \Phi \vdash \{c\} : \downarrow N$.

By inversion of $\Gamma; \Phi \models \{c\} : \downarrow N$, we have $\Gamma; \Phi \models c : N$. By the induction hypothesis applied to $\Gamma; \Phi \models c : N$, we have

1. $\Gamma \vdash N$, and hence, $\Gamma \vdash \downarrow N$;
2. $\Gamma; \Phi \vdash c : N$, which by Rule DTThunk implies $\Gamma; \Phi \vdash \{c\} : \downarrow N$.

Case 3. Rule ATReturn The proof is symmetric to the previous case (case 2).

Case 4. Rule ATPAnnot We are proving that if $\Gamma; \Phi \models (v : Q) : \mathbf{nf}(Q)$ then $\Gamma \vdash \mathbf{nf}(Q)$ and $\Gamma; \Phi \vdash (v : Q) : \mathbf{nf}(Q)$.

By inversion of $\Gamma; \Phi \models (v : Q) : \mathbf{nf}(Q)$, we have:

1. $\Gamma \vdash (v : Q)$, hence, $\Gamma \vdash Q$, and by corollary 14, $\Gamma \vdash \mathbf{nf}(Q)$;
2. $\Gamma; \Phi \models v : P$, which by the induction hypothesis implies $\Gamma \vdash P$ and $\Gamma; \Phi \vdash v : P$;
3. $\Gamma; \cdot \models Q \triangleright P \Rightarrow \cdot$, which by lemma 52 implies $\Gamma \vdash [\cdot] Q \triangleright_1 P$, that is $\Gamma \vdash Q \triangleright_1 P$.

To infer $\Gamma; \Phi \vdash (v : Q) : Q$, we apply Rule DTPAnnot to $\Gamma; \Phi \vdash v : P$ and $\Gamma \vdash Q \triangleright_1 P$. Then by Rule DTPEquiv, $\Gamma; \Phi \vdash (v : Q) : \mathbf{nf}(Q)$.

Case 5. Rule ATNAnnot The proof is symmetric to the previous case (case 4).

Case 6. Rule ATtLam We are proving that if $\Gamma; \Phi \models \lambda x : P.c : \mathbf{nf}(P \rightarrow N)$ then $\Gamma \vdash \mathbf{nf}(P \rightarrow N)$ and $\Gamma; \Phi \vdash \lambda x : P.c : \mathbf{nf}(P \rightarrow N)$.

By inversion of $\Gamma; \Phi \models \lambda x : P.c : \mathbf{nf}(P \rightarrow N)$, we have $\Gamma \vdash \lambda x : P.c$, which implies $\Gamma \vdash P$.

Also by inversion of $\Gamma; \Phi \models \lambda x : P.c : \mathbf{nf}(P \rightarrow N)$, we have $\Gamma; \Phi, x : P \models c : N$, applying induction hypothesis to which gives us:

1. $\Gamma \vdash N$, thus $\Gamma \vdash P \rightarrow N$, and by corollary 14, $\Gamma \vdash \mathbf{nf}(P \rightarrow N)$;
2. $\Gamma; \Phi, x : P \vdash c : N$, which by Rule DTtLam implies $\Gamma; \Phi \vdash \lambda x : P.c : P \rightarrow N$, and by Rule DTPEquiv, $\Gamma; \Phi \vdash \lambda x : P.c : \mathbf{nf}(P \rightarrow N)$.

Case 7. Rule ATTLam We are proving that if $\Gamma; \Phi \models \Lambda \alpha^+.c : \mathbf{nf}(\forall \alpha^+.N)$ then $\Gamma; \Phi \vdash \Lambda \alpha^+.c : \mathbf{nf}(\forall \alpha^+.N)$ and $\Gamma \vdash \mathbf{nf}(\forall \alpha^+.N)$.

By inversion of $\Gamma, \alpha^+; \Phi \models c : N$, we have $\Gamma \vdash \Lambda \alpha^+.c$, which implies $\Gamma, \alpha^+ \vdash c$.

Also by inversion of $\Gamma, \alpha^+; \Phi \models c : N$, we have $\Gamma, \alpha^+; \Phi \models c : N$. Obtaining the induction hypothesis to $\Gamma, \alpha^+; \Phi \models c : N$, we have:

1. $\Gamma, \alpha^+ \vdash N$, thus $\Gamma \vdash \forall \alpha^+.N$, and by corollary 14, $\Gamma \vdash \mathbf{nf}(\forall \alpha^+.N)$;
2. $\Gamma, \alpha^+; \Phi \vdash c : N$, which by Rule DTTTlam implies $\Gamma; \Phi \vdash \Lambda \alpha^+.c : \forall \alpha^+.N$, and by Rule DTPEquiv, $\Gamma; \Phi \vdash \Lambda \alpha^+.c : \mathbf{nf}(\forall \alpha^+.N)$.

Case 8. Rule ATVarLet We are proving that if $\Gamma; \Phi \models \mathbf{let} x = v; c : N$ then $\Gamma; \Phi \vdash \mathbf{let} x = v; c : N$ and $\Gamma \vdash N$.

By inversion of $\Gamma; \Phi \models \mathbf{let} x = v; c : N$, we have:

1. $\Gamma \vdash \mathbf{let} x = v; c$, which gives us $\Gamma \vdash v$ and $\Gamma \vdash c$.
2. $\Gamma; \Phi \models v : P$, which by the induction hypothesis implies $\Gamma \vdash P$ (and thus, $\Gamma \vdash \Phi, x : P$) and $\Gamma; \Phi \vdash v : P$;
3. $\Gamma; \Phi, x : P \models c : N$, which by the induction hypothesis implies $\Gamma \vdash N$ and $\Gamma; \Phi, x : P \vdash c : N$.

This way, $\Gamma; \Phi \vdash \mathbf{let} x = v; c : N$ holds by Rule DTVarLet.

Case 9. Rule ATAppLetAnn We are proving that $\Gamma; \Phi \models \mathbf{let} x : P = v(\vec{v}); c' : N$ then $\Gamma; \Phi \vdash \mathbf{let} x : P = v(\vec{v}); c' : N$ and $\Gamma \vdash N$.

By inversion, we have:

1. $\Gamma \vdash P$, hence, $\Gamma \vdash \Phi, x : P$
2. $\Gamma; \Phi \models v : \downarrow M$
3. $\Gamma; \Phi; \cdot \models \underline{M} \bullet \vec{v} \Rightarrow \uparrow Q \Rightarrow \Theta; SC_1$
4. $\Gamma; \Theta \models \uparrow Q \leq \uparrow P \Rightarrow SC_2$
5. $\Theta \vdash SC_1 \& SC_2 = SC$
6. $\Gamma; \Phi, x : P \models c' : N$

By the soundness of constraint merge (lemma 55), we have $\Theta \vdash SC$. Let us take $\hat{\sigma}$ such that $\Theta \vdash \hat{\sigma} : SC$ (it exists by lemma 58). Notice that by the soundness of constraint merge, $\Theta \vdash \hat{\sigma} : SC_1$ and $\Theta \vdash \hat{\sigma} : SC_2$.

By the induction hypothesis applied to $\Gamma; \Phi \models v : \downarrow M$, we have $\Gamma; \Phi \vdash v : \downarrow M$ and $\Gamma \vdash \downarrow M$ (and hence, $\Gamma; \Theta \vdash \underline{M}$).

By the induction hypothesis applied to $\Gamma; \Phi, x : P \models c' : N$, we have $\Gamma; \Phi, x : P \vdash c' : N$ and $\Gamma \vdash N$.

By the induction hypothesis applied to $\Gamma; \Phi; \cdot \models \underline{M} \bullet \vec{v} \Rightarrow \uparrow Q \Rightarrow \Theta; SC_1$, we have:

1. $\Gamma \vdash \supset \Theta$,
2. $\Gamma; \Theta \vdash \uparrow Q$,
3. $\Theta' \vdash SC_1$,
4. for any $\Theta' \vdash \hat{\sigma} : SC_1$, we have $\Gamma; \Phi \vdash [\hat{\sigma}] \underline{M} \bullet \vec{v} \Rightarrow [\hat{\sigma}] \uparrow Q$. In particular, it holds for the $\hat{\sigma}$ chosen above.

By the soundness of negative subtyping (??) applied to $\Gamma; \Theta \models \uparrow Q \leq \uparrow P \Rightarrow SC$, we have $\Gamma \vdash \uparrow[\hat{\sigma}]Q \leq_1 \uparrow P$.

To infer $\Gamma; \Phi \vdash \mathbf{let} x : P = v(\vec{v}); c' : N$, we apply the corresponding declarative rule Rule DTAAppLetAnn, where Q is $[\hat{\sigma}]Q$. Notice that all the premises were already shown to hold above:

1. $\Gamma \vdash P$ and $\Gamma; \Phi \vdash v : \downarrow M$ from the assumption,
2. $\Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow[\hat{\sigma}]Q$ holds since $[\hat{\sigma}]\uparrow Q = \uparrow[\hat{\sigma}]Q$,
3. $\Gamma \vdash \uparrow[\hat{\sigma}]Q \leq_1 \uparrow P$ by the soundness of negative subtyping,
4. $\Gamma; \Phi, x : P \vdash c' : N$ from the the induction hypothesis.

Case 10. Rule ATAppLet We are proving that if $\Gamma; \Phi \models \mathbf{let} x = v(\vec{v}); c' : N$ then $\Gamma; \Phi \vdash \mathbf{let} x = v(\vec{v}); c' : N$ and $\Gamma \vdash N$.

By the inversion, we have:

1. $\Gamma; \Phi \models v : \downarrow M$
2. $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow \uparrow Q \Rightarrow \Theta; SC$
3. $\mathbf{uv}(Q) \subseteq \mathbf{dom}(SC)$
4. $\mathbf{SC}|_{\mathbf{uv}(Q)}$ **singular with** $\hat{\sigma}_3$
5. $\Gamma; \Phi, x : [\hat{\sigma}_3]Q \models c' : N$

By the induction hypothesis applied to $\Gamma; \Phi \models v : \downarrow M$, we have $\Gamma; \Phi \vdash v : \downarrow M$ and $\Gamma \vdash \downarrow M$ (and thus, $\Gamma; \Theta \vdash M$).

By the induction hypothesis applied to $\Gamma; \Phi, x : [\hat{\sigma}_3]Q \models c' : N$, we have $\Gamma \vdash N$ and $\Gamma; \Phi, x : [\hat{\sigma}_3]Q \vdash c' : N$.

By the induction hypothesis applied to $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow \uparrow Q \Rightarrow \Theta; SC$, we have:

1. $\Gamma \vdash \Theta$
2. $\Gamma; \Theta \vdash \uparrow Q$
3. $\Theta \vdash SC$
4. for any $\Theta \vdash \hat{\sigma} : SC$, we have $\Gamma; \Phi \vdash [\hat{\sigma}]M \bullet \vec{v} \Rightarrow [\hat{\sigma}]\uparrow Q$, which, since M is ground means $\Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow[\hat{\sigma}]Q$.

To infer $\Gamma; \Phi \vdash \mathbf{let} x = v(\vec{v}); c' : N$, we apply the corresponding declarative rule Rule DTAAppLet. Let us show that the premises hold:

- $\Gamma; \Phi \vdash v : \downarrow M$ holds by the induction hypothesis;
- $\Gamma; \Phi, x : [\hat{\sigma}_3]Q \vdash c' : N$ also holds by the induction hypothesis, as noted above;
- Let us take an arbitrary substitution $\hat{\sigma}$ satisfying $\Theta \vdash \hat{\sigma} : SC$ (it exists by lemma 59). Then $\Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow[\hat{\sigma}]Q$ holds, as noted above;
- To show the uniqueness of $\uparrow[\hat{\sigma}]Q$, we assume that for some other type K holds $\Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow K$, that is $\Gamma; \Phi \vdash [\cdot]M \bullet \vec{v} \Rightarrow K$. Then by the completeness of typing (lemma 71), there exist N' , Θ' , and SC' such that
 1. $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow N' \Rightarrow \Theta'; SC'$ and
 2. there exists a substitution $\Theta' \vdash \hat{\sigma}' : SC'$ such that $\Gamma \vdash [\hat{\sigma}']N' \simeq_1^{\leq} K$.

By the determinicity of the typing algorithm (??), $\Gamma; \Phi; \cdot \models M \bullet \vec{v} \Rightarrow N' \Rightarrow \Theta'; SC'$, means that SC' is SC , Θ' is Θ , and N' is $\uparrow Q$. This way, $\Gamma \vdash [\hat{\sigma}']\uparrow Q \simeq_1^{\leq} K$ for a substitution $\Theta \vdash \hat{\sigma}' : SC$.

It is left to show that $\Gamma \vdash [\hat{\sigma}']\uparrow Q \simeq_1^{\leq} [\hat{\sigma}]\uparrow Q$, then by transitivity of the equivalence, we will have $\Gamma \vdash [\hat{\sigma}]\uparrow Q \simeq_1^{\leq} K$. Since $\Theta \vdash \hat{\sigma} : \mathbf{SC}|_{\mathbf{uv}(Q)}$ and $\Theta \vdash \hat{\sigma}' : \mathbf{SC}|_{\mathbf{uv}(Q)}$, and $\mathbf{SC}|_{\mathbf{uv}(Q)}$ **singular with** $\hat{\sigma}_3$, we have $\Theta \vdash \hat{\sigma} \simeq_1^{\leq} \hat{\sigma}_3 : \mathbf{dom}(\mathbf{SC}|_{\mathbf{uv}(Q)})$ and $\Theta \vdash \hat{\sigma}' \simeq_1^{\leq} \hat{\sigma}_3 : \mathbf{dom}(\mathbf{SC}|_{\mathbf{uv}(Q)})$. Then since $\mathbf{uv}(Q) \subseteq \mathbf{dom}(SC)$, we have $\mathbf{dom}(\mathbf{SC}|_{\mathbf{uv}(Q)}) = \mathbf{uv}(Q)$. This way, by transitivity and symmetry of the equivalence, $\Theta \vdash \hat{\sigma} \simeq_1^{\leq} \hat{\sigma}' : \mathbf{uv}(Q)$, which implies $\Gamma \vdash [\hat{\sigma}']\uparrow Q \simeq_1^{\leq} [\hat{\sigma}]\uparrow Q$.

Case 11. Rule ATUnpack We are proving that if $\Gamma; \Phi \models \mathbf{let}^3(\vec{\alpha}^-, x) = v; c' : N$ then $\Gamma; \Phi \vdash \mathbf{let}^3(\vec{\alpha}^-, x) = v; c' : N$ and $\Gamma \vdash N$.

By the inversion, we have:

1. $\Gamma; \Phi \models v : \exists \vec{\alpha}^-. P$
2. $\Gamma, \vec{\alpha}^-; \Phi, x : P \models c' : N$
3. $\Gamma \vdash N$

By the induction hypothesis applied to $\Gamma; \Phi \models v : \exists \vec{\alpha}^-. P$, we have $\Gamma; \Phi \vdash v : \exists \vec{\alpha}^-. P$ and $\exists \vec{\alpha}^-. P$ is normalized. By the induction hypothesis applied to $\Gamma, \vec{\alpha}^-; \Phi, x : P \models c' : N$, we have $\Gamma, \vec{\alpha}^-; \Phi, x : P \vdash c' : N$.

To show $\Gamma; \Phi \vdash \mathbf{let}^3(\vec{\alpha}^-, x) = v; c' : N$, we apply the corresponding declarative rule Rule DTUnpack. Let us show that the premises hold:

1. $\Gamma; \Phi \vdash v : \exists \alpha^{\rightarrow}. P$ holds by the induction hypothesis, as noted above,
2. $\mathbf{nf}(\exists \alpha^{\rightarrow}. P) = \exists \alpha^{\rightarrow}. P$ holds since $\exists \alpha^{\rightarrow}. P$ is normalized,
3. $\Gamma, \alpha^{\rightarrow}; \Phi, x : P \vdash c' : N$ also holds by the induction hypothesis,
4. $\Gamma \vdash N$ holds by the inversion, as noted above.

Case 12. Rule ATEmptyApp Then by assumption:

- $\Gamma \vdash^{\supseteq} \Theta$,
- $\Gamma; \Theta \vdash N$ is free from negative metavariables,
- $\Gamma; \Phi; \Theta \models N \bullet \cdot \Rightarrow \mathbf{nf}(N) \equiv \Theta; \cdot$.

Let us show the required properties:

1. $\Gamma \vdash^{\supseteq} \Theta$ holds by assumption,
2. $\Theta \subseteq \Theta$ holds trivially,
3. $\mathbf{nf}(N)$ is evidently normalized, $\Gamma; \Theta \vdash N$ implies $\Gamma; \Theta \vdash \mathbf{nf}(N)$ by corollary 14, and lemma 21 means that $\mathbf{nf}(N)$ is inherently free from negative metavariables,
4. $\Theta \vdash \cdot$ holds trivially,
5. for any $\Theta \vdash \hat{\sigma} : \cdot$, we have $\Gamma; \Phi \vdash [\hat{\sigma}]N \bullet \cdot \Rightarrow [\hat{\sigma}]N$. To show $\Gamma; \Phi \vdash [\hat{\sigma}]N \bullet \cdot \Rightarrow [\hat{\sigma}]\mathbf{nf}(N)$, we apply the corresponding declarative rule Rule DTEmptyApp. To show $\Gamma \vdash [\hat{\sigma}]N \simeq_1^P [\hat{\sigma}]\mathbf{nf}(N)$, we apply the following sequence: $N \simeq_1^P \mathbf{nf}(N)$ by lemma 22, then $[\hat{\sigma}]N \simeq_1^P [\hat{\sigma}]\mathbf{nf}(N)$ by lemma 36, then $\Gamma \vdash [\hat{\sigma}]N \simeq_1^P [\hat{\sigma}]\mathbf{nf}(N)$ by lemma 30.

Case 13. Rule ATArrowApp By assumption:

1. $\Gamma \vdash^{\supseteq} \Theta$,
2. $\Gamma; \Theta \vdash Q \rightarrow N$ is free from negative metavariables, and hence, so are Q and N ,
3. $\Gamma; \Phi; \Theta \models Q \rightarrow N \bullet v, \vec{v} \Rightarrow M \equiv \Theta'; SC$, and by inversion:
 - (a) $\Gamma; \Phi \vdash v : P$, and by the induction hypothesis applied to this judgment, we have $\Gamma; \Phi \vdash v : P$, and $\Gamma \vdash P$;
 - (b) $\Gamma; \Theta \models Q \geq P \equiv SC_1$, and by the soundness of subtyping: $\Theta \vdash SC$ and for any $\Theta \vdash \hat{\sigma} : SC_1$, we have $\Gamma \vdash [\hat{\sigma}]Q \geq_1 P$;
 - (c) $\Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rightarrow M \equiv \Theta'; SC_2$, and by the induction hypothesis applied to this judgment,
 - i. $\Gamma \vdash^{\supseteq} \Theta'$,
 - ii. $\Theta \subseteq \Theta'$,
 - iii. $\Gamma; \Theta' \vdash M$ is normalized and free from negative metavariables,
 - iv. $\Theta' \vdash SC_2$,
 - v. for any $\Theta' \vdash \hat{\sigma} : SC_2$, we have $\Gamma; \Phi \vdash [\hat{\sigma}]N \bullet \vec{v} \Rightarrow [\hat{\sigma}]M$;
 - (d) $\Theta \vdash SC_1 \& SC_2 = SC$.

Let us show the required properties:

1. $\Gamma \vdash^{\supseteq} \Theta'$ is shown above,
2. $\Theta \subseteq \Theta'$ is shown above,
3. $\Gamma; \Theta' \vdash M$ is normalized and free from negative metavariables, as shown above,
4. $\Theta' \vdash SC$ holds: $\Theta \vdash SC_1$ and $\Theta \subseteq \Theta'$ imply $\Theta' \vdash SC_1$, then we apply the soundness of constraint merge (lemma 55) to $\Theta' \vdash SC_1 \& SC_2$:
 - (a) $\Theta' \vdash SC_1$,
 - (b) for any $\Theta' \vdash \hat{\sigma} : SC$, $\Theta' \vdash \hat{\sigma} : SC_i$ holds;
5. suppose that $\Theta' \vdash \hat{\sigma} : SC$. Then to show $\Gamma; \Phi \vdash [\hat{\sigma}](Q \rightarrow N) \bullet v, \vec{v} \Rightarrow [\hat{\sigma}]M$, that is $\Gamma; \Phi \vdash [\hat{\sigma}]Q \rightarrow [\hat{\sigma}]N \bullet v, \vec{v} \Rightarrow [\hat{\sigma}]M$, we apply the corresponding declarative rule Rule DTArrowApp. Let us show the required premises:
 - (a) $\Gamma; \Phi \vdash v : P$ holds as shown above,
 - (b) $\Gamma \vdash [\hat{\sigma}]Q \geq_1 P$ holds by the soundness of subtyping as noted above, since $\Theta' \vdash \hat{\sigma} : SC$ implies $\Theta \vdash \hat{\sigma} : SC_1$.
 - (c) $\Gamma; \Phi \vdash [\hat{\sigma}]N \bullet \vec{v} \Rightarrow [\hat{\sigma}]M$ holds by the induction hypothesis as shown above, since $\Theta' \vdash \hat{\sigma} : SC$ implies $\Theta' \vdash \hat{\sigma} : SC_2$.

Case 14. Rule ATForallApp

By assumption:

1. $\Gamma \vdash^{\supseteq} \Theta$,

2. $\Gamma; \Theta \vdash \forall \vec{\alpha}^+. N$ is free from negative metavariables,
3. $\Gamma; \Phi; \Theta \models \forall \vec{\alpha}^+. N \bullet \vec{v} \Rightarrow M \equiv \Theta'; SC$, which by inversion means $\vec{v} \neq \cdot$, $\vec{\alpha}^+ \neq \cdot$, and $\Gamma; \Phi; \Theta, \vec{\alpha}^+ \{\Gamma\} \models [\vec{\alpha}^+ / \vec{\alpha}^+] N \bullet \vec{v} \Rightarrow M \equiv \Theta'; SC$. It is easy to see that the induction hypothesis is applicable to the latter judgment:
 - $\Gamma \vdash^\exists \Theta, \vec{\alpha}^+ \{\Gamma\}$ holds by $\Gamma \vdash^\exists \Theta$,
 - $\Gamma; \Theta, \vec{\alpha}^+ \{\Gamma\} \vdash [\vec{\alpha}^+ / \vec{\alpha}^+] N$ holds since $\Gamma; \Theta \vdash \forall \vec{\alpha}^+. N$. $[\vec{\alpha}^+ / \vec{\alpha}^+] N$ is normalized and free from negative metavariables since so is N ;

This way, by the inductive hypothesis applied to $\Gamma; \Phi; \Theta, \vec{\alpha}^+ \{\Gamma\} \models [\vec{\alpha}^+ / \vec{\alpha}^+] N \bullet \vec{v} \Rightarrow M \equiv \Theta'; SC$, we have:

- (a) $\Gamma \vdash^\exists \Theta'$,
- (b) $\Theta, \vec{\alpha}^+ \{\Gamma\} \subseteq \Theta'$,
- (c) $\Gamma; \Theta' \vdash M$ is normalized and free from negative metavariables,
- (d) $\Theta' \vdash SC$,
- (e) for any $\Theta' \vdash \hat{\sigma} : SC$, we have $\Gamma; \Phi \vdash [\hat{\sigma}][\vec{\alpha}^+ / \vec{\alpha}^+] N \bullet \vec{v} \Rightarrow [\hat{\sigma}] M$.

Let us show the required properties:

1. $\Gamma \vdash^\exists \Theta'$ is shown above;
2. $\Theta \subseteq \Theta'$ since $\Theta, \vec{\alpha}^+ \{\Gamma\} \subseteq \Theta'$;
3. $\Gamma; \Theta' \vdash M$ is normalized and free from negative metavariables, as shown above;
4. $\Theta' \vdash SC$ is shown above;
5. let us assume $\Theta' \vdash \hat{\sigma} : SC$. Then to show $\Gamma; \Phi \vdash [\hat{\sigma}] \forall \vec{\alpha}^+. N \bullet \vec{v} \Rightarrow [\hat{\sigma}] M$, we apply the corresponding declarative rule Rule DTForallApp with substitution $\Gamma \vdash \sigma : \vec{\alpha}^+$ defined in the following way: $[\sigma] \alpha_i^+ = [\hat{\sigma}] \hat{\alpha}_i^+$.
Let us show that the premises of Rule DTForallApp hold:
 - (a) $\Gamma \vdash \sigma : \vec{\alpha}^+$, i.e. $\Gamma \vdash [\sigma] \alpha_i^+$ holds since $\Theta' \vdash \hat{\sigma}$ and $\Gamma \vdash^\exists \Theta'$;
 - (b) $\Gamma; \Phi \vdash [\sigma][\hat{\sigma}] N \bullet \vec{v} \Rightarrow [\hat{\sigma}] M$ holds by rewriting it into $\Gamma; \Phi \vdash [\hat{\sigma}][\vec{\alpha}^+ / \vec{\alpha}^+] N \bullet \vec{v} \Rightarrow [\hat{\sigma}] M$ using equality $[\hat{\sigma}][\vec{\alpha}^+ / \vec{\alpha}^+] N = [\sigma][\hat{\sigma}] N$:
 - i. for $\alpha_i^+ \in \vec{\alpha}^+$, $[\hat{\sigma}][\vec{\alpha}^+ / \vec{\alpha}^+] \alpha_i^+ = [\hat{\sigma}] \hat{\alpha}_i^+ = [\sigma] \alpha_i^+ = [\sigma][\hat{\sigma}] \alpha_i^+$,
 - ii. for $\hat{\beta}^\pm \in \text{dom}(\hat{\sigma})$, $[\hat{\sigma}][\vec{\alpha}^+ / \vec{\alpha}^+] \hat{\beta}^\pm = [\hat{\sigma}] \hat{\beta}^\pm = [\sigma][\hat{\sigma}] \hat{\beta}^\pm$, where the latter equality holds since $\vec{\alpha}^+ \cap \Gamma = \emptyset$.
 - iii. $\vec{v} \neq \cdot$ and $\vec{\alpha}^+ \neq \cdot$ hold by assumption.

□

Lemma 71 (Completeness of Typing). *Suppose that $\Gamma \vdash \Phi$. For an inference tree T_1 ,*

- + If T_1 infers $\Gamma; \Phi \vdash v : P$ then $\Gamma; \Phi \models v : \mathbf{nf}(P)$
- If T_1 infers $\Gamma; \Phi \vdash c : N$ then $\Gamma; \Phi \models c : \mathbf{nf}(N)$
- Suppose that T_1 infers $\Gamma; \Phi \vdash [\hat{\sigma}] N \bullet \vec{v} \Rightarrow M$ for some $\Gamma \vdash^\exists \Theta$, $\Gamma; \Theta \vdash N$ (free from negative metavariables, that is $\hat{\alpha}^- \notin \mathbf{uv} N$), $\Theta \vdash \hat{\sigma}$, and $\Gamma \vdash M$. Then there exist M' , Θ' , and SC such that
 1. $\Gamma; \Phi; \Theta \models N \bullet \vec{v} \Rightarrow M' \equiv \Theta'; SC$ and
 2. for any $\Theta \vdash \hat{\sigma}$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\hat{\sigma}] N \bullet \vec{v} \Rightarrow M$, there exists $\hat{\sigma}'$ such that
 - (a) $\Theta' \vdash \hat{\sigma}' : SC$,
 - (b) $\Theta \vdash \hat{\sigma}' \simeq_1^< \hat{\sigma} : \text{dom}(\Theta)$, and
 - (c) $\Gamma \vdash [\hat{\sigma}'] M' \simeq_1^< M$.

Proof. We prove it by induction on $\text{metric}(T_1)$, mutually with the soundness of typing (lemma 70). Let us consider the last rule applied to infer the derivation.

Case 1. Rule DTThunk

Then we are proving that if $\Gamma; \Phi \vdash \{c\} : \downarrow N$ (inferred by Rule DTThunk) then $\Gamma; \Phi \models \{c\} : \mathbf{nf}(\downarrow N)$. By inversion of $\Gamma; \Phi \vdash \{c\} : \downarrow N$, we have $\Gamma; \Phi \vdash c : N$, which we apply the induction hypothesis to to obtain $\Gamma; \Phi \models c : \mathbf{nf}(N)$. Then by Rule ATThunk, we have $\Gamma; \Phi \models \{c\} : \downarrow \mathbf{nf}(N)$. It is left to notice that $\downarrow \mathbf{nf}(N) = \mathbf{nf}(\downarrow N)$.

Case 2. Rule DTReturn

The proof is symmetric to the previous case (case 1).

Case 3. Rule DTPAnnot

Then we are proving that if $\Gamma; \Phi \vdash (v : Q) : Q$ is inferred by Rule DTPAnnot then $\Gamma; \Phi \models (v : Q) : \mathbf{nf}(Q)$. By inversion, we have:

1. $\Gamma \vdash Q$;
2. $\Gamma; \Phi \vdash v : P$, which by the induction hypothesis implies $\Gamma; \Phi \models v : \mathbf{nf}(P)$;
3. $\Gamma \vdash Q \geq_1 P$, and by transitivity, $\Gamma \vdash Q \geq_1 \mathbf{nf}(P)$; Since Q is ground, we have $\Gamma; \cdot \vdash Q$ and $\Gamma \vdash [\cdot] Q \geq_1 \mathbf{nf}(P)$. Then by the completeness of subtyping (??), we have $\Gamma; \cdot \models Q \geq \mathbf{nf}(P) \Rightarrow SC$, where $\cdot \vdash SC$ (implying $SC = \cdot$). This way, $\Gamma; \cdot \models Q \geq \mathbf{nf}(P) \Rightarrow \cdot$.

Then we can apply Rule ATPAnnot to $\Gamma \vdash Q$, $\Gamma; \Phi \models v : \mathbf{nf}(P)$ and $\Gamma; \cdot \models Q \geq \mathbf{nf}(P) \Rightarrow \cdot$ to infer $\Gamma; \Phi \models (v : Q) : \mathbf{nf}(Q)$.

Case 4. Rule DTNAnnot

The proof is symmetric to the previous case (case 3).

Case 5. Rule DTtLam

Then we are proving that if $\Gamma; \Phi \vdash \lambda x : P.c : P \rightarrow N$ is inferred by Rule DTtLam, then $\Gamma; \Phi \models \lambda x : P.c : \mathbf{nf}(P \rightarrow N)$.

By inversion of $\Gamma; \Phi \vdash \lambda x : P.c : P \rightarrow N$, we have $\Gamma \vdash P$ and $\Gamma; \Phi, x : P \vdash c : N$. Then by the induction hypothesis, $\Gamma; \Phi, x : P \models c : \mathbf{nf}(N)$. By Rule ATtLam, we infer $\Gamma; \Phi \models \lambda x : P.c : \mathbf{nf}(P \rightarrow \mathbf{nf}(N))$. By idempotence of normalization (lemma 25), $\mathbf{nf}(P \rightarrow \mathbf{nf}(N)) = \mathbf{nf}(P \rightarrow N)$, which concludes the proof for this case.

Case 6. Rule DTTLam

Then we are proving that if $\Gamma; \Phi \vdash \Lambda \alpha^+.c : \forall \alpha^+.N$ is inferred by Rule DTTLam, then $\Gamma; \Phi \models \Lambda \alpha^+.c : \mathbf{nf}(\forall \alpha^+.N)$. Similar to the previous case, by inversion of $\Gamma; \Phi \vdash \Lambda \alpha^+.c : \forall \alpha^+.N$, we have $\Gamma, \alpha^+; \Phi \vdash c : N$, and then by the induction hypothesis, $\Gamma, \alpha^+; \Phi \models c : \mathbf{nf}(N)$. After that, application of Rule ATTLam, gives as $\Gamma; \Phi \models \Lambda \alpha^+.c : \mathbf{nf}(\forall \alpha^+.\mathbf{nf}(N))$.

It is left to show that $\mathbf{nf}(\forall \alpha^+.\mathbf{nf}(N)) = \mathbf{nf}(\forall \alpha^+.N)$. Assume $N = \forall \beta^+.M$ (where M does not start with \forall).

- Then by definition, $\mathbf{nf}(\forall \alpha^+.N) = \mathbf{nf}(\forall \alpha^+.\beta^+.M) = \forall \gamma^+.\mathbf{nf}(M)$, where $\mathbf{ord} \alpha^+.\beta^+ \text{ in } \mathbf{nf}(M) = \gamma^+$.
- On the other hand, $\mathbf{nf}(N) = \forall \gamma^{+\prime}.\mathbf{nf}(M)$, where $\mathbf{ord} \beta^+ \text{ in } \mathbf{nf}(M) = \gamma^{+\prime}$, and thus, $\mathbf{nf}(\forall \alpha^+.\mathbf{nf}(N)) = \mathbf{nf}(\forall \alpha^+.\gamma^{+\prime}.\mathbf{nf}(M)) = \forall \gamma^{+\prime\prime}.\mathbf{nf}(\mathbf{nf}(M)) = \forall \gamma^{+\prime\prime}.\mathbf{nf}(M)$, where $\mathbf{ord} \alpha^+.\gamma^{+\prime} \text{ in } \mathbf{nf}(\mathbf{nf}(M)) = \gamma^{+\prime\prime}$.

It is left to show that $\gamma^{+\prime\prime} = \gamma^+$.

$$\begin{aligned}
\gamma^{+\prime\prime} &= \mathbf{ord} \alpha^+.\gamma^{+\prime} \text{ in } \mathbf{nf}(\mathbf{nf}(M)) \\
&= \mathbf{ord} \alpha^+.\gamma^{+\prime} \text{ in } \mathbf{nf}(M) && \text{by idempotence (lemma 25)} \\
&= \mathbf{ord} \alpha^+ \cup \beta^+ \cap \mathbf{fv} \mathbf{nf}(M) \text{ in } \mathbf{nf}(M) && \text{by definition of } \gamma^{+\prime} \text{ and lemma 16} \\
&= \mathbf{ord} (\alpha^+ \cup \beta^+ \cap \mathbf{fv} \mathbf{nf}(M)) \cap \mathbf{fv} \mathbf{nf}(M) \text{ in } \mathbf{nf}(M) && \text{by corollary 11} \\
&= \mathbf{ord} (\alpha^+ \cup \beta^+) \cap \mathbf{fv} \mathbf{nf}(M) \text{ in } \mathbf{nf}(M) && \text{by set properties} \\
&= \mathbf{ord} \alpha^+.\beta^+ \text{ in } \mathbf{nf}(M) \\
&= \gamma^+
\end{aligned}$$

Case 7. Rule DTUnpack

Then we are proving that if $\Gamma; \Phi \vdash \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c : N$ is inferred by Rule DTUnpack, then $\Gamma; \Phi \models \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c : \mathbf{nf}(N)$.

By inversion of $\Gamma; \Phi \vdash \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c : N$, we have

1. $\mathbf{nf}(\exists \overrightarrow{\alpha^-}.P) = \exists \overrightarrow{\alpha^-}.P$,
2. $\Gamma; \Phi \vdash v : \exists \overrightarrow{\alpha^-}.P$, which by the induction hypothesis implies $\Gamma; \Phi \models v : \mathbf{nf}(\exists \overrightarrow{\alpha^-}.P)$, and hence, $\Gamma; \Phi \models v : \exists \overrightarrow{\alpha^-}.P$.
3. $\Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \vdash c : N$, and by the induction hypothesis, $\Gamma, \overrightarrow{\alpha^-}; \Phi, x : P \models c : \mathbf{nf}(N)$.
4. $\Gamma \vdash N$.

This way, we can apply Rule ATUnpack to infer $\Gamma; \Phi \models \mathbf{let}^{\exists}(\overrightarrow{\alpha^-}, x) = v; c : \mathbf{nf}(N)$.

Case 8. Rule DTPEquiv

Then we are proving that if $\Gamma; \Phi \vdash v : P'$ is inferred by Rule DTPEquiv, then $\Gamma; \Phi \models v : \mathbf{nf}(P')$. By inversion, $\Gamma; \Phi \vdash v : P$ and $\Gamma \vdash P \simeq_1^< P'$, and the metric of the tree inferring $\Gamma; \Phi \vdash v : P$ is less than the one inferring $\Gamma; \Phi \vdash v : P'$. Then by the induction hypothesis, $\Gamma; \Phi \models v : \mathbf{nf}(P)$.

By lemma 35 $\Gamma \vdash P \simeq_1^< P'$ implies $\mathbf{nf}(P) = \mathbf{nf}(P')$, and thus, $\Gamma; \Phi \models v : \mathbf{nf}(P)$ can be rewritten to $\Gamma; \Phi \models v : \mathbf{nf}(P')$.

Case 9. Rule DTVar

Then we are proving that $\Gamma; \Phi \vdash x : P$ implies $\Gamma; \Phi \models x : \mathbf{nf}(P)$. By inversion of $\Gamma; \Phi \vdash x : P$, we have $x : P \in \Phi$. Then Rule ATVar applies to infer $\Gamma; \Phi \models x : \mathbf{nf}(P)$.

Case 10. Rule DTVarLet

Then we are proving that $\Gamma; \Phi \vdash \mathbf{let} \ x = v(\vec{v}); c : N$ implies $\Gamma; \Phi \models \mathbf{let} \ x = v(\vec{v}); c : \mathbf{nf}(N)$.

By inversion of $\Gamma; \Phi \vdash \mathbf{let} \ x = v(\vec{v}); c : N$, we have

1. $\Gamma; \Phi \vdash v : P$, and by the induction hypothesis, $\Gamma; \Phi \models v : \mathbf{nf}(P)$.
2. $\Gamma; \Phi, x : P \vdash c : N$, and by lemma 69, since $\Gamma \vdash P \simeq_1^{\leq} \mathbf{nf}(P)$, we have $\Gamma; \Phi, x : \mathbf{nf}(P) \vdash c : N$. Then by the induction hypothesis, $\Gamma; \Phi, x : \mathbf{nf}(P) \models c : \mathbf{nf}(N)$.

Together, $\Gamma; \Phi \models v : \mathbf{nf}(P)$ and $\Gamma; \Phi, x : \mathbf{nf}(P) \models c : \mathbf{nf}(N)$ imply $\Gamma; \Phi \models \mathbf{let} \ x = v(\vec{v}); c : \mathbf{nf}(N)$ by Rule ATVarLet.

Case 11. Rule DTAAppLetAnn

Then we are proving that $\Gamma; \Phi \vdash \mathbf{let} \ x : P = v(\vec{v}); c : N$ implies $\Gamma; \Phi \models \mathbf{let} \ x : P = v(\vec{v}); c : \mathbf{nf}(N)$.

By inversion of $\Gamma; \Phi \vdash \mathbf{let} \ x : P = v(\vec{v}); c : N$, we have

1. $\Gamma \vdash P$
2. $\Gamma; \Phi \vdash v : \downarrow M$ for some ground M , which by the induction hypothesis means $\Gamma; \Phi \models v : \downarrow \mathbf{nf}(M)$
3. $\Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q$. By lemma 68, since $\Gamma \vdash M \simeq_1^{\leq} \mathbf{nf}(M)$, we have $\Gamma; \Phi \vdash [\cdot] \mathbf{nf}(M) \bullet \vec{v} \Rightarrow \uparrow Q$, which by the induction hypothesis means that there exist normalized M' , Θ , and SC_1 such that (noting that M is ground):
 - (a) $\Gamma; \Phi; \cdot \models \mathbf{nf}(M) \bullet \vec{v} \Rightarrow M' \equiv \Theta; SC_1$, where by the soundness, $\Gamma; \Theta \vdash M'$ and $\Theta \vdash SC_1$.
 - (b) for any $\Gamma \vdash M''$ such that $\Gamma; \Phi \vdash \mathbf{nf}(M) \bullet \vec{v} \Rightarrow M''$ there exists $\hat{\sigma}$ such that
 - i. $\Theta \vdash \hat{\sigma} : SC_1$, and
 - ii. $\Gamma \vdash [\hat{\sigma}] M' \simeq_1^{\leq} M''$,
 In particular, there exists $\Theta \vdash \hat{\sigma}_0 : SC_1$ such that $\Gamma \vdash [\hat{\sigma}_0] M' \simeq_1^{\leq} \uparrow Q$. Moreover, since uM' is normalized and free of negative metavariables, $M' = \uparrow Q_0$, and $\Gamma \vdash [\hat{\sigma}_0] Q_0 \simeq_1^{\leq} Q$.
4. $\Gamma \vdash \uparrow Q \leq_1 \uparrow P$, and by transitivity, since $\Gamma \vdash [\hat{\sigma}_0] \uparrow Q_0 \simeq_1^{\leq} \uparrow Q$, we have $\Gamma \vdash [\hat{\sigma}_0] \uparrow Q_0 \leq_1 \uparrow P$.
 Let us apply lemma 62 to $\Gamma \vdash [\hat{\sigma}_0] \uparrow Q_0 \leq_1 \uparrow P$ and obtain $\Theta \vdash SC_2$ such that
 - (a) $\Gamma; \Theta \models \uparrow Q_0 \leq \uparrow P \equiv SC_2$ and
 - (b) $\Theta \vdash \hat{\sigma}_0 : SC_2$.
5. $\Gamma; \Phi, x : P \vdash c : N$, and by the induction hypothesis, $\Gamma; \Phi, x : P \models c : \mathbf{nf}(N)$.

To infer $\Gamma; \Phi \models \mathbf{let} \ x : P = v(\vec{v}); c : \mathbf{nf}(N)$, we apply the corresponding algorithmic rule Rule ATAppLetAnn. Let us show that the premises hold:

1. $\Gamma \vdash P$,
2. $\Gamma; \Phi \models v : \downarrow \mathbf{nf}(M)$,
3. $\Gamma; \Phi; \cdot \models \mathbf{nf}(M) \bullet \vec{v} \Rightarrow \uparrow Q_0 \equiv \Theta; SC_1$,
4. $\Gamma; \Theta \models \uparrow Q_0 \leq \uparrow P \equiv SC_2$, and
5. $\Gamma; \Phi, x : P \models c : \mathbf{nf}(N)$ hold as noted above;
6. $\Theta \vdash SC_1 \& SC_2 = SC$ is defined by lemma 57, since $\Theta \vdash \hat{\sigma}_0 : SC_1$ and $\Theta \vdash \hat{\sigma}_0 : SC_2$.

Case 12. Rule DTAAppLet

By assumption, c is $\mathbf{let} \ x = v(\vec{v}); c'$. Then by inversion of $\Gamma; \Phi \vdash \mathbf{let} \ x = v(\vec{v}); c' : N$:

- $\Gamma; \Phi \vdash v : \downarrow M$, which by the induction hypothesis means $\Gamma; \Phi \models v : \downarrow \mathbf{nf}(M)$;
 - $\Gamma; \Phi \vdash M \bullet \vec{v} \Rightarrow \uparrow Q$ unique. Then by lemma 68, since $\Gamma \vdash M \simeq_1^{\leq} \mathbf{nf}(M)$, we have $\Gamma; \Phi \vdash \mathbf{nf}(M) \bullet \vec{v} \Rightarrow \uparrow Q$ and moreover, $\Gamma; \Phi \vdash \mathbf{nf}(M) \bullet \vec{v} \Rightarrow \uparrow Q$ unique (since symmetrically, $\mathbf{nf}(M)$ can be replaced back by M). Then the induction hypothesis applied to $\Gamma; \Phi \vdash [\cdot] \mathbf{nf}(M) \bullet \vec{v} \Rightarrow \uparrow Q$ implies that there exist M' , Θ , and SC such that (considering M is ground):
 1. $\Gamma; \Phi; \cdot \models \mathbf{nf}(M) \bullet \vec{v} \Rightarrow M' \equiv \Theta; SC$, which, by the soundness, implies, in particular that
 - (a) M' is normalized and free of negative metavariables,
 - (b) for any $\Theta \vdash \hat{\sigma} : SC$, we have $\Gamma; \Phi \vdash \mathbf{nf}(M) \bullet \vec{v} \Rightarrow [\hat{\sigma}] M'$, which, since $\Gamma; \Phi \vdash \mathbf{nf}(M) \bullet \vec{v} \Rightarrow \uparrow Q$ unique, means $\Gamma \vdash [\hat{\sigma}] M' \simeq_1^{\leq} \uparrow Q$.
- and

2. for any $\Gamma \vdash M''$ such that $\Gamma; \Phi \vdash \mathbf{nf}(M) \bullet \vec{v} \Rightarrow M''$, (and in particular, for $\Gamma \vdash \uparrow Q$) there exists $\hat{\sigma}_1$ such that
 - (a) $\Theta \vdash \hat{\sigma}_1 : SC$, and
 - (b) $\Gamma \vdash [\hat{\sigma}_1]M' \simeq_1^{\leq} M''$, and in particular, $\Gamma \vdash [\hat{\sigma}_1]M' \simeq_1^{\leq} \uparrow Q$. Since M' is normalized and free of negative metavariables, it means that $M' = \uparrow P$ for some P , that is $\Gamma \vdash [\hat{\sigma}_1]P \simeq_1^{\leq} Q$.
- $\Gamma; \Phi, x : Q \vdash c' : N$

To infer $\Gamma; \Phi \models \mathbf{let} \ x = v(\vec{v}); c' : \mathbf{nf}(N)$, let us apply the corresponding algorithmic rule (Rule ATAppLet):

1. $\Gamma; \Phi \models v : \downarrow \mathbf{nf}(M)$ holds as noted above;
2. $\Gamma; \Phi; \cdot \models \mathbf{nf}(M) \bullet \vec{v} \Rightarrow \uparrow P \equiv \Theta; SC$ holds as noted above;
3. To show $\mathbf{uv} P \subseteq \mathbf{dom}(SC)$ and $\mathbf{SC}|_{\mathbf{uv} P}$ **singular with** $\hat{\sigma}_0$ (for some $\hat{\sigma}_0$), we apply lemma 67. Let us show that the premise of this lemma holds.
As noted in 1b, for any $\hat{\sigma}$, $\Theta \vdash \hat{\sigma} : SC$ implies $\Gamma \vdash [\hat{\sigma}]M' \simeq_1^{\leq} \uparrow Q$, which is rewritten as $\Gamma \vdash [\hat{\sigma}]P \simeq_1^{\leq} Q$. And since $\Gamma \vdash [\hat{\sigma}']P \simeq_1^{\leq} Q$, we have $\Gamma \vdash [\hat{\sigma}]P \simeq_1^{\leq} [\hat{\sigma}']P$. It implies $\Theta \vdash \hat{\sigma} \simeq_1^{\leq} \hat{\sigma}' : \mathbf{uv} P$ by lemma 3.
4. Let us show $\Gamma; \Phi, x : [\hat{\sigma}_0]P \models c' : \mathbf{nf}(N)$. By the soundness of singularity (lemma 65), we have $\Theta \vdash \hat{\sigma}_0 : SC$, which by 1b means $\Gamma \vdash [\hat{\sigma}_0]M' \simeq_1^{\leq} \uparrow Q$, that is $\Gamma \vdash [\hat{\sigma}_0]P \simeq_1^{\leq} Q$, and thus, $\Gamma \vdash \Phi, x : Q \simeq_1^{\leq} \Phi, x : [\hat{\sigma}_0]P$.
Then by lemma 69, $\Gamma; \Phi, x : Q \vdash c' : N$ can be rewritten as $\Gamma; \Phi, x : [\hat{\sigma}_0]P \vdash c' : N$. Then by the induction hypothesis applied to it, $\Gamma; \Phi, x : [\hat{\sigma}_0]P \models c' : \mathbf{nf}(N)$ holds.

Case 13. Rule DTForallApp

Since N cannot be a metavariable, if $[\hat{\sigma}]N$ starts with \forall , so does N . This way, $N = \forall \alpha^+. N_1$. Then by assumption:

1. $\Gamma \vdash^{\supset} \Theta$
2. $\Gamma; \Theta \vdash \forall \alpha^+. N_1$ is free from negative metavariables, and then $\Gamma, \alpha^+; \Theta \vdash N_1$ is free from negative metavariables too;
3. $\Theta \vdash \hat{\sigma}$;
4. $\Gamma \vdash M$;
5. $\Gamma; \Phi \vdash [\hat{\sigma}] \forall \alpha^+. N_1 \bullet \vec{v} \Rightarrow M$, that is $\Gamma; \Phi \vdash (\forall \alpha^+. [\hat{\sigma}]N_1) \bullet \vec{v} \Rightarrow M$. Then by inversion there exists σ such that
 - (a) $\Gamma \vdash \sigma : \alpha^+$;
 - (b) $\vec{v} \neq \cdot$ and $\alpha^+ \neq \cdot$; and
 - (c) $\Gamma; \Phi \vdash [\sigma][\hat{\sigma}]N_1 \bullet \vec{v} \Rightarrow M$. Notice that σ and $\hat{\sigma}$ commute because the codomain of σ does not contain metavariables (and thus, does not intersect with the domain of $\hat{\sigma}$), and the codomain of $\hat{\sigma}$ is Γ and does not intersect with α^+ —the domain of σ .
Let us construct $N_0 = [\hat{\alpha}^+/\alpha^+]N_1$ and $\Theta, \hat{\alpha}^+\{\Gamma\} \vdash \hat{\sigma}_0$ defined as

$$\begin{cases} [\hat{\sigma}_0]\hat{\alpha}_i^+ = [\sigma]\alpha_i^+ & \text{for } \hat{\alpha}_i^+ \in \hat{\alpha}^+ \\ [\hat{\sigma}_0]\hat{\beta}^\pm = [\hat{\sigma}]\hat{\beta}^\pm & \text{for } \hat{\beta}^\pm \in \mathbf{dom}(\Theta) \end{cases}$$

Then it is easy to see that $[\hat{\sigma}_0][\hat{\alpha}^+/\alpha^+]N_1 = [\sigma][\hat{\sigma}]N_1$ because this substitution compositions coincide on $\alpha^+ \cup \mathbf{dom}(\Theta)$, their domain. In other words, $[\hat{\sigma}_0]N_0 = [\sigma][\hat{\sigma}]N_1$.

Then let us apply the induction hypothesis to $\Gamma; \Phi \vdash [\hat{\sigma}_0]N_0 \bullet \vec{v} \Rightarrow M$ and obtain M' , Θ' , and SC such that

- $\Gamma; \Phi; \Theta, \hat{\alpha}^+\{\Gamma\} \models N_0 \bullet \vec{v} \Rightarrow M' \equiv \Theta'; SC$ and
- for any $\Theta, \hat{\alpha}^+\{\Gamma\} \vdash \hat{\sigma}_0$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\hat{\sigma}_0]N_0 \bullet \vec{v} \Rightarrow M$, there exists $\hat{\sigma}'_0$ such that
 - i. $\Theta \vdash \hat{\sigma}'_0 : SC$,
 - ii. $\Theta, \hat{\alpha}^+\{\Gamma\} \vdash \hat{\sigma}'_0 \simeq_1^{\leq} \hat{\sigma}_0 : \mathbf{dom}(\Theta) \cup \hat{\alpha}^+$, and
 - iii. $\Gamma \vdash [\hat{\sigma}'_0]M' \simeq_1^{\leq} M$.

Let us take M' , Θ' , and SC from the induction hypothesis (5c) and show they satisfy the required properties.

1. to infer $\Gamma; \Phi; \Theta \models \forall \alpha^+. N_1 \bullet \vec{v} \Rightarrow M' \equiv \Theta'; SC$ we apply the corresponding algorithmic rule Rule ATForallApp, not that the required premises hold, as noted above:
 - (a) $\vec{v} \neq \cdot$, $\alpha^+ \neq \cdot$; and
 - (b) $\Gamma; \Phi; \Theta, \hat{\alpha}^+\{\Gamma\} \models [\hat{\alpha}^+/\alpha^+]N_1 \bullet \vec{v} \Rightarrow M' \equiv \Theta'; SC$ is obtained by unfolding the definition of N_0 in $\Gamma; \Phi; \Theta, \hat{\alpha}^+\{\Gamma\} \models N_0 \bullet \vec{v} \Rightarrow M' \equiv \Theta'; SC$ (5c).

2. Let us take an arbitrary $\Theta \vdash \hat{\sigma}$ and $\Gamma \vdash M$ and assume $\Gamma; \Phi \vdash [\hat{\sigma}] \forall \alpha^+ . N_1 \bullet \vec{v} \Rightarrow M$. Then the same reasoning as in 5c applies. In particular, we construct $\Theta, \hat{\alpha}^+ \{\Gamma\} \vdash \hat{\sigma}_0$ as an extension of $\hat{\sigma}$ and obtain $\Gamma; \Phi \vdash [\hat{\sigma}_0] N_0 \bullet \vec{v} \Rightarrow M$. It means, we can apply the property inferred from the induction hypothesis (5c) to obtain $\hat{\sigma}'_0$ such that
 - (a) $\Theta' \vdash \hat{\sigma}'_0 : SC$,
 - (b) $\Theta, \hat{\alpha}^+ \{\Gamma\} \vdash \hat{\sigma}'_0 \simeq_1^{\leq} \hat{\sigma}_0 : \mathbf{dom}(\Theta) \cup \hat{\alpha}^+$, and
 - (c) $\Gamma \vdash [\hat{\sigma}'_0] M' \simeq_1^{\leq} M$.

Let us show that the obtained $\hat{\sigma}'_0$ satisfies the required properties.

- (a) $\Theta' \vdash \hat{\sigma}'_0 : SC$ holds as shown,
- (b) $\Gamma \vdash [\hat{\sigma}'_0] M' \simeq_1^{\leq} M$ holds as shown,
- (c) $\Theta \vdash \hat{\sigma}_0 \simeq_1^{\leq} \hat{\sigma} : \mathbf{dom}(\Theta)$, holds. Let us take an arbitrary $\hat{\beta}^\pm \in \mathbf{dom}(\Theta) \subseteq \mathbf{dom}(\Theta) \cup \hat{\alpha}^+$. Then since $\Theta, \hat{\alpha}^+ \{\Gamma\} \vdash \hat{\sigma}'_0 \simeq_1^{\leq} \hat{\sigma}_0 : \mathbf{dom}(\Theta) \cup \hat{\alpha}^+$, we have $[\hat{\sigma}'_0] \hat{\beta}^\pm = [\hat{\sigma}_0] \hat{\beta}^\pm$ and by definition of $\hat{\sigma}_0$, $[\hat{\sigma}_0] \hat{\beta}^\pm = [\hat{\sigma}] \hat{\beta}^\pm$.

Case 14. Rule DTArrowApp

Since N cannot be a metavariable, if the shape of $[\hat{\sigma}]N$ is an arrow, so is the shape of N . This way, $N = Q \rightarrow N_1$. Then by assumption:

1. $\Gamma \vdash^\supset \Theta$;
2. $\Gamma; \Theta \vdash Q \rightarrow N_1$ is free from negative metavariables;
3. $\Theta \vdash \hat{\sigma}$;
4. $\Gamma \vdash M$;
5. $\Gamma; \Phi \vdash [\hat{\sigma}](Q \rightarrow N_1) \bullet v, \vec{v} \Rightarrow M$, that is $\Gamma; \Phi \vdash ([\hat{\sigma}]Q \rightarrow [\hat{\sigma}]N_1) \bullet v, \vec{v} \Rightarrow M$, and by inversion:
 - (a) $\Gamma; \Phi \vdash v : P$, and by the induction hypothesis, $\Gamma; \Phi \models v : \mathbf{nf}(P)$;
 - (b) $\Gamma \vdash [\hat{\sigma}]Q \geq_1 P$, which by transitivity (lemma 13) means $\Gamma \vdash [\hat{\sigma}]Q \geq_1 P'$, and then by completeness of subtyping (lemma 53), $\Gamma; \Theta \models Q \geq P' \equiv SC_1$, for some $\Theta \vdash SC_1$, and moreover, $\Theta \vdash \hat{\sigma} : SC_1$;
 - (c) $\Gamma; \Phi \vdash [\hat{\sigma}]N_1 \bullet \vec{v} \Rightarrow M$. Notice that the induction hypothesis is applicable to this case: $\Gamma; \Theta \vdash N_1$ is free from negative metavariables because so is $Q \rightarrow N_1$. This way, there exist M' , Θ' , and SC_2 such that
 - i. $\Gamma; \Phi; \Theta \models N_1 \bullet \vec{v} \Rightarrow M' \equiv \Theta'; SC_2$ and then by the soundness of typing (i.e. the induction hypothesis),
 - A. $\Theta \subseteq \Theta'$
 - B. $\Gamma; \Theta' \vdash M'$
 - ii. for any $\Theta \vdash \hat{\sigma}$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\hat{\sigma}]N_1 \bullet \vec{v} \Rightarrow M$, there exists $\hat{\sigma}'$ such that
 - A. $\Theta' \vdash \hat{\sigma}' : SC_2$,
 - B. $\Theta \vdash \hat{\sigma}' \simeq_1^{\leq} \hat{\sigma} : \mathbf{dom}(\Theta)$, and
 - C. $\Gamma \vdash [\hat{\sigma}']M' \simeq_1^{\leq} M$.

Let us take $\Theta \vdash \hat{\sigma}$ and M and construct $\Theta' \vdash \hat{\sigma}'$ by the induction hypothesis (5(c)ii). Then $\Theta' \vdash \hat{\sigma}' : SC_2$ and $\Theta' \vdash \hat{\sigma}' : SC_1$ holds and since $\Theta \vdash \hat{\sigma} : SC_1$ and $\Theta \vdash \hat{\sigma}' \simeq_1^{\leq} \hat{\sigma} : \mathbf{dom}(\Theta)$. Then by the completeness of constraint merge (lemma 57), $\Theta' \vdash SC_1 \& SC_2 = SC$ exists, $\Theta' \vdash SC$, and $\Theta' \vdash \hat{\sigma} : SC$.

To show the required properties, we take M' and Θ' from the induction hypothesis (5(c)ii), and SC defined above. Then

1. $\Gamma; \Phi; \Theta \models Q \rightarrow N_1 \bullet v, \vec{v} \Rightarrow M' \equiv \Theta'; SC$ is inferred by Rule ATArrowApp:
 - (a) $\Gamma; \Phi \models v : P'$ as noted above,
 - (b) $\Gamma; \Theta \models Q \geq P' \equiv SC_1$ as noted above,
 - (c) $\Gamma; \Phi; \Theta \models N_1 \bullet \vec{v} \Rightarrow M' \equiv \Theta'; SC_2$ as noted above;
2. let us take an arbitrary $\Theta \vdash \hat{\sigma}_0$ and $\Gamma \vdash M_0$ such that $\Gamma; \Phi \vdash [\hat{\sigma}_0](Q \rightarrow N_1) \bullet v, \vec{v} \Rightarrow M_0$. Then by inversion of $\Gamma; \Phi \vdash [\hat{\sigma}_0]Q \rightarrow [\hat{\sigma}_0]N_1 \bullet v, \vec{v} \Rightarrow M_0$, we have the same properties as in 5. In particular,
 - $\Gamma; \Phi \vdash [\hat{\sigma}_0]N_1 \bullet \vec{v} \Rightarrow M_0$. Then by 5(c)ii, there exists $\hat{\sigma}'$ such that
 - (a) $\Theta' \vdash \hat{\sigma}' : SC_2$,
 - (b) $\Theta \vdash \hat{\sigma}' \simeq_1^{\leq} \hat{\sigma}_0 : \mathbf{dom}(\Theta)$, and
 - (c) $\Gamma \vdash [\hat{\sigma}']M' \simeq_1^{\leq} M_0$.
 - $\Gamma \vdash [\hat{\sigma}_0]Q \geq_1 P'$ and by the completeness of subtyping (lemma 53), $\Theta \vdash \hat{\sigma}_0 : SC_1$.

This way,

- $\Theta \vdash \hat{\sigma}' \simeq_1^{\leq} \hat{\sigma}_0 : \mathbf{dom}(\Theta)$ holds as noted above;

- $\Theta' \vdash \hat{\sigma}' : SC_1$ holds because $\Theta \vdash \hat{\sigma}_0 : SC_1$ and $\Theta \vdash \hat{\sigma}' \simeq_1^{\leq} \hat{\sigma}_0 : \mathbf{dom}(\Theta)$, and $\Theta' \vdash \hat{\sigma}' : SC_1$ together with $\Theta' \vdash \hat{\sigma}' : SC_2$ implies $\Theta' \vdash \hat{\sigma}' : SC$ by the completeness of constraint merge (lemma 57); and
- $\Gamma \vdash [\hat{\sigma}'] M' \simeq_1^{\leq} M_0$ holds as noted above.

Case 15. Rule DTEmptyApp

By assumption:

1. $\Gamma \vdash^= \Theta$,
2. $\Gamma; \Theta \vdash N$,
3. $\Theta \vdash \hat{\sigma}$,
4. $\Gamma; \Phi \vdash [\hat{\sigma}] N \bullet \cdot \Rightarrow [\hat{\sigma}] N$.

Then we can apply the corresponding algorithmic rule Rule ATEmptyApp to infer $\Gamma; \Phi; \Theta \models N \bullet \cdot \Rightarrow \mathbf{nf}(N) \Leftarrow \Theta; \cdot$. Let us show the required properties. Let us take an arbitrary $\Theta \vdash \hat{\sigma}_1$ and $\Gamma \vdash M$ such that $\Gamma; \Phi \vdash [\hat{\sigma}_1] N \bullet \cdot \Rightarrow M$. Then we can take $\hat{\sigma}' = \hat{\sigma}_1$:

1. $\Theta \vdash \hat{\sigma}' : \cdot$ holds vacuously,
2. $\Theta \vdash \hat{\sigma}' \simeq_1^{\leq} \hat{\sigma}_1 : \mathbf{dom}(\Theta)$ holds by reflexivity of equivalence,
3. Let us show $\Gamma \vdash [\hat{\sigma}'] \mathbf{nf}(N) \simeq_1^{\leq} M$. Notice that $\Gamma; \Phi \vdash [\hat{\sigma}_1] N \bullet \cdot \Rightarrow M$ can only be inferred by Rule DTEmptyApp, and thus, $\Gamma \vdash [\hat{\sigma}_1] N \simeq_1^{\leq} M$. By corollary 17, $\Gamma \vdash [\hat{\sigma}_1] N \simeq_1^{\leq} [\hat{\sigma}_1] \mathbf{nf}(N)$, and then by transitivity, $\Gamma \vdash [\hat{\sigma}_1] \mathbf{nf}(N) \simeq_1^{\leq} M$, that is $\Gamma \vdash [\hat{\sigma}'] \mathbf{nf}(N) \simeq_1^{\leq} M$.

□