

1 The Vanilla System

First, we present the top-level system, which is easy to understand.

1.1 Grammar

P, Q	$::=$	positive types
	$ $	$a+$
	$ $	$\downarrow N$
	$ $	$\exists \alpha^-. P$
N, M	$::=$	negative types
	$ $	$a-$
	$ $	$\uparrow P$
	$ $	$\forall \alpha^+. N$
	$ $	$P \rightarrow N$

1.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_0^{\leq} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leq_0 M \quad \Gamma \vdash M \leq_0 N}{\Gamma \vdash N \simeq_0^{\leq} M} \quad \text{D0NDEF}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leq} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geq_0 Q \quad \Gamma \vdash Q \geq_0 P}{\Gamma \vdash P \simeq_0^{\leq} Q} \quad \text{D0PDEF}$$

$\boxed{\Gamma \vdash N \leq_0 M}$ Negative subtyping

$$\begin{array}{c} \overline{\Gamma \vdash a- \leq_0 a-} \quad \text{D0NVAR} \\ \frac{\Gamma \vdash P \simeq_0^{\leq} Q}{\Gamma \vdash \uparrow P \leq_0 \uparrow Q} \quad \text{D0SHIFTU} \\ \frac{\Gamma \vdash P \quad \Gamma \vdash [P/a+]N \leq_0 M \quad M \neq \forall \beta^+. M'}{\Gamma \vdash \forall \alpha^+. N \leq_0 M} \quad \text{D0FORALLL} \\ \frac{\Gamma, \alpha^+ \vdash N \leq_0 M}{\Gamma \vdash N \leq_0 \forall \alpha^+. M} \quad \text{D0FORALLR} \\ \frac{\Gamma \vdash P \geq_0 Q \quad \Gamma \vdash N \leq_0 M}{\Gamma \vdash P \rightarrow N \leq_0 Q \rightarrow M} \quad \text{D0ARROW} \end{array}$$

$\boxed{\Gamma \vdash P \geq_0 Q}$ Positive supertyping

$$\begin{array}{c} \overline{\Gamma \vdash a+ \geq_0 a+} \quad \text{D0PVAR} \\ \frac{\Gamma \vdash N \simeq_0^{\leq} M}{\Gamma \vdash \downarrow N \geq_0 \downarrow M} \quad \text{D0SHIFTD} \\ \frac{\Gamma \vdash N \quad \Gamma \vdash [N/a-]P \geq_0 Q \quad Q \neq \exists \alpha^-. Q'}{\Gamma \vdash \exists \alpha^-. P \geq_0 Q} \quad \text{D0EXISTSL} \\ \frac{\Gamma, \alpha^- \vdash P \geq_0 Q}{\Gamma \vdash P \geq_0 \exists \alpha^-. Q} \quad \text{D0EXISTSR} \end{array}$$

2 Multi-Quantified System

2.1 Grammar

P, Q	$::=$	multi-quantified positive types
	α^+	
	$\downarrow N$	
	$\exists \overrightarrow{\alpha^-}.P$	$P \neq \exists \dots$
	(P)	S
N, M	$::=$	multi-quantified negative types
	α^-	
	$\uparrow P$	
	$P \rightarrow N$	
	$\forall \overrightarrow{\alpha^+}.N$	$N \neq \forall \dots$
	(N)	S

2.2 Declarative Subtyping

$\boxed{\Gamma \vdash N \simeq_1^{\leq} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leq_1 M \quad \Gamma \vdash M \leq_1 N}{\Gamma \vdash N \simeq_1^{\leq} M} \quad \text{D1NDEF}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leq} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash Q \geq_1 P}{\Gamma \vdash P \simeq_1^{\leq} Q} \quad \text{D1PDEF}$$

$\boxed{\Gamma \vdash N \leq_1 M}$ Negative subtyping

$$\begin{array}{c} \overline{\Gamma \vdash \alpha^- \leq_1 \alpha^-} \quad \text{D1NVAR} \\ \frac{\Gamma \vdash P \simeq_1^{\leq} Q}{\Gamma \vdash \uparrow P \leq_1 \uparrow Q} \quad \text{D1SHIFTU} \\ \frac{\Gamma \vdash P \geq_1 Q \quad \Gamma \vdash N \leq_1 M}{\Gamma \vdash P \rightarrow N \leq_1 Q \rightarrow M} \quad \text{D1ARROW} \\ \frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leq_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leq_1 \forall \overrightarrow{\beta^+}.M} \quad \text{D1FORALL} \end{array}$$

$\boxed{\Gamma \vdash P \geq_1 Q}$ Positive supertyping

$$\begin{array}{c} \overline{\Gamma \vdash \alpha^+ \geq_1 \alpha^+} \quad \text{D1PVAR} \\ \frac{\Gamma \vdash N \simeq_1^{\leq} M}{\Gamma \vdash \downarrow N \geq_1 \downarrow M} \quad \text{D1SHIFTD} \\ \frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geq_1 Q'}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geq_1 \exists \overrightarrow{\beta^-}.Q} \quad \text{D1EXISTSL} \end{array}$$

2.3 Declarative Equivalence

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\begin{array}{c} \overline{\alpha^- \simeq_1^D \alpha^-} \quad (\text{VAR}^{-\simeq_1^D}) \\ \frac{P \simeq_1^D Q}{\uparrow P \simeq_1^D \uparrow Q} \quad (\uparrow \simeq_1^D) \\ \frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \rightarrow N \simeq_1^D Q \rightarrow M} \quad (\rightarrow \simeq_1^D) \end{array}$$

$$\frac{\{\vec{\alpha}^+\} \cap \mathbf{fv} M = \emptyset \quad \mu : (\{\vec{\beta}^+\} \cap \mathbf{fv} M) \leftrightarrow (\{\vec{\alpha}^+\} \cap \mathbf{fv} N) \quad N \simeq_1^D [\mu]M}{\forall \vec{\alpha}^+. N \simeq_1^D \forall \vec{\beta}^+. M} \quad (\forall \simeq_1^D)$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{\overline{\alpha^+ \simeq_1^D \alpha^+} \quad (\text{VAR}^+) \quad \frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M} \quad (\downarrow \simeq_1^D)}{\frac{\{\vec{\alpha}^+\} \cap \mathbf{fv} Q = \emptyset \quad \mu : (\{\vec{\beta}^+\} \cap \mathbf{fv} Q) \leftrightarrow (\{\vec{\alpha}^+\} \cap \mathbf{fv} P) \quad P \simeq_1^D [\mu]Q}{\exists \vec{\alpha}^+. P \simeq_1^D \exists \vec{\beta}^+. Q}} \quad (\exists \simeq_1^D)$$

3 Algorithm

3.1 Normalization

3.1.1 Ordering

$\boxed{\text{ord vars in } N = \vec{\alpha}}$

$$\begin{aligned} & \frac{\alpha^- \in \text{vars}}{\text{ord vars in } \alpha^- = \alpha^-} \quad (\text{VAR}_{\in}^-) \\ & \frac{\alpha^- \notin \text{vars}}{\text{ord vars in } \alpha^- = \cdot} \quad (\text{VAR}_{\notin}^-) \\ & \frac{\text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \uparrow P = \vec{\alpha}} \quad (\uparrow) \\ & \frac{\text{ord vars in } P = \vec{\alpha}_1 \quad \text{ord vars in } N = \vec{\alpha}_2}{\text{ord vars in } P \rightarrow N = \vec{\alpha}_1, (\vec{\alpha}_2 \setminus \{\vec{\alpha}_1\})} \quad (\rightarrow) \\ & \frac{\text{vars} \cap \{\vec{\alpha}^+\} = \emptyset \quad \text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \forall \vec{\alpha}^+. N = \vec{\alpha}} \quad (\forall) \end{aligned}$$

$\boxed{\text{ord vars in } P = \vec{\alpha}}$

$$\begin{aligned} & \frac{\alpha^+ \in \text{vars}}{\text{ord vars in } \alpha^+ = \alpha^+} \quad (\text{VAR}_{\in}^+) \\ & \frac{\alpha^+ \notin \text{vars}}{\text{ord vars in } \alpha^+ = \cdot} \quad (\text{VAR}_{\notin}^+) \\ & \frac{\text{ord vars in } N = \vec{\alpha}}{\text{ord vars in } \downarrow N = \vec{\alpha}} \quad (\downarrow) \\ & \frac{\text{vars} \cap \{\vec{\alpha}^-\} = \emptyset \quad \text{ord vars in } P = \vec{\alpha}}{\text{ord vars in } \exists \vec{\alpha}^-. P = \vec{\alpha}} \quad (\exists) \end{aligned}$$

$\boxed{\text{ord vars in } N = \vec{\alpha}}$

$$\overline{\text{ord vars in } \hat{\alpha}^- \{ \text{vars}' \} = \cdot} \quad (\text{UVar}^-)$$

$\boxed{\text{ord vars in } P = \vec{\alpha}}$

$$\overline{\text{ord vars in } \hat{\alpha}^+ \{ \text{vars}' \} = \cdot} \quad (\text{UVar}^+)$$

3.1.2 Quantifier Normalization

$$\boxed{\mathbf{nf}(N) = M}$$

$$\begin{array}{c} \overline{\mathbf{nf}(\alpha^-) = \alpha^-} \quad (\text{VAR}^-) \\ \frac{\mathbf{nf}(P) = Q}{\mathbf{nf}(\uparrow P) = \uparrow Q} \quad (\uparrow) \\ \frac{\mathbf{nf}(P) = Q \quad \mathbf{nf}(N) = M}{\mathbf{nf}(P \rightarrow N) = Q \rightarrow M} \quad (\rightarrow) \\ \frac{\mathbf{nf}(N) = N' \quad \mathbf{ord}\{\overrightarrow{\alpha^+}\} \text{ in } N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}(\forall \alpha^+. N) = \forall \alpha^+'. N'} \quad (\forall) \end{array}$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\begin{array}{c} \overline{\mathbf{nf}(\alpha^+) = \alpha^+} \quad (\text{VAR}^+) \\ \frac{\mathbf{nf}(N) = M}{\mathbf{nf}(\downarrow N) = \downarrow M} \quad (\downarrow) \\ \frac{\mathbf{nf}(P) = P' \quad \mathbf{ord}\{\overrightarrow{\alpha^-}\} \text{ in } P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}(\exists \alpha^-. P) = \exists \alpha^-' . P'} \quad (\exists) \end{array}$$

$$\boxed{\mathbf{nf}(N) = M}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^-\{vars\}) = \hat{\alpha}^-\{vars\}} \quad (\text{UVAR}^-)$$

$$\boxed{\mathbf{nf}(P) = Q}$$

$$\overline{\mathbf{nf}(\hat{\alpha}^+\{vars\}) = \hat{\alpha}^+\{vars\}} \quad (\text{UVAR}^+)$$

3.2 Unification

$$\boxed{N \stackrel{u}{\simeq} M \models \hat{\sigma}} \quad \text{Negative unification}$$

$$\begin{array}{c} \overline{\alpha^- \stackrel{u}{\simeq} \alpha^- \models \cdot} \quad \text{UNVAR} \\ \frac{P \stackrel{u}{\simeq} Q \models \hat{\sigma}}{\uparrow P \stackrel{u}{\simeq} \uparrow Q \models \hat{\sigma}} \quad \text{USHIFTU} \\ \frac{P \stackrel{u}{\simeq} Q \models \hat{\sigma}_1 \quad N \stackrel{u}{\simeq} M \models \hat{\sigma}_2}{P \rightarrow N \stackrel{u}{\simeq} Q \rightarrow M \models \hat{\sigma}_1 \ \& \ \hat{\sigma}_2} \quad \text{UARROW} \\ \frac{\overrightarrow{N \stackrel{u}{\simeq} M \models \hat{\sigma}}}{\forall \alpha^+. N \stackrel{u}{\simeq} \forall \alpha^+. M \models \hat{\sigma}} \quad \text{Uforall} \\ \frac{\mathbf{fv} N \subseteq vars}{\hat{\alpha}^-\{vars\} \stackrel{u}{\simeq} N \models \hat{\alpha}^- : \approx N} \quad \text{UNUvar} \end{array}$$

$$\boxed{P \stackrel{u}{\simeq} Q \models \hat{\sigma}} \quad \text{Positive unification}$$

$$\begin{array}{c} \overline{\alpha^+ \stackrel{u}{\simeq} \alpha^+ \models \cdot} \quad \text{UPvar} \\ \frac{N \stackrel{u}{\simeq} M \models \hat{\sigma}}{\downarrow N \stackrel{u}{\simeq} \downarrow M \models \hat{\sigma}} \quad \text{USHIFTD} \\ \frac{P \stackrel{u}{\simeq} Q \models \hat{\sigma}}{\exists \alpha^-. P \stackrel{u}{\simeq} \exists \alpha^-. Q \models \hat{\sigma}} \quad \text{UEXISTS} \\ \frac{\mathbf{fv} P \subseteq vars}{\hat{\alpha}^+\{vars\} \stackrel{u}{\simeq} P \models \hat{\alpha}^+ : \approx P} \quad \text{UPUvar} \end{array}$$

3.3 Algorithmic Subtyping

$\boxed{\Gamma \models N \leq M \Rightarrow \hat{\sigma}}$ Negative subtyping

$$\begin{array}{c}
\frac{}{\Gamma \models \alpha^- \leq \alpha^- \Rightarrow} \text{ANVAR} \\
\frac{\text{nf}(P) \stackrel{u}{\simeq} \text{nf}(Q) \Rightarrow \hat{\sigma}}{\Gamma \models \uparrow P \leq \uparrow Q \Rightarrow \hat{\sigma}} \text{AShiftU} \\
\frac{\Gamma \models P \geq Q \Rightarrow \hat{\sigma}_1 \quad \Gamma \models N \leq M \Rightarrow \hat{\sigma}_2}{\Gamma \models P \rightarrow N \leq Q \rightarrow M \Rightarrow \hat{\sigma}_1 \& \hat{\sigma}_2} \text{AArrow} \\
\frac{\Gamma, \vec{\beta}^+ \models [\hat{\alpha}^+ \{\Gamma, \vec{\beta}^+\} / \alpha^+] N \leq M \Rightarrow \hat{\sigma}}{\Gamma \models \forall \alpha^+. N \leq \forall \beta^+. M \Rightarrow \hat{\sigma} \setminus \hat{\alpha}^+} \text{AForall}
\end{array}$$

$\boxed{\Gamma \models P \geq Q \Rightarrow \hat{\sigma}}$ Positive supertyping

$$\begin{array}{c}
\frac{}{\Gamma \models \alpha^+ \geq \alpha^+ \Rightarrow} \text{APVAR} \\
\frac{\text{nf}(N) \stackrel{u}{\simeq} \text{nf}(M) \Rightarrow \hat{\sigma}}{\Gamma \models \downarrow N \geq \downarrow M \Rightarrow \hat{\sigma}} \text{AShiftD} \\
\frac{\Gamma, \vec{\beta}^- \models [\hat{\alpha}^- \{\Gamma, \vec{\beta}^-\} / \alpha^-] P \geq Q \Rightarrow \hat{\sigma}}{\Gamma \models \exists \alpha^-. P \geq \exists \beta^-. Q \Rightarrow \hat{\sigma}} \text{AExists} \\
\frac{\text{nf}(P) = P' \quad \text{vars}_1 = \text{fv } P' \setminus \text{vars} \quad \text{vars}_2 \text{ is fresh}}{\Gamma \models \hat{\alpha}^+ \{\text{vars}\} \geq P \Rightarrow (\hat{\alpha}^+ : \geq P' \vee [\text{vars}_2 / \text{vars}_1] P')} \text{APUVAR}
\end{array}$$

3.4 Unification Solution Merge

Unification solution is represented by a list of unification solution entries. Each entry restrict an unification variable in two possible ways: either stating that it must be equivalent to a certain type ($\hat{\alpha}^+ : \approx P$ or $\hat{\alpha}^- : \approx N$) or that it must be a (positive) supertype of a certain type ($\hat{\alpha}^+ : \geq P$).

Definition 1 (Matching Entries). *We call two entries matching if they are restricting the same unification variable.*

Two matching entries can be merged in the following way:

Definition 2.

$\boxed{e_1 \& e_2 = e_3}$ Unification Solution Entry Merge

$$\begin{array}{c}
\frac{}{(\hat{\alpha}^+ : \geq P) \& (\hat{\alpha}^+ : \geq Q) = (\hat{\alpha}^+ : \geq P \vee Q)} (\geq \& \geq) \\
\frac{\text{fv } P \cup \text{fv } Q \models P \geq Q \Rightarrow \hat{\sigma}'}{(\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \geq Q) = (\hat{\alpha}^+ : \approx P)} (\simeq \& \geq) \\
\frac{\text{fv } P \cup \text{fv } Q \models Q \geq P \Rightarrow \hat{\sigma}'}{(\hat{\alpha}^+ : \geq P) \& (\hat{\alpha}^+ : \approx Q) = (\hat{\alpha}^+ : \approx Q)} (\geq \& \simeq) \\
\frac{}{(\hat{\alpha}^+ : \approx P) \& (\hat{\alpha}^+ : \approx P) = (\hat{\alpha}^+ : \approx P)} (\simeq \& \simeq^+) \\
\frac{}{(\hat{\alpha}^- : \approx N) \& (\hat{\alpha}^- : \approx N) = (\hat{\alpha}^- : \approx N)} (\simeq \& \simeq^-)
\end{array}$$

To merge two unification solution, we merge each pair of matching entries, and unite the results.

Definition 3. $\hat{\sigma}_1 \& \hat{\sigma}_2 = \{e_1 \& e_2 \mid e_1 \in \hat{\sigma}_1, e_2 \in \hat{\sigma}_2, \text{ s.t. } e_1 \text{ matches with } e_2\}$

3.5 Least Upper Bound

$\boxed{P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\begin{array}{c}
\frac{}{\alpha^+ \vee \alpha^+ = \alpha^+} \text{ LUBVAR} \\
\frac{(\mathbf{fv} N \cup \mathbf{fv} M) \models \downarrow N \stackrel{a}{\simeq} \downarrow M \Rightarrow (P, \hat{\sigma}_1, \hat{\sigma}_2)}{\downarrow N \vee \downarrow M = \exists \alpha^-. [\overrightarrow{\alpha^-} / (\mathbf{uv} P)] P} \text{ LUBSHIFT} \\
\frac{\{\overrightarrow{\alpha^-}\} \cap \{\overrightarrow{\beta^-}\} = \emptyset}{\exists \alpha^-. P_1 \vee \exists \beta^-. P_2 = P_1 \vee P_2} \text{ LUBEXISTS}
\end{array}$$

3.6 Antiunification

$\boxed{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (Q, \hat{\sigma}_1, \hat{\sigma}_2)}$

$$\begin{array}{c}
\frac{}{\Gamma \models \alpha^+ \stackrel{a}{\simeq} \alpha^+ \Rightarrow (\alpha^+, \cdot, \cdot)} \text{ AUPVAR} \\
\frac{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (M, \hat{\sigma}_1, \hat{\sigma}_2)}{\Gamma \models \downarrow N_1 \stackrel{a}{\simeq} \downarrow N_2 \Rightarrow (\downarrow M, \hat{\sigma}_1, \hat{\sigma}_2)} \text{ AUPSHIFT} \\
\frac{\{\overrightarrow{\alpha^-}\} \cap \{\Gamma\} = \emptyset \quad \Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (Q, \hat{\sigma}_1, \hat{\sigma}_2)}{\Gamma \models \exists \alpha^-. P_1 \stackrel{a}{\simeq} \exists \alpha^-. P_2 \Rightarrow (\exists \alpha^-. Q, \hat{\sigma}_1, \hat{\sigma}_2)} \text{ AUPEXISTS}
\end{array}$$

$\boxed{\Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (M, \hat{\sigma}_1, \hat{\sigma}_2)}$

$$\begin{array}{c}
\frac{}{\Gamma \models \alpha^- \stackrel{a}{\simeq} \alpha^- \Rightarrow (\alpha^-, \cdot, \cdot)} \text{ AUNVAR} \\
\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (Q, \hat{\sigma}_1, \hat{\sigma}_2)}{\Gamma \models \uparrow P_1 \stackrel{a}{\simeq} \uparrow P_2 \Rightarrow (\uparrow Q, \hat{\sigma}_1, \hat{\sigma}_2)} \text{ AUNSHIFT} \\
\frac{\Gamma \models P_1 \stackrel{a}{\simeq} P_2 \Rightarrow (Q, \hat{\sigma}_1, \hat{\sigma}_2) \quad \Gamma \models N_1 \stackrel{a}{\simeq} N_2 \Rightarrow (M, \hat{\sigma}'_1, \hat{\sigma}'_2)}{\Gamma \models P_1 \rightarrow N_1 \stackrel{a}{\simeq} P_2 \rightarrow N_2 \Rightarrow (Q \rightarrow M, \hat{\sigma}_1 \cup \hat{\sigma}'_1, \hat{\sigma}_2 \cup \hat{\sigma}'_2)} \text{ AUNARROW} \\
\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \stackrel{a}{\simeq} M \Rightarrow (\hat{\alpha}_{\{N, M\}}^-, (\hat{\alpha}_{\{N, M\}}^- : \approx N), (\hat{\alpha}_{\{N, M\}}^- : \approx M))} \text{ AUNAU}
\end{array}$$

4 Proofs

4.1 Variable Ordering

Definition 4 (Collision free bijection). *We say that a bijection $\mu : A \leftrightarrow B$ between sets of variables is **collision free on sets P and Q** if and only if*

1. $\mu(P \cap A) \cap Q = \emptyset$
2. $\mu(Q \cap A) \cap P = \emptyset$

Lemma 1 (Soundness of variable ordering). *Variable ordering extracts precisely used free variables.*

- $\{\mathbf{ord} \text{ vars in } N\} \equiv \text{vars} \cap \mathbf{fv} N$ (as sets)
- + $\{\mathbf{ord} \text{ vars in } P\} \equiv \text{vars} \cap \mathbf{fv} P$ (as sets)

Proof. Straightforward mutual induction on $\mathbf{ord} \text{ vars in } N = \vec{\alpha}$ and $\mathbf{ord} \text{ vars in } P = \vec{\alpha}$ □

Corollary 1 (Additivity of ordering). *Variable ordering is additive (in terms of set union) with respect to its first argument.*

- $\{\mathbf{ord} (\text{vars}_1 \cup \text{vars}_2) \text{ in } N\} \equiv \{\mathbf{ord} \text{ vars}_1 \text{ in } N\} \cup \{\mathbf{ord} \text{ vars}_2 \text{ in } N\}$ (as sets)
- + $\{\mathbf{ord} (\text{vars}_1 \cup \text{vars}_2) \text{ in } P\} \equiv \{\mathbf{ord} \text{ vars}_1 \text{ in } P\} \cup \{\mathbf{ord} \text{ vars}_2 \text{ in } P\}$ (as sets)

Corollary 2 (Weakening of ordering). *Extending the first argument of the ordering with unused variables does not change the result.*

- $\mathbf{ord}(vars \cap \mathbf{fv} N) \mathbf{in} N = \mathbf{ord} vars \mathbf{in} N$
- + $\mathbf{ord}(vars \cap \mathbf{fv} P) \mathbf{in} P = \mathbf{ord} vars \mathbf{in} P$

Lemma 2 (Distributivity of renaming over variable ordering). *Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$.*

- *If μ is collision free on $vars$ and $\mathbf{fv} N$ then $[\mu](\mathbf{ord} vars \mathbf{in} N) = \mathbf{ord}([\mu]vars) \mathbf{in} [\mu]N$*
- + *If μ is collision free on $vars$ and $\mathbf{fv} P$ then $[\mu](\mathbf{ord} vars \mathbf{in} P) = \mathbf{ord}([\mu]vars) \mathbf{in} [\mu]P$*

Proof. Mutual induction on N and P .

Case 1. $N = \alpha^-$

let us consider four cases:

a. $\alpha^- \in A$ and $\alpha^- \in vars$

$$\begin{aligned}
 \text{Then } [\mu](\mathbf{ord} vars \mathbf{in} N) &= [\mu](\mathbf{ord} vars \mathbf{in} \alpha^-) \\
 &= [\mu]\alpha^- && \text{by Rule (Var}_{\epsilon}^+) \\
 &= \beta^- && \text{for some } \beta^- \in B \text{ (notice that } \beta^- \in [\mu]vars) \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} \beta^- && \text{by Rule (Var}_{\epsilon}^+), \text{ because } \beta^- \in [\mu]vars \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^-
 \end{aligned}$$

b. $\alpha^- \notin A$ and $\alpha^- \notin vars$

Notice that $[\mu](\mathbf{ord} vars \mathbf{in} N) = [\mu](\mathbf{ord} vars \mathbf{in} \alpha^-) = \cdot$ by Rule (Var_ε⁺). On the other hand, $\mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^- = \mathbf{ord} [\mu]vars \mathbf{in} \alpha^- = \cdot$. The latter equality is from Rule (Var_ε⁺), because μ is collision free on $vars$ and $\mathbf{fv} N$, so $\mathbf{fv} N \ni \alpha^- \notin \mu(A \cap vars) \cup vars \supseteq [\mu]vars$.

c. $\alpha^- \in A$ but $\alpha^- \notin vars$

Then $[\mu](\mathbf{ord} vars \mathbf{in} N) = [\mu](\mathbf{ord} vars \mathbf{in} \alpha^-) = \cdot$ by Rule (Var_ε⁺). To prove that $\mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^- = \cdot$, we apply Rule (Var_ε⁺). Let us show that $[\mu]\alpha^- \notin [\mu]vars$. Since $[\mu]\alpha^- = \mu(\alpha^-)$ and $[\mu]vars \subseteq \mu(A \cap vars) \cup vars$, it suffices to prove $\mu(\alpha^-) \notin \mu(A \cap vars) \cup vars$.

- (i) If there is an element $x \in A \cap vars$ such that $\mu x = \mu\alpha^-$, then $x = \alpha^-$ by bijectivity of μ , which contradicts with $\alpha^- \notin vars$. This way, $\mu(\alpha^-) \notin \mu(A \cap vars)$.
- (ii) Since μ is collision free on $vars$ and $\mathbf{fv} N$, $\mu(A \cap \mathbf{fv} N) \ni \mu(\alpha^-) \notin vars$.

d. $\alpha^- \notin A$ but $\alpha^- \in vars$

$\mathbf{ord} [\mu]vars \mathbf{in} [\mu]\alpha^- = \mathbf{ord} [\mu]vars \mathbf{in} \alpha^- = \alpha^-$. The latter is by Rule (Var_ε⁺), because $\alpha^- = [\mu]\alpha^- \in [\mu]vars$ since $\alpha^- \in vars$. On the other hand, $[\mu](\mathbf{ord} vars \mathbf{in} N) = [\mu](\mathbf{ord} vars \mathbf{in} \alpha^-) = [\mu]\alpha^- = \alpha^-$.

Case 2. $N = \uparrow P$

$$\begin{aligned}
 [\mu](\mathbf{ord} vars \mathbf{in} N) &= [\mu](\mathbf{ord} vars \mathbf{in} \uparrow P) \\
 &= [\mu](\mathbf{ord} vars \mathbf{in} P) && \text{by Rule } (\uparrow) \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]P && \text{by the induction hypothesis} \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} \uparrow [\mu]P && \text{by Rule } (\uparrow) \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]\uparrow P && \text{by the definition of substitution} \\
 &= \mathbf{ord} [\mu]vars \mathbf{in} [\mu]N
 \end{aligned}$$

Case 3. $N = P \rightarrow M$

$$\begin{aligned}
 [\mu](\mathbf{ord} vars \mathbf{in} N) &= [\mu](\mathbf{ord} vars \mathbf{in} P \rightarrow M) \\
 &= [\mu](\vec{\alpha}_1, (\vec{\alpha}_2 \setminus \{\vec{\alpha}_1\})) && \text{where } \mathbf{ord} vars \mathbf{in} P = \vec{\alpha}_1 \text{ and } \mathbf{ord} vars \mathbf{in} M = \vec{\alpha}_2 \\
 &= [\mu]\vec{\alpha}_1, [\mu](\vec{\alpha}_2 \setminus \{\vec{\alpha}_1\}) \\
 &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus [\mu]\{\vec{\alpha}_1\}) && \text{by induction on } \vec{\alpha}_2; \text{ the inductive step is similar to case 1. Notice that } \mu \text{ is} \\
 & && \text{collision free on } \{\vec{\alpha}_1\} \text{ and } \{\vec{\alpha}_2\} \text{ since } \{\vec{\alpha}_1\} \subseteq vars \text{ and } \{\vec{\alpha}_2\} \subseteq \mathbf{fv} N \\
 &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus \{[\mu]\vec{\alpha}_1\}) \\
 (\mathbf{ord} [\mu]vars \mathbf{in} [\mu]N) &= (\mathbf{ord} [\mu]vars \mathbf{in} [\mu]P \rightarrow [\mu]M) \\
 &= (\vec{\beta}_1, (\vec{\beta}_2 \setminus \{\vec{\beta}_1\})) && \text{where } \mathbf{ord} [\mu]vars \mathbf{in} [\mu]P = \vec{\beta}_1 \text{ and } \mathbf{ord} [\mu]vars \mathbf{in} [\mu]M = \vec{\beta}_2 \\
 & && \text{then by the induction hypothesis, } \vec{\beta}_1 = [\mu]\vec{\alpha}_1, \vec{\beta}_2 = [\mu]\vec{\alpha}_2, \\
 &= [\mu]\vec{\alpha}_1, ([\mu]\vec{\alpha}_2 \setminus \{[\mu]\vec{\alpha}_1\})
 \end{aligned}$$

Case 4. $N = \forall \alpha^+ . M$

$$\begin{aligned}
[\mu](\mathbf{ord\ vars\ in}\ N) &= [\mu]\mathbf{ord\ vars\ in}\ \forall \alpha^+ . M \\
&= [\mu]\mathbf{ord\ vars\ in}\ M \\
&= \mathbf{ord}\ [\mu]\mathbf{vars\ in}\ [\mu]M \quad \text{by the induction hypothesis} \\
(\mathbf{ord}\ [\mu]\mathbf{vars\ in}\ [\mu]N) &= \mathbf{ord}\ [\mu]\mathbf{vars\ in}\ [\mu]\forall \alpha^+ . M \\
&= \mathbf{ord}\ [\mu]\mathbf{vars\ in}\ \forall \alpha^+ . [\mu]M \\
&= \mathbf{ord}\ [\mu]\mathbf{vars\ in}\ [\mu]M
\end{aligned}$$

□

Lemma 3 (Completeness of variable ordering). *Variable ordering is invariant under equivalence.*

- For $N \simeq_1^D M$ and any vars, if $\mathbf{ord\ vars\ in}\ N = \vec{\alpha}_1$ and $\mathbf{ord\ vars\ in}\ M = \vec{\alpha}_2$, then $\vec{\alpha}_1 = \vec{\alpha}_2$ (as lists)
- + For $P \simeq_1^D Q$ and any vars, if $\mathbf{ord\ vars\ in}\ P = \vec{\alpha}_1$ and $\mathbf{ord\ vars\ in}\ Q = \vec{\alpha}_2$, then $\vec{\alpha}_1 = \vec{\alpha}_2$ (as lists)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

□

4.2 Normalizaion

Lemma 4. *Set of free variables is invariant under equivalence.*

- If $N \simeq_1^D M$ then $\mathbf{fv}\ N \equiv \mathbf{fv}\ M$ (as sets)
- + If $P \simeq_1^D Q$ then $\mathbf{fv}\ P \equiv \mathbf{fv}\ Q$ (as sets)

Proof. Straightforward mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$

□

Lemma 5. *Free variables are not changed by the normalization*

- $\mathbf{fv}\ N \equiv \mathbf{fv}\ \mathbf{nf}\ (N)$
- + $\mathbf{fv}\ P \equiv \mathbf{fv}\ \mathbf{nf}\ (P)$

Proof. By straightforward induction on $\mathbf{nf}\ (N) = M$.

□

Lemma 6 (Commutativity of normalization and renaming). *Normalization of a term commutes with renaming.*

Suppose that μ is a bijection between two sets of variables $\mu : A \leftrightarrow B$. Then

- $\mathbf{nf}\ ([\mu]N) = [\mu]\mathbf{nf}\ (N)$
- + $\mathbf{nf}\ ([\mu]P) = [\mu]\mathbf{nf}\ (P)$

Here equality means alpha-equivalence.

Proof. Mutual induction on N and P .

Case 1. $N = \alpha^-$

$\mathbf{nf}\ ([\mu]N) = \mathbf{nf}\ ([\mu]\alpha^-) = [\mu]\alpha^-$. The latter follows from the fact that $[\mu]\alpha^-$ is a variable, and thus, Rule (Var[−]) is applicable.
 $[\mu]\mathbf{nf}\ (N) = [\mu]\mathbf{nf}\ (\alpha^-) = [\mu]\alpha^-$.

Case 2. If the type is formed by \rightarrow , \uparrow , or \downarrow , the required equality follows from the congruence of the normalization and substitution, and the induction hypothesis. For example, if $N = P \rightarrow M$ then

$$\begin{aligned}
\mathbf{nf}\ ([\mu]N) &= \mathbf{nf}\ ([\mu](P \rightarrow M)) \\
&= \mathbf{nf}\ ([\mu]P \rightarrow [\mu]M) && \text{By the congruence of substitution} \\
&= \mathbf{nf}\ ([\mu]P) \rightarrow \mathbf{nf}\ ([\mu]M) && \text{By the congruence of normalization, i.e. Rule } (\rightarrow) \\
&= [\mu]\mathbf{nf}\ (P) \rightarrow [\mu]\mathbf{nf}\ (M) && \text{By the induction hypothesis} \\
&= [\mu](\mathbf{nf}\ (P) \rightarrow \mathbf{nf}\ (M)) && \text{By the congruence of substitution} \\
&= [\mu]\mathbf{nf}\ (P \rightarrow M) && \text{By the congruence of normalization} \\
&= [\mu]\mathbf{nf}\ (N)
\end{aligned}$$

Case 3. $N = \forall \alpha^{\vec{\alpha}^+}. M$

$$\begin{aligned} [\mu] \mathbf{nf}(N) &= [\mu] \mathbf{nf}(\forall \alpha^{\vec{\alpha}^+}. M) \\ &= [\mu] \forall \alpha^{\vec{\alpha}^+}. \mathbf{nf}(M) \quad \text{Where } \mathbf{ord} \{\alpha^{\vec{\alpha}^+}\} \text{ in } \mathbf{nf}(M) = \alpha^{\vec{\alpha}^+} \\ \mathbf{nf}([\mu]N) &= \mathbf{nf}([\mu] \forall \alpha^{\vec{\alpha}^+}. M) \\ &= \mathbf{nf}(\forall \alpha^{\vec{\alpha}^+}. [\mu]M) \quad \text{Assuming } \{\alpha^{\vec{\alpha}^+}\} \cap A = \emptyset \text{ and } \{\alpha^{\vec{\alpha}^+}\} \cap B = \emptyset \\ &= \forall \beta^{\vec{\beta}^+}. \mathbf{nf}([\mu]M) \quad \text{Where } \mathbf{ord} \{\alpha^{\vec{\alpha}^+}\} \text{ in } \mathbf{nf}([\mu]M) = \beta^{\vec{\beta}^+} \\ &= \forall \alpha^{\vec{\alpha}^+}. \mathbf{nf}([\mu]M) \quad \text{As } \beta^{\vec{\beta}^+} = \alpha^{\vec{\alpha}^+} \text{ (see below)} \end{aligned}$$

Notice that μ is free of collisions on $\{\alpha^{\vec{\alpha}^+}\}$ and $\mathbf{fv} \mathbf{nf}(M)$ because

$$\begin{aligned} \text{(i)} \quad & \mu(A \cap \{\alpha^{\vec{\alpha}^+}\}) \cap \mathbf{fv} \mathbf{nf}(M) = \emptyset \cap \mathbf{fv} \mathbf{nf}(M) = \emptyset \text{ and} \\ \text{(ii)} \quad & \mu(A \cap \mathbf{fv} \mathbf{nf}(M)) \cap \{\alpha^{\vec{\alpha}^+}\} \subseteq B \cap \{\alpha^{\vec{\alpha}^+}\} = \emptyset \\ \vec{\beta}^+ &= \mathbf{ord} \{\alpha^{\vec{\alpha}^+}\} \text{ in } \mathbf{nf}([\mu]M) \\ &= \mathbf{ord} \{\alpha^{\vec{\alpha}^+}\} \text{ in } [\mu] \mathbf{nf}(M) \quad \text{By the induction hypothesis} \\ &= \mathbf{ord} \{[\mu] \alpha^{\vec{\alpha}^+}\} \text{ in } [\mu] \mathbf{nf}(M) \quad \text{Since } \{\alpha^{\vec{\alpha}^+}\} \cap A = \emptyset \\ &= [\mu] \mathbf{ord} \{\alpha^{\vec{\alpha}^+}\} \text{ in } \mathbf{nf}(M) \quad \text{by lemma 2} \\ &= \mathbf{ord} \{\alpha^{\vec{\alpha}^+}\} \text{ in } \mathbf{nf}(M) \quad \text{Since } \{\mathbf{ord} \{\alpha^{\vec{\alpha}^+}\} \text{ in } \mathbf{nf}(M)\} \cap A \subseteq \{\alpha^{\vec{\alpha}^+}\} \cap A = \emptyset \\ &= \alpha^{\vec{\alpha}^+} \end{aligned}$$

To show alpha-equivalence of $[\mu] \forall \alpha^{\vec{\alpha}^+}. \mathbf{nf}(M)$ and $\forall \alpha^{\vec{\alpha}^+}. \mathbf{nf}([\mu]M)$, we can assume that $\{\alpha^{\vec{\alpha}^+}\} \cap A = \emptyset$, and $\{\alpha^{\vec{\alpha}^+}\} \cap B = \emptyset$. Then $[\mu] \forall \alpha^{\vec{\alpha}^+}. \mathbf{nf}(M) = \forall \alpha^{\vec{\alpha}^+}. [\mu] \mathbf{nf}(M) = \forall \alpha^{\vec{\alpha}^+}. \mathbf{nf}([\mu]M)$, the latter follows from the induction hypothesis.

Case 4. $P = \exists \alpha^{\vec{\alpha}^+}. Q$
 Same as for case 3.

□

Lemma 7 (Soundness of quantifier normalization).

$$\begin{aligned} - \quad & N \simeq_1^D \mathbf{nf}(N) \\ + \quad & P \simeq_1^D \mathbf{nf}(P) \end{aligned}$$

Proof. Mutual induction on $\mathbf{nf}(N) = M$ and $\mathbf{nf}(P) = Q$. Let us consider how this judgment is formed:

Case 1. (Var^-) and (Var^+)

By the corresponding equivalence rules.

Case 2. (\uparrow), (\downarrow), and (\rightarrow)

By the induction hypothesis and the corresponding congruent equivalence rules.

Case 3. (\forall), i.e. $\mathbf{nf}(\forall \alpha^{\vec{\alpha}^+}. N) = \forall \alpha^{\vec{\alpha}^+}. N'$

From the induction hypothesis, we know that $N \simeq_1^D N'$. In particular, by lemma 4, $\mathbf{fv} N \equiv \mathbf{fv} N'$. Then by lemma 1, $\{\alpha^{\vec{\alpha}^+}\} \equiv \{\alpha^{\vec{\alpha}^+}\} \cap \mathbf{fv} N' \equiv \{\alpha^{\vec{\alpha}^+}\} \cap \mathbf{fv} N$, and thus, $\{\alpha^{\vec{\alpha}^+}\} \cap \mathbf{fv} N' \equiv \{\alpha^{\vec{\alpha}^+}\} \cap \mathbf{fv} N$.

To prove $\forall \alpha^{\vec{\alpha}^+}. N \simeq_1^D \forall \alpha^{\vec{\alpha}^+}. N'$, it suffices to provide a bijection $\mu : \{\alpha^{\vec{\alpha}^+}\} \cap \mathbf{fv} N' \leftrightarrow \{\alpha^{\vec{\alpha}^+}\} \cap \mathbf{fv} N$ such that $N \simeq_1^D [\mu]N'$. Since these sets are equal, we take $\mu = id$.

Case 4. (\exists) Same as for case 3.

□

Lemma 8 (Completeness of quantified normalization). *Normalization returns the same representative for equivalent types.*

$$\begin{aligned} - \quad & \text{If } N \simeq_1^D M \text{ then } \mathbf{nf}(N) = \mathbf{nf}(M) \\ + \quad & \text{If } P \simeq_1^D Q \text{ then } \mathbf{nf}(P) = \mathbf{nf}(Q) \end{aligned}$$

(Here equality means alpha-equivalence)

Proof. Mutual induction on $N \simeq_1^D M$ and $P \simeq_1^D Q$.

Case 1. $(\forall \simeq_1^D)$

From the definition of the normalization,

- $\mathbf{nf}(\forall \vec{\alpha}^+. N) = \forall \vec{\alpha}^{+'}. \mathbf{nf}(N)$ where $\vec{\alpha}^{+'}$ is $\mathbf{ord} \{\vec{\alpha}^+\}$ in $\mathbf{nf}(N)$
- $\mathbf{nf}(\forall \vec{\beta}^+. M) = \forall \vec{\beta}^{+'}. \mathbf{nf}(M)$ where $\vec{\beta}^{+'}$ is $\mathbf{ord} \{\vec{\beta}^+\}$ in $\mathbf{nf}(M)$

Let us take $\mu : (\{\vec{\beta}^+\} \cap \mathbf{fv} M) \leftrightarrow (\{\vec{\alpha}^+\} \cap \mathbf{fv} N)$ from the inversion of the equivalence judgment. Notice that from lemmas 1 and 5, the domain and the codomain of μ can be written as $\mu : \{\vec{\beta}^{+'}\} \leftrightarrow \{\vec{\alpha}^{+'}\}$.

To show the alpha-equivalence of $\forall \vec{\alpha}^{+'}. \mathbf{nf}(N)$ and $\forall \vec{\beta}^{+'}. \mathbf{nf}(M)$, it suffices to prove that (i) $[\mu] \mathbf{nf}(M) = \mathbf{nf}(N)$ and (ii) $[\mu] \vec{\beta}^{+'} = \vec{\alpha}^{+'}$.

(i) $[\mu] \mathbf{nf}(M) = \mathbf{nf}([\mu]M) = \mathbf{nf}(N)$. The first equality holds by lemma 6, the second—by the induction hypothesis.

$$\begin{aligned}
 \text{(ii) } [\mu] \vec{\beta}^{+'} &= [\mu] \mathbf{ord} \{\vec{\beta}^+\} \text{ in } \mathbf{nf}(M) && \text{by the definition of } \vec{\beta}^{+'} \\
 &= [\mu] \mathbf{ord} (\{\vec{\beta}^+\} \cap \mathbf{fv} M) \text{ in } \mathbf{nf}(M) && \text{from lemma 5 and corollary 2} \\
 &= \mathbf{ord} [\mu](\{\vec{\beta}^+\} \cap \mathbf{fv} M) \text{ in } [\mu] \mathbf{nf}(M) && \text{by lemma 2, because } \{\vec{\alpha}^+\} \cap \mathbf{fv} N \cap \mathbf{fv} \mathbf{nf}(M) \subseteq \{\vec{\alpha}^+\} \cap \mathbf{fv} M = \emptyset \\
 &&& \text{and } \{\vec{\alpha}^+\} \cap \mathbf{fv} N \cap (\{\vec{\beta}^+\} \cap \mathbf{fv} M) \subseteq \{\vec{\alpha}^+\} \cap \mathbf{fv} M = \emptyset \\
 &= \mathbf{ord} [\mu](\{\vec{\beta}^+\} \cap \mathbf{fv} M) \text{ in } \mathbf{nf}(N) && \text{since } [\mu] \mathbf{nf}(M) = \mathbf{nf}(N) \text{ is proved} \\
 &= \mathbf{ord} (\{\vec{\alpha}^+\} \cap \mathbf{fv} N) \text{ in } \mathbf{nf}(N) && \text{because } \mu \text{ is a bijection between } \{\vec{\alpha}^+\} \cap \mathbf{fv} N \text{ and } \{\vec{\beta}^+\} \cap \mathbf{fv} M \\
 &= \mathbf{ord} \{\vec{\alpha}^+\} \text{ in } \mathbf{nf}(N) && \text{from lemma 5 and corollary 2} \\
 &= \vec{\alpha}^{+'} && \text{by the definition of } \vec{\alpha}^{+'}
 \end{aligned}$$

Case 2. $(\exists \simeq_1^D)$ Same as for case 1.

Case 3. Other rules are congruent, and thus, proved by the corresponding congruent alpha-equivalence rule, which is applicable by the induction hypothesis.

□