α, β	type variables
n, m, i, j	index variables

$\alpha^+,\ \beta^+$	$::=$			positive variable
	$\vert$	α^+		
$\alpha^-,\ \beta^-$	$::=$			negative variable
	$\vert$	α^-		
σ	$::=$			substitution
	$\vert$	$\cdot$		
	$\vert$	$P/a+$		
	$\vert$	$N/a-$		
	$\vert$	$\vec{P}/\overrightarrow{\alpha^+}$		
	$\vert$	$\vec{N}/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^-}}/\overrightarrow{\alpha^-}$		
	$\vert$	$\overrightarrow{\widetilde{\alpha^+}}/\overrightarrow{\alpha^+}$		
	$\vert$	$\overrightarrow{\widehat{\alpha^-}}/\overrightarrow{\alpha^-}$		
	$\vert$	$vars_1/vars_2$		
	$\vert$	$\overline{\sigma_i}^{\,i}$		concatenate
e	$::=$			entry of a unification solution
	$\vert$	$\widehat{\alpha}^+ :\approx P$		
	$\vert$	$\widehat{\alpha}^- :\approx N$		
	$\vert$	$\widehat{\alpha}^+ :\geqslant P$		
$\widehat{\sigma}$	$::=$			unification solution (substitution)
	$\vert$	$\cdot$		
	$\vert$	e		
	$\vert$	$\widehat{\sigma}\backslash\overrightarrow{\widetilde{\alpha^+}}$		
	$\vert$	$\widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha^-}}$		
	$\vert$	$\widehat{\sigma}\backslash\widehat{\alpha}^+$		
	$\vert$	$\widehat{\sigma}\backslash\widehat{\alpha}^-$		
	$\vert$	$\widehat{\sigma}_1 \cup \widehat{\sigma}_2$		
	$\vert$	$\overline{\widehat{\sigma}_i}^{\,i}$		concatenate
	$\vert$	$(\widehat{\sigma})$	S	
	$\vert$	$\widehat{\sigma}_1 \& \widehat{\sigma}_2$	M	
$P,\ Q$	$::=$			positive types
	$\vert$	$a+$		
	$\vert$	$\downarrow N$		
	$\vert$	$\exists \alpha^-.P$		
	$\vert$	$[\sigma]P$	M	
$N,\ M$	$::=$			negative types
	$\vert$	$a-$		
	$\vert$	$\uparrow P$		
	$\vert$	$\forall \alpha^+.N$		
	$\vert$	$P \to N$		
	$\vert$	$[\sigma]N$	M	

$\overrightarrow{\alpha^+}, \overrightarrow{\beta^+}$	$::=$			positive variable list
	$\vert$	$\cdot$		empty list
	$\vert$	α^+		a variable
	$\vert$	$\overline{\overrightarrow{\alpha^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\alpha^-}, \overrightarrow{\beta^-}$	$::=$			negative variables
	$\vert$	$\cdot$		empty list
	$\vert$	α^-		a variable
	$\vert$	$\overline{\overrightarrow{\alpha^-}_i}^{\,i}$		concatenate lists
P, Q	$::=$			multi-quantified positive types
	$\vert$	α^+		
	$\vert$	$\downarrow N$		
	$\vert$	$\exists\overrightarrow{\alpha^-}.P$		$P \neq \exists\ldots$
	$\vert$	$[\sigma]P$	M	
	$\vert$	$[\mu]P$	M	
	$\vert$	$P_1 \vee P_2$	M	
	$\vert$	$\mathbf{nf}\,(P')$	M	
N, M	$::=$			multi-quantified negative types
	$\vert$	α^-		
	$\vert$	$\uparrow P$		
	$\vert$	$P \to N$		
	$\vert$	$\forall\overrightarrow{\alpha^+}.N$		$N \neq \forall\ldots$
	$\vert$	$[\sigma]N$	M	
	$\vert$	$[\mu]N$	M	
	$\vert$	$\mathbf{nf}\,(N')$	M	
$\vec{P}$	$::=$			list of positive types
	$\vert$	$\cdot$		empty list
	$\vert$	P		a singel type
	$\vert$	$\overline{\vec{P}_i}^{\,i}$		concatenate lists
$\vec{N}$	$::=$			list of negative types
	$\vert$	$\cdot$		empty list
	$\vert$	N		a singel type
	$\vert$	$\overline{\vec{N}_i}^{\,i}$		concatenate lists
Γ	$::=$			declarative type context
	$\vert$	$\cdot$		empty context
	$\vert$	$vars$		
	$\vert$	$\overrightarrow{\alpha^+}$		list of variables
	$\vert$	$\overrightarrow{\alpha^-}$		list of variables
	$\vert$	$\overline{\Gamma_i}^{\,i}$		concatenate contexts
	$\vert$	(Γ)	S	
	$\vert$	$\Gamma_1 \cup \Gamma_2$		
$\vec{\alpha}$	$::=$			ordered positive or negative variables
	$\vert$	$\cdot$		empty list

	$\vert$	$\overrightarrow{\alpha^+}$		list of variables
	$\vert$	$\overrightarrow{\alpha^-}$		list of variables
	$\vert$	$\vec{\alpha}_1 \backslash \vec{\alpha}_2$		setminus
	$\vert$	$\overline{\vec{\alpha}_i}^{\,i}$		concatenate contexts
	$\vert$	$(\vec{\alpha})$	S	parenthesis
	$\vert$	$[\mu]\vec{\alpha}$	M	apply moving to list
	$\vert$	$\mathbf{ord}\ vars\ \mathbf{in}\ P$	M	
	$\vert$	$\mathbf{ord}\ vars\ \mathbf{in}\ N$	M	
	$\vert$	$\mathbf{ord}\ vars\ \mathbf{in}\ P$	M	
	$\vert$	$\mathbf{ord}\ vars\ \mathbf{in}\ N$	M	
$vars$	$::=$			set of variables
	$\vert$	$\varnothing$		empty set
	$\vert$	$\mathbf{fv}\ P$		free variables
	$\vert$	$\mathbf{fv}\ N$		free variables
	$\vert$	$\mathbf{fv}\ P$		free variables
	$\vert$	$\mathbf{fv}\ N$		free variables
	$\vert$	$vars_1 \cap vars_2$		set intersection
	$\vert$	$vars_1 \cup vars_2$		set union
	$\vert$	$vars_1 \backslash vars_2$		set complement
	$\vert$	$\mathbf{mv}\ P$		movable variables
	$\vert$	$\mathbf{mv}\ N$		movable variables
	$\vert$	$\mathbf{uv}\ N$		unification variables
	$\vert$	$\mathbf{uv}\ P$		unification variables
	$\vert$	$\mathbf{fv}\ N$		free variables
	$\vert$	$\mathbf{fv}\ P$		free variables
	$\vert$	$(vars)$	S	parenthesis
	$\vert$	Γ		context
	$\vert$	$\vec{\alpha}$		ordered list of variables
	$\vert$	$[\mu]vars$	M	apply moving to varset
μ	$::=$			
	$\vert$	$\cdot$		empty moving
	$\vert$	$\widetilde{\alpha}^+_1 \mapsto \widetilde{\alpha}^+_2$		Positive unit substitution
	$\vert$	$\widetilde{\alpha}^-_1 \mapsto \widetilde{\alpha}^-_2$		Positive unit substitution
	$\vert$	$\mu_1 \cup \mu_2$	M	Set-like union of movings
	$\vert$	$\overline{\mu_i}^{\,i}$		concatenate movings
	$\vert$	$\mu\vert_{vars}$	M	restriction on a set
	$\vert$	μ^{-1}	M	inversion
n	$::=$			cohort index
	$\vert$	0		
	$\vert$	$n+1$		
$\widetilde{\alpha}^+$	$::=$			positive movable variable
	$\vert$	$\widetilde{\alpha}^{+n}$		
$\widetilde{\alpha}^-$	$::=$			negative movable variable
	$\vert$	$\widetilde{\alpha}^{-n}$		

$\overrightarrow{\widetilde{\alpha}^+},\ \overrightarrow{\widetilde{\beta}^+}$	$::=$		positive movable variable list
	$\mid\ \cdot$		empty list
	$\mid\ \widetilde{\alpha}^+$		a variable
	$\mid\ \overrightarrow{\widetilde{\alpha^{+n}}}$		from a non-movable variable
	$\mid\ \overline{\overrightarrow{\widetilde{\alpha}^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\widetilde{\alpha}^-},\ \overrightarrow{\widetilde{\beta}^-}$	$::=$		negatiive movable variable list
	$\mid\ \cdot$		empty list
	$\mid\ \widetilde{\alpha}^-$		a variable
	$\mid\ \overrightarrow{\widetilde{\alpha^{-n}}}$		from a non-movable variable
	$\mid\ \overline{\overrightarrow{\widetilde{\alpha}^-}_i}^{\,i}$		concatenate lists
$P,\ Q$	$::=$		multi-quantified positive types with movable variables
	$\mid\ \alpha^+$		
	$\mid\ \widetilde{\alpha}^+$		
	$\mid\ \downarrow N$		
	$\mid\ \exists\overrightarrow{\alpha^-}.P$		
	$\mid\ [\sigma]P$	M	
	$\mid\ [\mu]P$	M	
$N,\ M$	$::=$		multi-quantified negative types with movable variables
	$\mid\ \alpha^-$		
	$\mid\ \widetilde{\alpha}^-$		
	$\mid\ \uparrow P$		
	$\mid\ P \to N$		
	$\mid\ \forall\overrightarrow{\alpha^+}.N$		
	$\mid\ [\sigma]N$	M	
	$\mid\ [\mu]N$	M	
$\widehat{\alpha}^+$	$::=$		positive unification variable
	$\mid\ \widehat{\alpha}^+$		
$\widehat{\alpha}^-$	$::=$		negative unification variable
	$\mid\ \widehat{\alpha}^-$		
	$\mid\ \widehat{\alpha}^-_{\{N,M\}}$		
$\overrightarrow{\widetilde{\alpha}^+},\ \overrightarrow{\widetilde{\beta}^+}$	$::=$		positive unification variable list
	$\mid\ \cdot$		empty list
	$\mid\ \widehat{\alpha}^+$		a variable
	$\mid\ \overrightarrow{\widehat{\alpha}^+\{vars\}}$		from a normal variable
	$\mid\ \overrightarrow{\widehat{\alpha}^+}$		from a normal variable, context unspecified
	$\mid\ \overline{\overrightarrow{\widetilde{\alpha}^+}_i}^{\,i}$		concatenate lists
$\overrightarrow{\widehat{\alpha}^-},\ \overrightarrow{\widehat{\beta}^-}$	$::=$		negative unification variable list
	$\mid\ \cdot$		empty list
	$\mid\ \widehat{\alpha}^-$		a variable
	$\mid\ \overrightarrow{\widehat{\alpha}^-\{vars\}}$		from a normal variable
	$\mid\ \overrightarrow{\widehat{\alpha}^-}$		from a normal variable, context unspecified
	$\mid\ \overline{\overrightarrow{\widehat{\alpha}^-}_i}^{\,i}$		concatenate lists

P, Q	$::=$			a positive algorithmic type (potentially with metavariables)
	$\|$	α^+		
	$\|$	$\widetilde{\alpha}^+$		
	$\|$	$\widehat{\alpha}^+\{vars\}$		
	$\|$	$\downarrow N$		
	$\|$	$\exists\overrightarrow{\alpha^-}.P$		
	$\|$	$[\sigma]P$	M	
	$\|$	$[\mu]P$	M	
	$\|$	$\mathbf{nf}\,(P')$	M	
N, M	$::=$			a negative algorithmic type (potentially with metavariables)
	$\|$	α^-		
	$\|$	$\widetilde{\alpha}^-$		
	$\|$	$\widehat{\alpha}^-\{vars\}$		
	$\|$	$\widehat{\alpha}^-$		
	$\|$	$\uparrow P$		
	$\|$	$P \rightarrow N$		
	$\|$	$\forall\overrightarrow{\alpha^+}.N$		
	$\|$	$[\sigma]N$	M	
	$\|$	$[\mu]N$	M	
	$\|$	$\mathbf{nf}\,(N')$	M	
terminals	$::=$			
	$\|$	$\exists$		
	$\|$	$\forall$		
	$\|$	$\uparrow$		
	$\|$	$\downarrow$		
	$\|$	$\rightarrow$		
	$\|$	$\leftrightarrow$		
	$\|$	$\in$		
	$\|$	$\notin$		
	$\|$	$\cdot$		
	$\|$	$\vdash$		
	$\|$	$\leqslant$		
	$\|$	$\geqslant$		
	$\|$	$\simeq$		
	$\|$	$\cup$		
	$\|$	$\cap$		
	$\|$	$\setminus$		
	$\|$	$\subseteq$		
	$\|$	$\mapsto$		
	$\|$	$\overset{u}{\simeq}$		
	$\|$	$\overset{a}{\simeq}$		
	$\|$	$\varnothing$		
	$\|$	$\models$		
	$\|$	$\dashv$		
	$\|$	$\neq$		
	$\|$	$\equiv_n$		
	$\|$	$\vee$		
	$\|$	$\Downarrow$		

formula	::=		
	|	*judgement*	
	|	$formula_1 \quad .. \quad formula_n$	
	|	$\mu : vars_1 \leftrightarrow vars_2$	
	|	μ **is bijective**	
	|	$\widehat{\sigma}$ **is functional**	
	|	$\widehat{\sigma}_1 \in \widehat{\sigma}_2$	
	|	$vars_1 \subseteq vars_2$	
	|	$vars_1 = vars_2$	
	|	$vars$ **is fresh**	
	|	$\alpha^- \notin vars$	
	|	$\alpha^+ \notin vars$	
	|	$\alpha^- \in vars$	
	|	$\alpha^+ \in vars$	
	|	if any other rule is not applicable	
	|	$N \neq M$	
	|	$P \neq Q$	
E1A	::=		
	|	$n \models N \simeq_1^A M \dashv \mu$	Negative multi-quantified type equivalence (algorit
	|	$n \models P \simeq_1^A Q \dashv \mu$	Positive multi-quantified type equivalence (algorith
A	::=		
	|	$\Gamma \models N \leqslant M \dashv \widehat{\sigma}$	Negative subtyping
	|	$\Gamma \models P \geqslant Q \dashv \widehat{\sigma}$	Positive supertyping
E1	::=		
	|	$N \simeq_1^D M$	Negative multi-quantified type equivalence
	|	$P \simeq_1^D Q$	Positive multi-quantified type equivalence
D1	::=		
	|	$\Gamma \vdash N \simeq_1^{\leqslant} M$	Negative equivalence on MQ types
	|	$\Gamma \vdash P \simeq_1^{\leqslant} Q$	Positive equivalence on MQ types
	|	$\Gamma \vdash N \leqslant_1 M$	Negative subtyping
	|	$\Gamma \vdash P \geqslant_1 Q$	Positive supertyping
D0	::=		
	|	$\Gamma \vdash N \simeq_0^{\leqslant} M$	Negative equivalence
	|	$\Gamma \vdash P \simeq_0^{\leqslant} Q$	Positive equivalence
	|	$\Gamma \vdash N \leqslant_0 M$	Negative subtyping
	|	$\Gamma \vdash P \geqslant_0 Q$	Positive supertyping
LUBF	::=		
	|	$P_1 \vee P_2 === Q$	
	|	**ord** $vars$ **in** $P === \vec{\alpha}$	
	|	**ord** $vars$ **in** $N === \vec{\alpha}$	
	|	**ord** $vars$ **in** $P === \vec{\alpha}$	
	|	**ord** $vars$ **in** $N === \vec{\alpha}$	
	|	**nf** $(N') === N$	

	|	$\mathbf{nf}\,(P') === P$	
	|	$\mathbf{nf}\,(N') === N$	
	|	$\mathbf{nf}\,(P') === P$	
	|	$\widehat{\sigma}_1 \& \widehat{\sigma}_2 === \widehat{\sigma}$	
LUB	::=		
	|	$P_1 \vee P_2 = Q$	Least Upper Bound (Least Common Supertype)
AU	::=		
	|	$\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)$	
	|	$\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)$	
Order	::=		
	|	$\mathbf{ord}\, vars\, \mathbf{in}\, N = vars'$	
	|	$\mathbf{ord}\, vars\, \mathbf{in}\, P = vars'$	
Nrm	::=		
	|	$\mathbf{nf}\,(N) = M$	
	|	$\mathbf{nf}\,(P) = Q$	
	|	$\mathbf{nf}\,(N) = M$	
	|	$\mathbf{nf}\,(P) = Q$	
SM	::=		
	|	$e_1 \& e_2 = e_3$	Unification Solution Entry Merge
	|	$\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3$	Merge unification solutions
U	::=		
	|	$N \overset{u}{\simeq} M \dashv \widehat{\sigma}$	Negative unification
	|	$P \overset{u}{\simeq} Q \dashv \widehat{\sigma}$	Positive unification
WF	::=		
	|	$\Gamma \vdash N$	Negative type well-formedness
	|	$\Gamma \vdash P$	Positive type well-formedness
	|	$\Gamma \vdash N$	Negative type well-formedness
	|	$\Gamma \vdash P$	Positive type well-formedness
judgement	::=		
	|	*E1A*	
	|	*A*	
	|	*E1*	
	|	*D1*	
	|	*D0*	
	|	*LUB*	
	|	*AU*	
	|	*Order*	
	|	*Nrm*	
	|	*SM*	
	|	*U*	
	|	*WF*	

$$
\begin{array}{lcl}
\mathit{user\_syntax} & ::= & \\
& | & \alpha \\
& | & n \\
& | & \alpha^{+} \\
& | & \alpha^{-} \\
& | & \sigma \\
& | & e \\
& | & \widehat{\sigma} \\
& | & P \\
& | & N \\
& | & \overrightarrow{\alpha^{+}} \\
& | & \overrightarrow{\alpha^{-}} \\
& | & P \\
& | & N \\
& | & \overrightarrow{P} \\
& | & \overrightarrow{N} \\
& | & \Gamma \\
& | & \overrightarrow{\alpha} \\
& | & \mathit{vars} \\
& | & \mu \\
& | & n \\
& | & \widetilde{\alpha}^{+} \\
& | & \widetilde{\alpha}^{-} \\
& | & \overrightarrow{\widetilde{\alpha}^{+}} \\
& | & \overrightarrow{\widetilde{\alpha}^{-}} \\
& | & P \\
& | & N \\
& | & \widehat{\alpha}^{+} \\
& | & \widehat{\alpha}^{-} \\
& | & \overrightarrow{\widetilde{\alpha}^{+}} \\
& | & \overrightarrow{\widehat{\alpha}^{-}} \\
& | & P \\
& | & N \\
& | & \mathit{terminals} \\
& | & \mathit{formula}
\end{array}
$$

$\boxed{n \models N \simeq_1^A M \dashv \mu}$ Negative multi-quantified type equivalence (algorithmic)

$$\frac{}{n \models \alpha^{-} \simeq_1^A \alpha^{-} \dashv \cdot}\ \textsc{E1ANVar}$$

$$\frac{n \models P \simeq_1^A Q \dashv \mu}{n \models \uparrow P \simeq_1^A \uparrow Q \dashv \mu}\ \textsc{E1AShiftU}$$

$$\frac{n \models P \simeq_1^A Q \dashv \mu_1 \quad n \models N \simeq_1^A M \dashv \mu_2 \quad \mu_1 \cup \mu_2\ \textbf{is bijective}}{n \models P \to N \simeq_1^A Q \to M \dashv \mu_1 \cup \mu_2}\ \textsc{E1AArrow}$$

$$\frac{n+1 \models [\overrightarrow{\widetilde{\alpha}^{+n}}/\overrightarrow{\alpha^{+}}]N \simeq_1^A [\overrightarrow{\widetilde{\beta}^{+n}}/\overrightarrow{\beta^{+}}]M \dashv \mu}{n \models \forall\overrightarrow{\alpha^{+}}.N \simeq_1^A \forall\overrightarrow{\beta^{+}}.M \dashv \mu|_{\mathbf{mv}\,M}}\ \textsc{E1AForall}$$

$$\frac{}{n \models \widetilde{\alpha}^{-n} \simeq_1^A \widetilde{\beta}^{-n} \dashv \widetilde{\beta}^{-n} \mapsto \widetilde{\alpha}^{-n}}\ \textsc{E1ANMVar}$$

$\boxed{n \models P \simeq_1^A Q \dashv \mu}$ Positive multi-quantified type equivalence (algorithmic)

$$\frac{}{n \models \alpha^+ \simeq_1^A \alpha^+ \dashv \cdot}\ \text{E1APVar}$$

$$\frac{n \models N \simeq_1^A M \dashv \mu}{n \models {\downarrow}N \simeq_1^A {\downarrow}M \dashv \mu}\ \text{E1AShiftD}$$

$$\frac{n+1 \models [\overrightarrow{\widetilde{\alpha}^{-n}}/\overrightarrow{\alpha^-}]P \simeq_1^A [\overrightarrow{\widetilde{\beta}^{-n}}/\overrightarrow{\beta^-}]Q \dashv \mu}{n \models \exists\overrightarrow{\alpha^-}.P \simeq_1^A \exists\overrightarrow{\beta^-}.Q \dashv \mu|_{\mathbf{mv}\,Q}}\ \text{E1AExists}$$

$$\frac{}{n \models \widetilde{\alpha}^{+n} \simeq_1^A \widetilde{\beta}^{+n} \dashv \widetilde{\beta}^{+n} \mapsto \widetilde{\alpha}^{+n}}\ \text{E1APMVar}$$

$\boxed{\Gamma \models N \leqslant M \dashv \widehat{\sigma}}$ Negative subtyping

$$\frac{}{\Gamma \models \alpha^- \leqslant \alpha^- \dashv \cdot}\ \text{ANVar}$$

$$\frac{\mathbf{nf}\,(P) \overset{u}{\simeq} \mathbf{nf}\,(Q) \dashv \widehat{\sigma}}{\Gamma \models {\uparrow}P \leqslant {\uparrow}Q \dashv \widehat{\sigma}}\ \text{AShiftU}$$

$$\frac{\Gamma \models P \geqslant Q \dashv \widehat{\sigma}_1 \quad \Gamma \models N \leqslant M \dashv \widehat{\sigma}_2}{\Gamma \models P \to N \leqslant Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2}\ \text{AArrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \models [\overrightarrow{\widehat{\alpha}^+\{\Gamma, \overrightarrow{\beta^+}\}/\alpha^+}]N \leqslant M \dashv \widehat{\sigma}}{\Gamma \models \forall\overrightarrow{\alpha^+}.N \leqslant \forall\overrightarrow{\beta^+}.M \dashv \widehat{\sigma}\backslash\overrightarrow{\widehat{\alpha}^+}}\ \text{AForall}$$

$\boxed{\Gamma \models P \geqslant Q \dashv \widehat{\sigma}}$ Positive supertyping

$$\frac{}{\Gamma \models \alpha^+ \geqslant \alpha^+ \dashv \cdot}\ \text{APVar}$$

$$\frac{\mathbf{nf}\,(N) \overset{u}{\simeq} \mathbf{nf}\,(M) \dashv \widehat{\sigma}}{\Gamma \models {\downarrow}N \geqslant {\downarrow}M \dashv \widehat{\sigma}}\ \text{AShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \models [\overrightarrow{\widehat{\alpha}^-\{\Gamma, \overrightarrow{\beta^-}\}/\alpha^-}]P \geqslant Q \dashv \widehat{\sigma}}{\Gamma \models \exists\overrightarrow{\alpha^-}.P \geqslant \exists\overrightarrow{\beta^-}.Q \dashv \widehat{\sigma}}\ \text{AExists}$$

$$\frac{\mathbf{nf}\,(P) = P' \quad vars_1 = \mathbf{fv}\,P'\backslash vars \quad vars_2\ \mathbf{is\ fresh}}{\Gamma \models \widehat{\alpha}^+\{vars\} \geqslant P \dashv (\widehat{\alpha}^+ :\geqslant P' \vee [vars_2/vars_1]P')}\ \text{APUVar}$$

$\boxed{N \simeq_1^D M}$ Negative multi-quantified type equivalence

$$\frac{}{\alpha^- \simeq_1^D \alpha^-}\ \text{E1NVar}$$

$$\frac{P \simeq_1^D Q}{{\uparrow}P \simeq_1^D {\uparrow}Q}\ \text{E1ShiftU}$$

$$\frac{P \simeq_1^D Q \quad N \simeq_1^D M}{P \to N \simeq_1^D Q \to M}\ \text{E1Arrow}$$

$$\frac{\overrightarrow{\alpha^+} \cap \mathbf{fv}\,M = \varnothing \quad \mu : (\overrightarrow{\beta^+} \cap \mathbf{fv}\,M) \leftrightarrow (\overrightarrow{\alpha^+} \cap \mathbf{fv}\,N) \quad N \simeq_1^D [\mu]M}{\forall\overrightarrow{\alpha^+}.N \simeq_1^D \forall\overrightarrow{\beta^+}.M}\ \text{E1Forall}$$

$\boxed{P \simeq_1^D Q}$ Positive multi-quantified type equivalence

$$\frac{}{\alpha^+ \simeq_1^D \alpha^+}\ \textsc{E1PVar}$$

$$\frac{N \simeq_1^D M}{\downarrow N \simeq_1^D \downarrow M}\ \textsc{E1ShiftD}$$

$$\frac{\overrightarrow{\alpha^-} \cap \mathbf{fv}\, Q = \varnothing \quad \mu : (\overrightarrow{\beta^-} \cap \mathbf{fv}\, Q) \leftrightarrow (\overrightarrow{\alpha^-} \cap \mathbf{fv}\, P) \quad P \simeq_1^D [\mu]Q}{\exists \overrightarrow{\alpha^-}.P \simeq_1^D \exists \overrightarrow{\beta^-}.Q}\ \textsc{E1Exists}$$

$\boxed{\Gamma \vdash N \simeq_1^{\leqslant} M}$ Negative equivalence on MQ types

$$\frac{\Gamma \vdash N \leqslant_1 M \quad \Gamma \vdash M \leqslant_1 N}{\Gamma \vdash N \simeq_1^{\leqslant} M}\ \textsc{D1NDef}$$

$\boxed{\Gamma \vdash P \simeq_1^{\leqslant} Q}$ Positive equivalence on MQ types

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash Q \geqslant_1 P}{\Gamma \vdash P \simeq_1^{\leqslant} Q}\ \textsc{D1PDef}$$

$\boxed{\Gamma \vdash N \leqslant_1 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash \alpha^- \leqslant_1 \alpha^-}\ \textsc{D1NVar}$$

$$\frac{\Gamma \vdash P \simeq_1^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_1 \uparrow Q}\ \textsc{D1ShiftU}$$

$$\frac{\Gamma \vdash P \geqslant_1 Q \quad \Gamma \vdash N \leqslant_1 M}{\Gamma \vdash P \to N \leqslant_1 Q \to M}\ \textsc{D1Arrow}$$

$$\frac{\Gamma, \overrightarrow{\beta^+} \vdash P_i \quad \Gamma, \overrightarrow{\beta^+} \vdash [\overrightarrow{P}/\overrightarrow{\alpha^+}]N \leqslant_1 M}{\Gamma \vdash \forall \overrightarrow{\alpha^+}.N \leqslant_1 \forall \overrightarrow{\beta^+}.M}\ \textsc{D1Forall}$$

$\boxed{\Gamma \vdash P \geqslant_1 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash \alpha^+ \geqslant_1 \alpha^+}\ \textsc{D1PVar}$$

$$\frac{\Gamma \vdash N \simeq_1^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_1 \downarrow M}\ \textsc{D1ShiftD}$$

$$\frac{\Gamma, \overrightarrow{\beta^-} \vdash N_i \quad \Gamma, \overrightarrow{\beta^-} \vdash [\overrightarrow{N}/\overrightarrow{\alpha^-}]P \geqslant_1 Q'}{\Gamma \vdash \exists \overrightarrow{\alpha^-}.P \geqslant_1 \exists \overrightarrow{\beta^-}.Q}\ \textsc{D1ExistsL}$$

$\boxed{\Gamma \vdash N \simeq_0^{\leqslant} M}$ Negative equivalence

$$\frac{\Gamma \vdash N \leqslant_0 M \quad \Gamma \vdash M \leqslant_0 N}{\Gamma \vdash N \simeq_0^{\leqslant} M}\ \textsc{D0NDef}$$

$\boxed{\Gamma \vdash P \simeq_0^{\leqslant} Q}$ Positive equivalence

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash Q \geqslant_0 P}{\Gamma \vdash P \simeq_0^{\leqslant} Q}\ \textsc{D0PDef}$$

$\boxed{\Gamma \vdash N \leqslant_0 M}$ Negative subtyping

$$\frac{}{\Gamma \vdash a- \leqslant_0 a-}\ \textsc{D0NVar}$$

$$\frac{\Gamma \vdash P \simeq_0^{\leqslant} Q}{\Gamma \vdash \uparrow P \leqslant_0 \uparrow Q} \quad \text{D0ShiftU}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash [P/a+]N \leqslant_0 M \quad M \neq \forall \beta^+.M'}{\Gamma \vdash \forall \alpha^+.N \leqslant_0 M} \quad \text{D0ForallL}$$

$$\frac{\Gamma, \alpha^+ \vdash N \leqslant_0 M}{\Gamma \vdash N \leqslant_0 \forall \alpha^+.M} \quad \text{D0ForallR}$$

$$\frac{\Gamma \vdash P \geqslant_0 Q \quad \Gamma \vdash N \leqslant_0 M}{\Gamma \vdash P \to N \leqslant_0 Q \to M} \quad \text{D0Arrow}$$

$\boxed{\Gamma \vdash P \geqslant_0 Q}$ Positive supertyping

$$\frac{}{\Gamma \vdash a+ \geqslant_0 a+} \quad \text{D0PVar}$$

$$\frac{\Gamma \vdash N \simeq_0^{\leqslant} M}{\Gamma \vdash \downarrow N \geqslant_0 \downarrow M} \quad \text{D0ShiftD}$$

$$\frac{\Gamma \vdash N \quad \Gamma \vdash [N/a-]P \geqslant_0 Q \quad Q \neq \exists \alpha^-.Q'}{\Gamma \vdash \exists \alpha^-.P \geqslant_0 Q} \quad \text{D0ExistsL}$$

$$\frac{\Gamma, \alpha^- \vdash P \geqslant_0 Q}{\Gamma \vdash P \geqslant_0 \exists \alpha^-.Q} \quad \text{D0ExistsR}$$

$\boxed{P_1 \vee P_2}$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P}$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N}$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ P}$

$\boxed{\mathbf{ord}\ vars\ \mathbf{in}\ N}$

$\boxed{\mathbf{nf}\,(N')}$

$\boxed{\mathbf{nf}\,(P')}$

$\boxed{\mathbf{nf}\,(N')}$

$\boxed{\mathbf{nf}\,(P')}$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2}$

$\boxed{P_1 \vee P_2 = Q}$ Least Upper Bound (Least Common Supertype)

$$\frac{}{\alpha^+ \vee \alpha^+ = \alpha^+} \quad \text{LUBVar}$$

$$\frac{(\mathbf{fv}\, N \cup \mathbf{fv}\, M) \models {\downarrow}N \overset{a}{\simeq} {\downarrow}M \dashv (P, \widehat{\sigma}_1, \widehat{\sigma}_2)}{{\downarrow}N \vee {\downarrow}M = \exists \overrightarrow{\alpha^-}.[\overrightarrow{\alpha^-}/\mathbf{uv}\, P]P} \quad \text{LUBShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \overrightarrow{\beta^-} = \varnothing}{\exists \overrightarrow{\alpha^-}.P_1 \vee \exists \overrightarrow{\beta^-}.P_2 = P_1 \vee P_2} \quad \text{LUBExists}$$

$\boxed{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \models \alpha^+ \overset{a}{\simeq} \alpha^+ \dashv (\alpha^+, \cdot, \cdot)} \quad \text{AUPVar}$$

$$\frac{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models {\downarrow}N_1 \overset{a}{\simeq} {\downarrow}N_2 \dashv ({\downarrow}M, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUPShift}$$

$$\frac{\overrightarrow{\alpha^-} \cap \Gamma = \varnothing \quad \Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models \exists \overrightarrow{\alpha^-}.P_1 \overset{a}{\simeq} \exists \overrightarrow{\alpha^-}.P_2 \dashv (\exists \overrightarrow{\alpha^-}.Q, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUPExists}$$

$\boxed{\Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}_1, \widehat{\sigma}_2)}$

$$\frac{}{\Gamma \models \alpha^- \overset{a}{\simeq} \alpha^- \dashv (\alpha^-, \cdot, \cdot)} \quad \text{AUNVar}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2)}{\Gamma \models {\uparrow}P_1 \overset{a}{\simeq} {\uparrow}P_2 \dashv ({\uparrow}Q, \widehat{\sigma}_1, \widehat{\sigma}_2)} \quad \text{AUNShift}$$

$$\frac{\Gamma \models P_1 \overset{a}{\simeq} P_2 \dashv (Q, \widehat{\sigma}_1, \widehat{\sigma}_2) \quad \Gamma \models N_1 \overset{a}{\simeq} N_2 \dashv (M, \widehat{\sigma}'_1, \widehat{\sigma}'_2)}{\Gamma \models P_1 \to N_1 \overset{a}{\simeq} P_2 \to N_2 \dashv (Q \to M, \widehat{\sigma}_1 \cup \widehat{\sigma}'_1, \widehat{\sigma}_2 \cup \widehat{\sigma}'_2)} \quad \text{AUNArrow}$$

$$\frac{\text{if any other rule is not applicable} \quad \Gamma \vdash N \quad \Gamma \vdash M}{\Gamma \models N \overset{a}{\simeq} M \dashv (\widehat{\alpha}^-_{\{N,M\}}, (\widehat{\alpha}^-_{\{N,M\}} :\approx N), (\widehat{\alpha}^-_{\{N,M\}} :\approx M))} \quad \text{AUNAU}$$

$\boxed{\mathbf{ord}\, vars\, \mathbf{in}\, N = vars'}$

$$\frac{\alpha^- \in vars}{\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^- = \alpha^-} \quad \text{ONVarIn}$$

$$\frac{\alpha^- \notin vars}{\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^- = \cdot} \quad \text{ONVarNIn}$$

$$\frac{}{\mathbf{ord}\, vars\, \mathbf{in}\, \widehat{\alpha}^-\{vars'\} = \cdot} \quad \text{ONUVar}$$

$$\frac{\mathbf{ord}\, vars\, \mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in}\, {\uparrow}P = \vec{\alpha}} \quad \text{OShiftU}$$

$$\frac{\mathbf{ord}\, vars\, \mathbf{in}\, P = \vec{\alpha}_1 \quad \mathbf{ord}\, vars\, \mathbf{in}\, N = \vec{\alpha}_2}{\mathbf{ord}\, vars\, \mathbf{in}\, P \to N = \vec{\alpha}_1, (\vec{\alpha}_2 \backslash \vec{\alpha}_1)} \quad \text{OArrow}$$

$$\frac{vars \cap \overrightarrow{\alpha^+} = \varnothing \quad \mathbf{ord}\, vars\, \mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in}\, \forall \overrightarrow{\alpha^+}. N = \vec{\alpha}} \quad \text{OForall}$$

$\boxed{\mathbf{ord}\, vars\, \mathbf{in}\, P = vars'}$

$$\frac{\alpha^+ \in vars}{\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^+ = \alpha^+} \quad \text{OPVarIn}$$

$$\frac{\alpha^+ \notin vars}{\mathbf{ord}\, vars\, \mathbf{in}\, \alpha^+ = \cdot} \quad \text{OPVarNIn}$$

$$\frac{}{\mathbf{ord}\, vars\, \mathbf{in}\, \widehat{\alpha}^+\{vars'\} = \cdot} \quad \text{OPUVar}$$

$$\frac{\mathbf{ord}\, vars\, \mathbf{in}\, N = \vec{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in}\, \downarrow N = \vec{\alpha}} \quad \text{OShiftD}$$

$$\frac{vars \cap \overrightarrow{\alpha^-} = \varnothing \quad \mathbf{ord}\, vars\, \mathbf{in}\, P = \vec{\alpha}}{\mathbf{ord}\, vars\, \mathbf{in}\, \exists \overrightarrow{\alpha^-}. P = \vec{\alpha}} \quad \text{OExists}$$

$\boxed{\mathbf{nf}\,(N) = M}$

$\boxed{\mathbf{nf}\,(P) = Q}$

$\boxed{\mathbf{nf}\,(N) = M}$

$$\frac{}{\mathbf{nf}\,(\alpha^-) = \alpha^-} \quad \text{NrmNVar}$$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^-\{vars\}) = \widehat{\alpha}^-\{vars\}} \quad \text{NrmNUVar}$$

$$\frac{\mathbf{nf}\,(P) = Q}{\mathbf{nf}\,(\uparrow P) = \uparrow Q} \quad \text{NrmShiftU}$$

$$\frac{\mathbf{nf}\,(P) = Q \quad \mathbf{nf}\,(N) = M}{\mathbf{nf}\,(P \to N) = Q \to M} \quad \text{NrmArrow}$$

$$\frac{\mathbf{nf}\,(N) = N' \quad \mathbf{ord}\, \overrightarrow{\alpha^+}\, \mathbf{in}\, N' = \overrightarrow{\alpha^+}'}{\mathbf{nf}\,(\forall \overrightarrow{\alpha^+}. N) = \forall \overrightarrow{\alpha^+}'. N'} \quad \text{NrmForall}$$

$\boxed{\mathbf{nf}\,(P) = Q}$

$$\frac{}{\mathbf{nf}\,(\alpha^+) = \alpha^+} \quad \text{NrmPVar}$$

$$\frac{}{\mathbf{nf}\,(\widehat{\alpha}^+\{vars\}) = \widehat{\alpha}^+\{vars\}} \quad \text{NrmPUVar}$$

$$\frac{\mathbf{nf}\,(N) = M}{\mathbf{nf}\,(\downarrow N) = \downarrow M} \quad \text{NrmShiftD}$$

$$\frac{\mathbf{nf}\,(P) = P' \quad \mathbf{ord}\, \overrightarrow{\alpha^-}\, \mathbf{in}\, P' = \overrightarrow{\alpha^-}'}{\mathbf{nf}\,(\exists \overrightarrow{\alpha^-}. P) = \exists \overrightarrow{\alpha^-}'. P'} \quad \text{NrmExists}$$

$\boxed{e_1 \& e_2 = e_3}$ Unification Solution Entry Merge

$$\frac{}{\widehat{\alpha}^+ :\geqslant P \& \widehat{\alpha}^+ :\geqslant Q = \widehat{\alpha}^+ :\geqslant P \vee Q} \quad \text{SMEPSupSup}$$

$$\frac{\mathbf{fv}\, P \cup \mathbf{fv}\, Q \vDash P \geqslant Q \dashv \widehat{\sigma}'}{\widehat{\alpha}^+ :\approx P \& \widehat{\alpha}^+ :\geqslant Q = \widehat{\alpha}^+ :\approx P} \quad \text{SMEPEqSup}$$

$$\frac{\mathbf{fv}\, P \cup \mathbf{fv}\, Q \vDash Q \geqslant P \dashv \widehat{\sigma}'}{\widehat{\alpha}^+ :\geqslant P \& \widehat{\alpha}^+ :\approx Q = \widehat{\alpha}^+ :\approx Q} \quad \text{SMEPSupEq}$$

$$\frac{}{\widehat{\alpha}^+ :\approx P \& \widehat{\alpha}^+ :\approx P = \widehat{\alpha}^+ :\approx P} \quad \text{SMEPEqEq}$$

$$\frac{}{\widehat{\alpha}^- :\approx N \& \widehat{\alpha}^- :\approx N = \widehat{\alpha}^- :\approx N} \quad \text{SMENEqEq}$$

$\boxed{\widehat{\sigma}_1 \& \widehat{\sigma}_2 = \widehat{\sigma}_3}$ Merge unification solutions

$$\frac{}{\cdot \& \widehat{\sigma} = \widehat{\sigma}} \quad \text{SMEmpty}$$

$$\frac{(\widehat{\alpha}^+ :\approx P) \in \widehat{\sigma}_2 \qquad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)} \quad \text{SMPEqEq}$$

$$\frac{(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \qquad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\geqslant P \vee Q, \widehat{\sigma}_3)} \quad \text{SMPSupSup}$$

$$\frac{(\widehat{\alpha}^+ :\approx Q) \in \widehat{\sigma}_2 \qquad \mathbf{fv}\, Q \cup \mathbf{fv}\, P \vDash Q \geqslant P \dashv \widehat{\sigma}' \qquad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\geqslant P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx Q, \widehat{\sigma}_3)} \quad \text{SMPSupEq}$$

$$\frac{(\widehat{\alpha}^+ :\geqslant Q) \in \widehat{\sigma}_2 \qquad \mathbf{fv}\, Q \cup \mathbf{fv}\, P \vDash P \geqslant Q \dashv \widehat{\sigma}' \qquad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^+) = \widehat{\sigma}_3}{(\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^+ :\approx P, \widehat{\sigma}_3)} \quad \text{SMPEqSup}$$

$$\frac{(\widehat{\alpha}^- :\approx N) \in \widehat{\sigma}_2 \qquad \widehat{\sigma}_1 \& (\widehat{\sigma}_2 \backslash \widehat{\alpha}^-) = \widehat{\sigma}_3}{(\widehat{\alpha}^- :\approx N, \widehat{\sigma}_1) \& \widehat{\sigma}_2 = (\widehat{\alpha}^- :\approx N, \widehat{\sigma}_3)} \quad \text{SMNEqEq}$$

$\boxed{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}$ Negative unification

$$\frac{}{\alpha^- \overset{u}{\simeq} \alpha^- \dashv \cdot} \quad \text{UNVar}$$

$$\frac{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\uparrow P \overset{u}{\simeq} \uparrow Q \dashv \widehat{\sigma}} \quad \text{UShiftU}$$

$$\frac{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}_1 \qquad N \overset{u}{\simeq} M \dashv \widehat{\sigma}_2}{P \to N \overset{u}{\simeq} Q \to M \dashv \widehat{\sigma}_1 \& \widehat{\sigma}_2} \quad \text{UArrow}$$

$$\frac{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\forall \overrightarrow{\alpha^+}. N \overset{u}{\simeq} \forall \overrightarrow{\alpha^+}. M \dashv \widehat{\sigma}} \quad \text{UForall}$$

$$\frac{\mathbf{fv}\, N \subseteq \mathit{vars}}{\widehat{\alpha}^-\{\mathit{vars}\} \overset{u}{\simeq} N \dashv \widehat{\alpha}^- :\approx N} \quad \text{UNUVar}$$

$\boxed{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}$ Positive unification

$$\frac{}{\alpha^+ \overset{u}{\simeq} \alpha^+ \dashv \cdot} \quad \text{UPVar}$$

$$\frac{N \overset{u}{\simeq} M \dashv \widehat{\sigma}}{\downarrow N \overset{u}{\simeq} \downarrow M \dashv \widehat{\sigma}} \quad \text{UShiftD}$$

$$\frac{P \overset{u}{\simeq} Q \dashv \widehat{\sigma}}{\exists \overrightarrow{\alpha^-}. P \overset{u}{\simeq} \exists \overrightarrow{\alpha^-}. Q \dashv \widehat{\sigma}} \quad \text{UExists}$$

$$\frac{\mathbf{fv}\, P \subseteq vars}{\widehat{\alpha}^{+}\{vars\} \overset{u}{\simeq} P \dashv \widehat{\alpha}^{+} :\approx P} \quad \textsc{UPUVar}$$

$\boxed{\Gamma \vdash N}$ Negative type well-formedness

$\boxed{\Gamma \vdash P}$ Positive type well-formedness

$\boxed{\Gamma \vdash N}$ Negative type well-formedness

$\boxed{\Gamma \vdash P}$ Positive type well-formedness

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Definition rules:        94 good    0 bad
Definition rule clauses: 165 good    0 bad
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