A Sufficient Condition for the Existence of Large Empty Convex Polygons*

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Abstract. Let P be a set of points in general position in the plane. We say that P is k-convex if no triangle determined by P contains more than k points of P in the interior. We say that a subset A of P in convex position forms an *empty polygon* (in P) if no point of $P \backslash A$ lies in the convex hull of A. We show that for any k, n there is an $N = N(k, n)$ such that any k-convex set of at least N points in general position in the plane contains an empty n-gon. We also prove an analogous statement in $\mathbb{R}^d$ for each odd $d \geq 3$.

1. Introduction

A set P of points in the plane is in *general position* if it contains no three points on a line. A finite set X of points is *in convex position* if the points of X are the vertices of a convex polygon.

In 1935 Erdős and Szekeres proved the following classical result:

Theorem 1 [4]. *For any $n \geq 3$, there is an integer $f(n)$ such that any set of at least $f(n)$ points in general position in the plane contains n points in convex position.*

We say that a subset A of P in convex position forms an *empty polygon* if no point of $P \backslash A$ lies in the convex hull of A. In 1975 Erdős [3] asked whether the following sharpening of the Erdős–Szekeres theorem is true: Is there a (smallest) finite $g(n)$ such that any set of at least $g(n)$ points in general position in the plane contains an empty

n-gon? Somewhat surprisingly, in 1983 Horton [7] gave a negative answer to this question for $n \geq 7$. It is easy to see that $g(3) = 3$, $g(4) = 5$. Harborth [6] proved that $g(5) = 10$. So far, the existence of $g(6)$ was neither proved nor disproved. In this paper we show that Erdős' question has an affirmative answer if we restrict ourselves to so-called *k*-convex sets.

We say that a finite planar point set P in general position is *k-convex* if no triangle determined by P contains more than k points of P in the interior. Thus, 0-convex sets are just sets in convex position. Károlyi et al. [8] proved that for any *n* there is an integer $N(n)$ such that any 1-convex set of at least N points in general position in the plane contains an empty *n*-gon. As far as we know, the 1-convexity is the first non-trivial condition which was proved to guarantee the existence of an empty *n*-gon in a large set of points in general position. Here we prove that for any fixed $k \geq 1$ the *k*-convexity is also sufficient:

Theorem 2. *For any k, n there is an $N = N(k, n)$ such that any k-convex set of at least N points in general position in the plane contains an empty n-gon.*

We conjecture that a statement analogous to Theorem 2 is true also in $\mathbb{R}^d$ for any $d \geq 2$. Besides $d = 2$, we can prove it for all odd $d \geq 3$ (see Theorem 3 below).

Let $d \geq 2$. We say that a set P of points in $\mathbb{R}^d$ is in *general position* if there are no $\leq d + 1$ affinely dependent points in P. We say that a finite set of points in $\mathbb{R}^d$ is *in convex position* if none of the points lies in the convex hull of the others. We say that a finite set P of points in general position in $\mathbb{R}^d$ is *k-convex* if no simplex determined by $d + 1$ points of P contains more than k points of P in the interior.

A notion analogous to empty polygons is the notion of holes. Let P be a set of points in $\mathbb{R}^d$, $d \geq 2$. An *n-hole* of P is any set X of n points of P in convex position such that the convex hull of X contains no other points of P.

Theorem 3. *For any odd $d \geq 3$ and for any k, n, there is an $N = N_d(k, n)$ such that any k-convex set of at least N points in general position in $\mathbb{R}^d$ contains an n-hole.*

We remark that for any fixed $d \geq 2$ (similarly as for $d = 2$), if we define $g_d(n)$ as the smallest number such that any set of at least $g_d(n)$ points in general position in $\mathbb{R}^d$ contains an *n*-hole, then $g_d(n)$ does not exist for *n* sufficiently large (see [9]).

Organization of the Paper.   The proof of Theorem 2 is given in Section 2. The proof of Theorem 3 is given in Section 3. An alternative proof of Lemma 4 (applied in the proof of Theorem 2) is given in Section 4. Section 5 contains concluding remarks and open problems.

2.   Proof of Theorem 2

Suppose $k \geq 0$ is fixed. Let P be a *k*-convex set of N points in general position in the plane, where N is very large.
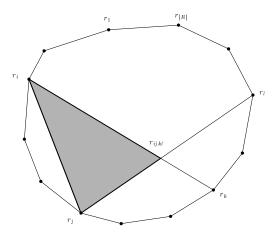
Fig. 1. The triangle $r_i r_j r_{ijkl}$.

Let $Q \subseteq P$ be the set of vertices of conv P. Any triangulation of conv $P = $ conv Q partitions the interior of conv P into $|Q| - 2$ triangles. Since P is k-convex, it follows that $|P| = N \leq |Q| + (|Q| - 2)k$. Consequently, $|Q| = \Omega(N)$.

We color $\binom{Q}{3}$ by the $k+1$ colors $0, 1, \ldots, k$ so that a triple $\{q_1, q_2, q_3\} \in \binom{Q}{3}$ is colored by color i if the interior of the triangle $q_1 q_2 q_3$ contains exactly i points of P. By Ramsey's theorem, Q contains a monochromatic subset R of size $\Omega(\log \log |Q|) = \Omega(\log \log N) = \Omega(\log^{[2]} N)$ ($\log^{[t]}$ denotes t iterations of log). Every triangle determined by R contains a constant number, c, of points of P in the interior.

Let $r_1, r_2, \ldots, r_{|R|}$ be the points of R listed in counterclockwise order. We could start the numbering from any vertex, but from now on the numbering is fixed. Consider the following coloring of $\binom{R}{4}$ by $c + 1$ colors $0, \ldots, c$. A 4-tuple $\{r_i, r_j, r_k, r_l\}, 1 \leq i < j < k < l \leq |R|$, is colored by color q if the triangle $r_i r_j r_{ijkl}$ contains exactly q points of P in the interior (where r_{ijkl} denotes the intersection point of the lines $r_i r_k$, $r_j r_l$, see Fig. 1).

Another application of Ramsey's theorem gives a monochromatic subset S of R of size $\Omega(\log^{[3]} |R|) = \Omega(\log^{[5]} N)$. Let $s_1, s_2, \ldots, s_{|S|}$ be the points of S listed in the same order as in the sequence $r_1, r_2, \ldots, r_{|R|}$. Then there are two integers $a, b \geq 0, a + b = c$, such that the following holds. If $s_i, s_j, s_k, s_l, 1 \leq i < j < k < l \leq |S|$, are four points of S, then the $2c = 2a + 2b$ points of P inside the quadrilateral $s_i s_j s_k s_l$ are partitioned by the diagonals into four subsets of sizes a, a, b, b in the way shown in Fig. 2 (here we use that S is monochromatic and that each triangle determined by $R \supseteq S$ contains exactly c points in the interior).

For $i = 2, \ldots, |S| - 1$, let T_i denote the triangle determined by the lines $s_i s_1$, $s_i s_{|S|}$, $s_{i-1} s_{i+1}$ (see Fig. 3). Here is the key lemma:

Lemma 4. *If $|S| \geq 6$, then $b = 0$ and the interior of each of the triangles T_i contains exactly $a = c$ points of P.*

Before proving Lemma 4, we derive Theorem 2 from it. Since conv S can be triangulated into $|S| - 2$ triangles, its interior contains exactly $(|S| - 2)c$ points of P.
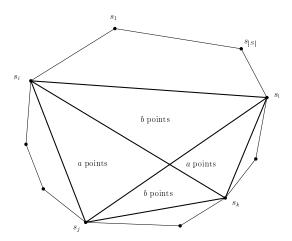
Fig. 2. Distribution of the points in every quadrilateral $s_i s_j s_k s_l$, $1 \le i < j < k < l \le |S|$.

Lemma 4 now shows that conv S contains no point of P outside of $T_2 \cup T_3 \cup \cdots \cup T_{|S|-1}$. It follows that if we choose a point $p_i \in P$ inside T_i closest to the line $s_{i-1} s_{i+1}$, then $s_1 p_2 p_3 \cdots p_{|S|-1} s_{|S|}$ is an empty $|S|$-gon in P. Since $|S| = \Omega(\log^{[5]} N)$ is at least n for N sufficiently large, Theorem 2 follows. It remains to prove Lemma 4.

We remark that Lemma 4 does not hold for $|S| = 5$—Fig. 4 shows a possible placement of points of P inside a pentagon for any $a \ge b \ge 0$, $a + b = c$.

Proof of Lemma 4. We first prove that $b = 0$. Figure 5 shows the two non-degenerate cases differing by the way the diagonals partition the interior of the hexagon $s_1 s_2 s_3 s_4 s_5 s_6$ (the remaining degenerate case when the diagonals $s_1 s_4$, $s_2 s_5$, $s_3 s_6$ have a common intersection point can be seen as a special case of any of them). We treat the two cases at the same time—they are very similar.
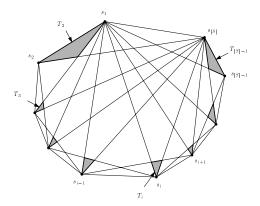

Fig. 3. The triangles T_i.
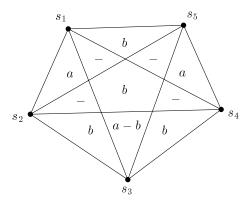
Fig. 4. Possible distribution of points inside the pentagon $s_1s_2s_3s_4s_5$, if $|S| = 5$, $a \geq b \geq 0$, $a + b = c$.

For each picture in Fig. 5, consider the regions R_1 through R_{12} (or R_{13}). For a region X, let $|X|$ denote the number of points of P lying in the interior of X.

We have $|R_1| = a$ (due to the partition of the points in the quadrilateral $s_1s_2s_3s_6$) and $|R_4| = b$ (quadrilateral $s_1s_2s_5s_6$). Consequently, we get $|R_3| = |R_5| = |R_{10}| = 0$ from $|R_3| + |R_4| + |R_5| + |R_{10}| \leq b$ (quadrilateral $s_1s_3s_4s_6$) and also $(|R_{13}| =)|R_2| = |R_8| = |R_9| = 0$ from $(|R_{13}|+)|R_1| + |R_2| + |R_8| + |R_9| = a$ (quadrilateral $s_1s_2s_4s_5$).

Analogously, $|R_7| = a$ and thus $|R_6| = |R_{11}| = |R_{12}| = 0$ follows from $(|R_{13}|+)$ $|R_6| + |R_7| + |R_{11}| + |R_{12}| = a$. Thus, $(|R_{13}|+)|R_8| + |R_9| + |R_{10}| + |R_{11}| + |R_{12}| = 0$ but this quantity also equals b (quadrilateral $s_1s_2s_5s_6$). Thus, $b = 0$ (and $a = c$).

We now easily show that $|T_i| = a$ for each i. We have $|T_2| = a$ (quadrilateral $s_1s_2s_3s_{|S|}$) and $|T_{|S|-1}| = a$ (quadrilateral $s_1s_{|S|-2}s_{|S|-1}s_{|S|}$). For $i = 3, \ldots, |S| - 2$, let U_i denote the triangle $s_{i-1}s_ix_i$, where x_i is the intersection point of the diagonals s_1s_i, $s_{i-1}s_{i+1}$ (see Fig. 6). Then $|U_i| = b = 0$ (quadrilateral $s_1s_{i-1}s_is_{i+1}$) and $|U_i \cup T_i| = a$ (quadrilateral $s_{i-1}s_is_{i+1}s_{|S|}$). Thus, $|T_i| = a$ for each i. □
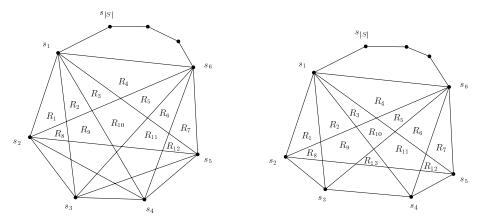

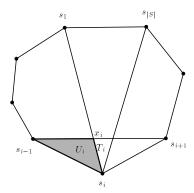**Fig. 5.** Possible partitions of the interior of the hexagon $s_1s_2s_3s_4s_5s_6$ and the regions R_i.

Fig. 6.   The region U_i.

3.   Proof of Theorem 3

Let $d \geq 3$ be odd. We may suppose that $n \geq d + 2$. Let P be a k-convex set of N points in general position in $\mathbb{R}^d$, where $N = N_d(k, n)$ is very large. By a generalization of the Erdős–Szekeres theorem to higher dimensions (e.g., see Exercise 7.3.6 of [5] or Corollary 3.8 of [2]), P contains a subset Q of N' vertices of a cyclic polytope in $\mathbb{R}^d$, where $N' = N'_d(k, n)$ is still large. Since P is k-convex, we can color $\binom{Q}{d+1}$ by the $k + 1$ colors $0, 1, \ldots, k$ so that a $(d + 1)$-tuple $\{q_1, \ldots, q_{d+1}\} \in \binom{Q}{d+1}$ is colored by color i if the interior of the simplex with the vertex set $\{q_1, \ldots, q_{d+1}\}$ contains exactly i points of P. If N' is sufficiently large, then, by Ramsey's theorem, Q contains a monochromatic subset R of size n. Every simplex determined by R contains a constant number, c, of points in the interior. We will show that $c = 0$.

Let $r_1, \ldots, r_n$ be the points of R listed in the "cyclic-polytope" order. Set

$$C := \operatorname{conv}\{r_1, \ldots, r_{d+2}\},$$

$$C_i := \operatorname{conv}\{r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_{d+2}\}, \qquad \text{for} \quad i = 1, \ldots, d + 2.$$

Lemma 5.   *The cyclic polytope C can be triangulated into simplices in the following two ways*: $C = C_1 \cup C_3 \cup \cdots \cup C_{d+2}$ *and* $C = C_2 \cup C_4 \cup \cdots \cup C_{d+1}$.

Proof.   If $i \neq j$ are of the same parity, then the interiors of C_i, C_j are disjoint because C_i, C_j lie on different sides of their common facet $\operatorname{conv}\{r_\alpha : 1 \leq \alpha \leq d + 2, \alpha \neq i, j\}$. It remains to show that $C \subseteq C_1 \cup C_3 \cup \cdots \cup C_{d+2}$ and $C \subseteq C_2 \cup C_4 \cup \cdots \cup C_{d+1}$.

The unique Radon partition of the set $R' := \{r_1, \ldots, r_{d+2}\}$ is $R' = R'_{\text{odd}} \cup R'_{\text{even}}$, where $R'_{\text{odd}} = \{r_1, r_3, \ldots, r_{d+2}\}$, $R'_{\text{even}} = \{r_2, r_4, \ldots, r_{d+1}\}$ (this is well known and follows from the fact that any other pair of disjoint subsets of R' is separated by a hyperplane). Algebraically,

$$\sum_{i \text{ odd}} \beta_i r_i = \sum_{i \text{ even}} \beta_i r_i,$$

$$\text{where} \quad \sum_{i \text{ odd}} \beta_i = \sum_{i \text{ even}} \beta_i = 1, \quad \beta_i \geq 0 \quad (i = 1, \ldots, d + 2).$$

Let $x \in C$. Then

$$x = \sum_{i=1}^{d+2} \alpha_i r_i, \qquad \text{where} \quad \sum_{i=1}^{d+2} \alpha_i = 1, \quad \alpha_i \geq 0 \quad (i = 1, \ldots, d+2).$$

If we sum up the equality $x = \sum_{i=1}^{d+2} \alpha_i r_i$ and a proper non-negative multiple of the equality

$$0 = \sum_{i \text{ even}} \beta_i r_i - \sum_{i \text{ odd}} \beta_i r_i,$$

we obtain an equality

$$x = \sum_{i=1}^{d+2} \alpha_i' r_i, \qquad \text{where} \quad \sum_{i=1}^{d+2} \alpha_i' = 1, \quad \alpha_i' \geq 0 \quad (i = 1, \ldots, d+2),$$

such that $\alpha_i' = 0$ holds for at least one odd index i. It follows that $x \in C_i$ and thus $C \subseteq C_1 \cup C_3 \cup \cdots \cup C_{d+2}$. Similarly we can prove that $C \subseteq C_2 \cup C_4 \cup \cdots \cup C_{d+1}$. $\square$

Since the interior of each of the simplices C_i contains the same number, c, of points of P, Lemma 5 gives $((d+3)/2)c = ((d+1)/2)c$ and thus $c = 0$. Thus, the interior of the convex hull of R contains no point of P. This concludes the proof of Theorem 3. $\square$

We remark that if d is even, then every triangulation of a cyclic polytope in $\mathbb{R}^d$ partitions the polytope into the same number, $\binom{n-d/2-1}{d/2}$, of simplices (because any two triangulations in this situation are connected by a finite number of flips that preserve the triangulation size). Therefore the above proof does not work if d is even.

4. Second Proof of Lemma 4

Here we give an alternative proof of Lemma 4. We believe that this proof (rather than the proof given in Section 2) might have a generalization in even dimensions which would consequently give an analogue of Theorems 2 and 3 in even dimensions.

We may suppose that $|S| = 2m$ is even. For $i = 1, 2, \ldots, 2m - 1$, let v_i be the point of intersection of the main diagonals $s_i s_{i \pm m}$, $s_{i+1} s_{i+1 \pm m}$ (we always choose "+" or "−" so that the obtained index lies in the range $\{1, 2, \ldots, 2m\}$) and let V_i be the triangle $s_i s_{i+1} v_i$. Further, put $v_{2m} := v_m$ and let V_{2m} be the triangle $s_{2m} s_1 v_{2m}$.

We first show that the triangles $V_1, \ldots, V_{2m}$ cover conv S. This is clearly true for $m = 2$ (we have $v_1 = v_2 = v_3 = v_4$ in this case). We further proceed by induction on m. Let $m \geq 3$. We denote by x the point of intersection of the diagonals $s_1 s_{m+1}$, $s_3 s_{m+3}$ (see Fig. 7). By the induction assumption, conv$(S \setminus \{s_2, s_{m+2}\})$ is covered by the $2m - 2$ triangles $\triangle s_1 s_3 x$, V_3, V_4, $\ldots$, V_m, $\triangle s_{m+1} s_{m+3} x$, V_{m+3}, $V_{m+4}, \ldots, V_{2m}$. The statement now follows from the following three facts (see Fig. 7): (a) conv $S = $ conv$(S \setminus \{s_2, s_{m+2}\}) \cup \triangle s_1 s_2 s_3 \cup \triangle s_{m+1} s_{m+2} s_{m+3}$, (b) $\triangle s_1 s_3 x \cup \triangle s_1 s_2 s_3$ is covered by $V_1 \cup V_2$, (c) $\triangle s_{m+1} s_{m+3} x \cup \triangle s_{m+1} s_{m+2} s_{m+3}$ is covered by $V_{m+1} \cup V_{m+2}$.

For each triangle V_i, we now find a smaller triangle V_i' contained in V_i such that the interiors of the triangles V_i, V_i' contain the same number of points of P. Then $V_i \setminus V_i'$ will
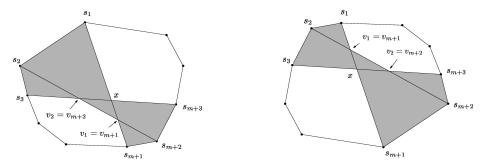
Fig. 7. The point x and the main diagonals $s_i s_{m+i}$, $i = 1, 2, 3$ (two combinatorially different cases). The shaded region is covered by $V_1 \cup V_2 \cup V_{m+1} \cup V_{m+2}$.

contain no point of P. Since conv $S = V_1 \cup \cdots \cup V_{2m}$, points of P inside conv S will lie in $V_1' \cup \cdots \cup V_{2m}'$.

For $i = 1, \ldots, m - 1$, we define V_i' as the intersection of the triangles $s_i s_{i+1} s_{2m}$ and $s_i s_{i+1} s_{i+2}$. For $i = m + 1, \ldots, 2m - 1$, we define V_i' as the intersection of the triangles $s_1 s_i s_{i+1}$ and $s_{i-1} s_i s_{i+1}$. Further, we define V_m' as the intersection of the triangles $s_{m-1} s_m s_{m+1}$ and $s_m s_{m+1} s_{m+2}$, and we define V_{2m}' as the intersection of the triangles $s_{2m} s_1 s_2$ and $s_{2m-1} s_{2m} s_1$ (see Fig. 8).

Clearly, if $i \in \{1, \ldots, 2m\}$, then $V_i' \subset V_i$ and, by the choice of S, the interiors of V_i', V_i contain the same number (a or b) of points of P. Thus, as argued above, all points of P inside conv S lie in $V_1' \cup \cdots \cup V_{2m}'$. By the choice of S, the interiors of V_m' and V_{2m}' contain b points each, and the interior of any other triangle V_i' contains a points of P. Since the triangle $s_1 s_2 s_{2m-1}$ contains V_1' and is disjoint from the interiors of all other triangles V_i', its interior contains a points of P. At the same time, it contains c points of
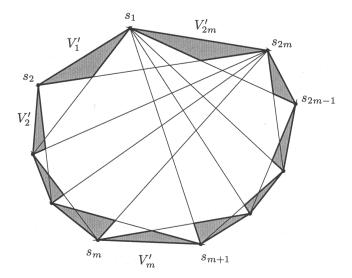

Fig. 8. The triangles V_i'.

P. It follows that $a = c$ and thus $b = 0$. The interior of V'_{2m} contains $b = 0$ points of P. Therefore, each triangle $s_1 s_i s_{2m}$, $i = 2, \ldots, 2m - 1$, contains its $a = c$ interior points belonging to P in its intersection with V'_i, which is the triangle T_i. $\qquad\square$

5. Concluding Remarks and Open Problems

5.1. *The Modular Version of the Erdős–Szekeres Theorem*

Károlyi et al. [8] studied 1-convex sets in relation to the following conjecture:

Conjecture 1 [1]. *For any $n \geq 3$ and $p \geq 2$, there exists an integer $B = B(n, p)$ such that every set P of at least B points in general position in the plane determines a convex n-gon such that the number of points of P in its interior is divisible by p.*

Conjecture 1 was proved in [1] for every $n \geq p + 2$, Károlyi et al. [8] proved it for every $n \geq 5p/6 + O(1)$. In their proof they applied the special case $k = 1$ of Theorem 2. In fact, Theorem 2 gives Conjecture 1 for all $n \geq 3p/4 + O(1)$. This is shown in [10]. In [10] we also discuss a d-dimensional version of Conjecture 1 and prove it if $d \geq 3$ is odd or $n \geq n_0(p, d)$.

5.2. *A Cap or an (Empty) Cup*

We believe that point sets with no large holes have a very special structure. Therefore, it is interesting to seek conditions ensuring the existence of a hole of any fixed size in every sufficiently large point set in $\mathbb{R}^d$. One such condition (k-convexity) is given in Theorems 2 and 3.

We say that a set X of n points in the plane, no two on a vertical line, is an *n-cap* (*n-cup*, respectively) if it is in convex position and all points of X lie on and above (on and below, respectively) the line connecting the leftmost point of X with the rightmost point of X (see Fig. 9).

In their classical paper [4], Erdős and Szekeres proved that any set of at least $\binom{2n-4}{n-2} + 1$ points in general position in the plane, no two of them on a vertical line, contains an n-cap or an n-cup. From this they concluded that Theorem 1 holds with $f(n) := \binom{2n-4}{n-2} + 1$.
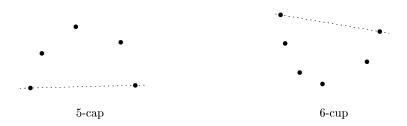
5-cap   6-cup

Fig. 9. A cap and a cup.

We conjecture that actually the following stronger result holds:

Conjecture 2.   *For any $n \geq 3$, there is an integer $f'(n)$ such that any set of at least $f'(n)$ points in general position in the plane, no two of them on a vertical line, contains an n-cap or an empty n-cup. (An empty n-cup is an n-cup which forms an empty n-gon.)*

Conjecture 2, if true, would imply that for any k and n any sufficiently large set of points in the plane with no k-cap contains an empty n-gon.

5.3.   *Strengthenings of Theorems 2 and 3*

Theorems 2 and 3 can be generalized in the following two ways:

- the general position is not necessary,
- the statement is true also if the considered set is a union of a fixed number of k-convex sets.

These generalizations may be combined in one theorem:

Theorem 6.   *If $d = 2$ or $d \geq 3$ is odd, then, for any three positive integers k, l, n, there is an $N = N(d, k, l, n)$ with the following property. Let $P = P_1 \cup P_2 \cup \cdots \cup P_l$ be a set of N points in $\mathbb{R}^d$ such that for each i the convex hull of any $d + 1$ points of P_i contains at most k other points of P_i. Then the set P has an n-hole.*

Proof.   Suppose that $d = 2$, that k, l, n are three positive integers, and that N is very large. Let $P = P_1 \cup P_2 \cup \cdots \cup P_l$ be as in the theorem. Without loss of generality, we may suppose that the sets P_i are pairwise disjoint and that $|P_1| \geq N/l$. The set P_1 cannot contain more than $k + 3$ points on a line (otherwise the convex hull of three of them would contain more than k other points of P_1). A greedy algorithm gives a set P_1' of N_1 points of P_1 in general position such that no segment determined by two points of P_1' contains other points of P_1 and N_1 is still very large. By the same proof as for Theorem 2, we can show that P_1' has a subset P_1'' forming an empty N_1'-gon in P_1, where N_1' is still very large. If conv P_1'' contains fewer than N_1'/n of points of $P_2 \cup \cdots \cup P_l$, then P_1'' contains n points forming an empty n-gon in $P_1 \cup \cdots \cup P_l$. Otherwise, we consider the points of $P \setminus P_1$ lying in conv P_1''. Their number is at least N_1'/n, which is still a very large number. If we repeat the same argument (at most) $l - 1$ times, we obtain an empty n-gon in $P_1 \cup \cdots \cup P_l$ formed by n points of one of the sets $P_2, \ldots, P_l$.

If $d \geq 3$ is odd, we can proceed similarly, applying Theorem 3 instead of Theorem 2. $\square$

5.4.   *A Density Statement Does Not Hold*

It does not suffice that 99% of the simplices determined by P are empty:

Claim 7.   *For any $d \geq 2$ and for any $\varepsilon > 0$, there is an integer $c = c(d, \varepsilon) \geq 3$ such that for any $N \geq d + 1$ there is a set P of N points in general position in $\mathbb{R}^d$ such*
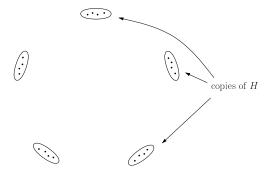
Fig. 10. $3/\varepsilon$ copies of H along a circle.

that more than $(1 - \varepsilon)\binom{|P|}{d+1}$ *simplices determined by* P *are* $(d + 1)$*-holes (i.e., empty simplices), but the set* P *does not contain a c-hole.*

Proof.   First, let $d = 2$. Suppose for simplicity that $3/\varepsilon$ is an integer and that N is divisible by $3/\varepsilon$. Let H be a set of $N/(3/\varepsilon) = \varepsilon N/3$ points in general position in the plane with no empty 7-gon (arbitrarily large sets with no empty 7-gon were constructed by Horton [7]). We shrink the second coordinates of points in H (without changing the combinatorial structure of H) so that all lines determined by H will be almost horizontal. Further, we take $3/\varepsilon$ copies of H rotated along a large circle (see Fig. 10). The union of these copies will be our set P. It contains no empty convex $(18/\varepsilon + 1)$-gon (an empty polygon cannot contain seven points from a copy of H). Moreover, each triple of points from distinct copies of H is empty. There are more than

$$\frac{3/\varepsilon - 1}{3/\varepsilon} \cdot \frac{3/\varepsilon - 2}{3/\varepsilon}\binom{n}{3} > (1 - \varepsilon)\binom{n}{3}$$

such triples. This completes the case $d = 2$. If $d > 2$, we proceed similarly, using the construction of [9] of arbitrarily large sets of points in general position in $\mathbb{R}^d$ with no c_d-hole, where c_d is a constant depending on d only. $\square$

Acknowledgment

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Note added in proof.   An alternative proof of Theorem 2 is given in the very recent manuscript by G. Kun and G. Lippner, "Large convex empty polygons in k-convex sets". The proof gives $N(k, n) \le 2^{(k+2)^{n+1}}$.