Thickness and Coarseness of Graphs

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§ 0. Introduction

Graphs in this paper are finite and undirected, without loops or parallel edges. The genus $\gamma(G)$ of a graph G denotes the lowest genus of any closed orientable surface in which G can be embedded, while Σ_k denotes the closed orientable surface of genus k — i.c. the sphere with k handles. An immersion $f\colon G \to \Sigma_k$ is a continuous map whose singularities occur in pairs of interior points of edges of G. If G is a graph, $V(G)$ denotes the set of vertices of G and $E(G)$ the set of edges.

The main results of the paper are Theorem 1.4, which shows that thickness of a graph may be defined in a purely geometric fashion; Corollary 3.3, which shows that the thickness of a graph is at most one more than its local crossing number; and Corollary 5.4, which shows that if H is the homomorphic image of G, then $\gamma(H) \leq \gamma(G)$. The latter result is due to Ringel, but no proof of it seems to exist in the literature.

An outline of the paper follows: In § 1 we define a geometric variant of thickness, called real thickness, and prove that the two notions coincide. We had already announced this result in [2]. In § 2 the concepts of linear thickness and real linear thickness are defined and their relationship is examined. We discuss in particular an application to circuit design.

In § 3 we introduce Ringel's concept of local crossing number and prove that the thickness of a graph is at most one more than its local crossing number by constructing a *crossing graph* G_f for any immersion f of G and describing properties of f in terms of properties of G_f.

Section 4 deals with a geometric variant to coarseness, called real coarseness, and we prove that the real coarseness is at least as great as the (ordinary) coarseness. We conjecture that the analogue of Theorem 1.4 holds — namely, that coarseness and real coarseness coincide.

Finally, in § 5, we investigate the genus, and coarseness of the homomorphic images of a graph G, and find that they are bounded above by the corresponding quantity for G. To do so, we note that for any homomorphism $f: G \to H$ which is connected (i.e. $f^{-1}(v)$ is connected for all $v \in V(H)$), there exists a subgraph G' of G and a commutative triangle

$$\begin{tikzcd} G \arrow[rr, "f"] & & H \\ & G' \arrow[ul, "i"] \arrow[ur, "f'"'] & \end{tikzcd}$$

where i is the inclusion of the subgraph G' into G, f' is the restriction of f to G', and f' is contractible in the sense that $f'^{-1}(v)$ is a tree for every $v \in V(H)$.

§ 1. Real Thickness

Let $f: G \to \Sigma_k$ be an immersion. We define the *thickness* of f, $\mathrm{th}(f)$, to be the least integer r such that there exist subgraphs $G_1, \ldots, G_r$ of G with $G_1 \cup \cdots \cup G_r = G$ and $f | G_i: G_i \to \Sigma_k$ an embedding, $1 \leq i \leq r$. Now we define the *real k-thickness* of G, $\mathrm{th}_k(G)$, by

$$\mathrm{th}_k(G) = \min\{\mathrm{th}(f) \,|\, f: G \to \Sigma_k \text{ an immersion}\}.$$

On the other hand, one may define (as in HARARY [1], p. 120) the *k-thickness* of G, $\vartheta_k(G)$, to be the least integer t such that there exist subgraphs $G_1, \ldots, G_t$ of G with $G_1 \cup \cdots \cup G_t = G$ and $\gamma(G_i) \leq k$, $1 \leq i \leq t$. This differs from the preceeding definition in that it requires the existence of embeddings $f_i: G_i \to \Sigma_k$ but does not insist that the embeddings be "compatable" in the sense that there exist an immersion $f: G \to \Sigma_k$ whose restriction to each G_i is an embedding.

Obviously, we have the trivial lemma:

Lemma 1.1. $\mathrm{th}_k(G) \geq \vartheta_k(G)$ *for all graphs G and all integers $k \geq 0$.*

To prove the converse of this lemma, we will need to make the embeddings f_i compatable. But, as ROY LEVOW has observed, that can always be done because of the following lemma.

Lemma 1.2. *Let $X = \{x_1, \ldots, x_n\} \subset \Sigma_k$ and let x, y be any two distinct points in $\Sigma_k - X$. Then there is a homeomorphism $h: \Sigma_k \to \Sigma_k$ such that $h(x) = y$ and $h(x_j) = x_j$, $1 \leq j \leq n$.*

Of course this lemma is true in a much broader context and with additional severe restrictions on the kind of homeomorphism allowed, but we shall content ourselves with a simple—and direct—proof.

Proof. Since $\Sigma_k - X$ is path connected, we may choose a simple arc α joining x and y with $\alpha \subset \Sigma_k - X$. Since α is contractible, we may find arbitrarily small closed neighborhoods of α which are (homeomorphic to) closed discs—that is, inside every neighborhood U of α, there exists a closed neighborhood β with $\alpha \subsetneqq \beta \subset U$ and a homeomorphism $b: \beta \to D = \{z \in C \mid |z| \leq 1\}$ from β onto the standard unit disc in the complex plane. Thus, in particular we have such a β and $b: \beta \to D$ when $U = \Sigma_k - X$.

Now it is easy to find a homeomorphism $h': D \to D$ carrying $b(x)$ to $b(y)$ and such that $h' | \dot{D} = 1$ where $\dot{D} = \{z \in C \mid |z| = 1\}$. We define the homeomorphism $h: \Sigma_k \to \Sigma_k$ as follows:

$$h(s) = \begin{cases} (b^{-1} \cdot h' \cdot b)(s) & s \in \beta \\ s & s \in \Sigma_k - \beta \end{cases}$$

The reader may easily verify that h is the required homeomorphism.

An obvious inductive application of the above lemma yields the proposition:

Proposition 1.3. Let G be a graph with subgraphs $G_1, \ldots G_r$ and let $f_i: G_i \to \Sigma_k$ be immersions, $1 \leq i \leq r$. Then there is a homeomorphism $h: \Sigma_k \to \Sigma_k$ such that, putting $g_i = h \circ f_i$, we have $g_i |_{V(G_i) \cap V(G_j)} = g_j |_{V(G_i) \cap V(G_j)}$ for all i, j, $1 \leq i, j \leq r$.

Now we can state the theorem toward which we have been aiming.

Theorem 1.4. *Let G be a graph. Then* $\mathrm{th}_k(G) = \vartheta_k(G)$.

Proof. By Lemma 1.2, it suffices to prove $th_k(G) \leq \vartheta_k(G)$. Let $G = G_1 \cup \cdots \cup G_r$ where $r = \vartheta_k(G)$, and let $f_i: G_i \to \Sigma_k$ be embeddings $1 \leq i \leq r$. We may assume, without loss of generality, that if $1 \leq i \neq j \leq r$, then $E(G_i) \cap E(G_j) = \emptyset$; for if some edge appears in both G_i and G_j we may delete it from G_i.

Now we may apply Proposition 1.3 to obtain embeddings $g_i: G_i \to \Sigma_k$ which agree on vertices. The condition that $E(G_i) \cap E(G_j) = \emptyset$ when $i \neq j$ allows us to define an immersion $g: G \to \Sigma_k$ by requiring

$$g | G_i = g_i, \quad 1 \leq i \leq r.$$

Thus, $\mathrm{th}_k(G) \leq th(g) = \vartheta_k(G)$.

§ 2. Linear Thickness

In this section we shall consider only immersions and embeddings in the euclidean plane R^2. As before, one may define (planar) thickness and real (planar) thickness which are, of course, nothing but o-thickness and real o-thickness, respectively. The advantage to dealing with the plane, rather than the sphere, is that we can bring in the concept of linearity.

If $f: G \to R^2$ is any immersion from a graph G into the plane, we say that f is *linear* if, given any two vertices v and w in G and an edge e joining them, the image $f(e)$ of e is a straight line segment joining $f(v)$ and $f(w)$. One may define the real linear thickness of G, $\mathrm{th}^{\mathrm{lin}}(G)$, to be

$$\mathrm{th}^{\mathrm{lin}}(G) = \min\{\mathrm{th}(f) \mid f: G \to R^2 \text{ linear immersion}\}$$

and there is a corresponding definition of $\vartheta^{\mathrm{lin}}(G)$ in terms of subgraphs which can be linearly embedded in the plane.

However, we have at our disposal the well-known theorem of WAGNER, FÁRY, and STEIN who all showed independently that

Theorem 2.1. *G is planar if and only if G can be linearly embedded in the plane.*

It is an immediate consequence of this theorem that

Corollary 2.2. $\vartheta^{\mathrm{lin}}(G) = \vartheta(G)$.

As in 1.2 we have the trivial remark:

Lemma 2.3. $\mathrm{th}^{\mathrm{lin}}(G) \geq \vartheta^{\mathrm{lin}}(G)$.

Unfortunately, Lemma 1.3, which permitted us to make a collection of immersions compatable, fails to preserve linearity. In fact, it seems implausible to us that the analogue of Theorem 1.4 holds, and we conjecture that it actually fails.

Conjecture 2.4. There exists a graph G with $\mathrm{th}^{\mathrm{lin}}(G) > \vartheta^{\mathrm{lin}}(G)$.

Since in micro-circuit or printed circuit design, one represents wires by straight lines, we have the following:

Application 2.5. If G is the graph corresponding to a circuit C, then $\vartheta^{\mathrm{lin}}(G) =$ minimum number of "blocks" or "sheets" necessary to accomodate C.

In micro-circuits even a length of wire may act as a circuit element. Thus, one wants to connect blocks by short wires. Hence, the appropriate invariant is $\mathrm{th}^{\mathrm{lin}}(G)$, rather $\vartheta^{\mathrm{lin}}(G)$.

§ 3. Local Crossing Number and Thickness

RINGEL has introduced in [4] the concept of local crossing numer. If $f: G \to \Sigma_k$ is an immersion, we define the *local crossing number*, $\lambda(f)$, to be the maximum number of edge crossings which occur on any one edge. Then we define the local crossing number of G on Σ_k to be

$$\lambda_k(G) = \min\{\lambda(f) \mid f: G \to \Sigma_k \text{ an immersion}\}.$$

Thus $\lambda_k(G) = 0$ if and only if $\gamma(G) \leq k$.

We want to relate local crossing number to thickness. First we shall need a definition. Let $f: G \to \Sigma_k$ be an immersion of G into a surface. We construct the *crossing graph* G_f of f follows: the vertices of G_f are the edges of G, and two vertices are joined in G_f if and only if the corresponding edges of G have a crossing (with respect to f). Thus, f is an embedding if and only if G_f is totally disconnected.

The following results relate f and G_f.

Proposition 3.1. Let $f: G \to \Sigma_k$ be an immersion. Then $\mathrm{th}(f) = \chi(G_f)$, where $\chi(G_f)$ denotes the chromatic number of G_f.

Proof. Suppost $G = G_1 \cup \cdots \cup G_r$, where $r = th(f)$, and $f|G_i \to \Sigma_k$ are embeddings. We may assume, as in the proof of 1.4, that $E(G_i) \cap E(G_j) = \emptyset$ if $i \neq j$. Then $V(G_f) = E(G) = E(G_1) \cup \cdots \cup E(G_r)$ so $\chi(G_f) \leq r = \mathrm{th}(f)$.

Conversely, suppose $s = \chi(G_f)$ and $V(G_f) = V_1 \cup \cdots \cup V_s$ where $V_i \cap V_j = \emptyset$ if $i \neq j$. Let E_i be the subset of $E(G)$ corresponding to V_i and let G_i be a subgraph of G with $E(G_i) = E_i$. Then $E(G) = E(G_1) \cup \cdots \cup E(G_s)$ and, if every vertex of G belongs to some edge of G, then $V(G) = V(G_1) \cup \cdots \cup V(G_s)$. If G has isolated vertices which belong to no edge, we can simply add these vertices to one of the G_i to insure that

$$G = G_1 \cup \cdots \cup G_s.$$

Since $E(G_i) = E_i$, we have $f|G_i$ an embedding. Therefore, $\mathrm{th}(f) \leq s = \chi(G_f)$.

Proposition 3.2. Let $f: G \to \Sigma_k$ be an immersion. Then $\lambda(f) = \Delta(G_f)$, where $\Delta(G_f)$ denotes the maximum valence of G_f.

Proof. Suppose $e \in E(G)$ and $v \in V(G_f)$ is the corresponding vertex. Obviously, the valence at v is precisely equal to the number of edge crossings on e.

Theorem 3.3. *Let $f: G \to \Sigma_k$ be an immersion. Then*

$$\mathrm{th}(f) \leq 1 + \lambda(f).$$

Proof. Consider the crossing graph G_f. It is well-known that for any graph G',

$$\chi(G') \leq 1 + \Delta(G'),$$

see, for example, [1], p. 128. But taking $G' = G_f$ and using Propositions 3.1 and 3.2 yield $\mathrm{th}(f) \leq 1 + \lambda(f)$, as required.

Corollary 3.3. Let $k \geq 0$. Then $\mathrm{th}_k(G) \leq 1 + \lambda_k(G)$.

Proof. Let $f: G \to \Sigma_k$ be an immersion with $\lambda(f) = \lambda_k(G)$. Then $\mathrm{th}_k(G) \leq \mathrm{th}(f) \leq 1 + \lambda(f) = 1 + \lambda_k(G)$.

Note that if H is any graph, then we can find a graph G, an integer $k \geq 0$, and an immersion $f: G \to \Sigma_k$ such that $G_f = H$. In fact, we can even make G connected. It is interesting to speculate, however, what graphs H can be realized in this fashion if we require f to be a minimal (with respect to edge crossings) immersion of G or if we choose the integer k ahead of time. In particular, we ask: What graphs H can be realized by an immersion of some graph G into the plane?

§ 4. Real Coarseness

Erdös has defined the *k-coarseness* of G, $c_k(G)$, as the greatest integer p for which there exist edge-disjoint subgraphs $G_1, \ldots, G_p$ of G with $G = G_1 \cup \cdots \cup G_p$ and $\lambda(G_j) > k$ (see [1], p. 121). We define the *coarseness*, $\mathrm{cs}(f)$, *of an immersion* $f: G \to \Sigma_k$ to be the greatest integer q for which there exist edgedisjoint subgraphs $G_1, \ldots, G_q$ of G with $G = G_1 \cup \cdots \cup G_q$ and $f | G_j$ not an embedding for any j. Now we define the *real k-coarseness*, $cs_k(G)$, of G to be

$$\mathrm{cs}_k(G) = \min \{\mathrm{cs}(f) \,|\, f: G \to \Sigma_k \text{ an immersion}\}.$$

Note that $\lambda(G) \leq k \Leftrightarrow \mathrm{c}_k(G) = 0 \Leftrightarrow \mathrm{cs}_k(G) = 0$.

Proposition 4.1. $\mathrm{cs}_k(G) \geq \mathrm{c}_k(G)$.

Proof. Suppose $\mathrm{c}_k(G) = p$. Then we can write $G = G_1 \cup \cdots \cup G_p$ as above. Let $f: G \to \Sigma_k$ be an immersion. Then $f | G_j$ is not an embedding since $\gamma(G_j) > k$. Thus $\mathrm{cs}(f) \geq p = \mathrm{c}_k(G)$. But f was arbitrary so we have $\mathrm{cs}_k(G) \geq \mathrm{c}_k(G)$.

We conjecture that the reverse inequality also holds.

Conjecture 4.2. $\mathrm{cs}_k(G) = \mathrm{c}_k(G)$.

§ 5. Epimorphisms of Graphs

Let $f: G \to H$ be a (connected) homomorphism of graphs (see, for example, Ore [4], p. 85). We call f an *epimorphism* if it is onto—that is, if every vertex or edge in H is the image of a vertex or edge in G. We call a homomorphism f *contractible* if, for every $v \in V(H)$, the subgraph $f^{-1}(v)$ of G is a tree (equivalently, $f^{-1}(v)$ is contractible).

Now suppose $f: G \to H$ is an epimorphism. We can find a subgraph G' of G such that $f' = f | G'$ is a contractible epimorphism. For simply consider the subgraphs $f^{-1}(v)$, $v \in V(H)$. In each such graph, delete edges until only a spanning tree is left, and let $G' = G$ minus all the deleted edges. Then $f | G'$ is still an epimorphism and $(f | G')^{-1}(v)$ is a tree so $f | G'$ is contractible. Thus, we have proved

Lemma 5.1. *Let $f: G \to H$ be an epimorphism. Then there is a subgraph G' of G such that $f|G'$ is a contractible epimorphism.*

Let ϱ be any integer invariant of graphs. We call ϱ *regular* if $\varrho(H) \leq \varrho(G')$ whenever there is a contractible epimorphism $f': G' \to H$ and $\varrho(G') \leq \varrho(G)$ whenever G' is a subgraph of G. An immediate consequence is

Lemma 5.2. *If ϱ is regular, then $\varrho(H) \leq \varrho(G)$ whenever there is an epimorphism $f: G \to H$.*

Let us consider some examples of regular invariants.

Lemma 5.3. *γ is regular.*

Proof. Suppose $f': G' \to H$ is a contractible epimorphism. Let $\varphi': G' \to \Sigma_k$ be an embedding. Then the subgraphs $f'^{-1}(v)$, $v \in V(H)$, being contractible, can all be shrunk to points without changing Σ_k. (Actually one shrinks small closed discs about each subgraph $f'^{-1}(v)$ to a point.) One obtains thereby an embedding of H in Σ_k. Therefore, $\gamma(H) \leq \gamma(G')$. Obviously, any embedding of G in Σ_k restricts to an embedding of G' in Σ_k so $\gamma(G') \leq \gamma(G)$.

Corollary 5.4. (Ringel) If $f: G \to H$ is an epimorphism, then $\gamma(H) \leq \gamma(G)$.

Remark 5.5. *ϑ_k is not regular.*

Since the inverse image of disjoint sets is disjoint, we have

Lemma 5.6. *c_k is regular.*

Corollary 5.7. *If $f: G \to H$ is an epimorphism and $k \geq 0$, then $c_k(G) \geqq c_k(H)$.*

Note, however, that cr_k, the crossing number on a surface of genus k, is *not* regular. For by "vertex-splitting", as in Tutte [5], one can take a graph H and obtain a new graph G with $\mathrm{cr}_k(G) < \mathrm{cr}_k(H)$. But clearly there is a (contractible) epimorphism $f: G \to H$.

Let G and G' be graphs. We say that G and G' are homeomorphic if their topological realizations are homeomorphic. This is equivalent to requiring that $\bar{G}$ and $\bar{G}'$ be isomorphic where $\bar{G}$ and $\bar{G}'$ are obtained from G and G', respectively, by surpressing all vertices of degree 2.

The following theorem, proved in Busacker and Saaty [0] p. 195, shows that k-thickness is *not* a topological invariant.

Theorem. *Let G be a graph and suppose $1 < \mathrm{th}_k(G)$. Then, for any integer t, $1 < t \leq \mathrm{th}_k(G)$, there is a graph G' homeomorphic to G and such that $th_k(G') = t$.*

In fact, G' is obtained from G by introducing vertices of degree 2.

However, it is obvious that genus and crossing number are topological invariants. Thus, an invariant may be topological with or without being regular.

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