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The Thickness of Graphs: A Survey

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Abstract. We give a state-of-the-art survey of the thickness of a graph from both a theoretical and a practical point of view. After summarizing the relevant results concerning this topological invariant of a graph, we deal with practical computation of the thickness. We present some modifications of a basic heuristic and investigate their usefulness for evaluating the thickness and determining a decomposition of a graph in planar subgraphs.

Key words: Thickness, maximum planar subgraph, branch and cut

1. Introduction

In VLSI circuit design, a chip is represented as a hypergraph consisting of nodes corresponding to macrocells and of hyperedges corresponding to the nets connecting the cells. A chip-designer has to place the macrocells on a printed circuit board (which usually consists of superimposed layers), according to several designing rules. One of these requirements is to avoid crossings, since crossings lead to undesirable signals. It is therefore desirable to find ways to handle wire crossings of the graph representing the chip.

In practice, crossing-wires must be laid out in different layers. There are two approaches for distributing the nets to the layers. According to the first method, one of the wires must change its layer with help of so-called *contact cuts* or *vias* whenever a crossing between two wires occurs. Unfortunately, the presence of too many contact cuts leads to an increase in area and consequently to a higher probability of faulty chips. Therefore, a requirement of this manufacturing method is to reduce crossings as much as possible.

If a large number of crossings is unavoidable, a second approach is appropriate. The representing graph is decomposed into planar subgraphs, each completely embedded in one layer, which is not used by the other planar subgraphs. Since no contact cuts are used, the manufacturing cost measure of this method is the number of layers. An application of this approach was given by Aggarwal, Klawe and Shor [3]. They proposed a layout-algorithm with a provably good layout-area. Since the algorithm needs a priori a decomposition of the graph in planar subgraphs, a graph-theoretic treatment seems helpful.

Indeed, both approaches have a graph-theoretic counterpart. In the first one we look for the minimum number of edge-crossings needed in a graph-embedding, the so-called *crossing number* $\nu(G)$ of a graph G. In the second approach, the minimum number of planar subgraphs, whose union is the original graph, is requested. This number is called the *thickness* $\theta(G)$ of a graph G.

Another application of the thickness is in the scheduling of multihop radio networks. Ramanathan and Lloyd [43, 44] gave approximation algorithms for the length is a schedule which is a function of the thickness of a graph.

The *thickness problem* is to determine the thickness of a graph. Unfortunately, the thickness is only known for restricted graph classes, e.g., complete and complete bipartite graphs. Therefore, we have to concentrate on bounds for the thickness. Besides, since the thickness problem is $\mathcal{NP}$-hard, there is little hope of devising an exact algorithm to evaluate the thickness of arbitrary graphs.

The paper is organized as follows. In Section 2 we deal with planar graphs and algorithms for computing a maximal planar subgraph. In Section 3 we list all known results concerning the thickness and in the next section we discuss some related problems. In Section 5 we analyze strategies of decomposing a graph in planar subgraphs. Finally, in the last section, we state some results which do not fit into the general pattern.

2. Planarity and Maximal Planarization

We assume familiarity with standard graph-theoretic terminology. For a survey, we refer the reader to Beineke and Wilson [15] or Harary [25]. As planarity is a basic concept for the thickness of a graph and maximal planarization algorithms are used in Section 5, we briefly list some results.

Among all graphs, *planar graphs* are of special interest. These graphs are the graphs that can be drawn in the plane without edge-crossings. Since planarity is a strong restriction on the structure of a graph, some problems can be solved efficiently for planar graphs even if they are intractable in the general case.

After detecting nonplanarity of a graph, using a standard planarity testing algorithm (see, e.g., Booth and Lueker [13] or Hopcroft and Tarjan [28]), it is often favorable to first extract a possibly large planar subgraph and treat the remaining edges independently. If no edge can be added to this subgraph without destroying planarity, the subgraph is called a *maximal planar subgraph*. A planar subgraph of greatest order is called a *maximum planar subgraph*.

There are two classical maximal planarization algorithms, one based on the Hopcroft and Tarjan planarity testing algorithm and the other based on a PQ-tree approach. The fastest algorithm using the first method is the $O(|E|\log|V|)$ algorithm of Cai, Han and Tarjan [17]. The history of maximal planarization algorithms using the PQ-approach is quite interesting [33, 34, 41] and appears to be still incomplete [36].

All these algorithms only guarantee that the resulting planar subgraph is a spanning tree together with an unknown small number of additional edges, which results in a worst-case ratio of a maximal planar subgraph to a maximum planar subgraph of $\frac{1}{3}$. An improved worst-case ratio of $\frac{4}{9}$ can be achieved using the algorithm of Călinescu et al. [16], which uses a so-called *triangle cactus* approach. Although determining a maximum planar subgraph is $\mathcal{NP}$-hard [37], the branch and cut algorithm of Jünger and Mutzel [29, 30] solves many problems to optimality (particularly problems on sparse graphs) or achieves a good approximation.

3. Theoretical Results

Beineke and Wilson [15] and Beineke [8] published earlier survey papers on the thickness of graphs. Since some interesting recent results are missing, in this section, we list all results that we are aware of. Proofs are omitted and interested readers are referred to the literature.

The thickness problem originated in 1961 in the following "Research Problem" by Harary [24], which came to him through Selfridge:

> "Prove or disprove the following conjecture: For any graph G with 9 points, G or its complementary graph $\overline{G}$ is nonplanar".

In the following year, Harary, Battle and Kodoma [10] and Tutte [46] independently gave a proof (they checked all planar subgraphs with nine nodes!) of Selfridge's conjecture. In other words, K_9 is not biplanar, i.e., not the union of two planar subgraphs. Generalizing the term of biplanarity, Tutte [47] defined the thickness of a graph. For notational convenience, we refer to the planar subgraphs in the following as layers.

Evidently, $\theta(G) = 1$ if and only if G is planar. A further observation is that an examination of the thickness can be restricted to biconnected graphs, since $\theta(G) = \max_j\{\theta(G_j)\}$ holds for the biconnected components $G_j (j = 1, 2, \ldots, b)$ of a graph G. Moreover, a subgraph of a graph must have a thickness not larger than the graph itself and consequently the thickness is a monotone topological invariant of a graph.

According to a corollary of Euler's polyhedron-formula, a planar graph $G = (V, E)$ has at most $3|V| - 6$ edges. This corollary is used to derive a lower bound for the thickness of a graph.

Theorem 3.1. *If $G = (V, E)$ is a graph with $|V| = n (n > 2)$ and $|E| = m$, then*

i) $\theta(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil$,
ii) $\theta(G) \geq \left\lceil \frac{m}{2n-4} \right\rceil$, *if G has no triangles.*

In the 1960's the cornerstone of the thickness-work on special graph classes was laid by Harary and Beineke, who published the first results on the thickness of complete [9] and complete bipartite graphs [11]. But the determination of a "nice" formula describing the thickness of complete graphs has a long history and was completed by Alekseev and Gončakov [2]. By the way, the question of whether

$\theta(K_{16}) = 3$ or $\theta(K_{16}) = 4$ gives rise to a little anecdote, since this question was the subject of a mathematical competition: Harary made a public offer of £10 to anyone who could compute the thickness of K_{16}. It lasted until 1972, when Jean Mayer, surprisingly a professor of french literature (!), won the prize by proving that $\theta(K_{16}) = 3$ [39].

We now list all known formulas describing the thickness of several graph classes. It is interesting to note that in the case of complete, complete bipartite graphs and hypercubes the lower bound of Theorem 3.1 is already the exact value. We start with the complete graphs.

Theorem 3.2 [2]. *The thickness of the complete graph K_n is*

$$\theta(k_n) = \left\lfloor \frac{n+7}{6} \right\rfloor, \quad \textit{for } n \neq 9, 10 \quad \textit{and} \quad \theta(K_9) = \theta(K_{10}) = 3.$$

As a by-product, the proof of Theorem 3.2 yields the following corollary.

Corollary 3.3 [2, 6]. *The thickness of the n-dimensional octahedron $K_{n(2)}$ is*

$$\theta(K_{n(2)}) = \left\lceil \tfrac{1}{3} n \right\rceil.$$

Along with their work on complete graphs, Beineke and Harary have computed the thickness of complete bipartite graphs in most cases.

Theorem 3.4 [11]. *The thickness of the complete bipartite graph $K_{m,n}$ is*

$$\theta(K_{m,n}) = \left\lceil \frac{m \cdot n}{2(m+n-2)} \right\rceil,$$

except possibly if m and n are both odd, $m \leq n$ and there is an integer k satisfying

$$n = \left\lfloor \frac{2k(m-2)}{m-2k} \right\rfloor.$$

As a corollary, we obtain the thickness of the regular complete bipartite graphs.

Corollary 3.5 [11]. *The thickness of the complete bipartite graph $K_{n,n}$ is*

$$\theta(K_{n,n}) = \left\lfloor \frac{n+5}{4} \right\rfloor.$$

Another graph class for which the thickness can be determined is that of the hypercubes, whose thickness was evaluated by Kleinert [35]. The hypercube Q_n is the graph whose vertices are the ordered n-tupels of 0's and 1's, two vertices being joined if and only if they differ in exactly one coordinate.

Theorem 3.6 [35]. *The thickness of the hypercube Q_n is*

$$\theta(Q_n) = \left\lfloor \frac{n+1}{4} \right\rfloor.$$

Recently, Jünger et al. [31] have shown that the thickness of a certain minor-excluded class of graphs is less than or equal to two. As a special case they obtained the following result.

Theorem 3.7 [31]. *If G is a graph without K_5-minors, then $\theta(G) \leq 2$.*

Moreover, graphs with thickness two have drawn some attention in the field of graph-drawing, where they are used in the study of so-called *rectangle-visibility* graphs [20, 27].

To our knowledge, the thickness of no other graph class has been settled yet. Moreover, we cannot expect to find a nice formula describing the thickness of an arbitrary graph, since the thickness problem was proven to be $\mathcal{NP}$-hard by Mansfield [38].

Hence, we turn to upper bounds. A simple consideration gives the order $O(n)$ of the thickness of a graph, since the formula of the thickness of complete graphs operates as an upper bound for arbitrary graphs. In the early 90's, two new results dealing with upper bounds were published. Dean, Hutchinson and Scheinerman [21] correlate the thickness of a graph with the number of edges and Halton [23] uses the maximal degree of a graph to compute an upper bound of the thickness of a graph. In the following theorem we summarize these three approaches.

Theorem 3.8. *If $G = (V, E)$ is a graph with $|V| = n$ $(n > 10)$, $|E| = m$ and maximal degree d, then*

i) [2] $\theta(G) \leq \left\lfloor \frac{n+7}{6} \right\rfloor$,

ii) [21] $\theta(G) \leq \left\lfloor \sqrt{\frac{m}{3}} + \frac{3}{2} \right\rfloor$,

iii) [23] $\theta(G) \leq \left\lceil \frac{d}{2} \right\rceil$.

Since graphs arising in practice are usually sparse and have a small maximal degree, Halton's attempt to relate the thickness of a graph to the maximal degree of the graph seems to be the most appropriate approach (see Section 5).

Halton also makes the following conjecture, which might influence the design of integrated circuits, since current chip-designers mainly use only two layers for designing a chip.

Conjecture 3.9 [23]. *Any graph of degree not exceeding six has thickness not exceeding two.*

4. Modifications of the Ground-Concept

In this section we will look at several modifications of the ground-concept of thickness, e.g., restrictions on the properties of each planar subgraph or considerations of the thickness on other surfaces.

Tutte [47] defined the *t-minimal* graphs to be the graphs of thickness t whose proper subgraphs have thickness less than t. According to Kuratowski's theorem there are infinitely many 2-minimal graphs, but they can easily be characterized as subdivisions of K_5 or $K_{3,3}$. The consideration of t-minimal graphs could be helpful in the investigation of the structure of a graph with given thickness. This originated in the fact that a graph G has thickness t $(t \geq 2)$ if and only if G contains a t-minimal, but no $(t+1)$-minimal subgraph. In other words, the t-minimal graphs are the forbidden subgraphs for the graphs of thickness not greater than t. Furthermore, in the same paper, Tutte derived the first results on t-minimal graphs and proved the existence of infinitely many t-minimal graphs satisfying several properties.

Hobbs and Grossman [26] extended these results. They derived the existence of another class of t-minimal graphs and showed that each t-minimal graph $(t \geq 2)$ is at least t-edge-connected. While the proofs of the existence of the classes of t-minimal graphs in [47] and [26] are non-constructive, Širáň and Horák [45] gave an explicit construction of an infinite number of t-minimal graphs of connectivity two, edge-connectivity t and minimal degree t. In addition, Beineke [5] proved that the $K_{2t-1,4t^2-10t+7}$ are t-minimal, as did Bouwer and Broere [4] for the $K_{4t-5,4t-5}$.

The thickness of a graph can be related to two other topological invariants of a graph: the crossing number $\nu(G)$ and the genus $\gamma(G)$ (see, e.g., Harary [25]) of a graph G. Whereas the simple formula $\theta(G) \leq \nu(G) + 1$ fulfills the first relation, the situation for the genus is not that easy. However, Asano [1] proved that $\theta(G) \leq \gamma(G) + 1$ holds, if G contains no triangle. Furthermore, he showed that a graph of genus 1 has thickness 2. Dean and Hutchinson [19] strengthened Asano's result in proving that $\theta(G) \leq 6 + \sqrt{2 \cdot \gamma(G) - 2}$.

In what follows we deal with restrictions on the shape of the planar subgraphs of the decomposition.

The *arboricity* $\Upsilon(G)$ of a graph G is the minimal number of forests whose union is G. In contrast to other topological invariants of a graph, the arboricity can be exactly determined using Nash-William's formula [40] $\Upsilon(G) = \max_{H \subseteq G} \left\lceil \frac{m_H}{n_H - 1} \right\rceil$ for an induced subgraph H of G with m_H edges and n_H nodes. Clearly, $\theta(G) \leq \Upsilon(G)$ and $\Upsilon(G) \leq 3 \cdot \theta(G)$, since a maximal planar subgraph is at most three times as large as a spanning tree. Analogously to their upper bound for the thickness, Dean, Hutchinson and Scheinerman [21] found an upper bound for the arboricity.

Another restriction on a layer is the outer-planarity. The *outerthickness* $\theta_o(G)$ of a graph G is the minimal number of outerplanar graphs whose union is G. In addition to the trivial relation $\theta(G) \leq \theta_o(G)$, Guy [22] has derived some results for the outerthickness of a graph.

The *tripartite* and *bipartite thickness* of a graph are defined almost analogously to the other modifications. Walther [48] and Wessel [49] gave some results for complete graphs.

Due to the application in the design of integrated circuits, the *degree-4 thickness* $\theta^4(G)$ of a graph G has been defined as the minimal number of planar subgraphs with maximal degree four, whose union is G. Using an explicit construction, Bose and Prabhu [14] computed the degree-4 thickness of complete graphs in almost all cases.

Theorem 4.1 [14]. *The degree-4 thickness of the complete graph K_n is*

i) $n > 5$: $\theta^4(K_n) = \left\lfloor \frac{n+3}{4} \right\rfloor$, *except if* $n = 4p + 1 (p \geq 3)$,

ii) $n \leq 5$: $\theta^4(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor$.

Moreover, they obtained the degree-4 thickness of the complete bipartite graphs, using an iterative method.

Theorem 4.2 [14]. *The degree-4 thickness of the complete bipartite graph $K_{m,n}$ is*

i) $\theta^4(K_{m,n}) = \left\lfloor \frac{m+5}{4} \right\rfloor$, *for* $m = n$,

ii) $\theta^4(K_{m,n}) = \left\lfloor \frac{m+3}{4} \right\rfloor$, *for* $m \leq n - 2$,

iii) $\theta^4(K_{m,n}) = \left\lfloor \frac{m+5}{4} \right\rfloor$, *for* $m = n - 1$, *except if* $m = 4r + 2 (r \geq 1)$.

A generalization of this concept is a node-restriction not only on four but also on any degree $k \in \mathbb{N}$. Bose and Prabhu have also analyzed this modification in the case of complete and complete bipartite graphs for small values of k.

The *book-thickness* $\theta_b(G)$ of a graph G is the smallest number n such that G has an n-book embedding, i.e., an arrangement of vertices in a line along the spine of the book and edges on the pages in such a way that edges residing on the same page do not cross. A survey can be found in [12]. The relations to outerthickness and thickness are given by $\theta_o(G) \leq \theta_b(G)$ and $\theta(G) \leq \left\lceil \frac{\theta_b(G)}{2} \right\rceil$.

Another modification of the ground-concept is made by considering the thickness on other surfaces. The *S-thickness* $\theta_S(G)$ of a graph G on a surface S is the minimal number of S-embeddable graphs whose union is G. Using Euler's generalized polyhedron-formula (see, e.g., Beineke and Wilson [15]), one derives a lower bound for the S-thickness similar to Theorem 3.1.

Beineke [7] has reported formulas for some surfaces by extending his constructions fo the planar cases. Independently, Ringel [42] found the "toroidal" thickness.

Theorem 4.3 [7]. *The S-thickness of the complete graph K_n $(n > 2)$ is*

projective plane: $\left\lfloor \frac{n+5}{6} \right\rfloor$,

torus: $\left\lfloor \frac{n+4}{6} \right\rfloor$,

double-torus: $\left\lfloor \frac{n+3}{6} \right\rfloor$.

Cases which still remain undetermined occur more frequently for complete bipartite graphs on other surfaces than in the planar case. However, if we restrict ourselves to the regular complete bipartite graphs, Beineke [7] has found the following formulas.

Theorem 4.4 [7]. *The S-thickness of the regular complete bipartite graph* $K_{n,n}$ *is*

$$\begin{aligned} \textit{torus}: &\ \left\lfloor \tfrac{n+3}{4} \right\rfloor, \\ \textit{double-torus}: &\ \left\lfloor \tfrac{n+3}{4} \right\rfloor, \\ \textit{triple-torus}: &\ \left\lfloor \tfrac{n+2}{4} \right\rfloor, \\ \textit{projective plane}: &\ \left\lfloor \tfrac{n+4}{4} \right\rfloor, \\ \textit{Klein bottle}: &\ \left\lfloor \tfrac{n+3}{4} \right\rfloor. \end{aligned}$$

5. Heuristic Approaches

In this section we deal with practical computation of the thickness of a graph. As previously mentioned, the thickness problem was proven to the $\mathcal{NP}$-hard by Mansfield [38]. Therefore, we cannot expect to find a polynomial algorithm and have to turn to heuristics.

A basic approach (and we did not find any other in the literature) is to extract iteratively a possibly large planar subgraph until the resulting graph is planar. The achieved decomposition in planar subgraphs is not necessarily minimal, but we at least get an upper bound for the thickness.

The different methods of computing a maximal or maximum planar subgraph introduced in Section 2 lead to different heuristics. The general outline in either case is as follows.

Heuristic Thick

Input: Graph $G = (V, E)$
Output: Upper bound $\theta'(G)$ of $\theta(G)$,
Decomposition of G in $\theta'(G)$ planar subgraphs

(1) Initialize: $G' := G, t := 1$
(2) while G' nonplanar do
  (2.1) $t := t + 1$
  (2.2) determine a maximal/maximum planar subgraph M of G'
  (2.3) $G' := G' - M$
(3) endwhile
(4) $\theta'(G) := t$

We compare the results of the following three heuristics for planarization. The Thick$_{HT}$ heuristic is based on the Cai, Han and Tarjan algorithm in an implementation of Jordan [32]. The basic algorithm for the Thick$_{PQ}$ heuristic is by Jayakumar, Thulasiraman and Swamy in an implementation of Winter [50]. Finally, the Thick$_{JM}$ heuristic is founded on the branch and cut algorithm of Jünger and Mutzel [30].

First, the three heuristics are compared on the complete and complete bipartite graphs, since the exact values of the thickness of these graphs are known. Table 1 and Table 2 show that the Thick$_{HT}$ and Thick$_{PQ}$ heuristics produce nearly the

Table 1. Complete graphs K_n

n	THICK_{HT}	THICK_{PQ}	THICK_{JM}	$\theta(K_n)$
10	3	3	3	3
15	4	4	4	3
20	5	6	5	4
25	7	7	6	5
30	8	8	7	6
35	9	9	8	7
40	10	11	9	7
45	11	12	10	8
50	13	13	11	9
55	15	14	12	10
60	15	15	13	11
65	17	16	14	12
70	18	18	15	12
75	19	19	16	13
80	21	20	17	14
85	23	22	18	15
90	23	23	19	16
95	24	24	20	17
100	26	26	21	17

Table 2. Complete bipartite graphs $K_{n,n}$

n	THICK_{HT}	THICK_{PQ}	THICK_{JM}	$\theta(K_{n,n})$
10	4	4	4	3
15	6	6	5	5
20	7	7	7	6
25	9	9	8	7
30	10	10	9	8
35	12	12	11	10
40	13	14	12	11
45	15	15	13	12
50	16	17	14	13

same values. For complete graphs, their values are about 38% away from the optimal solution (on average), whereas the values of the heuristic THICK_{JM} using the exact planarization algorithm, is only 20% away on average.

The results are much better for complete bipartite graphs. The obtained quality of the solutions is about 24% off the optimal solution for THICK_{HT} and THICK_{PQ}, whereas it is only 12% off for THICK_{JM}.

The resulting values of these graph classes are easy to compare with the exact values, but they lack expressiveness for the application. Shown in Table 3 are some results for real-world graphs originating in VLSI-design. Since the exact values are not known, we have to use the lower and upper bounds described in Theorems 3.1 and 3.8 in order to assess the achieved results.

Table 3. Special graphs

n	m	l.b.	Thick$_{HT}$	Thick$_{PQ}$	Thick$_{JM}$	u.b.
28	75	2	2	2	2	3
38	76	2	2	2	2	2
45	98	2	2	2	2	6
50	142	2	3	3	2	5
50	183	2	3	3	3	5
90	201	2	2	2	2	2
100	248	2	3	3	3	4
100	269	2	3	3	3	3
166	504	2	5	4	3	10
200	403	2	2	2	2	2
200	514	2	3	3	3	6
200	701	2	4	4	3	8
300	495	2	2	2	2	4
680	3103	2	4	5	4	15

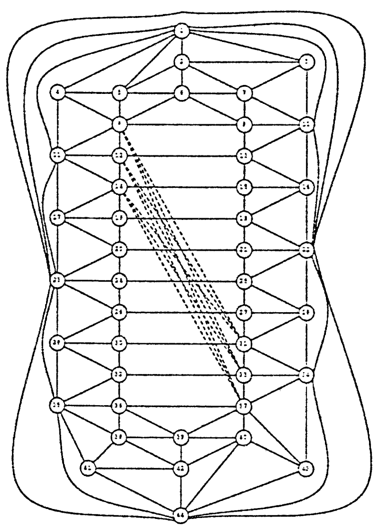

Fig. 1. Graph G

Here, our results indicate that the algorithm based on the HT- and PQ-algorithm produces a fairly good approximation of the thickness. Using the JM-algorithm produces only slightly better results.

Our experiments could lead to the conjecture that the choice of a maximum planar subgraph in each iteration of the heuristic would always result in a smaller thickness. By giving a counterexample, we show that this is not the case in general. Consider the graph depicted in Fig. 1.

After subtracting the maximum planar subgraph, we obtain the dotted-lined $K_{3,3}$. Since this graph is nonplanar, we get a value of 3. In Fig. 2, a decomposition of the same graph in two planar subgraphs can be seen.

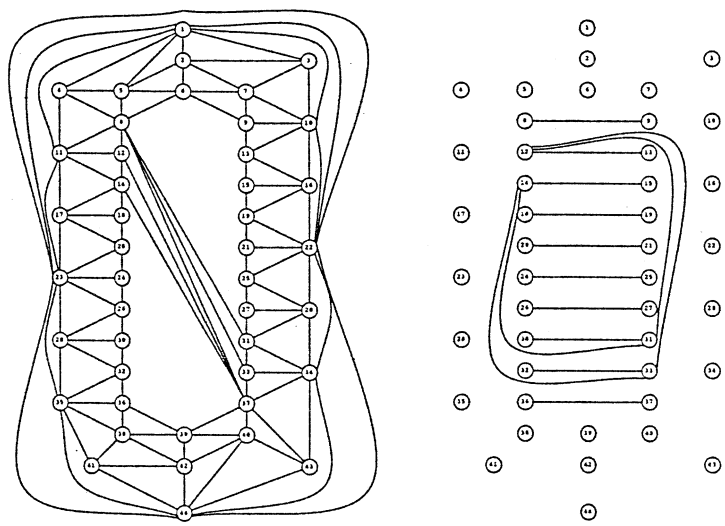

Fig. 2. Decomposition of G

Table 4. CPU-time in seconds for the graphs of Table 3

n	m	THICK$_{HT}$	THICK$_{PQ}$	THICK$_{JM}$
28	75	0.3	0.2	2
38	76	0.3	0.3	1002
45	98	0.4	0.4	1004
50	142	0.6	2	1004
50	183	0.6	3	2012
90	201	0.6	13	1006
100	248	1	5	1015
100	269	1	6	2013
166	504	3	11	2012
200	403	2	17	1032
200	514	4	39	2252
200	701	5	49	2090
300	495	4	12	1040
680	3103	103	111	17485

Nevertheless, on the average, the use of the JM-algorithm in each iteration of the heuristic yields a very good approximation. For sparse graph instances, the optimal value can usually be obtained, whereas in dense graph instances, values close to the optimum are obtained.

However, it should be noticed that one drawback of this method is the use of the time-consuming branch and cut algorithm. Table 4 gives an overview. Except for the last problem, the time bound for the branch and cut algorithm in each iteration was set to 1000 sec. computation time.

A characteristic of the layers is that they are not filled evenly. Often many edges adjacent to a node in the original graph are placed in the same layer.

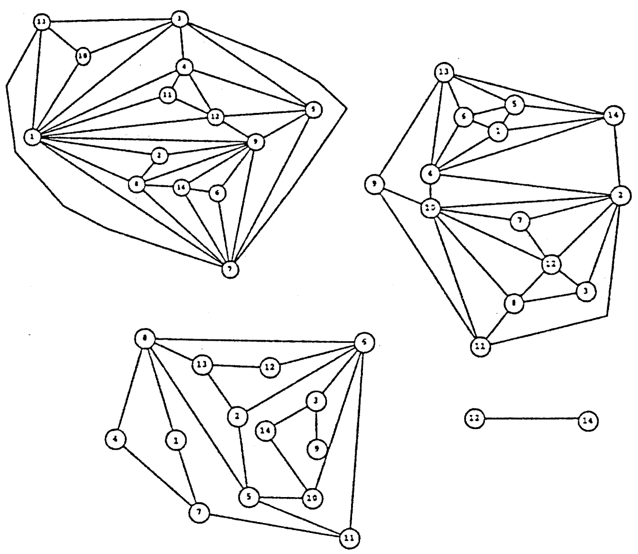

Fig. 3. Decomposition of K_{14} with THICK_{JM}

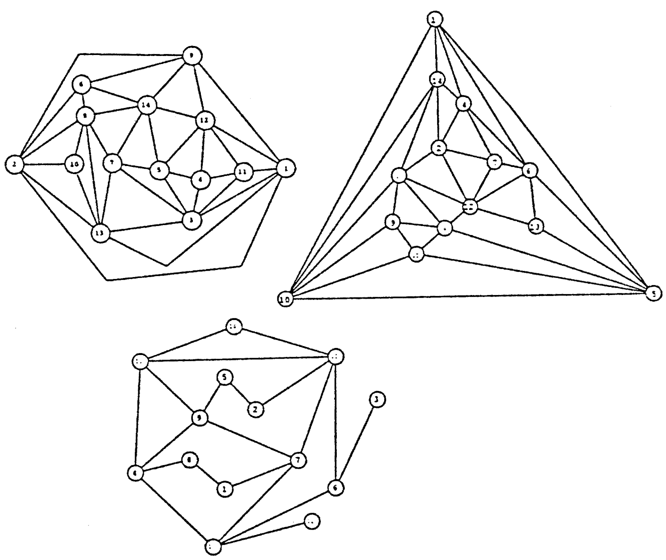

Fig. 4. Decomposition of K_{14} with THICK_{JM} and a degree restriction of 6

Figure 3 shows a decomposition of K_{14} achieved with the THICK_{JM} algorithm. Although the fourth layer consists of only one edge, we have a value of 4. Note, however, that node 1 has degree 10 in the first layer.

In order to avoid this situation, we carried out a degree-restriction for each node on all layers. With a restriction of at most 6 adjacent edges we got a decomposition of K_{14} in 3 layers, which can be seen in Fig. 4. Unfortunately, we did not find during our experiments a convenient rule for choosing a good node-restriction, so that we cannot recommend this approach in general.

6. Miscellaneous Results

This last section is devoted to some results which cannot be subsumed under a general topic. Chartrand, Geller and Hedetniemi [18] define a line-partition-number $\pi'_n(G)$ of a graph G, using a "Property P_n", which makes it possible to view the arboricity, the outerthickness and the thickness of a graph in a general framework, since $\pi'_2(G) = \Upsilon(G)$, $\pi'_3(G) = \theta_o(G)$, $\pi'_4(G) = \theta(G)$.

The *coarseness* $\chi(G)$ of a graph G was unintentionally defined by Erdös as the *maximum* number of edge-disjoint *nonplanar* subgraphs whose union is G (see Harary [25] for the whole story). This concept is in some sense opposite to the concept of thickness but it turned out that deriving formulas even for the complete graphs involves many exceptions. Since this problem does not yet have any applications we will not go into details here.

To complete this survey, we give an integer programming formulation of the thickness problem. We are encouraged by the fact that polyhedral combinatorics and branch and cut algorithms have been successfully applied to the maximum planar subgraph problem. Moreover, since the thickness value for practical problem instances is relatively small, only a few more variables are needed. The facet-defining inequalities occuring in the integer programming formulation for the maximum planar subgraph problem are the main ingredients of this formulation.

Consider a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ and let t be an upper bound for the thickness of G. The task is to assign each edge $e \in E$ to a subgraph $l \in \{1, 2, \ldots, t\}$ in such a way that all subgraphs are planar and the number of the planar subgraphs is minimized. We introduce *edge-variables* $y_{l,e}$ for $l = 1, 2, \ldots, t$ and $e = 1, 2, \ldots, m$ which indicate if edge e is assigned to layer $l (y_{l,e} = 1)$ or not $(y_{l,e} = 0)$. In addition, we need *layer-variables* x_l for $l = 1, 2, \ldots, t$, which indicate if layer l is required, i.e., the layer contains at least one edge. The integer programming formulation of the thickness problem can now be stated as follows:

$$
\begin{array}{lll}
\min \sum_{l=1}^{t} x_l & & \\
\text{s.t.} & \sum_{l=1}^{t} y_{l,e} = 1 & \text{for all } e = 1, 2, \ldots, m \\
& \sum_{e \in F_l} y_{l,e} \leq |F_l| - 1 & \text{for all subdivisions } F_l \text{ of } K_5 \text{ and } K_{3,3} \text{ on layer } l, \\
& & \text{for all } l = 1, 2, \ldots, t \\
& x_l \geq y_{l,e} & \text{for all } e = 1, 2, \ldots, m \text{ and } l = 1, 2, \ldots, t \\
& y_{l,e} \in \{0, 1\} & \text{for all } e = 1, 2, \ldots, m \text{ and } l = 1, 2, \ldots, t \\
& x_l \in \{0, 1\} & \text{for all } l = 1, 2, \ldots, t
\end{array}
$$

References

1. Asano, K.: On the genus and thickness of graphs, J. Comb. Theory Ser. B **43**, 287–292 (1987)
2. Alekseev, V.B., Gončakov, V.S.: The thickness of an arbitrary complete graph. Math. Sbornik **30(2)**, 187–202 (1976)

3. Aggarwal, A., Klawe, M., Shor, P.: Multilayer grid embeddings for VLSI. Algorithmica **1**, 129–151 (1991)
4. Bouwer, I.Z., Broere, I.: Note on t-minimal bipartite graphs. Canad. Math. Bull. **11**, 729–732 (1968)
5. Beineke, L.W.: Complete bipartite graphs: Decomposition into planar subgraphs. In: F. Harary: A seminar on graph theory. pp. 42–53. New York: 1967.
6. Beineke, L.W.: The decomposition of complete graphs into planar subgraphs. In: Graph theory and theoretical physics. pp. 139–153. New York: Academic Press, 1967
7. Beineke, L.W.: Minimal decompositions of complete graphs into subgraphs with embeddability properties. Canad. J. Math. **21**, 992–1000 (1969)
8. Beineke, L.W.: Fruited planes. Congressus Numerantium **63**, 127–138 (1988)
9. Beineke, L. W., Harary, F.: The thickness of the complete graph. Canad. J. Math. **17**, 850–859 (1965)
10. Battle, J., Harary, F., Kodoma, Y.: Every planar graph with nine points has a nonplanar complement. Bull. Amer. Math Soc. **68**, 569–571 (1962)
11. Beineke, L.W., Harary, F., Moon, J.W.: On the thickness of the complete bipartite graph. Proc. Camb. Phil. Soc. **60**, 1–5 (1964)
12. Bilski, T.: Embedding graphs in books: a survey. IEEE Proceedings-E, Vol. 139, No. 2, March 1992.
13. Booth, K.S., Lueker, G.S.: Testing for the consecutive ones property, Interval graphs and graph planarity testing using PQ-tree algorithms. J. of Computer and Systems Sciences **13**, 335–379 (1976)
14. Bose, N.K., Prabhu, K.A.: Thickness of graphs with degree constrained vertices. IEEE Trans. on Circuits and Systems **24**, 184–190 (1977)
15. Beineke, L., Wilson, R.: Selected topics in graph theory. pp 15–49. New York: Academic Press 1978
16. Călinescu, G., Fernandes, C.G., Finkler, U., Karloff, H.: A better approximation algorithm for the maximum planar subgraph problem, Proc. of the 7th Annual ACM-SIAM Symposium on Discrete Algorithms. pp. 16–25. SODA'96, Atlanta, Georgia, 1996
17. Cai, J., Han, X., Tarjan, R.E.: An $O(m \log n)$-Time algorithm for the maximal planar subgraph problem. SIAM J. Comput. **22**, 1142–1162 (1993)
18. Chatrand, G., Geller, D., Hedetniemi, S.: Graphs with forbidden subgraphs. J. Comb. Theory **10**, 12–41 (1971)
19. Dean, A.M., Hutchinson, J.P.: Relations among embedding parameters for graphs. In: Int'l Conf. on the Theory and Applications of Graphs, Kalamazoo, MI, 1988
20. Dean, A.M., Hutchinson, J.P.: Rectangle-visibility representations of bipartite graphs, Extended Abstract, Lecture Notes in Computer Science 894: R. Tamassia and I. Tollis: Graph Drawing. pp. 159–166. Berlin Heidelberg New York: Springer 1995
21. Dean, A.M., Hutchinson, J.P., Scheinerman, E.: On the thickness and arboricity of a graph. J. Comb. Theory Ser. B **52**, 147–151 (1991)
22. Guy, R.K.: Outerthickness and outercoarseness of graphs, in: Combinatorics, London Math. Soc. Lecture Notes 13, pp. 57–60. London: Cambridge University Press 1974
23. Halton, J.: On the thickness of graphs of given degree. Info. Sci. **54**, 219–238 (1991)
24. Harary, F.: Research problem. Bull. Amer. Math. Soc. **67**, 542 (1961)
25. Harary, F.: Graph Theory, Addison Wesely, 1969
26. Hobbs, A.M., Grossman, J.W.: Thickness and connectivity in graphs. J. Res. nat. Bur. Standards Sect. B **72**, 239–244 (1968)
27. Hutchinson, J., T. Shermer, Vince, A.: On Representations of some thickness-two graphs, Extended Abstract, Lecture Notes in Computer Science 1027, F. Brandenburg: Graph Drawing '95. pp. 324–332. Berlin Heidelberg New York: Springer 1996
28. Hopcroft, J., Tarjan, R.E.: Efficient planarity testing. J. ACM **21**, 549–568 (1974)
29. Jünger, M., Mutzel, P.: Solving the maximum planar subgraph problem by branch and cut. Proc. 3rd IPCO Conf. Erice, 1993

30. Jünger, M., Mutzel, P.: Maximum planar subgraph and nice embeddings: Practical layout tools, Algorithmica **16**, No. 1 33–59 (1996)
31. Jünger, M., Mutzel, P., Odenthal, T., Scharbrodt, M.: The thickness of a minor-excluded class of graphs, Discrete Mathematics, Special Volume on the Third Slovenian International Conference in Graph Theory, Bled (to appear 1997)
32. Jordan, E.: personal communication
33. Jayakumar, R., Thulasiraman, K., Swamy, M.N.S.: $O(n^2)$ Algorithms for graph planarization. IEEE Trans. on Computer-aided Design **8**, 257–267 (1989)
34. Kant, G.: An $O(n^2)$ Maximal planarization algorithm based on PQ-trees, Technical Report, RUU-CS-92-03, Dept. of Computer Science, Utrecht University, 1992
35. Kleinert, M.: Die Dicke des n-dimensionalen Würfel-Graphen. J. Comb. Theory **3**, 10–15 (1967)
36. Leipert, S.: The Problem of Computing a Maximal Planar Subgraph Using PQ-Trees is still not solved. Special proceedings for students pp. 95–110. ECMI'94, Kaiserslautern 1994
37. Liu, P.C., Geldmacher, R.C.: On the deletion of nonplanar edges of a graph. Proc. 10th S-E Conf. Combinatorics, Graph Theory, and Computing. pp. 727–738. FL: Boca Raton 1997
38. Mansfield, A.: Determining the thickness of graphs is NP-hard. Math. Proc. Cambridge Philos. Soc. **9**, 9–23 (1983)
39. Mayer, J.: Décomposition de K_{16} en Trois Graphes Planaires, J. Comb. Theory Ser. B **13**, 71 (1972)
40. Nash-Williams, C.St.J.A.: Edge-disjoint spanning trees of finite graphs. J. London Math. Soc. **36**, 445–450 (1961)
41. Ozawa, T., Takahashi, H.: A graph-planarization algorithm and its application to random graphs, Graph Theory and Algorithms, Lecture Notes in Computer Science **108**, 95–107 (1981)
42. Ringel, G.: Die torodiale Dicke des vollständigen Graphen. Math. Z. **87**, 19–26 (1965)
43. Ramanathan, S., Lloyd, E.L.: Scheduling algorithms for multihop radio networks. ACM SIGCOMM Computer Communication Review **22**, 211–222 (1992)
44. Ramanathan, S., Lloyd, E.L.: Scheduling algorithms for multihop radio networks, IEEE/ACM Transactions on Networking **1**, 166–177 (1993)
45. Širáň, J., Horák, P.: A construction of thickness-minimal graphs. Discr. Math. **64**, 262–268 (1987)
46. Tutte, W.T.: The non-biplanar character of the complete 9-graph. Canad. Math. Bull. **6**, 319–330 (1963)
47. Tutte, W.T.: The thickness of a graph. Indag. Math. **25**, 567–577 (1963)
48. Walther, H.: Über die Zerlegung des vollständigen Graphen in paare planare Graphen. In: H. Sachs: Beiträge zur Graphentheorie. pp. 189–205. Leipzig: Teubner 1968
49. Wessel, W.: On some variations of the thickness of a graph connected with colouring. In: Graphs and other combinatorial topics, Teubner Texte Math. **59** pp. 344–348, 1983
50. Winter, A.: Heuristiken für nicht-planare Graphen, Diplomarbeit Universität Passau, 1993

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