The chromatic number of the disjointness graph of the Double Chain

R. Fabila-Monroy[1], C. Hidalgo[2], J. Leaños[3], and M. Lomelí-Haro[4]

[1]Departamento de Matemáticas, Centro de Investigación y Estudios Avanzados del Instituto Politécnico Nacional. Ciudad de México, México.
ruyfabila@math.cinvestav.edu.mx

[2]Departamento de Matemáticas, Centro de Investigación y Estudios Avanzados del Instituto Politécnico Nacional. Ciudad de México, México.
chidalgo@math.cinvestav.edu.mx

[3]Unidad Académica de Matemáticas, UAZ. Zacatecas, México.
jleanos@matematicas.reduaz.mx

[4]Instituto de Física, Universidad Autónoma de San Luis Potosí. San Luis Potosí, SLP, México. lomeli@ifisica.uaslp.mx

M. L. gratefully acknowledges CONACyT for doctorate scholarship 163435.

October 9, 2018

Abstract

Let P be a set of n points in general position in the plane. Consider all the straight line segments with both endpoints in P. Suppose that these segments are colored with the rule that disjoint segments receive different colors. In this paper we show that if P is the point configuration known as the double chain, with k points in the upper convex chain and $l \geq k$ points in the lower convex chain, then $k+l-\left\lfloor\sqrt{2l+\frac{1}{4}}-\frac{1}{2}\right\rfloor$ colors are needed and that this number is sufficient.

Keywords: Chromatic number, Double chain, Edge disjointness graph

1 Introduction

Throughout this paper, P is a set of $n \geq 4$ points in general position in the plane. The *edge disjointness graph*, $D(P)$, of P is the graph whose vertices are all the closed straight line segments with endpoints in P; two of which are adjacent in $D(P)$ if and only if they are disjoint. The edge disjointness graph and other similar graphs were introduced by Araujo, Dumitrescu, Hurtado, Noy and Urrutia in [4], as geometric analogs of the well known Kneser graphs. The *Kneser graph*, $KG(n;k)$, has as vertices all the k-subsets of a set of n elements; two of which are adjacent in $KG(n;k)$ if they are disjoint.

The *chromatic number* of a graph G is the minimum number of colors needed to color its vertices so that adjacent vertices receive different colors; it is denoted by $\chi(G)$. In 1956, Kneser [11] posed the problem of finding the chromatic number of the Kneser graph. He conjectured that

$$\chi(KG(n;k)) = n - 2k + 2$$

for $n \geq 2k - 1$. The upper bound can be shown with simple combinatorial arguments. The lower bound was proved by Lovász in 1978 [12] using tools from algebraic topology (specifically the Borsuk-Ulam theorem). This is one of the earliest applications of Algebraic Topology to combinatorial problems. For a nice account of this connection see Matoušek's book [13].

Understandably, the chromatic number is a well studied parameter of the Kneser graph and its relatives. A general upper bound of

$$\chi(D(P)) \leq \min\left\{n - 2, n + \frac{1}{2} - \frac{\lfloor \log \log n \rfloor}{2}\right\}$$

was proved in [4]. They obtained it as follows. Let C_n be a set of n points in convex position in the plane. Let

$$f(n) := \chi(D(C_n)).$$

They showed that $f(n) \leq n - \frac{\lfloor \log_2 n \rfloor}{2}$. By the Erdős-Szekeres theorem [7], P has as subset of at least $m = \lfloor \log_2(n)/2 \rfloor$ points in convex position. The segments with endpoints in this subset are colored using $f(m)$ colors; the remaining segments are colored by deleting the remaining points one by one and in the process coloring all the segments with this point as an endpoint with the same new color.

The exact value of $f(n)$ has been computed [8]. It is now known that

$$f(n) = n - \left\lfloor \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rfloor.$$

Repeating the previous arguments, we have that

$$\chi(D(P)) \leq n - \left\lfloor \sqrt{\log n + \frac{1}{4}} - \frac{1}{2} \right\rfloor.$$

As far as we know $\{C_n\}_{n=1}^{\infty}$ is the only infinite family of point configurations[1] for which the exact value of the chromatic number of their disjointness graph has been computed. In this paper we compute the chromatic number of the disjointness graph of another infinite family of point configurations, called the double chain.

We now define this family. A *k-cup* is a set of k points in convex position in the plane such that its convex hull is bounded from above by a single segment. Similarly, an *l-cap* is a set of l points in convex position whose convex hull is bounded from below by a single segment.

Definition 1. *For $k \leq l$, the double chain is the union of a k-cup U and an l-cap L. Moreover, they satisfy the following.*

- *Every point of L is below every straight line determined by two points of U; and*
- *every point of U is above every straight line determined by two points of L.*

We denote this point set by $C_{k,l}$.

See Figure 1.

The double chain was first introduced by Hurtado, Noy and Urrutia in [10] as an example of a set of n points (in general position) whose flip graph of triangulations has diameter $\Theta(n^2)$. Since then the double chain has been used as an extremal example in various problems on point sets, see for example [1, 2, 3, 5, 6, 9].

[1]with different order types, that is.

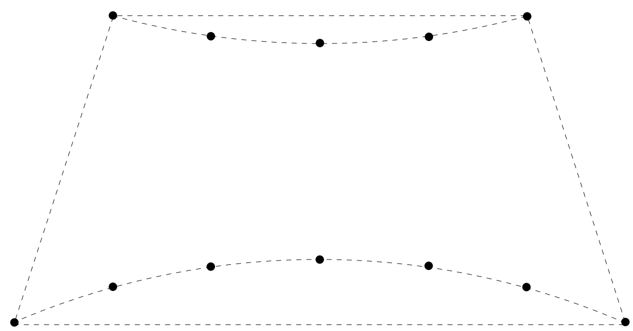

Figure 1: $C_{5,7}$.

In Theorem 1 we show that for $l \geq 3$.

$$\chi(D(C_{k,l})) = k + f(l).$$

Note that for n even and $k = l = n/2$, $C_{\frac{n}{2},\frac{n}{2}}$ is a set of n points for which

$$\chi\left(D\left(C_{\frac{n}{2},\frac{n}{2}}\right)\right) = n - \left\lfloor \sqrt{n+\frac{1}{4}} - \frac{1}{2} \right\rfloor \geq f(n) + c\sqrt{n},$$

for some positive constant c. So, to color the disjointness graph of $C_{\frac{n}{2},\frac{n}{2}}$ more colors are needed than to color the disjointness graph of C_n. We conjecture that in fact for every $n \geq 3$, and for every set P of n points

$$\chi(D(P)) \geq f(n).$$

2 Preliminary Results and Definitions

Before proceeding we present some results and definitions. A *geometric graph* is a graph whose vertices are points in the plane and whose edges are straight line segments joining these points. For exposition purposes we abuse notation and use P to refer to the complete geometric graph with vertex set equal to P. Thus, $\chi(D(P))$ is the minimum number of colors in an edge-coloring of P in which any two edges belonging to the same color class cross or are incident.

Let c be a proper vertex coloring[2] of $D(P)$. Let S be a color class of $D(P)$ in this coloring. We say that S is a *star* if all of its edges share a

[2]a coloring in which pairs of adjacent vertices receive different colors.

common vertex, which we call an *apex*. If S is not a star then it is *thrackle*. See Figure 2.

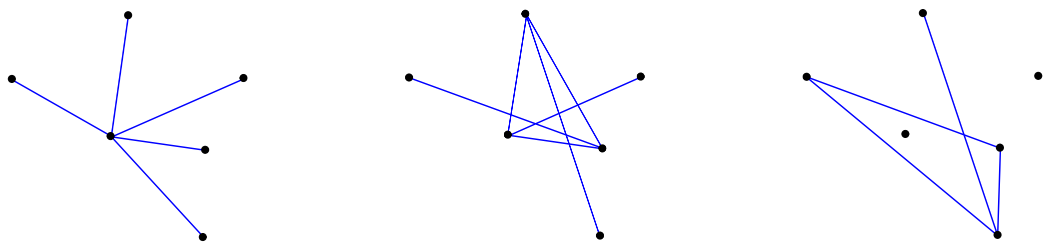

Figure 2: A star and two distinct thrackles of the same set of 6 points.

Proposition 1. *Let c be an optimal coloring of $D(P)$ and let $S_1, \ldots, S_r$ be stars of c with apices $v_1, \ldots, v_r$, respectively. Then*

$$\chi(D(P \setminus \{v_1, \ldots, v_r\})) = \chi(D(P)) - r.$$

Proof. Suppose for a contradiction that there exists a coloring of $\chi(D(P \setminus \{v_1, \ldots, v_r\}))$ with less than $\chi(D(P)) - r$ colors. Extend this coloring to a coloring of $D(P)$ by using a new different color for each S_i. This produces a coloring of $D(P)$ with less than $\chi(D(P))$ colors. $\square$

Let

$$g(n) := \max\left\{ i : i \in \mathbb{Z}^+, \binom{i}{2} \leq n \right\}. \tag{1}$$

In [8] it was observed, following Theorem 1, that

$$f(n) = n - g(n) + 1.$$

This implies the following result.

Proposition 2.

$$f(n+1) = \begin{cases} f(n) & \textit{if } n = \binom{i}{2} - 1 \textit{ for some positive integer } i \textit{ and} \\ f(n) + 1 & \textit{otherwise.} \end{cases}$$

Therefore, $f(n+k) - f(n) \leq k$, for every nonnegative integer k.

Proposition 3. *In every optimal coloring of $D(C_n)$ there is at most one chromatic class consisting of a single edge of P.*

Proof. Suppose for a contradiction that for some n there exists an optimal coloring c of $D(C_n)$ with two chromatic classes, S_1 and S_2, consisting of a single edge. Furthermore, suppose that n is the minimum such integer. The minimality of C_n and Proposition 1 imply that S_1 and S_2 are the only stars of c.

Let $T_1, \dots, T_k$ be the chromatic classes of c different from S_1 and S_2. Note that these are thrackles. In Theorem 2 of [8] it was shown that $T_1 \cup \cdots \cup T_k$ consists of at most $kn - \binom{k}{2}$ edges of C_n. Therefore, $\binom{n}{2} \leq kn - \binom{k}{2} + 2$. This implies that $(n-k)^2 \leq n + k + 4$. Since $k = f(n) - 2 = n - g(n) - 1$, we have that $(g(n)+1)^2 \leq 2n - (g(n)+1) + 4$. Rearranging terms in the previous inequality we arrive at $\binom{g(n)+1}{2} \leq n - g(n) + 1$. By the definition of $g(n)$, $\binom{g(n)+1}{2} > n$. Therefore, $g(n) < 1$, a contradiction. □

3 The Chromatic Number of $D(C_{k,l})$

It is relatively easy to find an optimal coloring of $D(C_{k,l})$.

Lemma 1. *For all positive integers $k \leq l$,*

$$\chi(D(C_{k,l})) \leq k + f(l).$$

Proof. Color the edges of L of $C_{k,l}$ with $f(l)$ colors. For each of the k vertices in U, color the edges incident to them, that have not been colored yet, with a new color. This yields a proper coloring of $D(C_{k,l})$ with $k + f(l)$ colors. □

The following lemma is needed to prove the lower bound on $\chi(D(C_{k,l}))$.

Lemma 2. *If $l \geq 3$, then $\chi(D(C_{1,l})) \geq f(l) + 1$.*

Proof. It can be verified by hand that $\chi(D(C_{1,3})) = f(3) + 1 = 2$. So assume that $l \geq 4$ and that the result holds for smaller values of l. Let c be an optimal coloring of $D(C_{1,l})$. We may assume that c, when restricted to L uses $f(l)$ colors, as otherwise we are done.

Suppose that c has a star with apex v. Then by Proposition 1, $D(C_{1,l} \setminus \{v\})$ can be properly colored with one color less. If v is the single point in U, then c uses at least $f(l) + 1$ colors. If v is in L, then by induction, c uses at least $\chi(D(C_{1,l-1})) + 1 = f(l-1) + 2$ colors. By Proposition 2 this is at least $f(l) + 1$.

Then we can assume that all chromatic classes of c are thrackles. This implies in particular that the edges incident to the single vertex u in U cannot all be of the the same color. Let e_1 and e_2 be two edges incident to u of different colors. Suppose that e_1 is colored *red* and e_2 is colored *blue*. Let v_1 and v_2 be their respective endpoints in L.

Since the *red* and *blue* edges are not stars, there exist f_1 and f_2 edges of L, of colors *red* and *blue*, respectively. Note that all the *red* edges of L must be incident to v_1 and all the *blue* edges of L must be incident to v_2. Since the *red* and the *blue* edges are not stars, this implies that there exist other edges incident to u of colors *red* and *blue*. Moreover, f_1 and f_2 are the only *red* and *blue* edges in L. Therefore, c when restricted to L is an optimal coloring of C_l in which two chromatic classes consist of a single edge. This contradicts Proposition 3. □

Lemma 3. *If* $l \geq 3$*, then* $\chi(D(C_{k,l})) \geq k + f(l)$.

Proof. Suppose for a contradiction that there exist k and l and a proper coloring c of $D(C_{k,l})$ with less than $k + f(l)$ colors. Furthermore suppose that k and l are such that $k + l$ is minimum. It can be checked by hand that the theorem holds for $k \leq l \leq 3$, and by Lemma 2 it holds for $k = 1$. Therefore, $k \geq 2$ and $l \geq 4$.

Suppose that c has a star with apex v. By Proposition 1, $D(C_{k,l} \setminus \{v\})$ can be colored with less than $k+f(l)-1$ colors. If v is in U then $C_{k,l} \setminus \{v\} = C_{k-1,l}$ and $D(C_{k-1,l})$ can be colored with less than $(k-1) + f(l)$ colors; this is a contradiction to our choice of k and l. If $k = l$, then we assume without loss of generality that v is in U and the previous aplplies. Now we are left with the case that v is in L and that $k < l$. Then $C_{k,l} \setminus \{v\} = C_{k,l-1}$ and, by Proposition 1 $D(C_{k,l-1})$ can be colored with less than $k + f(l) - 1$ colors. By Proposition 2, we know that $k + f(l) - 1 \leq k + f(l-1)$; this is a contradiction to our choice of k and l. Thus all the chromatic classes of c are thrackles.

Note that there are exactly four edges e_1, e_2, e_3 and e_4 in the convex hull of $C_{k,l}$. Let γ be the number of different colors received by these edges in c. Note that $\gamma = 2, 3$ or 4. If $\gamma = 2$ then at least one of these two chromatic classes is a star, and we are done.

Suppose that $\gamma = 4$. Let v_1, v_2, v_3 and v_4 be the set of endpoints of e_1, e_2, e_3 and e_4. Every edge of the same color as e_i must be incident to one of the endpoints of e_i. This implies that there are no edges with the same color as any of the e_i in $C_{k,l} \setminus \{v_1, v_2, v_3, v_4\}$. Therefore, c when restricted to $D(C_{k,l} \setminus \{v_1, v_2, v_3, v_4\})$ uses less than $k + f(l) - 4$ colors. Note that

$C_{k,l} \setminus \{v_1, v_2, v_3, v_4\} = C_{k-2,l-2}$; by Proposition 2, $k + f(l) - 4$ is at most $(k-2) + f(l-2)$; this a contradiction to our choice of k and l.

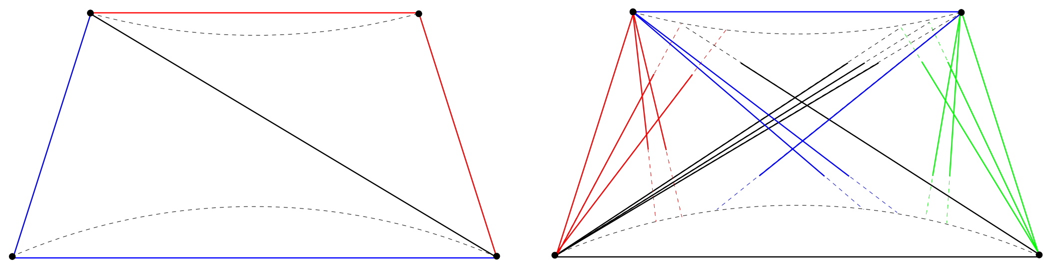

Figure 3: If $\gamma = 2$ (left), the diagonal must be colored *red* (then the *blue* chromatic class will be a star) or *blue* (then *red* chromatic class will be a star). If $\gamma = 4$ (right), and if we remove the 4 vertices in the convex hull, then we remove 4 colors.

Finally, suppose that $\gamma = 3$, then exactly two of the e_i are of the same color; moreover these edges share an endpoint. Without loss of generality assume that: these edges are e_1 and e_2; their common endpoint is v_3; and that the other endpoints of e_1 and e_2 are v_1 and v_2, respectively. Assume that e_1 and e_2 are colored *blue*. Since all the chromatic classes in c are thrackles then the edge v_1v_2 must also be colored *blue*. Let $S := U$ if v_3 is in U and let $S := L$ if v_3 is in L. Without loss of generality assume that v_1 is not in S.

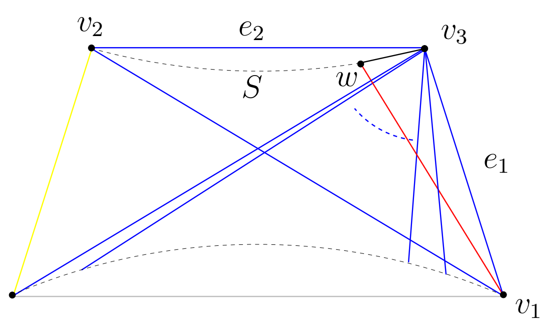

Figure 4: Case when $\gamma = 3$. If $c(v_3w)$ is *red*, then the *red* chromatic class is a star with apex w. Otherwise $C_{k,l} \setminus \{v_1, w, v_3\}$ would be a set that can be colored with less than $k + f(l) - 3$ colors. In both cases we have a contradiction.

Let w be the vertex different from v_2, that is adjacent to v_3 in the convex hull of S. Note that any other *blue* edge must be incident to v_3 and its other endpoint is not in S. Now we recolor *blue* all the edges incident with v_3 and having the other endpoint not in S. Note that the edge v_1w cannot be *blue*. Assume that v_1w is *red*. If v_3w is also colored *red*, then the *red* chromatic class is a star. Since this edge cannot be colored *blue*, assume that is colored *green*. Since no edge crosses v_3w, any other *green* edge must be incident to v_3 or w. Note that every *red* edge must be incident to v_1 or w. These observations together imply that c when restricted to $C_{k,l} \setminus \{v_1, w, v_3\}$ is a coloring $D(C_{k,l} \setminus \{v_1, w, v_3\})$ with less than $k + f(l) - 3$ colors. If $S = U$ then $C_{k,l} \setminus \{v_1, w, v_3\} = C_{k-2,l-1}$. By Proposition 2, $k + f(l) - 3 \leq (k-2) + f(l-1)$; this is a contradiction to our choice of k and l. If $S = L$ then $C_{k,l} \setminus \{v_1, w, v_3\} = C_{k-1,l-2}$. By Proposition 2, $k+f(l)-3 \leq (k-1)+f(l-2)$; this is a contradiction to our choice of k and l. The result follows. □

Summarizing, we have the following result.

Theorem 1. *For $l \geq 3$, $\chi(D(C_{k,l})) = k + f(l)$.*

4 Concluding Remarks

We have colored point sets by coloring the largest convex subset in an optimal way, and coloring the remaining edges in such a way that the chromatic classes they form are stars. Erdős and Szekeres [7] proved that every set of $\binom{2k-4}{k-2} + 1$ points in general position in the plane contains a subset of k points in convex position. Using Stirling's approximation this bound can be rephrased as follows.

Theorem. ***(Erdős and Szekeres 1935)*** *Every set of n points in general position in the plane has a subset of $\frac{1}{2}\log_2(n)$ points in convex position.*

This gives us the following bound on $L(n)$:

$$L(n) \leq f\left(\frac{1}{2}\log n\right) + \left(n - \frac{1}{2}\log n\right) = n - \left\lfloor \sqrt{\log n + \frac{1}{4}} - \frac{1}{2} \right\rfloor.$$

It is known that linear thrackles have at most n edges; we find maximal linear thrackles in point sets in convex position. We strongly believe that the lower bound for $\chi(n)$ is given by C_n. That is,

$$L(n) = f(n).$$

References

[1] O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, and B. Vogtenhuber. On the number of plane geometric graphs. *Graphs Combin.*, 23(suppl. 1):67–84, 2007.

[2] O. Aichholzer, W. Mulzer, and A. Pilz. Flip distance between triangulations of a simple polygon is NP-complete. *Discrete Comput. Geom.*, 54(2):368–389, 2015.

[3] O. Aichholzer, D. Orden, F. Santos, and B. Speckmann. On the number of pseudo-triangulations of certain point sets. *J. Combin. Theory Ser. A*, 115(2):254–278, 2008.

[4] G. Araujo, A. Dumitrescu, F. Hurtado, M. Noy, and J. Urrutia. On the chromatic number of some geometric type Kneser graphs. *Comput. Geom.*, 32(1):59–69, 2005.

[5] J. Cibulka, J. Kynčl, V. Mészáros, R. Stolař, and P. Valtr. Universal sets for straight-line embeddings of bicolored graphs. In *Thirty essays on geometric graph theory*, pages 101–119. Springer, New York, 2013.

[6] A. Dumitrescu, A. Schulz, A. Sheffer, and C. D. Tóth. Bounds on the maximum multiplicity of some common geometric graphs. *SIAM J. Discrete Math.*, 27(2):802–826, 2013.

[7] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[8] R. Fabila-Monroy, J. Jonsson, P. Valtr, and D. R. Wood. The exact chromatic number of the convex segment disjointness graph. *ArXiv e-prints*, Apr. 2018.

[9] A. García, M. Noy, and J. Tejel. Lower bounds on the number of crossing-free subgraphs of K_N. *Comput. Geom.*, 16(4):211–221, 2000.

[10] F. Hurtado, M. Noy, and J. Urrutia. Flipping edges in triangulations. *Discrete Comput. Geom.*, 22(3):333–346, 1999.

[11] M. Kneser. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 58, page 27. 1956. Aufgabe 360.

[12] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[13] J. Matoušek. *Using the Borsuk-Ulam theorem.* Universitext. Springer-Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler.