The exact chromatic number of
the convex segment disjointness graph[*]

In memory of Ferran Hurtado

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Abstract

Let P be a set of n points in strictly convex position in the plane. Let D_n be the graph whose vertex set is the set of all line segments with endpoints in P, where disjoint segments are adjacent. The chromatic number of this graph was first studied by Araujo, Dumitrescu, Hurtado, Noy, and Urrutia [2005] and then by Dujmović and Wood [2007]. Improving on their estimates, we prove the following exact formula:

$$\chi(D_n) = n - \left\lfloor \sqrt{2n + \tfrac{1}{4}} - \tfrac{1}{2} \right\rfloor.$$

1   Introduction

Throughout this paper, P is a set of n points in strictly convex position in the plane. The *convex segment disjointness graph*, denoted by D_n, is the graph whose vertex set is the set of all line segments with endpoints in P, where two vertices are adjacent if the corresponding segments are disjoint. Obviously D_n does not depend on the choice of P. Now assume that P consists of n evenly spaced points on a unit circle in the plane. The graph D_n was introduced by Araujo, Dumitrescu, Hurtado, Noy and Urrutia [1], who proved the following bounds on

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the chromatic number of D_n:

$$2 \left\lfloor \tfrac{1}{3}(n+1) \right\rfloor - 1 \leqslant \chi(D_n) < n - \tfrac{1}{2} \lfloor \log n \rfloor \ .$$

Both bounds were improved by Dujmović and Wood [10] to

$$\tfrac{3}{4}(n-2) \leqslant \chi(D_n) < n - \sqrt{\tfrac{1}{2}n} - \tfrac{1}{2}(\ln n) + 4 \ .$$

In this paper we prove matching upper and lower bounds, thus concluding the following exact formula for $\chi(D_n)$.

Theorem 1.

$$\chi(D_n) = n - \left\lfloor \sqrt{2n + \tfrac{1}{4}} - \tfrac{1}{2} \right\rfloor .$$

Equivalently, $\chi(D_n) = n - k$, where k is the unique integer satisfying $\binom{k+1}{2} \leqslant n < \binom{k+2}{2}$.

Theorem 1 is trivial for $n \leqslant 2$, so we henceforth assume that $n \geqslant 3$. The proof of the lower bound in Theorem 1 is based on the observation that each colour class in a colouring of D_n is a convex thrackle. We then prove that two maximal convex thrackles must share an edge in common. From this we prove a tight upper bound on the number of edges in the union of k convex thrackles. Theorem 1 quickly follows. These results are presented in Section 2. The proof of the upper bound in Theorem 1 is given by an explicit colouring, which we describe in Section 3.

2 Proof of Lower Bound

A *convex thrackle* on P is a geometric graph with vertex set P such that every pair of edges intersect; that is, they have a common endpoint or they cross. Observe that a geometric graph H on P is a convex thrackle if and only if $E(H)$ forms an independent set in D_n. A convex thrackle is *maximal* if it is edge-maximal. As illustrated in Figure 1, it is well known and easily proved that every maximal convex thrackle T consists of an odd cycle $C(T)$ together with some degree 1 vertices adjacent to vertices of $C(T)$. For each vertex v in $C(T)$, let $W_T(v)$ be the convex wedge with apex v, such that the boundary rays of $W_T(v)$ contain the neighbours of v in $C(T)$. Then every degree-1 vertex u of T lies in a unique wedge and the apex of this wedge is the only neighbour of u in T; see [8, Lemma 1] for a strengthening of these observations. See [2, 4–7, 9, 11–16, 18–23] for more on thrackles in general. Note that it is immediate from the above observations that every convex thrackle T satisfies $|E(T)| \leqslant |V(T)|$. Conways's famous thrackle conjecture says this property holds for all thrackles. Note that $C(T)$ is an example of a *musquash* [3, 17].

The following lemma is the heart of the proof of the lower bound in Theorem 1. We therefore include two proofs.

Lemma 2. *Let T_1 and T_2 be maximal convex thrackles on P. Let $C_1 := V(C(T_1))$ and $C_2 := V(C(T_2))$. Assume that $C_1 \cap C_2 = \emptyset$. Then there is an edge in $T_1 \cap T_2$, with one endpoint in C_1 and one endpoint in C_2.*
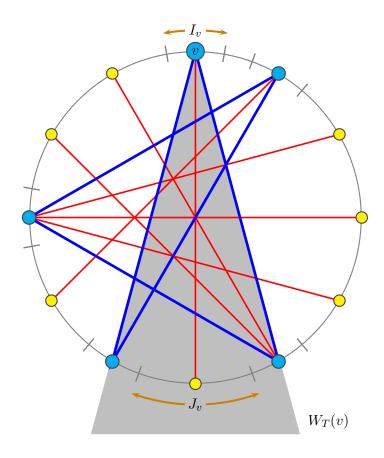
Figure 1: A maximal convex thrackle T with cycle $C(T)$ shown in blue.

Combinatorial Proof of Lemma 2. Define a directed bipartite multigraph H with bipartition $\{C_1, C_2\}$ as follows. For each vertex $u \in C_1$, add a *blue* arc uv to H, where v is the unique vertex in C_2 for which $u \in W_{T_1}(v)$. Similarly, for each vertex $u \in C_2$, add a *red* arc uv to H, where v is the unique vertex in C_1 for which $u \in W_{T_2}(v)$. Since $C_1 \cap C_2 = \emptyset$, every vertex of H has outdegree 1. Thus H contains a directed cycle Γ. By construction, vertices in H are not incident to an incoming and an outgoing edge of the same colour. Thus Γ alternates between blue and red arcs. The red edges of Γ form a matching as well as the blue edges, both of which are thrackles on the same set of points (namely, $V(\Gamma)$). However, there is only one matching thrackle on a set of points in convex position. Therefore Γ is a 2-cycle, which corresponds to an edge in $T_1 \cap T_2$, with one endpoint in C_1 and one endpoint in C_2. $\qquad\square$

Our second proof of Lemma 2 depends on the following topological notions. Let S^1 be the unit circle. For points $x, y \in S^1$, let $\overrightarrow{xy}$ be the clockwise arc from x to y in S^1. A $\mathbb{Z}_2$-*action* on S^1 is a homeomorphism $f : S^1 \to S^1$ such that $f(f(x)) = x$ for all $x \in S^1$. Say that f is *free* if $f(x) \neq x$ for all $x \in S^1$.

Lemma 3. *If f and g are free $\mathbb{Z}_2$-actions of S^1, then $f(x) = g(x)$ for some point $x \in S^1$.*

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Proof. Let $x_0 \in S^1$. If $f(x_0) = g(x_0)$ then we are done. Now assume that $f(x_0) \neq g(x_0)$. Without loss of generality, $x_0, g(x_0), f(x_0)$ appear in this clockwise order around S^1. Parameterise $\overrightarrow{x_0 g(x_0)}$ with a continuous injective function $p : [0,1] \to \overrightarrow{x_0 g(x_0)}$, such that $p(0) = x_0$ and $p(1) = g(x_0)$. Assume that $g(p(t)) \neq f(p(t))$ for all $t \in [0,1]$, otherwise we are done. Since g is free, $p(t) \neq g(p(t))$ for all $t \in [0,1]$. Thus $g(p([0,1])) = \overrightarrow{g(p(0))g(p(1))} = \overrightarrow{g(x_0)x_0}$. Also $f(p([0,1])) = \overrightarrow{f(x_0)f(p(1))}$, as otherwise $g(p(t)) = f(p(t))$ for some $t \in [0,1]$. This implies that $p(t), g(p(t)), f(p(t))$ appear in this clockwise order around S^1. In particular, with $t = 1$, we have $f(p(1)) \in \overrightarrow{x_0 g(x_0)}$. Thus $x_0 \in \overrightarrow{f(x_0)f(p(1))}$. Hence $x_0 = f(p(t))$ for some $t \in [0,1]$. Since f is a $\mathbb{Z}_2$-action, $f(x_0) = p(t)$. This is a contradiction since $p(t) \in \overrightarrow{x_0 g(x_0)}$ but $f(x_0) \notin \overrightarrow{x_0 g(x_0)}$. $\square$

Topological Proof of Lemma 2. Assume that P lies on S^1. Let T be a maximal convex thrackle on P. As illustrated in Figure 1, for each vertex u in $C(T)$, let (I_u, J_u) be a pair of closed intervals of S^1 defined as follows. Interval I_u contains u and bounded by the points of S^1 that are $\frac{1}{3}$ of the way towards the first points of P in the clockwise and anticlockwise direction from u. Let v and w be the neighbours of u in $C(T)$, so that v is before w in the clockwise direction from u. Let p be the endpoint of I_v in the clockwise direction from v. Let q be the endpoint of I_w in the anticlockwise direction from w. Then J_u is the interval bounded by p and q and not containing u. Define $f_T : S^1 \longrightarrow S^1$ as follows. For each $v \in C(T)$, map the anticlockwise endpoint of I_v to the anticlockwise endpoint of J_v, map the clockwise endpoint of I_v to the clockwise endpoint of J_v, and extend f_T linearly for the interior points of I_v and J_v, such that $f_T(I_v) = J_v$ and $f_T(J_v) = I_v$. Since the intervals I_v and J_v are disjoint, f_T is a free $\mathbb{Z}_2$-action of S^1.

By Lemma 3, there exists $x \in S^1$ such that $f_{T_1}(x) = y = f_{T_2}(x)$. Let $u \in C_1$ and $v \in C_2$ so that $x \in I_u \cup J_u$ and $x \in I_v \cup J_v$, where (I_u, J_u) and (I_v, J_v) are defined with respect to T_1 and T_2 respectively. Since $C_1 \cap C_2 = \emptyset$, we have $u \neq v$ and $I_u \cap I_v = \emptyset$. Thus $x \notin I_u \cap I_v$. If $x \in J_u \cap J_v$ then $y \in I_u \cap I_v$, implying $u = v$. Thus $x \notin J_u \cap J_v$. Hence $x \in (I_u \cap J_v) \cup (J_u \cap I_v)$. Without loss of generality, $x \in I_u \cap J_v$. Thus $y \in J_u \cap I_v$. If $I_u \cap J_v = \{x\}$ then x is an endpoint of both I_u and J_v, implying $u \in C_2$, which is a contradiction. Thus $I_u \cap J_v$ contains points other than x. It follows that $I_u \subset J_v$ and $I_v \subset J_u$. Therefore the edge uv is in both T_1 and T_2. Moreover one endpoint of uv is in C_1 and one endpoint is in C_2. $\square$

Theorem 4. *For every set P of n points in strictly convex position, the union of k maximal convex thrackles on P has at most $kn - \binom{k}{2}$ edges.*

Proof. For a set $\mathcal{T} = \{T_1, \ldots, T_k\}$ of k maximal convex thrackles on P, define $C_i := V(C(T_i))$ for $i \in [1, k]$, and let $r(\mathcal{T})$ be the set of triples (v, i, j) where $v \in C_i \cap C_j$ and $1 \leqslant i < j \leqslant k$. The proof proceeds by induction on $|r(\mathcal{T})|$.

First suppose that $r(\mathcal{T}) = \emptyset$. Thus $C_i \cap C_j = \emptyset$ for all distinct $T_i, T_j \in \mathcal{T}$. By Lemma 2, T_i and T_j have an edge in common, with one endpoint in C_i and one endpoint in C_j. Hence distinct pairs of thrackles have distinct edges in common. Since every maximal convex thrackle has n
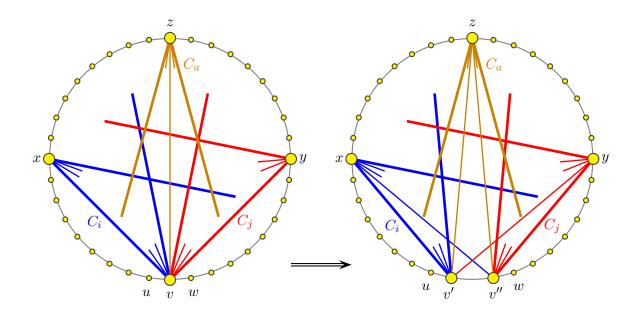
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Figure 2: Construction in the proof of Theorem 4.

edges and we overcount at least one edge for every pair, the total number of edges is at most $kn - \binom{k}{2}$.

Now assume that $r(\mathcal{T}) \neq \emptyset$. Thus there is a vertex v and a pair of distinct thrackles T_i and T_j, such that $v \in C_i \cap C_j$. We now modify $\mathcal{T}$ to create a new set $\mathcal{T}'$ of k convex thrackles, as illustrated in Figure 2. First, replace v by two consecutive vertices v' and v'' on P. Then, for each cycle C_ℓ with $v \in C_\ell$ and $\ell \neq j$ (which includes C_i), replace v by v' in T_ℓ, and add the edge xv'' to T_ℓ, where x is the vertex in C_ℓ for which v'' is inserted into $W_{T_\ell}(x)$. Now, replace v by v'' in T_j, and add the edge yv' to T_j, where y is the vertex in C_j for which v' is inserted into $W_{T_\ell}(y)$. Finally, for each cycle C_a with $v \notin C_a$, if z is the vertex in C_a with $v \in W_{T_a}(z)$, then replace the edge zv by zv' and zv'' in T_a. Let $\mathcal{T}'$ be the resulting set of thrackles. Then $(v, i, j) \notin r(\mathcal{T}')$, and every element of $r(\mathcal{T}')$ arises from an element of $r(\mathcal{T})$ (replacing v by v' or v'', as appropriate). Thus $r(\mathcal{T}') \leqslant r(\mathcal{T}) - 1$. Since one edge is added to each thrackle, the number of edges in $\mathcal{T}'$ equals the number of edges in $\mathcal{T}$ plus k. By induction, $\mathcal{T}'$ has at most $k(n+1) - \binom{k}{2}$ edges, implying $\mathcal{T}$ has at most $kn - \binom{k}{2}$ edges. $\qquad\square$

In the language of Dujmović and Wood [10], Theorem 4 says that every n-vertex graph with convex antithickness k has at most $kn - \binom{k}{2}$ edges.

We now show that Theorem 4 is best possible for all $n \geqslant 2k$. Let S be a set of k vertices in P with no two consecutive vertices in S. If $v \in S$ and x, v, y are consecutive in this order in P, then $T_v := \{vw : w \in P \setminus \{v\})\} \cup \{xy\}$ is a maximal convex thrackle, and $\{T_v : v \in S\}$ has exactly $kn - \binom{k}{2}$ edges in total.

Proof of Lower Bound in Theorem 1. If $\chi(D_n) = k$ then, there are k convex thrackles whose

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union is the complete geometric graph on P. Possibly add edges to obtain k maximal convex thrackles with $\binom{n}{2}$ edges in total. By Theorem 4, $\binom{n}{2} \leqslant kn - \binom{k}{2}$. The quadratic formula implies the result. $\qquad\square$

3   Proof of Upper Bound

Label the points of P by $1, 2, \ldots, n$ in clockwise order. Denote by ab the line segment between points $a, b \in P$ with $a < b$, which is a vertex of D_n. It will be convenient to adopt the matrix convention for indexing rows and columns in $\mathbb{Z}^2$. That is, row a is immediately below row $a-1$, column b is immediately to the right of column $b-1$, and (a, b) refers to the lattice point in row a and column b. Identify the vertex ab of D_n with the lattice point (a, b) where $a < b$, which we represent as a unit square in our figures. Define $\Omega_n = \{(i, j) \in \mathbb{Z}^2 : 1 \leqslant i < j \leqslant n\}$. We may consider $V(D_n) = \Omega_n$ represented as a triangle-shaped polyomino as illustrated in Figure 3(a).
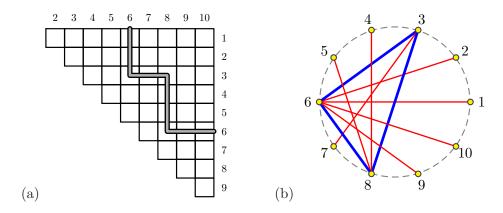

Figure 3: (a) A maximal independent set in D_{10} represented as a path in the polyomino Ω_{10}. (b) The corresponding maximal convex thrackle T. Turning points in the path correspond to vertices in $C(T)$.

Now, two distinct vertices (a, b) and (c, d) in D_n are adjacent if and only if $a \leqslant c \leqslant b \leqslant d$ or $c \leqslant a \leqslant d \leqslant b$. In particular, for (a, b) and (c, d) to be non-adjacent, (c, d) must lie in the nonshaded region in Figure 4. In particular, (c, d) cannot be strictly southwest or strictly northeast of (a, b). Moreover, $\max\{a, c\} \leqslant \min\{b, d\}$.

We conclude that every independent set S of D_n is a subset of some rectangle of the form $[1, r] \times [r, n]$ (with the southwest corner rr removed). Namely, choose $(a, b), (c, d) \in S$ such that a is maximal and d is minimal. Then $a' \leqslant a \leqslant d \leqslant b'$ for each $(a', b') \in S$. In fact, it is straightforward to show that each maximal independent set forms a path from $(1, r)$ to (r, n) for some $r \in \{2, \ldots, n-1\}$, where each step in the path is of the form $(i, j) \to (i, j+1)$ or $(i, j) \to (i+1, j)$. An example is given in Figure 3(a). Conversely, every such path is a maximal independent set. We refer to such a path as a *maximal thrackle path*; the corresponding set of line segments forms a maximal convex thrackle, as shown in Figure 3(b).
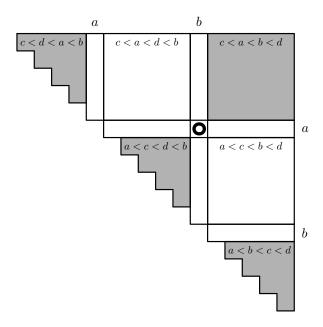
Figure 4: An element (c, d) is adjacent to (a, b) (marked with a thick circle) in the graph D_n if and only if (c, d) belongs to one of the shaded regions.
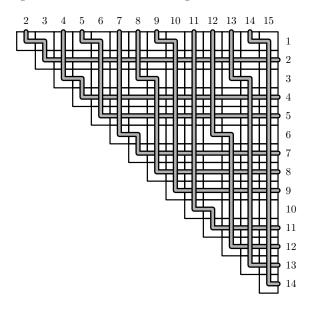

Figure 5: Ten thrackle paths covering Ω_{15}.

To summarize, the chromatic number of D_n equals the minimum number of maximal thrackle paths that cover Ω_n. For example, Figure 5 shows that it is possible to cover Ω_{15} with ten thrackle paths. As a consequence, $\chi(D_{15}) \leqslant 10$. Indeed, we have equality by the lower bound in Theorem 1.

For $k \geq 1$, define the following intervals:

$$\mathbb{N}_k := [\binom{k}{2} + 1, \binom{k+1}{2}] \quad \text{and} \quad \mathbb{N}'_k := [\binom{k}{2} + 1, \binom{k+1}{2} - 1].$$

Thus, $\mathbb{N}_1 = \{1\}$, $\mathbb{N}_2 = \{2, 3\}$, $\mathbb{N}_3 = \{4, 5, 6\}$, etc. The sets $\mathbb{N}_k$ form a partition of $\mathbb{N}$. Observe that $|\mathbb{N}_k| = k$ and $|\mathbb{N}'_k| = k - 1$ for each $k \geqslant 1$.

We now describe an infinite sequence of infinite paths covering the infinite polyomino $\Omega = \{(i, j) \in \mathbb{Z}^2 : 1 \leqslant i < j\}$. The final construction for Ω_n is then obtained as a restriction of the covering to the set Ω_n. For each $k \geq 2$ and for each $i \in \mathbb{N}'_k$, let P_i be the following path: start at $(1, i)$, walk south to

$$\left(\left(\frac{\binom{k+1}{2} - i + 1}{2} \right), i \right),$$

make one step east to

$$\left(\left(\frac{\binom{k+1}{2} - i + 1}{2} \right), i + 1 \right),$$

then walk south to $(i, i + 1)$, and finally walk east through all the points in the i-th row.

We now show that for each $j > 1$, the paths $P_1, \ldots, P_j$ cover all the points in the j-th column. Let $j \in \mathbb{N}_k$. If $j = \binom{k}{2} + 1$ then the path P_j covers the j-th column. If $j = \binom{k+1}{2}$ then the path P_{j-1} covers the j-th column. Now assume that $j \neq \binom{k}{2} + 1$ and $j \neq \binom{k+1}{2}$. Let $\ell := \binom{k+1}{2} - j$. The path P_j covers the topmost $\binom{\ell+1}{2}$ points in the j-th column. The next ℓ points of the j-th column lie in the rows $\binom{\ell+1}{2} + 1, \ldots, \binom{\ell+2}{2} - 1$. These rows are completely covered by the ℓ paths P_h where $h \in \mathbb{N}'_{\ell+1}$. The remaining bottom part of the j-th column from $(\binom{\ell+2}{2}, j)$ to $(j - 1, j)$ is covered by P_{j-1}.

Now consider the restriction of the paths $P_1, \ldots, P_n$ to the triangular polyomino Ω_n. Each intersection $P_i \cap \Omega_n$ is a maximal thrackle path in Ω_n. Let k be the unique integer satisfying $\binom{k+1}{2} \leqslant n < \binom{k+2}{2}$. Then the above construction gives a covering of the polyomino Ω_n by $n - k$ thrackle paths, since a path P_i exists for each $i \leqslant n$, except for the k values $i = \binom{2}{2}, \binom{3}{2}, \ldots, \binom{k+1}{2}$. The upper bound in Theorem 1 follows.

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