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Enumerating Order Types for Small Point Sets with Applications ⋆

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Abstract. Order types are a means to characterize the combinatorial properties of a finite point configuration. In particular, the crossing properties of all straight-line segments spanned by a planar n-point set are reflected by its order type. We establish a complete and reliable data base for all possible order types of size $n = 10$ or less. The data base includes a realizing point set for each order type in small integer grid representation. To our knowledge, no such project has been carried out before.

We substantiate the usefulness of our data base by applying it to several problems in computational and combinatorial geometry. Problems concerning triangulations, simple polygonalizations, complete geometric graphs, and k-sets are addressed. This list of applications is not meant to be exhaustive. We believe our data base to be of value to many researchers who wish to examine their conjectures on small point configurations.

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1. Introduction

Many questions in computational and combinatorial geometry are based on finite sets of points in the Euclidean plane. In fact, a finite point configuration is among the most basic geometric objects that lead to nontrivial problems – in a combinatorial, geometric, and algorithmic sense. For quite a large subclass, the combinatorial properties of the underlying point set rather than its metric properties already determine the problem. In particular, the *intersection properties* of all the straight-line segments spanned by the point set turn out to be of importance. Examples include the concepts of *triangulations* (maximal sets of pairwise noncrossing segments),

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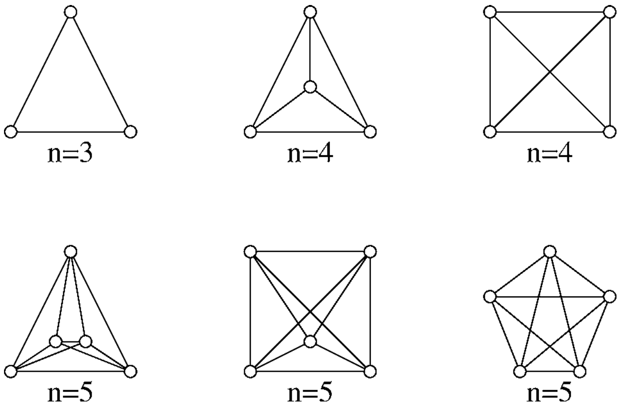

Figure 1. Inequivalent point sets of small size.

spanning trees (maximal acyclic sets of segments), *polygonalizations* (crossing-free Hamiltonian cycles), k-sets (subsets of k points separable by a straight line), and many others. For several problems, like counting all triangulations of a given point set [1], no efficient algorithms are known. For others, like for the number of different k-sets [15], the combinatorial complexity is still unsettled. Sometimes even the existence of a solution has not yet been established, such as the question of whether any two given n-point sets (with the same number of extreme points) can be triangulated in an isomorphic manner [3].

To gain insight into the structure of difficult problems, examples that are typical or extreme are often very helpful. Such examples are usually obtained by complete enumeration of all possible problem instances. For the questions mentioned above this means to investigate all 'different' sets of points, where difference is with respect to the crossing properties of the complete geometric graph spanned by the set. To this end it is necessary to have a data base of all different point sets for small size n. The aim of this work is to provide such a data base for $n \leq 10$ which is both complete and correct, and to describe some improved results obtained from it. To the knowledge of the authors, no such data base has been available before. A conference version of the present material appeared in [2].

It is well known that crossing properties are exactly reflected by the order type of a point set, introduced in Goodman and Pollack [22]. The *order type* of a set $\{p_1, \ldots, p_n\}$ of points in general position* is a mapping that assigns to each ordered triple i, j, k in $\{1, \ldots, n\}$ the orientation (either clockwise or counter-clockwise) of the point triple p_i, p_j, p_k. Two point sets S_1 and S_2 are said to be (*combinatorially*) *equivalent* if they exhibit the same order types. That is, there is a bijection between S_1 and S_2 such that any triple in S_1 agrees in orientation with the corresponding triple in S_2. Equivalently, two line segments spanned by S_1 cross if and only if the

* All point sets considered in this paper are assumed to be in *general position*, meaning that no three points are collinear. In addition, all line arrangements are assumed to be *simple*, that is, no three lines pass through the same point and no two lines are parallel.

two corresponding segments for S_2 do. See Figure 1: there is only one (combinarorial) way to place 3 points, two ways to place 4 points, and three to place 5. The situation changes drastically for larger size.

Loosely speaking, any two equivalent point sets S_1 and S_2 allow the same variety of combinatorial structures. For instance, they have the same number of extreme points, any triangulation of S_1 is also a valid triangulation of S_2, and so on. Thus for the above-mentioned problems instead of investigating all (infinitely many) point sets of given size, it is sufficient to deal only with inequivalent sets. Still to enumerate or even to count these sets is a highly nontrivial task. This is due to two reasons. First, the number of inequivalent point sets of size 10 is already in the millions (14 309 547 to be precise; see Table I). Second, there seems to be no direct way to generate all these sets, because each increase in size leads to types which cannot be obtained directly from sets of smaller size.

Let us point out some situations were the complete enumeration of all order types for small n leads to results for general problem size n.

The obvious case is when a counterexample can be provided that generalizes to larger n. There might exist counterexamples too large to be found by hand though small enough to be detected by checking all order types. For instance, a question raised by Arkin *et al.* [6] asks for how 'convex' the best polygonalization of an n-point set has to be in the worst case. More precisely, what is the minimum number $r(n)$ such that every n-point set admits a crossing-free Hamiltonian cycle with at most $r(n)$ reflex angles (i.e., angles larger than π)? The best known bounds are $\lfloor \frac{n}{4} \rfloor \leq r(n) \leq \lceil \frac{n}{2} \rceil$. By considering all possible order types of size 10 we found $r(10) = 3$ which shows that neither of these bounds is tight.

On the other hand, the nonexistence of small counterexamples gives some evidence for the truth of a conjecture. For example, our tests did not lead to a counterexample for the isomorphism conjecture on triangulations mentioned above. As a result, any two sets of 9 points (or less) admit an isomorphic triangulation provided their number of convex hull points is the same.

As another example, case analyses for problem instances of constant size are often encountered when proving some combinatorial property. This is particularly true for induction proofs if a sufficiently large induction basis needs to be found. We will describe in Subsection 3.2 an example where the quality of the initial values affects the asymptotic behavior of the solution.

Our approach to generate all order types makes use of the duality* of point sets (in general position) and (simple) line arrangements in the Euclidean plane. A *line arrangement* is the dissection of the plane induced by a set of n straight lines. As no direct way to enumerate these structures is known, we first produce all different arrangements of so-called pseudolines. These are combinatorial abstractions of line arrangements which retain their crossing properties. More precisely, a set of

* Any of the well-known point-line duality transforms used in computational geometry [14] may serve this purpose, though none of them leads to a bijection between order types and arrangements. We will show in Section 2 how to bypass this difficulty.

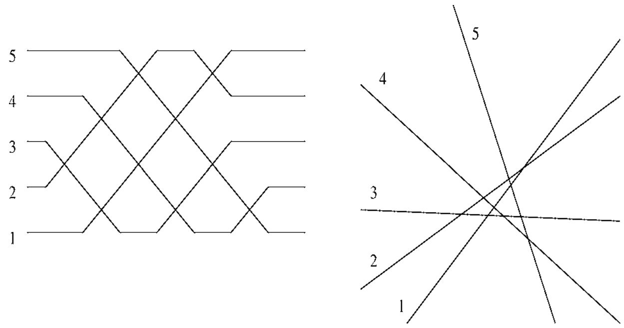

Figure 2. A wiring diagram that can be stretched.

pseudolines is a set of simple curves which pairwise cross at exactly one point. Handling pseudolines is relatively easy in view of their equivalent description by wiring diagrams; see, e.g., Goodman [21]. We can read off a corresponding *pseudo order type* from each pseudoline arrangement. (The intersection orders on all the pseudolines uniquely determine the orientation of all element triples.) Back in the primal setting, this leads to a list of candidates guaranteed to contain all different order types.

The main problem, however, is to identify all the *realizable* order types in this list, that is, those which can actually be realized by a set of points. Here we enter the realm of *oriented matroids*, an axiomatic combinatorial abstraction of geometric structures; see e.g., Björner *et al.* [11]. It is known that a pseudoline arrangement need not be *stretchable*, i.e., isomorphic to some straight line arrangement. In fact, there exist nonstretchable arrangements already for 8 pseudolines in the Euclidean plane. As a consequence, our candidate list will contain nonrealizable pseudo order types. Moreover, even if realizability has been decided for a particular candidate, how can we find a corresponding point set? We will provide answers to these questions, and a comparison to known related results, in the following section.

In some cases there is no need to identify the realizable order types. For instance, if a particular conjecture for point sets is true for all pseudo order types then it must be true for the subset of order types that actually can be realized by some set of points.

We remark that the situation gets conceptually and computationally easier in the *projective plane* where – unlike in the Euclidean plane – inequivalent order types directly correspond to nonisomorphic line arrangements, and isomorphism classes of pseudoline arrangements coincide with (reorientation classes of) rank 3 oriented matroids. Complete enumerations of the nonrealizable classes of these matroids have been done for sizes $n \leq 10$; see [11, 21]. The Euclidean order types can be derived from their projective counterparts. Combinatorial descriptions of the latter are not available to us, however. Apart from that, the hard problem is to construct their geometric realizations.

2. Order Type Enumeration

The first part of this section describes the approach we chose for generating the order type data base for all point sets of size $n \leq 10$. The second part gives a quick view of the output and reports on the representation of the realizing point sets as well as on the reliability of our data base.

2.1. THE APPROACH

Our strategy for computing all different order types proceeds in the following three steps. Each step is further detailed and explained below.

- Generating a candidate list $\mathcal{C}$ which is guaranteed to contain each pseudo order type – and thus each realizable order type – exactly once. We utilize wiring diagrams to calculate a unique representation (by a so-called λ-matrix) for each order type.
- Grouping the members of $\mathcal{C}$ into equivalence classes by correspondence to the same projective order type. In every class, either each or no order type is realizable. The numbers P_n of realizable projective classes of size n are known from the literature* for $n \leq 10$.
- Realizing all realizable order types in $\mathcal{C}$. For each member of $\mathcal{C}$ we try to recover a realizing point set from its λ-matrix. A counter is kept for the number of realizable projective classes all of whose members have been realized already. The process is terminated when this number reaches P_n.

2.1.1. *Generating the Candidate List*

Let S be a set of n points in the Euclidean plane. It is well known that the order type of S can be encoded by a so-called *λ-matrix*. For some fixed labelling of S, each entry $\lambda(i, j)$ of this matrix gives the number of points in S which lie to the left of the line ℓ_{ij} directed from point i to point j. Goodman and Pollack [22] introduced λ-matrices and showed that they are valid representations of order types: to know how many points lie on a fixed side of each line ℓ_{ij} is sufficient for knowing which points these are.

As the λ-matrix (but not the order type) of a set S depends on its labelling, attention may be restricted to a fixed matrix. For technical reasons, the smallest λ-matrix of S in lexicographical order is chosen. We call the corresponding labelling $p_1, \ldots, p_n$ of S a *natural ordering*, because p_1 then is a point on the convex hull of S and $p_2, \ldots, p_n$ appear in clockwise order as seen from p_1. For a point set in general position (and we only consider such sets in this paper) its natural ordering is unique – up to self-symmetry of the point set. Our first aim is to generate the λ-matrices for all naturally ordered sets of fixed size $n \leq 10$.

* We verified the numbers P_n by utilizing a known method – based on Grassmann–Pluecker relations – which identifies certain nonrealizable order types; see [11].

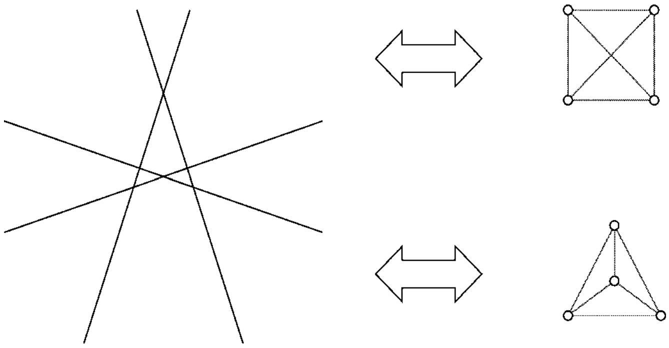

Figure 3. Arrangement and point sets of size 4.

This problem is computationally better to access in the dual setting. Using a suitable point-line duality [14], any set of n points can be transformed into a set of n straight lines. Unfortunately though, we cannot expect the transformation to yield a one-to-one correspondence between inequivalent objects. For instance, in the case $n = 4$ there exist two inequivalent order types but only a single line arrangement (up to isomorphism); see Figure 3. Moreover, for larger n several line arrangements may correspond to (different realizations of) the same order type. These difficulties can be bypassed by exploiting natural orderings as follows.

When using a duality transform which preserves the relative position of points and lines, each naturally ordered set $\{p_1, \ldots, p_n\}$ can be transformed into an ordered set $\{g_1, \ldots, g_n\}$ of *directed lines* with the following properties. (a) No line crossings lie to the left of g_1. (b) Each line g_i for $i > 1$ crosses g_1 from left to right. (c) For $1 \leq i < j \leq n$, the number of lines which have the crossing $g_i \cap g_j$ to their left is exactly $\lambda(i, j)$, the corresponding entry in the λ-matrix of $\{p_1, \ldots, p_n\}$. As a consequence, this λ-matrix can be read off from the line arrangement formed by $\{g_1, \ldots, g_n\}$.

More precisely, the order σ_i in which each directed line g_i crosses all other lines already determines the λ-matrix. Note that for g_1 this order is fixed, $\sigma_1 = 2, 3, \ldots, n$. (This follows from properties (a) to (c).) So, in order to generate all desired λ-matrices for n points, it suffices to generate all arrangements of $n - 1$ directed lines which exhibit different crossing orders $\sigma_2, \ldots, \sigma_n$. A superset thereof, namely all arrangements of $n - 1$ directed *pseudolines* with different crossing orders, can be computed relatively easily by means of their equivalent representation as *wiring diagrams*. We omit a precise definition of wiring diagrams and refer to [22] and Figure 2 instead. Note that a pseudoline arrangement may as well be represented by a λ-matrix by using property (c) above.

In summary, unique λ-matrix representations for all pseudo order types of size n are obtained efficiently in this way.

2.1.2. *Grouping into Projective Classes*

Given some point set realization on the sphere of a *projective* order type, a realizing point set for each Euclidean order type which belongs to that projective class can be obtained by rotation of the sphere and central projection on the plane. To group our order type list $\mathcal{C}$ into projective classes we use a combinatorial simulation of this process.

To this end, a *rotation* of a given (Euclidean) order type T is defined for each index i that corresponds to an extreme point for T. A rotation for index i simply reverses the orientation of all index triples that contain i. Its implementation in λ-matrix representation is straightforward. To be able to recognize identity of order types we lexicographically minimize each λ-matrix obtained from a rotation. Comparison of λ-matrices for identity then can be done quickly for each pair. All order types belonging to the same projective class can be obtained by rotation from each other. We use back-tracking to ensure an exhaustive enumeration within each class. In this way, each member of the list $\mathcal{C}$ is assigned to its projective class by comparison of λ-matrices.

These classes correspond exactly to the (reorientation classes of) rank 3 oriented matroids whose number is known for sizes $n \leq 10$; see Table I. Let us mention at this place that several multiple purpose methods for generating oriented matroids have been developed, the most recent ones being by Bokowski and Guedes de Oliveira [12] and by Finschi and Fukuda [18]. Our method is tailored for order types in the Euclidean plane and stands out by its simplicity, transparency, and efficiency.

2.1.3. *Finding the Realizing Point Sets*

Realizing a given order type means restoring a point set from a given λ-matrix that represents this order type. Deciding realizability (for order types in particular and oriented matroids in general) is an intriguing problem which is known to be NP-hard. Several heuristics have been developed; see [21] and references therein. For our case (rank 3 oriented matroids) a singly exponential algorithm exists but turned out to be too slow for our purposes. Instead, we used the following combination of methods with success.

In a first step, we applied an insertion strategy to obtain realizations of size n from realizations of size $n-1$, for $n \leq 10$. Suppose that, for each (realizable) order type T of size $n-1$, a corresponding point set $S(T)$ is available. Consider the arrangement $A(T)$ formed by $\binom{n-1}{2}$ lines, each passing through a pair of points in $S(T)$. Point sets of size n are now generated from $S(T)$ by placing an additional point in a cell of $A(T)$, for all possible cells. Carrying out this process for all order types of size $n-1$ leads to realizations for certain order types of size n. It is well known [11] that, in general, not all desired realizations are obtainable in this way; a principal difficulty with this approach is that the *geometry* of an $(n-1)$-point set $S(T)$ critically affects the *set of order types* of size n which actually get realized.

To increase effectiveness we restarted the insertion method after random (but order type preserving) perturbations of the $(n-1)$-point sets.

After having obtained certain realizations from the insertion method, we continue with scanning through the projective classes our candidate list $\mathcal{C}$ has been grouped into. If the number of classes having got at least one member realized coincides with P_n (the number of realizable projective classes) we continue to the next step below. Else we decide upon each completely unrealized class by a different method: we try to realize one of its members directly, starting from scratch with a simulated annealing method. (Only the case $n = 10$, and there only a very small fraction of classes – including the nonrealizable ones, of course – needed this treatment; see Subsection 2.2.) Moreover, and most important, we realized at least one member for each of the P_n realizable classes in one or the other way, and thus succeeded in distinguishing them from the nonrealizable ones.

Finally, each projective class T^o found to be realizable but still containing unrealized members is completed as follows. There is at least one member T of T^o whose realizing point set $S(T)$ has been computed. We calculate from $S(T)$ a spherical point set $S(T^o)$ realizing T^o, and derive realizing point sets for the yet unrealized members of T^o by rotation and projection of $S(T^o)$; compare Subsection 2.1.2.

2.2. OUTPUT AND RELIABILITY

Table I gives a quick overview of our results in comparison with other known results. Lines 1 to 3 are taken from [21]. Order types that are reorientations of each other (that is, obtainable from each other by reversing the orientation for all index triples) are counted only once. Recall that the number of projective (pseudo) order types is just the number of isomorphism classes of projective (pseudo)line arrangements. The tremendous growth of Euclidean order types can be seen in the lower part of the table. By the nature of our approach, we computed a combinatorial description for each of the objects counted in Table I, along with a geometric representation of this object if it is realizable.

Table II lists the numbers of Euclidean order types according to the number h of extreme points in the realizing point sets. The last line coincides with the respective line in Table I. Table III refers to counting reorientation pairs of order types separately. Note that reorientation of an order type (which geometrically means a reflection of the realizing point set) may or may not lead to its identical copy; thus reorientation leads to an increase only in the latter case.

It took 17 minutes on a 500 MHz Pentium III to generate all Euclidean pseudo order types of size $n = 9$, and to find realizing point sets for all but 13 of them, by using the insertion strategy followed by rotation within each projective class. The remaining 13 candidates turned out to belong to the same projective class (which therefore has to be the only existing nonrealizable projective class; cf. Table I, second line), witnessing completeness of the task. As had to be expected,

Table I. Number of different order types of size n.

n	4	5	6	7	8	9	10
Projective Pseudo Order Types	1	1	4	11	135	4 382	312 356
– thereof nonrealizable						1	242
= Projective Order Types	1	1	4	11	135	4 381	312 114
Euclidean Pseudo Order Types	2	3	16	135	3 315	158 830	14 320 182
– thereof nonrealizable						13	10 635
= Euclidean Order Types	2	3	16	135	3 315	158 817	14 309 547

Table II. Euclidean order types classified by extreme points.

h / n	4	5	6	7	8	9	10
3	1	1	6	49	1 178	55 235	4 876 476
4	1	1	6	59	1 468	70 475	6 319 019
5		1	3	22	570	28 232	2 628 738
6			1	4	90	4 552	450 176
7				1	8	311	33 969
8					1	11	1 146
9						1	22
10							1
$\sum$	2	3	16	135	3 315	158 817	14 309 547

Table III. As Table II, but respecting reorientation.

h / n	4	5	6	7	8	9	10
3	1	1	8	92	2 296	110 336	9 750 002
4	1	1	8	107	2 862	140 593	12 633 467
5		1	3	37	1 081	56 300	5 254 263
6			1	5	156	8 973	898 682
7				1	9	591	67 400
8					1	15	2 186
9						1	29
10							1
$\sum$	2	3	20	242	6 405	316 809	28 606 030

the situation turned out to be more complex for size $n = 10$. Generation of all pseudo order types plus partial realization by means of insertion took about 36 hours and left some 200 000 Euclidean pseudo-order types unrealized. Most of the corresponding projective classes got some member realized, however, and could be completed quickly by applying rotation. In particular, only 251 projective classes remained without any realized member. To try to realize a first member, we invoked our simulated annealing routine for all these classes. We were successful for 9 classes within 60 hours which finally completed this task. (The remaining 242 classes are indeed nonrealizable, by the nonrealizability check we applied in addition; see Subsection 2.1.)

Whenever computing realizing point sets, care was taken to avoid large coordinates. In addition, emphasis was laid on finding point coordinate descriptions more handsome than those calculated by the basic algorithm. In particular, the simulated annealing routine was used to post-process the point sets. In their final form, our point sets enjoy the following properties.

- *Compact grid representation.* For $n \leq 8$ all point sets are expressed with 1 byte coordinates, and for $n = 9$ and $n = 10$ with 16 bit (unsigned) integer coordinates. This should be contrasted with known negative results on the efficient grid embeddability of order types; see Goodman *et al.* [23]. Small integer coordinates ensure the efficiency and numerical stability of calculations based on our data base.
- *Coordinate uniqueness.* No two points in a set share the same x-coordinate or the same y-coordinate. Thus sorting the points by one of their coordinates is easy and unique. Moreover, when dualizing the points of a set into straight lines by common transforms, no parallelism will occur.
- *Resolution.* The minimum Euclidean distance between two points in a set is greater than 4.
- *'Very' general position.* Apart from being in general position, every set guarantees a normal distance of at least 1 between each point and each line through two other points. Thus any circle passing through three points in a set will have a reasonably small radius.
- *No cocircularities.* No four points in a set lie on a common circle. Therefore, in-circle tests as, e.g., those used in algorithms based on Delaunay triangulations [8] can be carried out uniquely.

We close this section by addressing the issue of reliability of our data base. The theoretical correctness of our approach has been argued for already. This raises the question of how reliable are our implementations. The algorithm used in Subsection 2.1.1 for generating all Euclidean pseudo-order types, and in Subsection 2.1.2 for grouping them into projective classes, respectively, are of purely combinatorial nature. Evidence for their correct implementation is gained from the correctly computed total numbers of projective pseudo order types (which have been known before; see Table I). As a byproduct, and being important in this context, the size (numbers of members) of each projective class is computed in a reliable way.

The critical part are the geometric computations in Subsection 2.1.3. We checked their correctness as follows. For each point set in its final grid form, we recalculated its λ-matrix and minimized it lexicographically. (In fact, we considered both the set and its reflected counterpart, and selected the smaller one of the two minimized matrices.) This involves only numerically reliable integer manipulations. By sorting the obtained matrices lexicographically, and checking for identity of neighbors, we ensured that our point sets represent pairwise different order types. Finally, we checked that the number E_n of Euclidean order types we have realized is correct, i.e., that no order type is missing. To this end, we added up the sizes (pre-computed in a combinatorial way as mentioned above) of all the projective classes that have their members realized as grid point sets, and compared to E_n. The number of these projective classes is P_n by the control flow of our algorithm, so no other projective class contains realizable Euclidean order types; compare Subsection 2.1.

In summary a complete, user-friendly, and reliable data base for all order types of sizes $n \leq 10$ has been obtained. The data base has been made public on the web.⋆ Due to space limitations, the grid point sets of size 10 are not accessible on-line but rather have been stored on a CD which is available from us upon request.

3. Applications and First Results

The present section reports on applications and first results obtained from our order type data base. The problems addressed here are mostly related to triangulations but the list is easily extended to questions of a different flavour now that the data base is available. We briefly address some of them at the end of this section.

3.1. ISOMORPHISM CONJECTURE FOR PLANAR TRIANGULATIONS

Informally speaking, the isomorphism conjecture for triangulations deals with the question of whether two point sets can be triangulated in the same fashion. Consider two equal-sized point sets $S_1 = \{p_1, \dots, p_n\}$ and $S_2 = \{q_1, \dots, q_n\}$ and two triangulations $T(S_1)$ and $T(S_2)$ thereof. Then $T(S_1)$ and $T(S_2)$ are called (*edge*) *isomorphic* iff for every edge $(p_i, p_j) \in T(S_1)$ there exists the edge $(q_i, q_j) \in T(S_2)$. Isomorphic triangulations belong to sets with the same number of extreme points because, by the Eulerian relation for planar graphs, the number of triangulation edges depends on the size of the convex hull. Still, an edge isomorphism may map extreme points to nonextreme ones, as in Figure 4. (If this effect is undesirable in a particular application, like morphing, the definition may be strengthened to an isomorphism between the face lattices formed by the triangles, edges, and vertices of the two triangulations.)

⋆ http://www.igi.TUGraz.at/oaich/triangulations/ordertypes.html

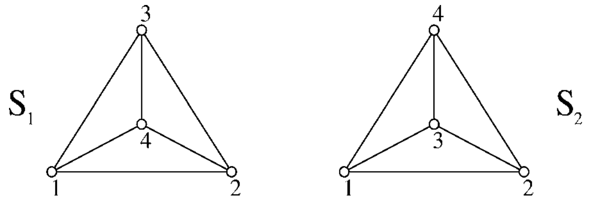

Figure 4. Edge-isomorphism is not 'hull-honest'.

If a bijection between two n-point sets S_1 and S_2 is fixed in advance, no isomorphic triangulations for S_1 and S_2 need to exist, even if the sets agree in the number of extreme points. It is an open problem whether the existence can be decided in polynomial time. See [27] where some necessary conditions are given. The problem becomes easier if S_1 and S_2 represent the cyclic order of the vertices of two simple polygons. In this case, the existence can be decided in time $O(n^3)$, and isomorphic triangulability can be forced by adding $O(n^2)$ extra points in each polygon; see [7]. An improved algorithm sensitive to the number of reflex angles of the polygons is given in [25].

On the other hand, for general n-point sets S_1 and S_2 with the same number of extreme points, it is conjectured that there always exists some bijection which admits isomorphic triangulations. A few results in this direction are known. Krasser [26] proved the conjecture true if the number of nonextreme points in either set is restricted to three or less. Moreover, it is not too hard to prove the existence of a *universal* point set, which is triangulable isomorphically with respect to every other set of the same size and convex hull size. A general affirmative answer would be a strong result, showing that any two point sets are 'topologically equivalent' in this sense. For results on the isomorphism conjecture known so far see [3].

It is an interesting challenge to investigate the conjecture numerically for sufficiently small n. We can make use of the following property: if the point sets S_1 and S_2 exhibit the same order type then every triangulation of S_1 has its counterpart in S_2. In principle, for each pair of point sets S_1, S_2 all (exponentially many) possible triangulations have to be computed (at least implicitly) and tested for isomorphism. In addition, the point indices describing the bijection have to be varied, which again gives an exponential growth of possibilities. This is doable for sizes $n \leq 8$ utilizing the order type data base described in Section 2. For $n = 9$ and $n = 10$, we tried out several heuristics for deciding isomorphic triangulability but did not obtain complete results for $n = 10$ yet. So, for the moment, we have to content ourselves with the following result.

OBSERVATION 3.1 *Let S_1 and S_2 be two n-point sets in general position in the plane which agree on the number of extreme points. Then for $n \leq 9$ there exist isomorphic triangulations for S_1 and S_2. This remains true even if a bijection between the extreme points of S_1 and S_2 is prescribed by a cyclic shift.*

Table IV. Exact extremum numbers of triangulations.

n	minimum	maximum
3	1	1
4	1	2
5	2	5
6	4	14
7	11	42
8	30	150
9	89	780
10	250	4550

It is worth mentioning that the conjecture is false if the assumption of general position of the point sets is dropped, i.e., when collinear points are allowed.

3.2. HOW MANY TRIANGULATIONS EVERY POINT SET MUST HAVE

Efficiently counting the number of triangulations of a given set of n points in the plane is an interesting open problem. The currently fastest method for counting is based on the concept of so-called triangulation paths, recently introduced in [1]. But still the running time shows exponential growth and computations are limited to $n \leq 40$.

On the other hand, no tight asymptotic bounds on the number of triangulations are known. The best examples for maximizing this number yield $8^{n-\Theta(\log n)}$ triangulations whereas the best known upper bound is much larger, $59^{n-\Theta(\log n)}$; see [28].

Even for small values of n no exact numbers are known. In view of the fact that point sets of the same order type admit equally many triangulations, we computed these numbers for all order types of size $n \leq 10$, using the method of [1]. We stress that this method actually needs the geometry of the realizing point sets (rather than just the λ-matrices) – a fact which is also true for other enumeration methods like reverse search; see [10].

OBSERVATION 3.2 *Let S be a set of n points in general position in the plane. From $n = 3$ to 10, Table* IV *gives exact lower and upper bounds on the number of triangulations of S.*

So, for example, every set of 8 points must have at least 30 triangulations, and there actually exists a set having that few triangulations. (The general position assumption – which has been adopted throughout – is crucial here.) This shows that point sets in convex position, whose number of triangulations is given by the Catalan numbers $C(n-2)$ (yielding 132 for $n = 8$), do not lead to the minimum.

This conjecture would have been plausible as it is true for crossing-free matchings and crossing-free spanning trees; see [20]. Though the conjecture was known to be false before, the results in Table IV may be useful in proving or disproving the extremal behavior of other point configurations. For instance, we could observe that the point sets leading to the lower bounds in Table IV obey a rather special structure.

In fact, and surprisingly, no good general lower bounds are known. We therefore seek for functions $t(n)$ such that every set of n points exhibits $\Omega(t(n))$ triangulations. A quick answer is $t(n) = 2^{(n-4)/6}$. This results from the fact that any triangulation on n points contains at least $(n-4)/6$ edges that can be flipped (i.e., replaced by the other diagonal of the corresponding convex quadrilateral) in an independent way; see [19]. A substantial improvement is based on the assertion on crossing families below which has been found by checking all order types for 10 points. A *crossing family* of size k of a point set S is a set of k line segments spanned by S which pairwise cross.

OBSERVATION 3.3 *Every set of $n \geq 10$ points in general position in the plane admits a crossing family of size* 3. *There exists a set of* 9 *points which does not have this property.*

This result improves over previously known lower and upper bounds and seems to be of separate interest. Aronov *et al*. [9] proved the existence of crossing families of size $\sqrt{n/12}$ for every set of n points. This evaluates to $n \geq 108$ for size 3. A smaller bound, $n \geq 37$, derives from a result by Tóth and Valtr [29]: among any $\binom{2c-5}{c-2} + 2$ points there are at least c points in convex position. Thus there must be a crossing family of size $\lfloor \frac{c}{2} \rfloor$.

Observation 3.3 is exploited in [4] to obtain the relation $t(n) \geq 3 \cdot t(n_1) \cdot t(n_2)$ for $n_1+n_2 = n+2$. For any base α, the function $t(n) = \frac{1}{3} \cdot \alpha^{n-2}$ is a solution. To get the recurrence started, values of α as large as possible are sought by examining all instances of small constant size. This is done by utilizing Table IV, in combination with other methods. The best values achieved up to now are $\alpha = 2 + \varepsilon$ for $\varepsilon < \frac{1}{80}$, that is, $t(n) > \frac{1}{13} \cdot (2 + \varepsilon)^n$.

3.3. SOME MORE APPLICATIONS

Applicability of our data base is by no means restricted to triangulation problems but rather includes various other questions concerning the crossing properties of geometric graphs. To aid the intuition of the reader, a few of them are briefly mentioned below. A more complete list of experiments carried out so far is given in the paper by Aichholzer and Krasser [5].

The (*rectilinear*) *crossing number problem* asks for the least number $c(n)$ of crossings every straight-line drawing of the complete graph K_n on n points in the plane must have. This problem has been studied first in [16]. In particular, the exact values of $c(n)$ have been known for $n \leq 9$. We were able to obtain $c(10) = 62$

which disproves a conjecture in [16].* Note that the *maximum* number of crossings of K_n is attained for n points in convex position and therefore is $\binom{n}{4}$. Note further that each crossing of K_n bijectively corresponds to a convex quadrilateral spanned by the underlying point set. Thus we have found the minimum number of convex quadrilaterals in a 10-point set as well.

Related is the (*rectilinear*) *Hamiltonian cycle problem*, which refers to the maximum number $h(n)$ of crossing-free Hamiltonian cycles a straight-line drawing of K_n can realize. In other words, what is the largest number of simple polygonalizations an n-point set does allow? Hayward [24] cites prior work on $h(n)$ and improves the lower bound. Exact values of $h(n)$ have been unknown for $n > 6$. We calculated these values for n up to 10 from our data base. As a result, the lower bounds given in [24] for $n = 7$ and 8 were found to be sharp whereas the contrary was true for $n = 9$ and 10. Note that the corresponding *minimization* problem is trivial: point sets in convex position admit a single crossing-free Hamiltonian cycle – the boundary of the convex hull.

A *k-set* of a planar n-point set S consists of k points which can be separated from S by a straight line. Finding upper and lower bounds on the maximum number $f_k(n)$ of k-sets is an intriguing problem which has been studied extensively; see, e.g., [15] and references therein. It is not hard to see that point sets of the same order type realize the same numbers of k-sets. For $n \leq 10$ the exact numbers $f_k(n)$ have been computed using the data base. The maximum number of $n/2$-sets (or halving lines) for 10 points, $f_5(10) = 13$, has been determined earlier; see Felsner [17].

Many more extremal properties of geometric graphs on $n \leq 10$ points are reported in [5]. Concepts addressed there include crossing-free spanning trees, crossing-free matchings, pseudo-triangulations, convex decompositions, spanned convex polygons, and several others.

4. Concluding Remarks

The goal of this work has been to provide a complete, valid, and user-friendly data base for order types of $n \leq 10$ points in the Euclidean plane, and to substantiate its usefulness to problems in computational and combinatorial geometry. Improved results on triangulation problems and on other questions have been obtained. As numerical stability is an important issue in geometrical algorithms, we have put particular emphasis on generating small and well-chosen grid representations of the realizing point sets. We believe our data base to be of use to many researchers who wish to examine their conjectures on small point configurations.

Clearly an extension of our project to point sets of larger size n is desirable. Apart from the obvious time and space limitations, a principal difficulty is that deciding realizability in a direct way is extremely costly; the algorithm described in [11] turned out being too slow already for $n = 8$. Heuristics have to be used

* The same result has been proved combinatorially (and independently) quite at the same time by Brodsky *et al*. [13].

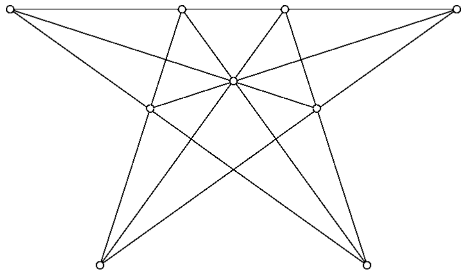

Figure 5. Order type with no grid representation.

for finding realizations, as well as for detecting non-realizability. Concerning our case $n = 10$, each order type could be classified in one way or the other. It remains unclear whether the gap of undecided order types can be closed for $n \geq 11$ as well. We are currently performing calculations for $n = 11$ with good progress.

While our approach can be adapted to cases of collinearities, it will cover only very small instances: the number of possible order types increases tremendously compared to the case of general position; see [21]. Moreover, it is theoretically impossible to obtain integer grid representations for all order types if the general position assumption is dropped, in view of the existence of non-rational oriented matroids for sizes $n > 8$. See [11] where Figure 5 is taken from.

References

1. Aichholzer, O.: The path of a triangulation, in *Proc. 15th Ann. ACM Sympos. Computational Geometry*, Miami Beach, USA, 1999, pp. 14–23.
2. Aichholzer, O., Aurenhammer, F. and Krasser, H.: Enumerating order types for small point sets with applications, In: *Proc. 17th Ann. ACM Sympos. Computational Geometry*, Medford, USA, 2001, pp. 11–18.
3. Aichholzer, O., Aurenhammer, F., Hurtado, F. and Krasser, H.: Towards compatible triangulations, In: *Proc. 7th Ann. Int. Computing and Combinatorics Conf. CoCOON-2001*, Guilin, China, Lecture Notes in Comput. Sci. 2108, Springer, New York, 2001, pp. 101–110. To appear in *Theoret. Comput. Sci*.
4. Aichholzer, O., Hurtado, F. and Noy, M.: On the number of triangulations every planar point set must have, In: *Proc. 13th Ann. Canadian Conference on Computational Geometry CCCG 2001*, Waterloo, Canada, 2001, pp. 13–16.
5. Aichholzer, O. and Krasser, H.: The point set order type data base: A collection of applications and results, In: *Proc. 13th Ann. Canadian Conference on Computational Geometry CCCG 2001*, Waterloo, Canada, 2001, pp. 17–20.
6. Arkin, E., Fekete, S., Hurtado, F., Mitchell, J., Noy, M., Sacristán, V. and Sethia, S.: On the reflexivity of point sets, In: *Proc. 10th Ann. Fall Workshop on Computational Geometry*, Stony Brook, NY, 2000.
7. Aronov, B., Seidel, R. and Souvaine, D.: On compatible triangulations of simple polygons, *Comput. Geom.* **3** (1993), 27–35.
8. Aurenhammer, F.: Voronoi diagrams – a survey of a fundamental geometric data structure, *ACM Comput. Surveys* **23** (1991), 345–405.
9. Aronov, B., Erdös, P., Goddard, W., Kleitman, D. J., Klugerman, M., Pach, J. and Schulman, L. J.: Crossing families, *Combinatorica* **14** (1994), 127–134.

10. Avis, D. and Fukuda, K.: Reverse search for enumeration, *Discrete Appl. Math.* **65** (1996), 618–632.
11. Björner, A., Las Vergnas, M., Sturmfels, B., White, N. and Ziegler, G.: *Oriented Matroids*, Cambridge Univ. Press, 1993.
12. Bokowski, J. and Guedes de Oliveira, A.: On the generation of oriented matroids, *Discrete Comput. Geom.* **24** (2000), 197–208.
13. Brodsky, A., Durocher, S. and Gether, E.: The rectilinear crossing number of K_{10} is 62, *Electron. J. Combinatorics* **8** (2001), Research Paper 23.
14. de Berg, M., van Krefeld, M., Overmars, M. and Schwarzkopf, O.: *Computational Geometry – Algorithms and Applications*, Springer, Berlin, 1997.
15. Dey, T. K.: Improved bounds for planar k-sets and related problems, *Discrete Comput. Geom.* **19** (1998), 373–382.
16. Erdös, P. and Guy, R. K.: Crossing number problems, *Amer. Math. Monthly* **88** (1973), 52–58.
17. Felsner, S.: On the number of arrangements of pseudolines, *Discrete Comput. Geom.* **18** (1997), 257–267.
18. Finschi, L. and Fukuda, K.: Generation of oriented matroids – a graph theoretical approach, *Discrete Comput. Geom.* **27** (2002), 117–136.
19. Galtier, J., Hurtado, F., Noy, M., Perennes, S. and Urrutia, J.: Simultaneous edge flipping in triangulations, Manuscript, Universitat Politecnica de Catalunya, Barcelona, Spain, 2000. http://www-ma2.upc.es/~hurtado/flipcorner.html
20. García, A., Noy, M. and Tejel, J.: Lower bounds on the number of crossing-free subgraphs of K_N, *Comput. Geom.* **16** (2000), 211–221.
21. Goodman, J. E.: Pseudoline arrangements, In: J. E. Goodman and J. O'Rourke (eds), *Handbook of Discrete and Computational Geometry*, CRC Press LLC, Boca Raton, NY, 1997.
22. Goodman, J. E. and Pollack, R.: Multidimensional sorting, *SIAM J. Comput.* **12** (1983), 484–507.
23. Goodman, J. E., Pollack, R. and Sturmfels, B.: Coordinate representation of order types requires exponential storage, In: *Proc. 21st Ann. ACM Sympos. Theory of Computing*, 1989, pp. 405–410.
24. Hayward, R. B.: A lower bound for the optimal crossing-free Hamiltonian cycle problem, *Discrete Comput. Geom.* **2** (1987), 327–343.
25. Kranakis, E. and Urrutia, J.: Isomorphic triangulations with small number of Steiner points, *Intertat. J. Comput. Geom. Appl.* **9** (1999), 171–180.
26. Krasser, H.: Kompatible Triangulierungen ebener Punktmengen, MS thesis, IGI-TU Graz, Austria, 1999.
27. Saalfeld, A.: Joint triangulations and triangulation maps, In: *Proc. 3rd Ann. ACM Sympos. Computational Geometry*, Waterloo, Canada, 1987, pp. 195–204.
28. Santos, F. and Seidel, R.: A better bound on the number of triangulations of a planar point set, Manuscript, 2000.
29. Tóth, G. and Valtr, P.: Note on Erdös–Szekeres theorem, *Discrete Comput. Geom.* **19** (1998), 457–459.