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The Book Thickness of a Graph

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The book thickness $bt(G)$ of a graph G is defined, its basic properties are delineated, and relations are given with other invariants such as thickness, genus, and chromatic number. A graph G has book thickness $bt(G) \leqslant 2$ if and only if it is a subgraph of a hamiltonian planar graph, but we conjecture that there are planar graphs with arbitrarily high book thickness.

1. INTRODUCTION

There are several geometric invariants which have been studied extensively for graphs—among them, genus and thickness. In this paper we introduce a new invariant defined by considering embeddings of graphs into the members of what seems to us to be a very natural class of objects.

For $n \geqslant 0$, an *n-book*, or a *book with n pages*, consists of a line L in 3-space (called the *spine*) together with n distinct half-planes (called *pages*) with L as their common boundary. We usually adopt the convention that L is the z axis (in the standard parameterization of euclidean 3-space) and so is oriented "up" and "down."

An *n-book embedding* is a topological embedding β of G in an n-book which carries each vertex into the spine and each edge into the interior of at most one page. Thus, G can be embedded in an n-book if and only if we can find points $x_1, \ldots, x_p$ corresponding to the vertices $v_1, \ldots, v_p$ of G and simple

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arcs a_{ij} corresponding to those pairs v_i, v_j which are adjacent such that the usual graph embedding conditions hold, and such that

(1) each x_i beongs to L;

(2) each arc a_{ij} is contained within L or else within at most one component of $B - L$.

Actually, in all but trivial situations, condition (2) implies (1).

The *book-thickness* $bt(G)$ of G is the smallest n such that G has an n-book embedding. In Section 2, we examine the basic properties of book embeddings. The next section deals with relations to other invariants, such as thickness, and we compute the book thickness of complete and some bipartite complete graphs. An upper bound for the chromatic number of a graph with specified book thickness is derived in Section 4, and machinery is developed to examine the book thickness of cubes. In the last section, we investigate the connection between genus and book thickness and formulate an interesting problem on the book thickness of planar graphs.

In the definition first contemplated for $bt(G)$, the strong condition (2) was not included. The reason for its addition is to make the study of book embeddings less trivial (see Theorem 5.4 at the end of the paper).

For any terms not defined here, see Harary [1]. Write $\lceil x \rceil$ for least integer k such that $x \leqslant K$.

2. Printing Cycles

Consider an n-book embedding of some graph G. The vertices of G occur in some specified order $v_1, \ldots, v_p$ from top to bottom along the spine and this sequence is called the *printing cycle* of the embedding. If any of the edges $[v_i, v_{i+1}]$, $1 \leqslant i \leqslant p - 1$, are not in G, we can add them by drawing them in the spine and if $[v_1, v_p]$ is not in G, we can add it to any of the n pages (assuming $n \geqslant 1$). Note that the printing cycle $v_1, \ldots, v_p$ for this n-book embedding of G' is a hamiltonian cycle, where G' results from G by adding the missing edges.

By including the edge $[v_1, v_p]$ in each page, we obtain the following result:

Lemma 2.1. *Let G be any graph. Then G has an n-book embedding ($n \geqslant 1$) with given hamiltonian printing cycle $v_1, \ldots, v_p$ if and only if $G = G_1 \cup \cdots \cup G_n$, where each G_i is isomorphic to a graph drawn in the plane as a polygon $v_1, \ldots, v_p$ with some inner diagonals, no two of which cross.*

This permits us to modify the printing cycle of a book embedding in two obvious ways.

LEMMA 2.2. *If G has an n-book embedding ($n \geqslant 1$) with printing cycle $v_1, \ldots, v_p$, then G also has an n-book embedding with printing cycle $v_2, \ldots, v_p, v_1$.*

LEMMA 2.3. *If G has an n-book embedding β ($n \geqslant 0$) with printing cycle $v_1, \ldots, v_p$, then G also has an n-book embedding β^- with printing cycle $v_p, \ldots, v_1$.*

In general, the number n of pages required for a book embedding of some graph G depends on the printing cycle. For example, if G is the 4-cycle C_4 with vertices v_1, v_2, v_3, v_4 forming the cycle in that order, then $n = 1$, while if the vertices are enumerated v_1, v_3, v_2, v_4, $n = 2$.

Let G be a graph and σ a listing of the vertices of G. The *σ-thickness* $bt(G, \sigma)$ of (G, σ) is the smallest integer n such that G has an n-book embedding with σ as printing cycle. The book thickness of G, $bt(G)$, is simply the minimum of $bt(G, \sigma)$ for σ any listing of the vertices.

In order to compute book thickness, it suffices to consider nonseparable graphs because of the next result.

THEOREM 2.4. *For any graph G, containing at least one cycle,*

$$bt(G) = \max\{bt(B_i) \mid B_i \text{ a block of } G, 1 \leqslant i \leqslant b\}.$$

Proof. Induct on b. The case $b = 1$ is trivial, so suppose the theorem holds for $b - 1$ blocks and let G be arbitrary with $b \geqslant 2$ blocks. Choose an endblock B which intersects the union G' of remaining blocks in a single vertex v. By using Lemmas 2.2 and 2.3, we can find disjoint book embeddings of B "on top" and G' "below" in an n-book (with $n = \max(bt(B), bt(G'))$ so that the bottom point of B and the top point of G' both correspond to v. Moving these two points together completes the embedding of G in an n-book. Now according to the inductive hypothesis, $\max(bt(B), bt(G')) = \max(bt(B), bt(G')) = \max\{bt(B_i) \mid B_i$ a block of G, $1 \leqslant i \leqslant b\}$. If G is disconnected, the proof is similar.

It is easy to characterize graphs with small book thickness.

THEOREM 2.5. *Let G be a connected graph. Then*

(i) *$bt(G) = 0$ if and only if G is a path;*

(ii) *$bt(G) \leqslant 1$ if and only f G is outerplanar;*

(iii) *$bt(G) \leqslant 2$ if and only if G is a subgraph of a hamiltonian planar graph.*

Since there are maximal planar graphs which are not hamiltonian, there are planar graphs with book thickness greater than 2 (see Section 5).

3. Relations with Other Invariants

The *thickness* $\theta(G)$ of G is the smallest number of planar subgraphs of G whose union is G. *Outerplanar thickness* $\theta_{op}(G)$ is defined analogously. It is not difficult to check that the following relations hold:

LEMMA 3.1. *For any graph G with $bt(G) \geqslant 1$,*

(i) $\theta_{op}(G) \leqslant bt(G)$;

(ii) $\theta(G) \leqslant \lceil bt(G)/2 \rceil$.

In general, neither of these inequalities is an equality. For example, $\theta_{op}(K_5) = 2$ since K_5 is the union of two edge-disjoint cycles but $bt(K_5) > 2$ by Theorem 2.5. The book thickness of K_{13} is 7 as we shall soon prove and it is known that $\theta(K_{13}) = 3$ (see, e.g., Harary [1, p. 120]), Therefore, $3 = \theta(K_{13}) < \lceil bt(K_{13})/2 \rceil = 4$.

Let $\alpha(G)$ denote the smallest number of vertices incident with every edge (point-line covering number).

LEMMA 3.2. *If G is any graph and σ any listing of the vertices, then $bt(G, \sigma) \leqslant \alpha(G)$.*

Proof. Put the vertices on the spine in the order σ. Now choose a set S of α vertices that cover the edges and assign a distinct page to each. At each vertex $v \in S$, we can accommodate as many edges as desired on the corresponding page.

An outerplanar graph with p vertices has at most $2p - 3$ edges [1, p. 107] and so we have

THEOREM 3.3. *Let G be a graph with p vertices and q lines. Then $bt(G) \geqslant \lceil (q - p)/(p - 3) \rceil$.*

Proof. Given an n-page book embedding of G, G has at most p edges in the printing cycle and at most $p - 3$ other edges on each page. Thus, $q \leqslant p + n(p - 3)$, which suffices.

Applying this result to K_m gives us one-half of the next theorem (see also Ollman [2]).

THEOREM 3.4. *For $m \geqslant 4$, $bt(K_m) = \lceil m/2 \rceil$.*

Proof. If m is even, say $m = 2k$, we must show $bt(K_{2k}) \leqslant k$. Then, since K_{2k-1} is a subgraph of K_{2k}, the result will be true for odd m as well. We indicate the pages of an embedding of K_{2k} in a k-book by taking the triangulated $2k$-gon in Fig. 1 and rotating it through k successive positions. Since no inner diagonal appears in more than one page, $k(2k - 3) + 2k$ accounts for all the edges of K_{2k}.

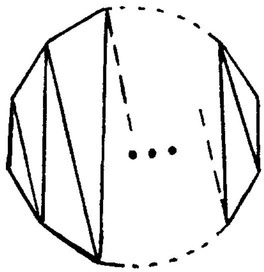

FIGURE 1

The next family of graphs of interest is the family of complete bipartite graphs $K(m, n)$. We assume $m \leqslant n$, and call the m vertices *blue*, and the n vertices *red*. By 3.2 we have $bt(K(m, n)) \leqslant \alpha(K(m, n)) = m$. It is natural to seek equality, in which case only $K(m, m)$ need be considered. By direct verification $bt(K(1, 1)) = 0$ (obviously exceptional), $bt(K(2, 2)) = 1$, and, since $K(3, 3)$ is well known to be nonplanar, $bt(K(3, 3)) \geqslant 3$. Apart from Lemma 3.2, the 3-book embedding $K(3, 3)$ in Fig. 2 shows that $bt(K(3, 3)) = 3$. We might therefore seek the equality $bt(K(m, m)) = m$ for $m \geqslant 3$. However, Fig. 3 shows that $bt(K(4, 4)) = 3$.

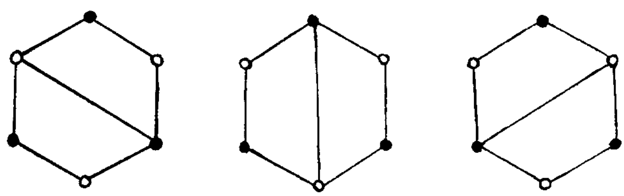

FIGURE 2

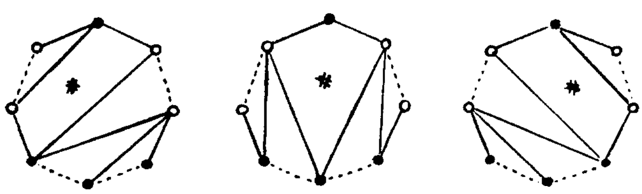

FIGURE 3

Using the lower bound previously obtained, and the fact that $K(m, n)$ has $m + n$ vertices and $m \cdot n$ edges, $bt(K(m, n)) \geqslant (mn - m - n)/(m + n - 3)$ or $bt(K(m, m)) \geqslant m/2$. This leaves a large gap in which to place the book thickness of $K(m, m)$.

By contrast, we can show the following

THEOREM 3.5. *For $m \leqslant n$ with $n \geqslant m^2 - m + 1$, we have $bt(K(m, n)) = m$.*

Proof. Arrange the $m + n$ vertices on the printing cycle in any order. The m blue vertices divide the cycle into m arcs, and an easy pigeon hole argment shows that for $n \geqslant m^2 - m + 1$ one of these arcs contains m red vertices. Let the labeling be chosen so that m blue vertices $v_1, \ldots, v_m$ and m red vertices $u_1, \ldots, u_m$ fall into a cyclic sequence $v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_m$. Then each edge $[v_i u_i]$ must be a diagonal in some page, but no two can be in the same page, since the endpoints alternate on the cycle. At least m pages are required.

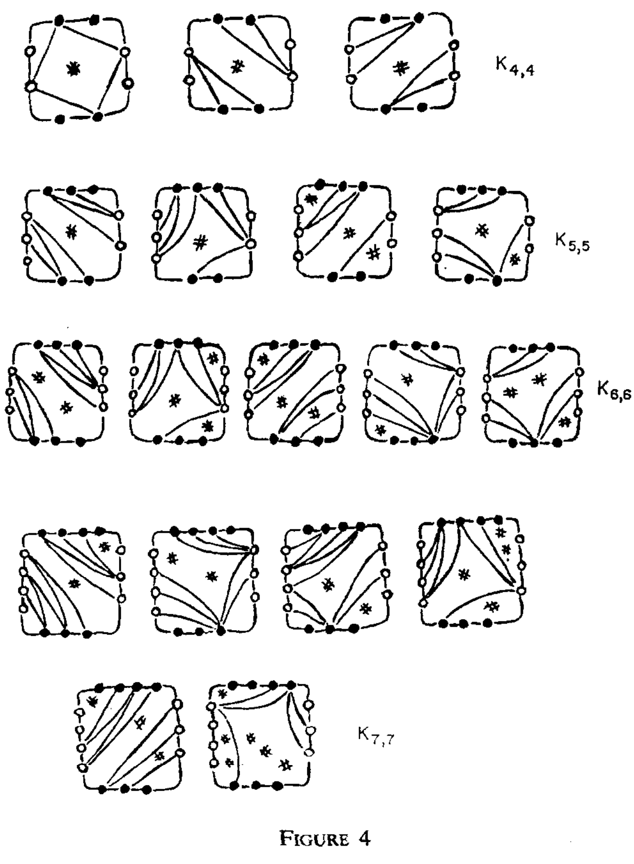

FIGURE 4

As a final result, we show that $bt(K(m, m)) = m$ fails for all $m \geqslant 4$. Consider these book embeddings in which red and blue vertices are grouped into two groups of nearly equal number (Fig. 4).

The cases for $m \geqslant 8$ can be reduced from m to $m - 4$ (see Fig. 5). Four pages are used to connect the eight vertices in the "corners" to all the others. The vertices next to them can be connected around the corners in the remaining pages to form a $(2m - 4)$-sided polygon.

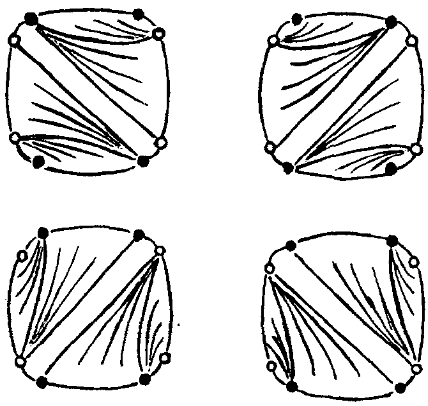

FIGURE 5

THEOREM 3.6. *For $m \geqslant 4$, we have $bt(K(m, m)) \leqslant m - 1$.*

The problem of determining $bt(K(m, m))$ could be rephrased as a combinatorial problem of "eccentric hosts" (following a problem of Seymour Schuster). A couple wishes to invite $m - 1$ other couples to a dinner party and seat them around a single table. Each of the men will want to talk with each of the women at some time during the evening, but it will generally not be possible to carry on a conversation across the table if another conversation is already going on that cuts across. The hosts plan to interrupt at intervals and suggest each guest become acquainted with a new person. They wish to allow maximum time for talking, and interrupt as few times as necessary in order to allow all the men to talk with all the women. Their job is easier because each man or woman is disposed to carry on as many conversations at once as is possible under the above limitations. How should they seat men and women around the table. This is essentially asking what arrangement of m blue and m red vertices on a printing cycle allows $K(m, m)$ to be embedded in a book with fewest pages.

4. Products and Chromatic Number

Let $\chi(G)$ denote as usual the chromatic number of G. By estimating the average degree in a graph G satisfying $bt(G) \leqslant k$, $k > 0$, we determine an upper bound on $\chi(G)$.

Let $d(G)$ denote the *average degree* of the verices of G; that is, $2q/p$ in a graph with p vertices and q edges. Hence, by a slight alteration of the proof of Theorem 3.3,

$$\begin{aligned} d &= 2q/p \\ &\leqslant 2((k+1)p - 3k)/p < 2k+2. \end{aligned}$$

We have proved

LEMMA 4.1. *For $k > 0$, if $bt(G) \leqslant k$, then $d(G) < 2k + 2$.*

Applying simple induction (see Szekeres-Wilf [3],) we get

THEOREM 4.2. *If $bt(G) \leqslant k$, then $\chi(G) \leqslant 2k + 2$.*

Note that there are graphs with book thickness k and chromatic number $2k$ (for example the complete graph of order $2k$) so the result just obtained cannot be very far from best possible. More specifically, we have shown that $2k \leqslant \sup\{\chi(G) \mid bt(G) \leqslant k\} \leqslant 2k + 2$. For certain values of k we can do a bit better. If $bt(G) \leqslant 1$, then G is outerplanar and hence $\chi(G) \leqslant 3$. If $bt(G) \leqslant 2$, then G is planar and so $\chi(G) \leqslant 5$. In fact, $\chi(G) \leqslant 4$, for $k \leqslant 2$ is equivalent to the 4-Color Theorem [4] by a theorem of Whitney [5].

The chromatic bound could be improved if for $k \geqslant 3$ and $bt(G) \leqslant k$, we could show G had a vertex v with $\deg(v) \leqslant 2k$. Query: Can we find a $(2k+1)$-regular graph G with $bt(G) = k$?

There is another interesting connection between book thickness and chromatic number, useful in seeking the book thickness of the cube $Q(d)$. Let the maximum value of $\deg(v)$ in H be $\Delta(H)$. Then a graph H is *dispersable* if there is a book embedding of H in a book with $\Delta(H)$ pages and an edge-coloring of H with $\Delta(H)$ colors so that all edges of one color lie on the same page. (Here an edge-coloring has the usual meaning: no two adjacent edges are colored the same.) For example, K_2 is dispersable.

THEOREM 4.3. *Let B be a dispersable bipartite graph and let G be arbitrary. Then $bt(G \times B) \leqslant bt(G) + \Delta(B)$.*

Proof. Since B is dispersable, let β be a book embedding of B in $\Delta(B)$ pages and let c be a $\Delta(B)$-edge coloring so that all edges of one color lie on one page. Furthermore, since B is bipartite, we can 2-color its vertices using colors 0 and 1. Now take all vertices of B colored 0 and replace them by a "small" book embedding φ of G in $bt(G)$ pages. The remaining vertices of B, which are colored 1, are replaced by the reversed embedding φ^{-} of G. Edges in $G \times B$ either lie within a single copy of G or join corresponding vertices of G in adjacent copies. But our assumption that B is bipartite means that all of the edges joining two adjacent copies of G can be accommodated on the appropriate page as concentric semicircles. G dispersable means that this process accomodates all edges of $G \times B$ in $bt(G) + \Delta(B)$ pages.

For example, if $G = K_5 - X$ and $B = P_3$ then β and φ are given as in Fig. 6. The resulting book embedding of $G \times B$ in $2 + 2 = 4$ pages is given in Fig. 7.

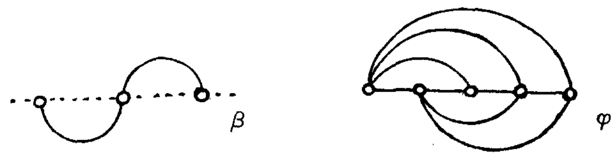

FIGURE 6

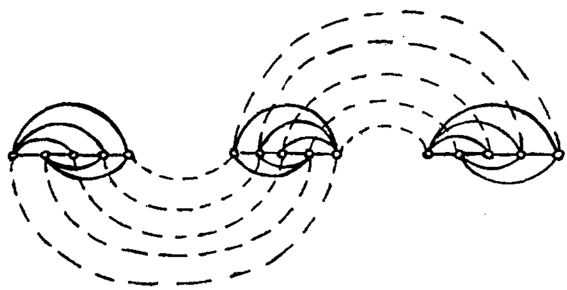

FIGURE 7

In view of this theorem, we would like to find dispersable bipartite graphs. It is interesting to note that we know of no bipartite graphs which are *not* dispersable. If a graph H is dispersable, then its edges can be $\Delta(H)$-colored. This latter property does hold for every bipartite graph B but we do not know that all of the edges of one color need be on a single page of a book embedding in $\Delta(B)$ pages.

Classes of bipartite graphs which can be shown to be dispersable include (i) $K(n, n)$ $(n \geqslant 1)$, (ii) C_{2m} $(m \geqslant 2)$, (iii) $Q(d)$ $(d \geqslant 0)$, (iv) trees. It follows from (iii) that $bt(Q(d)) \leqslant d$. In fact, we can improve this result by one.

THEOREM 4.4. *If $d \geqslant 1$, then $bt(Q(d)) \leqslant d - 1$.*

Proof. Clearly, $bt(Q(2)) = 1$ and $bt(Q(1)) = 0$. Since $Q(d) = Q(d - 1) \times K_2$, Theorem 4.3 and induction yields $bt(Q(d)) \leqslant d - 1$.

If $Q(d)$ is embedded in a k-book with a hamiltonian printing cycle, then each page consists of a polygon with inner diagonals and no triangles. Hence, each page has at most $\frac{1}{2}(2^d - 4) = 2^{d-1} - 2$ inner diagonals and so $k(2^{d-1} - 2) \geqslant d2^{d-1} - 2^d = (d-2)\,2^{d-1}$. Therefore, $k \geqslant (d-2)\,2^{d-1}(2^{d-1} - 2)^{-1}$. Since k is an integer and $2^{d-1}(2^{d-1} - 2)^{-1} > 1$ for $d \geqslant 3$, we have $k \geqslant d - 1$. Thus, we are led to conjecture that $bt(Q(d)) = d - 1$.

Unfortunately, there are minimal book embeddings of $Q(d)$ which do not have a hamiltonian printing cycle. In Fig. 8, for example, we give such an embedding of $Q(3)$.

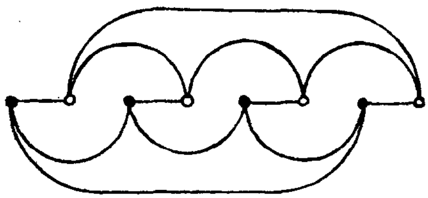

FIGURE 8

5. Book-Thickness and Genus

We have noted previously that if a maximal planar graph has book thickness 2, then it is hamiltonian. Since maximal planar graphs exist which are not hamiltonian, (see Fig. 9), not every planar graph has book thickness less than 3. In fact, we believe that the book thickness of planar graph can be made arbitrarily large.

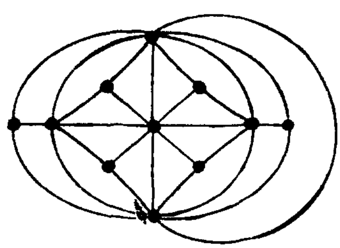

FIGURE 9

Define the *stellation* St(G) of G as the result of placing a new vertex in each face of planar G, and connecting it to each vertex around the face. If G is thought of as a polyhedron, the operation stellates each face with a small pyramid. Given K_3, the triangle graph; St(St(K_3)) is the maximal planar non-hamiltonian graph shown above. Whitney [4] proved that a maximal planar graph in which all triangles are face boundaries is hamiltonian. Stellation, however, when repeated, produces many triangles which are not of this type. Let $\mathrm{St}^n(G) = \mathrm{St}(\mathrm{St}^{n-1}(G))$.

CONJECTURE 5.1. For any maximal planar G, $bt(\mathrm{St}^n(G))$ can be made arbitrarily large by choosing n sufficiently large.

Replacing book thickness by another parameter, which is possibly related to it, an analogous result can be proved. Define the *hamilton ratio* $h(G)$ to be m/p if G is a maximal planar graph with p vertices and the circumference, or length of the longest cycle, is m. Clearly $h(G) \leqslant 1$, with equality iff G is hamiltonian.

THEOREM 5.2. *For a maximal planar* G, $h(\mathrm{St}^n(G)) \to 0$ *as* $n \to \infty$.

Proof. Let F, E, V be the numbers of faces, edges, and vertices of G. These quantities satisfy the Euler formula $F + V = E + 2$, and also $3F = 2E$, since all the faces are triangles. If the index $\Phi = V - 2$ is assigned to G, F, E, V are parametrically expressed

$$F = 2\Phi, \qquad E = 3\Phi, \qquad V = \Phi + 2.$$

Put $h = h(G)$, $h' = h(\mathrm{St}(G))$, and define other primed variables similarly for $\mathrm{St}(G)$. Without loss of generality, we can assume $\Phi \geqslant 6$, and the longest cycle of G has length at least five. For $\mathrm{St}(G)$ we compute

$$F' = 3F,\ E' = E + 3F,\ V' = V + F, \text{ or } V'/V = (3\Phi + 2)/(\Phi + 2) \geqslant 5/2.$$

Now let C' be a longest cycle of $\mathrm{St}(G)$, and v a vertex of degree 3. Then v must be one of the 2Φ new vertices of degree three. If C' goes through v, C' goes through two sides of one of the face triangles at v. Call the third side of this triangle the side *opposite* v. No edge will be a side opposite more than one vertex of degree three on C'. Change $\mathrm{St}(G)$ back to G by removing all the degree three vertices, and divert C' when necessary through the opposite side. The new cycle C is a cycle of G, and its length is at least half the length of C'. In summary, $\mathrm{St}(G)$ has at least $5/2$ as many vertices, but its longest cycle is at most twice as long as the longest cycle of G, thus $h' \leqslant (4/5)\, h$. Clearly, $h(\mathrm{St}^n(G))$ can be made as small as desired.

In the other direction, there is definitely no connection between genus $\gamma(G)$ and book thickness.

THEOREM 5.3. *There exist graphs* G *with* $bt(G) = 3$ *but* $\gamma(G)$ *arbitrarily large.*

Proof. The triangulations obtained by rotating the triangulation in Fig. 1 are edge-disjoint except for the bounding p-gon. If G is the union of three such triangulations, then G has p vertices and $4p$-9 edges. It follows from Euler's formula (see [1, p. 118]) that $\gamma(G) \geqslant (p - 3)/3$ but $bt(G) = 3$.

By this time, the reader may have wondered what happens to the idea of book embedding if the restrictions are loosened.

THEOREM 5.4. *If book embeddings were permitted to use more than one page for each edge, then every graph could be embedded in a 3-book.*

Proof. It is well known that any graph can be put in the plane with only a finite number of crossings. Treat the crossings as new vertices and distort the embedding to make the "new" vertices lie on a single straight line L. At each crossing make one of the modifications indicat3d below in Fig. 10. (Each pair of dots can be connected through a third page attached to the plane at L.)

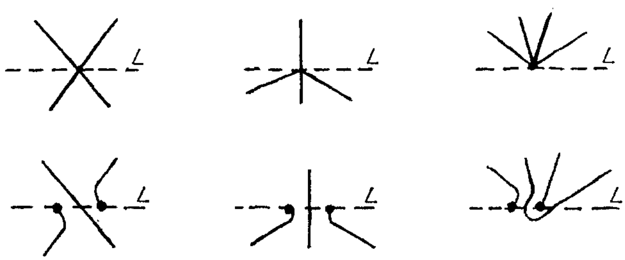

FIGURE 10

COROLLARY 5.5. *If G is any graph, there is a graph G' obtained by making suitable subdivisions of the edges of G, such that $bt(G') \leqslant 3$.*

This corollary may be viewed as another reason for aligning book embeddings with thickness and not with genus. It would be possible to introduce a new set of (j, k)-invariants (but we forbear) indicating the least number of subdivisions of edges of a specific graph G of book thickness k required to lower its book thickness to j. Keys [b] considers $bt(b)$ for certain graphs which are almost complete.

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