BOLYAI SOCIETY
MATHEMATICAL STUDIES, 25

Erdős Centennial
pp. 465–484.

The Beginnings of Geometric Graph Theory

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"... to ask the right question and
to ask it of the right person."
(Richard Guy)

Geometric graphs (topological graphs) are graphs drawn in the plane with possibly crossing straight-line edges (resp., curvilinear edges). Starting with a problem of Heinz Hopf and Erika Pannwitz from 1934 and a seminal paper of Paul Erdős from 1946, we give a biased survey of Turán-type questions in the theory of geometric and topological graphs. What is the maximum number of edges that a geometric or topological graph of n vertices can have if it contains no forbidden subconfiguration of a certain type? We put special emphasis on open problems raised by Erdős or directly motivated by his work.

1. Introduction

The term "geometric graph theory" is often used to refer to a large, amorphous body of research related to graphs defined by geometric means. Here we take a narrower view: by a *geometric graph* we mean a graph G drawn in the *plane* with possibly intersecting straight-line edges. If the edges are allowed to be arbitrary continuous curves connecting the vertices (points), then G is called a *topological graph*. Disregarding the particular way the graph is drawn, we obtain the "abstract" underlying graph of G, which is usually also denoted by G. We use the term *geometric graph theory* as a short form for "the theory of geometric and topological graphs."

*Supported by NSF Grant CCF-08-30272, by OTKA under EUROGIGA projects GraDR and ComPoSe 10-EuroGIGA-OP-003, and by Swiss National Science Foundation Grants 200020-144531 and 200021-137574.

In the past few decades, a number of exciting discoveries have been made in this field. Some of them have found interesting applications in graph drawing, in combinatorial and computational geometry, in additive number theory, and elsewhere. See, e.g., [5], [68], [95], [101], [27]. Many related contributions can be found in the proceedings of the annual symposia on graph drawing, published in Springer's Lecture Notes series in Computer Science (for instance, in [64]) and in two collections of papers [78], [79]. For surveys, see Chapter 14 in [80], Chapter 10 in [51], and Chapters 1 and 3 in [42].

Paul Erdős had a profound influence on the subject. On the occasion of his 100th birthday, we review the beginnings of geometric graph theory in the 1930s and 40s, which were also formative years in Erdős's personal and mathematical life. We use this as a starting point to give a short and biased survey of some research directions that can be traced back more or less directly to these early developments. We put special emphasis on open problems raised by Erdős and others, which had a large impact on the evolution of geometric graph theory.

2. A Problem in Jahresbericht – German Mathematics

In 1934, *Heinz Hopf* and *Erika Pannwitz*, Hopf's student at Friedrich Wilhelms University (today Humboldt University) in Berlin, posed the following problem in the problem section of *Jahresbericht der Deutschen Mathematiker-Vereinigung.*

Problem 1 [57]**.** Let $p_0, p_1, \ldots, p_{n-1}$, $p_n = p_0$ be n distinct points in the plane such that the distance conditions

$$d(p_i, p_j) \le 1 \quad (0 \le i < j < n),$$
$$d(p_i, p_{i+1}) = 1 \quad (i = 0, \ldots, n-1)$$

are satisfied. Prove that this is possible if and only if n is odd or $n = 2$.

Three solutions were subsequently published in 1935: by *W. Fenchel* (Copenhagen), by *J. W. Sutherland* (Cambridge) [43], and in the next issue of the journal, by *H. Baron* (Berlin) [9]. Other correct solutions were submitted by *A. E. Mayer* (Wien), *H. Baer* (Frankfurt a. M.), *L. Ehrlich* (Berlin), *J. Fox* (Brooklyn), *R. Frucht* (Triest), *L. Goeritz* (Rostock), *F. Gruber* (Vienna), *J. Juilfs* (Berlin), *R. Lauffer* (Graz), *E. Linés Escardó* (Madrid), *B. Neumann* (Cambridge), *L. Rédei* (Mezőtúr), *L. A. Santaló* (Madrid), *P. Scherk* (Göttingen), and *W. Schulz* (Berlin).

The "Annual Reports" of the German Mathematical Society were published, of course, in German. However, many solutions and articles were sent by mathematicians from other, non German speaking countries, mostly from Europe and from the United States. In the 1930s, German universities played a leading role in mathematics. From all over the world, many young talents (like Fox, Rédei, and Santaló) came to study in Berlin, München, Hamburg, Göttingen, and elsewhere. At the 1936 International Congress of Mathematicians held in Oslo, half of the plenary lectures were delivered in German [77]. When after a 14-year recess due to the war the next congress was held at Harvard University, only one of the 21 main lectures had a German title: it was the talk of Hopf, one of the original proposers of Problem 1. However, this time he did not arrive from Berlin, he was Professor at ETH Zürich. Fenchel, Frucht, Neumann, and Santaló had also fled Germany and built distinguished academic careers in Copenhagen, Valparaiso, Canberra, and Buenos Aires. They became leading experts in convexity, graph theory, group theory, and integral geometry. The lives of many of those who stayed in Germany were sidetracked: Pannwitz worked for the German Cryptography Service during the war and Juilfs became an SS Obersturmsführer. Between 1944 and 1951 the publication of *Jahresbericht* was halted.

Fenchel's elegant solution to Problem 1 was based on the following observation [43]. Connect two points, p_i and p_j, by a segment if their distance is equal to the diameter of the point set $P = \{p_0, \dots, p_{n-1}\}$ (which is, in our case, equal to 1). The resulting geometric graph is called the *diameter graph* (or the graph of diameters) associated with P. It follows from the triangle inequality that any two edges of the diameter graph either share an endpoint or cross each other. Suppose now that $n > 2$ and that P satisfies the properties in Problem 1. Since the diameter graph has no two disjoint edges, the segments p_0p_1 and p_2p_3 must lie in the same half-plane bounded by the line p_1p_2. Thus, p_0 and p_3 lie in the same half-plane. For the same reason, all edges $p_3p_4, p_4p_5, \dots, p_{n-1}p_0$ must cross the line p_1p_2, hence the elements of the sequence $p_3, p_4, \dots, p_n = p_0$ lie on alternating sides of the line p_1p_2. This is possible only if n is odd.

3. A Paper in the Monthly – Paul Erdős Enters the Scene

Erdős was one of the most successful problem solvers of *Középiskolai Matematikai Lapok*, an excellent Hungarian journal for high school students, founded in 1893. He had a lifelong passion for mathematical puzzles and spoke fluent German. In 1934, the same year, when the Hopf-Pannwitz problem appeared, Erdős received his doctorate at Péter Pázmány University (today Loránd Eötvös University), Budapest. Because of the increasingly anti-semitic atmosphere in Hungary, he accepted a fellowship arranged by *Louis J. Mordell*, and moved first to Manchaster and four years later to Princeton. He had access to the *Jahresbericht*, and it is almost certain that he came across Problem 1 shortly after it was published. We will see in the sequel that it inspired him to create a whole new area of research in discrete geometry.

The argument of Fenchel described in the previous section can be easily modified to yield the following statement. It first appeared in a classic paper of Erdős [29] published in the *American Mathematical Monthly* in 1946. He generously attributed the result to Hopf and Pannwitz, although in this form it does not appear in [57]: it was first formulated by him.

Theorem 2 [29]**.** *The number of edges of the graph of diameters induced by a set of n points in the plane is at most n. This bound can be attained for every $n > 2$.*

In the same paper, Erdős quoted Andrew Vázsonyi's conjecture from the mid-1930s (see also [31]), according to which the number of times the diameter (the maximum distance) can occur among n points in 3-space is at most $2n - 2$. This statement was proved independently by Grünbaum [52], Heppes [54], and Straszevicz [97]. All of these proofs used the notion of *ball polytopes*, that is, convex bodies obtained by taking the intersection of balls of equal radii. However, as was pointed out by Kupitz, Martini, and Perles [66], ball polytopes have some unpleasant features different from the properties of convex polytopes. In particular, their edge-skeletons need not be 3-connected. Therefore, making the above proofs precise requires a lengthy analysis. Half a century later, simpler proofs were found by Perlstein and Pinchasi [92] and by Swanepoel [99].

Theorem 3 ([52], [54], [97])**.** *The number of edges of the graph of diameters induced by a set of n points in 3-dimensional space is at most $2n - 2$. This bound can be attained for every $n > 3$.*

Erdős [29] also remarked that this statement has an interesting geometric corollary.

Corollary 4. *Every (finite) set of points in 3-dimensional space can be decomposed into 4 sets of smaller diameter.*

Indeed, it follows from Theorem 3 that the diameter graph associated with any finite set of points has a vertex of degree at most 3. Removing such a vertex, one can show by induction that the chromatic number of the diameter graph is at most 4. This is equivalent to Corollary 4. See also [26] and [55].

Corollary 4 is the $d = 3$ special case of Borsuk's conjecture [12] which states that any d-dimensional set of points can be decomposed into $d+1$ sets of smaller diameter. In 1993, Kahn and Kalai [59] (see also [76]) disproved Borsuk's conjecture for large values of d. Today the conjecture is known to fail in all dimensions $d \geq 65$. See [56] and [93] for a survey and [11] for a recent improvement.

As was reported by Erdős [32], a simple construction due to Lenz (1955) shows that, for a fixed $d \geq 4$, the number of times the diameter can occur among n points in d-dimensional space can grow quadratically in n. Indeed, let $k = \lfloor d/2 \rfloor$, and take k concentric unit circles in $\mathbb{R}^d$, in pairwise orthogonal planes. On each of these circles, pick $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ points very close to each other, so that their total number is n. The diameter of the resulting point set is $\sqrt{2}$, and the distance $\sqrt{2}$ occurs $\frac{1}{2}(1 - \frac{1}{k} + o(1))n^2$ times. Using the Erdős-Stone theorem [41], a cornerstone of extremal graph theory, Erdős proved that this construction is asymptotically best possible.

Theorem 5 [32]**.** *For a fixed $d \geq 4$, the maximum number of edges of the diameter graph of a set of n points in d-dimensional space is*

$$\frac{1}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor} + o(1)\right)n^2.$$

Erdős suggested that instead of estimating the number of occurrences of the largest distance, one can also investigate the frequency of the 2nd largest, 3rd largest, etc. distances determined by a set of n points. In particular, it was shown by Vesztergombi [111] (see also [38]) that the i-th largest distance among n points in the plane cannot occur more than $2in$ times. Morić and Pach [73] showed that for a fixed i, the number of times the i-th largest distance can occur among n points in 3-dimensional space is $O(n)$. The constant provided by the proof, hidden in the big-O notation, grows exponentially in i, which can probably be much improved. The nature of the problem again changes in dimension d larger than 3: the i-th largest distance can occur $\Omega(n^2)$ times.

Perhaps the most important contribution of Erdős's paper [29] in the *Monthly* was that he modified the Hopf-Pannwitz problem, as follows. Let $f_d(n)$ denote the the maximum number of times that *any* distance can occur among n points in d-dimensional space. Erdős [32] proved that for any $d \geq 4$, $f_d(n)$ is asymptotically equal to the maximum number of occurrences of the *diameter,* given in Theorem 5. The exact value of $f_4(n)$ for every n was determined by Brass [14]. Swanepoel [100] extended this result to every even $d \geq 4$, provided that n is sufficiently large depending on d. He also found the maximum number of times the diameter can occur among n points in d-dimensional space, for every $d \geq 4$ and for all sufficiently large n. For some other extensions of these results, see [39] and [7].

The asymptotic behavior of the functions $f_2(n)$ and $f_3(n)$ is still a mystery. Erdős [29] proved that $f_2(n) > n^{1+c/\log\log n}$ for a suitable constant $c > 0$, and conjectured that this bound is not far from being tight. However, the best known upper bound is still $f_2(n) = O(n^{4/3})$, which was established by Spencer, Szemerédi, and Trotter [96] thirty years ago. For alternative proofs, see [22], [101], and [88]. In 3-dimensional space, we have

$$cn^{4/3}\log\log n < f_3(n) < n^{3/2},$$

where $c > 0$ is a constant and $\alpha(n)$ is an extremely slowly growing function, closely related to the inverse of Ackermann's function. The lower and upper bounds were proved in [32] and [22], respectively. (With no danger of confusion, in different formulas we use the same letter c to denote different unrelated constants.)

Obviously, the number of distinct distances determined by n points in the plane is at least $\binom{n}{2}/f_2(n) > cn^{2/3}$. "Though I have thought to improve this result for many years – wrote Erdős in [29] – I have not been able to do so." After many small improvements ([75], [20], [21], [95], [102], [60], [61]), 65 years later Guth and Katz [53] got very close to verifying Erdős's conjecture:

Conjecture 6 (Erdős [29])**.** *The number of distinct distances determined by n points in the plane is at least $cn/\sqrt{\log n}$, for a suitable constant $c > 0$.*

If true, the order of magnitude of this bound cannot be improved, as shown by a $\sqrt{n} \times \sqrt{n}$ piece of the integer grid. In their breakthrough paper, using a framework set up by Elekes [28], Guth and Katz have established a $cn/\log n$ lower bound. In fact, Erdős [31], [33], [34], [35] also made a stronger conjecture, stating that any set of n points in the plane has an element from which there are at least $cn/\sqrt{\log n}$ distinct distances to the other points. It does not seem to be an easy task to adapt the Guth-Katz proof to estimate this quantity. So far the best lower bound is $cn^{0.864..}$, due to Katz and Tardos [61].

We close this section by another possible generalization of Theorem 2 to higher dimensions, different from Theorems 3 and 5.

Conjecture 7 (Z. Schur [94])**.** *For any positive integers d and n ($n > d$), the graph of diameters induced by a set of n points in d-dimensional space contains at most n complete subgraphs with d vertices.*

For $d = 3$, Schur's conjecture has been proved by Schur, Perles, Martini, and Kupitz [94]. In [74], it was shown Conjecture 7 would follow from the following statement.

Conjecture 8 [74]**.** *For any positive integers d and n $(n > d > 2)$, any two complete subgraphs of size d of the graph of diameters induced by a set of n points in d-dimensional space share at least $d - 2$ vertices.*

For $d = 3$, Conjecture 8 is true. In fact, Dolnikov [25] proved the stronger statement that the graph of diameters of a 3-dimensional point set contains no two disjoint odd cycles. For larger values of d, we have been unable to verify even the weaker conjecture that the graph of diameters contains no two vertex-disjoint cliques of size d.

For more results and open problems related to the subject of this section, see [15] and [40].

4. Dropping the Metric Restrictions – Geometric Graphs

Fenchel's solution [43] for the Hopf-Pannwitz problem (Problem 1) can be easily modified to establish a statement, a bit stronger than Theorem 2. Recall that a *geometric graph G* is a graph drawn in the plane by possibly crossing straight line edges. For simplicity, we assume throughout that no 3 vertices (points) of G are collinear. An *edge* of G is a *closed* segment connecting a pair of vertices. Therefore, the condition that no 2 edges are disjoint is equivalent to saying that any pair of edges share either an endpoint or an interior point. Of course, they cannot share more than one point, because of the assumption that no 3 vertices are collinear.

Theorem 9 (Erdős, Avital-Hanani [8], Kupitz [65], Perles)**.** *Every geometric graph of n vertices that does not contain 2 disjoint edges has at most n edges. This bound can be attained for every $n > 2$.*

This statement first appeared in print as Problem 3 at the end of a paper written by Shmuel Avital and Haim Hanani [8], which was published in *Gilyonot Le'matematika*, an Israeli journal for high school students and amateurs, edited by Joseph Gillis at Weizmann Institute, Rehovot. It is very likely that the authors heard the question from Paul Erdős. After being banned from entering the United States for 9 years, as an "undesirable alien," in 1955 Erdős was appointed a "Permanent Visiting Professor" at Technion, Haifa. Every year he spent at least one month in Israel, and Hanani was one one of his close friends and collaborators.

When Micha Perles (Hebrew University) was told about Theorem 9 roughly ten years after the publication of the Avital-Hanani paper, he found the following "proof from the Book:" Suppose that there is a spider sitting at each vertex v of the graph (web). It looks around and if it finds an edge e incident to v with the property that within the next 180-degree range in the clockwise direction there is no other edge, it walks to the middle of e and lays an egg. Otherwise, the spider stays at v and does not lay an egg. Notice that if G has no 2 disjoint edges, there will be no edge left without an egg. Therefore, the number of edges cannot exceed the number of spiders. Inspired by Perles, Yaakov Kupitz fully characterized all geometric graphs and point configurations for which equality holds in Theorems 9 and 2. (See also [67].) He has also found some interesting generalizations of Theorem 9, and these results constituted his master thesis [65].

It is a natural question to ask whether Theorem 9 can be generalized to topological graphs, that is, to graphs G drawn in the plane by possibly crossing curvilinear edges. It is clear that we need some additional assumptions on G, because it is easy to draw a complete topological graph in which every pair of edges intersect. We call a topological graph *simple* if every pair of edges have at most one point in common, which is either a common endpoint or a proper crossing. Two edges are not allowed to touch each other.

In the late 1960s, independently of the above developments, John Conway defined a *thrackle* as a simple topological graph, in which every pair of edges share precisely one point: an endpoint or a proper crossing. This term may have been first used in a commercial: fishermen referred to their entangled nets as being thrackled.

Conjecture 10 (Conway's thrackle conjecture [114])**.** *Every thrackle of n vertices has at most n edges. This bound can be attained for every $n > 2$.*

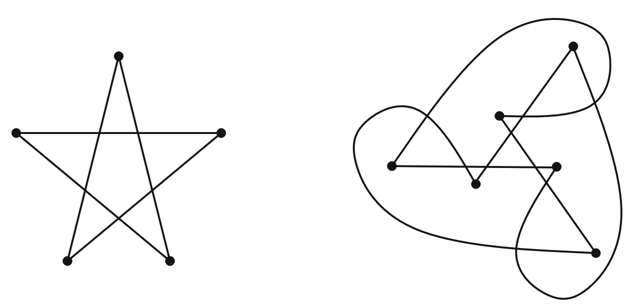

Fig. 1. C_5 and C_6 drawn as thrackles

The first linear upper bound on the number of edges of a thrackle of n vertices was established in [69]. It was improved by Cairns an Nikolayevsky [16]. The best known upper bound, $1.428n$, was proved in [48]. Apart from the case of straight-line thrackles (Theorem 9), Conway's conjecture is known to be true for *x-monotone* thrackles (for which any vertical line intersects every edge in at most one point) [87] and for *outerplanar* thrackles (whose vertices lie on a circle and all edges in its interior) [17]. Perhaps the next step would be to verify the conjecture for thrackles in which every edge is the union of at most 2 (or at most a bounded number of) x-monotone pieces.

Avital and Hanani [8] asked the question that at most how many edges can a geometric graph of n vertices have if it contains no k pairwise disjoint edges. For *convex geometric graphs*, that is, for geometric graphs whose vertices lie on a closed convex curve, Kupitz [65] proved that this maximum is equal to $(k-1)n$, for all $n > 2(k-1)$. For arbitrary geometric graphs, in the special case $k = 3$, the first linear upper bound (of roughly $6n$) was established by Alon and Erdős [6]. It was subsequently improved by O'Donnell and Perles (unpublished) and by Goddard, Katchalski, and Kleitman [50]. The following asymptotically tight bound was found by Černý [19].

Theorem 11 (Černý [19])**.** *Every geometric graph of n vertices which does not contain 3 disjoint edges has at most $2.5n$ edges. This bound is tight up to an additive constant.*

For larger values of k, the first linear upper bound, $O(k^4 n)$, for the number of edges of a geometric graph G with no k disjoint edges was given by Pach and Törőcsik [90]. After an initial improvement by G. Tóth and Valtr [107], Tóth [106] established the upper bound $|E(G)| \leq O(k^2 n)$; see also [112]. The following conjecture is perhaps too optimistic.

Conjecture 12. *The maximum number of edges of a geometric graph of n vertices that contains no k disjoint edges is $O(kn)$.*

It is perfectly possible that this conjecture remains true for simple topological graphs. However, in this case, even for $k = 3$, we do not have a linear upper bound in n on the number of edges. All we know is that, according to [89], the maximum number of edges of a simple topological graph with n vertices that contains no k disjoint edges is $n(\log n)^{O(k)}$. In particular, it follows that a *complete* simple topological graph with n vertices has $\Omega(\frac{\log n}{\log\log n})$ pairwise disjoint edges. Fox and Sudakov [47] improved this bound to $\Omega(\log^{1+\varepsilon} n)$, for a suitable $\varepsilon > 0$. Presently, the best known result in this direction is due to Suk [98].

Theorem 13 (Suk [98]). *Every complete simple topological graph of n vertices has $\Omega(n^{1/3})$ disjoint edges.*

An alternative proof of this bound was found by Fulek and Ruiz-Vargas [49]. If the strengthening of Conjecture 12 to all simple topological graphs is true, it immediately implies

Conjecture 14. *Every complete simple topological graph of n vertices has $\Omega(n)$ disjoint edges.*

For geometric graphs G (in fact, for topological graphs drawn with x-monotone edges), Conjecture 14 is obviously true. Ordering the vertices with respect to their x-coordinates and taking all edges between consecutive vertices, we obtain a non-selfintersecting Hamilton path in G. Taking every other edge of this path, we get a set of $\lfloor n/2 \rfloor$ pairwise disjoint edges. As far as I know, for complete simple topological graphs we do not have any lower bound for the size of the longest non-selfintersecting path, comparable to the one given by Suk's theorem (Theorem 13). The best bound I am aware of is $\Omega(\log^{1/6} n)$; see [86].

Conjecture 15. *There exists $\varepsilon > 0$ such that every complete simple topological graph on n vertices has a non-selfintersecting path of length at least n^{ε}.*

No example is known in which the size of the longest non-selfintersecting path is $o(n)$.

5. Relaxations of Planarity

For more than two decades starting from the 1940s, one of Erdős' contemporaries, György Hajós, made persistent efforts to settle the 4-color conjecture for planar graphs. He conjectured that every graph of chromatic number k contains a subdivision ("topological subgraph") of a complete graph with k vertices. For $k = 5$, this would of course imply the 4-color theorem. Unfortunately, we still do not know if Hajós' conjecture is true in this case. However, for $k \geq 7$, the conjecture was disproved by Catlin [18], and shortly after Erdős and Fajtlowicz [36] discovered that the conjecture combined with Turán's theorem [108] would imply that every graph G with at least constant times k^3 vertices has k vertices that induce either a complete subgraph or an empty subgraph in G. (See also [105].) However, in his classic note [30] written 30 years earlier, Erdős used the "probabilistic method" to prove the existence of graphs with $2^{k/2}$ vertices that do not have this property.

Nevertheless, a result much weaker than Hajós' conjecture, first proposed in the doctoral dissertation of Rudolf Halin, turned out to be true. Dirac [24] and Jung [58] observed that an idea of Wagner [113] can be used to establish the existence of a function $f(k)$ with the property that every graph with chromatic number at least $f(k)$ contains a subdivision of a complete graph K_k with k vertices. Surprisingly, Mader [70] found a much stronger result with a much simpler proof: There also exists a function $g(k)$ such that every graph of n vertices and more than $g(k)n$ edges contains a subdivision of K_k. (Every graph of chromatic number $f(k)$ contains a subgraph in which every vertex has degree at least $f(k) - 1$.) The correct order of magnitude of the function $g(k)$ was determined 30 years later by Komlós and Szemerédi [63] and by Bollobás and Thomason [10]: $g(k) = \Theta(k^2)$. This settled a conjecture of Erdős and Hajnal [37] and Mader [70]. Another famous result of this kind was conjectured by Dirac [23].

Theorem 16 (Mader [71])**.** *For every $n \geq 3$, the maximum number of edges that a graph with n vertices can have without containing a subdivision of K_5 is $3n - 6$.*

The above statements are usually discussed in the framework of "topological graph theory" (see [72]). They do not depend on the particular drawing of G. They describe "global" properties of graphs G with more edges than how many planar graphs can have, and one does not have much control of the size of the forced subdivisions. In what follows, we would like to discuss some problems related to "local" properties of geometric or topological graphs.

By Euler's theorem, if a geometric or topological graph G has more than $3n - 6$ edges, two of its edges must cross each other. (A *crossing* occurs when two edges share a common interior point.) In fact, if G has much more than $3n - 6$ edges, the number of crossings increases dramatically. Erdős and Guy conjectured, and Ajtai, Chvátal, Newborn, Szemerédi [5] and, independently, Leighton [68] proved that, if the number of edges, e, satisfies $e > 3n - 6$, there are at ce^3/n^2 crossings, where c is a suitable positive constant. The best known value of the constant $c > \frac{1024}{31827} > 0.032$ was found in [84].

What happens if, instead of a crossing pair of edges, we want to guarantee the existence of some larger configurations involving several crossings? What kind of *unavoidable* substructures must occur in every geometric or topological graph G having n vertices and more than Cn edges, for an appropriately large constant $C > 0$?

A geometric or topological graph is called *k-quasiplanar* if it contains no k pairwise crossing edges.

Conjecture 17. *For any positive integer k, there is a constant C_k such that the number of edges of any k-quasiplanar topological graph with n vertices is at most $C_k n$.*

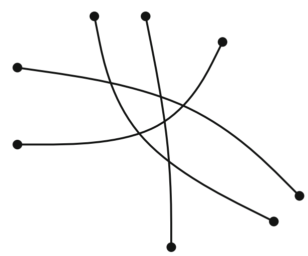

Fig. 2. Four pairwise crossing edges in a topological graph

For $k = 3$, for simple topological graphs (i.e., where every pair of edges cross at most once), Conjecture 17 was proved in [4]. Without the simplicity condition, the statement was first proved in [83]. The best known upper bound of roughly $8n$ was established by Ackerman and Tardos [3], who also proved that the maximum number of edges that a simple 3-quasiplanar topological graph can have is is $6.5n - O(1)$. For $k = 4$, the conjecture has been verified by Ackerman [1].

For larger values of k, Conjecture 17 is still open. The upper bound $n(\log n)^{O(k)}$ for the number of edges of a simple k-quasiplanar topological graph was first proved in [85], and then for all k-quasiplanar topological graphs in [83]. This was further improved to $n(\log n)^{O(\log k)}$ by Fox and Pach [44]. For simple topological graphs, presently the best known upper bound is $(n \log n)\alpha_k(n)$, where $\alpha_k(n)$ denotes an extremely slowly growing function related to the inverse of the Ackermann function. It was established in [45]. For k-quasiplanar geometric graphs and, more generally, for simple topological graphs whose edges are represented by x-monotone arcs, Valtr [109], [110] showed that the number of edges cannot exceed $c_k n \log n$. Extending Valtr's ideas, Fox, Pach, and Suk proved the following.

Theorem 18 [45]**.** *The number of edges of a k-quasiplanar topological graph with n vertices, the edges of which are represented by x-monotone arcs, is at most $2^{ck^6} n \log n$, for a suitable absolute constant c.*

Erdős raised the question whether every system of continuous arcs in the plane with no k pairwise intersecting members can be split into a constant number, c_k, of subsystems such that no two arcs belonging to the same subsystem intersect. He emphasized the first interesting special case, where $k = 3$ and the arcs are straight-line segments. A positive answer to Erdős' question would imply that Conjecture 17 is true. To see this, observe that no k members of the system of edges (open arcs) of a k-quasiplanar topological graph G intersect. If this system can be decomposed into c_k subsystems consisting of disjoint arcs, then one of these subsystems has at least $|E(G)|/c_k$ members. The corresponding edges form a planar subgraph of G, therefore we would obtain $|E(G)|/c_k \leq 3n - 6$, where $n \geq 3$ denotes the number of vertices of G. This would imply $|E(G)| = O_k(n)$, as required. However, Pawlik, Kozik, Krawczyk, Lasoń, Miczek, Trotter, and Walczak [91] constructed systems of n segments, no 3 of which are pairwise intersecting, such that they cannot be decomposed into fewer than $\log \log n$ subsystems of disjoint segments. Therefore, the answer to Erdős' question is no. It is interesting to observe that Conjecture 17 would also follow from the following weaker statement, which was not refuted by the construction of Pawlik *et al.*

Conjecture 19. *For any positive integer k, there is a constant $\varepsilon_k > 0$ with the property that every system on n continuous arcs (or segments) in the plane, no k of which are pairwise intersecting, has at least $\varepsilon_k n$ disjoint members.*

As the number of edges of a topological graph G with n vertices substantially exceeds the critical threshold $3n - 6$, more complicated crossing configurations appear. A $k \times l$ *grid* in G is a pair of disjoint subsets $E_1, E_2 \subset E(G)$ with $|E_1| = k$ and $|E_2| = l$ such that every edge in E_1 crosses all edges in E_2. It was proved in [81] that for any integer $k > 0$, there is a constant C_k such that every topological graph with n vertices and more than $C_k n$ edges has a $k \times k$ grid. See [46], for a different proof. The strongest result in this direction was proved by Tardos and Tóth [104]: There is a constant C_k such that in every topological graph with n vertices and more than $C_k n$ edges one can find 3 disjoint k-element sets of edges such that two of the subsets consist of edges incident to a vertex and every pair of edges from different subsets cross.

At first glance, one might believe that it is much easier to guarantee the existence of a $k \times k$ grid in "general position" in the sense that no pair of its edges share an endpoint. However, in this case the proof breaks down and we can only prove that every topological graph with n vertices and at least $C_k n \log^* n$ edges contains such a grid, where $\log^*$ denotes the iterated logarithm function [2].

Conjecture 20 (Ackerman, Fox, Pach, Suk [2])**.** *For any integers $k, l \geq 1$, there is a constant $C_{k,l}$ such that every topological graph with n vertices which contains no $k \times l$ grid with distinct vertices has at most $C_{k,l} n$ edges.*

This conjecture is known to be true for $l = 1$.

In lack of nontrivial examples (or counterexamples), one can formulate an even bolder conjecture. We call a $k \times l$ grid *natural* if it consists of a set of k disjoint (noncrossing) edges and a set of l disjoint edges with all $2(k + l)$ endpoints distinct, such that every edge in the first subset crosses every edge in the second. There are complete topological graphs in which every pair of edges cross, so they contain no natural 2×1 grid. Hence, to strengthen Conjecture 20, we have to make an additional distinction. For instance, we may restrict our attention to simple topological graphs or to geometric graphs.

Conjecture 21 [2]**.** *For any integers $k, l \geq 1$, there is a constant $C_{k,l}$ such that the number of edges of any simple topological graph with n vertices which contains no $k \times l$ natural grid is at most $C_{k,l}n$.*

Even for geometric graphs with no natural $k \times k$ grid, the best known upper bound for the number of edges is $O(k^2 n \log^2 n)$. For convex geometric graphs, the validity of the conjecture follows from [62]. In general, the only case in which Conjecture 21 has been verified is $k = 2$, $l = 1$ (see [2]).

We close this section with another relaxation of planarity, where we do have nontrivial constructions and we know that the number of edges forcing some crossing subconfigurations is superlinear. For any $k \geq 3$, a topological graph G is called *k-locally planar* if G has no selfintersecting path of length at most k. Roughly speaking, this means that the embedding of the graph is planar in a neighborhood of radius $k/2$ around any vertex. It was shown by Pach, Pinchasi, Tardos, and Tóth [82] that there exist 3-locally planar geometric graphs with n vertices and with at least constant times $n \log n$ edges. For larger values of k, Tardos [103] constructed a sequence of k-locally planar geometric graphs with n vertices and a superlinear number of edges (approximately n times the $\lfloor k/2 \rfloor$ times iterated logarithm of n). From the other direction, we only have a much weaker bound.

Theorem 22 [82]**.** *The number of edges of a 3-locally planar topological graph with n vertices is $O(n^{3/2})$.*

This result is probably far from being optimal. For 3-locally planar geometric graphs (and, more generally, for topological graphs with x-monotone edges) the $\Omega(n \log n)$ bound is known to be tight [82]. Boutin [13] showed that the number of edges of 3-locally planar convex geometric graph with n vertices is $O(n)$.

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