15.21 An experiment is to be conducted to compare the effectiveness of four household detergents. The response is to be the degree of stain removal from a section of clothing on a 10-point scale (1=no stain removed, 10=stain completely removed).

(a) Identify the experimental unit.

Answer: An experimental unit is the smallest unit of experimental material to which a treatment can be assigned; the experimental unit is thus determined by (15.3) the method of randomization.

Therefore, the unit is probably a single (whole) piece of clothing, ie a shirt or pants, since we can conduct this experiment by using detergent on one piece of clothing at a time.

(b) Identify the experimental factor(s), levels, and any factor-level combinations if present.

Answer: A factor is an explanatory variable to be studied in an investigation. The factor is the detergent being used. A factor level is a particular form of the factor. The level are the four different detergents used.

(c) Name two potential blocking factors.

Answer: The color of the clothing and the material of the clothing.

(d) Propose an experiment to accomplish the objectives of the study. How would you carry out the randomization?

Answer: We would have two observations (at least) for each treatment. To achieve this, assign each experimental unit a random number. We generate the random numbers from any continuous probability distribution. We sort these numbers in ascending order. Then assign them to the different detergents.

15.22 An experiment is to be carried out to determine the optimal combination of microwave oven setting for microwave popcorn. Cooking time has three possible settings (3,4, and 5 minutes) and cooking power has two settings (low power, high power). The response (to be minimized) is the number of burned plus the number of unpopped kernels.

(a) Identify the experimental unit.

Answer: We are measuring the popcorn in bag form, since we are probably cooking the popcorn a bag at a time.

(b) Identify the experimental factor(s), levels, and any factor-level combinations if present.

Answer: The factors are the microwave settings with time and power. They each have levels cooking time and cooking power. The 6 factor-level combos in table 1 below:

Table 1: Factor-level combinations

Factor	Time	Power
1	3 min	Low
2	4 min	Low
3	5 min	Low
4	3 min	High
5	4 min	High
6	5 min	High

(c) Name two potential blocking factors.

Answer: Two potential blocking factors include splitting by brand of popcorn and brand of microwave.

(d) Propose an experiment to accomplish the objectives of the study. How would you carry out the randomization?

Answer: Basically the same procedure as earlier in 15.21. Again, we have 2 observations of each treatment, more specifically factor-level combination. To randomly assign the treatments to the experimental units, we randomly order the treatments in accordance with a random number generator.

16.7 Productivity improvement. An economist compiled data on productivity improvements last year for a sample of firms producing electronic computing equipment. The firms were classified according to the level of their average expenditures for research and development in the past three years (low, moderate, high). The results of the study follow (productivity is measured on a scale from 0 to 100). Assume that the ANOVA model (16.2) is appropriate.

(a) Prepare aligned dot plots of the data. Do the factor level means appear to differ? Does the variability of the observations within each factor level appear to be approximately the same for all factor levels?

Answer: The means appear to differ.

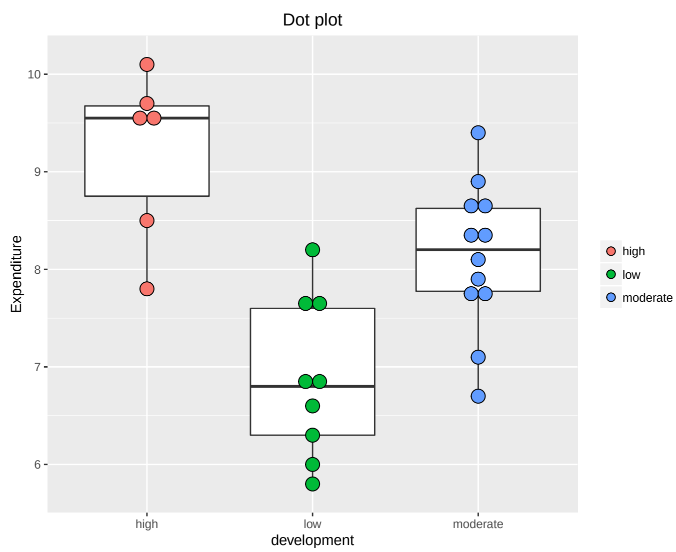

Figure 1: Dot plot for data

(b) Obtain the fitted values.

Answer: We use the ANOVA model

$$Y_{ij} = \mu_i + \varepsilon_{ij} \tag{1}$$

Where Y_{ij} is the value of the response variable in the jth trial for the ith factor level or treatment, μ_i are the parameters, and ε_{ij} are independent $N(0, \sigma^2)$.

The fitted value for observation Y_{ij}, denoted by $\hat{Y}_{ij}$ for regression models, is simply the corresponding factor level sample mean here:

$$\hat{Y}_{ij} = \overline{Y}_{i}.$$

In our case,

$$\text{low} = \hat{Y}_{1j} = \overline{Y}_{1.} = 6.878$$
$$\text{moderate} = \hat{Y}_{2j} = \overline{Y}_{2.} = 8.133$$
$$\text{high} = \hat{Y}_{3j} = \overline{Y}_{3.} = 9.2$$

(c) Obtain the residuals. Do they sum to zero in accord with (16.21).

Answer: The residuals are given by

$$e_{ij} = Y_{ij} - \hat{Y}_{ij} = Y_{ij} - \overline{Y}_{i} \tag{2}$$

Table 2: Residuals

Index $i \downarrow j \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
Low	0.722	1.32	-0.078	-1.078	0.022	-0.278	-0.578	0.822	-0.878			
Moderate	-1.433	-0.033	1.267	0.467	-0.333	-0.433	0.767	-0.233	0.167	0.567	-1.033	0.267
high	-0.7	0.5	0.9	-1.48	0.4	0.3						

In our case, The residuals sum to 1.15e-14, which is likely due to rounding error.

(d) Obtain the analysis of variance table.

Answer:

Table 3: Made with stargazer. ANOVA table

Response	DF	Sum Squares	Mean squares	F Value	Pr(>F)
Development	2	20.125	10.06	15.72	4.331e-05
Residuals	24	15.362	0.6401		
Total	26	35.487			

(e) Test whether or not the mean productivity improvement differs according to the level of research and development expenditures. Control the α risk as 0.05. State the alternatives, decision rule, and conclusion.

Answer: Our null and alternative hypothesis are as follows

$$H_0 = \mu_1 = \mu_2 = \mu_3$$
$$H_a = \text{not all } \mu_i \text{ are equal}$$

ie we are testing whether all the means should be the same. We control at $\alpha = 0.05$ risk. Since we know that F^* is distributed as $F(r-1, n_T - r)$ when H_0 holds and that large values of F^* lead to conclusion H_a, the appropriate decision rule to control the level of significance at α is

$$\begin{aligned} &\text{If} \quad F^* \leq F(1-\alpha; r-1, n_T - r), \quad \text{conclude } H_0 \\ &\text{If} \quad F^* \geq F(1-\alpha; r-1, n_T - r), \quad \text{conclude } H_a \end{aligned} \tag{3}$$

In our case, because $r = 3$ and $n_{\text{total}} = 27$, we find $F(0.96, 2, 24) = 3.403$. Our F^* statistic is given by

$$F^* = \frac{\text{MSTR}}{\text{MSE}} \tag{4}$$

From our ANOVA table

$$F^* = \frac{10.06}{0.6401} = 15.72 > 3.40$$

Therefore, we conclude the alternative hypothesis.

(f) What is the P-value of the test in part (e)? How does it support the conclusion reached in part (e)?

Answer:

(g) What appears to be the nature of the relationship between research and the development and productivity improvement?

<u>Answer:</u> It appears that higher level of research increases productivity improvement.

16.9 Rehabilitation therapy. A rehabilitation center researcher was interested in examining the relationship between physical fitness prior to surgery of persons undergoing corrective knee surgery and time required in physical therapy until successful rehabilitation. Patient records in the rehabilitation center were examined, and 24 male subjects ranging in age from 18 to 30 years who had undergone similar corrective knee surgery during the past year were selected from the study. The number of days required for successful completion pf physical therapy and the prior physical fitness status (below average, average, above average) for each patient follow. Assume that ANOVA model (16.2) is appropriate.

(a) Prepare aligned dot plots of the data. Do the factor level means appear to differ? Does the variability of the observations within each factor level appear to be approximately the same for all factor levels?

<u>Answer:</u> The means certainly appear to be different, whereas the variance appears pretty steady.

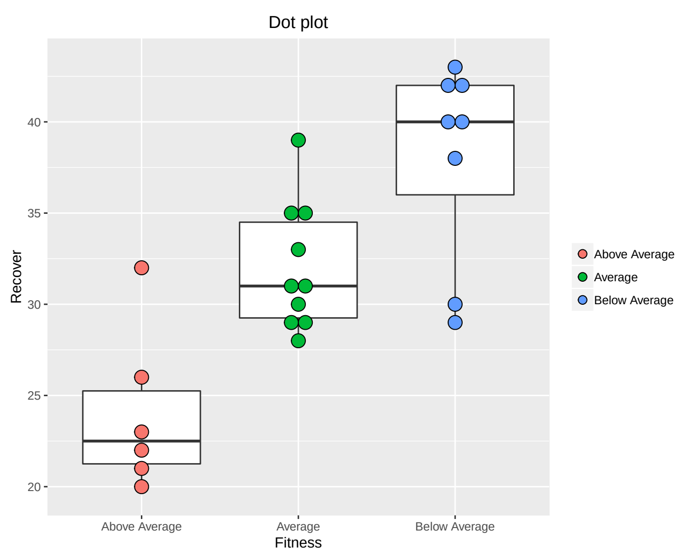

Figure 2: Dot plot for 169, with superimposed boxplots

(b) Obtain the fitted values

<u>Answer:</u> Same as before

$$\text{below average} = \hat{Y}_{1j} = \overline{Y}_{1.} = 38$$
$$\text{average} = \hat{Y}_{2j} = \overline{Y}_{2.} = 32$$
$$\text{above average} = \hat{Y}_{3j} = \overline{Y}_{3.} = 24$$

(c) Obtain the residuals. Do they sum to zero in accord with (16.21)?

<u>Answer:</u>

They do sum to zero as anticipated.

(d) Obtain the analysis of variance table.

<u>Answer:</u>

(e) Test whether or not the mean number of days required for successful rehabilitation is the same for the three fitness groups. Control the α risk at 0.01. State the alternatives, decision rule, and conclusion.

Table 4: Residuals

Index $i \downarrow j \rightarrow$	1	2	3	4	5	6	7	8	9	10
Below Average	-9	4	0	2	5	2	-8	4		
Average	-2	3	7	-4	-1	-1	-3	3	-3	1
Above Average	2	8	-3	-4	-1	-2				

Table 5: ANOVA table

Response	DF	Sum Squares	Mean squares	F Value	Pr(>F)
Fitness	2	672	336.00	16.962	4.129e-05
Residuals	21	416	19.81		
Total	23	1088			

Answer: Our null and alternative hypothesis are as follows

$$H_0 = \mu_1 = \mu_2 = \mu_3$$
$$H_a = \text{not all } \mu_i \text{ are equal}$$

ie we are testing whether all the means should be the same. We control at $\alpha = 0.05$ risk. Since we know that F^* is distributed as $F(r-1, n_T - r)$ when H_0 holds and that large values of F^* lead to conclusion H_a, the appropriate decision rule to control the level of significance at α is

$$\text{If} \quad F^* \le F(1-\alpha; r-1, n_T - r), \quad \text{conclude } H_0$$
$$\text{If} \quad F^* \ge F(1-\alpha; r-1, n_T - r), \quad \text{conclude } H_a$$

In our case, because $r = 3$ and $n_{\text{total}} = 24$, we find $F(0.99, 2, 21) = 5.78$. Our F^* statistic is given by

$$F^* = \frac{\text{MSTR}}{\text{MSE}}$$

From our ANOVA table

$$F^* = \frac{336}{19.71} = 16.91 > 5.78$$

Therefore, we conclude the alternative hypothesis.

(f) Obtain the P-value for the test in part (e). Explain how the same conclusion reached in part (e) can be obtained by knowing the P-value.

Answer: From the Anova table, the p-value is 4.129e-05, which yields the same conclusion as part (d). The p-value tells us the probability we observed what we observed given the null, which we thus reject.

(g) What appears to be the nature of the relationship between physical fitness status and duration of required physical therapy?

Answer: Better fitness levels pre surgery tends to mean less therapy is required after the surgery.

Relevant R-code for this question (also applies to 16.7 and 16.12)

```
data169 = read.table("http://www.stat.ufl.edu/~rrandles/sta4210/Rclassnotes/
data/textdatasets/KutnerData/Chapter%2016%20Data%20Sets/CH16PR09.txt")

data169$Recover = data169$V1
data169$Fitness = data169$V2
data169<-data169%>%
select(Recover, Fitness)
```

```
data169$Fitness[data169$Fitness ==1] = "Below␣Average"
data169$Fitness[data169$Fitness ==2] = "Average"
data169$Fitness[data169$Fitness ==3] = "Above␣Average"


data169%>%
ggplot(aes(x=Fitness, y=Recover, fill=factor(Fitness)))+geom_boxplot(fill='white')
+geom_dotplot(binaxis = "y", stackdir = "center")+
ggtitle('Dot␣plot')+xlab('Fitness')+ylab('Recover')+
theme(plot.title = element_text(hjust = 0.5))+theme(legend.title=element_blank())

#Fitted values##
data167

meanbelowavg=mean(data169$Recover[data169$Fitness=="Below␣Average"])
meanavg=mean(data169$Recover[data169$Fitness=="Average"])
meanaboveavg=mean(data169$Recover[data169$Fitness=="Above␣Average"])


resid169below=data169$Recover[data169$Fitness=="Below␣Average"]-meanbelowavg

resid169avg=data169$Recover[data169$Fitness=="Average"]-meanavg

resid169above=data169$Recover[data169$Fitness=="Above␣Average"]-meanaboveavg
sum(resid169below)+sum(resid169avg)+sum(resid169above)

anovastuff169=aov(Recover~Fitness,data=data169)

anovatable169=anova(anovastuff167)

alpha169=0.01
#number of levels
r169=length(unique(data169$Fitness))
#n total, total obervation
nt169=length(data169$Recover)

favl169=qf(1-alpha169, r169-1, nt169-r169)
fstat169=anovatable167[1,3]/anovatable167[2,3]
```

16.12 Premium distribution. A soft-drink manufacturer uses five agents (1,2,3,4,5) to handle premium distributions for its various products. The marketing director desired to study the timeliness with which the premiums are distributed. Twenty transactions for each agent were selected at random, and the time lapse (in days) for handling each transaction was determined. The results follow. Assume that ANOVA model (16.2) is appropriate.

(a) Prepare aligned box plots of the data. Do the factor level means appear to differ? Does the variability of the observations within each factor level appear to approximately the same for all factor levels?

<u>Answer:</u> Figure 3 answers this question. We see some variation based on agent.

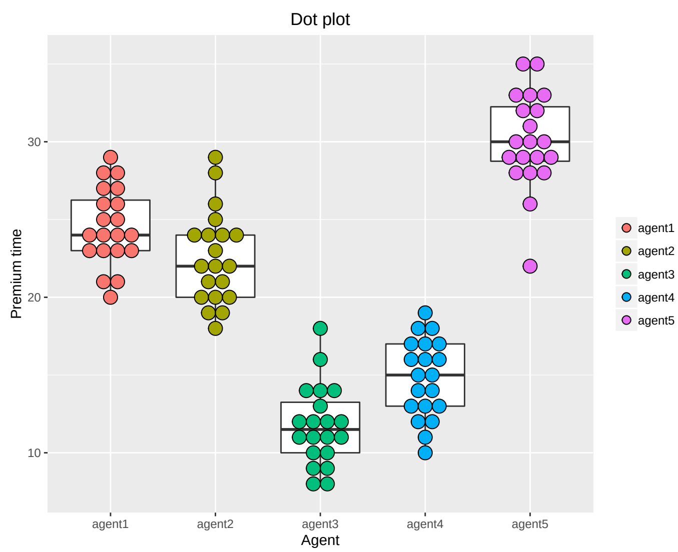

Figure 3: Dot plot for different agents.

(b) Obtain the fitted values.

Answer: Same as before

$$\text{agent } 1 = \hat{Y}_{1j} = \overline{Y}_1 = 24.55$$
$$\text{agent } 2 = \hat{Y}_{2j} = \overline{Y}_2 = 22.55$$
$$\text{agent } 3 = \hat{Y}_{3j} = \overline{Y}_3 = 11.75$$
$$\text{agent } 4 = \hat{Y}_{4j} = \overline{Y}_3 = 14.8$$
$$\text{agent } 5 = \hat{Y}_{5j} = \overline{Y}_3 = 30.1$$

(c) Obtain the residuals. Do they sum to zero in accord with (16.21)?

Answer: They sum more or less to zero, probably off due to rounding error (precisely they sum to -7.105e-14). The residuals are shown in table 6.

Table 6: Residuals

AGENT $i \downarrow j \rightarrow$	1	2	3	4	...	16	17	18	19	20
Agent 1	-0.55	-0.55	4.45	-4.55	...	-1.55	-1.55	2.45	1.45	0.45
Agent 2	-4.55	-2.55	-2.55	1.45	...	-0.55	-3.55	3.45	-0.55	-1.55
Agent 3	-1.75	-0.75	-3.75	0.25	...	-0.75	2.25	-2.75	-0.75	0.25
Agent 4	0.2	-1.8	3.2	1.2	...	2.2	1.2	2.2	-0.8	1.2
Agent 5	2.9	-8.1	-2.1	4.9	...	4.9	1.9	-4.1	-0.1	-1.1

(d) Obtain the analysis of variance table.

Answer:

Table 7: ANOVA table

Response	DF	Sum Squares	Mean squares	F Value	Pr(>F)
Agents	4	4430.1	1107.536	147.23	7.98 e-40
Residuals	95	714.65	7.52		
Total	99	5144.75			

(e) Test whether or not the mean time lapse differs for the five agents: use $\alpha = 0.10$. State the alternatives, decision rule, and conclusion.

Answer: Our null and alternative hypothesis are as follows

$$H_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$$
$$H_a = \text{not all } \mu_i \text{ are equal}$$

We have an F-value, $F(1-0.10, 5-1, 20-5) = 2.00$, and an F^* test statistic of (using the ANOVA table)

$$F^* = \frac{\text{MSTR}}{\text{MSE}} = 1107.52/7.522 = 147.23$$

We conclude the alternative.

(f) What is the P-value in part (e)? Explain how the same conclusion as in part (e) can be reached by knowing the P-value.

Answer: The p-value is essentially zero, so at $\alpha = 0.1$, we reject the null hypothesis.

(g) Based on the box plots obtained in part(a), does there appear to be much variation in the mean time lapse for the five agents? Is this variation necessarily the result of differences in the efficiency of operations of the five agents? Discuss.

Answer: There does appear to be significant variation across the agents.

We didn't sample the agents randomly, like in an experiment, we just analyzed a random subset of their clients. Therefore, this is still observational and we cannot conclude the variation is due to the efficiency of the individual agents.

16.17 Refer to problem 16.12, Suppose that 25% of all premium distributions are handled by agent 1, 20 % by agent 2, 20% by agent 3, 20 % by agent 4, and 15 % by agent 5.

(a) Obtain a point estimate of μ when the ANOVA model is expressed in the factor effects of formulation (16.62) and μ is defined by (16.65), with the weights being the proportions of premium distributions handled by each agent.

Answer: (16.62) restates the ANOVA model as

$$Y_{ij} = \mu. + \tau_i + \varepsilon_{ij}$$

where $\mu.$ is a constant component common to all observations, τ_i is the effect of the ith factor level, and the errors are normal around 0 and independent.

The weighted mean (16.65) is given by

$$\mu. = \sum_{i=1}^{r} w_i \mu_i \qquad \text{where} \quad \sum_{i=1}^{r} w_i = 1$$

Therefore,

$$\hat{\mu} = .25 \cdot \mu_1 + .2 \cdot \mu_2 + .2 \cdot \mu_3 + .2 \cdot \mu_4 + .15 \cdot \mu_5 = 20.47$$

Relevant R-code

```
weightedmean=.25*meanagent1+.2*meanagent2+.2*meanagent3+
.2*meanagent4+.15*meanagent5
```

(b) State the alternative for the test of equality of factor level means in terms of factor effects model (16.62) for the present case. Would this statement be affected if μ were defined according to (16.63)? Explain.

Answer: By (16.63), the unweighted mean, we have

$$\mu. = \frac{\sum_{i=1}^{r} \mu_i}{r}$$

We note that the treatment effect, $\tau_i = \mu_i - \mu.$. The null hypothesis is that $\tau_1 = \tau_2 = \ldots = \tau_5 = 0$ and the alternative is not all τ_i equal 0, ie the weighted mean equals the factor level means for each agent. Alternatively, we could just define the hypothesis test the same as in problem 16.12, where all the factor means we assume to be equal.

The statement would not be affected if $\mu.$ were defined by the unweighted case (16.63), because we are still comparing all the treatment effects based on a constant $\mu.$. The unweighted case is equivalently the weighted case with 5 equivalent weights (20 %) in this example.

16.25 Refer to problem 16.7. Obtain the power of the test in problem 16.7e if $\mu_1 = 7.0$, $\mu_2 = 8.0$, and $\mu_3 = 9.0$. Assume that $\sigma = 0.9$.

Answer:

(16.87b) defines $\mu_.$ as

$$\mu. = \frac{\sum n_i \mu_i}{n_T}$$

In our example, we get a value of 7.88.

$$\mu. = 7.88$$

The power of the F test for a single-factor study refers to the probability that the decision rule will lead to conclusion H_a, that the treatment means differ, when in fact H_a holds. Specifically, the power is given by the following expression for the cell means model (16.2)

$$\text{Power} = P\left\{F^* > F(1-\alpha; r-1, n_T - r)\right\} |\phi| \tag{5}$$

Where ϕ is the *non-centrality parameter*, that is, a measure of how unequal the treatment means μ_i are:

$$\phi = \frac{1}{\sigma}\sqrt{\frac{\sum_i n_i(\mu_i - \mu.)^2}{r}} \tag{6}$$

We get a value ϕ of

$$\phi = 2.4567$$

Then, the Power is from table. According to the table, we introduce v_1 and v_2, where v_1 is the number of degrees of freedom for the numerator of F^*, which is $r-1=2$ in this case. v_2 represents the degrees of freedom for the denominator of F^*, and $v_2 = n_T - r = 24$. At $\alpha - 0.05$ risk

$$\text{Power} = 1 - \beta \approx 0.96$$

This indicates there are 96 chances in 100 that the decision rule, based on the sample sizes employed, will lead to the detection of differences in the mean expenditure for the four fitness levels. Relevant R-code is listed below. Similar code is used for example 16.26, so this code will double for both.

```
#rename columns#
colnames(data167)=c('expenses', 'factors')

#lengths of each factor level
n1=length(data167$factors[data167$factors=='low'])
n2=length(data167$factors[data167$factors=='moderate'])
n3=length(data167$factors[data167$factors=='high'])

#total length
nt=length(data167$factors)

weight1=n1/nt
weight2=n2/nt
weight3=n3/nt
##we are given that mu1=7, mu2=8, mu3=9##

mu1=7
mu2=8
mu3=9

m1=(mu1-weightedmean167)^2
m2=(mu2-weightedmean167)^2
m3=(mu3-weightedmean167)^2



weightedmean167=weight1*7+weight2*8+weight3*9

sigma167=0.9
```

```
phi167=(1/sigma167)*sqrt((n1*m1+n2*m2+n3*m3)/r167)
```

16.26 Refer to problem 16.9. Obtain the power of the test in problem 16.9e if $\mu_1 = 37$, $\mu_2 = 35$, and $\mu_3 = 28$. Assume that $\sigma = 4.5$.

Answer: This is more or less an identical problem to 16.25. We find that $\mu. = 33.92$, and the non-centrality parameter is $\phi = 2.21$. We have $\nu_1 = 3-1 = 2$ and $\nu_2 = 24-3 = 21$. Using table B.11 in the textbook, at α risk of $\alpha = 0.01$, we find the power to be, putting ϕ roughly in between 2 and 2.5

$$1-\beta \approx 0.71$$

16.35 Refer to problem 16.12. Suppose that the sample sizes have not yet been determined but it has been decided to sample the same number of premium distributions for each agent. Assume that a reasonable planning value for the error standard deviation is $\sigma = 3.0$ days.

(a) What would be the required sample sizes if: (1) differences in the mean time lapse for the five agents are to be detected with probability 0.95 or more when the range of the treatment means is 3.75 days, and (2) the α risk is to be controlled at 0.10?

Answer: Since the range=3.75 days and $\sigma = 3$ days, we find, in accordance with equation (16.91)

$$\frac{\Delta}{\sigma} = \frac{3.75}{3} = 1.25 \tag{7}$$

With this value, we use table B.12. With $\alpha = 0.10$ and looking at power =0.95 (ie $\beta = 0.05$), with $r = 5$ because we have 5 agents, we find $n = 22$, the required sample size.

(b) Suppose the chief objective is to identify the best agent, i.e., the one with the smallest mean time lapse. The probability should be at least 0.90 that the best agent is recognized correctly when the mean time lapse for the second best agent differs by 1.0 day or more. What are the required sample sizes?

Answer: In this example, we use table B.13 developed by Bechhofer, enables us to determine the necessary sample sized so that with probability $1-\alpha$ the highest estimated treatment mean is from the treatment with the highest population mean. We need to specify the probability $1-\alpha$, the standard deviation σ, and the smallest difference λ between the highest and second highest treatment means. In table B.13, we use $\lambda\sqrt{n}/\sigma$, which helps determine the sample size to find best of r population means. Following similar procedure as in part (a) using table B.12 with $\beta = 0.10$ and $r = 5$, we find that

$$\frac{\lambda\sqrt{n}}{\sigma} = 2.6$$

Therefore, with $\lambda = 1$ and $\sigma = 3$, and therefore,

$$n = 7.8^2 = 60.84$$

Since n is an integer, then we find n should be 61 as minimum.

10 Consider a balanced incomplete block design (BIBD). Suppose there are t treatments, each appearing r times in the design, and there are b blocks, each with s units (i.e. block size=s). In addition, suppose g is the number of times that every treatment appears with every other treatment in the same block. Answer the following questions:

(a) Represent the total sample size N of this BIBD in terms of t, r, b, and s. Explain your answer.

Answer:

We can express the answer in two ways:

$$N = sb = tr$$

This is clear since r and s total instances for columns and rows respectively, and since t and b represent the total number of columns and rows respectively.

(b) Explain why the following equation has to be true for a BIBD:

$$gt(t-1) = bs(s-1)$$

Answer:
Using our result from part (a), with $sb = tr$, we the equation becomes

$$g(t-1) = r(s-1)$$

Now, we argue that these are equal. On the left hand side, we fix one treatment. After choosing one treatment, leaving $t-1$ available. We do this g times, because g is the number of times every treatment appears with a treatment in the same block. We use similar reasoning for the right hand side. We fix one row element, and then we have $s-1$ elements left to multiply by.