Unified Framework Notation

Manuel Arnold

Humboldt-Universität zu Berlin

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1 Introduction

This document is intended to facilitate a better understanding of the code used in the backend. Terminology and notation are not meant to be used in the frontend or in the paper.

Following Usami et al., I limit the scope to single-indicator models. Furthermore, I assume stationarity meaning that all models parameters are time-invariant.

I specify the unified framework in reticular action model (RAM) notation. In RAM notation, SEMs are described by a symmetrical $\mathbf{S}$ matrix of dimensions $V \times V$, an asymmetrical matrix $\mathbf{A}$ of dimensions $V \times V$, a filter matrix $\mathbf{F}$ of dimensions $M \times V$, and a mean vector $\mathbf{m}$ of length V, where M denotes the number of manifest variables and V the total number of variables including latent variables from the model.

Usami et al. use a element-wise notation of the unified framework. In the following, I will convert to Usami et al.'s notation to matrix notation so that is easier for programming. First, we look at the dimensions of the RAM matrices. Let $\mathbf{y}_t$ be a J-variate vector of manifest variables that are measured at the time points $t = 1 \ldots, T$. Thus, the total number of manifest variables is $JT = V$. The unified framework contains the following latent variables: $\mathbf{f}_t$ $(J \times 1)$ are latent true scores for the measurements $t = 1, \ldots, T$. $\mathbf{f}_t^\star$ $(J \times 1)$ are temporal deviation terms from group means that are measured at $t = 1, \ldots, T$. $\mathbf{R} = [\mathbf{I}^\top, \mathbf{S}^\top]^\top$ $(2J \times 1)$ are random effects, where $\mathbf{I}$ $(J \times 1)$ is a random intercept and $\mathbf{S}$ $(J \times 1)$ is a random slope. Finally, $\mathbf{C} = [\mathbf{A}^\top, \mathbf{B}^\top]^\top$ $(2J \times 1)$ are accumulating factors, where $\mathbf{A}$ $(J \times 1)$ has a constant effect over time and $\mathbf{B}$ $(J \times 1)$ has a changing effect over time. Hence, the total number of latent variables is $2JT + 4J = 2J(T + 2) = V$. I sort all variables in the following order: $\mathbf{y}_1, \ldots, \mathbf{y}_T, \mathbf{f}_1, \ldots, \mathbf{f}_T, \mathbf{f}_1^\star, \ldots, \mathbf{f}_T^\star, \mathbf{I}, \mathbf{S}, \mathbf{A}, \mathbf{B}$.

Table 1 summarizes the building blocks of the unified framework. I ignored the mean structure in my first draft. $\boldsymbol{\tau}_\mathbf{y}$ and $\boldsymbol{\tau}_\mathbf{A}$ are uninteresting in stationary models and could be replaced by centering. $\boldsymbol{\tau}_\mathbf{B}$ contains potentially interesting parameters. I suppose that $\mathbf{I}$ and $\mathbf{S}$ also have means but Usami et al. do not mention them. I am open to including a mean structure later. Because we only consider single-indicator models, $\mathbf{\Lambda_f}$ and $\mathbf{\Lambda_{f^\star}}$ are identity matrices of size $J \times J$.

Symbol	Usami[1]	Description	Used in model
$\boldsymbol{\tau}_\mathbf{y}$	μ_{xt}	observed means	CLPM, FCLPM, RICLPM, STARTS
$\mathbf{\Lambda_f}$	-	Factor loadings of $\mathbf{f}_t$	all
$\mathbf{\Lambda_{f^\star}}$	-	Factor loadings of $\mathbf{f}_t^\star$	all
$\mathbf{\Gamma}$	β_x, γ	ARCLP	all but LCS
$\mathbf{\Omega}$	same	residual var/cov	all
$\mathbf{\Psi}$	same	unique var/cov	FCLPM, STARTS, LCS
$\mathbf{\Phi_I}$	-	random intercepts var/cov	RICLPM, STARTS, LCMSR
$\mathbf{\Phi_S}$	-	random slopes var/cov	LCMSR
$\mathbf{\Phi_{I,S}}$	-	covariance between RE	LCMSR
$\boldsymbol{\tau}_\mathbf{A}$	-	constant accumulating factors mean	ALT, LCS
$\mathbf{\Phi_A}$	-	constant accumulating factors var/cov	ALT, LCS
$\boldsymbol{\tau}_\mathbf{B}$	-	changing accumulating factors mean	ALT
$\mathbf{\Phi_B}$	-	changing accumulating factors var/cov	ALT
$\mathbf{\Phi_{A,B}}$	-	covariance between CF	ALT
$\mathbf{\Gamma}^\star$	$\beta_x^\star$, $\gamma_x^\star$	LCS ARCLP	LCS
		Not used. See below.	

Table 1

[1]*: Notation used in Usami et al.*

The matrices $\mathbf{\Phi_I}$, $\mathbf{\Phi_S}$, $\mathbf{\Phi_A}$, and $\mathbf{\Phi_A}$ are symmetrical variance-covariance matrices and are structured similarly. Consider a bivariate process with two random intercepts named I_1 and I_2. Then, it follows that

$$\mathbf{\Phi_I} = \begin{bmatrix} \mathrm{Cov}(I_1, I_1) & \mathrm{Cov}(I_2, I_1) \\ \mathrm{Cov}(I_2, I_1) & \mathrm{Cov}(I_2, I_2) \end{bmatrix}$$

Conversely, $\mathbf{\Phi_{I,S}}$ and $\mathbf{\Phi_{I,S}}$ are not symmetrical. Both matrices have the following pattern (example given for a bivariate process with random intercepts and slopes):

$$\mathbf{\Phi_{I,S}} = \begin{bmatrix} \mathrm{Cov}(I_1, S_1) & \mathrm{Cov}(I_1, S_2) \\ \mathrm{Cov}(I_2, S_1) & \mathrm{Cov}(I_2, S_2) \end{bmatrix}$$

We use the endogenous version of the ALT model where the factor loadings of the constant accumulating factors $\mathbf{A}$ and the factor loadings of the changing accumulating factors $\mathbf{B}$ are constrained at the first time point $t = 1$. The factor loadings of $\mathbf{A}$ are constrained to $(\mathbf{I}_J - \mathbf{\Gamma})^{-1}$ at $t = 1$ and are $\mathbf{I}_J$ for $t \in 2, \ldots, T$. The factor loadings of $\mathbf{B}$ are constrained to $\mathbf{\Gamma}[(\mathbf{I}_J - \mathbf{\Gamma})(\mathbf{I}_J - \mathbf{\Gamma})]^{-1}$ at $t = 1$ and to $(t-1)\mathbf{I}_J$ for $t \in 2, \ldots, T$.

The LCS model can be respecified as an ALT model by removing the changing accumulating factors $\mathbf{B}$ from the ALT model and using $\mathbf{\Gamma}^\star = \mathbf{\Gamma} + \mathbf{I}_J$. Thus, it is not necessary to use $\mathbf{\Gamma}^\star$ in the backend.