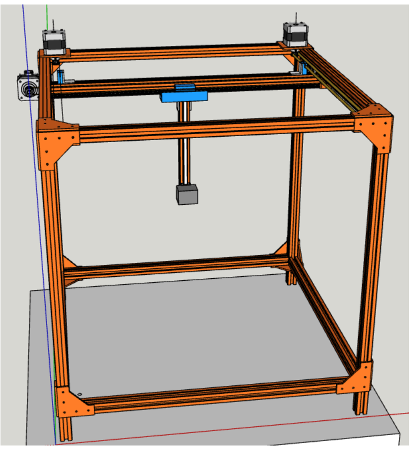
MATHEMATICAL MODEL OF A 3-DOF CARTESIAN ROBOT



Considering that the system under analysis is a Cartesian robot with three degrees of freedom (3-DOF), it is important to note that the Denavit–Hartenberg (DH) parameterization is not particularly significant in this case. This is because the robot does not include rotational joints; instead, it is composed exclusively of prismatic joints, which greatly simplifies the kinematic description. Therefore, a more direct and suitable mathematical model can be formulated for this type of architecture, based on the following key considerations:

Kinematics Models

Direct kinematic model (DKM):

1. Input variables
    - q1: Vertical displacement (Z-axis moving along Y)
    - q2: Horizontal displacement (Y-axis moving along X)
    - q3: Depth displacement (X-axis moving along Z)

2. End-Effector Position

$$[x\ y\ z]^T = [q1\ q2\ q3]^T$$

3. Homogeneous Transformation Matrix

$$T_0^E = \begin{bmatrix} 1 & 0 & 0 & q2 \\ 0 & 1 & 0 & q1 \\ 0 & 0 & 1 & q3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse kinematic Model (IKM):

To reach a desired end–effector position (x_d, y_d, z_d) , the Cartesian robot must generate the following prismatic joint displacements:

$$q_1 = y_d \quad ; \quad q_2 = x_d \quad ; \quad q_3 = z_d$$

Jacobian Matrix:

Since the robot is Cartesian, each joint produces a pure linear motion directly along one of the end-effector axes (X, Y, Z). Therefore, joint velocities map one-to-one to end-effector velocities without coupling. This results in a Jacobian equal to the 3×3 identity matrix.

$$J = \frac{\partial x}{\partial q} = I_3$$

Decoupled Dynamics (axis-by-axis equations):

X and Y axes:

$$m_x x'' + B_x x' + F_{cx} sgn(x') = Fx$$

Z axis:

$$m_z z'' + B_z z' + F_{cz} sgn(z') + m_z g = Fz$$

Where:

- F_i is the linear actuation force on each axis
- m_i is the effective moving mass on each axis (for Y, this usually includes the X-axis carriage)
- B_i is viscous friction
- F_{ci} is Coulomb friction
- g is gravitational acceleration

Torque Force Transmission relations:

Belt drive (X and Y axes):

$$F = \frac{\tau_{motor} N}{r} \Rightarrow \tau = \frac{Fr}{Nr}$$

Ball screw (Z axis):

$$\tau = \frac{FL}{2\pi\eta} \Rightarrow F = \frac{2\pi\eta\tau}{L}$$

Where:

- r= pulley radius (m)
- L= screw lead (m/rev)
- η = mechanical efficiency (0–1)
- N= gear/belt transmission ratio