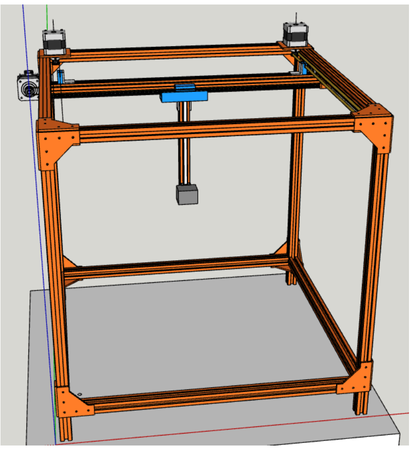
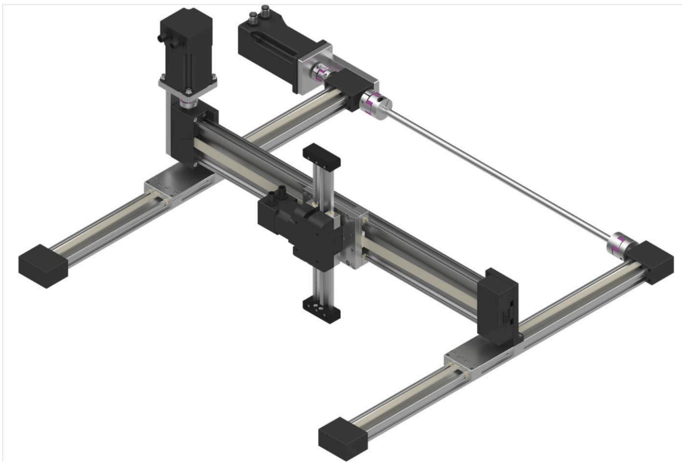
MATHEMATICAL MODEL OF A 3-DOF CARTESIAN ROBOT





Denavit–Hartenberg Parameter

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	d_1	0
2	-90	0	d_2	180
3	-90	0	d_3	-90
4	0	0	L_4	0

Where a_{i-1}, α_{i-1}, d_i and θ_i are the standard Denavit–Hartenberg parameters. The variables d_1, d_2 and d_3 correspond to the prismatic joint displacements, while L_4 represents the fixed end-effector offset.

Although the Denavit–Hartenberg parameters are included for completeness, the Cartesian nature of the robot allows the kinematic and dynamic models to be expressed directly in task space. Therefore, the following sections focus on a Cartesian formulation, which simplifies the analysis and highlights the decoupled behavior of each axis.

Kinematics Models

Direct kinematic model (DKM):

1. Input variables
   - q1: Vertical displacement (Z-axis moving along Y)
   - q2: Horizontal displacement (Y-axis moving along X)
   - q3: Depth displacement (X-axis moving along Z)

2. End-Effector Position

$$[x\ y\ z]^T = [q1\ q2\ q3]^T$$

3. Homogeneous Transformation Matrix

$$T_0^E = \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & 1 & 0 & q_1 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse kinematic Model (IKM):

To reach a desired end–effector position (x_d, y_d, z_d) , the Cartesian robot must generate the following prismatic joint displacements:

$$q_1 = y_d \quad ; \quad q_2 = x_d \quad ; \quad q_3 = z_d$$

Jacobian Matrix:

Since the robot is Cartesian, each joint produces a pure linear motion directly along one of the end-effector axes (X, Y, Z). Therefore, joint velocities map one-to-one to end-effector velocities without coupling. This results in a Jacobian equal to the 3×3 identity matrix.

$$J = \frac{\partial x}{\partial q} = I_3$$

Decoupled Dynamics (axis-by-axis equations):

X and Y axes:

$$m_x x'' + B_x x' + F_{cx} sgn(x') = Fx$$

Z axis:

$$m_z z'' + B_z z' + F_{cz} sgn(z') + m_z g = Fz$$

Where:

- F_i is the linear actuation force on each axis
- m_i is the effective moving mass on each axis (for Y, this usually includes the X-axis carriage)
- B_i is viscous friction
- F_{ci} is Coulomb friction
- g is gravitational acceleration

<u>Torque Force Transmission relations:</u>

Belt drive (X and Y axes):

$$F = \frac{\tau_{motor} N}{r} \Rightarrow \tau = \frac{Fr}{Nr}$$

Ball screw (Z axis):

$$\tau = \frac{FL}{2\pi\eta} \Rightarrow F = \frac{2\pi\eta\tau}{L}$$

Where:

- r= pulley radius (m)
- L= screw lead (m/rev)
- η = mechanical efficiency (0–1)
- N= gear/belt transmission ratio