Computational Photonics

– Matrix Method for Stratified Media –

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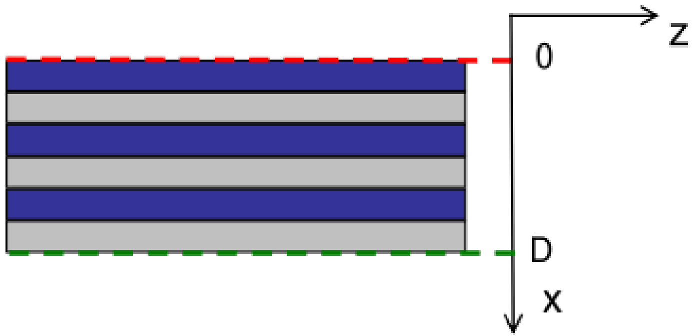
Motivation

Basic Structure



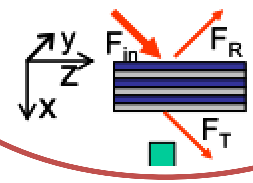
- 1D problem
- <u>here:</u> only homogeneous, isotropic materials

<u>possible tasks:</u>

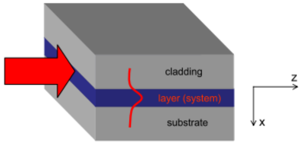
- transmission & reflection as a function of angle of incidence, wavelength, polarization
- EM fields in- & outside the structure
- dispersion relation & EM fields of guided waves (finite stack)
- dispersion relation & EM fields of Bloch modes (infinite structure)

this is our main focus now!

"scattering problem"



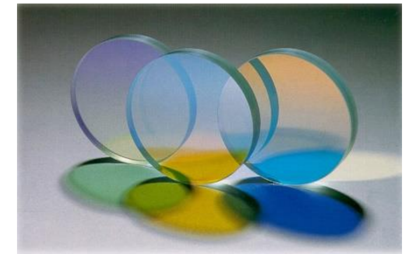
"eigenmode problem"



Applications

- anti-reflection (AR) coating
- Bragg mirrors
- Dichroic mirrors / beam splitters
- Fabry-Perot etalons
- Chirped mirrors
- Spectral filters

"scattering problem"



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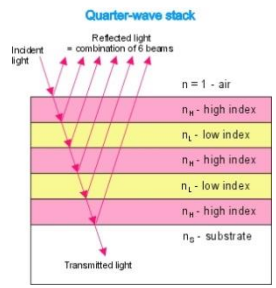

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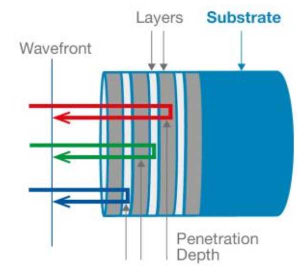

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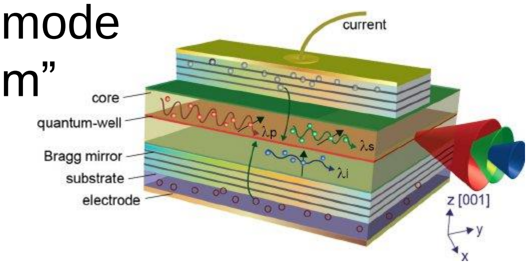
- Slab waveguide
- Multilayer (Bragg) waveguide

"eigenmode problem"



Bijlani et al. doi: 10.1063/1.4819736

Today: Learn how to solve the scattering problem!

... in 7 easy steps ;)

Solving the scattering problem

<u>0.) Recall basic equations:</u>

– Maxwell's curl equations in frequency domain:

$$\nabla \times \tilde{\mathbf{E}}(\mathbf{r}, \omega) = i\omega\mu_0 \tilde{\mathbf{H}}(\mathbf{r}, \omega)$$

$$\nabla \times \tilde{\mathbf{H}}(\mathbf{r}, \omega) = -i\omega\epsilon_0 \boxed{\epsilon}(\cancel{\mathbf{r}}, \omega) \cdot \tilde{\mathbf{E}}(\mathbf{r}, \omega)$$

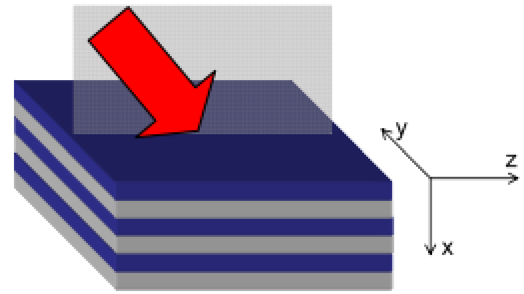
– ... yield Helmholtz equation
for isotropic, homogeneous media:

$$\Delta \tilde{\mathbf{E}}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \epsilon(\omega)\, \tilde{\mathbf{E}}(\mathbf{r}, \omega) = 0$$

– has to be solved for specific geometry!

Solving the scattering problem

1.) Exploiting symmetry:



- translational invariance in y & z direction!
- i.e. problem splits into 2 independent cases:

TE

$$\mathbf{E}_{\mathrm{TE}} = \begin{pmatrix} 0 \\ E_y \\ 0 \end{pmatrix}, \qquad \mathbf{H}_{\mathrm{TE}} = \begin{pmatrix} H_x \\ 0 \\ H_z \end{pmatrix}$$

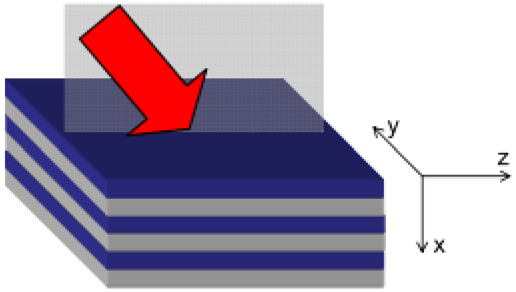
TM

$$\mathbf{H}_{\mathrm{TM}} = \begin{pmatrix} 0 \\ H_y \\ 0 \end{pmatrix}, \qquad \mathbf{E}_{\mathrm{TM}} = \begin{pmatrix} E_x \\ 0 \\ E_z \end{pmatrix}$$

main field component

only tangential components enter boundary conditions!

1.) Exploiting symmetry:



- so we can restrict the calculation to the tangential components!
- relationship between them given by Maxwell curl eq.

TE

$$\mathbf{E}_{\mathrm{TE}} = \begin{pmatrix} 0 \\ E_y \\ 0 \end{pmatrix}, \qquad \mathbf{H}_{\mathrm{TE}} = \begin{pmatrix} H_x \\ 0 \\ H_z \end{pmatrix}$$

$$i\omega\mu_0\,\tilde{H}_z(\mathbf{r},\omega) = \frac{\partial \tilde{E}_y(\mathbf{r},\omega)}{\partial x}$$

TM

$$\mathbf{H}_{\mathrm{TM}} = \begin{pmatrix} 0 \\ H_y \\ 0 \end{pmatrix}, \qquad \mathbf{E}_{\mathrm{TM}} = \begin{pmatrix} E_x \\ 0 \\ E_z \end{pmatrix}$$

$$-i\omega\epsilon_0\,\tilde{E}_z(\mathbf{r},\omega) = \frac{1}{\epsilon(\omega)} \cdot \frac{\partial \tilde{H}_y(\mathbf{r},\omega)}{\partial x}$$

2.) Solve Helmholtz equation inside a homogeneous layer:

- We know: plane waves are the analytic solution for homogeneous media!
- Ansatz:

TE	**TM**
$E_y(x)\exp(ik_z z - i\omega t)$	$H_y(x)\exp(ik_z z - i\omega t)$
$H_z(x)\exp(ik_z z - i\omega t)$	$E_z(x)\exp(ik_z z - i\omega t)$

- same structure: restrict to main field component
- plugging into Helmholtz equation...

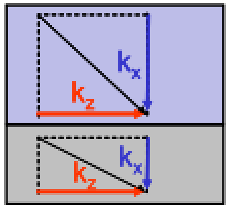
2.) Solve Helmholtz equation inside a single homogeneous layer:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c^2}\epsilon(\omega) - k_z^2\right] E_y(x) = 0 \qquad \textbf{(TE)}$$

– Solutions are up- and downward propagating plane waves with the dispersion relation:

$$k_x^2 + k_z^2 = \frac{\omega^2}{c^2}\epsilon(\omega)$$

– k_z is <u>preserved</u> in every layer, k_x <u>varies</u> with material!



3.) Deriving the transfer matrix inside a single layer:

- goal: connecting **E** & **H** fields at arbitrary x-position with fields at x=0
- solution for TE:

$$E_y(x) = C_1 \cos(k_x x) + C_2 \sin(k_x x)$$

$$i\omega\mu_0 H_z(x) = \frac{\partial E_y(x)}{\partial x} = k_x \left[-C_1 \sin(k_x x) + C_2 \cos(k_x x)\right]$$

- boundary conditions:

$$C_1 = E_y(0), \qquad C_2 = \frac{1}{k_x} \cdot \left.\frac{\partial E_y(x)}{\partial x}\right|_{x=0}$$

- **TM**: do the same again for $H_y(x), -i\omega\epsilon_0 E_z(x) = \frac{1}{\epsilon(\omega)} \cdot \frac{\partial H_y}{\partial x}$

3.) Deriving the transfer matrix inside a single layer:

– we have successfully connected the main field component and its derivative at any x-position in the layer with the fields at x=0:

TE:

$$E_y(x) = \cos(k_x x)\, E_y(0) + \frac{1}{k_x} \sin(k_x x)\, \left.\frac{\partial E_y(x)}{\partial x}\right|_{x=0}$$

$$\frac{\partial E_y(x)}{\partial x} = -k_x \sin(k_x x)\, E_y(0) + \cos(k_x x)\, \left.\frac{\partial E_y(x)}{\partial x}\right|_{x=0}$$

TM:

$$H_y(x) = \cos(k_x x)\, H_y(0) + \frac{\epsilon(\omega)}{k_x} \sin(k_x x) \frac{1}{\epsilon(\omega)} \left.\frac{\partial H_y(x)}{\partial x}\right|_{x=0}$$

$$\frac{1}{\epsilon(\omega)} \frac{\partial H_y(x)}{\partial x} = -\frac{k_x}{\epsilon(\omega)} \sin(k_x x)\, H_y(0) + \cos(k_x x) \frac{1}{\epsilon(\omega)} \left.\frac{\partial H_y(x)}{\partial x}\right|_{x=0}$$

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3.) Deriving the transfer matrix inside a single layer:

- the form allows to combine both polarizations into a unified matrix treatment if we introduce auxiliary fields F, G and q:

$$\begin{bmatrix} F(x) \\ G(x) \end{bmatrix} = \begin{bmatrix} \cos(k_x x) & \frac{1}{qk_x}\sin(k_x x) \\ -qk_x \sin(k_x x) & \cos(k_x x) \end{bmatrix} \begin{bmatrix} F(0) \\ G(0) \end{bmatrix}$$

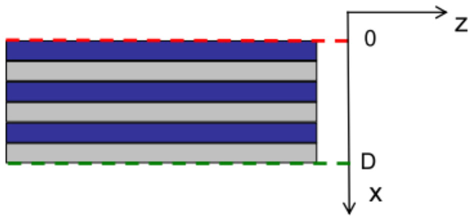
- different meanings in the two polarizations:

TE	TM
$F(x) = E_y(x)$	$F(x) = H_y(x)$
$G(x) = q\frac{\partial F(x)}{\partial x} = i\omega\mu_0 H_z(x)$	$G(x) = q\frac{\partial F(x)}{\partial x} = -i\omega\epsilon_0 E_z(x)$
$q = 1$	$q = 1/\epsilon(\omega)$

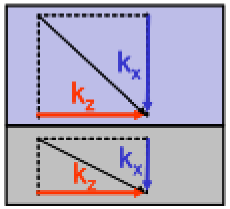
4.) Linking all layers together:

- *every* homogeneous layer with thickness d_i has its *own* matrix (index i):

$$\hat{m}_i = \begin{bmatrix} \cos\left(k_x^{(i)} d_i\right) & \frac{1}{q_i k_x^{(i)}} \sin\left(k_x^{(i)} d_i\right) \\ -q_i k_x^{(i)} \sin\left(k_x^{(i)} d_i\right) & \cos\left(k_x^{(i)} d_i\right) \end{bmatrix}$$



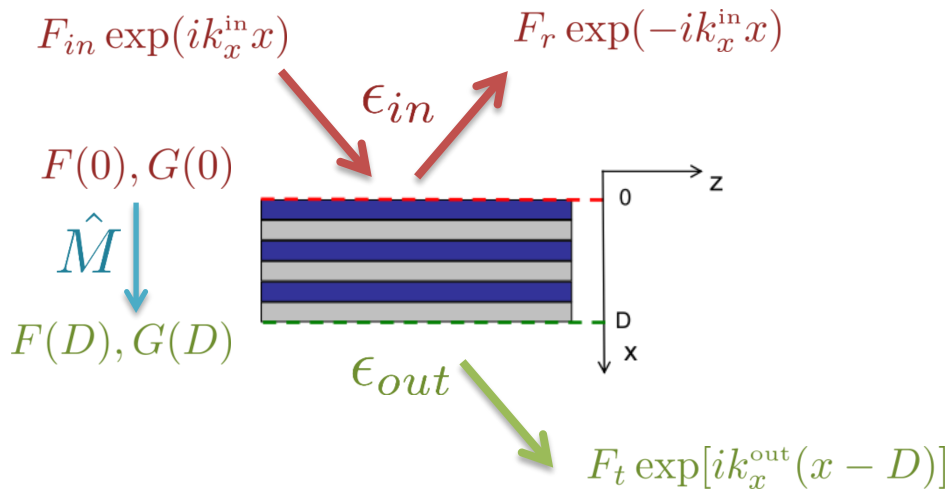
with: $q_i = \begin{cases} 1 & (TE) \\ 1/\epsilon_i(\omega) & (TM) \end{cases} \qquad \left[k_x^{(i)}\right]^2 = \left(\frac{2\pi}{\lambda_0}\right)^2 \epsilon_i(\omega) - k_z^2$



- since transverse fields need to be continuous, whole stack is described by simple matrix multiplication:

$$\hat{M} = \prod_i \hat{m}_i \qquad \begin{bmatrix} F(D) \\ G(D) \end{bmatrix} = \hat{M} \begin{bmatrix} F(0) \\ G(0) \end{bmatrix}$$

5.) Connecting fields in stack to incident, reflected, transmitted field:



$$\begin{bmatrix} F_t \\ iq_{\text{out}} k_x^{\text{out}} F_t \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} F_{in} + F_r \\ iq_{\text{in}} k_x^{\text{in}} (F_{in} - F_r) \end{bmatrix}$$

6.) Solving for complex reflection and transmission amplitude coefficients:

$$r = \frac{F_r}{F_{in}} = \frac{q_{\mathrm{in}} k_x^{\mathrm{in}} M_{22} - q_{\mathrm{out}} k_x^{\mathrm{out}} M_{11} - i(M_{21} + q_{\mathrm{in}} k_x^{\mathrm{in}} q_{\mathrm{out}} k_x^{\mathrm{out}} M_{12})}{q_{\mathrm{in}} k_x^{\mathrm{in}} M_{22} + q_{\mathrm{out}} k_x^{\mathrm{out}} M_{11} + i(M_{21} - q_{\mathrm{in}} k_x^{\mathrm{in}} q_{\mathrm{out}} k_x^{\mathrm{out}} M_{12})}$$

$$t = \frac{F_t}{F_{in}} = \frac{2 q_{\mathrm{in}} k_x^{\mathrm{in}}}{q_{\mathrm{in}} k_x^{\mathrm{in}} M_{22} + q_{\mathrm{out}} k_x^{\mathrm{out}} M_{11} + i(M_{21} - q_{\mathrm{in}} k_x^{\mathrm{in}} q_{\mathrm{out}} k_x^{\mathrm{out}} M_{12})}$$

with:

$$q_i = \begin{cases} 1 & (TE) \\ 1/\epsilon_i(\omega) & (TM) \end{cases} \qquad k_x^{(i)} = \left[\left(\frac{2\pi}{\lambda_0}\right)^2 \epsilon_i(\omega) - k_z^2\right]^{\frac{1}{2}}$$

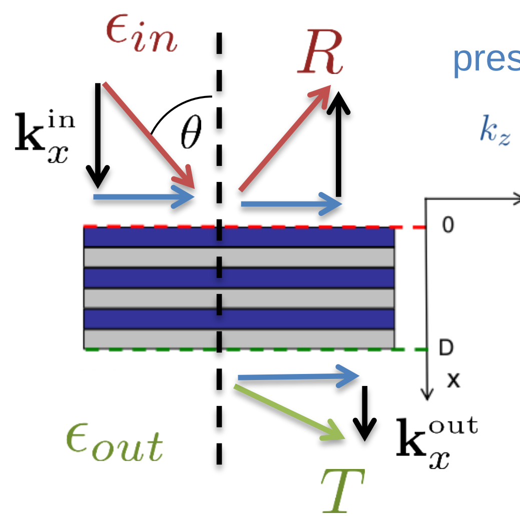
7.) Calculating transmitted / reflected intensity:

– reflectivity:

$$R = \frac{I_r}{I_{\text{in}}} = |r|^2$$

– transmissivity:

$$T = \frac{I_t}{I_{\text{in}}} = \frac{q_{\text{out}} \mathrm{Re}\left(k_x^{\text{out}}\right)}{q_{\text{in}} \mathrm{Re}\left(k_x^{\text{in}}\right)} |t|^2$$



preserved component:

$$k_z = \frac{2\pi}{\lambda_0} \sqrt{\epsilon_{in}} \sin(\theta)$$

$$k_x^{(i)} = \left[\left(\frac{2\pi}{\lambda_0} \right)^2 \epsilon_i(\omega) - k_z^2 \right]^{\frac{1}{2}}$$

note: R and T make strict sense just in cases where $\epsilon_{\text{in}}, \epsilon_{\text{out}}$ are purely real (lossless surrounding media)

Summary of this lecture

How to solve the scattering problem?

Summary

1. given parameters: polarization, $\lambda_0, \epsilon_i, d_i, \theta$
2. calculate $k_z = \frac{2\pi}{\lambda_0}\sqrt{\epsilon_{in}}\,\sin(\theta)$ $\quad k_x^{(i)} = \left[\left(\frac{2\pi}{\lambda_0}\right)^2 \epsilon_i(\omega) - k_z^2\right]^{\frac{1}{2}}$
3. calculate layer matrices:

$$\hat{m}_i = \begin{bmatrix} \cos\left(k_x^{(i)} d_i\right) & \frac{1}{q_i k_x^{(i)}} \sin\left(k_x^{(i)} d_i\right) \\ -q_i k_x^{(i)} \sin\left(k_x^{(i)} d_i\right) & \cos\left(k_x^{(i)} d_i\right) \end{bmatrix} \qquad q_i = \begin{cases} 1 & (TE) \\ 1/\epsilon_i(\omega) & (TM) \end{cases}$$

4. calculate stack matrix by matrix multiplication:

$$\hat{M} = \prod_i \hat{m}_i$$

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5. calculate complex reflection / transmission coefficients:

$$r = \frac{F_r}{F_{in}} = \frac{q_{\text{in}} k_x^{\text{in}} M_{22} - q_{\text{out}} k_x^{\text{out}} M_{11} - i(M_{21} + q_{\text{in}} k_x^{\text{in}} q_{\text{out}} k_x^{\text{out}} M_{12})}{q_{\text{in}} k_x^{\text{in}} M_{22} + q_{\text{out}} k_x^{\text{out}} M_{11} + i(M_{21} - q_{\text{in}} k_x^{\text{in}} q_{\text{out}} k_x^{\text{out}} M_{12})}$$

$$t = \frac{F_t}{F_{in}} = \frac{2 q_{\text{in}} k_x^{\text{in}}}{q_{\text{in}} k_x^{\text{in}} M_{22} + q_{\text{out}} k_x^{\text{out}} M_{11} + i(M_{21} - q_{\text{in}} k_x^{\text{in}} q_{\text{out}} k_x^{\text{out}} M_{12})}$$

6. calculate reflectivity / transmissivity:

$$R = \frac{I_r}{I_{\text{in}}} = |r|^2$$

$$T = \frac{I_t}{I_{\text{in}}} = \frac{q_{\text{out}} \operatorname{Re}\left(k_x^{\text{out}}\right)}{q_{\text{in}} \operatorname{Re}\left(k_x^{\text{in}}\right)} |t|^2$$

- We have solved the problem formally in an analytic rigorous way
- Computer programs just help us to deal with the many matrix multiplications in this case, which could, however, also be done with pen and paper
- We can now write programs to do scans over wavelengths, different incidence angles or configurations or even optimize a specific structure design

Next lecture

- Examples
  (you will also do some in the seminar tasks)
- Calculating the EM fields
- Alternative formulations / symmetry properties
- Guided modes in stratified media