05-Report

July 13, 2020

1 Sorting: Homework 2

1.1 Exercise 1

- Generalize the `SELECT` algorithm to deal also with repeated values and prove that it still belongs to $\mathcal{O}(n)$.

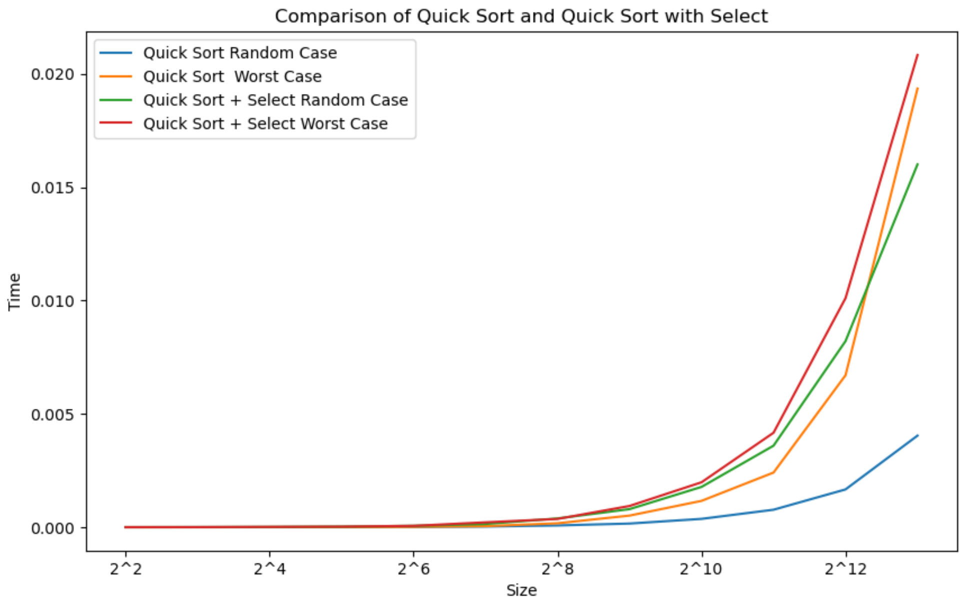
Solution:

One who wants to generalize the `select` algorithm to deal with repeated values must change the `partition`. The idea (Introduction to Algorithms, 7-2 *Quicksort with equal element values*) is dividing array into three parts. First part includes the elements smaller than pivot, second part includes the equal elements and lastly third part includes the elements which are greater than pivot. You can find the new implementation in `src\select.c`. It can be easily deduced that complexity of new partition algorithm is as same as previous one which is $\prime(n)$ so complexity of `Select` algorithm won't change.

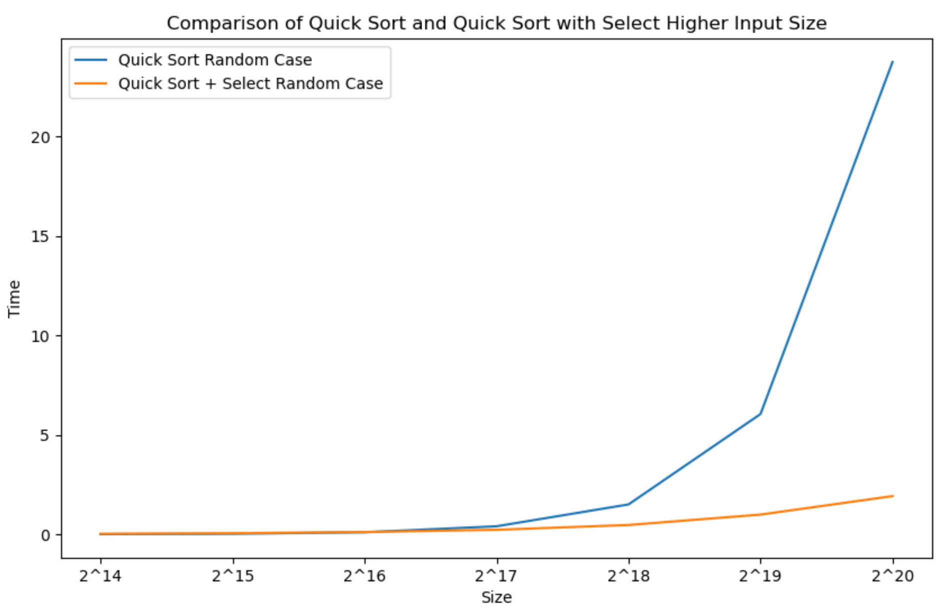
1.2 Exercise 2

- Download the latest version of the code from [https://github.com/albertocasagrande/AD_sorting] and
  - Implement the `SELECT` algorithm of Ex. 1.
    - It is implemented in `src\select.c`.
  - Implement a variant of the `QUICK SORT` algorithm using above mentioned `SELECT` to identify the best pivot for partitioning.
    - It is implemented in `src\select.c` by using new partition.
  - Draw a curve to represent the relation between the input size and the execution-time of the two variants of `QUICK SORT` (i.e, those of Ex. 2 and Ex. 1 31/3/2020) and discuss about their complexities.

Below you can find the comparasion of `quick_sort` and `quick sort + select` for random and worst cases. It can be observed that `quick_sort` performs better.



However below you can find the comparasion for higher input sizes and in this case, `quick sort + select` outperform `quick_sort`.



1.3 Exercise 3

- (Ex. 9.3-1) In the algorithm `SELECT`, the input elements are divided into chunks of 5. Will the algorithm work in linear time if they are divided into chunks of 7? What about chunks of 3?

Solution:

We still know that the median of medians is less than at least 4 elements from half of the $\lceil n/7 \rceil$ so the number elements at least greater than m (median) is for chunks $\lceil n/7 \rceil$ is:

$$4\big(\lceil \frac{1}{2} \lceil \frac{n}{7} \rceil \rceil - 2\big) \geq \frac{2n}{7} - 8$$

An upper bound for the # of elements smaller or equal to m is: $n - (\frac{2n}{7} - 8) = \frac{5n}{7} + 8$

So the recurrence becomes:

$$T_s(n) = T(\lceil n/7 \rceil) + T(5n/7 + 8) + \Theta(n)$$

We can solve this by substitution method and show that it is linear. Selecting cn and $c'n$ representatives of $/(n)$ and $\Theta(n)$ and assume that $T_s(m) \geq cm$ for all $m < n$.

$$\begin{align} T(n) &\leq c\lceil n/7 \rceil + c(5n/7 + 8) + c'n \tag{1} \\ &\leq c(n/7 + 1) + c(5n/7 + 8) + c'n \tag{2} \\ &\leq 6/7cn + 9c + c'n \tag{3} \end{align}$$

Hence, $T(n) \leq cn$ for $c \geq 14c'$ when $n \geq 126$ and $T(n) \in /(n)$

For groups of 3, the number of elements greater than m, and the number of elements less than m, is at least

$$2\big(\lceil \frac{1}{2} \lceil \frac{n}{3} \rceil \rceil - 2\big) \geq \frac{n}{3} - 4$$

So the recurrence becomes:

$$T_s(n) = T(\lceil n/3 \rceil) + T(2n/3 + 4) + \Theta(n)$$

We can solve this by substitution method and show that it is non-linear. We guess that $T(n) > cn$

$$\begin{align} T(n) &\geq c\lceil n/3 \rceil + c(2n/3 + 4) + c'n \tag{4} \\ &\geq c(n/3) + c(2n/3 + 4) + c'n \tag{5} \\ &\geq cn + 4c + c'n \tag{6} \\ &\geq cn + 4c \tag{7} \end{align}$$

therefore we have that it grows more quickly than linear.

1.4 Exercise 4

- (Ex. 9.3-5) Suppose that you have a “black-box” worst-case linear-time subroutine to get the position in A of the value that would be in position $n/2$ if A was sorted. Give a simple, linear-time algorithm that solves the selection problem for an arbitrary position i.

A : Array, p,r are the subarray indices, i is the position.

```
def Select(A, l, r, i)
    if l = r
        return A[l]
    m <- MEDIAN(A,p,r)
    p <- PARTITION(A,m)
    k <- p - l + 1  // length of the first half of the original array
    // i is half the length of the array, return the element coming from the median finding bla
    if i = k
        return A[p]
    // i is less than half the length of the original array, recurse on the first half
    elseif i < k
        return SELECT(A,l,p-1,i)
    // i is greater than half the length of the original array, recurse on the second half
    return SELECT(A,p+1,r,i-k)
```

The recurrence for the worst-case running time is $T(n) \leq T(n/2) + \prime(n) \in \prime(n)$

1.5 Exercise 5

- Solve the following recursive equations by using both the recursion tree and the substitution method:

First two questions were solved during the lectures, so solutions will be provided from lecture notes.

1.5.1 a) $T_1(n) = 2 * T_1(n/2) + \mathcal{O}(n)$

Recursion Tree

In recursion tree, we have 2^i nodes for level i and height of tree is $\log n$ cost for each level can be calculated with $2^i \frac{cn}{2^i}$ so total cost can be represented like following.

$T_1(n) \leq \sum_{i=0}^{\log n} 2^i \frac{cn}{2^i}$ which equals to $cn \log n \in \mathcal{O}(n \log n)$

Substitution Method

We guess that $T_1(n) \in \mathcal{O}(n \log n)$. We select representatives for $\mathcal{O}(n \log n)$ and $\mathcal{O}(n)$ as $cn \log n$ and $c'n$

$$
\begin{aligned}
T_1(n) &= 2 * T_1(n/2) + c'n && (8)\\
&\leq 2\frac{cn}{2} logn/2 + c'n && (9)\\
&\leq cn\log n - cn\log 2 + c'n \text{ (if } cn\log 2 + c'n \text{ is smaller or equal to zero)} && (10)\\
&\leq cn\log n && (11)\\
T_1(n) &\leq cn\log n \in \mathcal{O}(n\log n) && (12)
\end{aligned}
$$

1.5.2 b) $T_2(n) = T_2(\lceil n/2\rceil) + T_2(\lfloor n/2\rfloor) + \Theta(1)$

Recursion Tree

In this recursion tree which is not complete because of ceiling and floor operations. Height of left side is smaller than $\log 2n$ and height of right is larger than $\log \frac{n}{2}$ By choosing c as a represantative of $\Theta(1)$ we have the following boundaries for cost.

$$
\begin{aligned}
T_2(n) &\geq \sum_{i=0}^{\log \frac{n}{2}} c2^i && (13)\\
&\geq c\frac{2^{log\frac{n}{2}+1} - 1}{2-1} && (14)\\
&\geq c(2^{\log n - \log 2 + 1} - 1) && (15)\\
&\geq cn - c \in \Omega(n) && (16)
\end{aligned}
$$

$$
\begin{aligned}
T_2(n) &\leq \sum_{i=0}^{\log 2n} c2^i && (17)\\
&\leq c(2^{log2n+1} - 1) && (18)\\
&\leq c(2^{\log n + 2} - 1) && (19)\\
&\leq 4cn - c \in \mathcal{O}(n) && (20)
\end{aligned}
$$

So $T_2(n) \in \Theta(n)$

Substitution Method

First we guess that $T_2(n) \in \mathcal{O}(n)$ and we select representatives for $\mathcal{O}(n)$ and $\Theta(1)$ as cn and 1.

By assuming $\forall m < n$ and $T_2(m) \leq cm$ we want to prove that $T_2(n) \leq cn$

$$
\begin{aligned}
T_2(n) &= T_2(\lceil n/2\rceil) + T_2(\lfloor n/2\rfloor) + 1 && (21)\\
T_2(n) &\leq c\lceil n/2\rceil + c\lfloor n/2\rfloor + 1 && (22)\\
&\leq cn + 1 && (23)
\end{aligned}
$$

which doesn't prove $T_2(n) \leq cn$ because $cn + 1 \nleq cn$

The problem is that we selected the wrong representative for $\mathcal{O}(n)$ so we choose a new representative for $\mathcal{O}(n)$ as $cn - d$

Again by assuming $\forall m < n$ and $T_2(m) \leq cm - d$ we want to prove that $T_2(n) \leq cn - d$

$$\begin{aligned} T_2(n) &= T_2(\lceil n/2 \rceil) + T_2(\lfloor n/2 \rfloor) + 1 && (24) \\ T_2(n) &\leq c\lceil n/2 \rceil - d + c\lfloor n/2 \rfloor - d + 1 && (25) \\ &\leq cn - 2d + 1 && (26) \\ & && (27) \end{aligned}$$

If $1 - d \leq 0$ then $T_2(n) \leq cn - d$ and we can say that $T_2(n) \in \mathcal{O}(n)$

Now we guess that $T_2(n) \in \Omega(n)$ and cn, 1 are the representatives for $\Omega(n)$ and $\Theta(1)$.

$$\begin{aligned} T_2(n) &\geq c\lceil n/2 \rceil - d + c\lfloor n/2 \rfloor - d + 1 && (28) \\ &\geq cn + 1 \geq cn \text{ (so we can prove that } T_2 \in \Omega(n)) && (29) \end{aligned}$$

Hence we can say that $T_2(n) \in \Theta(n)$

1.5.3 c) $T_3(n) = 3 * T_3(n/2) + \mathcal{O}(n)$

Recursion Tree

Here for every level, we have 3^i nodes where i is the level and for each level cost is $3^i \frac{cn}{2^i}$. Height of the tree is $\log n$. So the total cost is:

$$\begin{aligned} T_3(n) &\leq cn \sum_{i=0}^{\log n} \big(\frac{3}{2}\big)^i && (30) \\ &= cn\Big(\frac{\big(\frac{3}{2}\big)^{\log n} - 1}{\frac{3}{2} - 1}\Big) = 2c(n^{\log 3} - n) && (31) \end{aligned}$$

Hence we can say that $T_3(n) \in \mathcal{O}(n^{\log 3})$

Substitution Method

We guess that $T_3(n) \in \mathcal{O}(n^{\log 3})$ and we select representatives for $mathcalO(n^{\log 3})$ and $mathcalO(n)$ as $cn^{\log 3}$ and $c'n$.

By assuming $\forall m < n$ and $T_3(m) \leq cm^{\log 3}$ we want to prove that $T_3(n) \leq cn^{\log 3}$

$T_3(n) \leq 3(c(n/2)^{\log 3}) + c'n = cn^{\log 3} + c'n$ from here we can't prove our assumption so we have to change our representative like previous exercise. Therefore new representative is $cn^{\log 3} - dn$

$T_3(n) \leq 3(c(n/2)^{\log 3} - dn/2) + c'n = cn^{\log 3} - dn/2 + c'n$ so for $-dn/2 + c'n \leq 0-> d \geq 2c'$ we prove that $T_3(n) \in \mathcal{O}(n^{\log 3})$

1.5.4 d) $T_4(n) = 7 * T_4(n/2) + \Theta(n^2)$

Recursion Tree

Here for every level, we have 7^i nodes where i is the level and for each level cost is $7^i c\big(\frac{n}{2^i}\big)^2$. Height of the tree is $\log n$. So the total cost is:

$$
\begin{aligned}
T_4(n) \leq\geq cn^2 \sum_{i=0}^{\log n} (\frac{7}{4})^i && (32)\\
= cn^2\Big(\frac{(\frac{7}{4})^{\log n} - 1}{\frac{7}{4} - 1}\Big) && (33)\\
= \frac{4}{3}cn^2\Big((\frac{7}{4})^{\log n} - 1\Big) && (34)\\
= \frac{4}{3}c\Big(n^2\big(n\big)^{\log\frac{7}{4}} - n^2\Big) && (35)\\
= \frac{4}{3}c\Big(\big(n\big)^{\log 7} - n^2\Big) && (36)
\end{aligned}
$$

Hence we can say that $T_4(n) \in \Theta(n^{\log 7})$

Substitution Method

We guess that $T_4(n) \in \mathcal{O}(n^{\log 7})$ and we select representatives for $\mathcal{O}(n^{\log 7})$ and $\Theta(n^2)$ as $cn^{\log 7} - dn^2$ and $c'n^2$.

By assuming $\forall m < n$ and $T_4(m) \leq cm^{\log 7} - dm^2$ we want to prove that $T_4(n) \leq cn^{\log 7} - dn^2$

$T_4(n) \leq 7c(\frac{n}{2})^{\log 7} - d(\frac{n}{2})^2 + c'n^2 = cn^{\log 7} - \frac{dn^2}{4} + c'n^2$ So for $-\frac{dn^2}{4} + c'n^2 \leq 0 -> d \geq 4c'$ we prove that $T_4(n) \in \mathcal{O}(n^{\log 7})$

Now for lower boundary we need to update our representative as $cn^{\log 7}$ so we have $T_4(n) \geq 7c(\frac{n}{2})^{\log 7} + c'n^2 = cn^{\log 7} + c'n^2$ which is $\geq$ than $cn^{\log 7}$ so we can say that $T_4(n) \in \Omega(n^{\log 7})$.

Hence we proved that $T_4(n) \in \Theta(n^{\log 7})$