Report

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1 Binary Heaps: Homework

1.1 Exercise 1

- Implement the array-based representation of binary heap together with the functions HEAP MIN, REMOVE MIN, HEAPIFY, BUILD HEAP, DECREASE KEY, and INSERT VALUE.

Solution: Array- based representation of binary heap has been already implemented in lecture 6, lecture 7 and lecture 8. All the implementations can be found in `binheap.c`.

Firstly, representation of binary heaps are defined and this struct includes the following properties:

- `void *A`: This is the array used to store heap nodes
- `num_of_elem`: This is the number of nodes in the heap
- `max_size`: This is the maximum number of size
- `key_size`: Size of the key type
- `leq`: This is the heap total order which is user defined (total_order_type)
- `max_order_value`: This is the maximum value stored in the heap

Also common heap operations are implemented as well as useful ones such as `heapify`, `build_heap`, `insert`, `find_max` and `extract`.

1.2 Exercise 2

- Implement an iterative version of HEAPIFY.

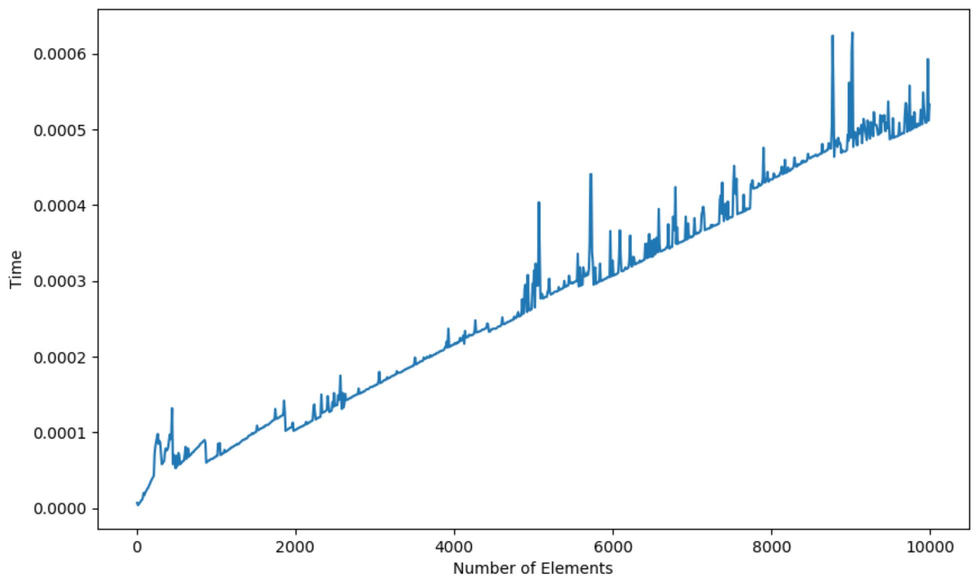
Solution: Again iterative version of heapify has been implemented in lectures 6,7 and 8. Function takes two arguments `binheap_type *H` and `node`. Destination node is containing the minimum among the nodes its children.

Idea is correcting a single violation of the heap property in a sub-trees root, of course by making necessary comparisons (`total_order` is used, good candidates are chosen by checking right and left children) and swaps.

1.3 Exercise 3

- Test the implementation on a set of instances of the problem and evaluate the execution time.

Solution: In the folder tests, `performance_test.c` file has been written. It is a simple code which measures the time to insert random value to heap and produce a file named `output.txt`. One can easily compile the `performance_test.c` by using following command `gcc performance_test.c -I /usr/local/lib ../libbinheap.so -o insert_test.o`.



Above you can find the the asymptotic complexity of the `insert_value` function.

1.4 Exercise 4

- (Ex. 6.1-7) Show that, with the array representation, the leaves of a binary heap containing n nodes are indexed by $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, ...n$.

Solution: Consider the node indexed by $\lfloor n/2 \rfloor + 1$ and I want to show that index $\lfloor n/2 \rfloor + 1$ is a leaf which can be proved if we can show the index of the left child is larger than the number of elements in the heap.

$$LEFT(\lfloor n/2 \rfloor + 1) = 2(\lfloor n/2 \rfloor + 1) > 2(n/2 - 1) + 2 = n - 2 + 2 = n$$

Since the index of the left child is larger than the number of elements in the heap we can say that the node is leaf (since it doesn't have children). Same procedure can be applied for larger indices.

1.5 Exercise 5:

- (Ex. 6.2-6) Show that the worst-case running time of `HEAPIFY` on a binary heap of size n is $\Omega(logn)$. (Hint: For a heap with n nodes, give node values that cause `HEAPIFY` to be called recursively at every node on a simple path from the root down to a leaf.)

Solution: Assume there is the smallest value at the root (worst case scenario for max heap but one can easily adapt this idea to min heap). Since it is the smallest value, it has to be swapped through each level of the heap until it became a leaf. In this case , heapify will be called h times (height of tree) since the height of heap is logn, it has worst-case time $\Omega(logn)$.

1.6 Exercise 6:

- (Ex. 6.3-3) Show that there are at most $\lceil n/2^{h+1} \rceil$ nodes of height h in any n-element binary heap.

Solution: We know that from (Exercise 4) leaves of a heap are nodes indexed by $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, ...n$. Those elements are the second half of the heap (if n is odd also middle element). Hence number of leaves in a heap is $\lceil n/2 \rceil$. Let's prove this by induction:

- Assume that n_h is the number of nodes at height h. The upper bound holds for the base since $n_0 = \lceil n/2 \rceil$ which is exactly the number of leaves in a heap of size n.
- Now let's check it for $h - 1$. Note that if n_{h-1} is even each node at height h has exactly two children, which implies that $n_h = n_{h-1}/2 = \lfloor n_{h-1}/2 \rfloor$. If n_{h-1} is odd, one node at height h has one child and remaining has two children, which implies that $n_h = \lfloor n_{h-1}/2 \rfloor + 1 = \lceil n_{h-1}/2 \rceil$. So we have:

$$n_h = \lceil n_{h-1}/2 \rceil \leq \lceil 1/2 . \lceil n/2^{(h-1)+1} \rceil \rceil = \lceil 1/2 . \lceil n/2^h \rceil \rceil = \lceil n/2^{h+1} \rceil$$

which implies that it holds for h.