Homework 1

Exercise 1 $\mathbb{E}(y) = \sum_{j \in J} f(j)\, p(j)$ (Discrete case) where $\sum_{j=1}^{N} f(j) = \frac{N \cdot (N+1)}{2}$ and $p(j) = \frac{1}{N}$

$$\mathbb{E}(y) = \frac{N(N+1)}{2} \cdot \frac{1}{N} \longrightarrow \boxed{\mathbb{E}(y) = \frac{N+1}{2}}$$

Exercise 2 $\mathbb{E}(x) = \int_X f(x)\, x\, dx$ (Continious case) where $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-N)^2}{2\sigma^2}}$

$$\mathbb{E}(x) = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-N)^2}{2\sigma^2}} dx$$

Let's simplify the problem $N=0, \sigma=1$ (standard normal)

$$\mathbb{E}(x) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} x e^{-\frac{x^2}{2}} dx$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{2\pi}} \left[-e^{-\frac{x^2}{2}}\right]_{-n}^{n}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{2\pi}} \left[-e^{-\frac{n^2}{2}} - \left(-e^{-\frac{(-n)^2}{2}}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n \to \infty} 0$$

$$= 0$$

For general case apply transformation $y = \frac{x-N}{\sigma} \longrightarrow dy = \frac{dx}{\sigma}$ and $x = \sigma y + N$

$$E(x) = \int_{-\infty}^{\infty} (N + \sigma y) \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} \sigma dy$$

$$= N \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} \sigma dy + \sigma \int_{-\infty}^{\infty} y \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2}} \sigma dy$$

$$= N \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}_{\text{p.d.f of standard normal dist}} + \sigma \underbrace{\int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}_{\text{we calculated before}}$$

$$= N\, 1 + \sigma\, 0$$

$$\boxed{= N}$$ //

Exercise 3 If X and Y independent $\underset{p(x|y)\,p(y)}{p(x,y)} = p(x)\, p(y) \Rightarrow p(x|y) = p(x)$

$$p(x) = \sum_{y \in Y} p(x,y) = \frac{x}{30} + \frac{x+1}{30} + \frac{x+2}{30} + \frac{x+3}{30} = \frac{2x+3}{15}$$

$$p(y) = \sum_{x \in X} p(x,y) = \frac{y}{30} + \frac{y+1}{30} + \frac{y+2}{30} = \frac{y+1}{10}$$

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{(x+y)}{30} \div \frac{y+1}{10} = \frac{x+y}{30} \cdot \frac{10}{y+1} = \frac{x+y}{3y+3}$$

So $p(x|y) \neq p(x)$ hence we cannot say X and Y are independent

Exercise 4 $f(y) = \int_X f(x,y)\, dx$

$$f(y) = \int_0^{1-y} 24xy\, dx = \left.\frac{24x^2y}{2}\right|_0^{1-y} = \left.12x^2y\right|_0^{1-y}$$

$$= 12y(1-y)^2 \quad (0<y<1)$$
$$0 \quad \text{, otherwise} //$$

$$f(x|y=\tfrac{1}{2}) = \frac{f(x, y=1/2)}{f(y=1/2)} = \frac{24xy}{12y(1-y)^2} = \frac{2x}{(1-y)^2} = \frac{2x}{(1-\frac{1}{2})^2}$$

$$= 8x \quad (0<x<1/2)$$
$$0 \quad \text{, otherwise} //$$

Exercise 5 $f(x) = \int_y f(x,y)\,dy$

$$\text{Beta}(\alpha_1, \alpha_2 + \alpha_3) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\cdot\Gamma(\alpha_2+\alpha_3)} \cdot x^{\alpha_1-1} \cdot (1-x)^{\alpha_2+\alpha_3}$$

$$f(x) = \int_{-\infty}^{\infty} \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\cdot\Gamma(\alpha_2)\,\Gamma(\alpha_3)} \cdot x^{\alpha_1-1} \cdot y^{\alpha_2-1} \cdot (1-x-y)^{\alpha_3-1}$$

$$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\alpha_3)} \cdot x^{\alpha_1-1} \cdot \int_0^{1-x} y^{\alpha_2-1} \cdot (1-x-y)^{\alpha_3-1}\,dy \quad (y = (1-x)u)$$

$$= \int_0^{1-x} y^{\alpha_2-1}(1-x-y)^{\alpha_3-1} = (1-x)^{\alpha_2+\alpha_3-1} \underbrace{\int_0^1 u^{\alpha_2-1}(1-u)^{\alpha_3-1}\,du}_{B(\alpha_2,\alpha_3) = \frac{\Gamma(\alpha_2)\,\Gamma(\alpha_3)}{\Gamma(\alpha_2+\alpha_3)}}$$

$$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2+\alpha_3)}\, x^{\alpha_1-1}\,(1-x)^{\alpha_2+\alpha_3-1} \sim B(\alpha_1, \alpha_2+\alpha_3) \;//$$