Optimal Operation of Macroscopic Transient Gas Transport Networks

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Abstract

In this paper we introduce a tri-level MILP formulation for the optimal operation of macroscopic transient gas transport networks. In this context, macroscopic stands for the usage of simplified graph representations for the originally complex intersection areas, which we call network stations. The capabilities of a station, in particular the increase of gas pressure through compression, are modelled by purpose-built artificial arcs. Their interplay is described using the concepts of flow directions and simple states. For each station and each timestep we choose a predefined flow direction, which determines where gas has to enter and leave. Additionally, we choose a fitting simple state, which is characterized by two subsets of artificial arcs: Arcs that must be used and arcs that cannot be used for controlling the gas flow. The main simplifications made to derive a MILP formulation are a linearization of the so-called Euler equations, a linear approximation of the power equation for compressor machines, and an overestimation of the compression capabilities of a station by enabling simultaneous parallel and sequential compression. The objective of the optimization problem is to ensure that all demands are satisfied, while the usage of non-technical and technical measures is minimized. The motivation for the design of this model is the NAVI project conducted in the GasLab of the Research Campus MODAL. Given a prognosis for the future gas demands and supplies as well as an estimation of the inflow pressure at the entries of the network, the goal of this decision support system is to equip the dispatchers every couple of minutes with control recommendations guaranteeing security of supply while maximizing stability.

1 Introduction

TODO

2 Mathematical Model

In this section, we define our macroscopic transient gas flow model. We describe the entities of the underlying network and introduce variables and constraints representing their behaviour. Additionally, we explain concepts describing the interplay between different network elements and derive mathematical models for them. In the remainder of this paper, a gas network is modelled as a directed graph $G = (\mathcal{V}, \mathcal{A})$, where $\mathcal{V}$ denotes the set of *nodes* and $\mathcal{A}$ the set of *arcs*.

2.1 Timesteps and Granularity

Additionally, we are given a set of *timesteps* $\mathcal{T}_0 := \{0, \dots, k\}$ together with a monotonically increasing function $\tau : \mathcal{T}_0 \to \mathbb{N}$, called *granularity*. W.l.o.g. we assume that $\tau(0) = 0$. In this context, $\tau(t)$ represents the number of seconds that have passed until time step $t \in \mathcal{T}_0$ w.r.t. timestep 0. For notational purposes we define $\mathcal{T} := \mathcal{T}_0 \setminus \{0\}$.

2.2 Boundary Values

Furthermore, $\mathcal{V}^+ \subseteq \mathcal{V}$ and $\mathcal{V}^- \subseteq \mathcal{V}$ denote the *sources* and *sinks* of the network, respectively, and we assume w.l.o.g. that $\mathcal{V}^+ \cap \mathcal{V}^- = \emptyset$. While $\mathcal{V}^{\mathrm{b}} := \mathcal{V}^+ \cup \mathcal{V}^-$ is called the set of *boundary nodes*, $\mathcal{V}^0 := \mathcal{V} \setminus \mathcal{V}^{\mathrm{b}}$ denotes the set of *inner nodes*.

For each boundary node $v \in \mathcal{V}^{\mathrm{b}}$ and each timestep $t \in \mathcal{T}$ we are given a so-called *boundary value* $D_{v,t} \in \mathbb{R}$. They represent the future requirements in terms of supply (inflow), when $v \in \mathcal{V}^+$ is an entry and we have $D_{v,t} \in \mathbb{R}_{\geq 0}$, and demand (outflow), when $v \in \mathcal{V}^-$ is an exit and we have $D_{v,t} \in \mathbb{R}_{\leq 0}$. The boundary values may be adjusted dynamically to ensure the feasibility of the model. Thus, for each boundary node $v \in \mathcal{V}^{\mathrm{b}}$ and $t \in \mathcal{T}$ we introduce two continuous slack variables $\sigma^{d+}_{v,t}, \sigma^{d-}_{v,t} \in \mathbb{R}_{\geq 0}$. The actual boundary values, which are then considered in the model, are established through the additional variables $d_{v,t} \in \mathbb{R}_{\geq 0}$ for each source $v \in \mathcal{V}^+$ and $d_{v,t} \in \mathbb{R}_{\leq 0}$ for each sink $v \in \mathcal{V}^-$ and constraints

$$d_{v,t} + \sigma^{d+}_{v,t} - \sigma^{d-}_{v,t} = D_{v,t} \quad \forall v \in \mathcal{V}^{\mathrm{b}}, \forall t \in \mathcal{T}. \tag{1}$$

The so-called boundary value slack variables $\sigma^{d+}_{v,t}$ and $\sigma^{d-}_{v,t}$ are associated with the cost parameter $w^{\sigma\text{-d}} \in \mathbb{R}_{\geq 0}$. The physical unit for the boundary values and the variables introduced in this subsection is $\frac{kg}{s}$.

2.3 Pressures and Pressure Bounds

Additionally, for each node $v \in \mathcal{V}$ we are given an initial non-negative pressure value, which we denote by $p_{v,0} \in \mathbb{R}_{\geq 0}$. Furthermore, we introduce pressure variables $p_{v,t} \in [\underline{p}_{v,t}, \bar{p}_{v,t}] \subseteq \mathbb{R}_{\geq 0}$ for each point in time $t \in \mathcal{T}$. Here $\underline{p}_{v,t} \in \mathbb{R}_{\geq 0}$ is a lower and $\bar{p}_{v,t} \in \mathbb{R}_{\geq 0}$ is an upper bound on the pressure at node v and time t. These bounds are called *technical pressure bounds*.

For each boundary node $v \in \mathcal{V}^{\mathrm{b}}$ and each point in time $t \in \mathcal{T}$ we are additionally given so-called *inflow pressure bounds* $\underline{p}^{\mathrm{act}}_{v,t} \in \mathbb{R}_{\geq 0}$ and $\bar{p}^{\mathrm{act}}_{v,t} \in \mathbb{R}_{\geq 0}$. These bounds are tighter than the technical pressure bounds and have to be respected if a boundary node has initial nonzero supply or demand, but they may be relaxed using slack to ensure feasibility. Thus, we introduce two continuous variables $\sigma^{p+}_{v,t} \in [0, \underline{p}^{\mathrm{act}}_{v,t} - \underline{p}_{v,t}]$ and $\sigma^{p-}_{v,t} \in [0, \bar{p}_{v,t} - \bar{p}^{\mathrm{act}}_{v,t}]$ as well as constraints

$$p_{v,t} + \sigma^{p+}_{v,t} \geq \underline{p}^{\mathrm{act}}_{v,t} \quad \forall v \in \mathcal{V}^{\mathrm{b}} \text{ with } D_{v,t} \neq 0, \forall t \in \mathcal{T} \text{ and} \tag{2}$$

$$p_{v,t} - \sigma^{p-}_{v,t} \leq \bar{p}^{\mathrm{act}}_{v,t} \quad \forall v \in \mathcal{V}^{\mathrm{b}} \text{ with } D_{v,t} \neq 0, \forall t \in \mathcal{T}. \tag{3}$$

The so-called inflow pressure slack variables $\sigma^{p+}_{v,t}$ and $\sigma^{p-}_{v,t}$ are associated with the cost parameter $w^{\sigma\text{-p}} \in \mathbb{R}_{\geq 0}$. The physical unit for all parameters, bounds and variables introduced is *bar*.

2.4 Massflows

Next, we introduce variables representing massflow on the arcs in $\frac{kg}{s}$. Therefore, the arc set is partitioned into four sets $\mathcal{A} = \mathcal{A}^{\mathrm{va}} \,\dot\cup\, \mathcal{A}^{\mathrm{rg}} \,\dot\cup\, \mathcal{A}^{\mathrm{pi}} \,\dot\cup\, \mathcal{A}^{\mathrm{ar}}$, where $\mathcal{A}^{\mathrm{va}}$ denotes the set of *valves*, $\mathcal{A}^{\mathrm{rg}}$ the set of *regulators* (often synonymously called *control valves* in the literature), $\mathcal{A}^{\mathrm{pi}}$ the set of *pipes*, and $\mathcal{A}^{\mathrm{ar}}$ the set of so-called *artificial arcs*. The artificial arcs are further partitioned into *monodirected* arcs $\mathcal{A}^{\text{ar-mo}}$ and *bidirected* arcs $\mathcal{A}^{\text{ar-bi}}$, i.e., $\mathcal{A}^{\mathrm{ar}} = \mathcal{A}^{\text{ar-mo}} \,\dot\cup\, \mathcal{A}^{\text{ar-bi}}$. We allow parallel and anti-parallel arcs, but we do not allow loops.

For each $a \in \mathcal{A}^{\mathrm{rg}} \cup \mathcal{A}^{\text{ar-mo}}$ and each timestep $t \in \mathcal{T}$ we introduce a variable $q_{a,t} \in [0, \bar{q}_{a,t}]$ representing the massflow on the corresponding arc in forward direction. On the other hand, for valves and bidirected artificial arcs we add two variables $q^{\rightarrow}_{a,t}, q^{\leftarrow}_{a,t} \in [0, \bar{q}_{a,t}]$ representing massflow in forward direction and backward direction on arc $a \in \mathcal{A}^{\mathrm{va}} \cup \mathcal{A}^{\text{ar-bi}}$, respectively. Further, for each pipe $a = (\ell, r) \in \mathcal{A}^{\mathrm{pi}}$ and each timestep $t \in \mathcal{T}$ we introduce two variables $q_{\ell,a,t}, q_{r,a,t} \in [-\bar{q}_{a,t}, \bar{q}_{a,t}]$ representing the massflow into a at ℓ and out of a at r. Note that negative variable values represent massflow out of a at ℓ and into a at r, respectively. In all of the previous definitions, $\bar{q}_{a,t}$ represents the *technical upper flow bound* for a at time t.

Finally, for timestep $t = 0$ and each of the variables introduced above we are given an initial massflow value, which is denoted analogously having index 0.

2.5 Massflow Conservation

Next, for all nodes $v \in \mathcal{V}$ we introduce massflow conservation equations. For each inner node $v \in \mathcal{V}^0$ and each timestep $t \in \mathcal{T}$ the amount of flow entering v has to leave it and for a boundary node $v \in \mathcal{V}^{\mathrm{b}}$ supply or demand must be satisfied. Hence, we have

$$\begin{aligned}
& - \sum_{a=(\ell,v)\in\mathcal{A}^{\mathrm{va}}} q^{\rightarrow}_{a,t} + \sum_{a=(\ell,v)\in\mathcal{A}^{\mathrm{va}}} q^{\leftarrow}_{a,t} + \sum_{a=(v,r)\in\mathcal{A}^{\mathrm{va}}} q^{\rightarrow}_{a,t} - \sum_{a=(v,r)\in\mathcal{A}^{\mathrm{va}}} q^{\leftarrow}_{a,t} \\
& - \sum_{a=(\ell,v)\in\mathcal{A}^{\mathrm{rg}}} q_{a,t} + \sum_{a=(v,r)\in\mathcal{A}^{\mathrm{rg}}} q_{a,t} - \sum_{a=(\ell,v)\in\mathcal{A}^{\mathrm{pi}}} q_{v,a,t} + \sum_{a=(v,r)\in\mathcal{A}^{\mathrm{pi}}} q_{v,a,t} \\
& - \sum_{a=(\ell,v)\in\mathcal{A}^{\text{ar-bi}}} q^{\rightarrow}_{a,t} + \sum_{a=(\ell,v)\in\mathcal{A}^{\text{ar-bi}}} q^{\leftarrow}_{a,t} + \sum_{a=(v,r)\in\mathcal{A}^{\text{ar-bi}}} q^{\rightarrow}_{a,t} - \sum_{a=(v,r)\in\mathcal{A}^{\text{ar-bi}}} q^{\leftarrow}_{a,t} \\
& - \sum_{(\ell,v)\in\mathcal{A}^{\text{ar-mo}}} q_{a,t} + \sum_{(v,r)\in\mathcal{A}^{\text{ar-mo}}} q_{a,t} = d_{v,t} \qquad \forall v \in \mathcal{V}^{\mathrm{b}},\, \forall t \in \mathcal{T} \qquad (4)
\end{aligned}$$

for each boundary node $v \in \mathcal{V}^{\mathrm{b}}$. For each inner node $v \in \mathcal{V}^0$ we introduce the same constraints except for the right side hand being 0.

2.6 Valves

Valves are network elements that can be used to link or unlink network parts by either creating a connection between the two corresponding endnodes or by disconnecting them. Thereby, a valve can be in one of two possible states. Either it is open, which implies that the pressure values at both ends are equal and massflow is allowed in arbitrary direction (one can think of the endnodes being merged). Or it is closed, implying that there is no massflow and the pressure values are independent or, as we synonymously call it, decoupled.

Thus, let $a = (\ell, r) \in \mathcal{A}^{\mathrm{va}}$ be a valve in G. For each step in time $t \in \mathcal{T}$ we introduce an additional binary variable $z_{a,t} \in \{0, 1\}$ indicating whether the valve is open or not. The behaviour described above can then be modelled using the following constraints

$$
\begin{aligned}
p_{\ell,t} - p_{r,t} &\le (1 - z_{a,t})(\bar{p}_{\ell,t} - \underline{p}_{r,t}) && \forall a = (\ell, r) \in \mathcal{A}^{\mathrm{va}},\ \forall t \in \mathcal{T} && (5)\\
p_{\ell,t} - p_{r,t} &\ge (1 - z_{a,t})(\underline{p}_{\ell,t} - \bar{p}_{r,t}) && \forall a = (\ell, r) \in \mathcal{A}^{\mathrm{va}},\ \forall t \in \mathcal{T} && (6)\\
q^{\rightarrow}_{a,t} &\le \bar{q}_{a,t} z_{a,t} && \forall a \in \mathcal{A}^{\mathrm{va}},\ \forall t \in \mathcal{T} && (7)\\
q^{\leftarrow}_{a,t} &\le \bar{q}_{a,t} z_{a,t} && \forall a \in \mathcal{A}^{\mathrm{va}},\ \forall t \in \mathcal{T}. && (8)
\end{aligned}
$$

2.7 Regulators

Regulators can be seen as the continuous equivalent of valves. Besides being completely open or closed, regulators can also be partially open. Thereby, they generate friction due to which the gas pressure is decreased in flow direction. To model this behaviour, consider some $a = (\ell, r) \in \mathcal{A}^{\mathrm{rg}}$. For each $t \in \mathcal{T}$ we add constraints

$$
p_{\ell,t} - p_{r,t} \ge 0 \quad \forall t \in \mathcal{T}. \tag{9}
$$

Note that in our model the pressure at r can never be greater than the pressure at ℓ, i.e., we do not model so-called flap traps here. This mechanism closes the regulator if a pressure at r gets greater than the pressure at l and makes flow in the backward direction impossible. The reason for not including this mechanism in our model is that all regulators are considered to be connections to distribution parts of the network, i.e., parts only consisting of pipes, inner nodes and exits, which are usually not at a higher pressure level than the upstream transportation network.

2.8 Pipes

One-dimensional gas flow in cylindric pipelines is usually described by the so-called Euler equations, a set of nonlinear hyperbolic partial differential equations, together with the equation of state for real gases. In our model, for which we assume isothermality, we use the linearized formulation of Hennings [2].

Let $a = (\ell, r) \in \mathcal{A}^{\mathrm{pi}}$ be a pipe in G. We describe the flow on a with two types of constraints, which we introduce for each timestep.

$$
\begin{aligned}
&\frac{2\,R_s\,(\tau(t) - \tau(t-1))\,T\,z_a}{L_a\,A_a}\,(q_{r,a,t} - q_{\ell,a,t})\\
&\qquad + p_{\ell,t} - p_{\ell,t-1} + p_{r,t} - p_{r,t-1} = 0 \ \forall t \in \mathcal{T} && (10)\\
&p_{r,t} - p_{\ell,t} + \frac{\lambda_a\,L_a}{4\,A_a\,D_a}\,(|v_{\ell,0}|\,q_{\ell,a,t} + |v_{r,0}|\,q_{r,a,t})\\
&\qquad + \frac{g\,s_a\,L_a}{2\,R_s\,T\,z_a}(p_{\ell,t} + p_{r,t}) = 0 \ \forall t \in \mathcal{T} && (11)
\end{aligned}
$$

The parameters within these constraints are the specific gas constant R_s, the length of the pipe L_a in m, the area of the pipe A_a in m^2, the gas temperature T in K, the compressibility factor z_a of the gas in the pipe, the friction factor of the pipe λ_a, the diameter D_a of the pipe in m, the initial absolute velocities $|v_{\ell,0}|, |v_{\ell,0}|$ of the gas flows at ℓ and r in $\frac{m}{s}$, the gravitational acceleration $g = 9.81\frac{m}{s^2}$, and the slope of the pipe $s_a = \frac{h_r - h_\ell}{L_a}$, where h_ℓ and h_r are the altitude at ℓ and r, respectively. The first constraint (10) is derived from the continuity equation of the Euler equations and captures the transient behaviour of the flow through the pipe. The second (11), derived from the momentum equation, determines the pressure loss due to friction and the height difference of the endnodes.

For the friction factor we use the formula of Nikuradse [1][3]. Furthermore, we assume the compressibility factor of the gas in the pipe to be constant and determine it as the average of the compressibility factors at both endnodes using their initial pressure values and the formula of Papay [4]. Another crucial simplification to derive this linear formulation is the fixation of the absolute velocities in the friction-based pressure difference term of the momentum equation to the absolute velocities in timestep 0. If the velocity of the massflow in- or decreases significantly, we might under- or overestimate the friction loss, respectively.

2.9 Gas Network Stations

Within G there exist $m \in \mathbb{N}$ subgraphs $G_i = (\mathcal{V}_i, \mathcal{A}_i^{\mathrm{ar}})$ called *gas network stations*, which consist of inner nodes and artificial arcs only, i.e., $\mathcal{V}_i \subseteq \mathcal{V}^0$ and $\mathcal{A}_i^{\mathrm{ar}} \subseteq \mathcal{A}^{\mathrm{ar}}$ for all $i \in \{1, \ldots, m\}$. Each artificial arc is contained in exactly one station and each inner node is contained in at most one station, i.e., $\mathcal{A}_i^{\mathrm{ar}} \cap \mathcal{A}_j^{\mathrm{ar}} = \emptyset$ and $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ holds for $i, j \in \{1, \ldots, m\}$ with $i \neq j$ and we have $\mathcal{A}^{\mathrm{ar}} = \bigcup_{i=1}^{m} \mathcal{A}_i^{\mathrm{ar}}$.

The node set $\mathcal{V}_i$ can be further partitioned into so-called *fence nodes* $\mathcal{V}_i^{\mathrm{fn}}$ and *artificial nodes* $\mathcal{V}_i^{\mathrm{ar}}$, i.e., $\mathcal{V}_i := \mathcal{V}_i^{\mathrm{fn}} \,\dot{\cup}\, \mathcal{V}_i^{\mathrm{ar}}$. A node $v \in \mathcal{V}_i$ is a fence node if it is connected to at least one arc outside the gas network station, i.e., if $\delta(v) \cap (\mathcal{A}^{\mathrm{pi}} \cup \mathcal{A}^{\mathrm{rg}} \cup \mathcal{A}^{\mathrm{va}}) \neq \emptyset$. Otherwise, if a node is only incident to artificial arcs, i.e., if $\delta(v) \subseteq \mathcal{A}_i^{\mathrm{ar}}$, it is an artificial node.

Additionally, $\mathcal{F}_i \subseteq \mathcal{P}(\mathcal{V}_i^{\mathrm{fn}}) \times \mathcal{P}(\mathcal{V}_i^{\mathrm{fn}})$ denotes the set of so-called *flow directions* of gas network station G_i. A flow direction $f = (f^+, f^-) \in \mathcal{F}_i$ consists of its entry fence nodes $f^+ \subseteq \mathcal{V}_i^{\mathrm{fn}}$ and its exit fence nodes $f^- \subseteq \mathcal{V}_i^{\mathrm{fn}}$ and it holds that $f^+ \cap f^- = \emptyset$.

Furthermore, the set $\mathcal{S}_i \subseteq \mathcal{P}(\mathcal{F}_i) \times \mathcal{P}(\mathcal{A}_i^{\mathrm{ar}}) \times \mathcal{P}(\mathcal{A}_i^{\mathrm{ar}})$ containing the so-called *simple states* is given for each gas network station G_i. A single simple state $s = (s^f, s^{on}, s^{off}) \in \mathcal{S}_i$ is composed of the set of flow directions s^f it supports as well as the set of its active s^{on} and its inactive artificial arcs s^{off}.

2.10 Controlling Gas Network Stations

In each timestep $t \in \mathcal{T}_0 := \{0, \ldots, k\}$ three types of control decisions have to be made for a gas network station G_i. These decisions impact each other and can be put into a hierarchical order. Here, we describe this order in a top to bottom fashion and introduce the variables and constraints modelling the decisions and their interplay.

First of all, exactly one flow direction $f \in \mathcal{F}_i$ has to be chosen for each G_i. Given this flow direction, one must additionally choose exactly one simple state

$s \in \mathcal{S}_i$ which supports this flow direction, i.e., $f \in s^f$ has to hold. Given a decision on the simple state, all arcs in s^{on} must be activated, while the inactive arcs s^{off} cannot be used. All remaining artificial arcs $a \in \mathcal{A}_i^{\text{ar}} \setminus (s^{on} \cup s^{off})$, which we call optional arcs in the following, may be active, but do not have to.

Thus, for each timestep $t \in \mathcal{T}_0$ we introduce binary variables $x_{f,t} \in \{0,1\}$ for each flow direction $f \in \mathcal{F}_i$, $x_{s,t} \in \{0,1\}$ for each simple state $s \in \mathcal{S}_i$, as well as $x_{a,t} \in \{0,1\}$ for each artificial link $a \in \mathcal{A}_i^{\text{ar}}$ all indicating whether the corresponding entity is active at that point in time or not. Furthermore, for each navi station G_i we add the following constraints

$$
\begin{aligned}
\sum_{f \in \mathcal{F}_i} x_{f,t} &= 1 && \forall t \in \mathcal{T}_0 && (12)\\
\sum_{s \in \mathcal{S}_i} x_{s,t} &= 1 && \forall t \in \mathcal{T}_0 && (13)\\
\sum_{f \in s^f} x_{f,t} &\geq x_{s,t} && \forall s \in \mathcal{S}_i,\ \forall t \in \mathcal{T} && (14)\\
x_{s,t} &\leq x_{a,t} && \forall s \in \mathcal{S}_i,\ \forall a \in s^{on},\ \forall t \in \mathcal{T}_0 && (15)\\
1 - x_{s,t} &\geq x_{a,t} && \forall s \in \mathcal{S}_i,\ \forall a \in s^{off},\ \forall t \in \mathcal{T}_0. && (16)
\end{aligned}
$$

While constraints (12) and (13) ensure that exactly one flow direction and one simple state are chosen for each timestep $t \in \mathcal{T}_0$, (14) guarantees that the chosen simple state supports the chosen flow direction. Additionally, constraints (15) and (16) make sure that the artificial arcs corresponding to the simple state are active or not, respectively. No condition is imposed on the optional arcs.

Next, in order to account for changes over time w.r.t. flow directions, simple states, or artificial links we introduce additional binary variables. For each station G_i and each timestep $t \in \mathcal{T}$ we have $\delta_{f,t} \in \{0,1\}$ for each $f \in \mathcal{F}_i$, $\delta_{s,t} \in \{0,1\}$ for each $s \in \mathcal{S}_i$, and $\delta_{a,t}^{\text{on}}, \delta_{a,t}^{\text{off}} \in \{0,1\}$ for each $a \in \mathcal{A}_i^{\text{ar}}$. Furthermore, we add constraints

$$
\begin{aligned}
x_{f,t-1} - x_{f,t} + \delta_{f,t} &\geq 0 && \forall f \in \mathcal{F}_i,\ \forall t \in \mathcal{T} && (17)\\
x_{s,t-1} - x_{s,t} + \delta_{s,t} &\geq 0 && \forall s \in \mathcal{S}_i,\ \forall t \in \mathcal{T} && (18)\\
x_{a,t-1} - x_{a,t} + \delta_{a,t}^{\text{on}} - \delta_{a,t}^{\text{off}} &= 0 && \forall a \in \mathcal{A}_i^{\text{ar}},\ \forall t \in \mathcal{T}. && (19)
\end{aligned}
$$

While $\delta_{f,t}, \delta_{s,t}, \delta_{a,t}^{\text{on}}$ indicate whether or not a flow direction, simple state, or artificial link has been switched on in timestep t, $\delta_{a,t}^{\text{off}}$ additionally indicates whether or not an artificial link has been switched off. For the flow directions and simple states we do not need such a variable, since we know that exactly one of them is active in each timestep, but in the case of optional artificial arcs this does not apply. All variables $\delta_{f,t}$ are associated with an individual cost parameter $w^f \in \mathbb{R}_{\geq 0}$, the variables $\delta_{s,t}$ with $w^s \in \mathbb{R}_{\geq 0}$, and the variables $\delta_{a,t}^{\text{on}}$ as well as $\delta_{a,t}^{\text{off}}$ are assigned a common cost parameter $w^a \in \mathbb{R}_{\geq 0}$.

2.11 Flow Direction Related Constraints

Activating a flow direction imposes certain conditions on the massflow and pressure values w.r.t. a gas network station G_i. Most importantly, for a flow direction $f = (f^+, f^-) \in \mathcal{F}_i$ no outflow is allowed at its entry and no inflow at its exit fence nodes. Though, it is allowed that there is no flow at all, which is the condition that must hold for all other fence nodes $v \in \mathcal{V}_i^{\mathrm{fn}} \setminus (f^+ \cup f^-)$. Furthermore, for some of the fence nodes there exist additional pressure bounds if a flow direction is chosen in which they serve as exits. And finally, there exist conditions on the sums of absolute amounts of flow of subsets of fence nodes, which have to be satisfied in order to activate certain flow directions.

2.11.1 In- and Outflow Constraints

First of all, for each fence node $v \in \mathcal{V}_i$ and each point in time $t \in \mathcal{T}_0$ we introduce two continuous variables $q^{\mathrm{in}}_{v,t}, q^{\mathrm{out}}_{v,t} \in \mathbb{R}_{\geq 0}$ that, together with the following constraint, account for the total in- or outflow from outside the station, respectively

$$\begin{aligned} &\sum_{(\ell,v)\in\mathcal{A}^{\mathrm{ar}}} q_{a,t} - \sum_{(v,r)\in\mathcal{A}^{\mathrm{ar}}} q_{a,t} + \sum_{(\ell,v)\in\mathcal{A}^{\mathrm{ar\text{-}bi}}} q^{\rightarrow}_{a,t} \\ - &\sum_{(\ell,v)\in\mathcal{A}^{\mathrm{ar\text{-}bi}}} q^{\leftarrow}_{a,t} - \sum_{(v,r)\in\mathcal{A}^{\mathrm{ar\text{-}bi}}} q^{\rightarrow}_{a,t} + \sum_{(v,r)\in\mathcal{A}^{\mathrm{ar\text{-}bi}}} q^{\leftarrow}_{a,t} = q^{\mathrm{out}}_{v,t} - q^{\mathrm{in}}_{v,t}. \end{aligned} \tag{20}$$

Note that one could alternatively sum up the massflow values of the incident pipes, regulators, and valves on the left hand side and switch the signs of the variables on the right hand side of the equation. This is because flow conservation holds at the fence nodes, since $\mathcal{V}_i^{\mathrm{fn}} \subseteq \mathcal{V}^0$. Next, for each flow direction $f = (f^+, f^-) \in \mathcal{F}_i$ we introduce the following constraints:

$$q^{\mathrm{in}}_{v,t} + \overline{q}^{\mathrm{in}}_{v,t}\, x_{f,t} \leq \overline{q}^{\mathrm{in}}_{v,t} \quad \forall f \in \mathcal{F}_i,\, \forall v \in V_i \setminus f^+,\, \forall t \in \mathcal{T} \tag{21}$$

$$q^{\mathrm{out}}_{v,t} + \overline{q}^{\mathrm{out}}_{v,t}\, x_{f,t} \leq \overline{q}^{\mathrm{out}}_{v,t} \quad \forall f \in \mathcal{F}_i,\, \forall v \in V_i \setminus f^-,\, \forall t \in \mathcal{T}. \tag{22}$$

Here, $\overline{q}^{\mathrm{in}}_{v,t}$ and $\overline{q}^{\mathrm{out}}_{v,t}$ are upper and lower bounds on the maximum possible in- and outflow, respectively, which can be derived from constraints (20) together with the above mentioned counterpart. If a flow direction is active, $q^{\mathrm{in}}_{v,t}$ can be nonzero (w.r.t. ε) for the entry and $q^{\mathrm{out}}_{v,t}$ for the exit fence groups only. We relax the corresponding constraints by adding some slack ε to the right hand side in order to account for noise in the initial parameters and for other physical effects like, for example, the usage of linepack.

2.11.2 Exit Pressure Bounds

Furthermore, for some fence nodes $v \in \mathcal{V}_i^{\mathrm{fn}}$ there exists an additional upper pressure bound $\bar{p}^{\mathrm{exit}}_v$, which is tighter than its technical upper bound and has to be respected if a flow direction $f = (f^+, f^-) \in \mathcal{F}_i$ is active, for which v is an exit fence node, i.e., for which $v \in f^-$. This can be modelled via the following constraints

$$p_{v,t} \leq \bar{p}_{v,t} + x_{f,t}\,(\bar{p}^{\mathrm{exit}}_v - \bar{p}_{v,t}) \quad \forall f \in \mathcal{F}_i \text{ with } v \in f^-,\, \forall t \in \mathcal{T}. \tag{23}$$

2.11.3 Flow Direction Conditions

Finally, for each navi station, there exists a set of so-called flow direction conditions $\mathcal{W}_i \subseteq \mathcal{F}_i \times \mathcal{P}(\mathcal{V}_i^{\mathrm{fn}}) \times \mathcal{P}(\mathcal{V}_i^{\mathrm{fn}})$, demanding that the sum of the absolute in- and outflows of the first set of fence nodes is less than or equal than the sum of the in- and outflows of the second set in order to activate the corresponding flow direction. Hence, for each $w = (f, \mathcal{V}_{w_1}, \mathcal{V}_{w_2}) \in \mathcal{W}_i$ we introduce

$$\sum_{v \in \mathcal{V}_{w_2}} (q^{\text{in}}_{v,t} + q^{\text{out}}_{v,t}) - \sum_{v \in \mathcal{V}_{w_1}} (q^{\text{in}}_{v,t} + q^{\text{out}}_{v,t}) \geq -M_{w,t} + x_{f,t}\, M_{w,t} \quad \forall t \in \mathcal{T}_0 \tag{24}$$

where $M_{w,t} := \sum_{v \in \mathcal{V}_{w_1} \cap f^+} \overline{q}^{\text{in}}_{v,t} + \sum_{v \in \mathcal{V}_{w_1} \cap f^-} \overline{q}^{\text{out}}_{v,t}$.

2.12 Artificial Arcs

The set of artificial arcs can be further partitioned into four disjoint subsets $\mathcal{A}^{\text{ar}} = \mathcal{A}^{\text{ar-sc}} \dot{\cup} \mathcal{A}^{\text{ar-rg}} \dot{\cup} \mathcal{A}^{\text{ar-co}} \dot{\cup} \mathcal{A}^{\text{ar-cb}}$. Here, $\mathcal{A}^{\text{ar-sc}}$ denotes the set of so-called *shortcuts*, $\mathcal{A}^{\text{ar-rg}}$ the set of so-called *regulating arcs*, $\mathcal{A}^{\text{ar-co}}$ the set of so-called *compressor arcs*, and $\mathcal{A}^{\text{ar-cb}}$ the set of so-called *combined arcs*. Further, we denote the set of *pressure increasing arcs* by $\mathcal{A}^{\text{ar-pr}} = \mathcal{A}^{\text{ar-co}} \cup \mathcal{A}^{\text{ar-cb}}$. The sets $\mathcal{A}^{\text{ar-sc}}_i \subseteq \mathcal{A}^{\text{ar-sc}}$, $\mathcal{A}^{\text{ar-rg}}_i \subseteq \mathcal{A}^{\text{ar-rg}}$, $\mathcal{A}^{\text{ar-co}}_i \subseteq \mathcal{A}^{\text{ar-co}}$, $\mathcal{A}^{\text{ar-cb}}_i \subseteq \mathcal{A}^{\text{ar-cb}}$, and $\mathcal{A}^{\text{ar-pr}}_i \subseteq \mathcal{A}^{\text{ar-pr}}$ describe the corresponding entities contained in gas network station G_i.

In this section we explain how the artificial arcs are modelled and their different capabilities when controlling the gas flow through the station. But first, we shortly explain the difference between bidirected and monodirected arcs.

2.12.1 Bidirected Arcs

In contrast to the monodirected arcs, massflow is possible into both directions on bidirected arcs. Further, their capabilities, for example the compression of gas, are also applicable in both directions. Thus, in our model we first decide into which direction the massflow is going at each point in time.

Therefore, for each bidirected arc $a \in \mathcal{A}^{\text{ar-bi}}$ and each timestep $t \in \mathcal{T}_0$ we introduce two binary variables $x^{\rightarrow}_{a,t}, x^{\leftarrow}_{a,t} \in \{0,1\}$ encoding the direction of the flow and add constraints

$$x^{\rightarrow}_{a,t} + x^{\leftarrow}_{a,t} = x_{a,t} \quad \forall a \in \mathcal{A}^{\text{ar-bi}}, \forall t \in \mathcal{T} \tag{25}$$

to the model. Given this decision, bidirected arcs are modelled analogously to their monodirected counterparts using the corresponding variable.

2.12.2 Shortcuts

All shortcuts are bidirected arcs and massflow is possible into both directions. They can conceptually be seen as the equivalent of valves (see Section 2.6) inside a station and are used to connect network parts if the corresponding pressure levels are equal. Thus, for each shortcut $a = (\ell, r) \in \mathcal{A}^{\text{ar-sc}}$ we add constraints

$$p_{\ell,t} - p_{r,t} \leq (1 - x_{a,t})(\bar{p}_{\ell,t} - \underline{p}_{r,t}) \quad \forall t \in \mathcal{T}, \tag{26}$$

$$p_{\ell,t} - p_{r,t} \geq (1 - x_{a,t})(\underline{p}_{\ell,t} - \bar{p}_{r,t}) \quad \forall t \in \mathcal{T}, \tag{27}$$

$$q^{\rightarrow}_{a,t} \leq \bar{q}_{a,t} x^{\rightarrow}_{a,t} \quad \forall t \in \mathcal{T}, \tag{28}$$

$$q^{\leftarrow}_{a,t} \leq \bar{q}_{a,t} x^{\leftarrow}_{a,t} \quad \forall t \in \mathcal{T}. \tag{29}$$

If a shortcut is active at time $t \in \mathcal{T}$, i.e., if $x_{a,t} = 1$, the pressures at ℓ and r have to be equal and massflow can go into an arbitrary direction with an arbitrary value, i.e., there may be be forward flow $q^{\rightarrow}_{a,t} \in [0, \bar{q}_{a,t}]$ or backward flow $q^{\leftarrow}_{a,t} \in [0, \bar{q}_{a,t}]$ depending on the decision made in constraint (25). If the shortcut is not active, the pressure values are decoupled and there is no flow.

2.12.3 Regulating Arcs

Regulating arcs can conceptually be seen as the equivalent of regulators (see Section 2.7) inside a gas network station. They are used to decrease the gas pressure in the direction of the massflow, which is needed if, for example, gas enters a distribution network, which is technically not suited for the high pressure of the transportation network. Hence, for regulating arcs $a = (\ell, r) \in \mathcal{A}^{\text{ar-rg}}$ we introduce the following constraints

$$\begin{aligned} p_{\ell,t} - p_{r,t} &\geq (1 - x_{a,t})(\underline{p}_{\ell,t} - \bar{p}_{r,t}) \quad && \forall t \in \mathcal{T}, && (30)\\ q^{\rightarrow}_{a,t} &\leq \bar{q}_{a,t} x_{a,t} && \forall t \in \mathcal{T}. && (31) \end{aligned}$$

If a regulating arc is active at some point in time $t \in \mathcal{T}$, i.e., if $x_{a,t} = 1$, the pressure at ℓ has to be greater or equal than the pressure at r. Otherwise, the pressure values are decoupled and there is no massflow. For bidirected regulating arcs $a = (\ell, r) \in \mathcal{A}^{\text{ar-rg}} \cap \mathcal{A}^{\text{ar-bi}}$, we derive an analogous set of constraints using $x^{\rightarrow}_{a,t}$ and $x^{\leftarrow}_{a,t}$ instead of $x_{a,t}$.

2.12.4 Pressure Increasing Arcs

The pressure increasing arcs $\mathcal{A}^{\text{ar-pr}}$, i.e., the compressor arcs $\mathcal{A}^{\text{ar-co}}$ and the combined arcs $\mathcal{A}^{\text{ar-cb}}$, are key elements when it comes to control a macroscopic gas network. They are able to compress gas and thereby increase its pressure, which makes up for pressure loss due to friction in the pipes or height differences that have to be overcome.

In our model, one can conceptually think of one (big) compressor machine being installed at each arc $a \in \mathcal{A}^{\text{ar-pr}}_i$ of each gas network station G_i. The maximum power it has available for compression $\tilde{\pi}_{a,t} \in \mathbb{R}_{\geq 0}$, the maximum amount of massflow that can pass through it $\tilde{q}_{a,t} \in \mathbb{R}_{\geq 0}$, and its maximum compression ratio $\tilde{r}_{a,t} \in [1, \infty)$ are dynamically determined in each timestep through an assignment of approximations of real-world compressor machines, simply called *machines* in the following, and a linear combination of their corresponding values.

Thus, for each station G_i we are given a set of machines $\mathcal{M}_i$ and each machine $m \in \mathcal{M}_i$ possesses an associated power value $P_{m,t} \in \mathbb{R}_{\geq 0}$, a maximum massflow $Q_{m,t} \in \mathbb{R}_{\geq 0}$, and a maximum compression ratio $R_{m,t} > 1$ for each timestep $t \in \mathcal{T}$. Further, for each pressure increasing arc $a \in \mathcal{A}^{\text{ar-pr}}_i$ there exists a subset of machines $\mathcal{M}^a_i \subseteq \mathcal{M}_i$ that can potentially be assigned to it and a maximum number of assignable machines $M^{\max}_a$. Since each machine can be assigned to at most one compressing link in each timestep $t \in \mathcal{T}$, we introduce binary variables $y_{m,a,t} \in \{0, 1\}$ indicating whether machine $m \in \mathcal{M}_i$ is assigned to arc $a \in \mathcal{A}^{\text{ar-pr}}_i$ or not, and add constraints

$$\begin{aligned} \sum_{a \in \mathcal{A}^{\text{ar-pr}}_i \,:\, m \in \mathcal{M}^a_i} y_{m,a,t} &\leq 1 && \forall m \in \mathcal{M}_i,\ \forall t \in \mathcal{T}, && (32)\\ \sum_{m \in \mathcal{M}^a_i} y_{m,a,t} &\leq M^{\max}_a x_{a,t} \quad && \forall a \in \mathcal{A}^{\text{ar-pr}}_i,\ \forall t \in \mathcal{T}. && (33) \end{aligned}$$

In the real world, machines are set up in parallel, sequentially or in some combination of the first two. By setting them up in parallel, a maximum amount of massflow can be compressed, while the compression ratio may not be as high as in the sequential case. Here, the compression ratios of the machines are multiplied with each other, but the maximum amount of possible massflow is limited. In our model, we refrain from choosing a setup for the machines and overestimate the capabilities of a pressure increasing arcs in that sense, that we assume that the maximum amount of flow (parallel setting) and the highest compression ratio (sequential setting) are available at the same time. Thus, we add the following constraints

$$\sum_{m \in \mathcal{M}_i^a} P_{j,t}\, y_{m,a,t} = \tilde{\pi}_{a,t} \quad \forall a \in \mathcal{A}_i^{\text{ar-pr}},\ \forall t \in \mathcal{T}, \tag{34}$$

$$\sum_{m \in \mathcal{M}_i^a} Q_{j,t}\, y_{m,a,t} = \tilde{q}_{a,t} \quad \forall a \in \mathcal{A}_i^{\text{ar-pr}},\ \forall t \in \mathcal{T}, \tag{35}$$

$$1 + \sum_{m \in \mathcal{M}_i^a} (R_{j,t} - 1)\, y_{m,a,t} = \tilde{r}_{a,t} \quad \forall a \in \mathcal{A}_i^{\text{ar-pr}},\ \forall t \in \mathcal{T}. \tag{36}$$

The first constraint (34) determines the power available on arc $a \in \mathcal{A}^{\text{ar-pr}}$ by adding up the maximum power of the assigned machines. Analogously, the second constraint (35) determines the maximum amount of massflow that can be compressed. On the other hand, the third constraint (36) is a (conservative) approximation of the maximum compression ratio, which is used in order to avoid non-linear constraints.

Finally, the connection between pressure difference, the amount of massflow passing through a compressor machine, and the power necessary to realize it is given by the non-linear *power equation* for compressor machines

$$\tilde{\pi}_{a,t} \geq \pi_{a,t} = \frac{q_{a,t}}{\eta_{\text{ad}}} R_s T z_l \frac{\kappa}{\kappa - 1} \left[\left(\frac{p_{r,t}}{p_{\ell,t}} \right)^{\frac{\kappa-1}{\kappa}} - 1 \right],$$

where $\pi_{a,t} \in \mathbb{R}_{\geq 0}$ is the variable representing the necessary power when a massflow of $q_{a,t}$ with initial pressure $p_{\ell,t}$ shall be compressed up to $p_{r,t}$. Here, η_{ad} is the adiabatic efficiency of the compression, which we assume to be constant for all existing compressor machines, and $\kappa = 1,296$ [1].

To avoid introducing this non-linear constraint we determine a linear approximation as follows. For each artificial compressing link $a \in \mathcal{A}^{\text{ar-pr}}$ and each $t \in \mathcal{T}$, we sample 100.000 points $(p_{\ell,t}, p_{r,t}, \pi_{a,t}) \in [\underline{p}_{\ell,t}, \bar{p}_{\ell,t}] \times [\underline{p}_{r,t}, \bar{p}_{r,t}] \times [\frac{\pi_{a,t}^{\max}}{4}, \pi_{a,t}^{\max}]$, where $\pi_{a,t}^{\max}$ is the maximum possible power for a at t derived from (32) and (33), such that $p_{\ell,t} \leq p_{r,t}$ and determine the corresponding massflow $q_{a,t}$ using the original power equation. To the resulting set of 4-tuples we apply an ordinary least-squares method and determine coefficients $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ for a linear approximation, which gives rise to constraints

$$\alpha_0 + \alpha_1\, p_{\ell,t} + \alpha_2\, p_{r,t} + \alpha_3\, q_{a,t} \leq \pi_{a,t} + (1 - x_{a,t})(\alpha_0 + \alpha_1\, \underline{p}_{\ell,t} + \alpha_2\, \bar{p}_{r,t}), \tag{37}$$

$$\alpha_0 + \alpha_1\, p_{\ell,t} + \alpha_2\, p_{r,t} + \alpha_3\, q_{a,t} \geq \pi_{a,t} + (1 - x_{a,t})(\alpha_0 + \alpha_1\, \bar{p}_{\ell,t} + \alpha_2\, \underline{p}_{r,t}), \tag{38}$$

where we assume that $\alpha_1 \in \mathbb{R}_{\leq 0}$ and $\alpha_2 \in \mathbb{R}_{\geq 0}$ (otherwise we use the corresponding other bound for the coefficients of $x_{a,t}$ on the right hand sides). If the pressure increasing arc is active, it has to respect this linear approximation. Otherwise, there is no flow and the pressures at both ends are decoupled. Finally, we add the following set of constraints

$$\pi_{a,t} \leq \tilde{\pi}_{a,t} \qquad \forall a \in \mathcal{A}_i^{\text{ar-pr}}, \forall t \in \mathcal{T} \tag{39}$$

$$q_{a,t} \leq \tilde{q}_{a,t} \qquad \forall a \in \mathcal{A}_i^{\text{ar-pr}}, \forall t \in \mathcal{T} \tag{40}$$

$$p_{\ell,0}\tilde{r}_{a,t} - p_{r,t} \geq (1 - x_{a,t})(p_{\ell,0} - \bar{p}_{r,t}) \quad \forall a \in \mathcal{A}_i^{\text{ar-pr}}, \forall t \in \mathcal{T}. \tag{41}$$

The first two constraints (39) and (40) ensure that the massflow and power used for compression do not violate the upper bounds given by the machine assignments. Finally, the outgoing pressure is bounded by the product of the initial ingoing pressure at $t = 0$ and the current maximum compression ratio (41) if the corresponding arc is active. Using the pressure variables instead would again result in nonlinear constraints.

2.12.5 Compressor Arcs

Besides constraints (32) - (41), for each compressor arc $a = (\ell, r) \in \mathcal{A}^{\text{ar-co}}$ and each timestep $t \in \mathcal{T}$ we add constraints

$$p_{\ell,t} - p_{r,t} \leq (1 - x_{a,t})(\bar{p}_{\ell,t} - \underline{p}_{r,t}) \qquad \forall t \in \mathcal{T} \tag{42}$$

$$r_{a,t}^{\max} p_{\ell,t} - p_{r,t} \geq (1 - x_{a,t})(r_{a,t}^{\max} \underline{p}_{\ell,t} - \bar{p}_{r,t}) \quad \forall t \in \mathcal{T}. \tag{43}$$

If the arc is active at some point in time $t \in \mathcal{T}$, i.e., $x_{a,t} = 1$, the pressure at ℓ has to be smaller than or equal to the pressure at r. Further, we bound $p_{r,t}$ by $r_{a,t}^{\max} p_{\ell,t}$ where $r_{a,t}^{\max}$ is the maximum possible compression ratio of a at time t, which can be derived from constraints (33 and (41). If it is not active, the pressure values are decoupled and there is no massflow due to constraints (40) and (35).

Further, there may be an additional upper bound on the pressure at node r, which has to be respected if the arc is active. Let $\bar{p}_{r,t}^{\text{out}}$ denote this upper bound. We can model this requirement by:

$$p_{r,t} \leq \bar{p}_{r,t} - x_{a,t}(\bar{p}_{r,t} - \bar{p}_{r,t}^{\text{out}}) \quad \forall t \in \mathcal{T} \tag{44}$$

If such a bound is given, we shrink the sample space described in the previous section, accordingly.

2.12.6 Combined Arcs

A combined arc $a = (\ell, r) \in \mathcal{A}_i^{\text{ar-cb}}$ can be used as a regulating or a compressing arc. Hence, we first of all introduce two binary decision variables encoding in which mode it is activated.

$$x_{a,t}^{\text{rg}} + x_{a,t}^{\text{cp}} = x_{a,t} \quad \forall a \in \mathcal{A}_i^{\text{ar-cb}}, \forall t \in \mathcal{T} \tag{45}$$

All constraints (32) - (41), where $x_{a,t}$ is replaced by $x_{a,t}^{\text{cp}}$ in (33), (37), (38), and (41), are added for each combined arcs except for (40), which is replaced by

$$q_{a,t} \leq \tilde{q}_{a,t} + \bar{q}_{a,t} x_{a,t}^{\text{rg}} \quad \forall a \in \mathcal{A}_i^{\text{ar-cb}}, \forall t \in \mathcal{T}. \tag{46}$$

To capture the behaviour as regulating arc, we add constraints

$$p_{\ell,t} - p_{r,t} \geq (1 - x^{\mathrm{rg}}_{a,t})(\underline{p}_{\ell,t} - \bar{p}_{r,t}) \quad \forall t \in \mathcal{T} \tag{47}$$

analogously to (9), while for the pressure increasing arc we additionally have

$$\begin{aligned} p_{\ell,t} - p_{r,t} &\leq (1 - x^{\mathrm{cp}}_{a,t})(\bar{p}_{\ell,t} - \underline{p}_{r,t}) && \forall t \in \mathcal{T} && (48) \\ r^{\max}_{a,t} p_{\ell,t} - p_{r,t} &\geq (1 - x^{\mathrm{cp}}_{a,t})(r^{\max}_{a,t} \underline{p}_{\ell,t} - \bar{p}_{r,t}) && \forall t \in \mathcal{T}, && (49) \end{aligned}$$

analogously to (42) and (43). Further, as for the compressing arcs, there may be an additional upper bound on the pressure at node r, if compression is used. Thus, we add

$$p_{r,t} \leq \bar{p}_{r,t} - x^{\mathrm{cp}}_{a,t}(\bar{p}_{r,t} - \bar{p}^{\mathrm{out}}_{r,t}) \quad \forall t \in \mathcal{T}, \tag{50}$$

similar to (44), and also shrink the sample space from the previous section, accordingly.

2.13 Objectives

As mentioned in the introduction, we solve a hierarchical MIP formulation consisting of three levels, i.e., a tri-level mixed-integer linear program. The first level controls the slack variables for the inflow pressure bounds, while the second level controls the slack variables for the boundary values. The goal of both levels is to minimize the sum of the corresponding slack variables. The third level, which can be seen as the level actually controlling the network while the other two only ensure feasibility, controls the remaining variables and minimizes the total cost, which is given as the weighted sum of flow direction, simple state and (optional) auxiliary link changes. Thus, we can write the complete program as

$$\begin{aligned} &\min_{\sigma^p} \sum_{t \in \mathcal{T}} \sum_{v \in \mathcal{V}^{\mathrm{b}}} (\sigma^{p+}_{v,t} + \sigma^{p-}_{v,t}) \\ &\quad \min_{\sigma^d} \sum_{t \in \mathcal{T}} \sum_{v \in \mathcal{V}^{\mathrm{b}}} (\sigma^{d+}_{v,t} + \sigma^{d-}_{v,t}) \\ &\quad\quad \min_{\delta} \sum_{t \in \mathcal{T}} (\sum_{f \in \mathcal{F}} w^f \delta_{f,t} + \sum_{s \in \mathcal{S}} w^s \delta_{s,t} + \sum_{a \in \mathcal{A}^{\mathrm{ar}}} w^a (\delta^{\mathrm{on}}_{a,t} + \delta^{\mathrm{off}}_{a,t})) \\ &\quad\quad\quad \text{s.t. } (1) - (50) \end{aligned}$$

3 Heuristics

Next, we introduce two heuristics based on the MIP model as defined in Section 2. Thereby, we first try to find a solution for the lower level problem with all slack variables being fixed to 0. If no solution is found, we release the boundary value slack variables and resolve. Finally, if again no feasible solution is found, the inflow pressure slack variables are released.

First, we introduce a rolling horizon approach. We start by solving the MIP model for timesteps $\mathcal{T}_0 = \{0, 1\}$ only. Then, in each of the following $(n-1)$ iterations we solve the MIP for the model with an additional timestep being added w.r.t. the previous iteration and the binary decisions for all but the newly added timestep being fixed to the solution value from the previous iteration.

The second heuristic initially solves a special Min-Cost-Flow problem defined on $G = (\mathcal{V}, \mathcal{A})$ for each timestep $t \in \mathcal{T}$. Analyzing the in- and outflows of the fence nodes of each station in the optimal solutions, we reduce the number of possible flow directions by fixing the binary variables of flow directions for this timestep to 0, if they are not consistent with the MCF solution.

3.1 Rolling-Horizon Heuristic

The first idea to determine a feasible solution for the model from Section 2 is to use a rolling horizon approach, i.e., to iteratively create the MIP model with an additional timestep, solve it, and fix the binary decision variables of the MIP of the next iteration to the values of the best solution found in the previous iteration. This procedure is described in Algorithm 1.

Algorithm 1: Rolling Horizon Heuristic (RHH)

Input : Graph $G = (\mathcal{V}, \mathcal{A})$ with corresponding parameters, timelimit ϕ
Output: Feasible solution SOL_n or `UNSUCCESSFUL`

for $k \leftarrow 0$ **to** n **do**
  $\mathrm{MIP}_k \leftarrow$ model for $\mathcal{T}_0 := \{0, \dots, k\}$
  for $i \leftarrow 0$ **to** $k-1$ **do**
    fix binary variables for timestep i in MIP_k to SOL_{k-1}
  end
  solve MIP_k with timelimit $\frac{\phi}{n}$
  if *MIP_k is infeasible or no feasible solution is found* **then**
    return `UNSUCCESSFUL`
  else
    $\mathrm{SOL}_k \leftarrow$ best solution found for MIP_k
  end
end

return SOL_n

Here, for each MIP we set a timelimit of $\frac{\phi}{n}$. If the model is infeasible or no feasible solution has been found, we stop the algorithm and return `UNSUCCESSFUL`. Otherwise, the heuristic terminates with a feasible solution.

3.2 A Min-Cost-Flow Based Heuristic

The idea behind the second heuristic (MBH) is to decrease the number of binary variables w.r.t. the flow directions in the original MIP formulation by fixing a subset of them to 0 for each timestep. Due to the hierarchical structure of the decisions in a station, further fixations, for example for binary variables corresponding to simple states, are implied. In order to exclude certain flow directions, we solve a Min-Cost-Flow problem on the underlying graph for each timestep and analyze the in- and outflows at the fence nodes.

Next, consider the directed graph $G = (\mathcal{V}, \mathcal{A})$. For each arc $a \in \mathcal{A}^{\text{rg}} \cup \mathcal{A}^{\text{ar-mo}}$ we introduce a non-negative flow variable $f_a \in \mathbb{R}_{\geq 0}$ and for each $a \in \mathcal{A}^{\text{va}} \cup \mathcal{A}^{\text{pi}} \cup \mathcal{A}^{\text{ar-bi}}$ we introduce two non-negative flow variables $f_a^{\rightarrow}, f_a^{\leftarrow} \in \mathbb{R}_{\geq 0}$ describing the forward and the backward flow, respectively. Further, consider some $t \in \mathcal{T}$ and w.l.o.g. we have $\sum_{v \in \mathcal{V}^+} D_{v,t} \geq \sum_{v \in \mathcal{V}^-} |D_{v,t}| > 0$ and let $\chi_t := \frac{\sum_{v \in \mathcal{V}^+} D_{v,t}}{\sum_{v \in \mathcal{V}^-} |D_{v,t}|}$. Additionally, recall that for each pipe $a \in \mathcal{A}$ we are given its length $\ell_a \in \mathbb{R}_{\geq 0}$. Finally, the MCF we solve for each timestep t can be formulated as the following linear program (LP)

$$\min \sum_{a \in \mathcal{A}^{\text{pi}}} L_a (f_a^{\rightarrow} + f_a^{\leftarrow}) \tag{51}$$

$$\begin{aligned}
&\sum_{(v,r) \in \mathcal{A}^{\text{rg}} \cup \mathcal{A}^{\text{ar-mo}}} f_a + \sum_{(v,r) \in \mathcal{A}^{\text{va}} \cup \mathcal{A}^{\text{pi}} \cup \mathcal{A}^{\text{ar-bi}}} (f_a^{\rightarrow} - f_a^{\leftarrow}) \\
- &\sum_{(\ell,v) \in \mathcal{A}^{\text{rg}} \cup \mathcal{A}^{\text{ar-mo}}} f_a + \sum_{(\ell,v) \in \mathcal{A}^{\text{va}} \cup \mathcal{A}^{\text{pi}} \cup \mathcal{A}^{\text{ar-bi}}} (f_a^{\leftarrow} - f_a^{\rightarrow}) = 0 \qquad \forall v \in \mathcal{V}^0 \qquad (52)\\
&\sum_{(v,r) \in \mathcal{A}^{\text{rg}} \cup \mathcal{A}^{\text{ar-mo}}} f_a + \sum_{(v,r) \in \mathcal{A}^{\text{va}} \cup \mathcal{A}^{\text{pi}} \cup \mathcal{A}^{\text{ar-bi}}} (f_a^{\rightarrow} - f_a^{\leftarrow}) \\
- &\sum_{(\ell,v) \in \mathcal{A}^{\text{rg}} \cup \mathcal{A}^{\text{ar-mo}}} f_a + \sum_{(\ell,v) \in \mathcal{A}^{\text{va}} \cup \mathcal{A}^{\text{pi}} \cup \mathcal{A}^{\text{ar-bi}}} (f_a^{\leftarrow} - f_a^{\rightarrow}) = D_{v,t} \qquad \forall v \in \mathcal{V}^+ \qquad (53)\\
&\sum_{(v,r) \in \mathcal{A}^{\text{rg}} \cup \mathcal{A}^{\text{ar-mo}}} f_a + \sum_{(v,r) \in \mathcal{A}^{\text{va}} \cup \mathcal{A}^{\text{pi}} \cup \mathcal{A}^{\text{ar-bi}}} (f_a^{\rightarrow} - f_a^{\leftarrow}) \\
- &\sum_{(\ell,v) \in \mathcal{A}^{\text{rg}} \cup \mathcal{A}^{\text{ar-mo}}} f_a + \sum_{(\ell,v) \in \mathcal{A}^{\text{va}} \cup \mathcal{A}^{\text{pi}} \cup \mathcal{A}^{\text{ar-bi}}} (f_a^{\leftarrow} - f_a^{\rightarrow}) = \chi_t D_{v,t} \qquad \forall v \in \mathcal{V}^-. \qquad (54)
\end{aligned}$$

Note that if there is no supply or demand in a timestep, we define the right hand sides of all constraints to be 0. Now, given an optimal solution, for each gas network station G_i and each of its fence nodes $v \in \mathcal{V}_i^{\text{fn}}$ we check whether there is in- or outflow w.r.t. to G_i at v. If for

$$f_v := \sum_{(v,r) \in \mathcal{A}^{\text{ar-mo}}} f_a + \sum_{(v,r) \in \mathcal{A}^{\text{ar-bi}}} (f_a^{\rightarrow} - f_a^{\leftarrow}) - \sum_{(\ell,v) \mathcal{A}^{\text{ar-mo}}} f_a + \sum_{(\ell,v) \mathcal{A}^{\text{ar-bi}}} (f_a^{\leftarrow} - f_a^{\rightarrow})$$

we have $f_v \in \mathbb{R}_{\geq 0} \setminus [0, \varepsilon)$ we call v a MCF entry fence node and if $f_v \in \mathbb{R}_{\leq 0} \setminus (\varepsilon, 0]$ we call it an MCF exit fence node. The idea to use some $\varepsilon \in \mathbb{R}_{\geq 0}$ in this definition is that small in- and outflows could be realized by using the gas stored in adjacent pipelines, i.e., linepack.

Next, we call a flow direction $f = (f^+, f^-)$ MCF valid for station G_i and timestep t, if for each MCF entry fence node v we have $v \in f^+$ and for each MCF exit fence node v we have $v \in f^-$. As the final step of the heuristic, we solve the MIP model with some timelimit ϕ as described in Section 2, where we fix for each station G_i and each timestep $t \in \mathcal{T}$ all binary variables corresponding to flow directions which are not MCF valid to 0 if there exists at least one MCF valid flow direction. Otherwise, we do not apply any fixations for this timestep and station.

4 Solution Approach

TODO

5 Computational Results

5.1 Solution Smoothening

When using a simplex algorithm within the branch and bound process to solve our MIP model, we observed that the pressure values of the nodes of a gas network station as well as the absolute in- and outflows of its fence nodes may differ extremely from one timestep to another, even though there was no change in the flow direction or simple state. Thus, in the following postprocessing step we smooth a given solution while keeping all binary decision fixed.

To do this, we introduce the following linear program (LP). Given a feasible solution, we take the MIP model and fix all binary variables as well as all slack variables to their corresponding values. Further, for each gas network station we denote the set of timesteps for which no change happened w.r.t. to the flow direction and simple state by $\mathcal{T}_i := \{t \in \mathcal{T} \,|\, \delta_{f,t} = 0 \text{ for all } f \in \mathcal{F}_i \text{ and } \delta_{s,t} = 0 \text{ for all } s \in \mathcal{S}_i\}$. For each station G_i, each fence node $v \in \mathcal{V}_i^{\text{fn}}$, and each $t \in \mathcal{T}_i$ we then introduce four continuous variables $\delta^{q+}_{v,t}, \delta^{q-}_{v,t} \in \mathbb{R}_{\geq 0}$ and $\delta^{p+}_{v,t}, \delta^{p-}_{v,t} \in \mathbb{R}_{\geq 0}$, while for the artificial nodes $v \in \mathcal{V}_i^{\text{ar}}$ we only introduce the ladder two ones. Finally, for each station G_i the following constraints on the nodes are added

$$p_{v,t} - p_{v,t-1} + \delta^{p+}_{v,t} - \delta^{p-}_{v,t} = 0 \quad \forall v \in \mathcal{V}_i, \forall t \in \mathcal{T}_i \tag{55}$$

$$q^{\text{in}}_{v,t} - q^{\text{in}}_{v,t-1} + q^{\text{out}}_{v,t} - q^{\text{out}}_{v,t-1} + \delta^{q+}_{v,t} - \delta^{q-}_{v,t} = 0 \quad \forall v \in \mathcal{V}_i^{\text{fn}}, \forall t \in \mathcal{T}_i \tag{56}$$

Both, the pressure smoothening and the flow smoothening variable, contribute to the objective function with cost $w^{\text{sm-p}}$ and $w^{\text{sm-q}}$, respectively, i.e., the objective function of the resulting LP is

$$\min \sum_{i=1}^{m} \sum_{t \in \mathcal{T}_i} (w^{\text{sm-p}} \sum_{v \in \mathcal{V}_i} (\delta^{p+}_{v,t} + \delta^{p-}_{v,t}) + w^{\text{sm-q}} \sum_{v \in \mathcal{V}_i^{\text{fn}}} (\delta^{q+}_{v,t} + \delta^{q-}_{v,t})). \tag{57}$$

6 Conclusion and Outlook

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