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Controlling transient gas flow in complex pipeline intersection areas

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Abstract

The research on optimization of transient gas transport is still in the early stages. The optimization model presented in this paper includes a variety of different network elements. Especially compressors are described in great detail, comparable to already existing models for the less complex stationary case. In this paper, we concentrate on those parts of the network, which contain the majority of active elements, i.e., elements for which control decisions have to be taken, and are typically located at the intersection of major transportation pipelines. Due to the proximity of these elements, connecting pipes are rather short. Therefore, we use a linear approximation of the equations describing the physics of gas flow in pipelines and can formulate the optimization problem as a mixed-integer program (MIP). The majority of the discrete decisions is determined using a specialized algorithm, which repeatedly solves a stationary variant of the MIP model. Finally, we obtain a transient solution of the problem using a rolling horizon approach.

1 Introduction

Throughout the past years, the mathematics of gas transport have been an intensively studied topic. While natural gas was, is and will be one of the major energy source in Europe, making the efficient and save transport a field of high economical and political relevance, the task is also challenging from a mathematical point of view. On the one hand, the physics of gas flow are described by the Euler Equations, a set of nonlinear hyperbolic partial differential equations (PDEs), which yield even in simplified versions highly complex and computationally challenging constraints. On the other hand, controlling the single elements like valves or compressors to operate the overall network involves a vast amount of discrete decisions, making the problem also hard from a combinatorial point of view.

Historically, research focused first on the simulation of gas flow, i.e. dealing with the partial differential equations given all the discrete decisions, which has been studied for many decades already, see for example [3] and the references therein. In the recent years, the optimization of gas transport including also the combinatorial aspects gained more and more attention. In [21] a general overview over optimization problems related to natural gas is given, which includes but is not restricted to the transport of gas. Most of the corresponding literature so far considers the stationary gas transport problem, which searches for one stable network state, making an algebraic description of the gas flow possible. An overview of state-of-the-art approaches for the stationary case can be found in [13] resp. [20], which consider a huge amount of element detail and large real-world instances. The more challenging variant of the problem, which will also be the subject of this paper, is the transient gas transport problem, where the goal is to find a set of control decisions on the elements over a future time horizon. Here, research is still in somewhat early stages.

One of the first publications on transient gas transport optimization was [6], who are solving the problem of minimizing the compressor fuel costs. In their approach a combination of mixed-integer (linear) problems (MIPs), in which the non-linear constraints arising from discretizing the PDE constraints are approximating with piece-wise linear functions, and non-linear problems (NLPs) are solved in an alternating way to find a solution to the overall mixed-integer non-linear problem (MINLP) within a chosen approximation error. Other approaches tackle transient transport optimization problems, but neglect the discrete nature of some of the elements and therefore purely optimize over continuous variables, i.e. solve NLP problems. We mention as example the work of [24] and [15], who decide on the compression ratios of compressors, while minimizing again their fuel consumption. Very recently a few more studies on transient gas network optimization has been published. In [10] a specialized branching rule is used to solve a MINLP formulation of the problem with the objective to minimize fuel consumption. Another approach combining different specialized solving techniques is presented in [9], where iteratively a MIP model and NLP model are solved for each single time step, which is achieved by using a special discretization of the Euler Equations. The objective is here to comply with a set of future pressure and flow values given at the boundary nodes. At last we mention [4] who are considering the maximization of temporarily stored gas in the network while maintaining a feasible transient control of the elements. They also introduce a new discretization of the Euler Equations, which results in a formulation close to the algebraic form of the stationary model and use this to obtain globally optimal solutions. The problem is solved again by alternating between solving a MIP model, which is obtained by replacing the non-linear constraints by piece-wise linear function, and a NLP model, in which the discrete decisions from the MIP are already fixed.

Because of the complexity of the overall problem, the approaches of the above mentioned publications have all been tested on rather small instances in terms of network sizes. Furthermore, the implemented models for the single elements have been relatively simple compared to the amount of detail included in work on the stationary problem like in [13], with the exception of the non-linear PDE constraint pipelines. In contrast to this, we will study in this paper the problem of transient gas network optimization featuring lot of different elements, which have rather detailed substructures, comparable to previous stationary work. To do so, we will concentrate on those network areas containing the majority of active elements, which are mainly the intersection areas of major transportation pipelines. At these places the gas in compressed, regulated, and redirected depending in the current needs. We call these network areas *navi stations*. Due to the proximity of the elements in a navi station, pipes are very short and hence have a much smaller impact on the overall model. Therefore, we restrict ourselves to a linear and thus simple pipe model introduced in [11]. For each navi station we are given an initial state as well as future demands in terms of inflow and pressure at the boundaries. The goal is then to find a feasible control of all the network elements over time, which can be interpreted as a recommendation for network operators on how to control the network in the future, similar to a navigation system. The overall objective is to meets the demands as best as possible while trying to minimize the number of needed control changes in the single elements.

The rest of the paper is organized as follows: In Section 2 we will describe the mathematical models for all used elements and simultaneously formulate a corresponding MIP model. The preprocessing needed to convert the given compressor data into a linear description of the feasible operating range will be introduced in Section 3. However, even when using a linearized pipe model, solving the resulting MIP is quiet challenging. We therefore propose a different solution approach in Section 4 based on solving slightly adjusted versions of the presented MIP. We finish with the conclusion in Section 5.

2 Mathematical model

As usual in gas network optimization the gas network is modelled as a directed graph $G = (\mathcal{V}, \mathcal{A})$ in which the arcs $\mathcal{A}$ represent the different network elements and nodes $\mathcal{V}$ represent the junctions of arcs. We split $\mathcal{A}$ into individual sets $\mathcal{A} = \mathcal{A}^{\text{pi}} \cup \mathcal{A}^{\text{va}} \cup \mathcal{A}^{\text{rs}} \cup \mathcal{A}^{\text{rg}} \cup \mathcal{A}^{\text{cs}}$ for the network elements considered in this paper, i.e. pipes, resistors, valves, regulators, and compressor stations respectively. Note that regulators are also often named control valves in the literature, e.g. see [7] or [13]. In a similar fashion we split the node set $\mathcal{V} = \mathcal{V}^{\text{b}} \cup \mathcal{V}^{0}$ into boundary nodes and inner nodes respectively. Here, boundary nodes $\mathcal{V}^{\text{b}}$ represent those having inflow and pressure level demands value for the future time steps. We define the set of considered time steps as $\mathcal{T}_0 := \{0, \dots, k\}$ where $\mathcal{T} := \mathcal{T}_0 \setminus \{0\}$ are the future time steps having demand conditions at boundary nodes. Associated with each time step t is a value $\tau(t)$ representing the time difference in seconds from t to the initial time step 0 representing the time of the initial state.

The most important quantities we will consider to describe the gas flow are the pressure $p_{v,t}$ at each node $v \in \mathcal{V}$ and time $t \in \mathcal{T}_0$ as well as the flow $q_{a,t}$ from l to r on each non-pipe arc $(l, r) = a \in \mathcal{A} \setminus \mathcal{A}^{\text{pi}}$ and time $t \in \mathcal{T}_0$. For a pipe $(l, r) = a \in \mathcal{A}^{\text{pi}}$, we have two flow variables $q_{l,a,t}$ representing the inflow into the pipe at end node l and $q_{r,a,t}$ representing the outflow out of the pipe at end node r, similar to the model of [7] on pipes or the one of [6] on all arcs. As last important flow quantity we have the inflow at a boundary node $d_{v,t}$ entering the network for each $v \in \mathcal{V}^{\text{b}}$ and $t \in \mathcal{T}_0$. Although we are given flow demands for the future, we allow to deviate from these and hence have to have a variable capturing the actual inflow value.

We assume that there are bounds on all stated quantities, so upper and lower bounds $\bar{p}_{v,t}$ and $\underline{p}_{v,t}$ on the pressure at each node $v \in \mathcal{V}$, upper and lower bounds $\bar{q}_{v,t}$ and $\underline{q}_{v,t}$ on the flow on each non-pipe arc $a \in \mathcal{A} \setminus \mathcal{A}^{\text{pi}}$ respectively the inflow and outflow of each pipe $a \in \mathcal{A}^{\text{pi}}$ as well as upper and lower bounds $\bar{d}_{v,t}$ and $\underline{d}_{v,t}$ on the inflow at each boundary node $v \in \mathcal{V}^{\text{b}}$ to exist for each point in time $t \in \mathcal{T}_0$. Note that while the pressure is always positive the flow can be negative as well representing flow in the opposite direction, for example negative inflow at a boundary node represents flow out of the network at this node.

In the following we will describe each of the elements and at the same time introduce a MIP formulation in terms of variables and constraints. This MIP will not be solved directly, but is the basis for the three variants used in our overall solution algorithm described in Section 4.

2.1 Pipes

Gas flow in pipelines is for operational purposes modeled as one-dimensional flow in a straight line pipe of cylindric shape. When assuming a constant gas temperature T, the isothermal *Euler Equations*[18] consisting of *Continuity Equation* and *Momentum Equation* describe this flow in a pipe a as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0 \tag{1}$$

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial p}{\partial x} + \frac{\partial \rho v^2}{\partial x} + \frac{\lambda_a}{2D_a}|v|v\rho + g s_a \rho = 0. \tag{2}$$

Here x denotes the position in the pipe in meters, t the current time in seconds, ρ the density of the gas, v its velocity, D_a the diameter of the pipe and g the gravitational acceleration and λ_a the friction factor of the pipe, which we assume to depend on pipe characteristics only, see Section 2.1.1. With $s_a \in [-1, 1]$ we denote the slope of the pipe, i.e. the quotient of the acceleration increase between the pipes endpoints and the length L_a of the pipe. In order to complete the equation system describing the state variable p, ρ, and v we add the equation of state for real gases establishing the connection between p and ρ as

$$p = \rho R_s T z_a. \tag{3}$$

The two new quantities are the specific gas constant R_s, depending on the molar mass of the gas mixture which we assume to be a given constant, and the compressibility factor z_a, which we again assume to be constant and only to depend on pipe characteristics, see again Section 2.1.1.

In the following, we will drop the terms $\partial_t(\rho v)$ and $\partial_x(\rho v^2)$ as they contribute only little to the equation under normal operating conditions [7][18]. In addition, we reformulate the pipe flow equations in terms of the quantities we are interested in, i.e. pressure and mass flow q, which is defined using the cross sectional area $A_a = D_a^2 \frac{\pi}{4}$ of the cylindric pipe a as

$$q = A_a \rho v. \tag{4}$$

Then we can write (1) and (2) as

$$\frac{\partial p}{\partial t} + \frac{R_s T z_a}{A_a} \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial p}{\partial x} + \frac{\lambda_a R_s T z_a}{2 D_a A_a^2} \frac{|q|q}{p} + \frac{g s_a}{R_s T z_a} p = 0.$$

For discretization, we use the implicit box scheme introduced by [6], resp. [14], where the time domain is the already defined set of time steps $\mathcal{T}_0$ and the space is discretized along the length L_a of pipe $a = (l, r)$. Using the above introduced notation of flow into and out of a pipe as well as the function τ we are able to write the discretized model for two adjacent time point t_0 and t_1 as

$$p_{l,t_1} + p_{r,t_1} - p_{l,t_0} - p_{r,t_0} + \frac{2R_sTz_a(\tau(t_1) - \tau(t_0))}{L_aA_a}(q_{r,a,t_1} - q_{l,a,t_1}) = 0 \tag{5}$$

$$p_{r,t_1} - p_{l,t_1} + \frac{\lambda_a R_s T z_a L_a}{4D_aA_a^2}\left(\frac{|q_{l,a,t_1}|q_{l,a,t_1}}{p_{l,t_1}} + \frac{|q_{r,a,t_1}|q_{r,a,t_1}}{p_{r,t_1}}\right) + \frac{gs_aL_a}{2R_sTz_a}(p_{l,t_1} + p_{r,t_1}) = 0. \tag{6}$$

In one final step, we will linearize the Momentum Equation (6) as proposed in [11] by fixing the absolute velocity $|v|$, which is according to (3) and (4) defined as $|v| = |\frac{R_sTz_a}{A_a}\frac{q}{p}| = \frac{R_sTz_a}{A_a}\frac{|q|}{p}$, to a predefined constant in the friction term. We do this to be able to model the pipe flow in a MIP context. Also, since navi stations are usually clustered in a small geographic area pipes are relatively short in comparison to the rest of the network. Since the friction based pressure reduction is depending on the pipe length the corresponding term has much less impact than usual making the relative error stated in [11] less relevant for the overall accuracy. However, we discuss further aspects of how to deal with this error in the outlook. Using the constant absolute velocity in the friction term, the final equations for each pipe $(l, r) = a \in \mathcal{A}^{\mathrm{pi}}$ and all adjacent time points $t_0, t_1 \in \mathcal{T}_0$ are:

$$p_{l,t_1} + p_{r,t_1} - p_{l,t_0} - p_{r,t_0} + \frac{2R_sTz_a(\tau(t_1) - \tau(t_0))}{L_aA_a}(q_{r,a,t_1} - q_{l,a,t_1}) = 0 \tag{7}$$

$$p_{r,t_1} - p_{l,t_1} + \frac{\lambda_a L_a}{4D_aA_a}(|v_{l,a}|q_{l,a,t_1} + |v_{r,a}|q_{r,a,t_1}) + \frac{gs_aL_a}{2R_sTz_a}(p_{l,t_1} + p_{r,t_1}) = 0 \tag{8}$$

The constant $|v_{x,a}|$ for one of the end nodes $x \in \{l, r\}$ of pipe $a = (l, r)$ is determined based on the flow and pressure values of the given initial state, i.e.

$$|v_{x,a}| = \frac{R_sTz_a}{A_a}\frac{|q_{x,a,0}|}{p_{x,0}}.$$

2.1.1 Friction and compressibility factor

The friction factor λ, more specific the *Darcy–Weisbach friction factor*, describes the pressure drop on a pipe a caused by frictional forces and depends on the diameter D_a and integral roughness k_a of the pipe, as well as the current flow q and the dynamic viscosity η of the gas. For turbulent gas flow the most accurate description is given by the implicit Colebrook-White equation[5][8]. There exist a series of different explicit approximation formulas typically depending on the *Reynolds Number*, which describes the amount of turbulence of the flow. We use the formula of Nikuradse[16][8], which assumes infinite turbulence and makes the friction factor only depending on the constant diameter D_a and integral roughness k_a of the pipe:

$$\lambda_a = \left(2\log_{10}\left(\frac{D_a}{k_a}\right) + 1.138\right)^{-2}.$$

For the compressibility factor z we use the approximation formula developed by Papay[19][22], which by being valid up to 150 bar fits well to our considered pressure range and is given as

$$z(p) = 1 - 3.52 \frac{p}{p^{\mathrm{c}}} e^{-2.26 \frac{T}{T^{\mathrm{c}}}} + 0.247 \left(\frac{p}{p^{\mathrm{c}}} \right)^2 e^{-1.878 \frac{T}{T^{\mathrm{c}}}}.$$

Apart from the pressure p and gas temperature T it depends on the gas mixture depended pseudo-critical pressure p^{c} and temperature T^{c}, which we assume to be given constants. The constant compressibility factor z_a per pipe $a = (l, r)$ is then determined as average of the corresponding values derived from the initial state pressure values of the pipes end nodes, i.e. by $z_a = (z(p_{l,0}) + z(p_{r,0}))/2$.

2.2 Resistors

Resistors are artificial elements to model points of high friction in the network caused by all sorts of special elements like for example measuring equipment or complex local piping, which is not captured by the other considered element types but needs to be taken into account. The pressure drop induces by a resistor arc $(l, r) = a \in \mathcal{A}^{\mathrm{rs}}$ for time $t \in \mathcal{T}$ is defined by the *Darcy-Weisbach equation*[8]:

$$p_{l,t} - p_{r,t} = \frac{\zeta_a R_s T z_a}{2 A_a^2} \left(\frac{|q_{a,t}| q_{a,t}}{p_{\mathrm{in},t}} \right)$$

Here the friction factor is called *drag factor* ζ_a of the resistor and is given as a dimensionless parameter of the element. The compressibility factor z_a is determined in the same way as described for pipes, see Section 2.1.1. Also note, that the formula is flow direction dependent, where $p_{\mathrm{in},t}$ is either $p_{l,t}$ or $p_{r,t}$ depending on $q_{a,t}$ being positive or negative (for $q_{a,t} = 0$ holds $p_{l,t} = p_{r,t}$).

We use the same simplification for resistors as we already used for pipes and linearize the model by assuming a constant velocity $|v| = \frac{R_s T z_a}{A_a} \frac{|q|}{p}$, which also includes the flow direction dependent pressure value. The equations for each arc $(l, r) = a \in \mathcal{A}^{\mathrm{rs}}$ and time $t \in \mathcal{T}$ then reads

$$p_{l,t} - p_{r,t} = \frac{\zeta_a |v_a|}{2 A_a} q_{a,t}. \tag{9}$$

The constant velocity value is again calculated based on the initial element state and is defined as average of the two velocities using the pressure from the corresponding resistors end node as

$$|v_l| = \frac{R_s T z_a}{A_a} \frac{|q_{a,0}|}{p_{l,0}}$$
$$|v_r| = \frac{R_s T z_a}{A_a} \frac{|q_{a,0}|}{p_{r,0}}$$
$$|v_a| = \frac{|v_l| + |v_r|}{2}$$

2.3 Valves

Valves are active elements allowing to dynamically connect or disconnects two nodes by being open or closed respectively and thereby changing the network topology. This is captured by the binary variable $m^{\mathrm{op}}_{a,t}$. The consequence of this decision in terms of no flow for a closed valve and equal pressure at the end nodes of an open valve can for a valve arc $(l,r) = a \in \mathcal{A}^{\mathrm{va}}$ and some time $t \in \mathcal{T}$ be written as

$$p_{l,t} - p_{r,t} \leq (1 - m^{\mathrm{op}}_{a,t})(\bar{p}_{l,t} - \underline{p}_{r,t}) \tag{10}$$
$$p_{l,t} - p_{r,t} \geq (1 - m^{\mathrm{op}}_{a,t})(\underline{p}_{l,t} - \bar{p}_{r,t}) \tag{11}$$
$$q_{a,t} \leq (m^{\mathrm{op}}_{a,t})\bar{q}_{a,t} \tag{12}$$
$$q_{a,t} \geq (m^{\mathrm{op}}_{a,t})\underline{q}_{a,t}. \tag{13}$$

2.4 Regulators

A regulator or control valve is a valve with variable opening degree, used to reduce the pressure in flow direction. Like a regular valve it has the capability of changing the network topology by disconnecting its two end nodes in the closed mode. In addition it can be completely open, which is called the bypass mode, and in active mode reduce the pressure. For each mode there is a binary variable, from which exactly one is equal to 1 at a time, i.e. the regulator always has to have a unique mode

$$1 = m^{\mathrm{cl}}_{a,t} + m^{\mathrm{by}}_{a,t} + m^{\mathrm{ac}}_{a,t} \qquad \forall a \in \mathcal{A}^{\mathrm{rg}} \quad \forall t \in \mathcal{T} \tag{14}$$

The implications of each of the modes can be modeled by the following constraints for each arc $a \in \mathcal{A}^{\mathrm{rg}}$ and all times $t \in \mathcal{T}$ as

$$p_{l,t} - p_{r,t} \leq +(1 - m^{\mathrm{by}}_{a,t})(\bar{p}_{l,t} - \underline{p}_{r,t}) \tag{15}$$
$$p_{l,t} - p_{r,t} \geq +(1 - m^{\mathrm{by}}_{a,t} - m^{\mathrm{ac}}_{a,t})(\underline{p}_{l,t} - \bar{p}_{r,t}) \tag{16}$$
$$q_a \leq (1 - m^{\mathrm{cl}}_{a,t})\bar{q}_{a,t} \tag{17}$$
$$q_a \geq 0. \tag{18}$$

Note that the flow is always positive, even in bypass mode, since regulators have a flap trap, which prevent flow against the topological orientation.

2.5 Compressor stations

The final of the five different element types are the compressor stations. They are the most important elements as they increase the pressure in the network and thereby control the flow of the gas in the network. However, they are also the most complex elements having an own substructure and a lot of operation restrictions.

Structure of a compressor station Similar to regulators a compressor station $(l,r) = a \in \mathcal{A}^{\mathrm{cs}}$ has three different modes: Bypass and closed, which work in the same way as the corresponding modes for regulators, and the active mode, in which the pressure is increased in flow direction. For the active compression, the compressor station has a set of associated compressor units $\mathcal{U}_a$ to use. These are the actual pressure increasing elements, each with a separate operating range. In the compressor station, these compressor units can be combined in series and/or parallel to allow reacting to situations of different compression requirements. The set of all allowed series-parallel compressor unit combinations is called the set of configurations $\mathcal{C}_a$ for a compressor station $a \in \mathcal{A}^{\text{cs}}$, from which exactly one active configuration has to be chosen if the compressor station is in active mode. For each of these configurations $c \in \mathcal{C}_a$, we create a polytope in the space (p_l, p_r, q) describing the feasible operating range of the compressor station using configuration c. This polytope is described as a set of hyperplanes $\mathcal{H}_c = \{(w, x, y, z) \in \mathbb{R}^4\}$ encoding inequalities of the form $w \cdot p_l + x \cdot p_r + y \cdot q + z \leq 0$. The creation of the feasible operating range of the configurations of the compressor stations is described in Section 3.

Compressor station model To model the above described constraints we use a disjunctive formulation, whose LP relaxation was proved to be equal to the convex hull of its feasible points and also to be the compactest possible formulation for the convex hull in terms of number of constraints and variables[1][2]. For this we introduce for each configuration $c \in \mathcal{C}_a$ of a compressor station $(l, r) = a \in \mathcal{A}^{\text{cs}}$ as well as the bypass and closed mode corresponding binary "selection" variables $m^{\text{cf}}_{c,a,t}$, $m^{\text{by}}_{a,t}$, and $m^{\text{cl}}_{a,t}$ respectively and in addition a corresponding separate set of pressure and flow variables. The activation variables will force the pressure and flow variable of all not selected configurations or modes to be zero and only enforce the constraints of the selected configuration or mode. The introduced pressure and flow variables are the following ones:

$$
\begin{array}{llllr}
p^{\text{by}}_{a,t} & q^{\text{by}}_{a,t} & & & \text{bypass mode variables} \\
p^{\text{l-cl}}_{a,t} & p^{\text{r-cl}}_{a,t} & & & \text{closed mode variables} \\
p^{\text{l-cf}}_{c,a,t} & p^{\text{r-cf}}_{c,a,t} & q^{\text{cf}}_{c,a,t} & \forall c \in \mathcal{C}_a & \text{configuration variables}
\end{array}
$$

Note that we only have one p value for bypass mode, since here $p_l = p_r$ holds. Also there is no q variable for the closed mode, since $q = 0$ holds in this case anyway. Furthermore, all introduced variables have bounds equal to the corresponding original pressure and flow bounds $\bar{p}_{l,t}$, $\underline{p}_{l,t}$, $\bar{p}_{r,t}$, $\underline{p}_{r,t}$, $\bar{q}_{a,t}$, $\underline{q}_{a,t}$ of the compressor station $(l, r) = a \in \mathcal{A}^{\text{cs}}$, possibly enlarged to include zero. We indicate these bounds by the variable symbol combined with an overscore resp. underscore.

We are now able to state the constraints as for all $(l, r) = a \in \mathcal{A}^{\text{cs}}$ and $t \in \mathcal{T}$:

$$1 = \sum_{c \in \mathcal{C}_a} m^{\text{cf}}_{c,a,t} + m^{\text{by}}_{a,t} + m^{\text{cl}}_{a,t} \tag{19}$$

$$p_{l,t} = p^{\text{by}}_{a,t} + p^{\text{l-cl}}_{a,t} + \sum_{c \in \mathcal{C}_a} p^{\text{l-cf}}_{c,a,t} \tag{20}$$

$$p_{r,t} = p^{\text{by}}_{a,t} + p^{\text{r-cl}}_{a,t} + \sum_{c \in \mathcal{C}_a} p^{\text{r-cf}}_{c,a,t} \tag{21}$$

$$q_{a,t} = q^{\text{by}}_{a,t} + \sum_{c \in \mathcal{C}_a} q^{\text{cf}}_{c,a,t} \tag{22}$$

$$\underline{p}^{\text{l-cf}}_{c,a,t} m^{\text{cf}}_{c,a,t} \leq p^{\text{l-cf}}_{c,a,t} \leq \bar{p}^{\text{l-cf}}_{c,a,t} m^{\text{cf}}_{c,a,t} \quad \forall c \in \mathcal{C}_a \tag{23}$$

$$\underline{p}^{\text{r-cf}}_{c,a,t} m^{\text{cf}}_{c,a,t} \leq p^{\text{r-cf}}_{c,a,t} \leq \bar{p}^{\text{r-cf}}_{c,a,t} m^{\text{cf}}_{c,a,t} \quad \forall c \in \mathcal{C}_a \tag{24}$$

$$\underline{q}^{\text{cf}}_{c,a,t} m^{\text{cf}}_{c,a,t} \leq q^{\text{cf}}_{c,a,t} \leq \bar{q}^{\text{cf}}_{c,a,t} m^{\text{cf}}_{c,a,t} \quad \forall c \in \mathcal{C}_a \tag{25}$$

$$\underline{p}^{\text{by}}_{a,t} m^{\text{by}}_{a,t} \leq p^{\text{by}}_{a,t} \leq \bar{p}^{\text{by}}_{a,t} m^{\text{by}}_{a,t} \tag{26}$$

$$\underline{q}^{\text{by}}_{a,t} m^{\text{by}}_{a,t} \leq q^{\text{by}}_{a,t} \leq \bar{q}^{\text{by}}_{a,t} m^{\text{by}}_{a,t} \tag{27}$$

$$\underline{p}^{\text{l-cl}}_{a,t} m^{\text{cl}}_{a,t} \leq p^{\text{l-cl}}_{a,t} \leq \bar{p}^{\text{l-cl}}_{a,t} m^{\text{cl}}_{a,t} \tag{28}$$

$$\underline{p}^{\text{r-cl}}_{a,t} m^{\text{cl}}_{a,t} \leq p^{\text{r-cl}}_{a,t} \leq \bar{p}^{\text{r-cl}}_{a,t} m^{\text{cl}}_{a,t} \tag{29}$$

$$w \cdot p^{\text{l-cf}}_{c,a,t} + x \cdot p^{\text{r-cf}}_{c,a,t} + y \cdot q^{\text{cf}}_{c,a,t} + z m^{\text{cf}}_{c,a,t} \leq 0 \quad \forall (w,x,y,z) \in \mathcal{H}_c \quad \forall c \in \mathcal{C}_a \tag{30}$$

Note that the underscore and overscore variables represent lower bound and upper bound values of the corresponding variables.

2.6 Nodes

The nodes represent no technical elements but rather establish the connections between them. While the pressure coupling is realized by using the same pressure variables of the nodes in the constraints of all incident arcs, the mass flow values of the arcs are connected by the flow conservation in each node, meaning that the sum of incoming flows should match the sum of outgoing flows resulting in the constraints for all $t \in \mathcal{T}$:

$$\begin{aligned} &\sum_{(u,v)=a \in \mathcal{A}^{\text{pi}}} q_{r,a,t} - \sum_{(v,u)=a \in \mathcal{A}^{\text{pi}}} q_{l,a,t} \\ &+ \sum_{(u,v)=a \in \mathcal{A} \setminus \mathcal{A}^{\text{pi}}} q_{a,t} - \sum_{(v,u)=a \in \mathcal{A} \setminus \mathcal{A}^{\text{pi}}} q_{a,t} + d_{v,t} = 0 \qquad \forall v \in \mathcal{V}^{\text{b}} \end{aligned} \tag{31}$$

$$\begin{aligned} &\sum_{(u,v)=a \in \mathcal{A}^{\text{pi}}} q_{r,a,t} - \sum_{(v,u)=a \in \mathcal{A}^{\text{pi}}} q_{l,a,t} \\ &+ \sum_{(u,v)=a \in \mathcal{A} \setminus \mathcal{A}^{\text{pi}}} q_{a,t} - \sum_{(v,u)=a \in \mathcal{A} \setminus \mathcal{A}^{\text{pi}}} q_{a,t} = 0 \qquad \forall v \in \mathcal{V}^{0} \end{aligned} \tag{32}$$

2.7 Navi station

In addition to the constraints imposed by the single elements used in the network, also the navi station itself has a certain set of restrictions to the set of feasible solutions.

Navi station structure Most important are the *operation modes* $\mathcal{O}$ of the navi station from which exactly one has to be selected at each time point, and which determine the modes and configurations of all valves and compressor station and thereby prescribe the controlling capabilities. Note that not all possible mode combinations of the different elements have to be valid operation modes of the navi station. In addition to the operation mode a *flow direction* of the navi station has to be chosen from the set of flow directions of the navi station $\mathcal{F}$. The restrictions here are, that each flow direction of the station prescribes the flow patterns in terms of inflow, outflow or no flow over the boundary nodes of the station, which has to match the actual flow patterns, and that the flow direction has to fit to the selected operation mode of the navi station, where the set of feasible operation mode flow direction pairs is $\mathcal{OF}$.

Operation modes model We introduce binary variables $m^{\mathrm{om}}_{o,t}$ for each $o \in \mathcal{O}$ and $t \in \mathcal{T}_0$ representing if the operation mode has been selected, resp. if the navi station in the given operation mode at this point in time. Furthermore we define the following function $M(o,a)$ mapping operation modes to the modes and configurations they define for valves and compressor stations:

$$
\begin{aligned}
M(o,a) &:= x \text{ where } x \text{ is the mode or configuration of arc } a \\
&\qquad \text{in operation mode } o \quad \forall o \in \mathcal{O} \quad \forall a \in \mathcal{A}^{\mathrm{va}} \cup \mathcal{A}^{\mathrm{cs}} \\
\text{with} \qquad & x \in \{\text{“op”}, \text{“cl”}\} \text{ if } a \in \mathcal{A}^{\mathrm{va}} \\
& x \in \{\text{“by”}, \text{“cl”}\} \cup \mathcal{C}_a \text{ if } a \in \mathcal{A}^{\mathrm{cs}}
\end{aligned}
$$

Note, that we assume w.l.o.g. that all valves to be determined by the navi station operation modes. In really there are also valves, whose mode is already given over time and cannot be changed. However, these can be handled by preprocessing and either shrunken for open valve or just removed for closed ones.

Using $M(o,a)$ we can then state the operation mode related constraints for all $t \in \mathcal{T}$:

$$\sum_{o \in \mathcal{O}} m^{\mathrm{om}}_{o,t} = 1 \tag{33}$$

$$m^{\mathrm{op}}_{a,t} = \sum_{o \in \mathcal{O}: M(o,a) = \text{“op”}} m^{\mathrm{om}}_{o,t} \quad \forall a \in \mathcal{A}^{\mathrm{va}} \tag{34}$$

$$m^{\mathrm{by}}_{a,t} = \sum_{o \in \mathcal{O}: M(o,a) = \text{“by”}} m^{\mathrm{om}}_{o,t} \quad \forall a \in \mathcal{A}^{\mathrm{cs}} \tag{35}$$

$$m^{\mathrm{cl}}_{a,t} = \sum_{o \in \mathcal{O}: M(o,a) = \text{“cl”}} m^{\mathrm{om}}_{o,t} \quad \forall a \in \mathcal{A}^{\mathrm{cs}} \tag{36}$$

$$m^{\mathrm{cf}}_{c,a,t} = \sum_{o \in \mathcal{O}: M(o,a) = c} m^{\mathrm{om}}_{o,t} \quad \forall c \in \mathcal{C}_a \quad \forall a \in \mathcal{A}^{\mathrm{cs}} \tag{37}$$

Operation mode unavailability Certain navi station operation modes are not available at specific points in time. The basis for this is the non-availability of compressor units over time, which is part of the model input data.

As explained in Section 2.5, a configuration $c \in \mathcal{C}_a$ of some compressor station $a \in \mathcal{A}^{\mathrm{cs}}$ represents the series and/or parallel combination of a subset of the compressor stations compressor units. Hence, the unavailability of a certain compressor unit at time t results in the unavailability of all configurations which use this unit. On the next level, each navi station operation mode defines the mode and (for the active mode) the configuration of each compressor station in the navi station. Hence, all navi station operation modes using a configuration for a compressor station which is unavailable for time t will be unavailable for t, too. To implement this in the model, we just fix the variables $m^{\text{om}}_{o,t}$ for the corresponding operation mode o and time points t to zero, i.e. remove them from the model.

The unavailability of a compressor unit may not be aligned with the set of discrete time points $\mathcal{T}_0$, i.e. the unavailability period may start or stop in between two adjacent time points. To be able to tell which of the two points is then effected by this, we have to establish an interpretation for the operation mode of a navi station between two time steps. Therefore, we define that if a navi station has the operation mode A at the discrete time point t, then we also assume the station to have operation mode A in the following time interval up to the next discrete time point $t+1$.

From this definition follows, that if a navi station operation mode is not available for some $t \in (k, k+1)$ with $k, k+1 \in \mathcal{T}_0$, then the station mode is not available for time k but potentially for time $k+1$. Since for time $t \in (k, k+1)$ the active mode is determine by the time k, we only have to mark it unavailable there.

Operation mode transition times If the operation mode of a navi station is changed from mode A to some other mode B, the transition takes a given amount of time $\theta(A, B)$ which is given for each possible combination of operation modes as part of the input data. While in transition between the two modes, the navi station acts as follows: Assume the transition starts at time t_0, then for $t \in \left[t_0, t_0 + \frac{\theta(A,B)}{2}\right)$ the station uses mode A while for $t \in \left[t_0 + \frac{\theta(A,B))}{2}, t_0 + \theta(A, B)\right)$ the station uses mode B. In other words, the station stays in mode A until reaching the middle of the transition period and then changes to mode B. While being in transition, the whole transition period is blocked for other changes, i.e. two transition periods should not overlap. Since we are only able to change navi station operation modes at discrete time points, we assume that for each transition the middle point is in $\mathcal{T}_0$. This is also in line with our interpretation of navi station operation modes in between discrete time points.

In Figure 1 we see an example of a navi station mode sequence with corresponding transition times. The time points t_X represent the first $t \in \mathcal{T}_0$ in which station mode X is active, which is according to the interpretation above also the first discrete time point t in which X is active. In this example, there would be a conflict in the transition times $\theta(C, D)$ from mode C to mode D and $\theta(D, E)$ from mode D to mode E, since the two time periods overlap. Note that this is true, although for each single transition period the navi station has the correct operation mode for each point in time, i.e. first mode in the first half of the period and the second mode in the second half.

We will *not* cover the transition time restrictions in the MIP model, but will make sure our solutions respect these in a different way, see Section 4.2. For this we only need a way to check for a given sequence of station modes if all the transition times are valid, i.e. the corresponding periods do not overlap,

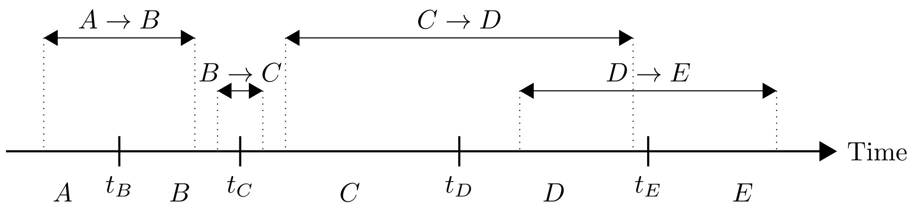


Figure 1: Transition time example with 5 station modes and 4 transitions, from which two are in conflict. The time point t_X represent the first discrete time point in $\mathcal{T}_0$ in which mode X is active.

which we do as follows: For each navi station operation mode in the sequence, we check if the time of the mode being active is at least as big as the sum of adjacent transition time parts, i.e. the sum of half the time of the transition into that mode and half of the time of the transition out of that mode. In the given example, we would have to check for mode C if $\tau(t_D) - \tau(t_C) \geq \frac{\theta(B,C)}{2} + \frac{\theta(C,D)}{2}$ holds, where $\tau(t)$ represent the time different of a time step t to the initial state time. If yes, the transition periods can never overlap, since they are centered around the time points in which the mode change happens.

Note, that for the last mode in the sequence, we do not need to do any checks at all. Here, we do not know how long the station mode is going to be active in the future and assume it is active long enough to comply with the given transition time, to have the possibility to apply navi station mode changes also at the very last time step. For the first mode, a similar assumption would be too optimistic, since then the desired navi station mode change would have already been triggered before we even start our time horizon. However, there is no known transition into the first mode, so we only have to check if half of the time of the transition into the next mode fits into the active period of the first one.

Flow directions model Similar to the ones for operations modes, we also introduce binary variables $m^{\text{fd}}_{f,t}$ representing the selection of flow direction $f \in \mathcal{F}$ at time $t \in \mathcal{T}_0$. Furthermore, we have to state the connection between the chosen flow direction and the actual boundary node inflow pattern. Therefore, we represent each flow direction f as tuple of the set of boundary nodes having inflow into the station f^+ and the set of boundary nodes having outflow out of the station f^-. Hence, using the power set $\mathcal{P}$ a flow direction f is defined as

$$(f^+, f^-) = f \in \mathcal{F} \subseteq \mathcal{P}(\mathcal{V}^{\text{b}}) \times \mathcal{P}(\mathcal{V}^{\text{b}}) \text{ where } f^+ \cap f^- = \emptyset$$

Note that if $v \notin f^+$ and $v \notin f^-$ for flow direction (f^+, f^-) the inflow of node v is zero.

Using the inflow variable $d_{v,t}$ for boundary node v and time t we can define the flow direction constraints for each $t \in \mathcal{T}$ as:

$$\sum_{f \in \mathcal{F}} m^{\text{fd}}_{f,t} = 1 \tag{38}$$

$$m^{\text{om}}_{o,t} \leq \sum_{(o,f) \in \mathcal{OF}} m^{\text{fd}}_{f,t} \quad \forall o \in \mathcal{O} \tag{39}$$

$$d_{v,t} \geq (1 - \sum_{f=(f^+,f^-) \in \mathcal{F}: v \notin f^-} m^{\text{fd}}_{f,t}) \underline{d}_{v,t} \quad \forall v \in \mathcal{V}^{\text{b}} \tag{40}$$

$$d_{v,t} \leq (1 - \sum_{f=(f^+,f^-) \in \mathcal{F}: v \notin f^+} m^{\text{fd}}_{f,t}) \bar{d}_{v,t} \quad \forall v \in \mathcal{V}^{\text{b}} \tag{41}$$

Flow direction exit pressures Apart from the consequences the flow direction choice has onto the corresponding boundary node inflows, there is also an upper pressure bound $\bar{p}^{\text{exit}}_v$ given for each boundary node $v \in \mathcal{V}^{\text{b}}$, which is only active if the node is in the outflow set of the currently active flow direction, i.e. serving as exit of the navi station. The corresponding constraint is the following for each time $t \in \mathcal{T}$

$$p_{v,t} \leq \bar{p}^{\text{exit}}_v + (1 - \sum_{f=(f^+,f^-) \in \mathcal{F}: v \in f^-} m^{\text{fd}}_{f,t})(\bar{p}_{v,t} - \bar{p}^{\text{exit}}_v) \quad \forall v \in \mathcal{V}^{\text{b}} \tag{42}$$

Flow direction conditions As last constraints concerning flow directions, there exist in some navi stations a special set of conditions $\mathcal{W}$ concerning the flow over sets of boundary nodes, which have to be met for a flow direction to be active. There are two variants of conditions, which we state for $\sim \in \{\geq, \leq\}$ being one of the supported mathematical operators as:

Flow value conditions in which the condition $w = (f, \mathcal{V}^{w_1}, \sim, M_w)$ states that the flow over a set of boundary nodes $\mathcal{V}^{w_1}$ has to be smaller resp. bigger than a given constant M_w if f is selected

Comparison conditions in which the condition $w = (f, \mathcal{V}^{w_1}, \sim, \mathcal{V}^{w_2})$ states that the flow over a set of boundary nodes $\mathcal{V}^{w_1}$ has to be smaller resp. bigger than the flow over a second set of boundary nodes $\mathcal{V}^{w_2}$ if f is selected

Note, that the flow over a set of boundary nodes $\mathcal{V}^w$ for time t is defined as $\sum_{v \in \mathcal{V}^w} |d_{v,t}|$, which is potentially a non-linear expression due to the absolute value. However, each boundary node set $\mathcal{V}^w$ which is part of some $w \in \mathcal{W}$ is known to be either a subset of f^+ or a subset of f^- of the corresponding flow direction (f^+, f^-). For this reason, we always know the sign of the flow over $\mathcal{V}^w$ in advance and hence using the following function definition for easier notation

$$\text{sgn} : \mathcal{F} \times \mathcal{V}^{\text{b}} \to \{-1, 1\}, \ \big((f^+, f^-), v\big) \to \begin{cases} 1 & \text{if } v \in f^+ \\ -1 & \text{if } v \in f^- \end{cases}$$

we can write the flow value conditions for each $t \in \mathcal{T}$ as

$$\sum_{v \in \mathcal{V}^{w_1}} \text{sgn}(f, v) d_{v,t} \geq M_w + (1 - m^{\text{fd}}_{f,t}) C_1 \qquad \forall (f, \mathcal{V}^{w_1}, \geq, M_w) \in \mathcal{W} \tag{43}$$

$$\sum_{v \in \mathcal{V}^{w_1}} \text{sgn}(f, v) d_{v,t} \leq M_w + (1 - m^{\text{fd}}_{f,t}) C_2 \qquad \forall (f, \mathcal{V}^{w_1}, \leq, M_w) \in \mathcal{W} \tag{44}$$

while the comparison conditions can be written for each $t \in \mathcal{T}$ as

$$\begin{aligned}
\sum_{v \in \mathcal{V}^{w_1}} \operatorname{sgn}(f,v) d_{v,t} - \sum_{v \in \mathcal{V}^{w_2}} \operatorname{sgn}(f,v) d_{v,t}& \\
\geq (1 - m^{\text{fd}}_{f,t}) C_3 & \qquad \forall (f, \mathcal{V}^{w_1}, \geq, \mathcal{V}^{w_2}) \in \mathcal{W} \qquad (45)\\
\sum_{v \in \mathcal{V}^{w_1}} \operatorname{sgn}(f,v) d_{v,t} - \sum_{v \in \mathcal{V}^{w_2}} \operatorname{sgn}(f,v) d_{v,t}& \\
\leq (1 - m^{\text{fd}}_{f,t}) C_4 & \qquad \forall (f, \mathcal{V}^{w_1}, \leq, \mathcal{V}^{w_2}) \in \mathcal{W}. \qquad (46)
\end{aligned}$$

Here C_1-C_4 denote big-M constants, which can be set as follows:

$$\begin{aligned}
C_1 &= \sum_{v \in \mathcal{V}^{w_1} : v \in f^+} \max(0, \underline{d}_{v,t}) - \sum_{v \in \mathcal{V}^{w_1} : v \in f^-} \min(0, \bar{d}_{v,t}) - M_w \\
C_2 &= \sum_{v \in \mathcal{V}^{w_1} : v \in f^+} \max(0, \bar{d}_{v,t}) - \sum_{v \in \mathcal{V}^{w_1} : v \in f^-} \min(0, \underline{d}_{v,t}) - M_w \\
C_3 &= \sum_{v \in \mathcal{V}^{w_1} : v \in f^+} \max(0, \underline{d}_{v,t}) - \sum_{v \in \mathcal{V}^{w_1} : v \in f^-} \\
&\quad - \left(\sum_{v \in \mathcal{V}^{w_2} : v \in f^+} \max(0, \bar{d}_{v,t}) - \sum_{v \in \mathcal{V}^{w_2} : v \in f^-} \min(0, \underline{d}_{v,t}) \right) \\
C_4 &= \sum_{v \in \mathcal{V}^{w_1} : v \in f^+} \max(0, \bar{d}_{v,t}) - \sum_{v \in \mathcal{V}^{w_1} : v \in f^-} \min(0, \underline{d}_{v,t}) \\
&\quad - \left(\sum_{v \in \mathcal{V}^{w_2} : v \in f^+} \max(0, \underline{d}_{v,t}) - \sum_{v \in \mathcal{V}^{w_2} : v \in f^-} \min(0, \bar{d}_{v,t}) \right)
\end{aligned}$$

2.8 Scenario and initial state

For the future we are given scenario values for the boundaries of the station in terms of pressure and inflow. While we are given one pressure value $\hat{p}_{v,t}$ per boundary node $v \in \mathcal{V}^{\text{b}}$ for each future time point $t \in \mathcal{T}$, the flow demands are only given for sets of boundary nodes, the so called *fence groups* of the navi station, which form the set $\mathcal{FG}$. For each set $g \in \mathcal{FG}$, which can also consist of only a single boundary node, and each future time point $t \in \mathcal{T}$ the sum of inflows should be equal to the given demand value $\hat{d}_{g,t}$.

However, we do not require strict obedience of the given values $\hat{p}_{v,t}$ and $\hat{d}_{g,t}$, but instead allow to deviate from them, which in return will be punished in the objective function. The deviation is captured in the *slack variables* $\sigma^{p+}_{v,t}$ and $\sigma^{p-}_{v,t}$ for the positive resp. negative difference of the pressure value of boundary node v at future time t from the given demand $\hat{p}_{v,t}$, as well as the variables $\sigma^{d+}_{v,t}$ and $\sigma^{d-}_{v,t}$ capturing the positive and negative contribution to the difference of the inflow demand $\hat{d}_{g,t}$ of fence group $g \in \mathcal{FG}$ of each boundary node v in the fence group g and each future time step t.

The described relations can be modeled for each future time step $t \in \mathcal{T}$ as:

$$\hat{p}_{v,t} = p_{v,t} - \sigma^{p+}_{v,t} + \sigma^{p-}_{v,t} \quad \forall v \in \mathcal{V}^{\text{b}} \tag{47}$$

$$\hat{d}_{g,t} = \sum_{v \in g} \left(d_{v,t} - \sigma^{d+}_{v,t} + \sigma^{d-}_{v,t} \right) \quad \forall g \in \mathcal{FG} \tag{48}$$

The second set of prescribed values are those of the initial state. Here we are given values for the initial pressures $p_{v,0}$ for each node $v \in \mathcal{V}$, the in- and outflow values $q_{l,a,0}$ and $q_{r,a,0}$ for each pipe $(l,r) = a \in \mathcal{A}^{\text{pi}}$, the flow values $q_{a,0}$ for each non-pipe arc $a \in \mathcal{A} \setminus \mathcal{A}^{\text{pi}}$, as well as the modes of all valves, regulators, compressors stations, the corresponding configuration for all active compressor stations and the operation mode of the navi station itself, which determine the values of the variables $m^{\text{op}}_{a,0}$, $m^{\text{by}}_{a,0}$, $m^{\text{cl}}_{a,0}$, $m^{\text{ac}}_{a,0}$ for each corresponding arc a, the values of $m^{\text{cf}}_{c,a,0}$ for each compressor station $a \in \mathcal{A}^{\text{cs}}$ and corresponding configuration $c \in \mathcal{C}_a$ as well as the values $m^{\text{om}}_{o,0}$ for each navi station operation mode $o \in \mathcal{O}$. All these variables are actually parameters of the model and fixed to the corresponding value.

2.9 Objective

As already describe above, the objective function should punish deviation from the given future scenario, while simultaneously favor those solutions with a stable control of the single elements. While the first part can easily be described by using the slack variables introduced in Section 2.8, we still need to define a measure for the stability. To do so, we first quantify the discrete changes of the control in binary variables, i.e. the change of the navi station into a new operation mode at time t in variable δ^{om}_t as well as the change into a new mode of regulator r at time t in variable $\delta^{\text{rg}}_{a,t}$. Furthermore, we capture the start of compressor unit u at time t in the variable $\delta^{\text{us}}_{u,t}$, since starting a compressor is a very time and energy intensive action and should therefore be avoided if possible. The mode changes of valves and compressor stations are not tracked separately, since each of the elements can only change its mode by changing the operation mode of the whole navi station. When denoting by $\mathcal{C}_u$ the set of configurations of the containing compressor station which uses compressor unit u, we enable the described variable behavior using the following constraints for each future time step $t \in \mathcal{T}$:

$$\delta^{\text{om}}_t \geq m^{\text{om}}_{o,t} - m^{\text{om}}_{o,t-1} \quad \forall o \in \mathcal{O} \tag{49}$$
$$\delta^{\text{om}}_t \leq 2 - m^{\text{om}}_{o,t} - m^{\text{om}}_{o,t-1} \quad \forall o \in \mathcal{O} \tag{50}$$
$$\delta^{\text{rg}}_{r,t} \geq m^{\text{cl}}_{r,t} - m^{\text{cl}}_{r,t-1} \quad \forall r \in \mathcal{A}^{\text{rg}} \tag{51}$$
$$\delta^{\text{rg}}_{r,t} \leq 2 - m^{\text{cl}}_{r,t} - m^{\text{cl}}_{r,t-1} \quad \forall r \in \mathcal{A}^{\text{rg}} \tag{52}$$
$$\delta^{\text{rg}}_{r,t} \geq m^{\text{by}}_{r,t} - m^{\text{by}}_{r,t-1} \quad \forall r \in \mathcal{A}^{\text{rg}} \tag{53}$$
$$\delta^{\text{rg}}_{r,t} \leq 2 - m^{\text{by}}_{r,t} - m^{\text{by}}_{r,t-1} \quad \forall r \in \mathcal{A}^{\text{rg}} \tag{54}$$
$$\delta^{\text{rg}}_{r,t} \geq m^{\text{ac}}_{r,t} - m^{\text{ac}}_{r,t-1} \quad \forall r \in \mathcal{A}^{\text{rg}} \tag{55}$$
$$\delta^{\text{rg}}_{r,t} \leq 2 - m^{\text{ac}}_{r,t} - m^{\text{ac}}_{r,t-1} \quad \forall r \in \mathcal{A}^{\text{rg}} \tag{56}$$
$$\delta^{\text{us}}_{u,t} \geq \sum_{c \in \mathcal{C}_u} m^{\text{cf}}_{c,a,t} - \sum_{c \in \mathcal{C}_u} m^{\text{cf}}_{c,a,t-1} \quad \forall u \in \mathcal{U}_a \forall a \in \mathcal{A}^{\text{cs}} \tag{57}$$

In order to even obtain a smooth operation for network situations without discrete mode switching we will also add variables tracking the change of the operation point of single elements, i.e. their corresponding changes in flow, incoming pressure and outgoing pressure. We do this for all elements with an actual operation point, i.e. regulators and compressor stations in active mode, while ignoring times in which the navi station operation mode or regulator mode has just been changed. The variables $\delta_{a,t}^{\text{rg-pl}}$, $\delta_{a,t}^{\text{rg-pr}}$, $\delta_{a,t}^{\text{rg-q}}$ representing changes of the incoming pressure, outgoing pressure and flow of an active regulator a respectively $\delta_{a,t}^{\text{cs-pl}}$, $\delta_{a,t}^{\text{cs-pr}}$, $\delta_{a,t}^{\text{cs-q}}$ representing the corresponding value changes of an active compressor station a can be established using the following constraints for each $(l,r) = a \in \mathcal{A}^{\text{rg}}$ and each $t \in \mathcal{T}$

$$p_{l,t} - p_{l,t-1} \le \delta_{a,t}^{\text{rg-pl}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{a,t}^{\text{rg}})(\bar{p}_{l,t} - \underline{p}_{l,t-1}) \tag{58}$$
$$p_{l,t-1} - p_{l,t} \le \delta_{a,t}^{\text{rg-pl}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{a,t}^{\text{rg}})(\bar{p}_{l,t-1} - \underline{p}_{l,t}) \tag{59}$$
$$p_{r,t} - p_{r,t-1} \le \delta_{a,t}^{\text{rg-pr}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{a,t}^{\text{rg}})(\bar{p}_{r,t} - \underline{p}_{r,t-1}) \tag{60}$$
$$p_{r,t-1} - p_{r,t} \le \delta_{a,t}^{\text{rg-pr}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{a,t}^{\text{rg}})(\bar{p}_{r,t-1} - \underline{p}_{r,t}) \tag{61}$$
$$q_{a,t} - q_{a,t-1} \le \delta_{a,t}^{\text{rg-q}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{a,t}^{\text{rg}})(\bar{q}_{a,t} - \underline{q}_{a,t-1}) \tag{62}$$
$$q_{a,t-1} - q_{a,t} \le \delta_{a,t}^{\text{rg-q}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{a,t}^{\text{rg}})(\bar{q}_{a,t-1} - \underline{q}_{a,t}) \tag{63}$$

respectively for each $(l,r) = a \in \mathcal{A}^{\text{cs}}$ and each $t \in \mathcal{T}$

$$p_{l,t} - p_{l,t-1} \le \delta_{a,t}^{\text{cs-pl}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{t}^{\text{om}})(\bar{p}_{l,t} - \underline{p}_{l,t-1}) \tag{64}$$
$$p_{l,t-1} - p_{l,t} \le \delta_{a,t}^{\text{cs-pl}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{t}^{\text{om}})(\bar{p}_{l,t-1} - \underline{p}_{l,t}) \tag{65}$$
$$p_{r,t} - p_{rl,t-1} \le \delta_{a,t}^{\text{cs-pr}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{t}^{\text{om}})(\bar{p}_{r,t} - \underline{p}_{r,t-1}) \tag{66}$$
$$p_{r,t-1} - p_{r,t} \le \delta_{a,t}^{\text{cs-pr}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{t}^{\text{om}})(\bar{p}_{r,t-1} - \underline{p}_{r,t}) \tag{67}$$
$$q_{a,t} - q_{a,t-1} \le \delta_{a,t}^{\text{cs-q}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{t}^{\text{om}})(\bar{q}_{a,t} - \underline{q}_{a,t-1}) \tag{68}$$
$$q_{a,t-1} - q_{a,t} \le \delta_{a,t}^{\text{cs-q}} + (m_{a,t}^{\text{by}} + m_{a,t}^{\text{cl}} + \delta_{t}^{\text{om}})(\bar{q}_{a,t-1} - \underline{q}_{a,t}). \tag{69}$$

Note that we needed to define the upper bound constraints for the discrete change variables δ_t^{om} and $\delta_{r,t}^{\text{rg}}$ to allow them to be only if there really is a discrete change. Otherwise it might have been possible to set the change variable to 1 although there is no actual discrete change and thereby avoid high costs imposed by the continuous change variables, which is not a desired behavior.

Finally, we are able to state our objective function, which minimizes the weighted sum of the change variable and the slack variables defined in Section 2.8

as

$$
\begin{aligned}
&\min \text{ objective} := \\
&\sum_{t\in\mathcal{T}} \Bigg(\sum_{v\in\mathcal{V}^{\mathrm{b}}} w^{\sigma\text{-p}} \cdot (\sigma^{p+}_{v,t} + \sigma^{p-}_{v,t}) + w^{\sigma\text{-d}} \cdot (\sigma^{d+}_{v,t} + \sigma^{d-}_{v,t}) \\
&\quad + w^{\mathrm{om}} \cdot \delta^{\mathrm{om}}_{t} \\
&\quad + \sum_{a\in\mathcal{A}^{\mathrm{rg}}} w^{\mathrm{rg}} \cdot \delta^{\mathrm{rg}}_{a,t} \\
&\quad + \sum_{u\in\mathcal{U}_a, a\in\mathcal{A}^{\mathrm{cs}}} w^{\mathrm{us}} \cdot \delta^{\mathrm{us}}_{u,t} \\
&\quad + \sum_{a\in\mathcal{A}^{\mathrm{rg}}} w^{\text{rg-pl}} \cdot \delta^{\text{rg-pl}}_{a,t} + w^{\text{rg-pr}} \cdot \delta^{\text{rg-pr}}_{a,t} + w^{\text{rg-q}} \cdot \delta^{\text{rg-q}}_{a,t} \\
&\quad + \sum_{a\in\mathcal{A}^{\mathrm{cs}}} w^{\text{cs-pl}} \cdot \delta^{\text{cs-pl}}_{a,t} + w^{\text{cs-pr}} \cdot \delta^{\text{cs-pr}}_{a,t} + w^{\text{cs-q}} \cdot \delta^{\text{cs-q}}_{a,t} \Bigg),
\end{aligned} \tag{70}
$$

where the w^* parameters denote the corresponding positive weights given to the single quantities.

2.10 Final model

Putting everything together, we can formulate our problem in the following transient gas flow model $\mathscr{P}$:

$$
\begin{array}{lll}
 & \min & (70) \\
\text{s.t.} & \forall t\in\mathcal{T} & (7)-(8) \quad \forall a\in\mathcal{A}^{\mathrm{pi}} \\
 & & (9) \quad \forall a\in\mathcal{A}^{\mathrm{rs}} \\
 & & (10)-(13),(34) \quad \forall a\in\mathcal{A}^{\mathrm{va}} \\
 & & (14)-(18),(51)-(56),(58)-(63) \quad \forall a\in\mathcal{A}^{\mathrm{rg}} \\
 & & (19)-(22),(26)-(29),(35)-(36),(64)-(69) \quad \forall a\in\mathcal{A}^{\mathrm{cs}} \\
\mathscr{P} & & (23)-(25),(37) \quad \forall a\in\mathcal{A}^{\mathrm{cs}} \quad \forall c\in\mathcal{C}_a \\
 & & (30) \quad \forall a\in\mathcal{A}^{\mathrm{cs}} \quad \forall c\in\mathcal{C}_a \quad \forall (w,x,y,z)\in\mathcal{H}_c \\
 & & (31),(40)-(42),(47) \quad \forall v\in\mathcal{V}^{\mathrm{b}} \\
 & & (32) \quad \forall v\in\mathcal{V}^{0} \\
 & & (33),(38) \\
 & & (39),(49)-(50) \quad \forall o\in\mathcal{O} \\
 & & (43)\vee(44)\vee(45)\vee(46) \quad \forall w\in\mathcal{W} \\
 & & (48) \quad \forall g\in\mathcal{FG} \\
 & & (57) \quad \forall a\in\mathcal{A}^{\mathrm{cs}} \quad u\in\mathcal{U}_a
\end{array}
$$

Note, that we apply the constraints only starting from time step 1 explicitly excluding the initial time step 0. We do this, since the initial pressure and flow values as well as the initial modes described in Section 2.8 are not guaranteed to fit our model and just serve as a starting point for the calculations.

3 Feasible operating range of compressor station configurations

As already explained in Section 2.5 each compressor station arc $a \in \mathcal{A}^{\mathrm{cs}}$ has an inherent substructure. It represents a set of compressor units $\mathcal{U}_a$, which are the actual compressing elements and can be combined in a series and/or parallel fashion, to either allow for a higher compression ratio, a higher flow rate or a mixture of both. The set of all feasible series-parallel combinations is called the set of configurations $\mathcal{C}_a$ of a compressor station. For each of these configurations, we will describe in this section how to obtain their polytope description given by the set of hyperplanes $\mathcal{H}_c$, which is used in the model as described in Section 2.5. The polytope of a configuration is created based on the polytopes of the compressor units, which unify the corresponding feasible operation ranges with a maximum power restriction. Note that we drop the time index in this section for the ease of notation.

3.1 Feasible operating range for a single compressor unit

A compressor unit is a combination of a single compressor machine (or just a compressor), which increases the gas pressure in flow direction, and a corresponding drive, providing the power needed to run the compressor. For each compressor machine we are given a feasible operating range as polytope in the space $(\frac{p_r}{p_l}, Q)$, where the volumetric flow rate Q is given as

$$Q = q/\rho_l, \quad \text{with } \rho_l = \frac{p_l}{R_s T z_a}.$$

Usually, the feasible operating range, sometimes also called "characteristic diagram" or "performance curve", is given as area in the dimensions (H_{ad}, Q) restricted by a set of possibly concave quadratic curves, see e.g. [8][17]. The quantity H_{ad} denotes the *specific change in adiabatic enthalpy* and is defined as

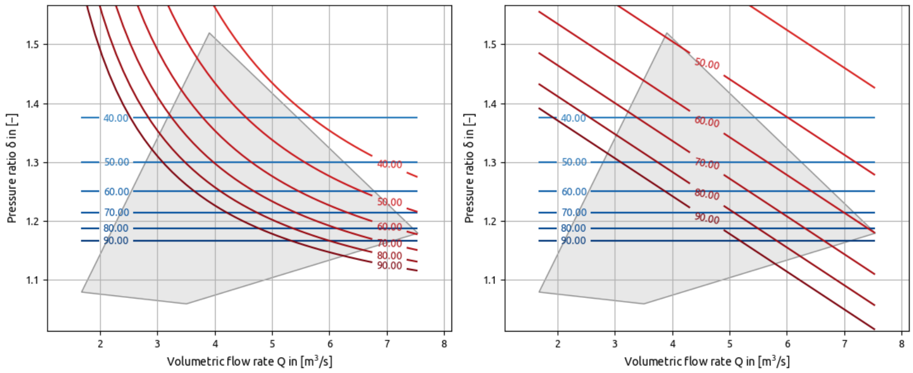
$$H_{\mathrm{ad}} = R_s T z_a \frac{\kappa}{\kappa - 1} \left[\left(\frac{p_r}{p_l} \right)^{\frac{\kappa-1}{\kappa}} - 1 \right], \tag{71}$$

using for the *isentropic exponent* κ the constant value 1.296, as stated in [8]. The transformation of such a feasible operating range using H_{ad} into the format used here is easily doable, since there is a unique transformation from H_{ad} to $\frac{p_r}{p_l}$ obtained by simply rearranging (71). The diagram then just has to be linearized by approximation or relaxation to obtain the polytope description in the desired space.

In addition to the feasible operating range polytope, each compressor machine is given an upper bound on the absolute pressure increase $\bar{\Delta}_p \geq p_r - p_l$ and an upper bound on the maximum power to use $\bar{P}$ based on the power of the compressor drive. The power needed for compression depends on the above defined H_{ad} as well as the mass flow and is given as

$$P = \frac{q H_{\mathrm{ad}}}{\eta_{\mathrm{ad}}} = \frac{q}{\eta_{\mathrm{ad}}} R_s T z_a \frac{\kappa}{\kappa - 1} \left[\left(\frac{p_r}{p_l} \right)^{\frac{\kappa-1}{\kappa}} - 1 \right]. \tag{72}$$

Here η_{ad} denotes the *adiabatic efficiency* of the compression, which in theory depends on the actual point of operation in the feasible region but is here assumed to be a given constant per compressor unit. An example of a feasible operating range of a compressor unit is given in Figure 2a, where different levels of the maximum pressure difference bound and the maximum power bound are given based on different values for the incoming pressure p_l.



(a) Original power bound (b) Linearized power bound

Figure 2: The feasible operating range of a compressor unit. The grey region shows the operating range given as a polytope. The blue lines represent the upper bound on the absolute pressure increase $\bar{\Delta}_p$, the red lines illustrate the power bound $\bar{P}$, respectively drawn for different values of the incoming pressure p_l. While the left picture shows the original non-linear non-convex power bound, the right pictures shows the linearized version, see Section 3.2

.

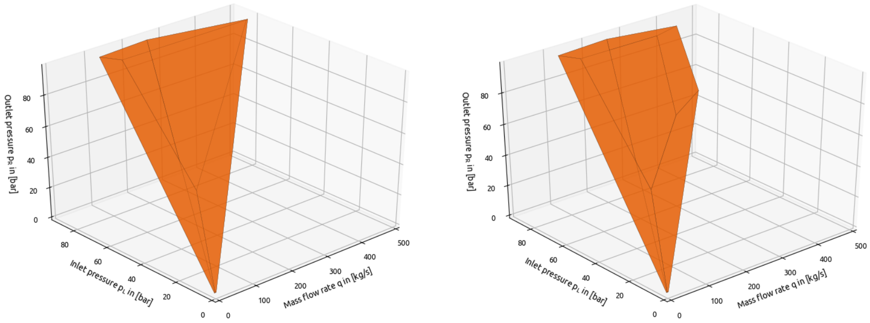
Ignoring the power bound for a moment, we will now lift the feasible operating range into the (p_l, p_r, q) space we are interested in. Therefore, we first transform each of the faces $a_0 + a_1 Q + a_2 \frac{p_r}{p_l} \leq 0$ of the original polytope into constraints of the higher dimensional space using the equation of state for real gases (3):

$$\begin{aligned} & a_0 + a_1 Q + a_2 \frac{p_r}{p_l} \leq 0 \\ \Leftrightarrow \quad & a_0 + a_1 \frac{q}{\rho_L} + a_2 \frac{p_r}{p_l} \leq 0 \\ \Leftrightarrow \quad & a_0 + a_1 \frac{q R_s T z_a}{p_l} + a_2 \frac{p_r}{p_l} \leq 0 \\ \Leftrightarrow \quad & a_0 p_l + a_1 R_s T z_a q + a_2 p_r \leq 0 \\ \Leftrightarrow \quad & \tilde{a}_0 p_l + \tilde{a}_1 p_r + \tilde{a}_2 q \leq 0. \end{aligned}$$

To bound the polyhedron described by the new constraints, we add the restriction of the absolute pressure difference as well as two pressure bounds of the end nodes of the compressor station arc $(l, r) = a \in \mathcal{A}^{\text{cs}}$ containing this machine

$$\begin{aligned} p_r - p_l &\leq \bar{\Delta}_p \\ p_l &\geq \underline{p}_l \\ p_r &\leq \bar{p}_r. \end{aligned}$$

A picture of the three dimensional polytope resulting from the feasible operating range of Figure 2 can be seen in Figure 3a.



(a) Without power bound

(b) With power bound

Figure 3: The feasible operating range of a compressor unit in the space (p_l, p_r, q), computed from the two dimensional operating range shown in Figure 2. While the left picture shows the lifted polytope based on the original feasible operating range, the maximum absolute pressure difference and the end nodes pressure bounds, the right pictures also includes the linearized power bound, see Section 3.2 .

3.2 Power bound linearization

Until now, we have ignored the maximum power constraint that restricts the value of the available power P for compression in the three dimensional feasible operation range polytope. Figure 2a shows the constraint $P \leq \bar{P}$ cuts into the original two dimensional feasible operating range in a non-linear and non-convex fashion. The same holds for the feasible operating range representation in (p_l, p_r, q). In the following, we are going to derive a linear approximation to this constraint that can then be added to the operating range polytope.

Therefore, we generate a set of N random sample points from within the three dimensional operating range polytope, which we represent by the vectors $\mathbf{p_l}, \mathbf{p_r}, \mathbf{q} \in \mathbb{R}^N$. For each point, we then determine the corresponding compressor power using Equation (72) and store these values again in a vector $\mathbf{P} \in \mathbb{R}^N$. The goal is now to obtain a linear approximation of the power function, i.e. an approximation of the form

$$\mathbf{P} \approx a_0 + a_1 \mathbf{p_l} + a_2 \mathbf{p_r} + a_3 \mathbf{q}.$$

We achieve this by applying an ordinary least-squares method in order to determine the coefficients of the linear function as

$$\min_{a_0, a_1, a_2, a_3} || \, \mathbf{P} - (a_0 + a_1 \mathbf{p_l} + a_2 \mathbf{p_r} + a_3 \mathbf{q}) \, ||^2 \, .$$

Finally, we can formulate the new linearized power bound constraint based on the obtained solution values $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3)$ as

$$\bar{a}_0 + \bar{a}_1 p_l + \bar{a}_2 p_r + \bar{a}_3 q \leq \bar{P},$$

which is then added to the three dimensional polyhedron to create the final three dimensional polytope description of the feasible operating range of a compressor unit. An example the final polytope is illustrated in Figure 3b, while the linearized power bound projected to the original two dimensional operating range can be seen in Figure 2b.

3.3 Feasible operating range for a compressor station configuration

In the final step, we will now create the polytope description for each configuration by combining the polytopes of the used compressor units. The procedure was originally described in [12], while we exactly follow the steps of the variant described in [23].

Each configuration c is given as a serial sequence $S_1, \dots, S_{n_c}$ of parallel compressor machine arrangements, where combining compressors in series allows for higher output pressures by multi-step compression while parallel compression increases the throughput in terms of flow. We call such a parallel machine arrangement *stage* and denote by $\mathcal{U}_S$ the set of compressor units combined in stage $S \in \{S_1, \dots, S_{n_c}\}$.

We will now start with definition the feasible operation range polytope P_S of such a stage $S \in \{S_1, \dots, S_{n_c}\}$. Denoting by P_u the corresponding polytopes determine above for each compressor unit $u \in \mathcal{U}_S$, we can described P_S as

$$
\begin{aligned}
P_S := \{ \quad (p_l, p_r, q) \quad | \quad \forall u \in \mathcal{U}_S \quad & \exists (p_{l,u}, p_{r,u}, q_u) \in P_u \\
\text{with } p_l &= p_{l,u} \quad \forall u \in \mathcal{U}_S, \\
p_r &= p_{r,u} \quad \forall u \in \mathcal{U}_S, \\
q &= \sum_{u \in \mathcal{U}_S} q_u \quad \}.
\end{aligned}
$$

In words, a valid operation point of the stage is represented by each unit operating at the same incoming and outgoing pressures, while the the mass flows add up.

In a similar fashion, we can then define the polytope P_c for the overall configuration. Here, the mass flow through all the stages stays the same, while the outgoing pressure of some stage S_i in the sequence has to match the incoming pressure of the subsequent stage S_{i+1}. Using this logic P_c can be defined as

$$
\begin{aligned}
P_c := \{ \quad (p_l, p_r, q) \quad | \quad \forall S \in \{S_1, \dots, S_{n_c}\} \quad & \exists (p_{l,S}, p_{r,S}, q_S) \in P_S \\
\text{with } p_l &= p_{l,S_1}, \\
p_r &= p_{r,S_{n_c}}, \\
p_{r,S_i} &= p_{l,S_{i+1}} \quad \forall i \in \{1, \dots, n_c - 1\}, \\
q &= q_S \quad \forall S \in \{S_1, \dots, S_{n_c}\} \quad \}.
\end{aligned}
$$

Finally, the set of hyperplanes $\mathcal{H}_c$ used in Section 2.5 to define the feasible operating range of the configuration c are simply the facets of the polytope P_c representing the feasible operating range.

Note, that due to the symmetric polytope creation the parallel compressor units of a stage do not need a specific order. In contrast, the serial stage sequence is indeed important since in general two sequences using the same stages but in a different order will have different operating ranges.

4 Specialized navi station algorithm

The MIP model $\mathscr{P}$ presented in Section 2.10 turns out to be quite challenging according to our experiments even though we are only considering parts of the original gas network. We therefore created a specialized algorithm to solve the problem for navi stations. The baseline insight of it is that the elements in navi stations are all very close to each other, making the corresponding pipes inside the station relatively short. That means, their capability to store gas, which is often referred to as *linepack*, is insignificant in comparison to the long pipelines in between the navi stations. Thus there is not possibility to "prepare for the future" in terms of preparing for upcoming critical demand situations by already transporting gas from or to right network areas, at least not inside the navi station itself. The station basically has to adjust its control to what is needed at a point in time in terms of pressure and inflow demands at its boundaries.

This led us to the idea of splitting the time coupled model $\mathscr{P}$ into individual stationary models to determine the best operation mode for each individual time step. We then use this information to find a sequence of operation modes over time which enables us to meet the boundary demands for each time step as good as possible while respecting also the transition time condition, which has not been modeled in $\mathscr{P}$ explicitly, see the corresponding part of Section 2.7. Note however, that our algorithm does not have the guarantee to find a globally optimal solution anymore, which we would have been the case when directly solving $\mathscr{P}$ enhanced by a proper description of the operation mode transition time constraints.

Since our goal is to find a feasible solution for the presented model $\mathscr{P}$ of Section 2.10, we will not only have to determine the navi station operation modes, but also all other involved quantities. Hence, after determining the navi station operation modes we still need to calculate a transient version of $\mathscr{P}$, which for example also takes care of minimizing the differences in the operation points of the single elements, see Section 2.9. However, since we will prescribe the obtained operation modes here the costs resulting from the operation point differences have a lower priority. This works fine, as long as the operation mode depending objective function weights w^{om} for changing to a new operation mode and w^{us} for starting a new compressor unit by choosing a operation mode with a corresponding active compressor station configuration dominate the other weights, which we assume to be the case.

In this Section we will first introduce in 4.1 the three different variants of the model $\mathscr{P}$ we will use in our algorithm. Afterward, we will present our algorithm to determine navi station operation modes in 4.2 and finish with the so-called smoothening procedure in 4.3, in which we will find the feasible transient solution to our problem.

4.1 Model variants

As mentioned above, we will not solve the complete transient model $\mathscr{P}$ directly, but use the following variants of it in our overall solution approach.

$\mathscr{P}^{\text{s}}$ – **Stationary model** For the first variant we solve a **s**tationary version of the model, do determine the best operation mode for one independent time step t. This mainly effects the pipe model. In the stationary case a pipe has no longer the possibility to store gas, since the incoming and outgoing pipe flows are balanced, i.e. $q_{l,a,t} = q_{r,a,t} = q_{a,t}$ for all pipes $(l, r) = a \in \mathcal{A}^{\mathrm{pi}}$. This is due to the Continuity Equation, in which $\partial_t p = 0$ holds as for all time dependent derivatives resulting in the mass flow balance. Hence, the Continuity Equations is no longer part of the model and the stationary Momentum Equation (8) for pipe $(l, r) = a \in \mathcal{A}^{\mathrm{pi}}$ and the stationary time step t we are considering can be stated as

$$p_{r,t} - p_{l,t} + \frac{\lambda_a L_a}{4 D_a A_a} \left(|v_{l,a}| + |v_{r,a}|\right) q_{a,t} + \frac{g s_a L_a}{2 R_s T z_a} \left(p_{l,t} + p_{r,t}\right) = 0.$$

Note that we still calculate the fixed velocity based on the initial pressure and flow values from time step 0, since we are looking for a feasible solution for the original model $\mathscr{P}$ in the end.

For all other elements the constraints only consider exactly one time step and we therefore simply apply them for the one time step t of the stationary case. The only other part to adjust is the objective function, where we keep the penalties $\sigma^{p+}_{v,t}$, $\sigma^{p-}_{v,t}$, $\sigma^{d+}_{v,t}$, and $\sigma^{d-}_{v,t}$ for each boundary node $v \in \mathcal{V}^{\mathrm{b}}$ since they are defined based on each individual time step t. In addition, we will keep tracking the change to a new navi station operation mode in variable δ^{om}_t as well as the start of new compressor units using $\delta^{\mathrm{us}}_{u,t}$ for all $u \in \mathcal{U}_a\, a \in \mathcal{A}^{\mathrm{cs}}$ by calling the model with the parameter `prevMode` representing the navi station operation mode of the previous time step. The other variables tracking changes in regulator modes $\delta^{\mathrm{rg}}_{a,t}$ as well as the current point of operation of regulators and compressor stations as $\delta^{\text{rg-pl}}_{a,t}$, $\delta^{\text{rg-pr}}_{a,t}$, $\delta^{\text{rg-q}}_{a,t}$, $\delta^{\text{cs-pl}}_{a,t}$, $\delta^{\text{cs-pr}}_{a,t}$, and $\delta^{\text{cs-q}}_{a,t}$ for all $a \in \mathcal{A}^{\mathrm{rg}}$ resp. $a \in \mathcal{A}^{\mathrm{cs}}$ will be remove from the model, as well as the constraints (51)-(56) resp. (58)-(69) defining their behavior. The final stationary objective function for the time step t under consideration reads as

$$\begin{aligned}
\min \quad & \sum_{v \in \mathcal{V}^{\mathrm{b}}} w^{\sigma\text{-p}} \cdot (\sigma^{p+}_{v,t} + \sigma^{p-}_{v,t}) + w^{\sigma\text{-d}} \cdot (\sigma^{d+}_{v,t} + \sigma^{d-}_{v,t}) \\
& + w^{\mathrm{om}} \cdot \delta^{\mathrm{om}}_t \\
& + \sum_{u \in \mathcal{U}_a, a \in \mathcal{A}^{\mathrm{cs}}} w^{\mathrm{us}} \cdot \delta^{\mathrm{us}}_{u,t}
\end{aligned}$$

$\mathscr{P}^{\mathbf{f}}$ – Transient with fixed operation modes and flow directions For this variant, we are given a **f**ixed operation mode o_t and flow direction f_t for all future time steps $t \in \mathcal{T}$. This is used in our algorithm to finally determine all transient quantities, after the decision for a navi station operation mode and flow direction for each time step has been made. By fixing the corresponding variables $m^{\mathrm{om}}_{o_t,t}$ and $m^{\mathrm{fd}}_{f_t,t}$ to 1 for all future time steps $t \in \mathcal{T}$, the majority of binary variables can be replaced by constants, since the operation modes already decides about valve and compressor station modes, as well as the configuration of all active compressor stations. Only the binary variables $m^{\mathrm{ac}}_{a,t}$, $m^{\mathrm{by}}_{a,t}$ and $m^{\mathrm{cl}}_{a,t}$ for the mode of a regulator $a \in \mathcal{A}^{\mathrm{rg}}$ are still to be decided.

In addition, a lot of implicating big-M constraints can already be resolved, i.e. we can remove the current formulation and just add the implied constraints, if the corresponding condition is fulfilled. Examples for this are the constraints (10)-(13) describing the valve behavior or the flow direction conditions (43)-(46). Furthermore, we do no longer need to use a disjunctive model, but can apply the corresponding constraints (20)-(30) of the active mode and/or configuration of time step t directly to the variables $p_{l,t}$, $p_{r,t}$ and $q_{a,t}$ for each compressor station $(l, r) = a \in \mathcal{A}^{\mathrm{cs}}$.

$\mathscr{P}^{\mathbf{sf}}$ **– Stationary with fixed operation mode** As a last variant we basically combine the two variants above and use the **s**tationary version of the model with already **f**ixed operation mode. Note, that in contrast to model $\mathscr{P}^{\mathrm{f}}$ the flow direction of the navi station is not already given, which results in more binary variables and still to decide big-M constraints. However, this variant still results in a very small and rather simple model and we can therefore solve it very often to test the appropriateness of a given operation mode for a certain time step.

4.2 Determining navi station operation modes

Our algorithm to determine the operation modes of the navi station is split into two steps: First, we create an initial solution by a greedy, forward oriented procedure presented in 4.2.1. We then in a second step improved this solution by testing, if certain operation modes can be replaced by similar ones to find a better sequence of operation modes over time. This second step is described in 4.2.2.

4.2.1 Initial solution creation

To find a first feasible sequence of navi station operation modes over time, we follow a rather simple idea. In order to keep the number of needed operation mode changes small, we determine an operation mode for time step t by first testing the used operation mode of the previous time step $t-1$ using $\mathscr{P}^{\mathrm{sf}}$. Only if this previously used operation mode does not yield a sufficiently cheap solution in terms if the objective function value, we use the general stationary model $\mathscr{P}^{\mathrm{s}}$ to determine the best operation mode for t. By this mechanic, we also reduce the amount of calls to the $\mathscr{P}^{\mathrm{s}}$ model, which are in general much more expensive in terms of computing time then calls to the $\mathscr{P}^{\mathrm{sf}}$ model. A detailed description is given as Algorithm 1.

There are a few things to note about Algorithm 1. First, the parameter `prevMode` given to the calls for solving the models $\mathscr{P}^{\mathrm{sf}}$ in line 5 and $\mathscr{P}^{\mathrm{s}}$ in line 13 is the navi station operation mode of the previously time step, which we need to determine the operation mode change and compressor unit start variables δ_t^{om} resp. $\delta_{u,t}^{\mathrm{us}}$ for some unit $u \in \mathcal{U}_a\, a \in \mathcal{A}^{\mathrm{cs}}$ and $t \in \mathcal{T}$ as explained in Section 4.1. In the call to $\mathscr{P}^{\mathrm{s}}$ we furthermore give the parameter `validModes`, which replaces the set of valid navi station operation modes $\mathcal{O}$. We also call the functions `modeAvailable` and `transitionsWork` in Algorithm 1. These refer to the navi station operation mode unavailability and the transition time restriction introduced in Section 2.7, where `modeAvailable` checks for a given operation mode o and time t if o is available at t and `transitionsWork` performs the checks described in Section 2.7 to test if a given navi station operation mode sequence is valid regarding the corresponding transition times. In addition, Algorithm 1 uses a function called `notSoonInfeasible`. Here, we check if choosing a new operation mode for time t would result in an infeasibility at one of the subsequent time steps caused by a combination of operation mode unavailability and too long transition times. More specifically, we check if the new operation mode for time

Data: Operation mode o_0 of the navi station in the initial state
Result: A list of navi station operation modes for each time $t \in \mathcal{T}_0$

```
1  operationModes ⟵ list()
2  operationModes.add(o_0)
3  for t ∈ 𝒯 do
4      oldMode ⟵ operationModes.last()
       // call 𝒫^sf with fixed operation mode oldMode for time t
5      oldModeFeasible, oldModeCost
         ⟵ 𝒫^sf (oldMode, t, prevMode = oldMode)
6      if oldModeFeasible
         and modeAvailable(oldMode,t)
         and oldModeCost < w^om then
           // operation mode of t − 1 is also good for t
7          operationModes.add(oldMode)
8      else
           // operation mode of t − 1 is NOT good for t
           // ⇒ search best possible valid stationMode for t
9          validModes ⟵ list()
10         for o ∈ 𝒪 do
11             if modeAvailable(o,t)
                 and transitionsWork(concat(operationModes, list(o)))
                 and notSoonInfeasible(t, newMode = o, oldMode = oldMode)
               then
12                 validModes.append(o)
           // find best from validModes by calling 𝒫^s for time t
13         bestMode, bestModeFeasible, bestModeCost
             ⟵ 𝒫^s (t, 𝒪 = validModes, prevMode = oldMode)
14         if not bestModeFeasible then abort without solution
           // bestMode is best choice for t
15         operationModes.add(bestMode)
16 return operationModes
```

Algorithm 1: Initial solution creation

t will become unavailable in one of the future time steps and if yes, if there is enough time left to transition into another operation mode until then, also taking into account the time needed by the transition from the old operation mode at time $t-1$ to this new operation mode at time t. This look into the near future turned out to be necessary according to our computational experiments in order to avoid that Algorithm 1 gets stuck in infeasible situations. However, apart from this check, the algorithm only consider the very time point t it is currently searching a new operation mode for.

Two other lines in the algorithm need some further explanation. In line 6 we decide if the previous navi station operation mode is good enough for the current time step by comparing its stationary objective function value against the cost of an operation mode change w^{om}. If the objective function value is indeed smaller we know that the previous operation mode is the best option considering this individual time step given the chosen operation modes for the past time steps, since each other operation mode would at least have to pay the penalty of w^{om}

for changing the operation mode. As a last point to mention, Algorithm 1 may abort without a feasible solution in line 14. This is no proof of infeasibility, since in theory cases are possible, where we abort although a feasible solution exists. However, in all of our test cases we never aborted the algorithm at this point after we added to check using the `notSoonInfeasible` function. Furthermore, the combination of transition times and unavailable operation modes can lead to very hard to find feasible solutions, which makes the design of an algorithm performing reasonably fast on average but guaranteeing to find all feasible solutions a challenge. Therefore, we leave this problem open for future research.

4.2.2 An improvement heuristic

After we have found a feasible solution using Algorithm 1, we now look for further improvements of it. The way we implemented it, Algorithm 1 only considers individual time steps to decide which operation mode to choose for each time step. However, we can easily imagine a situation in which the operation mode o_1 found by $\mathscr{P}^{\mathrm{s}}$ is best for time step t, but another operation mode o_2 is slightly better for all subsequent time steps and would have been the overall better choice at time t. We might even be able to avoid operation mode changes, if o_1 would become unavailable in the future, while o_2 has only slightly worse objective function values, but stays available.

To deal with these situations, we created for a given feasible solution the improvement heuristic stated as Algorithm 2. Here the idea is, to identify all sequences of identical operation modes over time in the solution. We call these sequences stable phases or just *phases* of a feasible solution represented by a sequence of navi station operation modes over time. Obviously, the switch from one phase to the subsequent one happens if the navi station operation mode changes to a new mode. For each of these phases we then check if we can replace the operation mode of the whole phase with a similar one being more beneficial in terms of the objective function value.

We obtain these similar navi station operation modes from the call of the function `convexCombination`, which is the key feature of Algorithm 2. To define it, we use the the function $M(o,a)$ returning the mode or active configuration of a valve or compressor station a in operation mode o, see Section 2.7. Furthermore, we denote by $\mathcal{U}(x)$ the compressor units used in mode or configuration $x \in \{\text{“by”}, \text{“cl”}\} \cup \mathcal{C}_a$ for some compressor station $a \in \mathcal{A}^{\mathrm{cs}}$, where $\mathcal{U}(\text{“by”}) = \mathcal{U}(\text{“cl”}) = \emptyset$. Then we first define the function `convexCombination` on a tuple (x,y) with $x, y \in \{\text{“by”}, \text{“cl”}\} \cup \mathcal{C}_a$ as

$$\begin{aligned}\texttt{convexCombination}(x,y) := \{x,y\} \cup \\ \Big\{c \in \mathcal{C}_a \quad | \quad \forall u \in \mathcal{U}(x) \cap \mathcal{U}(y) : u \in \mathcal{U}(c) \\ \wedge \forall u \in \mathcal{U}(c) : u \in \mathcal{U}(x) \cup \mathcal{U}(y) \Big\}.\end{aligned}$$

Note, that $\texttt{convexCombination}(x,y) \subseteq \{\text{“by”}, \text{“cl”}\} \cup \mathcal{C}_a$ holds. Then we are ready to define `convexCombination` on a tuple (o_1, o_2) of navi stations operation

Data: A sequence S of valid navi station operation modes over time
Result: A valid sequence S^* of navi station operation modes with $\text{obj}(S^*) \leq \text{obj}(S)$

backwards $\longleftarrow$ True
while *not having two iterations without improvements* **do**
  changeTimes $\longleftarrow$ list of times t with $S[t-1] \neq S[t]$
  if *backwards* **then** reverse(changeTimes)
  for $t \in$ *changeTimes* **do**
    if $S[t-1] = S[t]$ **then** continue
    if *backwards* **then**
      phaseToReplace $\longleftarrow$ phase ending at $S[t-1]$
    else
      phaseToReplace $\longleftarrow$ phase starting from $S[t]$
    S^{best}, bestImprovement $\longleftarrow$ list(), 0.0
    for *newMode* $\in$ *convexCombination($S[t-1]$, $S[t]$)* **do**
      $S^{\text{new}} \longleftarrow S$.replace(phaseToReplace, newMode)
      improvement $\longleftarrow$ obj(S) - obj(S^{new})
      if *allModesAvailable(S^{new})*
        and *transitionsWork(S^{new})*
        and *improvement* $>$ *bestImprovement* **then**
        S^{best}, bestImprovement $\longleftarrow$ S^{new}, improvement
    if *bestImprovement* > 0 **then**
      `// Found improvement in this interation!`
      $S \longleftarrow S^{\text{best}}$
  backwards $\longleftarrow$ not backwards
$S^* \longleftarrow S$

Algorithm 2: Improvement heuristic

modes as

$$\begin{aligned}\texttt{convexCombination}(o_1, o_2) := \Big\{ & o \in \mathcal{O} \mid \\ & \big(\forall a \in \mathcal{A}^{\text{va}} : M(o,a) = M(o_1, a) \quad \vee \quad \forall a \in \mathcal{A}^{\text{va}} : M(o,a) = M(o_2, a)\big) \\ \wedge \quad & \forall a \in \mathcal{A}^{\text{cs}} : M(o,a) \in \texttt{convexCombination}\big(M(o_1,a), M(o_2,a)\big)\Big\}.\end{aligned}$$

Note here, that while we allow a compressor station a to have a configuration using a compressor unit set "in between" the used compressor unit set of the configurations used in o_1 and o_2 for a, we only allow the exact valve mode combination used in o_1 or the one used in o_2. The reason for this is, that a valve mode combination enables a very specify set of paths through each navi station, and it is very unlikely that a valve mode set obtained from combining the modes used in the two given operation modes yields operation modes, which are able to handle the same demand situation. Since, each of the modes obtained by calling `convexCombination` is tested in Algorithm 2, we hereby restrict the result set to the most promising candidates.

Apart from calling `convexCombination`, Algorithm 2 uses the two functions `transitionsWork`, which works in the same way as described for Algorithm 1 above and `allModesAvailable`, which is similar to `modeAvailable` from Algorithm 1, but instead of checking the availability of a given operation mode o for time t checks the availability of a whole sequence of operation modes at the times corresponding to the position in the sequence. Furthermore, we evaluated the objective function value of a sequence of operation modes using the function `obj` by successively calling $\mathscr{P}^{\mathrm{sf}}$ for each operation mode and time corresponding to its position in the sequence. If one of the models turns out to be infeasible, the returned objective function value will be infinity.

Finally, we note that we decided to start the algorithm in the backwards oriented mode, where we test to replace the operation mode of a phase with an operation mode obtained by combining this operation mode with the one of the subsequent phase, since the initial solution is obtained by Algorithm 1 which is operated in a forward direction. Furthermore, we highlight that Algorithm 2 has the potential to reduce the total number of needed operation mode changes, since the two original operation modes are always part of the result of `convexCombination`. In addition, it is possible that the an operation mode change from the loop of line 5 has already been removed in the previous iteration by replacing one of the involved operation modes with the other one, which makes the check in line 6 necessary.

4.3 Transient solution smoothening

As a final step of our specialized navi station algorithm, we solve the transient model variant $\mathscr{P}^{\mathrm{f}}$ with fixed navi station operation modes and flow directions. We obtain both from the stationary model solutions for each time step created in the previous steps of the algorithm. We expect the transient solution states to be more similar in general and in case of changing conditions to be more smooth compared to the series of stationary solution states. This is due to the missing penalty of operation point changes in the independent stationary models, which may result in considerable different solution states, for example in terms of the overall pressure level, even if the demand situation as well as the determined navi station operation mode and flow direction are the same.

In our computational experiments we observed that even though most of the binary decision variables of $\mathscr{P}$ are fixed in $\mathscr{P}^{\mathrm{f}}$, only a limited number of time steps can be solve for large navi stations. Therefore, we use a *rolling horizon* approach to solve $\mathscr{P}^{\mathrm{f}}$, which is described in Algorithm 3. Here, we specify a small fixed time horizon size h, which represent the number of time steps to solve in model $\mathscr{P}^{\mathrm{f}}$ including the time step for the given initial state. We then solve a series of models $\mathscr{P}^{\mathrm{f}}$, while always fixing the earliest time step and shifting the time horizon by 1 in each iteration. In the function call to solve $\mathscr{P}^{\mathrm{f}}$ in Algorithm 3, we give the subsequence of navi station operation modes and flow directions, corresponding to and also encoding the current time horizon to solve. Furthermore, we specify the state to use as fixed initial state.

The main benefit of this method is, that increasing the size $|\mathcal{T}_0 := \{0, \ldots, k\}|$ of the overall time horizon only increases the number of equally sized and therefore similar complex MIP models to solve rather then increasing the complexity of the model, which may lead to an exponential increase in runtime.

Data: A sequence S of tuples of navi station operation modes and flow directions over time as well as a time horizon size h

Result: A set of transient solution states for all $t \in \mathcal{T}_0$

if $h \geq |\mathcal{T}_0|$ **then**
    `// overall time horizon covered by smoothening time horizon`
    return $\mathscr{P}^{\mathrm{f}}$ (S, initialState = initialState)
solutionStates $\longleftarrow$ list()
solutionStates.add(initialState)
currTime $\longleftarrow$ 0
while *currTime* + $h \leq |\mathcal{T}_0|$ **do**
    $S^{\mathrm{h}} \longleftarrow S$.slice(currTime, currTime + h)
    thisTimeStates $\longleftarrow$ $\mathscr{P}^{\mathrm{f}}$ (S^{h}, initialState = solutionStates.last())
    if *currTime* + h = —$\mathcal{T}_0$ — **then**
        solutionStates.addAll(thisTimeStates)
    else
        solutionStates.add(thisTimeStates.first())
    currTime $\longleftarrow$ currTime + 1
return solutionStates;

Algorithm 3: Transient smoothening

5 Conclusion

In this paper we presented the transient gas network transport problem on so-called navi stations, which represent the intersection points of major transportation pipelines and contain the majority of active elements to control the network. We introduced a mixed-integer programming model for the problem including a complex model for compressor stations as well as additional variables and constraint for the navi station itself. For the pipes, a linear approximation was used, since they are short and therefore have less overall impact in navi stations. To solve the problem, we developed a specialized algorithm, exploiting the high dependency of the operation mode choice on the corresponding demands in each individual time step caused by the short pipes. Therefore, we determine the operation modes of the navi station based on solving a stationary version of the presented MIP. Here, we also satisfy the previously excluded transition time constraints of the operation modes. In order to obtain a feasible solution for the original MIP model, we finally solve the original MIP with fixed navi station operation modes in a rolling horizon fashion.

There are a lot of different possibilities to continue the research. To increase the model correctness, the approximative linearization of the friction in pipes and resistors as well as the maximum power bound of compressor units should be replaced by their original non-linear versions, turning the model into a MINLP and thereby increasing the overall complexity. From a theoretical point of view, extending Algorithm 1 to be able to guarantee finding a feasible solution without losing overall performance in terms of execution time would greatly improve it's robustness. Finally, real-world gas network operation is a complicated business with a never-ending list of special elements and extra constraints, which can still be added to our model. As examples we name ramp-up and cool down times for compressor units as well as target value based control of regulators and compressor stations.

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