Métodos Matemáticos **Name:** *Sergio Montoya*
Examen Final
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1. **Teorema del Residuo** Primero, encuentra los polos de la función y su orden. Luego determina las singularidades dentro de la región que cierra la curva. Luego de eso aplica el teorema del residuo

$$\oint_C f(x)\,dx = 2\pi i \cdot \mathrm{Res}(f,\alpha)$$
.

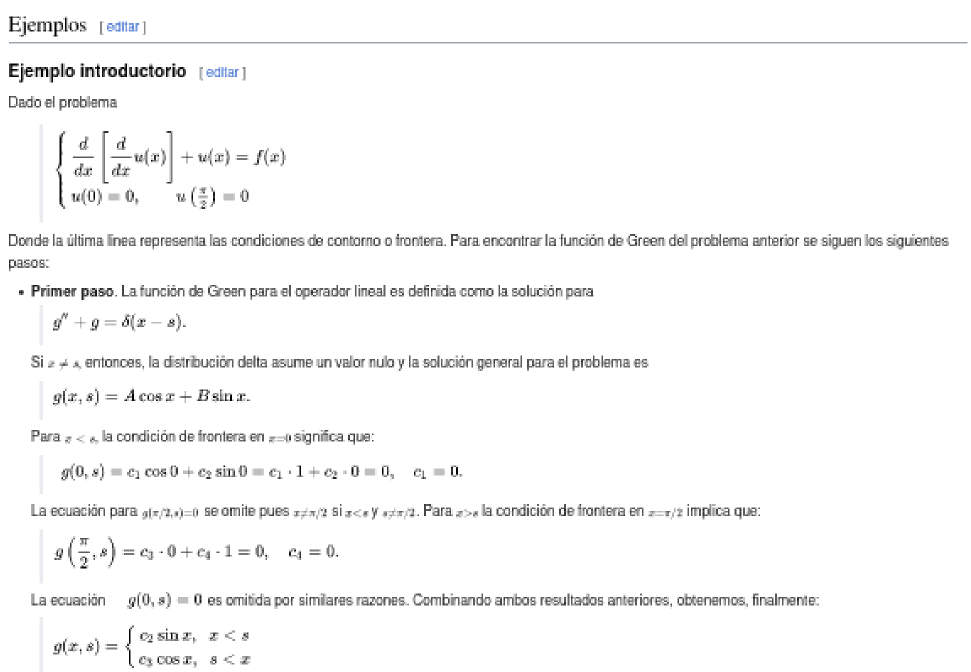
El residuo puede valer:

(a) Polo Simple: $\lim_{n\to z_0}(z-z_0)\,f(z)$

(b) Polo de Orden N : $[(N-1)!]^{-1}\lim_{z\to z_0}\frac{d^{N-1}}{dz^{N-1}}(z-z_0)^N d(z)$

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)}$$
.

2. Delta de Dirac y Funciones de Green



Ejemplos [editar]

Ejemplo introductorio [editar]

Dado el problema

$$\begin{cases} \dfrac{d}{dx}\left[\dfrac{d}{dx}u(x)\right] + u(x) = f(x) \\ u(0) = 0, \qquad u\left(\frac{\pi}{2}\right) = 0 \end{cases}$$

Donde la última linea representa las condiciones de contorno o frontera. Para encontrar la función de Green del problema anterior se siguen los siguientes pasos:

- **Primer paso**. La función de Green para el operador lineal es definida como la solución para

$$g'' + g = \delta(x-s).$$

Si $x \neq s$, entonces, la distribución delta asume un valor nulo y la solución general para el problema es

$$g(x,s) = A\cos x + B\sin x.$$

Para $x < s$, la condición de frontera en $x=0$ significa que:

$$g(0,s) = c_1\cos 0 + c_2\sin 0 = c_1\cdot 1 + c_2\cdot 0 = 0, \quad c_1 = 0.$$

La ecuación para $g(\pi/2,s)=0$ se omite pues $x\neq\pi/2$ si $x<s$ y $s\neq\pi/2$. Para $x>s$ la condición de frontera en $x=\pi/2$ implica que:

$$g\left(\frac{\pi}{2},s\right) = c_3\cdot 0 + c_4\cdot 1 = 0, \quad c_4 = 0.$$

La ecuación $g(0,s)=0$ es omitida por similares razones. Combinando ambos resultados anteriores, obtenemos, finalmente:

$$g(x,s) = \begin{cases} c_2\sin x, & x<s \\ c_3\cos x, & s<x \end{cases}$$

Figure 1: green1.png

- **Segundo paso.** A continuación, vamos a encontrar c_2 y c_3. Debemos asegurar la continuidad de la función de green para el intervalo escogido. Cuando $x = s$ se tiene que:

  $$c_2 \sin s = c_3 \cos s.$$

  También debemos asegurar la discontinuidad de la primera derivada por integración de la ecuación diferencial de $x=s-\epsilon$ a $x=s+\epsilon$ y tomando el límite cuando ϵ tiende a cero. Por lo cual, derivando la igualdad anterior y garantizando la discontinuidad de esta, tenemos:

  $$c_3 \cdot [-\sin s] - c_2 \cdot \cos s = 1$$

  En la cual se iguala a 1 pues $p(x) = 1$. Resolvemos para las constantes. c_2 y c_3 obteniendo:

  $$c_2 = -\cos s \quad ; \quad c_3 = -\sin s$$

  Entonces, la función de Green es:

  $$g(x,s) = \begin{cases} -\cos(s)\sin(x) & x < s, \\ -\sin(s)\cos(x) & s < x \end{cases}$$

- **Solución final**, recopilando los resultados anteriores tenemos que la solución final al problema planteado es:

  $$u(x) = \int_0^{\pi/2} f(s) g(x,s)\, ds = -\int_0^x f(s) \sin s \cos x\, ds - \int_x^{\pi/2} f(s) \cos s \sin x\, ds$$

Dicha solución existe para cualquier función $f \in L^1([0,\pi/2])$ integrable en el intervalo $[0,\pi/2]$.

Figure 2: green2.png

3. Parseval y seno y cose Fourier

$$\|f\|^2_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi} |f(x)|^2\, dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

where the Fourier coefficients c_n of f are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

The result holds as stated provided f is a square-integrable function or, more generally, in Lp space $L^2[-\pi,\pi]$. A similar result is the Plancherel theorem, which asserts that the integral of the square of the Fourier transform of a function is equal to the integral of the square of the function itself. In one-dimension, for $f \in L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

Figure 3: parseval.png

From Euler's rule, we have,

$$x(t) = \sin \omega_0 t = \left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right]$$

Then, from the definition of Fourier transform, we have,

$$F[\sin \omega_0 t] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sin \omega_0\ t\ e^{-j\omega t} dt$$

$$\Rightarrow\ X(\omega) = \int_{-\infty}^{\infty} \left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] e^{-j\omega t} dt$$

$$\Rightarrow\ X(\omega) = \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt - \int_{-\infty}^{\infty} e^{-j\omega_0 t} e^{-j\omega t} dt\right]$$

$$= \frac{1}{2j}\{F[e^{j\omega_0 t}] - F[e^{-j\omega_0 t}]\}$$

Figure 4: sin1.png

Since, the Fourier transform of complex exponential function is given by,

$$F[e^{j\omega_0 t}] = 2\pi\delta(\omega - \omega_0)\ \text{ and }\ F[e^{-j\omega_0 t}] = 2\pi\delta(\omega + \omega_0)$$

$$\therefore\ X(\omega) = \frac{1}{2j}[2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)]$$

$$\Rightarrow\ X(\omega) = -j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Therefore, *the Fourier transform of the sine wave is,*

$$F[\sin \omega_0\ t] = -j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Or, it can also be represented as,

$$\sin \omega_0\ t \overset{FT}{\leftrightarrow} -j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Figure 5: sin2.png

From Euler's rule, we have,

$$\cos\omega_0 t = \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

Then, from the definition of Fourier transform, we have,

$$F[\cos\omega_0 t] = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \cos\omega_0 t e^{-j\omega t}dt$$

$$\Rightarrow\ X(\omega) = \int_{-\infty}^{\infty} \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] e^{-j\omega t}dt$$

$$\Rightarrow\ X(\omega) = \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{j\omega_0 t}e^{-j\omega t}dt + \int_{-\infty}^{\infty} e^{-j\omega_0 t}e^{-j\omega t}dt\right]$$

$$= \frac{1}{2}\{F[e^{j\omega_0 t}] + F[e^{-j\omega_0 t}]\}$$

$$\Rightarrow\ X(\omega) = \frac{1}{2}[2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)]$$

$$\Rightarrow\ X(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

Figure 6: cos1.png

Therefore, *the Fourier transform of cosine wave function is,*

$$F[\cos\omega_0 t] = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

Or, it can also be represented as,

$$\cos\omega_0 t \overset{FT}{\leftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

Figure 7: cos2.png

4. Fourier

Tabla de Propiedades de la transformada de Fourier

$$\mathbb{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \partial t$$

$$\mathbb{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{+j\omega t} \partial \omega$$

Linealidad	$\mathbb{F}[\alpha f(t) + \beta g(t)] = \alpha F(\omega) + \beta G(\omega)$
Dualidad	$\mathbb{F}[f(t)] = F(\omega) \rightarrow \mathbb{F}[F(t)] = 2\pi f(-\omega)$
Cambio de escala	$\mathbb{F}[f(at)] = \frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
Transformada de la conjugada	$\mathbb{F}[f^*(t)] = F^*(-\omega)$
Translacion en el tiempo	$\mathbb{F}[f(t - t_0)] = e^{-j\omega t_0} F(\omega)$
Translacion en frecuencia	$\mathbb{F}[e^{+j\omega_0 t} f(t)] = F(\omega - \omega_0)$
Derivacion en el tiempo	$\mathbb{F}\left[\frac{\partial^n f(t)}{\partial t^n}\right] = (j\omega)^n F(\omega)$
Derivacion en la frecuencia	$\mathbb{F}[(-jt)^n f(t)] = \frac{\partial^n F(\omega)}{\partial \omega^n}$
Transformada de la integral	$\mathbb{F}\left[\int_{-\infty}^{t} f(\tau) \partial \tau\right] = \frac{F(\omega)}{j\omega} + \pi F(0) \delta(\omega)$

Figure 8: fourier1.png

5. Laplace

Tabla 1. Pares de transformadas elementales

$f(x)$	$F(p) = L[f(x)]$
1	$\frac{1}{p}$
x	$\frac{1}{p^2}$
x^n	$\frac{n!}{p^{n+1}}$
e^{ax}	$\frac{1}{p-a}$
$\operatorname{sen} ax$	$\frac{a}{p^2+a^2}$
$\cos ax$	$\frac{p}{p^2+a^2}$
$\operatorname{sh} ax$	$\frac{a}{p^2-a^2}$
$\operatorname{ch} ax$	$\frac{p}{p^2-a^2}$

Figure 9: laplace1.png

$f(t)$	$F(s) = \mathcal{L}[f(t)]$		Formula
$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0$	A
$f(t) = e^{at}$	$F(s) = \frac{1}{(s-a)}$	$s > a$	B
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0$	C
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2 + a^2}$	$s > 0$	D
$f(t) = \cos(at)$	$F(s) = \frac{s}{s^2 + a^2}$	$s > 0$	E
$f(t) = \sinh(at)$	$F(s) = \frac{a}{s^2 - a^2}$	$s > \|a\|$	F
$f(t) = \cosh(at)$	$F(s) = \frac{s}{s^2 - a^2}$	$s > \|a\|$	G
$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s-a)^{(n+1)}}$	$s > a$	H
$f(t) = e^{at}\sin(bt)$	$F(s) = \frac{b}{(s-a)^2 + b^2}$	$s > a$	I
$f(t) = e^{at}\cos(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2 + b^2}$	$s > a$	J
$f(t) = e^{at}\sinh(bt)$	$F(s) = \frac{b}{(s-a)^2 - b^2}$	$s - a > \|b\|$	K
$f(t) = e^{at}\cosh(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2 - b^2}$	$s - a > \|b\|$	L

Figure 10: laplace2.png

6. Funciones de Bessel

Bessel functions, first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel, are canonical solutions $y(x)$ of Bessel's differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - \alpha^2\right) y = 0$$

Figure 11: bessel1.png

Type	First kind	Second kind
Bessel functions	J_α	Y_α
Modified Bessel functions	I_α	K_α
Hankel functions	$H_\alpha^{(1)} = J_\alpha + iY_\alpha$	$H_\alpha^{(2)} = J_\alpha - iY_\alpha$
Spherical Bessel functions	j_n	y_n
Spherical Hankel functions	$h_n^{(1)} = j_n + iy_n$	$h_n^{(2)} = j_n - iy_n$

Figure 12: bessel1.png

Bessel functions of the first kind: J_α [edit]

Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of Bessel's differential equation. For integer or positive α, Bessel functions of the first kind are finite at the origin ($x = 0$); while for negative non-integer α, Bessel functions of the first kind diverge as x approaches zero. It is possible to define the function by x^α times a Maclaurin series (note that α need not be an integer, and non-integer powers are not permitted in a Taylor series), which can be found by applying the Frobenius method to Bessel's equation:[4]

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha},$$

Figure 13: bessel1.png

For non-integer α, the functions $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent, and are therefore the two solutions of the differential equation. On the other hand, for integer order n, the following relationship is valid (the gamma function has simple poles at each of the non-positive integers):[5]

$$J_{-n}(x) = (-1)^n J_n(x).$$

Figure 14: bessel1.png

Bessel's integrals [edit]

Another definition of the Bessel function, for integer values of n, is possible using an integral representation:[6]

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\tau - x\sin\tau)\,d\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n\tau - x\sin\tau)}\,d\tau,$$

which is also called Hansen-Bessel formula.[7]

This was the approach that Bessel used,[8] and from this definition he derived several properties of the function. The definition may be extended to non-integer orders by one of Schläfli's integrals, for $\mathrm{Re}(x) > 0$:[6][9][10][11][12]

$$J_\alpha(x) = \frac{1}{\pi}\int_0^\pi \cos(\alpha\tau - x\sin\tau)\,d\tau - \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-x\sinh t - \alpha t}\,dt.$$

Figure 15: bessel1.png

Bessel functions of the second kind: Y_α [edit]

The Bessel functions of the second kind, denoted by $Y_\alpha(x)$, occasionally denoted instead by $N_\alpha(x)$, are solutions of the Bessel differential equation that have a singularity at the origin ($x = 0$) and are multivalued. These are sometimes called **Weber functions**, as they were introduced by H. M. Weber (1873), and also **Neumann functions** after Carl Neumann.[15]

For non-integer α, $Y_\alpha(x)$ is related to $J_\alpha(x)$ by

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

Figure 16: bessel1.png

Ejemplo expansión de Bessel para $p^{(k)}$: Suponemos que

$$f(\rho) = \sum_{n=1}^{\infty} A_{v_n} J_v \left[X_{v_n} \frac{\rho}{b} \right].$$

con

$$\begin{aligned}
A_{v_n} &= \frac{2}{b^2 \left[J_{v+1}(X_{v_n}) \right]^2} \int_0^b \rho \left(\rho^k \right) J_v \left(X_{v_n} \frac{\rho}{b} \right) d\rho \\
\frac{d\left[Y^n J_n \right]}{dy} &= Y^n J_{n-1} \\
A_{v_n} &= \frac{2}{b^2 \left[J_{v+1}(X_{v_n}) \right]^2} \int_0^b \left(\rho^{k+1} \right) J_v \left(X_{v_n} \frac{\rho}{b} \right) d\rho \\
Y &= X_{v_n} \frac{\rho}{b} \\
\frac{dY}{d\rho} &= \frac{X_{v_n}}{b} \leftrightarrow d\rho = \frac{b}{X_{v_n}} \\
\rho &= \frac{Yb}{X_{v_n}} \\
A_{v_n} &= \frac{2}{b^2 \left[J_{v+1}(X_{v_n}) \right]^2} \int_0^{X_{v_n}} \left(\frac{yb}{X_{v_n}} \right)^{k+1} J_v(y) \frac{b}{x_{v_n}} dY \\
&= \frac{2}{b^2 \left[J_{v+1}(X_{v_n}) \right]^2} \int_0^{X_{v_n}} \frac{y^{k+1} b^{k+2}}{X_{v_n}^{k+2}} J_v(Y)\, dY \\
&= \frac{2b^k}{\left[J_{v+1}(X_{v_n}) \right]^2 X_{v_n}^{(k+2)}} \int_0^{X_{v_n}} Y^{k+1} J_v(Y)\, dY
\end{aligned}$$

Si $v = k$

$$\begin{aligned}
\frac{d\left[Y^k J_k(v) \right]}{dY} &= Y^{k+1} J_k(V) \\
A_{k_n} &= \frac{2b^k}{\left[J_{v+1}(X_{v_n}) \right]^2 X_{v_n}^{(k+2)}} \int_0^{X_{v_n}} Y^{k+1} J_v(Y)\, dY \\
&= \frac{2b^k}{\left[J_{v+1}(X_{v_n}) \right]^2 X_{v_n}^{(k+2)}} \left[Y^k J_k(Y) \right]_0^{X_{k_n}} \\
&= \frac{2b^k}{\left[J_{v+1}(X_{v_n}) \right]^2 X_{v_n}^{(k+2)}} \left[X_{k_n}^k J_k(x_{k_n}) - 0 \right] \\
&= \frac{2b^k}{\left[J_{v+1}(X_{v_n}) \right]^2 X_{v_n}^{(k+2)}} \frac{J_k(X_{k_n})}{X_{k_n}^2} \\
f(\rho) = \rho^k &= \sum_{n=1}^{\infty} \frac{2b^k}{\left[J_{v+1}(X_{v_n}) \right]^2 X_{v_n}^{(k+2)}} \frac{J_k(X_{k_n})}{X_{k_n}^2} J_k \left[X_{k_n} \frac{\rho}{b} \right].
\end{aligned}$$

7. Laplace Ejemplo

Tenemos la función:

$$f\left(x\right)=\begin{cases}V_0 & 0<x\leq 1\\ -V_0 & -1\leq x<0\end{cases}.$$

Decimos entonces que la expresión de f en polinomios de Legendre es:

$$f\left(x\right)=\sum_{n=0}^{\infty}a_nP_n\left(x\right).$$

Donde los coeficientes están determinados por:

$$\begin{aligned}a_n&=\frac{2n+1}{2}\int_{-1}^{1}f\left(x\right)P_n\left(x\right)dx\\&=\frac{2n+1}{2}\left[\int_{-1}^{0}\left(-V_0\right)P_ndx+\int_0^1V_0P_ndx\right]\\&=\frac{2n+1}{2}V_0\left[\int_0^1P_n\left(x\right)dx-\int_{-1}^{0}P_n\left(x\right)dx\right].\end{aligned}$$

Ahora bien, se tienen dos casos, P_n par o impar:

- Para P_n impar, $P_n\left(-x\right)=-P_n\left(x\right)\implies\int_{-1}^{0}P_n\left(x\right)dx=-\int_0^1P_n\left(x\right)dx$
- Para P_n par, $P_n\left(-x\right)=P_n\left(x\right)\implies\int_{-1}^{0}P_n\left(x\right)dx=\int_0^1P_n\left(x\right)dx$

Con lo cual que podemos ver que los términos pares se contrarrestan. Por lo cual, tenemos que:

$$a_n=\frac{2n+1}{2}V_0\left[\int_0^1P_n\left(x\right)dx-\int_{-1}^{0}P_n\left(x\right)dx\right]=\left(2n+1\right)V_0\int_0^1P_n\left(x\right)dx.$$

Luego:

$$f\left(x\right)=\sum_{n=0}^{\infty}\left(2n+1\right)V_0\int_0^1P_n\left(x\right)dxP_n\left(x\right).$$

Ademas, recordemos que P_n es impar. Con esto entonces calculemos:

$$\begin{aligned}\int_0^1P_1\left(x\right)dx&=\int_0^1xdx=\frac{x^2}{2}|_0^1=\frac{1}{2}\\\int_0^1P_3\left(x\right)dx&=\int_0^1\frac{1}{2}\left(5x^3-3x\right)dx\\&=\frac{1}{2}\left[\frac{5x^4}{4}-\frac{3x^2}{2}\right]_0^1\\&=\frac{1}{4}\left(\frac{5}{2}-3\right)\\\implies f\left(x\right)&=\frac{3}{2}V_0x-\frac{7}{8}V_0\left(5x^3-3x\right)+\ldots.\end{aligned}$$