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Lista 1
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1 P.4

1.1 b

Let W be a subset of $\mathbb{F}^3$, with $\mathbb{F}$ being $\mathbb{R}$ or $\mathbb{C}$, caracterized by

$$W = \{(x, y, z) \in \mathbb{F}^3 | x \in \mathbb{Z}\} \tag{1}$$

then W is not a sub-space of $\mathbb{F}^3$ because there exists a vector $v = (1, 0, 0)$ and a number (i.e. π) such that $v \in W$ but $\pi v = (\pi, 0, 0) \notin W$

1.2 c

considering that the polinom 0 has $grau(0) < 2$ then the subset

$$W = \{p(t) \in P_n(\mathbb{F}) | grau(p) \geq 2\} \tag{2}$$

cannot be a subspace of $P(\mathbb{F})$

2 P.5

2.1 a

Let W be a subset of $M(n, \mathbb{F})$, with $\mathbb{F}$ being $\mathbb{R}$ or $\mathbb{C}$, caracterized by

$$W = \left\{ A \in M(n, \mathbb{F}) | tr(A) = \sum_{i=1}^{n} a_{ii} = 0 \right\} \tag{3}$$

.

Then we can assert

$$\forall A_1, A_2 \in W, \forall \alpha \in \mathbb{F} : tr(\alpha A_1) = 0 \land tr(A_1 + A_2) = 0 \tag{4}$$

.

Let $A_1, A_2 \in W$ and $\alpha \in \mathbb{F}$. Then

$$tr(\alpha A_1) = \sum_{i=1}^{n} \alpha a_{ii} = \alpha \sum_{i=1}^{n} a_{ii} = \alpha tr(A) = \alpha 0 = 0$$

And

$$tr(A_1 + A_2) = \sum_{i=1}^{n} a_{ii}^1 + a_{ii}^2 = \left(\sum_{i=1}^{n} a_{ii}^1 \right) + \left(\sum_{i=1} a_{ii}^2 \right) = tr(A_1) + tr(A_2) = 0 + 0 = 0$$

therefore, W is a subspace of $M(n, \mathbb{F})$

2.2 b

Let W be a subset of $M(n, \mathbb{F})$, with $\mathbb{F}$ being $\mathbb{R}$ or $\mathbb{C}$, caracterized by

$$W = \{A \in M(n, \mathbb{F}) | \det(A) = 0\} \tag{5}$$

.

We can proof that W is not a subspace of $M(n, \mathbb{F})$ by providing two matrixes M_1, M_2 that have $\det(M_i) = 0$ but $M_1 + M_2 \neq 0$. Let's take an $n > 1$ (because $n \leq 1$ would have this subspace just because the only posible matrix with $\det(M) = 0$ is (0)) and take the matrix with only ones in the diagonal except for a_{nn} and the matrix with all 0 except in the element $a_n n$. You can see that

$$\begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}$$

Now you get the identity matrix, and therefore $\det(I) \neq 0$ so it can not be a subspace.

3 P.9

3.1 a

$$\begin{aligned}
&\implies \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 2 & -1 & 0 \end{array}\right) \\
(-1)(I) + (II) &\implies \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -1 & 0 \\ 0 & -2 & -2 & 1 & 0 & 0 \end{array}\right) \\
(-\frac{1}{2})(II) &\implies \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 \end{array}\right) \\
(-1)(II) + (I) &\implies \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{3}{2} & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 \end{array}\right) \\
&\begin{cases} x + \frac{3}{2}w - z = 0 \\ y + z - \frac{w}{2} = 0 \end{cases} \\
&\begin{cases} x = -\frac{3}{2}w + z \\ y = -z + \frac{w}{2} \end{cases} \\
&\left\{ \left(-\frac{3}{2}w + z, -z + \frac{w}{2}, z, w, 0 \right) \middle| w, z \in \mathbb{F} \right\} \\
&\left[\left(-\frac{3}{2}, \frac{1}{2}, 0, 1, 0 \right), (1, -1, 1, 0, 0) \right] \\
&\dim \left(\left[\left(-\frac{3}{2}, \frac{1}{2}, 0, 1, 0 \right), (1, -1, 1, 0, 0) \right] \right) = 2
\end{aligned}$$

3.2 b

$$\implies \left(\begin{array}{cccc|c} 4 & 3 & -1 & 1 & 0 \\ 1 & -1 & 2 & -1 & 0 \end{array}\right)$$
$$\implies \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 4 & 3 & -1 & 1 & 0 \end{array}\right)$$
$$\implies \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 0 & 7 & -9 & 5 & 0 \end{array}\right)$$
$$\implies \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{9}{7} & \frac{5}{7} & 0 \end{array}\right)$$
$$\implies \begin{cases} x - y + 2z - t = 0 \\ y - \frac{9}{7}z + \frac{5}{7}t = 0 \end{cases}$$
$$\implies \begin{cases} x - y + 2z - t = 0 \\ y = \frac{9}{7}z - \frac{5}{7}t \end{cases}$$
$$\implies \begin{cases} x - \frac{9}{7}z + \frac{5}{7}t + 2z - t = 0 \\ y = \frac{9}{7}z - \frac{5}{7}t \end{cases}$$
$$\implies \begin{cases} x = -\frac{5}{7}z + \frac{2}{7}t \\ y = \frac{9}{7}z - \frac{5}{7}t \end{cases}$$
$$\implies \left\{(-\frac{5}{7}z + \frac{2}{7}t, \frac{9}{7}z - \frac{5}{7}t, z, t) \middle| z, t \in \mathbb{R}\right\}$$
$$\implies \left[\left(-\frac{5}{7}, \frac{9}{7}, 1, 0\right), \left(\frac{2}{7}, -\frac{5}{7}, 0, 1\right)\right]$$
$$\implies \dim\left(\left[\left(-\frac{5}{7}, \frac{9}{7}, 1, 0\right), \left(\frac{2}{7}, -\frac{5}{7}, 0, 1\right)\right]\right) = 2$$

4 P.12

$$S \implies \left(\begin{array}{cccc|c} 3 & 0 & 2 & 0 & 0 \\ -6 & 3 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 9 & -3 & 3 & 1 & 0 \\ -9 & 5 & -1 & 1 & 0 \end{array}\right)$$
$$(\frac{1}{3})(I) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ -6 & 3 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 9 & -3 & 3 & 1 & 0 \\ -9 & 5 & -1 & 1 & 0 \end{array}\right)$$
$$((6)(I) + (II)) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 3 & 3 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 9 & -3 & 3 & 1 & 0 \\ -9 & 5 & -1 & 1 & 0 \end{array}\right)$$

$$((-9)(I)+(IV)) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 3 & 3 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 1 & 0 \\ -9 & 5 & -1 & 1 & 0 \end{array}\right)$$

$$((9)(I)+(V)) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 3 & 3 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 1 & 0 \\ 0 & 5 & 5 & 1 & 0 \end{array}\right)$$

$$((III) \leftrightarrow (II)) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 3 & 3 & -1 & 0 \\ 0 & -3 & -3 & 1 & 0 \\ 0 & 5 & 5 & 1 & 0 \end{array}\right)$$

$$((III)+(IV)) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 3 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & 1 & 0 \end{array}\right)$$

$$((-3)(II)+(III)) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & 1 & 0 \end{array}\right)$$

$$((-5)(II)+(V)) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right)$$

$$((-1)(III)+(V)) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$((-\frac{1}{4})(III)+(V)) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$((-1)(III)+(II)) \implies \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

With this we demonstrate that this subspace has dimention 3.

$$\Big\{x = 12y = 7\tfrac{2}{3}x + y = 15z = 7$$

As you can see this equations area already solved and it actually works!

5 P.14

5.1 a

$$
\begin{aligned}
W_1 \cap W_2 &= \{(x,y,z,t) \in \mathbb{R}^4 | x+y=0; z-t=0; x-y-z+t=0\} \\
W_1 \cap W_2 &\implies \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \end{array}\right) \\
(-1)(I)+(III) &\implies \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -2 & -1 & 1 & 0 \end{array}\right) \\
(II) \leftrightarrow (III) &\implies \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \\
(-\frac{1}{2})(II) &\implies \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \\
(-1)(II)+(I) &\implies \left(\begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \\
(-\frac{1}{2})(III)+(II) &\implies \left(\begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \\
(\frac{1}{2})(III)+(I) &\implies \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \\
&\implies \begin{cases} x = 0 \\ y = 0 \\ z = t \end{cases} \\
&\implies [(0,0,1,1)]
\end{aligned}
$$

Meaning that the dimention is 1.

5.2 b

Let's start by getting W_1 and W_2

$$
\begin{aligned}
W_1 &\implies \begin{cases} x = -y \\ z = t \end{cases} \\
W_1 &= [(1,-1,0,0),(0,0,1,1)] \\
W_2 &\implies x = y+z-t \\
W_2 &= [(1,1,0,0),(1,0,1,0),(-1,0,0,1)]
\end{aligned}
$$

We can develop all the steps necessary whoever we can take a simpler approach knowing that we have the dimention of $W_1 \cap W_2$ and that is:

$$
\begin{aligned}
\dim(W_1+W_2) &= \dim(W_1)+\dim(W_2)-\dim(W_1 \cap W_2) \\
&= 2+3-1 \\
&= 4
\end{aligned}
$$

and knwoing that both subspaces are in $\mathbb{R}^4$ we know that $W_1+W_2 = \mathbb{R}^4$

5.3 c

It is not a direct sum, we know that by getting the dimention of $W_1 \cap W_2$. Given that $\dim(W_1 \cap W_2) \neq 0$ we know that it is not a direct sum.

5.4 d

As we discuss above it is, and that helps a lot by removing the need for calculating $W_1 + W_2$ as we know it have dimention 4 and that made it mandatory to be $\mathbb{R}^4$ express in, probably, a different basis.

6 P.21

6.1 a

$$\begin{aligned}(x+1)^2 &= x^2 + x + 1 \\ &= 1 \cdot x^2 + 1 \cdot x + 1 \cdot 1\end{aligned}$$

this already shows the coordinates in $\mathscr{B}$ (Long live the canon basis!)

6.2 b

6.2.1 $P_{\mathscr{C} \to \mathscr{B}}$

$$\begin{aligned}(x+1)^2 = x^2 + x + 1 &\implies (1,1,1) \\ x + 4 &\implies (4,1,0) \\ 3x &\implies (0,3,0) \\ P_{\mathscr{C} \to \mathscr{B}} &= \begin{pmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix}\end{aligned}$$

Check:

$$\begin{aligned}a(x+1)^2 + b(x+4) + c(3x) &= ax^2 + ax + a + bx + 4b + 3cx \\ &= ax^2 + (a + b + 3c)x + (a + 4b) \\ P_{\mathscr{C} \to \mathscr{B}} v &= \begin{pmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{\mathscr{C}} \\ &= \begin{pmatrix} a + 4b \\ a + b + 3c \\ a \end{pmatrix}_{\mathscr{B}} \\ &= ax^2 + (a + b + 3c)x + (a + 4b) \square\end{aligned}$$

6.2.2 $P_{\mathscr{B}\to\mathscr{C}}$

$$
\begin{aligned}
P_{\mathscr{B}\to\mathscr{C}} &= P_{\mathscr{C}\to\mathscr{B}}^{-1} \\
P_{\mathscr{C}\to\mathscr{B}} &\implies \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \\
(-4)(I)+(II) &\implies \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -4 & -4 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \\
(II) \leftrightarrow (III) &\implies \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & -3 & -4 & -4 & 1 & 0 \end{array}\right) \\
(\frac{1}{3})(II) &\implies \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & -3 & -4 & -4 & 1 & 0 \end{array}\right) \\
(-1)(II)+(I) &\implies \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & -3 & -4 & -4 & 1 & 0 \end{array}\right) \\
(3)(II)+(III) &\implies \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & -4 & -4 & 1 & 1 \end{array}\right) \\
(-\frac{1}{4})(III) &\implies \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 1 & -\frac{1}{4} & -\frac{1}{4} \end{array}\right) \\
(-1)(III)+(I) &\implies \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{12} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 1 & -\frac{1}{4} & -\frac{1}{4} \end{array}\right) \\
P_{\mathscr{B}\to\mathscr{C}} &= \begin{pmatrix} 0 & \frac{1}{4} & -\frac{1}{12} \\ 0 & 0 & \frac{1}{3} \\ 1 & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}
\end{aligned}
$$

Check:

$$
\begin{aligned}
a(x+1)^2 + b(x+4) + c(3x) &= ax^2 + ax + a + bx + 4b + 3cx \\
&= ax^2 + (a+b+3c)x + (a+4b) \\
P_{\mathscr{B}\to\mathscr{C}} v &= \begin{pmatrix} 0 & \frac{1}{4} & -\frac{1}{12} \\ 0 & 0 & \frac{1}{3} \\ 1 & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} a+4b \\ a+b+3c \\ a \end{pmatrix}_{\mathscr{B}} \\
&= \begin{pmatrix} a \\ \frac{a}{4} + b - \frac{a}{4} \\ -\frac{a}{12} - \frac{b}{3} + \frac{a}{3} + \frac{b}{3} + c - \frac{a}{4} \end{pmatrix}_{\mathscr{C}} \\
&= \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{\mathscr{C}} \\
&= a(x+1)^2 + b(x+4) + c(3x)
\end{aligned}
$$

6.2.3 $P_{\mathscr{D}\to\mathscr{B}}$

$$
\begin{aligned}
1-x^2 &\implies (1,0,-1)\\
x(x+1)=x^2+x &\implies (0,1,1)\\
x^2+1 &\implies (1,0,1)\\
P_{\mathscr{D}\to\mathscr{B}} &= \begin{pmatrix}1&0&-1\\0&1&1\\1&0&1\end{pmatrix}
\end{aligned}
$$

Check:

$$
\begin{aligned}
a(1-x^2)+b(x(x+1))+c(x^2+1) &= a-ax^2+bx^2+bx+cx^2+c\\
&= -ax^2+bx^2+cx^2+bx+c+a\\
&= (-a+b+c)x^2+bx+(c+a)\\
P_{\mathscr{C}\to\mathscr{B}}v &= \begin{pmatrix}1&0&-1\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}_{\mathscr{D}}\\
&= \begin{pmatrix}a+c\\b\\-a+b+c\end{pmatrix}_{\mathscr{B}}\\
&= (-a+b+c)x^2+bx+(a+c)\square
\end{aligned}
$$

6.2.4 $P_{\mathscr{B}\to\mathscr{D}}$

$$
\begin{aligned}
P_{\mathscr{B}\to\mathscr{D}} &= P^{-1}_{\mathscr{D}\to\mathscr{B}}\\
P_{\mathscr{D}\to\mathscr{B}} &\implies \left(\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&1&1&0&1&0\\1&0&1&0&0&1\end{array}\right)\\
(-1)(I)+(III) &\implies \left(\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&1&1&0&1&0\\0&0&2&-1&0&1\end{array}\right)\\
(\frac{1}{2})(III) &\implies \left(\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&1&1&0&1&0\\0&0&1&-\frac{1}{2}&0&\frac{1}{2}\end{array}\right)\\
(1)(III)+(I) &\implies \left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{2}&0&\frac{1}{2}\\0&1&1&0&1&0\\0&0&1&-\frac{1}{2}&0&\frac{1}{2}\end{array}\right)\\
(-1)(III)+(II) &\implies \left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{2}&0&\frac{1}{2}\\0&1&0&\frac{1}{2}&1&-\frac{1}{2}\\0&0&1&-\frac{1}{2}&0&\frac{1}{2}\end{array}\right)\\
P_{\mathscr{B}\to\mathscr{D}} &= \begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&1&-\frac{1}{2}\\-\frac{1}{2}&0&\frac{1}{2}\end{pmatrix}
\end{aligned}
$$

Check:

$$\begin{aligned}
a(1-x^2)+b(x(x+1))+c(x^2+1) &= a-ax^2+bx^2+bx+cx^2+c \\
&= -ax^2+bx^2+cx^2+bx+c+a \\
&= (-a+b+c)x^2+bx+(c+a) \\
P_{\mathscr{B}\to\mathscr{D}}v &= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a+c \\ b \\ -a+b+c \end{pmatrix}_{\mathscr{B}} \\
&= \begin{pmatrix} \frac{a}{2}+\frac{c}{2}+\frac{b}{2}+\frac{a}{2}-\frac{b}{2}-\frac{c}{2} \\ b \\ \frac{a}{2}+\frac{c}{2}-\frac{b}{2}-\frac{a}{2}+\frac{b}{2}+\frac{c}{2} \end{pmatrix}_{\mathscr{C}} \\
&= \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{\mathscr{C}} \\
&= a(1-x^2)+b(x(x+1))+c(x^2+1)
\end{aligned}$$

7 c

The coordinate for $a(x)$ is trivial for $\mathscr{C}$ because it's an element of the basis so it is only $(1,0,0)$. Now for the next part we have

$$\begin{aligned}
P_{\mathscr{B}\to\mathscr{D}}v &= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}_{\mathscr{B}} \\
&= \begin{pmatrix} \frac{1}{2}+\frac{1}{2}-\frac{1}{2} \\ 1 \\ \frac{1}{2}-\frac{1}{2}+\frac{1}{2} \end{pmatrix}_{\mathscr{D}} \\
&= \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}_{\mathscr{D}}
\end{aligned}$$

Check:

$$\begin{aligned}
\frac{1}{2}(1-x^2)+1(x(x+1))+\frac{1}{2}(x^2+1) &= -\frac{x^2}{2}+\frac{1}{2}+x^2+x+\frac{x^2}{2}+\frac{1}{2} \\
&= x^2+x+1\square
\end{aligned}$$

8 P.23

8.1 c

It is (just beacause it is described as a linear combination of coordinates in $\mathbb{R}^2$). However, let's check it.

$$\begin{aligned}
T((x_1,y_1)+\alpha(x_2,y_2)) &= T(x_1+\alpha x_2, y1+\alpha y_2) \\
&= (x_1+\alpha x_2+y_1+\alpha y_2, x_1+\alpha x_2-y_1-\alpha y_2, y_1+\alpha y_2) \\
T(x_1,y_1)+\alpha T(x_2,y_2) &= (x_1+y_1, x_1-y_1, y_1)+(\alpha x_2+\alpha y_2, \alpha x_2-\alpha y_2, \alpha y_2) \\
&= (x_1+\alpha x_2+y_1+\alpha y_2, x_1+\alpha x_2-y_1-\alpha y_2, y_1+\alpha y_2) \\
T((x_1,y_1)+\alpha(x_2,y_2)) &= T(x_1,y_1)+\alpha T(x_2,y_2)
\end{aligned}$$

Meaning that it is a linear transformation.

8.2 d

Let's first check how it looks for a polinomial

$$\begin{aligned} T(at^2+bt+c) &= t^2(at^2+bt+c)'' \\ &= t^2(2at+b)' \\ &= 2at^2 \end{aligned}$$

Now let's check if this is linear (it is beacuse it is basically applying a linear transformation two times, but lets check it).

$$\begin{aligned} T((at^2+bt+c)+\alpha(dt^2+et+f)) &= T((a+\alpha d)t^2+(b+\alpha e)t+(c+\alpha f)) \\ &= 2(a+\alpha d)t^2 \\ T(at^2+bt+c)+\alpha T(dt^2+et+f) &= 2at^2+2\alpha dt^2 \\ &= 2(a+\alpha d)t^2 \\ T((at^2+bt+c)+\alpha(dt^2+et+f)) &= T(at^2+bt+c)+\alpha T(dt^2+et+f) \end{aligned}$$

It is a Linear transformation

9 P.27

It is actually quite simple to make this problem, what we need is a linear combination of x, y and z that equals to 0 when the coordinates solve this equations:

$$\begin{cases} x+y+z=0 \\ x-2y-3z=0 \end{cases}$$

But that equations already solve to 0 (and are 2). So we can take the transformation as

$$T(x,y,z) = (x+y+z, x-2y-3z)$$

This already gave us the transformation, now let's check what is the image by first representhing this as the sum of vectors.

$$T(x,y,z) = x(1,1) + y(1,-2) + z(1,-3)$$

And now we should check what is the space that this generates

$$
\begin{aligned}
T(x,y,z) &\implies \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & -3 & 0 \end{array}\right) \\
(-1)(I)+(II) &\implies \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right) \\
(-1)(I)+(III) &\implies \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & -4 & 0 \end{array}\right) \\
(-\frac{1}{3})(II) &\implies \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 0 \end{array}\right) \\
(-1)(II)+(I) &\implies \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 0 \end{array}\right) \\
(4)(II)+(III) &\implies \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)
\end{aligned}
$$

This actually mean that it generates $\mathbb{R}^2$

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$$
\begin{aligned}
F(1,2,1) &= (1+1, 2-2) = (2,0) \\
F(0,1,1) &= (0+1, 1-2) = (1,-1) \\
F(0,3,-1) &= (0-1, 3+2) = (-1,5)
\end{aligned}
$$

Now we need to represent that in the base $\mathscr{C}$

$$
\begin{aligned}
(2,0) &= \alpha_1(1,5) + \beta_1(2,-1) \\
(1,-1) &= \alpha_2(1,5) + \beta_2(2,-1) \\
(-1,5) &= \alpha_3(1,5) + \beta_3(2,-1) \\
&\implies \left(\begin{array}{cc|ccc} 1 & 2 & 2 & 1 & -1 \\ 5 & -1 & 0 & -1 & 5 \end{array}\right) \\
(-5)(I)+(II) &\implies \left(\begin{array}{cc|ccc} 1 & 2 & 2 & 1 & -1 \\ 0 & -11 & -10 & -6 & 10 \end{array}\right) \\
(-\frac{1}{11})(II) &\implies \left(\begin{array}{cc|ccc} 1 & 2 & 2 & 1 & -1 \\ 0 & 1 & \frac{10}{11} & \frac{6}{11} & -\frac{10}{11} \end{array}\right) \\
(-2)(II)+(I) &\implies \left(\begin{array}{cc|ccc} 1 & 0 & \frac{2}{11} & -\frac{1}{11} & \frac{9}{11} \\ 0 & 1 & \frac{10}{11} & \frac{6}{11} & -\frac{10}{11} \end{array}\right) \\
[F]^{\mathscr{C}}_{\mathscr{B}} &= \frac{1}{11}\begin{pmatrix} 2 & -1 & 9 \\ 10 & 6 & -10 \end{pmatrix}
\end{aligned}
$$