1.

Partimos desde la definición
$$P_{ij}^{(\alpha)} := \frac{n_\alpha}{|G|}\sum_{g\in G} D_{ij}^{(\alpha)}(g^{-1})r(g)$$

Y por tanto

$$\begin{aligned}
P_{ij}^{(\alpha)}P_{kl}^{(\beta)} &= \left(\frac{n_\alpha}{|G|}\sum_{g\in G} D_{ij}^{(\alpha)}(g^{-1})r(g)\right)\left(\frac{n_\beta}{|G|}\sum_{h\in G} D_{kl}^{(\beta)}(h^{-1})r(h)\right)\\
&= \frac{n_\alpha n_\beta}{|G|^2}\sum_{g,h\in G} D_{ij}^{(\alpha)}(g^{-1})D_{kl}^{(\beta)}(h^{-1})r(g)r(h)\\
&= \frac{n_\alpha n_\beta}{|G|^2}\sum_{g,h\in G} D_{ij}^{(\alpha)}(g^{-1})D_{kl}^{(\beta)}(h^{-1})r(gh)
\end{aligned}$$

Ahora podemos tomar una nueva variable $x = gh \implies h = g^{-1}x$ y por tanto

$$\begin{aligned}
P_{ij}^{(\alpha)}P_{kl}^{(\beta)} &= \frac{n_\alpha n_\beta}{|G|^2}\sum_{g,x\in G} D_{ij}^{(\alpha)}(g^{-1})D_{kl}^{(\beta)}((g^{-1}x)^{-1})r(x)\\
&= \frac{n_\alpha n_\beta}{|G|^2}\sum_{g,x\in G} D_{ij}^{(\alpha)}(g^{-1})D_{kl}^{(\beta)}(x^{-1}g)r(x)\\
D_{kl}^{(\beta)}(x^{-1}g) &= \sum_{m=1}^{n_\beta} D_{km}^{(\beta)}(x^{-1})D_{ml}^{(\beta)}(g)\\
P_{ij}^{(\alpha)}P_{kl}^{(\beta)} &= \frac{n_\alpha n_\beta}{|G|^2}\sum_{g,x\in G} D_{ij}^{(\alpha)}(g^{-1})\sum_{m=1}^{n_\beta} D_{km}^{(\beta)}(x^{-1})D_{ml}^{(\beta)}(g)r(x)\\
&= \frac{n_\alpha n_\beta}{|G|^2}\sum_{x\in G} r(x)\sum_{m=1}^{n_\beta} D_{km}^{(\beta)}(x^{-1})\left(\sum_{g\in G} D_{ml}^{(\beta)}(g)D_{ij}^{(\alpha)}(g^{-1})\right)\\
\sum_{g\in G} D_{ml}^{(\beta)}(g)D_{ij}^{(\alpha)}(g^{-1}) &= \frac{|G|}{n_\alpha}\delta_{\alpha\beta}\delta_{il}\delta_{mj}\\
P_{ij}^{(\alpha)}P_{kl}^{(\beta)} &= \frac{n_\alpha n_\beta}{|G|^2}\sum_{x\in G} r(x)\sum_{m=1}^{n_\beta} D_{km}^{(\beta)}(x^{-1})\left(\frac{|G|}{n_\alpha}\delta_{\alpha\beta}\delta_{il}\delta_{mj}\right)
\end{aligned}$$

Dado que tenemos δ_{mj} entonces esa sumatoria se reduce a simplemente el caso $m = j$

$$\begin{aligned}
P_{ij}^{(\alpha)}P_{kl}^{(\beta)} &= \delta_{\alpha\beta}\delta_{il}\frac{n_\beta}{|G|}\sum_{x\in G} D_{kj}^{(\beta)}(x^{-1})r(x)\\
&= \delta_{\alpha\beta}\delta_{il}P_{kj}^{(\beta)}
\end{aligned}$$

Esto ya es el resultado pues sabemos que en caso de que $\alpha \neq \beta$ entonces todo se va a 0. Por lo tanto
$$P_{ij}^{(\alpha)}P_{kl}^{(\beta)} = \delta_{\alpha\beta}\delta_{il}P_{kj}^{(\alpha)}$$

2.

2.1. $P^2 = P$.

$$\left(P^{(\alpha)}\right)^2 = \sum_{k,j=1}^{n_\alpha} P^{\alpha}_{kk} P^{\alpha}_{jj}$$

$$P^{(\alpha)}_{kk} P^{(\alpha)}_{jj} = \delta_{\alpha\alpha} \delta_{kj} P^{(\alpha)}_{kj}$$

$$\left(P^{(\alpha)}\right)^2 = \sum_{k,j=1}^{n_\alpha} \delta_{kj} P^{(\alpha)}_{kj}$$

$$\left(P^{(\alpha)}\right)^2 = \sum_{k=1}^{n_\alpha} P^{(\alpha)}_{kk}$$

$$\left(P^{(\alpha)}\right)^2 = P^{\alpha}$$

2.2. $P^* = P$.