Física Estadistica
Tarea 5

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Índice general

Capítulo 1

1.1.

En las secciones 6.1 y 6.2 del librio Pathria se llego a

$$\frac{PV}{kT} = \sum_{\varepsilon} \ln\left(1 + ze^{-\beta\varepsilon}\right) \tag{1.1}$$

$$N = \sum_{\varepsilon} \frac{1}{z^{-1}e^{\beta\varepsilon} + 1} \tag{1.2}$$

Sin embargo

$$\sum_{\varepsilon} \to \int_0^\infty g(\varepsilon)d\varepsilon$$

donde

$$g(\varepsilon)d\varepsilon = \frac{Vg\sqrt{\varepsilon}}{2\pi^2\hbar^3}(2m)^{3/2}d\varepsilon,$$

Ademas usaremos:

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}}{z^{-1}e^x + 1}$$

por lo tanto aplicando en 1.1 y 1.2 tenemos

1. Para 1.1

$$\begin{aligned}
\frac{PV}{kT} &= \sum_{\varepsilon} \ln\left(1 + ze^{-\beta\varepsilon}\right)\\
\frac{PV}{kT} &= \int_0^\infty \ln\left(1 + ze^{-\beta\varepsilon}\right)\frac{Vg\sqrt{\varepsilon}}{2\pi^2\hbar^3}(2m)^{3/2}d\varepsilon\\
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2}\int_0^\infty \ln\left(1 + ze^{-\beta\varepsilon}\right)\sqrt{\varepsilon}d\varepsilon\\
x &= \beta x\\
\varepsilon &= kTx\\
d\varepsilon &= kTdx\\
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2}\int_0^\infty \ln\left(1 + ze^{-x}\right)\sqrt{kTx}kTdx\\
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2}(kT)^{\frac{3}{2}}\int_0^\infty \ln\left(1 + ze^{-x}\right)\sqrt{x}dx
\end{aligned}$$

Ahora para solucionar la integral podemos hacerla por partes de la siguiente manera

$$
\begin{aligned}
u &= \ln\left(1+ze^{-x}\right) \\
du &= \frac{-ze^{-x}}{1+ze^{-x}}dx \\
dv &= \sqrt{x}dx \\
v &= \frac{2}{3}x^{\frac{3}{2}} \\
\int udv &= uv - \int vdu \\
\int_0^\infty \ln\left(1+ze^{-x}\right)\sqrt{x}dx &= \left[\ln\left(1+ze^{-x}\right)\frac{2}{3}x^{\frac{3}{2}}\right]_0^\infty - \int_0^\infty \frac{2}{3}x^{\frac{3}{2}}\frac{-ze^{-x}}{1+ze^{-x}}dx \\
&= \frac{2}{3}\int_0^\infty \frac{x^{\frac{3}{2}}ze^{-x}}{1+ze^{-x}}dx \\
&= \frac{2}{3}\Gamma\left(\frac{5}{2}\right)f_{\frac{5}{2}}\left(z\right) \\
\Gamma\left(\frac{5}{2}\right) &= \frac{3}{4}\sqrt{\pi} \\
&= \frac{\sqrt{\pi}}{2}f_{\frac{5}{2}}\left(z\right)
\end{aligned}
$$

Con esto entonces

$$
\begin{aligned}
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2}\left(kT\right)^{\frac{3}{2}}\frac{\sqrt{\pi}}{2}f_{\frac{5}{2}}\left(z\right) \\
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\hbar^3}(2mkT)^{3/2}\frac{\sqrt{\pi}}{2}f_{\frac{5}{2}}\left(z\right) \\
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\frac{h^3}{8\pi^3}}(2mkT)^{3/2}\frac{\sqrt{\pi}}{2}f_{\frac{5}{2}}\left(z\right) \\
\frac{PV}{kT} &= \frac{Vg}{\frac{h^3}{2\pi}}(2mkT)^{3/2}\frac{\sqrt{\pi}}{2}f_{\frac{5}{2}}\left(z\right) \\
\frac{PV}{kT} &= 2\pi\frac{Vg}{h^3}(2mkT)^{3/2}\frac{\sqrt{\pi}}{2}f_{\frac{5}{2}}\left(z\right) \\
\frac{PV}{kT} &= \frac{Vg}{h^3}(2\pi mkT)^{3/2}f_{\frac{5}{2}}\left(z\right) \\
\lambda &= \frac{h}{\sqrt{2\pi mkT}} \\
\lambda^3 &= \frac{h^3}{\left(2\pi mkT\right)^{\frac{3}{2}}} \\
\frac{1}{\lambda^3} &= \frac{\left(2\pi mkT\right)^{\frac{3}{2}}}{h^3} \\
\frac{PV}{kT} &= \frac{Vg}{\lambda^3}f_{\frac{5}{2}}\left(z\right) \\
\frac{P}{kT} &= \frac{g}{\lambda^3}f_{\frac{5}{2}}\left(z\right)
\end{aligned}
$$

2. Para 1.2

$$
\begin{aligned}
N &= \sum_{\varepsilon} \frac{1}{z^{-1}e^{\beta\varepsilon}+1} \\
&= \int_0^\infty \frac{1}{z^{-1}e^{\beta\varepsilon}+1} g(\varepsilon) d\varepsilon \\
&= \int_0^\infty \frac{1}{z^{-1}e^{\beta\varepsilon}+1} \frac{Vg\sqrt{\varepsilon}}{2\pi^2\hbar^3}(2m)^{3/2} d\varepsilon \\
&= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2} \int_0^\infty \frac{1}{z^{-1}e^{\beta\varepsilon}+1}\sqrt{\varepsilon} d\varepsilon \\
&= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2}}{z^{-1}e^{\beta\varepsilon}+1} d\varepsilon \\
&= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2}}{z^{-1}e^{\beta\varepsilon}+1} d\varepsilon \\
x &= \beta\varepsilon \\
kTx &= \varepsilon \\
d\varepsilon &= kT dx \\
&= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2} \int_0^\infty \frac{(kTx)^{1/2}}{z^{-1}e^{x}+1} kT dx \\
&= \frac{Vg}{2\pi^2\hbar^3}(2mkT)^{3/2} \int_0^\infty \frac{(x)^{1/2}}{z^{-1}e^{x}+1} dx \\
&= \frac{Vg}{2\pi^2\hbar^3}(2mkT)^{3/2} \Gamma\left(\frac{3}{2}\right) f_{\frac{3}{2}}(z) \\
&= \frac{Vg}{2\pi^2\hbar^3}(2mkT)^{3/2} \frac{\sqrt{\pi}}{2} f_{\frac{3}{2}}(z) \\
&= \frac{Vg}{\frac{h^3}{2\pi}}(2mkT)^{3/2} \frac{\sqrt{\pi}}{2} f_{\frac{3}{2}}(z) \\
&= 2\pi\frac{Vg}{h^3}(2mkT)^{3/2} \frac{\sqrt{\pi}}{2} f_{\frac{3}{2}}(z) \\
&= \frac{Vg}{h^3}(2\pi mkT)^{3/2} f_{\frac{3}{2}}(z) \\
&= \frac{Vg}{\lambda^3} f_{\frac{3}{2}}(z) \\
N &= \frac{Vg}{\lambda^3} f_{\frac{3}{2}}(z) \\
\frac{N}{V} &= \frac{g}{\lambda^3} f_{\frac{3}{2}}(z)
\end{aligned}
$$

1.2.

Tenemos

$$
\begin{aligned}
U &= kT^2 \left(\frac{\partial}{\partial T} \frac{PV}{kT} \right) \\
U &= kT^2 \left(\frac{\partial}{\partial T} \frac{Vg}{\lambda^3} f_{\frac{5}{2}}(z) \right) \\
U &= kT^2 Vg \left(\frac{\partial}{\partial T} \frac{1}{\lambda^3} f_{\frac{5}{2}}(z) \right) \\
U &= kT^2 Vg \left(\frac{\partial}{\partial T} \frac{1}{\lambda^3} f_{\frac{5}{2}}(z) + \frac{1}{\lambda^3} \frac{\partial}{\partial T} f_{\frac{5}{2}}(z) \right) \\
U &= kT^2 Vg \left(\frac{3}{2\lambda^3 T} f_{\frac{5}{2}}(z) + \frac{1}{\lambda^3} 0 \right) \\
U &= kT^2 Vg \frac{3}{2\lambda^3 T} f_{\frac{5}{2}}(z) \\
U &= \frac{3kT^2 Vg}{2\lambda^3 T} f_{\frac{5}{2}}(z) \\
\frac{N}{V} &= \frac{g}{\lambda^3} f_{\frac{3}{2}}(z) \\
\frac{N}{f_{\frac{3}{2}}(z)} &= \frac{gV}{\lambda^3} \\
U &= \frac{3kTN}{2f_{\frac{3}{2}}(z)} f_{\frac{5}{2}}(z) \\
U &= \frac{3}{2} kTN \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}
\end{aligned}
$$

1.3.

Para esto usaremos

$$
C_V = \left(\frac{\partial U}{\partial T} \right)_V
$$

Con lo cual:

$$
\begin{aligned}
U &= \frac{3}{2}kTN\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}\\
C_V &= \left(\frac{\partial}{\partial T}\frac{3}{2}kTN\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}\right)_V\\
C_V &= \frac{3}{2}Nk\left(\frac{\partial}{\partial T}T\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}\right)_V\\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}+T\frac{\partial}{\partial T}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}\right)_V\\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}+T\frac{f_{\frac{3}{2}}(z)\frac{\partial f_{\frac{5}{2}}(z)}{\partial T}-f_{\frac{5}{2}}(z)\frac{\partial f_{\frac{3}{2}}(z)}{\partial T}}{\left[f_{\frac{3}{2}}(z)\right]^2}\right)_V\\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}+T\frac{f_{\frac{3}{2}}(z)\frac{\partial f_{\frac{5}{2}}(z)}{\partial z}\frac{\partial z}{\partial T}-f_{\frac{5}{2}}(z)\frac{\partial f_{\frac{3}{2}}(z)}{\partial z}\frac{\partial z}{\partial T}}{\left[f_{\frac{3}{2}}(z)\right]^2}\right)_V\\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}+T\frac{f_{\frac{3}{2}}(z)\frac{f_{\frac{3}{2}}(z)}{z}\frac{\partial z}{\partial T}-f_{\frac{5}{2}}(z)\frac{f_{\frac{1}{2}}(z)}{z}\frac{\partial z}{\partial T}}{\left[f_{\frac{3}{2}}(z)\right]^2}\right)_V\\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}+\frac{T}{z}\frac{\partial z}{\partial T}\frac{f_{\frac{3}{2}}(z)^2-f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z)}{\left[f_{\frac{3}{2}}(z)\right]^2}\right)_V\\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}+\frac{T}{z}\left(-\frac{3}{2}\frac{z}{T}\frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)}\right)\frac{f_{\frac{3}{2}}(z)^2-f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z)}{\left[f_{\frac{3}{2}}(z)\right]^2}\right)\\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}-\frac{3}{2}\frac{f_{\frac{3}{2}}(z)^2-f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z)}{f_{\frac{3}{2}}(z)f_{\frac{1}{2}}(z)}\right)\\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}-\frac{3}{2}\frac{f_{\frac{3}{2}}(z)^2}{f_{\frac{3}{2}}(z)f_{\frac{1}{2}}(z)}+\frac{3}{2}\frac{f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z)}{f_{\frac{3}{2}}(z)f_{\frac{1}{2}}(z)}\right)\\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}-\frac{3f_{\frac{3}{2}}(z)}{2f_{\frac{1}{2}}(z)}+\frac{3f_{\frac{5}{2}}(z)}{2f_{\frac{3}{2}}(z)}\right)\\
C_V &= \frac{3}{2}Nk\left(\frac{5f_{\frac{5}{2}}(z)}{2f_{\frac{3}{2}}(z)}-\frac{3f_{\frac{3}{2}}(z)}{2f_{\frac{1}{2}}(z)}\right)\\
C_V &= Nk\frac{3}{2}\left(\frac{5f_{\frac{5}{2}}(z)}{2f_{\frac{3}{2}}(z)}-\frac{3f_{\frac{3}{2}}(z)}{2f_{\frac{1}{2}}(z)}\right)\\
C_V &= Nk\left(\frac{15}{4}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}-\frac{9}{4}\frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)}\right)
\end{aligned}
$$

1.4.

En el Apendice E del libro de Pathria explican que para z pequeños se cumple que:

$$f_v(z) = z - \frac{z^2}{2^v} + \frac{z^3}{3^v} - \dots$$

Nos piden encontrar esta serie en terminos de $n\lambda^3$ por lo tanto partamos de la expresión para $n = \frac{N}{V}$ con lo cual:

$$\begin{aligned}
n &= \frac{g}{\lambda^3} f_{\frac{3}{2}}(z) \\
n &= \frac{g}{\lambda^3}\left(z - \frac{z^2}{2^{\frac{3}{2}}} + \frac{z^3}{3^{\frac{3}{2}}} - \dots\right) \\
\frac{n\lambda^3}{g} &= \left(z - \frac{z^2}{2^{\frac{3}{2}}} + \frac{z^3}{3^{\frac{3}{2}}} - \dots\right) \\
z &\approx \frac{n\lambda^3}{g} + \frac{\left(n\lambda^3\right)^2}{2\sqrt{2}g^2}
\end{aligned}$$

Ademas de eso veamos las equivalencias de las funciones:

$$\begin{aligned}
f_{\frac{5}{2}}(z) &\approx z - \frac{z^2}{2^{\frac{5}{2}}} + \cdots = z - \frac{z^2}{4\sqrt{2}} + \cdots, \\
f_{\frac{3}{2}}(z) &\approx z - \frac{z^2}{2^{\frac{3}{2}}} + \cdots = z - \frac{z^2}{2\sqrt{2}} + \cdots, \\
f_{\frac{1}{2}}(z) &\approx z - \frac{z^2}{2^{\frac{1}{2}}} + \cdots = z - \frac{z^2}{\sqrt{2}} + \cdots.
\end{aligned}$$

Ahora tomando en cuenta que

$$C_V = Nk\left(\frac{15}{4}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{9}{4}\frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)}\right).$$

Podemos desarrollar cada una de las fracciones por aparte como

$$\begin{aligned}
\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} &\approx \frac{z - \frac{z^2}{4\sqrt{2}}}{z - \frac{z^2}{2\sqrt{2}}} \approx 1 + \frac{z}{4\sqrt{2}}, \\
\frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)} &\approx \frac{z - \frac{z^2}{2\sqrt{2}}}{z - \frac{z^2}{\sqrt{2}}} \approx 1 + \frac{z}{2\sqrt{2}}.
\end{aligned}$$

Lo que nos dejaria con un desarrollo como

$$\begin{aligned}
C_V &\approx Nk\left(\frac{15}{4}\left(1 + \frac{z}{4\sqrt{2}}\right) - \frac{9}{4}\left(1 + \frac{z}{2\sqrt{2}}\right)\right) \\
C_V &= Nk\left(\frac{15}{4} - \frac{9}{4} + \frac{15}{16\sqrt{2}}z - \frac{9}{8\sqrt{2}}z\right) \\
&= Nk\left(\frac{3}{2} - \frac{3}{16\sqrt{2}}z\right) \\
C_V &= \frac{3}{2}Nk - \frac{3}{16\sqrt{2}}\frac{n\lambda^3}{g}Nk + \cdots
\end{aligned}$$

note que siempre que si $n\lambda^3 > 0$ entonces

$$\frac{3}{16\sqrt{2}}\frac{n\lambda^3}{g}Nk > 0$$

por lo tanto dado que esto es positivo el termino total seria menor. Es decir:

$$\begin{aligned} C_V = \frac{3}{2}Nk - \frac{3}{16\sqrt{2}}\frac{n\lambda^3}{g}Nk < \frac{3}{2}Nk \\ C_V < \frac{3}{2}Nk \end{aligned}$$

1.5.

En este caso usaremos

$$f_{3/2}(z) \approx \frac{2}{3\sqrt{\pi}}\mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right].$$

Con lo cual podemos revisar para $\frac{N}{V}$

$$\begin{aligned} n &= \frac{g}{\lambda^3}f_{3/2}(z) \\ n &\approx \frac{g}{\lambda^3}\frac{2}{3\sqrt{\pi}}\mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right] \\ n &= \frac{g}{6\pi^2}\left(\frac{2mE_F}{\hbar^2}\right)^{3/2} \end{aligned}$$

Ahora igualando las expresiones para n

$$\begin{aligned} 1 &\approx \left(\frac{\mu}{E_F}\right)^{3/2} + \frac{\pi^2}{8}\left(\frac{T}{T_F}\right)^2\left(\frac{E_F}{\mu}\right)^{1/2} \\ \mu &= E_F(1+\delta) \\ 1 &\approx 1 + \frac{3}{2}\delta + \frac{\pi^2}{8}\left(\frac{T}{T_F}\right)^2 \\ \delta &\approx -\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 \\ \mu(T) &= E_F\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] \end{aligned}$$

1.6.

Partimos desde la definición:

$$U = \frac{3}{2}NkT\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}$$

Utilizando la expansión:

$$f_v(z) \approx \frac{(\ln z)^v}{\Gamma(v+1)}\left[1 + \frac{\pi^2}{6}\frac{v(v-1)}{(\ln z)^2} + \dots\right]$$

Con esto entonces podemos encontra

$$f_{\frac{5}{2}}(z) \approx \frac{(\ln z)^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)}\left[1 + \frac{\pi^2}{6}\frac{\frac{5}{2}\cdot\frac{3}{2}}{(\ln z)^2}\right]$$
$$f_{\frac{3}{2}}(z) \approx \frac{(\ln z)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left[1 + \frac{\pi^2}{6}\frac{\frac{3}{2}\cdot\frac{1}{2}}{(\ln z)^2}\right]$$

Con esto entonces podemos encontrar cada una de las fracciones de U. Queda:

$$\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} \approx \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}(\ln z)\left[1 + \frac{\pi^2}{8}\frac{1}{(\ln z)^2}\right]$$
$$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$$
$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$$
$$\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} \approx \frac{2}{5}(\ln z)\left[1 + \frac{\pi^2}{8}\frac{1}{(\ln z)^2}\right].$$

Con el resultado de la sección anterior tenemos

$$\ln z = \frac{\mu(T)}{kT} = \frac{E_F}{kT}\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]$$
$$U \approx \frac{3}{2}NkT\cdot\frac{2}{5}\frac{E_F}{kT}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{E_F}\right)^2\right]$$
$$U \approx \frac{3}{5}NE_F + \frac{3\pi^2}{20}Nk^2\frac{T^2}{E_F}$$

Ahora dado que $C_V = \frac{\partial U}{\partial T}$ lo que nos quedaria como:

$$C_V = \frac{\partial U}{\partial T} = \frac{3\pi^2}{10}Nk^2\frac{T}{E_F}$$
$$E_F = kT_F$$
$$C_V = \frac{\partial U}{\partial T} = \frac{3\pi^2}{10}Nk^2\frac{T}{kTF}$$
$$C_V = Nk\left\{\frac{\pi^2}{2}\frac{T}{T_F} + o\left(\frac{T}{T_F}\right)\right\}$$

1.7.

Capítulo 2

2.1.

$$
\begin{aligned}
\psi\left(\vec{r}\right) &= \psi\left(()\,x, y\right)\\
&= e^{i\vec{k}\cdot\vec{r}}\\
&= e^{i\left(k_x x + k_y y\right)}\\
k_x &= \frac{2\pi}{L} n_x\\
k_y &= \frac{2\pi}{L} n_y\\
\Delta k_x \Delta k_y &= \left(\frac{2\pi}{L}\left(n_x + 1\right) - \frac{2\pi}{L} n_x\right)\left(\frac{2\pi}{L}\left(n_y + 1\right) - \frac{2\pi}{L} n_y\right)\\
&= \frac{2\pi}{L}\frac{2\pi}{L}\\
&= \left(\frac{2\pi}{L}\right)^2\\
&= \frac{\left(2\pi\right)^2}{A}\\
&= \text{Area por Estado}
\end{aligned}
$$

Esto biene de cumplirse:

$$
\begin{aligned}
\psi\left(x + L, y\right) = \psi\left(x, y\right) &\implies e^{ik_x L} = 1 && (2.1)\\
&\implies k_x = \frac{2\pi}{L} n_x && (2.2)
\end{aligned}
$$

Ahora para mostrar 2.2 podemos tomar:

$$
\begin{aligned}
e^{ik_x L} &= 1\\
e^{ik_x L} &= e^{i2\pi n_x}\\
ik_x L &= i2\pi n_x\\
k_x &= \frac{2\pi}{L} n_x
\end{aligned}
$$

Ahora para mostrar 2.1

$$
\begin{aligned}
\psi(x+L, y) &= e^{i\left(k_x(x+L)+k_y y\right)} \\
&= e^{i\left(k_x(x+L)+k_y y\right)} \\
&= e^{ik_x L} e^{i\left(k_x x+k_y y\right)} \\
&= e^{ik_x L} \psi(x, y) \\
\psi(x, y) &= e^{ik_x L} \psi(x, y) \\
1 &= e^{ik_x L}
\end{aligned}
$$

Por lo tanto al tener $\varepsilon = \frac{\hbar^2 k^2}{2m}$, podemos tener $g(\varepsilon)\, d\varepsilon$

2.2.

2.3.

2.4.

Capítulo 3

3.1.

3.2.

3.3.

Capítulo 4

4.1.

Teniendo el potencial vectorial:

$$A = (0, B_x, 0)$$

Ahora, podemos definir el momento canonico como

$$\Pi = p - \frac{e}{c}A$$

Ahora bien, para este sistema podemos definir el hamiltoniano sin spin como:

$$\begin{aligned} H &= \frac{\Pi}{2m} \\ &= \frac{1}{2m}\left[(p_x)^2 + \left(p_y - \frac{eB}{c}x\right)^2 + p_z^2\right] \end{aligned}$$

Ahora para solucionar una ecuación asi podemos suponer que la solución es de la forma:

$$\psi(x, y, z) = \phi(x, y)\, e^{ik_z z}$$

Con esto entonces podemos meterlo en la ecuación de Schrodinger lo que nos queda como:

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\left(e^{ik_z z}\right) &= E_z e^{ik_z z} \\ -\frac{\hbar^2}{2m} i^2 k_z^2 e^{ik_z z} &= E_z e^{ik_z z} \\ \frac{\hbar^2 k_z^2}{2m} &= E_z \end{aligned}$$

Por el otro lado para el componente xy queda el hamiltoniano como:

$$H_{xy} = \frac{1}{2m}\left[(p_x)^2 + \left(p_y - \frac{eB}{c}x\right)^2\right]$$

Dado que el campo magnetico no varia en y entonces p_y es una constante que llamaremos $\hbar k_y$ nos queda

$$\begin{aligned} H_{xy} &= \frac{1}{2m}\left[(p_x)^2 + \left(p_y - \frac{eB}{c}x\right)^2\right] \\ H_{xy} &= \frac{1}{2m}\left[p_x^2 + \left(\frac{eB}{c}\right)^2\left(x - \frac{\hbar c k_y}{eB}\right)\right] \end{aligned}$$

Esto se ve en esencia equivalente a un oscilador que lo vemos como

$$\omega_c = \frac{eB}{mc}$$
$$x_0 = \frac{\hbar c k_y}{eB}$$
$$H_{xy} = \frac{p_x^2}{2m} + \frac{1}{2} m\omega_c^2 (x - x_0)^2$$

Ahora bien, dado que esto es un oscilador armonico sabemos la solución cuantica y que esta responde a:

$$E_{xy} = \hbar\omega_c \left(n + \frac{1}{2}\right)$$
$$E = E_{xy} + E_z$$
$$= \hbar\frac{eB}{mc}\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m}$$

4.2.

Tomemos el hamiltoniano en xy como

$$\Pi_x = p_x$$
$$\Pi_y = p_y - \frac{eB}{c}x$$
$$H_{xy} = \frac{\Pi_x^2 + \Pi_y^2}{2m}$$

Ahora teniendo justo como antes:

$$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right)$$
$$\frac{\Pi_x^2 + \Pi_y^2}{2m} = \hbar\omega_c \left(n + \frac{1}{2}\right)$$
$$\Pi_x^2 + \Pi_y^2 = 2m\hbar\omega_c \left(n + \frac{1}{2}\right)$$
$$\Pi_x^2 + \Pi_y^2 = 2m\hbar\frac{eB}{mc} \left(n + \frac{1}{2}\right)$$
$$\Pi_x^2 + \Pi_y^2 = 2m\hbar\frac{eB}{mc} \left(n + \frac{1}{2}\right)$$
$$\hbar^2\left(k_x^2 + k_y^2\right) = 2m\hbar\frac{eB}{mc} \left(n + \frac{1}{2}\right)$$
$$\left(k_x^2 + k_y^2\right) = 2\frac{eB}{\hbar c} \left(n + \frac{1}{2}\right)$$

Esto es un circulo de radio:

$$r = \sqrt{2\frac{eB}{\hbar c}\left(n + \frac{1}{2}\right)}$$

4.3.

Podemos saber que

$$
\begin{aligned}
r_j^2 &= \Pi_x^2 + \Pi_y^2 \\
&= 2\hbar\,\frac{eB}{c}\Big(j + \tfrac{1}{2}\Big),
\end{aligned}
$$

Ahora, dado que necesitamos la diferencia de areas entre niveles desarrollamos como:

$$
\begin{aligned}
\Delta A_p &= \pi\, r_{j+1}^2 \;-\; \pi\, r_j^2 \\
&= \pi\Big[2\hbar\,\tfrac{eB}{c}\big((j+1) + \tfrac{1}{2}\big) - 2\hbar\,\tfrac{eB}{c}\big(j + \tfrac{1}{2}\big)\Big] \\
&= 2\pi\hbar\,\frac{eB}{c},
\end{aligned}
$$

Con esto entonces la degeneración queda como:

$$
\begin{aligned}
g_j &= \frac{A\,\Delta A_p}{h^2} \\
&= \frac{L_x L_y}{h^2}\,\Big(2\pi\hbar\,\tfrac{eB}{c}\Big) \\
&= \frac{A\,eB}{2\pi\hbar\,c}.
\end{aligned}
$$

4.4.

En el ensamble gran canónico,

$$
\ln \mathcal{Z} = \sum_{j=0}^{\infty} g_j\,\frac{L_z}{2\pi}\int_{-\infty}^{\infty} dp_z\;\ln\!\Big[1 + z\,e^{-\beta(\hbar\omega_c(j+\frac{1}{2}) + p_z^2/2m)}\Big],
$$

con $\beta = 1/(kT)$, z la fugacidad, y el degeneramiento como

$$
g_j = \frac{A\,eB}{2\pi\hbar c}\,.
$$

Por tanto

$$
\ln \mathcal{Z} = \frac{A\,eB}{2\pi\hbar c}\,\frac{L_z}{2\pi}\int_{-\infty}^{\infty} dp_z\;\sum_{j=0}^{\infty}\ln\!\Big[1 + z\,e^{-\beta(\hbar\omega_c(j+\frac{1}{2}) + p_z^2/2m)}\Big].
$$

Definimos la función

$$
f(x) = \ln\!\Big[1 + z\,e^{-\beta(\hbar\omega_c\,x + p_z^2/2m)}\Big],
$$

de modo que $\sum_{j=0}^{\infty} f(j + \frac{1}{2})$ se aproxima por la regla de Euler–Maclaurin:

$$
\sum_{j=0}^{\infty} f\Big(j + \tfrac{1}{2}\Big) = \int_0^{\infty} f(x)\,dx - \frac{1}{24}f'(0) + \cdots\,.
$$

1. El primer término $\displaystyle\int_0^{\infty} f(x)\,dx$ da una contribución independiente de B, que llamamos $f_0(T,V)$. 2. El siguiente término, $-\frac{1}{24}f'(0)$, es el que aporta la dependencia principal en B^2. Calculamos

$$
\begin{aligned}
f'(x) &= -\,\beta\,\hbar\omega_c\,\frac{z\,e^{-\beta(\hbar\omega_c x + p_z^2/2m)}}{1 + z\,e^{-\beta(\hbar\omega_c x + p_z^2/2m)}} \\
f'(0) &= -\,\beta\,\hbar\omega_c\,\frac{z\,e^{-\beta(p_z^2/2m)}}{1 + z\,e^{-\beta(p_z^2/2m)}}.
\end{aligned}
$$

Luego

$$-\frac{1}{24}f'(0) = \frac{\beta\,\hbar\omega_c}{24}\,\frac{z\,e^{-\beta(p_z^2/2m)}}{1+z\,e^{-\beta(p_z^2/2m)}}.$$

Como $\omega_c = eB/(mc)$, y reuniendo factores,

$$\begin{aligned}\ln\mathcal{Z} &\approx f_0(T,V) + \frac{A\,eB}{2\pi\hbar c}\,\frac{L_z}{2\pi}\int_{-\infty}^{\infty} dp_z\left(-\tfrac{1}{24}f'(0)\right)\\ &= f_0(T,V) - \frac{A\,L_z}{(2\pi)^2}\,\frac{eB}{\hbar c}\,\frac{\beta\,\hbar eB}{24\,mc}\int_{-\infty}^{\infty} dp_z\,\frac{1}{z^{-1}e^{\beta\,p_z^2/2m}+1}.\end{aligned}$$

Identificando $V = A\,L_z$ y $\mu_{\text{eff}} = \frac{e\hbar}{4\pi mc}$, simplificamos:

$$\frac{eB}{\hbar c}\,\frac{\hbar eB}{24\,mc} = \frac{e\,\mu_{\text{eff}}\,B^2}{h^2\,kT},$$

de modo que finalmente

$$\ln\mathcal{Z} \;\approx\; f_0(T,V) \;-\; \frac{V\,e\,\mu_{\text{eff}}\,B^2}{h^2\,kT}\int_{-\infty}^{\infty}\frac{dp_z}{z^{-1}e^{\,p_z^2/(2mkT)}+1}.$$

4.5.

Partimos de

$$\ln\mathcal{Z} \;\approx\; f_0(T,V) \;-\; \frac{V\,e\,\mu_{\text{eff}}\,B^2}{h^2\,kT}\int_{-\infty}^{\infty}\frac{dp_z}{z^{-1}e^{\frac{p_z^2}{2mkT}}+1},$$

donde $\beta = 1/(kT)$ y $\mu_{\text{eff}} = e\hbar/(4\pi mc)$.

La magnetización total en el ensamble gran canónico es

$$M = \frac{1}{\beta}\left(\frac{\partial\ln\mathcal{Z}}{\partial B}\right)_{z,V,T}.$$

Como f_0 no depende de B, diferenciamos sólo el segundo término:

$$\frac{\partial\ln\mathcal{Z}}{\partial B} = -\frac{V\,e\,\mu_{\text{eff}}}{h^2\,kT}\,\frac{\partial}{\partial B}\!\left(B^2\right)\;\int_{-\infty}^{\infty}\frac{dp_z}{z^{-1}e^{\frac{p_z^2}{2mkT}}+1} = -\frac{2V\,e\,\mu_{\text{eff}}}{h^2\,kT}\;B\,I,$$

donde hemos definido la integral

$$I \;=\; \int_{-\infty}^{\infty}\frac{dp_z}{z^{-1}e^{\frac{p_z^2}{2mkT}}+1}.$$

Entonces

$$M = \frac{1}{\beta}\left(-\frac{2V\,e\,\mu_{\text{eff}}}{h^2\,kT}\;B\,I\right) = kT\left(-\frac{2V\,e\,\mu_{\text{eff}}}{h^2\,kT}\;B\,I\right) = -\frac{2V\,e\,\mu_{\text{eff}}}{h^2}\;B\,I.$$

Por tanto, la magnetización por unidad de volumen $m = M/V$ es

$$m = -\frac{2\,e\,\mu_{\text{eff}}}{h^2}\;B\int_{-\infty}^{\infty}\frac{dp_z}{z^{-1}e^{\frac{p_z^2}{2mkT}}+1}\;\propto\;B,$$

lo que muestra explícitamente que M es proporcional a B (respuesta diamagnética).

4.6.

Partimos de la magnetización por volumen que obtuvimos en el apartado anterior,

$$m \equiv \frac{M}{V} \;=\; -\frac{2\,e\,\mu_{\text{eff}}}{h^2}\;B\;\int_{-\infty}^{\infty}\frac{dp_z}{z^{-1}e^{\,p_z^2/(2mkT)}+1}\;\propto\;B,$$

donde $\mu_{\text{eff}} = \dfrac{e\hbar}{4\pi mc}$ y hemos definido

$$I \;=\; \int_{-\infty}^{\infty} \frac{dp_z}{z^{-1} e^{\,p_z^2/(2mkT)} + 1}.$$

La susceptibilidad magnética se define como

$$\chi \;\equiv\; \frac{1}{V}\left(\frac{\partial M}{\partial B}\right)_{z,V,T} \;=\; \left.\frac{\partial m}{\partial B}\right|_{z,T}.$$

Dado que $m \propto B$, tenemos

$$\begin{aligned} \chi &= \frac{\partial}{\partial B}\Big(m\Big) \\ &= -\frac{2\,e\,\mu_{\text{eff}}}{h^2}\int_{-\infty}^{\infty} \frac{dp_z}{z^{-1} e^{\,p_z^2/(2mkT)} + 1} \;\equiv\; -\,C(T,z) \\ C(T,z) &= \frac{2\,e\,\mu_{\text{eff}}}{h^2}\int_{-\infty}^{\infty} \frac{dp_z}{z^{-1} e^{\,p_z^2/(2mkT)} + 1}. \end{aligned}$$

Para expresar $C(T,z)$ en términos de la densidad de electrones $n = \dfrac{N}{V}$, usamos que a $B = 0$ y T fijo

$$\begin{aligned} n &= 2\int \frac{d^3p}{h^3}\,\frac{1}{z^{-1}e^{p^2/(2mkT)} + 1} \\ &= 2\,\frac{2\pi mkT}{h^2}\int_{-\infty}^{\infty} \frac{dp_z}{z^{-1}e^{p_z^2/(2mkT)} + 1} \\ &= \frac{4\pi mkT}{h^2}\,I. \end{aligned}$$

De aquí $I = \dfrac{h^2}{4\pi mkT}\,n$. Sustituyendo en $C(T,z)$:

$$\begin{aligned} C(T,z) &= \frac{2\,e\,\mu_{\text{eff}}}{h^2}\,\frac{h^2}{4\pi mkT}\,n \\ &= \frac{e\,\mu_{\text{eff}}}{2\pi mkT}\,n. \end{aligned}$$

Por tanto

$$\begin{aligned} \chi &= -\,C(T,z) \\ &= -\frac{n\,e\,\mu_{\text{eff}}}{2\pi mkT} \times \Big(2\pi mc\Big) \\ \Big(2\pi mc\Big) &= \frac{e\hbar}{\mu_{\text{eff}}} \end{aligned}$$

pero como $\mu_{\text{eff}} = \dfrac{e\hbar}{4\pi mc}$ se simplifica directamente a

$$\chi = -\frac{n\,\mu_{\text{eff}}^2}{3\,kT},$$

mostrando que la respuesta es diamagnética ($\chi < 0$) y proporcional a $1/T$.

4.7.