Mecánica
Tarea 2

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Contents

Chapter 1

1.1 Random Examples

Definition 1.1.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say

$$\lim_{n\to\infty} s_n = s$$

where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$

$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Question 1

Is the set $x-$axis$\setminus\{$Origin$\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Note:-

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

Claim 1.1.1 Topology

Topology is cool

Example 1.1.1 (Open Set and Close Set)

Open Set:
- ϕ
- $\bigcup\limits_{x\in X} B_r(x)$ (Any $r > 0$ will do)
- $B_r(x)$ is open

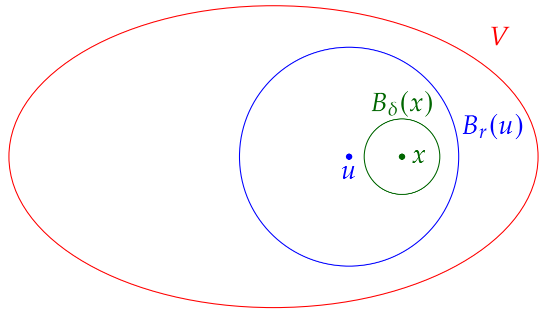
Closed Set:
- $X,\ \phi$
- $\overline{B_r(x)}$

$x-$axis $\cup$ $y-$axis

Theorem 1.1.1

If $x \in$ open set V then $\exists$ $\delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leqslant d(u, x) + d(x, y) < d + \delta < r$$

😊

Corollary 1.1.1

By the result of the proof, we can then show...

Lenma 1.1.1

Suppose $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 1.1.1

$1 + 1 = 2$.

1.2 Random

Definition 1.2.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\|$ $V \to \mathbb{R}_{\geqslant 0}$ satisfying

1. $\|x\| = 0 \iff x = 0 \; \forall \; x \in V$
2. $\|\lambda x\| = |\lambda| \|x\| \; \forall \; \lambda \in \mathbb{R}(\text{or } \mathbb{C}),\; x \in V$
3. $\|x + y\| \leqslant \|x\| + \|y\| \; \forall \; x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.
• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\geqslant 0}$) where 2 becomes $\|\lambda x\| = |\lambda| \|x\|$ $\forall \; \lambda \in \mathbb{C}$, $x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

Example 1.2.1 ($p-$Norm)

$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geqslant 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_m|^p\right)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.

Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$

For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

Question 2

Prove that triangle inequality is true if $p \geqslant 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property ③ for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\begin{aligned}
&\sum_i (x_i + y_i)^2 \leqslant \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2}\right)^2 \\
\implies &\sum_i (x_i^2 + 2x_iy_i + y_i^2) \leqslant \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2 \\
\implies &\left[\sum_i x_iy_i\right]^2 \leqslant \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]
\end{aligned}$$

So in other words prove $\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$ where

$$\langle x, y\rangle = \sum_i x_iy_i$$

Note:-

- $\|x\|^2 = \langle x, x\rangle$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle \cdot, \cdot\rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx + x', y\rangle = r\langle x, y\rangle + \langle x', y\rangle \text{ and similarly for second slot}$$

  Here in $\langle x, y\rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y\rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}
\langle x - \lambda y, x - \lambda y\rangle &= \langle x, x - \lambda y\rangle - \lambda\langle y, x - \lambda y\rangle \\
&= \langle x, x\rangle - \lambda\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle \\
&= \langle x, x\rangle - 2\lambda\langle x, y\rangle + \lambda^2\langle y, y\rangle
\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y\rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y\rangle = \sum_i \overline{x_i}y_i$$

Then we still have $\langle x, x\rangle \geqslant 0$

1.3 Algorithms

Algorithm 1: what

```
Input: This is some input
Output: This is some output
/* This is a comment */
some code here;
x ← 0;
y ← 0;
if x > 5 then
    x is greater than 5 ;                    // This is also a comment
else
    x is less than or equal to 5;
end
foreach y in 0..5 do
    y ← y + 1;
end
for y in 0..5 do
    y ← y − 1;
end
while x > 5 do
    x ← x − 1;
end
return Return something here;
```

Chapter 2

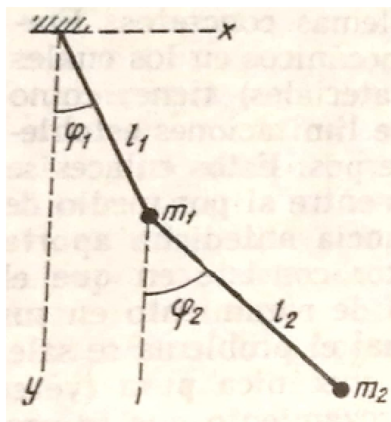


Figure 2.1: Péndulo Doble

En este caso, tenemos $3 \cdot N = 3 \cdot 2 = 6$ coordenadas cartesianas. Sin embargo, tenemos las siguientes ligaduras:

1. $z = 0$
2. $l_1 = cte$
3. $l_2 = cte$

Por lo cual tenemos $3N - n = 3 \cdot 2 - 4 = 2$. Por lo tanto tenemos dos grados de libertad. Con esto entonces definamos las coordenadas generalizadas de la manera en la que nos propone la imagen 2.1. Con lo cual nos queda:

$$
\begin{aligned}
x_1 &= l_1 \cdot \sin(\varphi_1) \implies \dot{x_1} = l_1 \dot{\varphi_1} \cos(\varphi_1) \\
y_1 &= -l_1 \cdot \cos(\varphi_1) \implies \dot{y_1} = l_1 \dot{\varphi_1} \sin(\varphi_1).
\end{aligned}
$$

Dado que este primer caso es esencialmente un triangulo en donde estamos calculando los dos catetos (Note que el valor de y es negativo)

Ahora bien, de manera similar, para la masa 2 esto seria como calcular estos mismos catetos. Sin embargo, debe iniciar desde los valores de m_1

$$
\begin{aligned}
x_2 &= l_1 \cdot \sin(\varphi_1) + l_2 \cdot \sin(\varphi_2) \implies \dot{x_2} = l_1 \dot{\varphi_1} \cos(\varphi_1) + l_2 \dot{\varphi_2} \cos(\varphi_2) \\
y_2 &= -l_1 \cdot \cos(\varphi_1) - l_2 \cdot \cos(\varphi_2) \implies \dot{y_2} = l_1 \dot{\varphi_1} \sin(\varphi_1) + l_2 \dot{\varphi_2} \sin(\varphi_2) \\
&.
\end{aligned}
$$

Con esto entonces, podemos calcular la energía cinética que es:

$$\begin{aligned}
T_1 &= \frac{1}{2} m_1 v_1^2 \\
&= \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) \\
&= \frac{1}{2} m_1 \left(l_1^2 \dot{\varphi_1}^2 \cos^2(\varphi_1) + l_1^2 \dot{\varphi_1}^2 \sin^2(\varphi)\right) \\
&= \frac{1}{2} m_1 \left(l_1^2 \dot{\varphi_1}^2\right)\left(\cos^2(\varphi_1) + \sin^2(\varphi_1)\right) \\
&= \frac{1}{2} m_1 \left(l_1^2 \dot{\varphi_1}^2\right).
\end{aligned}$$

Y

$$\begin{aligned}
T_2 &= \frac{1}{2} m_2 v_2^2 \\
&= \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right) \\
&= \frac{1}{2} m_2 \left(\left(l_1 \dot{\varphi}_1 \cos(\varphi_1) + l_2 \dot{\varphi}_2 \cos(\varphi_2)\right)^2 + \left(l_1 \dot{\varphi}_1 \sin(\varphi_1) + l_2 \dot{\varphi}_2 \sin(\varphi_2)\right)^2\right) \\
\left(l_1 \dot{\varphi}_1 \cos(\varphi_1) + l_2 \dot{\varphi}_2 \cos(\varphi_2)\right)^2 &= l_1^2 \dot{\varphi_1}^2 \cos^2(\varphi_1) + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1)\cos(\varphi_2) + l_2^2 \dot{\varphi_2}^2 \cos^2(\varphi_2) \\
\left(l_1 \dot{\varphi}_1 \sin(\varphi_1) + l_2 \dot{\varphi}_2 \sin(\varphi_2)\right)^2 &= l_1^2 \dot{\varphi_1}^2 \sin^2(\varphi_1) + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \sin(\varphi_1)\sin(\varphi_2) + l_2^2 \dot{\varphi_2}^2 \sin^2(\varphi_2) \\
l_1^2 \dot{\varphi_1}^2 \sin^2(\varphi_1) + l_1^2 \dot{\varphi_1}^2 \cos^2(\varphi_1)^2 &= l_1^2 \dot{\varphi_1}^2 \left(\sin^2(\varphi_1) + \cos^2(\varphi_1)\right) \\
&\implies l_1^2 \dot{\varphi_1}^2 \\
l_2^2 \dot{\varphi_2}^2 \sin^2(\varphi_2) + l_2^2 \dot{\varphi_2}^2 \cos^2(\varphi_2)^2 &= l_2^2 \dot{\varphi_2}^2 \left(\sin^2(\varphi_2) + \cos^2(\varphi_2)\right) \\
&\implies l_2^2 \dot{\varphi_2}^2 \\
2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1)\cos(\varphi_2) + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \sin(\varphi_1)\sin(\varphi_2) &= 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \left(\cos(\varphi_1)\cos(\varphi_2) + \sin(\varphi_1)\sin(\varphi_2)\right) \\
&\implies 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \\
&= \frac{1}{2} m_2 \left(l_1^2 \dot{\varphi_1}^2 + l_2^2 \dot{\varphi_2}^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)\right).
\end{aligned}$$

Ahora bien, para el caso de la energía potencial tenemos

$$\begin{aligned}
V_1 &= m_1 g y_1 \\
&= -m_1 g l_1 \cos(\varphi_1) \\
V_2 &= m_2 g y_2 \\
&= -m_2 g \left(l_1 \cos(\varphi_1) + l_2 \cos(\varphi_2)\right).
\end{aligned}$$

Ahora bien, tenemos entonces que:

$$\begin{aligned}
T &= T_1 + T_2 \\
&= \frac{1}{2} m_1 \left(l_1^2 \dot{\varphi}_1^2\right) + \frac{1}{2} m_2 \left(l_1^2 \dot{\varphi_1}^2 + l_2^2 \dot{\varphi_2}^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)\right) \\
&= \frac{1}{2} l_1^2 \dot{\phi_1}^2 (m_1 + m_2) + \frac{1}{2} m_2 l_2^2 \dot{\phi_2}^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2).
\end{aligned}$$

Y

$$\begin{aligned}
V &= V_1 + V_2 \\
&= -m_1 g l_1 \cos(\varphi_1) - m_2 g l_1 \cos(\varphi_1) - m_2 g l_2 \cos(\varphi_2) \\
&= -g l_1 \cos(\varphi_1)(m_1 + m_2) - m_2 g l_2 \cos(\varphi_2).
\end{aligned}$$

Por lo tanto dado que $L = T - V$ nos queda:

$$
\begin{aligned}
L &= T - V\\
L &= \frac{1}{2} l_1^2 \dot{\phi_1}^2 (m_1 + m_2) + \frac{1}{2} m_2 l_2^2 \dot{\phi_2}^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) - (-g l_1 \cos(\varphi_1)(m_1 + m_2) - m_2 g l_2 \cos(\varphi_2))\\
&= \frac{1}{2} l_1^2 \dot{\phi_1}^2 (m_1 + m_2) + \frac{1}{2} m_2 l_2^2 \dot{\phi_2}^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + g l_1 \cos(\varphi_1)(m_1 + m_2) + m_2 g l_2 \cos(\varphi_2).
\end{aligned}
$$