Mecanica Cuantica
Tarea 5

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2.1

Para mostrar que esta normalizado sumamos cada coeficiente y mostramos que esto equivale a 1

$$
\begin{aligned}
|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 &= 1 \\
\left|\frac{\sqrt{2}}{4}\right|^2 + \left|\frac{2i}{4}\right|^2 + \left|-\frac{i}{4}\right|^2 + \left|\frac{3}{4}e^{i\frac{\pi}{3}}\right|^2 &= 1 \\
\frac{2}{16} + \frac{4}{16} + \frac{1}{16} + \left|\frac{3}{4}\right|^2 \left|e^{i\frac{\pi}{3}}\right|^2 &= 1 \\
\frac{2}{16} + \frac{4}{16} + \frac{1}{16} + \frac{9}{16}\left|\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right|^2 &= 1 \\
\frac{2}{16} + \frac{4}{16} + \frac{1}{16} + \frac{9}{16}\left(\sqrt{\cos^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{3}\right)}\right)^2 &= 1 \\
\frac{2}{16} + \frac{4}{16} + \frac{1}{16} + \frac{9}{16}(1)^2 &= 1 \\
\frac{2}{16} + \frac{4}{16} + \frac{1}{16} + \frac{9}{16} &= 1 \\
\frac{2+4+1+9}{16} &= 1 \\
1 &= 1
\end{aligned}
$$

2.2

Para encontrar la energia podemos usar la ecuación 4.2.27 de las notas de clase en donde sabemos que los estados se pueden encontrar como:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

Por lo tanto las energias son:

$$
\begin{aligned}
E_n &= \left(n + \frac{1}{2}\right)\hbar\omega \\
E_0 &= \left(0 + \frac{1}{2}\right)\hbar\omega \\
&= \frac{1}{2}\hbar\omega \\
E_1 &= \left(1 + \frac{1}{2}\right)\hbar\omega \\
&= \left(\frac{3}{2}\right)\hbar\omega \\
E_2 &= \left(2 + \frac{1}{2}\right)\hbar\omega \\
&= \left(\frac{5}{2}\right)\hbar\omega \\
E_3 &= \left(3 + \frac{1}{2}\right)\hbar\omega \\
&= \left(\frac{7}{2}\right)\hbar\omega
\end{aligned}
$$

Ahora bien, las probabilidades son:

$$
\begin{aligned}
P_n &= \left|\left\langle n|\psi\right\rangle\right|^2 \\
&= |c_n|^2
\end{aligned}
$$

Esto ya lo calculamos en la sección anterior por lo que sabemos que serian:

$$
\begin{aligned}
P_0 &= \frac{2}{16} \\
P_1 &= \frac{4}{16} \\
P_2 &= \frac{1}{16} \\
P_3 &= \frac{9}{16}
\end{aligned}
$$

2.3

Para calcular

$$
\langle E\rangle = \sum_{n=0}^{3} P_n E_n
$$

Tomando los resultados de la sección anterior tenemos:

$$
\begin{aligned}
\langle E\rangle &= P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3 \\
&= \frac{2}{16}\left(\frac{1}{2}\hbar\omega\right) + \frac{4}{16}\left(\frac{3}{2}\hbar\omega\right) + \frac{1}{16}\left(\frac{5}{2}\hbar\omega\right) + \frac{9}{16}\left(\frac{7}{2}\hbar\omega\right) \\
&= \left(\frac{2}{32}\hbar\omega\right) + \left(\frac{12}{32}\hbar\omega\right) + \left(\frac{5}{32}\hbar\omega\right) + \left(\frac{63}{32}\hbar\omega\right) \\
&= \left(\frac{2 + 12 + 5 + 63}{32}\hbar\omega\right) \\
&= \left(\frac{82}{32}\hbar\omega\right) \\
&= \left(\frac{41}{16}\hbar\omega\right)
\end{aligned}
$$

Chapter 3

Chapter 4

4.1

Para solucionar esto partimos desde:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a_- + a_+\right);\ p = i\sqrt{\frac{m\omega\hbar}{2}}\left(a_+ - a_-\right)$$

Ahora bien, tomemos que:

$$\begin{aligned}
\langle\alpha|a_-|\alpha\rangle &= \langle\alpha|\alpha|\alpha\rangle \\
&= \alpha\langle\alpha|\alpha\rangle \\
&= \alpha \\
\langle\alpha|a_+|\alpha\rangle &= \langle\alpha|\alpha^*|\alpha\rangle \\
&= \alpha^*\langle\alpha|\alpha\rangle \\
&= \alpha^*
\end{aligned}$$

Por lo tanto

$$\begin{aligned}
x &= \sqrt{\frac{\hbar}{2m\omega}}\left(a_- + a_+\right) \\
\langle x\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|\left(a_- + a_+\right)|\alpha\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha + \alpha^*\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}2\Re\left(\alpha\right) \\
&= \sqrt{\frac{4\hbar}{2m\omega}}\Re\left(\alpha\right) \\
&= \sqrt{\frac{2\hbar}{m\omega}}\Re\left(\alpha\right)
\end{aligned}$$

Para $\langle p \rangle$

$$\begin{aligned}
p &= i\sqrt{\frac{m\omega\hbar}{2}}\left(a_+ - a_-\right) \\
\langle p \rangle &= i\sqrt{\frac{m\omega\hbar}{2}}\langle \alpha | \left(a_+ - a_-\right) | \alpha \rangle \\
&= i\sqrt{\frac{m\omega\hbar}{2}}\left(\alpha^* - \alpha\right) \\
&= i\sqrt{\frac{m\omega\hbar}{2}}\left(-2i\Im\left(\alpha\right)\right) \\
&= \sqrt{\frac{4m\omega\hbar}{2}}\Im\left(\alpha\right) \\
&= \sqrt{2m\omega\hbar}\Im\left(\alpha\right)
\end{aligned}$$

Ahora con los casos de $\langle x^2 \rangle$ y $\langle p^2 \rangle$
Primero miremos lo siguiente:

$$\begin{aligned}
x^2 &= \frac{\hbar}{2m\omega}\left(a_- + a_+\right)^2 \\
&= \frac{\hbar}{2m\omega}\left(a_-^2 + a_+^2 + a_-a_+ + a_+a_-\right) \\
p^2 &= -\frac{m\omega\hbar}{2}\left(a_+ - a_-\right)^2 \\
&= -\frac{m\omega\hbar}{2}\left(a_+^2 + a_-^2 - a_+a_- - a_-a_+\right)
\end{aligned}$$

Por lo tanto vamos a necesitar:

$$\begin{aligned}
\langle \alpha | a_-^2 | \alpha \rangle &= \alpha\langle \alpha | a_- | \alpha \rangle \\
&= \alpha^2 \\
\langle \alpha | a_+^2 | \alpha \rangle &= \alpha^*\langle \alpha | a_- | \alpha \rangle \\
&= \left(\alpha^*\right)^2 \\
\langle \alpha | a_+a_- | \alpha \rangle &= \alpha\langle \alpha | a_+ | \alpha \rangle \\
&= \alpha\alpha^*\langle \alpha | \alpha \rangle \\
&= |\alpha|^2 \\
\langle \alpha | a_-a_+ | \alpha \rangle &= \langle \alpha | a_+a_- + 1 | \alpha \rangle \\
&= \alpha\langle \alpha | a_+ | \alpha \rangle + \langle \alpha | 1 | \alpha \rangle \\
&= \alpha\alpha^*\langle \alpha | \alpha \rangle + 1\langle \alpha | \alpha \rangle \\
&= |\alpha|^2 + 1
\end{aligned}$$

Ya con esto podemos pasar a calcular

1. $\langle x^2 \rangle$

$$
\begin{aligned}
x^2 &= \frac{\hbar}{2m\omega}\left(a_-^2 + a_+^2 + a_-a_+ + a_+a_-\right) \\
\langle x^2 \rangle &= \frac{\hbar}{2m\omega}\langle\alpha|\left(a_-^2 + a_+^2 + a_-a_+ + a_+a_-\right)|\alpha\rangle \\
&= \frac{\hbar}{2m\omega}\left(\langle\alpha\left|a_-^2\right|\alpha\rangle + \langle\alpha\left|a_+^2\right|\alpha\rangle + \langle\alpha\left|a_-a_+\right|\alpha\rangle + \langle\alpha\left|a_+a_-\right|\alpha\rangle\right) \\
&= \frac{\hbar}{2m\omega}\left(\alpha^2 + (\alpha^*)^2 + |\alpha|^2 + |\alpha|^2 + 1\right) \\
&= \frac{\hbar}{2m\omega}\left(\alpha^2 + (\alpha^*)^2 + 2|\alpha|^2 + 1\right) \\
&= \frac{\hbar}{2m\omega}\left(2\Re(\alpha)^2 - 2\Im(\alpha)^2 + 2\Re(\alpha)^2 + 2\Im(\alpha)^2 + 1\right) \\
&= \frac{\hbar}{2m\omega}\left(4\Re(\alpha)^2 + 1\right) \\
&= \frac{2\hbar}{m\omega}\Re(\alpha)^2 + \frac{\hbar}{2m\omega}
\end{aligned}
$$

2. $\langle p^2 \rangle$

$$
\begin{aligned}
p^2 &= -\frac{m\omega\hbar}{2}\left(a_+^2 + a_-^2 - a_+a_- - a_-a_+\right) \\
\langle p^2 \rangle &= -\frac{m\omega\hbar}{2}\langle\alpha\left|a_+^2 + a_-^2 - a_+a_- - a_-a_+\right|\alpha\rangle \\
&= -\frac{m\omega\hbar}{2}\left(\langle\alpha\left|a_+^2\right|\alpha\rangle + \langle\alpha\left|a_-^2\right|\alpha\rangle - \langle\alpha\left|a_+a_-\right|\alpha\rangle - \langle\alpha\left|a_-a_+\right|\alpha\rangle\right) \\
&= -\frac{m\omega\hbar}{2}\left(\alpha^2 + (\alpha^*)^2 - |\alpha|^2 - \left(|\alpha|^2 + 1\right)\right) \\
&= -\frac{m\omega\hbar}{2}\left(2\Re(\alpha)^2 - 2\Im(\alpha)^2 - |\alpha|^2 - |\alpha|^2 - 1\right) \\
&= -\frac{m\omega\hbar}{2}\left(2\Re(\alpha)^2 - 2\Im(\alpha)^2 - 2\Re(\alpha)^2 - 2\Im(\alpha)^2 - 1\right) \\
&= -\frac{m\omega\hbar}{2}\left(-4\Im(\alpha)^2 - 1\right) \\
&= \frac{m\omega\hbar}{2}\left(4\Im(\alpha)^2 + 1\right) \\
&= \frac{m\omega\hbar}{2}4\Im(\alpha)^2 + \frac{m\omega\hbar}{2} \\
&= 2m\omega\hbar\Im(\alpha)^2 + \frac{m\omega\hbar}{2}
\end{aligned}
$$

Por lo tanto los resultados son:

1. $\langle x \rangle = \sqrt{\frac{2\hbar}{m\omega}}\Re(\alpha)$
2. $\langle p \rangle = \sqrt{2m\omega}\Im(\alpha)$
3. $\langle x^2 \rangle = \frac{2\hbar}{m\omega}\Re(\alpha)^2 + \frac{\hbar}{2m\omega}$
4. $\langle p^2 \rangle = 2m\omega\hbar\Im(\alpha)^2 + \frac{m\omega\hbar}{2}$

4.2

En este caso tenemos:

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

Por lo tanto veamos cuanto es $\langle x\rangle^2$

$$\langle x\rangle^2 = \left(\sqrt{\frac{2\hbar}{m\omega}}\Re\left(\alpha\right)\right)^2$$
$$\langle x\rangle^2 = \frac{2\hbar}{m\omega}\Re\left(\alpha\right)^2$$

Con esto entonces

$$\begin{aligned}\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2}\\ &= \sqrt{\frac{2\hbar}{m\omega}\Re\left(\alpha\right)^2 + \frac{\hbar}{2m\omega} - \frac{2\hbar}{m\omega}\Re\left(\alpha\right)^2}\\ &= \sqrt{\frac{\hbar}{2m\omega}}\end{aligned}$$

Por el otro lado

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$$

con

$$\begin{aligned}\langle p\rangle^2 &= \left(\sqrt{2m\omega\hbar}\Im\left(\alpha\right)\right)^2\\ &= \left(\sqrt{2m\omega\hbar}\Im\left(\alpha\right)\right)^2\\ &= 2m\omega\hbar\Im\left(\alpha\right)^2\end{aligned}$$

De nuevo calculemos

$$\begin{aligned}\sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2}\\ &= \sqrt{2m\omega\hbar\Im\left(\alpha\right)^2 + \frac{m\omega\hbar}{2} - 2m\omega\hbar\Im\left(\alpha\right)^2}\\ &= \sqrt{\frac{m\omega\hbar}{2}}\end{aligned}$$

Ahora al final:

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\cdot\sqrt{\frac{m\omega\hbar}{2}}$$
$$\sigma_x\sigma_p = \sqrt{\frac{\hbar^2 m\omega}{2\cdot 2m\omega}}$$
$$\sigma_x\sigma_p = \sqrt{\frac{\hbar^2}{4}}$$
$$\sigma_x\sigma_p = \frac{\hbar}{2}$$

4.3

Apliquemos a_- de la siguiente manera

$$\begin{aligned} a_- \left|\alpha\right\rangle &= \alpha \sum_n c_n \left|n\right\rangle \\ &= \sum_n c_n \sqrt{n} \left|n-1\right\rangle \end{aligned}$$

Dado que son esencialmente los mismos podemos hacer

$$\begin{aligned} \alpha \sum_n c_n \left|n\right\rangle &= \sum_n c_n \sqrt{n} \left|n-1\right\rangle \\ \sum_n \alpha c_n \left|n\right\rangle &= \sum_n c_n \sqrt{n} \left|n-1\right\rangle \\ \sum_n \alpha c_n \left|n\right\rangle &= \sum_n c_{n+1} \sqrt{n-1} \left|n\right\rangle \\ \alpha c_n &= c_{n+1} \sqrt{n-1} \\ \frac{\alpha}{\sqrt{n-1}} c_n &= c_{n+1} \end{aligned}$$

Dada esta definición recursiva podemos reducirla hasta

$$\frac{\alpha^n}{\sqrt{n!}} c_0 = c_n$$

4.4

Tenemos:

$$\begin{aligned} \left\langle\alpha|\alpha\right\rangle &= \sum_{n=0}^{\infty} \left|c_n\right|^2 = 1 \\ \sum_{n=0}^{\infty} \left|\frac{\alpha^n}{\sqrt{n!}}\right|^2 &= \left|c_0\right|^2 \sum_{n=0}^{\infty} \frac{\left|\alpha\right|^{2n}}{n!} \\ \sum_{n=0}^{\infty} \left|\frac{\alpha^n}{\sqrt{n!}}\right|^2 &= \left|c_0\right|^2 \sum_{n=0}^{\infty} \frac{\left(\left|\alpha\right|^2\right)^n}{n!} \end{aligned}$$

Esta es una serie exponencial conocida:

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

Por lo tanto

$$\begin{aligned} \left|c_0\right|^2 \sum_{n=0}^{\infty} \frac{\left|\alpha\right|^{2n}}{n!} &= \left|c_0\right|^2 e^{\left|\alpha\right|^2} \\ \left|c_0\right|^2 e^{\left|\alpha\right|^2} &= 1 \\ \left|c_0\right|^2 &= e^{-\left|\alpha\right|^2} \\ c_0 &= e^{-\left|\alpha\right|^2/2} \end{aligned}$$

4.5

En los punto anterior definimos que:

$$|\alpha\rangle = \sum_n c_n |n\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} c_0 |n\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} |n\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Con esto entonces, podemos pasar a $|\alpha(t)\rangle$ de la manera en la que nos dicen esto queda como

$$|\alpha(t)\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-iE_n t/\hbar} |n\rangle$$

Ahora apliquemos a_-

$$\begin{aligned}
a_- |\alpha(t)\rangle &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-iE_n t/\hbar} \sqrt{n} |n-1\rangle \\
a_- |\alpha(t)\rangle &= e^{-|\alpha|^2/2} \sum_{n+1} \frac{\alpha^{n+1}}{\sqrt{n+1!}} \sqrt{n+1} e^{-iE_{n+1} t/\hbar} |n\rangle \\
a_- |\alpha(t)\rangle &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \alpha e^{-iE_{n+1} t/\hbar} |n\rangle \\
E_n &= \hbar\omega \left(n + \frac{1}{2}\right) \\
\implies E_{n+1} &= \hbar\omega \left(n + \frac{3}{2}\right) \\
\implies e^{-iE_{n+1} t/\hbar} &= e^{-i\omega t} e^{-iE_n t/\hbar} \\
a_- |\alpha(t)\rangle &= e^{-|\alpha|^2/2} \alpha \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega t} e^{-iE_n t/\hbar} |n\rangle \\
a_- |\alpha(t)\rangle &= e^{-|\alpha|^2/2} \alpha e^{-i\omega t} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-iE_n t/\hbar} |n\rangle \\
\alpha e^{-i\omega t} &= \alpha(t) \\
a_- |\alpha(t)\rangle &= \alpha(t) e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-iE_n t/\hbar} |n\rangle \\
a_- |\alpha(t)\rangle &= \alpha(t) |\alpha(t)\rangle
\end{aligned}$$

Esto en esencia quiere decir que este operador oscila de manera coherente con un oscilador clasico de periodo ω. Cosa que para ser honestos tiene sentido pues estamos minimizando la incertidumbre.

4.6

Verifiquemos cada caso:

$$
\begin{aligned}
a_- |0\rangle &= 0 \cdot |0\rangle \\
\alpha &= 0 \\
|\alpha\rangle &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\
|\alpha = 0\rangle &= e^{-|0|^2/2} \sum_n \frac{0^n}{\sqrt{n!}} |n\rangle \\
|\alpha = 0\rangle &= \sum_n 0 |n\rangle \\
|\alpha = 0\rangle &= |0\rangle
\end{aligned}
$$

Por ultimo

1. $\langle x\rangle = \sqrt{\frac{2\hbar}{m\omega}} \Re(0) = 0$
2. $\langle p\rangle = \sqrt{2m\omega\hbar} \Im(0) = 0$
3. $\langle x^2\rangle = \frac{2\hbar}{m\omega} \Re(0)^2 + \frac{\hbar}{2m\omega} = \frac{\hbar}{2m\omega}$
4. $\langle p^2\rangle = 2m\omega\hbar \Im(0)^2 + \frac{m\omega\hbar}{2} = \frac{m\omega\hbar}{2}$

Por lo tanto:

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{\frac{\hbar}{2m\omega} - 0} \\
&= \sqrt{\frac{\hbar}{2m\omega}} \\
\sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\
&= \sqrt{\frac{m\omega\hbar}{2} - 0} \\
&= \sqrt{\frac{m\omega\hbar}{2}} \\
\sigma_x\sigma_p &= \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\frac{m\omega\hbar}{2}} \\
\sigma_x\sigma_p &= \sqrt{\frac{\hbar^2}{4}} \\
\sigma_x\sigma_p &= \frac{\hbar}{2}
\end{aligned}
$$

Por lo tanto si es un estado coherente con $\alpha = 0$