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Tarea Laplace
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1. *a*) Sea $f(x) = 1$ Luego,

$$F(p) = \int_0^\infty e^{-px} f(x)\, dx = \int_0^\infty e^{-px} dx = -\frac{1}{p} e^{-px}|_0^\infty.$$

Nótese que: Si $x \to \infty$ entonces $e^{-px} \to 0$ (para $p > 0$ y si $x = 0, e^{-px} = 1$ $\implies F(p) = -\frac{1}{p}(0-1) = \frac{1}{p}$

b) Sea $f(x) = x$ Luego

$$\begin{aligned} F(p) &= \int_0^\infty e^{-px} f(x)\, dx \\ &= \int_0^\infty e^{-px} x dx \\ u &= x \\ du &= dx \\ F(p) &= \left[x \left(-\frac{1}{p} e^{-px} \right) \right]_0^\infty - \int_0^\infty \left(-\frac{1}{p} e^{-px} \right) dx \\ &= \int_0^\infty \frac{1}{p} e^{-px} dx \\ &= \frac{1}{p^2} \end{aligned}$$

.

c) Sea $f(x) = x^n$. Luego,

$$\begin{aligned} F(p) &= \int_0^\infty x^n e^{-px} dx = In(p) \\ I(p) &= \int_0^\infty e^{-px} dx = \frac{1}{p} \end{aligned}$$

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Ahora bien, tomemos $u = x^n \implies du = nx^{(n-1)}dx \wedge dv = e^{-px}dx \implies v = -\frac{1}{p}e^{-px}$.

$$\begin{aligned} I_n(p) &= \int_0^\infty x^n e^{-px} dx = \left[-\frac{x^n}{p} e^{-px} \right]_0^\infty + \frac{n}{p} \int_0^\infty x^{(n-1)} e^{-px} dx \\ &= \frac{n}{p} \int_0^\infty x^{(n-1)} e^{-px} dx \\ I_n(p) &= \frac{n}{p} I_{n-1}(p) \\ I_n(p) &= \frac{n}{p} \cdot \frac{(n-1)}{p} \cdot I_{n-2}(p) = \ldots = \frac{n!}{p^n} I_0(p) = \frac{n!}{p^{n+1}}. \end{aligned}$$

d) Sea $f(x) = e^{ax}$. Luego,

$$F(p) = \int_0^\infty e^{ax}e^{-px}dx = \left[\frac{1}{a-p}e^{(a-p)x}\right]_0^\infty = \frac{1}{p-a}.$$

e) Sea $f(x) = \sin(ax)$. Luego,

$$F(p) = \int_0^\infty \sin(ax)\, e^{-px}dx.$$

Tomemos $u = \sin(ax)$ y $dv = e^{-px}dx$. Entonces,

$$\begin{aligned}
I &= \int_0^\infty \sin(ax)\, e^{-px}dx \\
&= \left[-\frac{1}{p}\sin(ax)\, e^{-px}\right]_0^\infty + \frac{1}{p}\int_0^\infty a\cos(ax)\, e^{-px}dx \\
I &= \frac{a}{p}\left[\left[-\frac{1}{p}\cos(ax)\, e^{-px}\right]_0^\infty + \frac{1}{p}\int_0^\infty (-a\sin(ax))\, e^{-px}dx\right] \\
&= -\frac{a^2}{p^2}I \implies I + \frac{a^2}{p^2}I = 0 \\
I &= \frac{a}{(p^2+a^2)}
\end{aligned}$$

.

f) Sea $f(x) = \cos(ax)$. Luego,

$$\begin{aligned}
F(p) &= \int_0^\infty \cos(ax)\, e^{-px}dx = \int_0^\infty e^{-px}\left(\frac{e^{iax}+e^{-iax}}{2}\right)dx \\
&= \frac{1}{2}\int_0^\infty \left(e^{(ia-p)x} + e^{(-ia-p)x}\right)dx \\
&= \int_0^\infty e^{(ia-p)x}dx + \int_0^\infty e^{(-ia-p)x}dx \\
\int_0^\infty e^{(ia-p)x}dx &= \left[\frac{1}{ia-p}e^{(ia-p)x}\right]_0^\infty = \frac{1}{ia-p} \\
\int_0^\infty e^{(-ia-p)x}dx &= \left[\frac{1}{-ia-p}e^{(-ia-p)x}\right]_0^\infty = \frac{1}{-ia-p} \\
2F(p) &= \frac{1}{ia-p} + \frac{1}{-ia-p} = \frac{-p-p}{(ia)^2-p^2} \\
&= \frac{-2p}{(p^2+a^2)\cdot(-1)} \\
F(p) &= \frac{1}{2}\frac{2p}{p^2+a^2} = \frac{p}{p^2+a^2}
\end{aligned}$$

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g) $\sinh(ax)$: Nótese que:

$$\sinh(ax)=\frac{e^{ax}-e^{-ax}}{2}\implies L\{\sinh(ax)\}=L\left\{\frac{e^{ax}-e^{-ax}}{2}\right\}=\frac{1}{2}\left[L\{e^{ax}\}-L\left\{e^{-ax}\right\}\right]$$
$$=\frac{1}{2}\left(\frac{1}{p-a}-\frac{1}{p+a}\right)=\frac{1}{2}\left(\frac{p+a-p+a}{p^2-a^2}\right)$$
$$=\frac{1}{2}\left(\frac{2a}{p^2-a^2}\right)=\frac{1}{2}\left(\frac{2a}{p^2-a^2}\right)=\frac{a}{p^2-a^2}$$

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h) Transformada de $\cosh(ax)$: Nótese que

$$\cosh(ax)=\frac{e^{ax}+e^{-ax}}{2}\implies L\{\cosh(ax)\}=L\left\{\frac{e^{ax}+e^{-ax}}{2}\right\}=\frac{1}{2}\left[L\{e^{ax}\}+L\left\{e^{-ax}\right\}\right]$$
$$=\frac{1}{2}\left(\frac{1}{p-a}+\frac{1}{p+a}\right)=\frac{1}{2}\left(\frac{(p+a)+(p-a)}{(p-a)(p+a)}\right)$$
$$=\frac{1}{2}\left(\frac{p+a+p-a}{p^2-a^2}\right)=\frac{p}{p^2-a^2}$$

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2.

3. a) Sea $f(x)=x^5+\cos(2x)$ entonces

$$\begin{aligned}L\left\{x^5=\cos(2x)\right\}&=L\left\{x^5\right\}+L\{\cos(2x)\}\\&=\frac{5!}{p^6}+\frac{p}{p^2+2^2}\\&=\frac{5!}{p^6}+\frac{p}{p^2+4}.\end{aligned}$$

b) Sea $f(x)=2e^{3x}-\sin(5x)$ entonces

$$\begin{aligned}L\{f(x)\}&=L\left\{2e^{3x}\right\}-L\{\sin(5x)\}\\&=2L\left\{e^{3x}\right\}-L\{\sin(5x)\}\\&=2\frac{1}{p-3}-\frac{5}{p^2+25}\end{aligned}$$

.

c) Sea x^5e^{-2x} entonces,

$$L\{f(x)\}=\frac{d^5}{ds^5}\left(\frac{1}{s-2}\right)=\frac{120}{(s-2)^6}.$$

4. *a*) Sea $\frac{1}{p^2+p}$. Luego,

$$\frac{1}{p^2+p}=\frac{1}{p\left(p+1\right)}=\frac{A}{p}+\frac{B}{p+1};\ 1=A\left(p+1\right)+Bp.$$

para $p=0:$

$$1=A\left(0+1\right)\implies A=1.$$

para $p=-1:$

$$1=B\left(-1\right)\implies B=-1.$$

por lo tanto

$$\begin{aligned}\implies \frac{1}{p\left(p+1\right)}&=\frac{1}{p}-\frac{1}{p+1}\\ \implies L^{-1}\left\{\frac{1}{p}\right\}&=1\\ \implies L^{-1}\left\{\frac{1}{p+1}\right\}&=e^{-x}\\ \implies f\left(x\right)&=1-e^{-x}.\end{aligned}$$

b) Sea $\frac{4}{p^3}+\frac{6}{p^2+9}$. Luego,

$$\begin{aligned}\frac{4}{p^3}&=4\frac{1}{p^3}\implies 4L^{-1}\left\{\frac{1}{p^3}\right\}=4\frac{x^2}{2}=2x^2\\ \frac{6}{p^2+4}&=6\cdot\frac{1}{p^2+2^2}\implies 6L^{-1}\left\{\frac{1}{p^2+2^2}\right\}=6\frac{\sin\left(2x\right)}{2}=3\sin\left(2x\right)\\ f\left(x\right)&=2x^2+3\sin\left(2x\right).\end{aligned}$$

c) Sea $\frac{p+3}{p^2+2p+5}$. Luego,

$$\begin{aligned}\frac{p+3}{p^2+2p+5}&=\frac{p+1+2}{\left(p+1\right)^2+4}\\ &=\frac{p+1}{\left(p+1\right)^2+4}+\frac{2}{\left(p+1\right)^2+4}\\ L^{-1}\left\{\frac{p+1}{\left(p+1\right)^2+4}\right\}+L^{-1}\left\{\frac{2}{\left(p+1\right)^2+4}\right\}&=e^{-x}\cos\left(2x\right)+2e^{-x}\sin\left(2x\right)\\ &=e^{-x}\left(\cos\left(2x\right)+2\sin\left(2x\right)\right).\end{aligned}$$

5. *a*)

$$\begin{aligned}
\mathscr{L}\left\{y''\right\}-4\mathscr{L}\left\{y'\right\}+4\mathscr{L}\left\{y\right\}&=0\\
\mathscr{L}\left\{y''\right\}&=p^2y\left(p\right)-py\left(0\right)-y'\left(0\right)=p^2Y\left(p\right)-3\\
\mathscr{L}\left\{y'\right\}&=py\left(p\right)-y\left(0\right)=pY\left(p\right)\\
\mathscr{L}\left\{y\right\}&=Y\left(p\right)\\
p^2Y\left(p\right)-3-4\left(PY\left(p\right)\right)+4\left(Y\left(p\right)\right)&=0\\
\left(p^2-4p+4\right)Y\left(p\right)&=3\\
Y\left(p\right)\left(p-2\right)^2&=3\\
\implies Y\left(p\right)&=3\left(p-2\right)^{-2}\\
\implies 3\mathscr{L}^{-1}\left\{\frac{1}{\left(p-2\right)^2}\right\}&=3te^{2t}\\
\implies y\left(t\right)&=3te^{2t}.
\end{aligned}$$

b) Sea $y''+2y'+5y=3e^{-x}\sin\left(x\right)$ con $y\left(0\right)=0, y'\left(0\right)=3$. Luego,

$$\begin{aligned}
\mathscr{L}\left\{y''\right\}+2\mathscr{L}\left\{y'\right\}+5\mathscr{L}\left\{y\right\}&=3\mathscr{L}\left\{e^{-x}\sin\left(x\right)\right\}\\
\mathscr{L}\left\{y''\right\}&=p^2y\left(p\right)-py\left(0\right)-y'\left(0\right)=p^2Y\left(p\right)-3\\
\mathscr{L}\left\{y'\right\}&=py\left(p\right)-y\left(0\right)=pY\left(p\right)\\
\mathscr{L}\left\{y\right\}&=Y\left(p\right)\\
p^2Y\left(p\right)-3+2\left(pY\left(p\right)\right)+5Y\left(p\right)&=3\mathscr{L}\left\{e^{-x}\sin\left(x\right)\right\}\\
\left(p^2+2p+5\right)Y\left(p\right)-3&=3\mathscr{L}\left\{e^{-x}\sin\left(x\right)\right\}\\
\mathscr{L}\left\{f\left(t\right)\right\}=F\left(0\right)\implies\mathscr{L}\left\{e^{at}f\left(t\right)\right\}&=F\left(s-a\right)\\
\mathscr{L}\left\{\sin\left(x\right)\right\}&=\frac{1}{p^2+1}\\
\implies\mathscr{L}\left\{e^{-x}\sin\left(x\right)\right\}&=\frac{1}{\left(p+1\right)^2+1}\\
\implies Y\left(p\right)&=\frac{3}{p^2+2p+5}+\frac{3}{\left[\left(p+1\right)^2+1\right]\left(p^2+2p+5\right)}\\
&=\frac{3}{\left(p+1\right)^2+4}+\frac{3}{\left(\left(p+1\right)^2+1\right)\left(\left(p+1\right)^2+4\right)}\\
\implies y&=\frac{3e^{-t}}{2}\sin\left(2t\right)+3\mathscr{L}^{-1}\left\{\frac{1}{\left(\left(p+1\right)^2+1\right)\left(\left(p+1\right)^2+4\right)}\right\}\\
\implies y\left(t\right)&=3e^{-t}\left[\frac{\sin\left(2t\right)}{2}+\mathscr{L}^{-1}\left\{\frac{1}{\left(p^2+1\right)\left(p^2+4\right)}\right\}\right]\\
\implies y\left(t\right)&=\frac{3e^{-t}\sin\left(2t\right)}{2}+3e^{-t}\left(\frac{1}{3}\sin\left(t\right)-\frac{1}{6}\sin\left(2t\right)\right)\\
&=e^{-t}\sin\left(2t\right)+e^{-t}\sin\left(t\right)\\
y\left(t\right)&=e^{-t}\left(\sin\left(2t\right)+\sin\left(t\right)\right).
\end{aligned}$$

6.

7.

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