Métodos Matemáticos
Tarea 4

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Contents

Chapter 1

Arfken: 19.1.1

En este caso tenemos la definición:

$$\Delta p = \int_0^{2\pi} \left[f(x) - \frac{a_0}{2} - \sum_{n=1}^{p} \left(a_n \cos(nx) + b_n \sin(nx) \right) \right]^2 dx.$$

Por lo tanto:

$$\begin{aligned} 0 = \frac{\partial \Delta p}{\partial a_n} &= -2 \int_0^{2\pi} \left[f(x) - \frac{a_0}{2} - \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right) \right] \cos(nx)\, dx \\ &= -2 \int_0^{2\pi} f(x) \cos(nx)\, dx + 2\pi a_n. \end{aligned}$$

Por otro lado:

$$\begin{aligned} 0 = \frac{\partial \Delta p}{\partial b_n} &= -2 \int_0^{2\pi} \left[f(x) - \frac{a_0}{2} - \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right) \right] \sin(nx)\, dx \\ &= -2 \int_0^{2\pi} f(x) \sin(nx)\, dx + 2\pi b_n. \end{aligned}$$

Chapter 2

Arfken: 19.1.5

Tenemos:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(s)\cos(ns)\,ds.$$

Lo cual implica que

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin(nx)\cos(nx)}{n}dx$$
$$a_n = 0;\;\; n \geq 0.$$

Por otro lado:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(s)\sin(ns)\,ds \\
&= \frac{1}{2\pi}\left[\int_0^{\pi}(\pi - x)\sin(nx)\,dx - \int_{-\pi}^{0}(\pi + x)\sin(nx)\,dx\right] \\
&= \frac{1}{2\pi}\left[-\frac{\pi}{n}\cos(nx) + \frac{x}{n}\cos(nx)\right]_0^{\pi} - \frac{1}{2\pi n}\int_0^{\pi}\cos(nx)\,dx \\
&- \frac{1}{2\pi n}\int_{-\pi}^{0}\cos(nx)\,dx - \frac{1}{2\pi}\left[-\frac{\pi}{n}\cos(nx) - \frac{x}{n}\cos(nx)\right]_{-\pi}^{0} \\
&= \frac{1}{n} - \left.\frac{\sin(nx)}{2\pi n^2}\right|_0^{\pi} - \left.\frac{\sin(nx)}{2\pi n^2}\right|_{-\pi}^{0} = \frac{1}{n}.
\end{aligned}$$

Chapter 3

Arfken: 19.1.10

En este caso tenemos:

$$f(x) = \begin{cases} 4x(1-x) & 0 \le x \le 1 \\ 4x(1+x) & -1 \le x \le 0 \end{cases} = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

Ahora necesitamos encontrar b_n

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x)\sin(nx\pi)\,dx;\ \ n = 1, 2, \dots \\ &= \frac{1}{\pi}\left[\int_{-1}^{0} 4x(1+x)\sin(n\pi x)\,dx + \int_{0}^{1} 4x(1-x)\sin(n\pi x)\,dx\right] \\ &= \frac{1}{\pi}\left[\int_{-1}^{0} \left(4x+4x^2\right)\sin(n\pi x)\,dx + \int_{0}^{1} \left(4x-4x^2\right)\sin(n\pi x)\,dx\right] \\ &= \frac{1}{\pi}\left[\int_{-1}^{0} 4x\sin(n\pi x)\,dx + \int_{-1}^{0} 4x^2\sin(n\pi x)\,dx\right. \\ &\left.+\int_{0}^{1} 4x\sin(n\pi x)\,dx - \int_{0}^{1} 4x^2\sin(n\pi x)\,dx\right]. \end{aligned}$$

Con esto ya podemos simplemente notar que con n par tendríamos que su valor es 0 pues estas funciones serian pares. Por otro lado, quitando $x^2\sin(n\pi x)$ por ser pares queda entonces para n impar queda entonces

$$\begin{aligned} b_n &= 8\left(\frac{2}{n\pi} - \frac{6}{n^3\pi^3}\right) \\ &= \frac{32}{n^3\pi^3}. \end{aligned}$$

Chapter 4

Arfken: 19.2.17

4.1 Parte a

En este caso necesitamos b_n lo cual es:

$$b_n = \frac{2}{L}\int_0^L \delta\left(x-a\right)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Lo cual por la definición de δ nos queda como

$$b_n = \frac{2}{L}\sin\left(\frac{n\pi a}{L}\right).$$

Ahora, reemplazando en la definición de serie seno de fourier nos queda:

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\\
&= \sum_{n=1}^{\infty}\frac{2}{L}\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\\
&= \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right).
\end{aligned}$$

4.2 Parte b

En este caso al hacer la integral tenemos:

$$\begin{aligned}
\delta\left(x-a\right) &= \sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\\
\int_0^x \delta\left(x-a\right)dx &= \int_0^x\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx\\
\begin{cases} 0 & 0\le x<a\\ 1 & a<x<L\end{cases} &= \sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\int_0^x\sin\left(\frac{n\pi x}{L}\right)dx\\
&= \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\left(\frac{L}{n\pi}\right)\left[1-\cos\left(\frac{n\pi x}{L}\right)\right]\\
&= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right)-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi x}{L}\right).
\end{aligned}$$

Chapter 5

Arfken: 19.2.20

Ahora queda:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{d^2 b_n(t)}{dt^2} \sin\left(\frac{n\pi x}{L}\right) &= v^2 \sum_{n=1}^{\infty} b_n(t) \left(\frac{n^2\pi^2}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right) \\
\frac{d^2 b_n(t)}{dt^2} + v^2 \frac{n^2\pi^2}{L^2} b_n(t) &= 0 \\
b_n(t) &= A_n \cos\left(\frac{n\pi v}{L} t\right) + B_n \sin\left(\frac{n\pi v}{L} t\right) \\
u(x,0) &= \sum_{n=1}^{\infty} b_n(0) \sin\left(\frac{n\pi x}{L}\right) = f(x) \\
b_n(0) &= A_n \\
A_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\
\frac{\partial u(x,t)}{\partial t} &= \sum_{n=1}^{\infty} \left(-A_n \frac{n\pi v}{L} \sin\left(\frac{n\pi v}{L} t\right) + B_n \frac{n\pi v}{L} \cos\left(\frac{n\pi v}{L} t\right)\right) \sin\left(\frac{n\pi x}{L}\right) \\
&\text{Con } t = 0 \\
\sum_{n=1}^{\infty} B_n \frac{n\pi v}{L} \sin\left(\frac{n\pi x}{L}\right) &= g(x) \\
B_n &= \frac{2}{n\pi v} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.
\end{aligned}$$

Con esto entonces ya encontramos cuanto es b_n y por lo tanto llegamos a la respuesta correcta.

Chapter 6

Tellez: 3.5.1

Para iniciar sacamos los valores de a_n

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx \\
&= \frac{1}{\pi}\left[-\cos(x)\right]_0^{\pi} \\
&= \frac{2}{\pi}.
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi} \left(\sin(1+n)x + \sin\left(1-\frac{n}{x}\right)\right)dx \\
&= -\frac{1}{2\pi}\left[\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]_0^{\pi} \\
&= -\frac{1}{2\pi}\left[\frac{(-1)^{1+n}}{1+n} + \frac{(-1)^{1-n}}{1-n} - \left(\frac{1}{1+m} + \frac{1}{1-m}\right)\right] \\
&= \frac{1}{\pi}\frac{(-1)^n + 1}{(1-n^2)} \quad (n \neq 1).
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(x)\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi} \sin(2\pi)\,dx \\
&= \frac{1}{2\pi}\left(\frac{1-\cos(2x)}{2}\right)_0^{\pi} \\
&= 0.
\end{aligned}
$$

Ahora sacamos los valores de b_n

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi} (\cos(n-1)x - \cos(n+1)x)\,dx \\
&= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} \\
&= 0 \quad (n \neq 1).
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(x)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} \frac{1-\cos(2x)}{2} \\
&= \frac{1}{2\pi}\left[x - \frac{\sin(2x)}{2}\right]_0^{\pi} \\
&= \frac{1}{2\pi} + \pi \\
&= \frac{1}{2}.
\end{aligned}
$$

Ahora juntando todo:

$$
f(x) = \frac{1}{\pi} + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{1}{\pi}\frac{(-1)^n + 1}{(1-n^2)}\frac{1}{2}\sin(x).
$$

Chapter 7

Tellez: 3.5.5

En este caso asuma $u(x,t) = X(t)T(t)$

Ahora poniendo esto en la ecuación queda:

$$\begin{aligned}\ddot{X}(x)T(t) &= \frac{1}{c^2}X(x)\ddot{T}(t)\\ \frac{\ddot{X}(x)T(t)}{X(x)T(t)} &= \frac{1}{c^2}\frac{X(x)\ddot{T}(t)}{X(x)T(t)}\\ \frac{\ddot{X}(x)}{X(x)} &= \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = -\lambda.\end{aligned}$$

Con esto entonces encontramos dos ecuaciones:

$$\begin{aligned}\ddot{X}(x) + \lambda X(x) &= 0\\ \ddot{T}(t) + \lambda c^2 T(t) &= 0.\end{aligned}$$

Ahora resolviendo para $X(x)$ tenemos que usar $X(0) = 0$ y $X(L) = 0$ y teniendo

$$X(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right).$$

Con lo cual

$$\begin{aligned}X(0) &= A\sin(0) + B\cos(0)\\ 0 &= B.\end{aligned}$$

Por lo tanto $X(x) = A\sin\left(\sqrt{\lambda}x\right)$ con lo que queda

$$\begin{aligned}X(L) &= A\sin\left(\sqrt{\lambda}x\right)\\ 0 &= \sin\left(\sqrt{\lambda}x\right)\\ \sqrt{\lambda}L &= n\pi\\ \lambda &= \frac{n^2\pi^2}{L^2}.\end{aligned}$$

Con lo que queda

$$X_n(x) = A_n\sin\left(\frac{n\pi x}{L}\right).$$

Y ahora probando con $T(t)$ queda

$$\ddot{T}(t) + \frac{n^2\pi^2c^2}{L^2}T(t) = 0.$$

Con la solución general

$$T_n(t) = B_n \cos\left(\frac{n\pi c}{L}t\right) + C_n \sin\left(\frac{n\pi c}{L}t\right).$$

y tomando $\frac{\partial u}{\partial t}(x,0) = 0$ entonces

$$\begin{aligned}
T_n(t) &= B_n \cos\left(\frac{n\pi c}{L}t\right) + C_n \sin\left(\frac{n\pi c}{L}t\right)\\
\dot{T}_n(0) &= -B_n \sin(0) + C_n \cos(0)\\
0 &= C_n.
\end{aligned}$$

Con lo cual queda

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c}{L}t\right).$$