Física Estadistica
Tarea 5

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19 de mayo de 2025

Índice general

Capítulo 1

1.1.

En las secciones 6.1 y 6.2 del librio Pathria se llego a

$$\frac{PV}{kT} = \sum_{\varepsilon} \ln\left(1 + ze^{-\beta\varepsilon}\right) \tag{1.1}$$

$$N = \sum_{\varepsilon} \frac{1}{z^{-1}e^{\beta\varepsilon} + 1} \tag{1.2}$$

Sin embargo

$$\sum_{\varepsilon} \to \int_0^\infty g(\varepsilon)d\varepsilon$$

donde

$$g(\varepsilon)d\varepsilon = \frac{Vg\sqrt{\varepsilon}}{2\pi^2\hbar^3}(2m)^{3/2}d\varepsilon,$$

Ademas usaremos:

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}}{z^{-1}e^x + 1}$$

por lo tanto aplicando en 1.1 y 1.2 tenemos

1. Para 1.1

$$\begin{aligned}
\frac{PV}{kT} &= \sum_{\varepsilon} \ln\left(1 + ze^{-\beta\varepsilon}\right) \\
\frac{PV}{kT} &= \int_0^\infty \ln\left(1 + ze^{-\beta\varepsilon}\right) \frac{Vg\sqrt{\varepsilon}}{2\pi^2\hbar^3}(2m)^{3/2}d\varepsilon \\
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2}\int_0^\infty \ln\left(1 + ze^{-\beta\varepsilon}\right)\sqrt{\varepsilon}d\varepsilon \\
x &= \beta x \\
\varepsilon &= kTx \\
d\varepsilon &= kTdx \\
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2}\int_0^\infty \ln\left(1 + ze^{-x}\right)\sqrt{kTx}kTdx \\
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2}(kT)^{\frac{3}{2}}\int_0^\infty \ln\left(1 + ze^{-x}\right)\sqrt{x}dx
\end{aligned}$$

Ahora para solucionar la integral podemos hacerla por partes de la siguiente manera

$$
\begin{aligned}
u &= \ln\left(1 + ze^{-x}\right) \\
du &= \frac{-ze^{-x}}{1 + ze^{-x}} dx \\
dv &= \sqrt{x} dx \\
v &= \frac{2}{3} x^{\frac{3}{2}} \\
\int u dv &= uv - \int v du \\
\int_0^\infty \ln\left(1 + ze^{-x}\right) \sqrt{x} dx &= \left[\ln\left(1 + ze^{-x}\right) \frac{2}{3} x^{\frac{3}{2}}\right]_0^\infty - \int_0^\infty \frac{2}{3} x^{\frac{3}{2}} \frac{-ze^{-x}}{1 + ze^{-x}} dx \\
&= \frac{2}{3} \int_0^\infty \frac{x^{\frac{3}{2}} ze^{-x}}{1 + ze^{-x}} dx \\
&= \frac{2}{3} \Gamma\left(\frac{5}{2}\right) f_{\frac{5}{2}}\left(z\right) \\
\Gamma\left(\frac{5}{2}\right) &= \frac{3}{4} \sqrt{\pi} \\
&= \frac{\sqrt{\pi}}{2} f_{\frac{5}{2}}\left(z\right)
\end{aligned}
$$

Con esto entonces

$$
\begin{aligned}
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\hbar^3} (2m)^{3/2} \left(kT\right)^{\frac{3}{2}} \frac{\sqrt{\pi}}{2} f_{\frac{5}{2}}\left(z\right) \\
\frac{PV}{kT} &= \frac{Vg}{2\pi^2\hbar^3} (2mkT)^{3/2} \frac{\sqrt{\pi}}{2} f_{\frac{5}{2}}\left(z\right) \\
\frac{PV}{kT} &= \frac{Vg}{2\pi^2 \frac{h^3}{8\pi^3}} (2mkT)^{3/2} \frac{\sqrt{\pi}}{2} f_{\frac{5}{2}}\left(z\right) \\
\frac{PV}{kT} &= \frac{Vg}{\frac{h^3}{2\pi}} (2mkT)^{3/2} \frac{\sqrt{\pi}}{2} f_{\frac{5}{2}}\left(z\right) \\
\frac{PV}{kT} &= 2\pi \frac{Vg}{h^3} (2mkT)^{3/2} \frac{\sqrt{\pi}}{2} f_{\frac{5}{2}}\left(z\right) \\
\frac{PV}{kT} &= \frac{Vg}{h^3} (2\pi mkT)^{3/2} f_{\frac{5}{2}}\left(z\right) \\
\lambda &= \frac{h}{\sqrt{2\pi mkT}} \\
\lambda^3 &= \frac{h^3}{(2\pi mkT)^{\frac{3}{2}}} \\
\frac{1}{\lambda^3} &= \frac{(2\pi mkT)^{\frac{3}{2}}}{h^3} \\
\frac{PV}{kT} &= \frac{Vg}{\lambda^3} f_{\frac{5}{2}}\left(z\right) \\
\frac{P}{kT} &= \frac{g}{\lambda^3} f_{\frac{5}{2}}\left(z\right)
\end{aligned}
$$

2. Para 1.2

$$
\begin{aligned}
N &= \sum_{\varepsilon} \frac{1}{z^{-1}e^{\beta\varepsilon}+1} \\
&= \int_0^{\infty} \frac{1}{z^{-1}e^{\beta\varepsilon}+1} g(\varepsilon) d\varepsilon \\
&= \int_0^{\infty} \frac{1}{z^{-1}e^{\beta\varepsilon}+1} \frac{Vg\sqrt{\varepsilon}}{2\pi^2\hbar^3}(2m)^{3/2} d\varepsilon \\
&= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2} \int_0^{\infty} \frac{1}{z^{-1}e^{\beta\varepsilon}+1} \sqrt{\varepsilon} d\varepsilon \\
&= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2} \int_0^{\infty} \frac{\varepsilon^{1/2}}{z^{-1}e^{\beta\varepsilon}+1} d\varepsilon \\
&= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2} \int_0^{\infty} \frac{\varepsilon^{1/2}}{z^{-1}e^{\beta\varepsilon}+1} d\varepsilon \\
x &= \beta\varepsilon \\
kTx &= \varepsilon \\
d\varepsilon &= kTdx \\
&= \frac{Vg}{2\pi^2\hbar^3}(2m)^{3/2} \int_0^{\infty} \frac{(kTx)^{1/2}}{z^{-1}e^{x}+1} kTdx \\
&= \frac{Vg}{2\pi^2\hbar^3}(2mkT)^{3/2} \int_0^{\infty} \frac{(x)^{1/2}}{z^{-1}e^{x}+1} dx \\
&= \frac{Vg}{2\pi^2\hbar^3}(2mkT)^{3/2} \Gamma\left(\frac{3}{2}\right) f_{\frac{3}{2}}(z) \\
&= \frac{Vg}{2\pi^2\hbar^3}(2mkT)^{3/2} \frac{\sqrt{\pi}}{2} f_{\frac{3}{2}}(z) \\
&= \frac{Vg}{\frac{h^3}{2\pi}}(2mkT)^{3/2} \frac{\sqrt{\pi}}{2} f_{\frac{3}{2}}(z) \\
&= 2\pi\frac{Vg}{h^3}(2mkT)^{3/2} \frac{\sqrt{\pi}}{2} f_{\frac{3}{2}}(z) \\
&= \frac{Vg}{h^3}(2\pi mkT)^{3/2} f_{\frac{3}{2}}(z) \\
&= \frac{Vg}{\lambda^3} f_{\frac{3}{2}}(z) \\
N &= \frac{Vg}{\lambda^3} f_{\frac{3}{2}}(z) \\
\frac{N}{V} &= \frac{g}{\lambda^3} f_{\frac{3}{2}}(z)
\end{aligned}
$$

1.2.

Tenemos

$$U = kT^2 \left(\frac{\partial}{\partial T} \frac{PV}{kT} \right)$$

$$U = kT^2 \left(\frac{\partial}{\partial T} \frac{Vg}{\lambda^3} f_{\frac{5}{2}}(z) \right)$$

$$U = kT^2 Vg \left(\frac{\partial}{\partial T} \frac{1}{\lambda^3} f_{\frac{5}{2}}(z) \right)$$

$$U = kT^2 Vg \left(\frac{\partial}{\partial T} \frac{1}{\lambda^3} f_{\frac{5}{2}}(z) + \frac{1}{\lambda^3} \frac{\partial}{\partial T} f_{\frac{5}{2}}(z) \right)$$

$$U = kT^2 Vg \left(\frac{3}{2\lambda^3 T} f_{\frac{5}{2}}(z) + \frac{1}{\lambda^3} 0 \right)$$

$$U = kT^2 Vg \frac{3}{2\lambda^3 T} f_{\frac{5}{2}}(z)$$

$$U = \frac{3kT^2 Vg}{2\lambda^3 T} f_{\frac{5}{2}}(z)$$

$$\frac{N}{V} = \frac{g}{\lambda^3} f_{\frac{3}{2}}(z)$$

$$\frac{N}{f_{\frac{3}{2}}(z)} = \frac{gV}{\lambda^3}$$

$$U = \frac{3kTN}{2f_{\frac{3}{2}}(z)} f_{\frac{5}{2}}(z)$$

$$U = \frac{3}{2} kTN \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}$$

1.3.

Para esto usaremos

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V$$

Con lo cual:

$$
\begin{aligned}
U &= \frac{3}{2}kTN\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} \\
C_V &= \left(\frac{\partial}{\partial T}\frac{3}{2}kTN\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}\right)_V \\
C_V &= \frac{3}{2}Nk\left(\frac{\partial}{\partial T}T\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}\right)_V \\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} + T\frac{\partial}{\partial T}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}\right)_V \\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} + T\frac{f_{\frac{3}{2}}(z)\frac{\partial f_{\frac{5}{2}}(z)}{\partial T} - f_{\frac{5}{2}}(z)\frac{\partial f_{\frac{3}{2}}(z)}{\partial T}}{\left[f_{\frac{3}{2}}(z)\right]^2}\right)_V \\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} + T\frac{f_{\frac{3}{2}}(z)\frac{\partial f_{\frac{5}{2}}(z)}{\partial z}\frac{\partial z}{\partial T} - f_{\frac{5}{2}}(z)\frac{\partial f_{\frac{3}{2}}(z)}{\partial z}\frac{\partial z}{\partial T}}{\left[f_{\frac{3}{2}}(z)\right]^2}\right)_V \\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} + T\frac{f_{\frac{3}{2}}(z)\frac{f_{\frac{3}{2}}(z)}{z}\frac{\partial z}{\partial T} - f_{\frac{5}{2}}(z)\frac{f_{\frac{1}{2}}(z)}{z}\frac{\partial z}{\partial T}}{\left[f_{\frac{3}{2}}(z)\right]^2}\right)_V \\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} + \frac{T}{z}\frac{\partial z}{\partial T}\frac{f_{\frac{3}{2}}(z)^2 - f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z)}{\left[f_{\frac{3}{2}}(z)\right]^2}\right)_V \\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} + \frac{T}{z}\left(-\frac{3}{2}\frac{z}{T}\frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)}\right)\frac{f_{\frac{3}{2}}(z)^2 - f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z)}{\left[f_{\frac{3}{2}}(z)\right]^2}\right) \\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{3}{2}\frac{f_{\frac{3}{2}}(z)^2 - f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z)}{f_{\frac{3}{2}}(z)f_{\frac{1}{2}}(z)}\right) \\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{3}{2}\frac{f_{\frac{3}{2}}(z)^2}{f_{\frac{3}{2}}(z)f_{\frac{1}{2}}(z)} + \frac{3}{2}\frac{f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z)}{f_{\frac{3}{2}}(z)f_{\frac{1}{2}}(z)}\right) \\
C_V &= \frac{3}{2}Nk\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{3f_{\frac{3}{2}}(z)}{2f_{\frac{1}{2}}(z)} + \frac{3f_{\frac{5}{2}}(z)}{2f_{\frac{3}{2}}(z)}\right) \\
C_V &= \frac{3}{2}Nk\left(\frac{5f_{\frac{5}{2}}(z)}{2f_{\frac{3}{2}}(z)} - \frac{3f_{\frac{3}{2}}(z)}{2f_{\frac{1}{2}}(z)}\right) \\
C_V &= Nk\frac{3}{2}\left(\frac{5f_{\frac{5}{2}}(z)}{2f_{\frac{3}{2}}(z)} - \frac{3f_{\frac{3}{2}}(z)}{2f_{\frac{1}{2}}(z)}\right) \\
C_V &= Nk\left(\frac{15}{4}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{9}{4}\frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)}\right)
\end{aligned}
$$

1.4.

En el Apendice E del libro de Pathria explican que para z pequeños se cumple que:

$$f_v(z) = z - \frac{z^2}{2^v} + \frac{z^3}{3^v} - \dots$$

Nos piden encontrar esta serie en terminos de $n\lambda^3$ por lo tanto partamos de la expresión para $n = \frac{N}{V}$ con lo cual:

$$\begin{aligned}
n &= \frac{g}{\lambda^3} f_{\frac{3}{2}}(z) \\
n &= \frac{g}{\lambda^3}\left(z - \frac{z^2}{2^{\frac{3}{2}}} + \frac{z^3}{3^{\frac{3}{2}}} - \dots\right) \\
\frac{n\lambda^3}{g} &= \left(z - \frac{z^2}{2^{\frac{3}{2}}} + \frac{z^3}{3^{\frac{3}{2}}} - \dots\right) \\
z &\approx \frac{n\lambda^3}{g} + \frac{\left(n\lambda^3\right)^2}{2\sqrt{2}g^2}
\end{aligned}$$

Ademas de eso veamos las equivalencias de las funciones:

$$\begin{aligned}
f_{\frac{5}{2}}(z) &\approx z - \frac{z^2}{2^{\frac{5}{2}}} + \cdots = z - \frac{z^2}{4\sqrt{2}} + \cdots , \\
f_{\frac{3}{2}}(z) &\approx z - \frac{z^2}{2^{\frac{3}{2}}} + \cdots = z - \frac{z^2}{2\sqrt{2}} + \cdots , \\
f_{\frac{1}{2}}(z) &\approx z - \frac{z^2}{2^{\frac{1}{2}}} + \cdots = z - \frac{z^2}{\sqrt{2}} + \cdots .
\end{aligned}$$

Ahora tomando en cuenta que

$$C_V = Nk\left(\frac{15}{4}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{9}{4}\frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)}\right).$$

Podemos desarrollar cada una de las fracciones por aparte como

$$\begin{aligned}
\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} &\approx \frac{z - \frac{z^2}{4\sqrt{2}}}{z - \frac{z^2}{2\sqrt{2}}} \approx 1 + \frac{z}{4\sqrt{2}}, \\
\frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)} &\approx \frac{z - \frac{z^2}{2\sqrt{2}}}{z - \frac{z^2}{\sqrt{2}}} \approx 1 + \frac{z}{2\sqrt{2}}.
\end{aligned}$$

Lo que nos dejaria con un desarrollo como

$$\begin{aligned}
C_V &\approx Nk\left(\frac{15}{4}\left(1 + \frac{z}{4\sqrt{2}}\right) - \frac{9}{4}\left(1 + \frac{z}{2\sqrt{2}}\right)\right) \\
C_V &= Nk\left(\frac{15}{4} - \frac{9}{4} + \frac{15}{16\sqrt{2}}z - \frac{9}{8\sqrt{2}}z\right) \\
&= Nk\left(\frac{3}{2} - \frac{3}{16\sqrt{2}}z\right) \\
C_V &= \frac{3}{2}Nk - \frac{3}{16\sqrt{2}}\frac{n\lambda^3}{g}Nk + \cdots
\end{aligned}$$

note que siempre que si $n\lambda^3 > 0$ entonces

$$\frac{3}{16\sqrt{2}}\frac{n\lambda^3}{g}Nk > 0$$

por lo tanto dado que esto es positivo el termino total seria menor. Es decir:

$$\begin{aligned} C_V = \frac{3}{2}Nk - \frac{3}{16\sqrt{2}}\frac{n\lambda^3}{g}Nk &< \frac{3}{2}Nk \\ C_V &< \frac{3}{2}Nk \end{aligned}$$

1.5.

En este caso usaremos

$$f_{3/2}(z) \approx \frac{2}{3\sqrt{\pi}}\mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right].$$

Con lo cual podemos revisar para $\frac{N}{V}$

$$\begin{aligned} n &= \frac{g}{\lambda^3}f_{3/2}(z) \\ n &\approx \frac{g}{\lambda^3}\frac{2}{3\sqrt{\pi}}\mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right] \\ n &= \frac{g}{6\pi^2}\left(\frac{2mE_F}{\hbar^2}\right)^{3/2} \end{aligned}$$

Ahora igualando las expresiones para n

$$\begin{aligned} 1 &\approx \left(\frac{\mu}{E_F}\right)^{3/2} + \frac{\pi^2}{8}\left(\frac{T}{T_F}\right)^2\left(\frac{E_F}{\mu}\right)^{1/2} \\ \mu &= E_F(1+\delta) \\ 1 &\approx 1 + \frac{3}{2}\delta + \frac{\pi^2}{8}\left(\frac{T}{T_F}\right)^2 \\ \delta &\approx -\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 \\ \mu(T) &= E_F\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] \end{aligned}$$

1.6.

Partimos desde la definición:

$$U = \frac{3}{2}NkT\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}$$

Utilizando la expansión:

$$f_v(z) \approx \frac{(\ln z)^v}{\Gamma(v+1)} \left[1 + \frac{\pi^2}{6} \frac{v(v-1)}{(\ln z)^2} + \dots\right]$$

Con esto entonces podemos encontra

$$f_{\frac{5}{2}}(z) \approx \frac{(\ln z)^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)} \left[1 + \frac{\pi^2}{6} \frac{\frac{5}{2} \cdot \frac{3}{2}}{(\ln z)^2}\right]$$

$$f_{\frac{3}{2}}(z) \approx \frac{(\ln z)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} \left[1 + \frac{\pi^2}{6} \frac{\frac{3}{2} \cdot \frac{1}{2}}{(\ln z)^2}\right]$$

Con esto entonces podemos encontrar cada una de las fracciones de U. Queda:

$$\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} \approx \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} (\ln z) \left[1 + \frac{\pi^2}{8} \frac{1}{(\ln z)^2}\right]$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} \approx \frac{2}{5} (\ln z) \left[1 + \frac{\pi^2}{8} \frac{1}{(\ln z)^2}\right].$$

Con el resultado de la sección anterior tenemos

$$\ln z = \frac{\mu(T)}{kT} = \frac{E_F}{kT} \left[1 - \frac{\pi^2}{12} \left(\frac{T}{T_F}\right)^2\right]$$

$$U \approx \frac{3}{2} NkT \cdot \frac{2}{5} \frac{E_F}{kT} \left[1 + \frac{\pi^2}{8} \left(\frac{kT}{E_F}\right)^2\right]$$

$$U \approx \frac{3}{5} NE_F + \frac{3\pi^2}{20} Nk^2 \frac{T^2}{E_F}$$

Ahora dado que $C_V = \frac{\partial U}{\partial T}$ lo que nos quedaria como:

$$C_V = \frac{\partial U}{\partial T} = \frac{3\pi^2}{10} Nk^2 \frac{T}{E_F}$$

$$E_F = kT_F$$

$$C_V = \frac{\partial U}{\partial T} = \frac{3\pi^2}{10} Nk^2 \frac{T}{kTF}$$

$$C_V = Nk \left\{\frac{\pi^2}{2} \frac{T}{T_F} + o\left(\frac{T}{T_F}\right)\right\}$$

1.7.

Capítulo 2

2.1.

2.2.

2.3.

2.4.

Capítulo 3

3.1.

Partimos desde

$$\chi = \frac{2n\mu^{*2}}{\left(\frac{\partial\mu_0(xN)}{\partial x}\right)_{x=1/2}}$$

Tenemos que considerar que segun Pathria en la ecuación 8.1.34

3.2.

3.3.

Capítulo 4

4.1.

4.2.

4.3.

4.4.

4.5.

4.6.

4.7.