# Mecanica Cuantica
# Tarea 5

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# Chapter 1

# Chapter 2

## 2.1

Para mostrar que esta normalizado sumamos cada coeficiente y mostramos que esto equivale a 1

$$
\begin{aligned}
|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 &= 1 \\
\left|\frac{\sqrt{2}}{4}\right|^2 + \left|\frac{2i}{4}\right|^2 + \left|-\frac{i}{4}\right|^2 + \left|\frac{3}{4}e^{i\frac{\pi}{3}}\right|^2 &= 1 \\
\frac{2}{16} + \frac{4}{16} + \frac{1}{16} + \left|\frac{3}{4}\right|^2 \left|e^{i\frac{\pi}{3}}\right|^2 &= 1 \\
\frac{2}{16} + \frac{4}{16} + \frac{1}{16} + \frac{9}{16}\left|\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right|^2 &= 1 \\
\frac{2}{16} + \frac{4}{16} + \frac{1}{16} + \frac{9}{16}\left(\sqrt{\cos^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{3}\right)}\right)^2 &= 1 \\
\frac{2}{16} + \frac{4}{16} + \frac{1}{16} + \frac{9}{16}(1)^2 &= 1 \\
\frac{2}{16} + \frac{4}{16} + \frac{1}{16} + \frac{9}{16} &= 1 \\
\frac{2+4+1+9}{16} &= 1 \\
1 &= 1
\end{aligned}
$$

## 2.2

Para encontrar la energia podemos usar la ecuación 4.2.27 de las notas de clase en donde sabemos que los estados se pueden encontrar como:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

Por lo tanto las energias son:

$$
\begin{aligned}
E_n &= \left(n + \frac{1}{2}\right)\hbar\omega \\
E_0 &= \left(0 + \frac{1}{2}\right)\hbar\omega \\
&= \frac{1}{2}\hbar\omega \\
E_1 &= \left(1 + \frac{1}{2}\right)\hbar\omega \\
&= \left(\frac{3}{2}\right)\hbar\omega \\
E_2 &= \left(2 + \frac{1}{2}\right)\hbar\omega \\
&= \left(\frac{5}{2}\right)\hbar\omega \\
E_3 &= \left(3 + \frac{1}{2}\right)\hbar\omega \\
&= \left(\frac{7}{2}\right)\hbar\omega
\end{aligned}
$$

Ahora bien, las probabilidades son:

$$
\begin{aligned}
P_n &= |\langle n|\psi\rangle|^2 \\
&= |c_n|^2
\end{aligned}
$$

Esto ya lo calculamos en la sección anterior por lo que sabemos que serian:

$$
\begin{aligned}
P_0 &= \frac{2}{16} \\
P_1 &= \frac{4}{16} \\
P_2 &= \frac{1}{16} \\
P_3 &= \frac{9}{16}
\end{aligned}
$$

## 2.3

Para calcular

$$\langle E\rangle = \sum_{n=0}^{3} P_n E_n$$

Tomando los resultados de la sección anterior tenemos:

$$
\begin{aligned}
\langle E \rangle &= P_0 E_0 + P_1 E_1 + P_2 E_2 + P_3 E_3 \\
&= \frac{2}{16}\left(\frac{1}{2}\hbar\omega\right) + \frac{4}{16}\left(\frac{3}{2}\hbar\omega\right) + \frac{1}{16}\left(\frac{5}{2}\hbar\omega\right) + \frac{9}{16}\left(\frac{7}{2}\hbar\omega\right) \\
&= \left(\frac{2}{32}\hbar\omega\right) + \left(\frac{12}{32}\hbar\omega\right) + \left(\frac{5}{32}\hbar\omega\right) + \left(\frac{63}{32}\hbar\omega\right) \\
&= \left(\frac{2 + 12 + 5 + 63}{32}\hbar\omega\right) \\
&= \left(\frac{82}{32}\hbar\omega\right) \\
&= \left(\frac{41}{16}\hbar\omega\right)
\end{aligned}
$$

# Chapter 3

# Chapter 4

## 4.1

Para solucionar esto partimos desde:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a_- + a_+\right);\ p = i\sqrt{\frac{m\omega}{2}}\left(a_+ - a_-\right)$$

Ahora bien, tomemos que:

$$\begin{aligned}
\langle\alpha|a_-|\alpha\rangle &= \langle\alpha|\alpha|\alpha\rangle \\
&= \alpha\langle\alpha|\alpha\rangle \\
&= \alpha \\
\langle\alpha|a_+|\alpha\rangle &= \langle\alpha|\alpha^*|\alpha\rangle \\
&= \alpha^*\langle\alpha|\alpha\rangle \\
&= \alpha^*
\end{aligned}$$

Por lo tanto

$$\begin{aligned}
x &= \sqrt{\frac{\hbar}{2m\omega}}\left(a_- + a_+\right) \\
\langle x\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|\left(a_- + a_+\right)|\alpha\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha + \alpha^*\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}2\Re\left(\alpha\right) \\
&= \sqrt{\frac{4\hbar}{2m\omega}}\Re\left(\alpha\right) \\
&= \sqrt{\frac{2\hbar}{m\omega}}\Re\left(\alpha\right)
\end{aligned}$$

Para $\langle p\rangle$

$$\begin{aligned}
p &= i\sqrt{\frac{m\omega}{2}}\left(a_+ - a_-\right)\\
\langle p\rangle &= i\sqrt{\frac{m\omega}{2}}\langle\alpha|\left(a_+ - a_-\right)|\alpha\rangle\\
&= i\sqrt{\frac{m\omega}{2}}\left(\alpha^* - \alpha\right)\\
&= i\sqrt{\frac{m\omega}{2}}\left(-2i\Im\left(\alpha\right)\right)\\
&= \sqrt{\frac{4m\omega}{2}}\Im\left(\alpha\right)\\
&= \sqrt{2m\omega}\Im\left(\alpha\right)
\end{aligned}$$

Ahora con los casos de $\langle x^2\rangle$ y $\langle p^2\rangle$
Primero miremos lo siguiente:

$$\begin{aligned}
x^2 &= \frac{\hbar}{2m\omega}\left(a_- + a_+\right)^2\\
&= \frac{\hbar}{2m\omega}\left(a_-^2 + a_+^2 + a_-a_+ + a_+a_-\right)\\
p^2 &= -\frac{m\omega}{2}\left(a_+ - a_-\right)^2\\
&= -\frac{m\omega}{2}\left(a_+^2 + a_-^2 - a_+a_- - a_-a_+\right)
\end{aligned}$$

Por lo tanto vamos a necesitar:

$$\begin{aligned}
\langle\alpha|a_-^2|\alpha\rangle &= \alpha\langle\alpha|a_-|\alpha\rangle\\
&= \alpha^2\\
\langle\alpha|a_+^2|\alpha\rangle &= \alpha^*\langle\alpha|a_-|\alpha\rangle\\
&= \left(\alpha^*\right)^2\\
\langle\alpha|a_+a_-|\alpha\rangle &= \alpha\langle\alpha|a_+|\alpha\rangle\\
&= \alpha\alpha^*\langle\alpha|\alpha\rangle\\
&= |\alpha|^2\\
\langle\alpha|a_-a_+|\alpha\rangle &= \langle\alpha|a_+a_- + 1|\alpha\rangle\\
&= \alpha\langle\alpha|a_+|\alpha\rangle + \langle\alpha|1|\alpha\rangle\\
&= \alpha\alpha^*\langle\alpha|\alpha\rangle + 1\langle\alpha|\alpha\rangle\\
&= |\alpha|^2 + 1
\end{aligned}$$

Ya con esto podemos pasar a calcular

1. $\langle x^2 \rangle$

$$\begin{aligned}
x^2 &= \frac{\hbar}{2m\omega}\left(a_-^2 + a_+^2 + a_-a_+ + a_+a_-\right) \\
\langle x^2 \rangle &= \frac{\hbar}{2m\omega}\langle \alpha | \left(a_-^2 + a_+^2 + a_-a_+ + a_+a_-\right) |\alpha\rangle \\
&= \frac{\hbar}{2m\omega}\left(\langle \alpha \left| a_-^2 \right| \alpha\rangle + \langle \alpha \left| a_+^2 \right| \alpha\rangle + \langle \alpha \left| a_-a_+ \right| \alpha\rangle + \langle \alpha \left| a_+a_- \right| \alpha\rangle\right) \\
&= \frac{\hbar}{2m\omega}\left(\alpha^2 + (\alpha^*)^2 + |\alpha|^2 + |\alpha|^2 + 1\right) \\
&= \frac{\hbar}{2m\omega}\left(\alpha^2 + (\alpha^*)^2 + 2|\alpha|^2 + 1\right) \\
&= \frac{\hbar}{2m\omega}\left(2\Re(\alpha)^2 - 2\Im(\alpha)^2 + 2\Re(\alpha)^2 + 2\Im(\alpha)^2 + 1\right) \\
&= \frac{\hbar}{2m\omega}\left(4\Re(\alpha)^2 + 1\right) \\
&= \frac{2\hbar}{m\omega}\Re(\alpha)^2 + \frac{\hbar}{2m\omega}
\end{aligned}$$

2. $\langle p^2 \rangle$

$$\begin{aligned}
p^2 &= -\frac{m\omega}{2}\left(a_+^2 + a_-^2 - a_+a_- - a_-a_+\right) \\
\langle p^2 \rangle &= -\frac{m\omega}{2}\langle \alpha \left| a_+^2 + a_-^2 - a_+a_- - a_-a_+ \right| \alpha\rangle \\
&= -\frac{m\omega}{2}\left(\langle \alpha \left| a_+^2 \right| \alpha\rangle + \langle \alpha \left| a_-^2 \right| \alpha\rangle - \langle \alpha \left| a_+a_- \right| \alpha\rangle - \langle \alpha \left| a_-a_+ \right| \alpha\rangle\right) \\
&= -\frac{m\omega}{2}\left(\alpha^2 + (\alpha^*)^2 - |\alpha|^2 - \left(|\alpha|^2 + 1\right)\right) \\
&= -\frac{m\omega}{2}\left(2\Re(\alpha)^2 - 2\Im(\alpha)^2 - |\alpha|^2 - |\alpha|^2 - 1\right) \\
&= -\frac{m\omega}{2}\left(2\Re(\alpha)^2 - 2\Im(\alpha)^2 - 2\Re(\alpha)^2 - 2\Im(\alpha)^2 - 1\right) \\
&= -\frac{m\omega}{2}\left(-4\Im(\alpha)^2 - 1\right) \\
&= \frac{m\omega}{2}\left(4\Im(\alpha)^2 + 1\right) \\
&= \frac{m\omega}{2}4\Im(\alpha)^2 + \frac{m\omega}{2} \\
&= 2m\omega\Im(\alpha)^2 + \frac{m\omega}{2}
\end{aligned}$$

Por lo tanto los resultados son:

1. $\langle x \rangle = \sqrt{\frac{2\hbar}{m\omega}}\Re(\alpha)$
2. $\langle p \rangle = \sqrt{2m\omega}\Im(\alpha)$
3. $\langle x^2 \rangle = \frac{2\hbar}{m\omega}\Re(\alpha)^2 + \frac{\hbar}{2m\omega}$
4. $\langle p^2 \rangle = 2m\omega\Im(\alpha)^2 + \frac{m\omega}{2}$

## 4.2

En este caso tenemos:

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

Por lo tanto veamos cuanto es $\langle x \rangle^2$

$$\langle x \rangle^2 = \left( \sqrt{\frac{2\hbar}{m\omega}} \Re(\alpha) \right)^2$$
$$\langle x \rangle^2 = \frac{2\hbar}{m\omega} \Re(\alpha)^2$$

Con esto entonces

$$\begin{aligned} \sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \sqrt{\frac{2\hbar}{m\omega} \Re(\alpha)^2 + \frac{\hbar}{2m\omega} - \frac{2\hbar}{m\omega} \Re(\alpha)^2} \\ &= \sqrt{\frac{\hbar}{2m\omega}} \end{aligned}$$

Por el otro lado

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

con

$$\begin{aligned} \langle p \rangle^2 &= \left( \sqrt{2m\omega} \Im(\alpha) \right)^2 \\ &= \left( \sqrt{2m\omega} \Im(\alpha) \right)^2 \\ &= 2m\omega \Im(\alpha)^2 \end{aligned}$$

De nuevo calculemos

$$\begin{aligned} \sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\ &= \sqrt{2m\omega \Im(\alpha)^2 + \frac{m\omega}{2} - 2m\omega \Im(\alpha)^2} \\ &= \sqrt{\frac{m\omega}{2}} \end{aligned}$$

Ahora al final:

$$\sigma_x \sigma_p = \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\frac{m\omega}{2}}$$
$$\sigma_x \sigma_p = \sqrt{\frac{\hbar^2 m\omega}{2 \cdot 2m\omega}}$$
$$\sigma_x \sigma_p = \sqrt{\frac{\hbar^2}{4}}$$
$$\sigma_x \sigma_p = \frac{\hbar}{2}$$

## 4.3

Apliquemos $a_-$ de la siguiente manera

$$\begin{aligned} a_- |\alpha\rangle &= \alpha \sum_n c_n |n\rangle \\ &= \sum_n c_n \sqrt{n} |n-1\rangle \end{aligned}$$

Dado que son esencialmente los mismos podemos hacer

$$\begin{aligned} \alpha \sum_n c_n |n\rangle &= \sum_n c_n \sqrt{n} |n-1\rangle \\ \sum_n \alpha c_n |n\rangle &= \sum_n c_n \sqrt{n} |n-1\rangle \\ \sum_n \alpha c_n |n\rangle &= \sum_n c_{n+1} \sqrt{n-1} |n\rangle \\ \alpha c_n &= c_{n+1} \sqrt{n-1} \\ \frac{\alpha}{\sqrt{n-1}} c_n &= c_{n+1} \end{aligned}$$

Dada esta definición recursiva podemos reducirla hasta

$$\frac{\alpha^n}{\sqrt{n!}} c_0 = c_n$$

## 4.4

Tenemos:

$$\begin{aligned} \langle\alpha|\alpha\rangle &= \sum_{n=0}^{\infty} |c_n|^2 = 1 \\ \sum_{n=0}^{\infty} \left| \frac{\alpha^n}{\sqrt{n!}} \right|^2 &= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \\ \sum_{n=0}^{\infty} \left| \frac{\alpha^n}{\sqrt{n!}} \right|^2 &= |c_0|^2 \sum_{n=0}^{\infty} \frac{\left(|\alpha|^2\right)^n}{n!} \end{aligned}$$

Esta es una serie exponencial conocida:

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

Por lo tanto

$$\begin{aligned} |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} &= |c_0|^2 e^{|\alpha|^2} \\ |c_0|^2 e^{|\alpha|^2} &= 1 \\ |c_0|^2 &= e^{-|\alpha|^2} \\ c_0 &= e^{-|\alpha|^2/2} \end{aligned}$$

## 4.5

En los punto anterior definimos que:

$$|\alpha\rangle = \sum_n c_n |n\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} c_0 |n\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} |n\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$