Física Estadistica
Tarea 4

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Contents

Chapter 1

1.1 A

Para construir una función de onda simetrica dado que son independientes tenemos

$$\left|u_{j_1}(q_i)\right\rangle \left|u_{j_2}(q_i)\right\rangle \left|u_{j_3}(q_i)\right\rangle$$

Ahora debemos hacer que esta función de onda sea simetrica ante cualquier permutación. Tome en cuenta que todas las permutaciones posibles son $3! = 6$. Estas permutaciones son:

1. $|u_{j_1}(q_1)\rangle|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle$ (permutación identidad)
2. $|u_{j_2}(q_1)\rangle|u_{j_1}(q_2)\rangle|u_{j_3}(q_3)\rangle$ (intercambio de partículas 1 y 2)
3. $|u_{j_3}(q_1)\rangle|u_{j_2}(q_2)\rangle|u_{j_1}(q_3)\rangle$ (intercambio de partículas 1 y 3)
4. $|u_{j_1}(q_1)\rangle|u_{j_3}(q_2)\rangle|u_{j_2}(q_3)\rangle$ (intercambio de partículas 2 y 3)
5. $|u_{j_2}(q_1)\rangle|u_{j_3}(q_2)\rangle|u_{j_1}(q_3)\rangle$ (permutación cíclica $1 \to 2 \to 3 \to 1$)
6. $|u_{j_3}(q_1)\rangle|u_{j_1}(q_2)\rangle|u_{j_2}(q_3)\rangle$ (permutación cíclica $1 \to 3 \to 2 \to 1$)

Ademas, tenemos que normalizar con $\frac{1}{\sqrt{3!}} = \frac{1}{\sqrt{6}}$ con lo cual la función de onda nos queda como:

$$
\begin{aligned}
|\psi_S\rangle = \frac{1}{\sqrt{6}}[\\
&|u_{j_1}(q_2)\rangle|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle+ \\
&|u_{j_1}(q_2)\rangle|u_{j_3}(q_3)\rangle|u_{j_2}(q_2)\rangle+ \\
&|u_{j_2}(q_2)\rangle|u_{j_1}(q_1)\rangle|u_{j_3}(q_3)\rangle+ \\
&|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle|u_{j_1}(q_1)\rangle+ \\
&|u_{j_3}(q_2)\rangle|u_{j_1}(q_1)\rangle|u_{j_2}(q_2)\rangle+ \\
&|u_{j_3}(q_2)\rangle|u_{j_2}(q_2)\rangle|u_{j_1}(q_1)\rangle \\
].
\end{aligned}
$$

1.2 B

Tomemos P_{12} Con lo cual:

$$
P_{12}|\psi_S\rangle = \frac{1}{\sqrt{6}}[
|u_{j_2}(q_2)\rangle|u_{j_1}(q_1)\rangle|u_{j_3}(q_3)\rangle+
|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle|u_{j_1}(q_1)\rangle+
|u_{j_1}(q_2)\rangle|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle+
|u_{j_1}(q_2)\rangle|u_{j_3}(q_3)\rangle|u_{j_2}(q_2)\rangle+
|u_{j_3}(q_2)\rangle|u_{j_2}(q_2)\rangle|u_{j_1}(q_1)\rangle+
|u_{j_3}(q_2)\rangle|u_{j_1}(q_1)\rangle|u_{j_2}(q_2)\rangle
].
$$

Con lo cual podemos simplemente reorganizar hasta

$$
P_{12}|\psi_S\rangle = \frac{1}{\sqrt{6}}[
|u_{j_1}(q_1)\rangle|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle+
|u_{j_1}(q_1)\rangle|u_{j_3}(q_3)\rangle|u_{j_2}(q_2)\rangle+
|u_{j_2}(q_2)\rangle|u_{j_1}(q_1)\rangle|u_{j_3}(q_3)\rangle+
|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle|u_{j_1}(q_1)\rangle+
|u_{j_3}(q_2)\rangle|u_{j_1}(q_1)\rangle|u_{j_2}(q_2)\rangle+
|u_{j_3}(q_2)\rangle|u_{j_2}(q_2)\rangle|u_{j_1}(q_1)\rangle
].
$$

Que es en esencia:

$$
P_{12}|\psi_s\rangle = +|\psi_s\rangle
$$

Ahora para el siguiente caso tome P_{123}

$$
P_{123}|\psi_S\rangle = \frac{1}{\sqrt{6}}[
|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle|u_{j_1}(q_1)\rangle+
|u_{j_2}(q_2)\rangle|u_{j_1}(q_1)\rangle|u_{j_3}(q_3)\rangle+
|u_{j_3}(q_3)\rangle|u_{j_2}(q_2)\rangle|u_{j_1}(q_1)\rangle+
|u_{j_3}(q_3)\rangle|u_{j_1}(q_1)\rangle|u_{j_2}(q_2)\rangle+
|u_{j_1}(q_1)\rangle|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle+
|u_{j_1}(q_1)\rangle|u_{j_3}(q_3)\rangle|u_{j_2}(q_2)\rangle
].
$$

Que de nuevo podemos reorganilzarlo hasta que nos quede como:

$$
P_{123}|\psi_S\rangle = \frac{1}{\sqrt{6}}[
|u_{j_1}(q_1)\rangle|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle+
|u_{j_1}(q_1)\rangle|u_{j_3}(q_3)\rangle|u_{j_2}(q_2)\rangle+
|u_{j_2}(q_2)\rangle|u_{j_1}(q_1)\rangle|u_{j_3}(q_3)\rangle+
|u_{j_2}(q_2)\rangle|u_{j_3}(q_3)\rangle|u_{j_1}(q_1)\rangle+
|u_{j_3}(q_2)\rangle|u_{j_1}(q_1)\rangle|u_{j_2}(q_2)\rangle+
|u_{j_3}(q_2)\rangle|u_{j_2}(q_2)\rangle|u_{j_1}(q_1)\rangle
].
$$

1.3 C

Para esto tomemos la matriz:

$$\begin{pmatrix} |u_a(q_1)\rangle & |u_b(q_1)\rangle & |u_c(q_1)\rangle \\ |u_a(q_2)\rangle & |u_b(q_2)\rangle & |u_c(q_2)\rangle \\ |u_a(q_3)\rangle & |u_b(q_3)\rangle & |u_c(q_3)\rangle \end{pmatrix}$$

Ahora nuestra función antisimetrica seria:

$$|\psi_A\rangle = \frac{1}{\sqrt{6}} det\left(\begin{pmatrix} |u_a(q_1)\rangle & |u_b(q_1)\rangle & |u_c(q_1)\rangle \\ |u_a(q_2)\rangle & |u_b(q_2)\rangle & |u_c(q_2)\rangle \\ |u_a(q_3)\rangle & |u_b(q_3)\rangle & |u_c(q_3)\rangle \end{pmatrix}\right)$$

Con esto entonces: el resultado es:

$$det\left(\begin{pmatrix} |u_a(q_1)\rangle & |u_b(q_1)\rangle & |u_c(q_1)\rangle \\ |u_a(q_2)\rangle & |u_b(q_2)\rangle & |u_c(q_2)\rangle \\ |u_a(q_3)\rangle & |u_b(q_3)\rangle & |u_c(q_3)\rangle \end{pmatrix}\right) =$$

$$\begin{aligned} &|u_a(q_1)\rangle\left(|u_b(q_2)\rangle|u_c(q_3)\rangle - |u_b(q_3)\rangle|u_c(q_2)\rangle\right) - \\ &|u_b(q_1)\rangle\left(|u_a(q_2)\rangle|u_c(q_3)\rangle - |u_a(q_3)\rangle|u_c(q_2)\rangle\right) + \\ &|u_c(q_1)\rangle\left(|u_a(q_2)\rangle|u_b(q_3)\rangle - |u_a(q_3)\rangle|u_b(q_2)\rangle\right) \\ &= |u_a(q_1)\rangle|u_b(q_2)\rangle|u_c(q_3)\rangle - |u_a(q_1)\rangle|u_b(q_3)\rangle|u_c(q_2)\rangle - \\ &|u_b(q_1)\rangle|u_a(q_2)\rangle|u_c(q_3)\rangle + |u_b(q_1)\rangle|u_a(q_3)\rangle|u_c(q_2)\rangle + \\ &|u_c(q_1)\rangle|u_a(q_2)\rangle|u_b(q_3)\rangle - |u_c(q_1)\rangle|u_a(q_3)\rangle|u_b(q_2)\rangle \end{aligned}$$

Lo que queda como:

$$\begin{aligned} |\psi\rangle = \frac{1}{\sqrt{6}}[\\ &|u_a(q_1)\rangle|u_b(q_2)\rangle|u_c(q_3)\rangle - \\ &|u_a(q_1)\rangle|u_b(q_3)\rangle|u_c(q_2)\rangle - \\ &|u_b(q_1)\rangle|u_a(q_2)\rangle|u_c(q_3)\rangle + \\ &|u_b(q_1)\rangle|u_a(q_3)\rangle|u_c(q_2)\rangle + \\ &|u_c(q_1)\rangle|u_a(q_2)\rangle|u_b(q_3)\rangle - \\ &|u_c(q_1)\rangle|u_a(q_3)\rangle|u_b(q_2)\rangle \\ &]. \end{aligned}$$

1.4 D

Ahora con esto podemos aplicar las permutaciones:

$$\begin{aligned} P_{12}|\psi\rangle = \frac{1}{\sqrt{6}}[\\ &|u_a(q_2)\rangle|u_b(q_1)\rangle|u_c(q_3)\rangle - \\ &|u_a(q_2)\rangle|u_b(q_3)\rangle|u_c(q_1)\rangle - \\ &|u_b(q_2)\rangle|u_a(q_1)\rangle|u_c(q_3)\rangle + \\ &|u_b(q_2)\rangle|u_a(q_3)\rangle|u_c(q_1)\rangle + \\ &|u_c(q_2)\rangle|u_a(q_1)\rangle|u_b(q_3)\rangle - \\ &|u_c(q_2)\rangle|u_a(q_3)\rangle|u_b(q_1)\rangle \\ &]. \end{aligned}$$

Lo cual lo podemos reorganizar hasta tener:

$$P_{12}|\psi\rangle = -|\psi\rangle$$

Chapter 2

2.1 A

Dado que estamos en el Gran Canonico podemos usar la función de partición:

$$Z = \sum_{n=0}^{\ell} e^{\beta(\mu-\varepsilon)n} = \frac{1 - e^{\beta(\mu-\varepsilon)(\ell+1)}}{1 - e^{\beta(\mu-\varepsilon)}}$$

Ademas usaremos que:

$$\langle n_e \rangle = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z$$

Por lo tanto necesitamos desarrollar:

$$\begin{aligned}
\ln Z &= \ln\left(\frac{1 - e^{\beta(\mu-\varepsilon)(\ell+1)}}{1 - e^{\beta(\mu-\varepsilon)}}\right) \\
&= \ln\left(1 - e^{\beta(\mu-\varepsilon)(\ell+1)}\right) - \ln\left(1 - e^{\beta(\mu-\varepsilon)}\right)
\end{aligned}$$

Ahora sacando la derivada:

$$\begin{aligned}
\frac{\partial}{\partial\mu}Z &= \frac{\partial}{\partial\mu}\left[\ln\left(1 - e^{\beta(\mu-\varepsilon)(\ell+1)}\right) - \ln\left(1 - e^{\beta(\mu-\varepsilon)}\right)\right] \\
&= \frac{\partial}{\partial\mu}\ln\left(1 - e^{\beta(\mu-\varepsilon)(\ell+1)}\right) - \frac{\partial}{\partial\mu}\ln\left(1 - e^{\beta(\mu-\varepsilon)}\right) \\
&= \frac{-\beta(\ell+1)e^{\beta(\mu-\varepsilon)(\ell+1)}}{1 - e^{\beta(\mu-\varepsilon)(\ell+1)}} + \frac{\beta e^{\beta(\mu-\varepsilon)}}{1 - e^{\beta(\mu-\varepsilon)}}
\end{aligned}$$

Ahora debemos multiplicar por $\frac{1}{\beta}$

$$\begin{aligned}
\frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z &= \frac{1}{\beta}\left(\frac{-\beta(\ell+1)e^{\beta(\mu-\varepsilon)(\ell+1)}}{1 - e^{\beta(\mu-\varepsilon)(\ell+1)}} + \frac{\beta e^{\beta(\mu-\varepsilon)}}{1 - e^{\beta(\mu-\varepsilon)}}\right) \\
&= \frac{1}{\beta}\beta\left(\frac{-(\ell+1)e^{\beta(\mu-\varepsilon)(\ell+1)}}{1 - e^{\beta(\mu-\varepsilon)(\ell+1)}} + \frac{e^{\beta(\mu-\varepsilon)}}{1 - e^{\beta(\mu-\varepsilon)}}\right) \\
&= \frac{e^{\beta(\mu-\varepsilon)}}{1 - e^{\beta(\mu-\varepsilon)}} - \frac{(\ell+1)e^{\beta(\mu-\varepsilon)(\ell+1)}}{1 - e^{\beta(\mu-\varepsilon)(\ell+1)}}
\end{aligned}$$

Ahora, para que estos resultados sean los planteados en el enunciado podemos simplemente poner:

$$
\begin{aligned}
\langle n_\varepsilon \rangle &= \frac{e^{\beta(\mu-\varepsilon)}}{1-e^{\beta(\mu-\varepsilon)}} - \frac{(\ell+1)e^{\beta(\mu-\varepsilon)(\ell+1)}}{1-e^{\beta(\mu-\varepsilon)(\ell+1)}} \\
&= \frac{e^{\beta(\mu-\varepsilon)}}{1-e^{\beta(\mu-\varepsilon)}}\frac{e^{\beta(\varepsilon-\mu)}}{e^{\beta(\varepsilon-\mu)}} - \frac{(\ell+1)e^{\beta(\mu-\varepsilon)(\ell+1)}}{1-e^{\beta(\mu-\varepsilon)(\ell+1)}}\frac{e^{\beta(\varepsilon-\mu)(\ell+1)}}{e^{\beta(\varepsilon-\mu)(\ell+1)}} \\
&= \frac{e^{\beta(\mu-\varepsilon)}}{1-e^{\beta(\mu-\varepsilon)}}\frac{e^{-\beta(\mu-\varepsilon)}}{e^{\beta(\varepsilon-\mu)}} - \frac{(\ell+1)e^{\beta(\mu-\varepsilon)(\ell+1)}}{1-e^{\beta(\mu-\varepsilon)(\ell+1)}}\frac{e^{-\beta(\mu-\varepsilon)(\ell+1)}}{e^{\beta(\varepsilon-\mu)(\ell+1)}} \\
&= \frac{e^{\beta(\mu-\varepsilon)}e^{-\beta(\mu-\varepsilon)}}{e^{\beta(\varepsilon-\mu)}-e^{\beta(\mu-\varepsilon)}e^{-\beta(\mu-\varepsilon)}} - \frac{(\ell+1)e^{\beta(\mu-\varepsilon)(\ell+1)}e^{-\beta(\mu-\varepsilon)(\ell+1)}}{e^{\beta(\varepsilon-\mu)(\ell+1)}-e^{\beta(\mu-\varepsilon)(\ell+1)}e^{-\beta(\mu-\varepsilon)(\ell+1)}} \\
&= \frac{1}{e^{\beta(\varepsilon-\mu)}-1} - \frac{(\ell+1)}{e^{\beta(\varepsilon-\mu)(\ell+1)}-1} \square
\end{aligned}
$$

2.2 B

Para esto inicemos por hacer $\ell = 1$

$$
\begin{aligned}
\langle n_\varepsilon \rangle &= \frac{1}{e^{\beta(\varepsilon-\mu)}-1} - \frac{(1+1)}{e^{\beta(\varepsilon-\mu)(1+1)}-1} \\
&= \frac{e^{2\beta(\varepsilon-\mu)}-1-\left(2e^{\beta(\varepsilon-\mu)}-2\right)}{\left(e^{\beta(\varepsilon-\mu)}-1\right)\left(e^{2\beta(\varepsilon-\mu)}-1\right)} \\
&= \frac{e^{2\beta(\varepsilon-\mu)}-2e^{\beta(\varepsilon-\mu)}+1}{\left(e^{\beta(\varepsilon-\mu)}-1\right)\left(e^{2\beta(\varepsilon-\mu)}-1\right)} \\
&= \frac{\left(e^{\beta(\varepsilon-\mu)}-1\right)^2}{\left(e^{\beta(\varepsilon-\mu)}-1\right)\left(e^{2\beta(\varepsilon-\mu)}-1^2\right)} \\
&= \frac{\left(e^{\beta(\varepsilon-\mu)}-1\right)^2}{\left(e^{\beta(\varepsilon-\mu)}-1\right)\left(e^{\beta(\varepsilon-\mu)}-1\right)\left(e^{\beta(\varepsilon-\mu)}+1\right)} \\
&= \frac{\left(e^{\beta(\varepsilon-\mu)}-1\right)^2}{\left(e^{\beta(\varepsilon-\mu)}-1\right)^2\left(e^{\beta(\varepsilon-\mu)}+1\right)} \\
&= \frac{1}{e^{\beta(\varepsilon-\mu)}+1} \square
\end{aligned}
$$

2.3 C

Para esto usemos

$$
\lim_{\ell\to\infty} \frac{1}{e^{\beta(\varepsilon-\mu)}-1} - \frac{(\ell+1)}{e^{\beta(\varepsilon-\mu)(\ell+1)}-1}
$$

Lo que podemos mostrar como:

$$
\begin{aligned}
\lim_{\ell\to\infty} \frac{1}{e^{\beta(\varepsilon-\mu)}-1} - \frac{(\ell+1)}{e^{\beta(\varepsilon-\mu)(\ell+1)}-1} &= \frac{1}{e^{\beta(\varepsilon-\mu)}-1} - \lim_{\ell\to\infty} \frac{(\ell+1)}{e^{\beta(\varepsilon-\mu)(\ell+1)}-1} \\
&= \frac{1}{e^{\beta(\varepsilon-\mu)}-1} - \lim_{\ell\to\infty} \frac{(\infty+1)}{e^{\beta(\varepsilon-\mu)(\infty+1)}-1} \\
&= \frac{1}{e^{\beta(\varepsilon-\mu)}-1} - \lim_{\ell\to\infty} \frac{\infty}{e^{\infty}-1} \\
&= \frac{1}{e^{\beta(\varepsilon-\mu)}-1} - 0 \\
&= \frac{1}{e^{\beta(\varepsilon-\mu)}-1} \square
\end{aligned}
$$

Con esto ya mostramos lo que se deseaba

Chapter 3

3.1 A

Con $n_1 = n_2 = n_3 = 0$ se da que $\varepsilon = \frac{1}{2}\hbar(\omega_1 + \omega_2 + \omega_3)$
con esto entonces:

$$\begin{aligned}
g(\varepsilon) &= \int_0^\infty \int_0^\infty \int_0^\infty d(\varepsilon - \hbar\omega_1 n_1 - \hbar\omega_2 n_2 - \hbar\omega_3 n_3) dn_1 dn_2 dn_3 \\
x_i &= \hbar\omega_i n_i \\
dn_i &= \frac{dx_i}{\hbar\omega_i} \\
g(\varepsilon) &= \int_0^\infty \int_0^\infty \int_0^\infty d(\varepsilon - \hbar\omega_1 n_1 - \hbar\omega_2 n_2 - \hbar\omega_3 n_3) \frac{dx_1 dx_2 dx_3}{\hbar^3\omega_1\omega_2\omega_3} \\
g(\varepsilon) &= \int_0^\infty \int_0^\infty \int_0^\infty d(\varepsilon - x_1 - x_2 - x_3) \frac{dx_1 dx_2 dx_3}{\hbar^3\omega_1\omega_2\omega_3} \\
g(\varepsilon) &= \frac{1}{\hbar^3\omega_1\omega_2\omega_3} \int_0^\infty \int_0^\infty \int_0^\infty d(\varepsilon - x_1 - x_2 - x_3) dx_1 dx_2 dx_3 \\
g(\varepsilon) &= \frac{1}{\hbar^3\omega_1\omega_2\omega_3} \frac{\varepsilon^2}{2} \\
\omega_0 &= (\omega_1\omega_2\omega_3)^{\frac{1}{3}} \\
\omega_0^3 &= (\omega_1\omega_2\omega_3) \\
g(\varepsilon) &= \frac{1}{\hbar^3\omega_0^3} \frac{\varepsilon^2}{2} \square
\end{aligned}$$

3.2 B

Para comenzar

$$\Psi = -kT \sum_{\varepsilon_r} \ln(1 - e^{-\beta(\varepsilon_r - \mu)})$$
$$\Psi = -kT \sum_{\varepsilon_r} \ln(1 - e^{-\beta\varepsilon_r} e^{\beta\mu})$$
$$\Psi = -kT \int_0^\infty g(\varepsilon) \ln(1 - e^{-\beta\varepsilon_r} e^{\beta\mu}) d\varepsilon$$
$$\Psi = -kT \int_0^\infty \left(\frac{\varepsilon^2}{2(\hbar\omega_0)^3}\right) \ln(1 - e^{-\beta\varepsilon_r} e^{\beta\mu}) d\varepsilon$$
$$x = \beta\varepsilon$$
$$\varepsilon = \frac{x}{\beta} = k_b T x$$
$$d\varepsilon = \frac{dx}{\beta} = k_b T dx$$
$$\Psi = -\frac{kT}{2(\hbar\omega_0)^3} \int_0^\infty (kT)^2 x^2 \ln(1 - e^{-x} e^{\beta\mu}) kT dx$$
$$\Psi = -\frac{(kT)^4}{2(\hbar\omega_0)^3} \int_0^\infty x^2 \ln(1 - e^{-x} e^{\beta\mu}) dx$$
$$\int_0^\infty x^2 \ln(1 - e^{-x} e^{\beta\mu}) dx = -2g_4(z)$$
$$\Psi = \frac{(kT)^4}{(\hbar\omega_0)^3} g_4(z)$$

3.3 C

El numero promedio es:

$$N(\mu, T) = \left(\frac{\partial\psi}{\partial\mu}\right)_T = \left(\frac{\partial}{\partial\mu} \frac{(kT)^4}{(\hbar\omega_0)^3} g_4(e^{\beta\mu})\right)_T$$

Con lo cual:

$$N = \frac{(kT)^4}{(\hbar\omega_0)^3} \left(\frac{\partial g_4(e^{\beta\mu})}{\partial\mu}\right)_T$$
$$N = \left(\frac{(kT)^4}{(\hbar\omega_0)^3} g_3(e^{\beta\mu})\beta\right)_T$$
$$N = \left(\frac{(kT)^3}{(\hbar\omega_0)^3} g_3(e^{\beta\mu})\right)_T$$

3.4 D

Para un N fijo el potencial quimico es inversamente proporcional a la temperatura hasta llegar a un condensado Bose-Einstein con $\mu = 0$ por lo que:

$$N = \left(\frac{kT}{\hbar\omega_0}\right)^3 g_3(1)$$
$$T_c = \left(\frac{N}{\zeta(3)}\right)^{\frac{1}{3}} \frac{\hbar\omega_0}{k}$$

3.5 E

$$1 = \frac{N_e + N_0}{N}$$
$$\frac{N_0}{N} = 1 - \frac{N_e}{N}$$
$$\frac{N_0}{N} = 1 - \frac{T^3}{T_c^3}$$

3.6 F

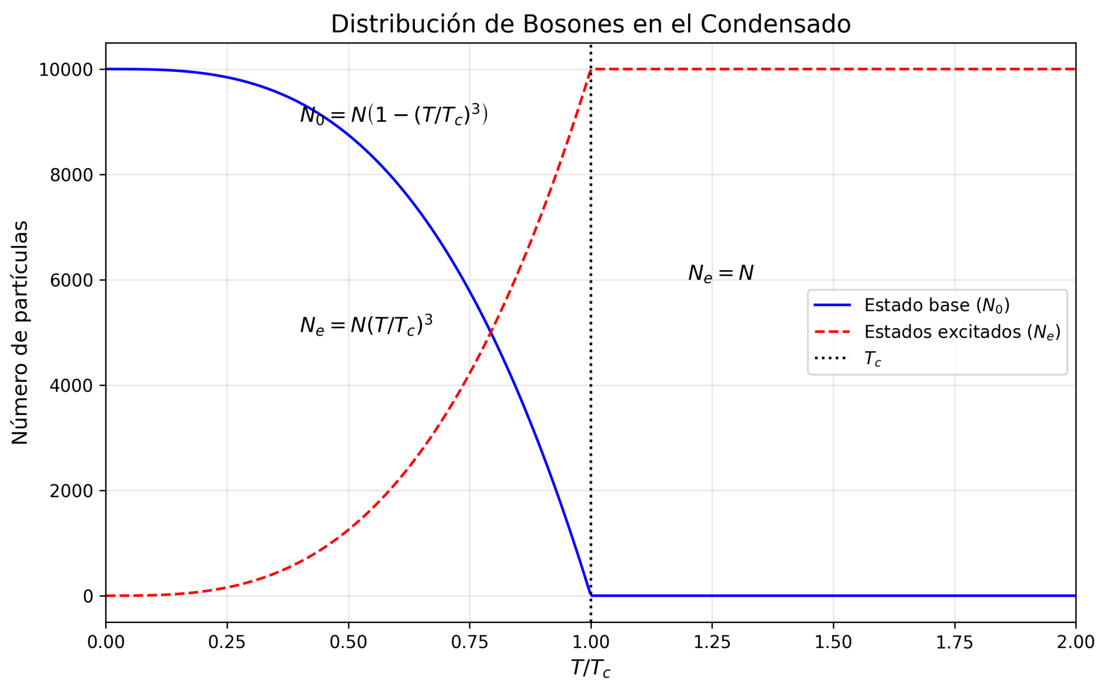

Figure 3.1: Distribución de bosones en el estado base (N_0) y estados excitados (N_e) en función de T/T_c.

Que fue obtenido con el codigo

```
import numpy as np
import matplotlib.pyplot as plt

N = 1e4
Tc = 1

T_ratios = np.linspace(0, 2, 500)

N0 = np.piecewise(T_ratios,
                  [T_ratios < 1, T_ratios >= 1],
                  [lambda x: N*(1 - x**3), 0])

Ne = np.piecewise(T_ratios,
                  [T_ratios < 1, T_ratios >= 1],
                  [lambda x: N*x**3, N])

plt.figure(figsize=(10, 6))
```

```
plt.plot(T_ratios, N0, 'b-', label='Estado base ($N_0$)')
plt.plot(T_ratios, Ne, 'r--', label='Estados excitados ($N_e$)')
plt.axvline(x=1, color='k', linestyle=':', label=r'$T_c$')

plt.xlabel(r'$T/T_c$', fontsize=12)
plt.ylabel('Numero de particulas', fontsize=12)
plt.title('Distribucion de Bosones en el Condensado', fontsize=14)
plt.legend()
plt.grid(alpha=0.3)
plt.xlim(0, 2)
plt.ylim(-0.05*N, 1.05*N)

plt.text(0.4, 0.9*N, r'$N_0 = N\left(1 - (T/T_c)^3\right)$', fontsize=12)
plt.text(1.2, 0.6*N, r'$N_e = N$', fontsize=12)
plt.text(0.4, 0.5*N, r'$N_e = N(T/T_c)^3$', fontsize=12)

plt.savefig('n_0_N.png', dpi=300, bbox_inches='tight')
```

3.7 G

3.8 H

3.8.1 $T \leq T_c$

Para hacer esto podemos partir desde:

$$U = \int_0^\infty \varepsilon g(\varepsilon) \frac{1}{e^{\frac{\varepsilon}{kT}} - 1} d\varepsilon$$

Dado el $g(\varepsilon)$ ya encontrado podemos poner:

$$\begin{aligned}
U &= \int_0^\infty \varepsilon g(\varepsilon) \frac{1}{e^{\frac{\varepsilon}{kT}} - 1} d\varepsilon \\
&= \int_0^\infty \frac{\varepsilon^3}{2(\hbar\omega_0)^3} \frac{1}{e^{\frac{\varepsilon}{kT}} - 1} d\varepsilon \\
&= \frac{1}{2(\hbar\omega_0)^3} \int_0^\infty \frac{\varepsilon^3}{e^{\frac{\varepsilon}{kT}} - 1} d\varepsilon \\
x &= \frac{\varepsilon}{kT} \\
\varepsilon &= xkT \\
d\varepsilon &= kT dx \\
U &= \frac{1}{2(\hbar\omega_0)^3} \int_0^\infty \frac{(xkT)^3}{e^x - 1} kT dx \\
U &= \frac{1}{2(\hbar\omega_0)^3} \int_0^\infty (kT)^4 \frac{x^3}{e^x - 1} dx \\
U &= \frac{(kT)^4}{2(\hbar\omega_0)^3} \int_0^\infty \frac{x^3}{e^x - 1} dx
\end{aligned}$$

Ahora, el resultado de la integral

$$\begin{aligned}
\int_0^\infty \frac{x^3}{e^x - 1} dx &= \Gamma(4)\zeta(3) \\
&= 6\zeta(3)
\end{aligned}$$

Ahora teniendo

$$T_c = \frac{\hbar\omega_0}{k}\left(\frac{N}{\zeta(3)}\right)^{\frac{1}{3}}$$

Con lo cual podemos reorganizarnos

$$\begin{aligned}
T_c &= \frac{\hbar\omega_0}{k}\left(\frac{N}{\zeta(3)}\right)^{1/3} \\
\hbar\omega_0 &= kT_c\left(\frac{N}{\zeta(3)}\right)^{-1/3} \\
(\hbar\omega_0)^3 &= k^3T_c^3\left(\frac{N}{\zeta(3)}\right)^{-1} \\
U &= \frac{3(kT)^4\zeta(4)}{(\hbar\omega_0)^3} \\
U &= \frac{3(kT)^4\zeta(4)}{k^3T_c^3\left(\frac{N}{\zeta(3)}\right)^{-1}} = \frac{3(kT)^4\zeta(4)N}{k^3T_c^3\zeta(3)} \\
U &= \frac{3kT^4\zeta(4)N}{k^3T_c^3\zeta(3)} = \frac{3T^4N\zeta(4)}{k^2T_c^3\zeta(3)} \\
U &= \frac{3T^4\zeta(4)}{k^2T_c^3\zeta(3)}\cdot\zeta(3)\left(\frac{kT_c}{\hbar\omega_0}\right)^3 \\
U &= 3\left(\frac{T}{T_c}\right)^4\frac{\zeta(4)}{\zeta(3)} \\
U(T \leqslant T_c) &= 3\left(\frac{T}{T_c}\right)^4\frac{\zeta(4)}{\zeta(3)}
\end{aligned}$$

3.8.2 $T \geq T_c$

En este caso la diferencia mas relevante es que:

$$U = \int_0^\infty \varepsilon g(\varepsilon)\frac{1}{z^{-1}e^{\frac{\varepsilon}{kT}} - 1}d\varepsilon$$

Lo cual nos permite hacer exactamente el mismo desarrollo que antes cambiando la integral donde:

$$\begin{aligned}
\int_0^\infty \frac{x^3}{z^{-1}e^x - 1}dx &= \Gamma(4)g_4(z) \\
&= 6g_4(z)
\end{aligned}$$

Con esto entonces:

$$\begin{aligned}
U &= \frac{6(kT)^4}{2(\hbar\omega_0)^3}g_4(z) = \frac{3(kT)^4}{(\hbar\omega_0)^3}g_4(z) \\
(\hbar\omega_0)^3 &= \frac{k^3T_c^3\zeta(3)}{N} \\
U &= \frac{3(kT)^4}{\frac{k^3T_c^3\zeta(3)}{N}}g_4(z) = 3\left(\frac{T}{T_c}\right)^4\frac{g_4(z)}{\zeta(3)} \\
U(T \geqslant T_c) &= 3\left(\frac{T}{T_c}\right)^4\frac{g_4(z)}{\zeta(3)}
\end{aligned}$$

3.9 I

3.9.1 $T < T_c$

En este caso tenemos

$$U(T) = 3\left(\frac{T}{T_c}\right)^4 \frac{\zeta(4)}{\zeta(3)} NkT_c$$

Con esto entonces:

$$\begin{aligned} C_V &= \frac{\partial U}{\partial T} \\ &= 12\frac{\zeta(4)}{\zeta(3)} Nk\left(\frac{T}{T_c}\right)^3 \end{aligned}$$

3.9.2 $T < T_c$

La energía incluye la fugacidad z:

$$U(T \geqslant T_c) = 3\left(\frac{T}{T_c}\right)^4 \frac{g_4(z)}{\zeta(3)} NkT_c.$$

Con esto entonces queda:

$$C_V = \frac{\partial U}{\partial T} = 12\frac{g_4(z)}{\zeta(3)} Nk\left(\frac{T}{T_c}\right)^3 - 9\frac{g_3(z)^2}{\zeta(3)g_2(z)} Nk\left(\frac{T}{T_c}\right)^3 \frac{dz}{dT}$$

3.9.3 T_c

Límite $T \to T_c^-$

$$C_V(T_c^-) = 12\frac{\zeta(4)}{\zeta(3)} Nk.$$

Límite $T \to T_c^+$

$$C_V(T_c^+) = 12\frac{\zeta(4)}{\zeta(3)} Nk - 9\frac{\zeta(3)}{\zeta(2)} Nk.$$

Discontinuidad:

$$\Delta C_V = C_V(T_c^+) - C_V(T_c^-) = -9\frac{\zeta(3)}{\zeta(2)} Nk \approx -6.577Nk.$$

3.9.4 Grafico

Este grafico lo podemos hacer con codigo como sigue:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import zeta

# Constantes
zeta3 = zeta(3)
zeta4 = zeta(4)
zeta2 = np.pi**2 / 6

Nk = 1

def cv_low_T(T_ratio):
    return 12 * (zeta4 / zeta3) * Nk * T_ratio**3

def cv_high_T(T_ratio):
    term1 = 12 * (zeta4 / zeta3) * T_ratio**3
```

```python
    term2 = 9 * (zeta3**2 / (zeta2 * zeta3)) * T_ratio**3
    return term1 - term2

def cv_free_boson(T_ratio):
    return 12 * (zeta4 / zeta3) * Nk * T_ratio**3 * 0.8

T_ratios_low = np.linspace(0, 1, 100)
T_ratios_high = np.linspace(1, 2, 100)

cv_low = cv_low_T(T_ratios_low)
cv_high = cv_high_T(T_ratios_high)
cv_free = cv_free_boson(np.concatenate([T_ratios_low, T_ratios_high]))

plt.figure(figsize=(10, 6))

plt.plot(T_ratios_low, cv_low, 'b-', label=r'$T \leq T_c$ (Potencial Armonico)')
plt.plot(T_ratios_high, cv_high, 'r-', label=r'$T \geq T_c$ (Potencial Armonico)')

plt.plot(np.concatenate([T_ratios_low, T_ratios_high]), cv_free, 'g:', label='Gas de 
    Bosones Libres')

plt.plot([1, 1], [cv_low[-1], cv_high[0]], 'ko', markersize=8, markerfacecolor='none')
plt.vlines(1, cv_high[0], cv_low[-1], colors='k', linestyles='dashed')

plt.xlabel(r'$T/T_c$', fontsize=12)
plt.ylabel(r'$C_V / Nk$', fontsize=12)
plt.title('Calor Especifico a Volumen Constante', fontsize=14)
plt.legend()
plt.grid(alpha=0.3)
plt.xlim(0, 2)
plt.ylim(0, 15)

plt.savefig('calor_especifico.png', dpi=300, bbox_inches='tight')
```

Con lo cual producimos esta grafica:

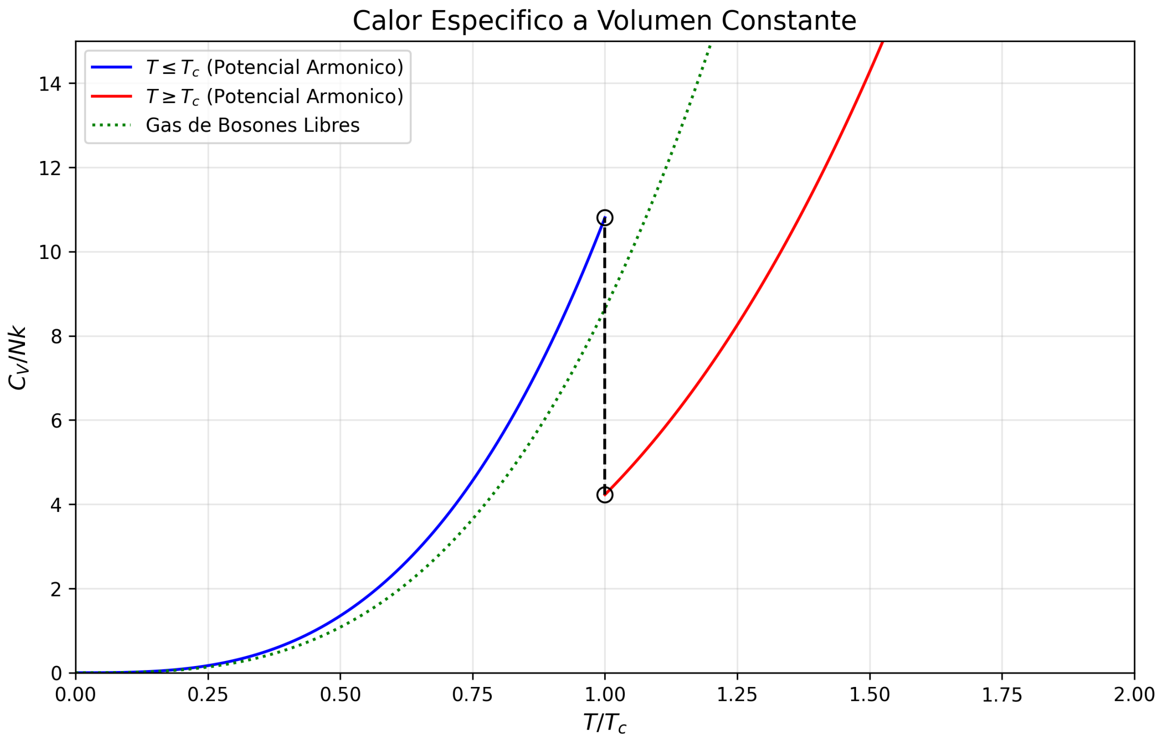

Figure 3.2: Calor específico a volumen constante para el sistema en un potencial armónico y comparación con gas libre.

3.10 J

3.10.1 i

1. $\omega_0 = 2\pi \times 100\,\text{Hz} = 200\pi\,\text{rad/s}$.
2. $\hbar = 1.054 \times 10^{-34}\,\text{J·s}$, $k_B = 1.38 \times 10^{-23}\,\text{J/K}$.
3. $N = 2 \times 10^4$, $\zeta(3) \approx 1.202$.

   Reemplazando queda

$$T_c = \frac{(1.054 \times 10^{-34})(200\pi)}{1.38 \times 10^{-23}} \left(\frac{2 \times 10^4}{1.202}\right)^{1/3} \approx 122\,\text{nK}.$$

   Ahora para λ a T_c Para ^{87}Rb ($m \approx 1.44 \times 10^{-25}\,\text{kg}$):

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{mk_BT_c}} \approx 0.54\,\mu\text{m}.$$

   Para la discontinuidad queda como:

$$\Delta C_V \approx -6.577Nk_B = -6.577 \times (2 \times 10^4) \times 1.38 \times 10^{-23} \approx -1.82 \times 10^{-18}\,\text{J/K}.$$

3.10.2 ii

^{87}Rb

Usando el articulo Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor disponible en Science en https://www.science.org/doi/10.1126/science.269.5221.198

1. $T_c \approx 170\,\mathrm{nK}$.
2. $\lambda \approx 1\,\mu\mathrm{m}$.

^{23}Na

Basandonos en el articulo Bose-Einstein Condensation in a Gas of Sodium Atoms https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.75.3969

1. $T_c \approx 500\,\mathrm{nK}$ (experimentos de Ketterle, 1995).

   Con lo cual nuestros resultados parecen estar apuntando en la dirección correcta.

3.10.3 iii